Lyapunov Stability of Differential Inclusions with Lipschitz Cusco Perturbations of Maximal Monotone Operators

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Abstract

We give new criteria for weak and strong invariant closed sets for differential inclusions in $\mathbb{R}^n$, and which are simultaneously governed by Lipschitz Cusco mapping and by maximal monotone operators. Correspondingly, we provide different characterizations for the associated strong Lyapunov functions and pairs. The resulting conditions only depend on the data of the system, while the invariant sets are assumed to be closed, and the Lyapunov pairs are assumed to be only lower semi-continuous.

1 Introduction

In this paper, we investigate (weak and strong) invariant closed sets $S \subset \mathbb{R}^n$ with respect to the following differential inclusion, given in $\mathbb{R}^n$,

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \text{ a.e. } t \geq 0, \ x(0) = x_0 \in \text{cl}(\text{dom } A), \tag{1}$$

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where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a Lipschitz Cusco multifunction; that is, a Lipschitz continuous set-valued mapping with nonempty, convex and compact values, and $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator. Hence, the values of the right-hand side may by empty or bounded. There is no restriction on the initial condition x_0 that can be any point in the closure of the domain of A, possibly not a point of the domain of definition of A. The sets S are required to satisfy the following condition

$$\Pi_S(x) \subset S \cap \operatorname{dom} A, \ \forall x \in \operatorname{dom} A, \tag{2}$$

where Π_S refers to the projection operator on S. This condition has been used in many works; see, for instance, [10], where the author is concerned with flow invariance characterizations for differential equations, with right-hand-sides given by nonlinear semigroup generators in the sense of Crandall- Liggett (see [20]). It is clear that condition (2) holds whenever $S \subset \operatorname{dom} A$. When dealing with weak invariant closed sets, we shall require some usual boundedness conditions on the invariant set, relying on the minimal norm section of the maximal monotone operator A.

Equivalently, we also characterize (strong) Lyapunov functions and, more generally, a-Lyapunov pairs associated to the differential inclusion above. Our criteria are given by means only of the data of the system, represented by the multifunction F and the operator A, together with first-order approximations of the invariant sets candidates, using Bouligand tangent cones or, equivalently, Fréchet or proximal normal cones, and first-order (general) derivatives of Lyapunov functions candidates, using directional derivatives, Fréchet or proximal subdifferentials.

There is a long and rich history on the invariance and Lyapunov functions theory, which deals with different variants of differential inclusion (1). We refer the reader to [16, 24] and references therein for more details on this subject. In the current paper, our aim is to gather in one framework two different kinds of dynamic systems that were studied separately in the literature, at least in what concerns Lyapunov stability. The first kind of these dynamic systems is governed exclusively by Cusco multifunctions, giving rise to a natural extension of the classical differential equations, stated in the form

$$\dot{x}(t) \in F(x(t)), \ \text{a.e. } t \geq 0, \ x(0) = x_0 \in \mathbb{R}^n. \tag{3}$$

The consideration of differential inclusions rather than differential equations allows more useful existence theorems, as revealed by Filippov's theory for differential equations with discontinuous right-hand-sides [26]. Stability of such systems; namely, the study of Lyapunov functions and invariant sets, has been extensively studied and investigated especially during the nineties by many authors; see, for example, [16, 17, 24] , as well as [7, 8, 27] (see, also, the references therein). For instance, complete characterizations for invariant closed sets for (3) can be found in [16] in the finite-dimensional setting, and in [17] for Hilbert spaces. It is worth recalling that only the upper semicontinuity of the Cusco mapping F is required for the weak invariance, while Lipschitz continuity is used for the strong invariance (see [17]). The results in [16] also have been adapted in [19] to the following more general differential inclusion (for $T \in [0, +\infty]$)

$$\dot{x}(t) \in F(t, x(t)) - \mathrm{N}_{C(t)}(x(t)), \ \text{a.e. } t \in [0, T], \ x(0) = x_0 \in C(0), \tag{4}$$

where $C(t)$ is a uniformly prox-regular sets in $\mathbb{R}^n$ and $\mathrm{N}_{C(t)}$ is the associated normal cone.

The other kind of systems that is covered by (1) concerns differential inclusions governed by maximal monotone operators or, more generally, by (single-valued) Lipschitz continuous perturbations of maximal monotone operators, written as

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \text{cl}(\text{dom } A). \tag{5}$$

This system can be seen as perturbations of the ordinary differential equation $\dot{x}(t) = f(x(t))$, where A could represent some associated control action. In this single-valued and Lipschitz continuous setting, weak and strong invariance coincide since differential inclusion (1) possesses a unique solution. Compared to (3), the right-hand-side in (5) can be unbounded, or even empty. A typical example of (5) occurs when A is the Fenchel subdifferential of a proper, lower semicontinuous convex function (see [1]). System (5) has been extensively studied; namely, regarding existence, regularity and properties of the solutions [11], while Lyapunov stability of such systems has been initiated in [31]; see, also, [2, 4, 5] for recent contributions on the subject. Different criteria using the semi-group generated by the operator A can also be found in [30], where Lyapunov functions are characterized as viscosity-type solutions of Hamilton-Jacobi equations, and in [12], using implicit tangent cones associated to the invariant sets candidates.

It is worth observing that (1) is a special case of the following more general differential inclusion

$$\dot{x}(t) \in F(t, x(t)) - A(t)(x(t)), \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \text{cl}(\text{dom } A(0)(\cdot)), \tag{6}$$

where A and F are also allowed to move in an appropriate way with respect to the time variable, and satisfy some natural continuity and measurability conditions. Existence of solution of (6) have been also studied in [6, 34] among others. In particular, [6] considers similar systems as the one in (6), with A being independent of t, and requiring strong assumptions on the multifunction F. In [34] the authors assume that F is a single-valued mapping, Lipschitz continuous with respect to the second variable, while the minimal section mapping of the maximal monotone operators $A(t)$ is assumed uniformly bounded.

In [23–25], the authors provide many characterizations for weak and strong invariant sets, associated to the following differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad \text{a.e. } t \geq 0,$$

for a multivalued-mapping F with the so-called one-sided Lipschitz condition ([21]; see also [22], and [33] for other extensions). The standard hypotheses in these papers require that the mapping F has non-empty and compact values. In our case, as a sum of a Lipschitz continuous mapping and a monotone operator, the right-hand side in (1) also defines a one-sided Lipschitz mapping, but, due to the general nature of the maximal monotone operator A, it may have empty and unbounded values.

Other interesting criteria for weak invariance results are obtained in [13] for (1) in the setting of Banach spaces, and by assuming that F is only upper semi-continuous. The criteria used in the last work are given by means of the so-called A-quasi-tangents that involve the semigroup generated by the operator A.

The main (strong-) invariance result of this work is given in Theorem 2, where we prove that a closed set S is strong invariant for (1) if and only if

$$(v - A(x)) \cap \text{T}_S(x) \neq \emptyset, \quad \forall v \in F(x), \forall x \in S \cap \text{dom } A,$$

if and only if

$$\sup_{\xi \in \text{N}_S(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0, \quad \forall x \in S \cap \text{dom } A.$$

Therefore, only the data of (1) are evoked for the characterization of strong invariant sets, namely, the operator A and the mapping F, together with the geometric structure of the set S, which is modeled by the normal and the tangent cones. In particular, when $A \equiv 0$, the above characterizations reduce to the ones obtained in [16], and when F is single-valued, we recover the results of [4] and [5] (at least in the finite-dimensional setting). The characterization of weak invariant sets, which is given in Theorem 3, is also new.

The above invariance results are translated into characterizations of Lyapunov functions and pairs in Theorem 4, where, among many characterizations, we prove that a given pair of lower semi-continuous functions V, W, is a strong Lyapunov pair for (1) if and only if, for all $x \in \operatorname{dom} V$,

$$\sup_{\xi \in \partial_P V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle + W(x) \le 0 \tag{7}$$

and

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \le 0.$$

The last condition is superfluous when the operator A is locally bounded, as we show in Corollary 1.

Consequently, for being small, compared to the Fréchet or the Mordukhovich subdifferentials (or even the gradient for differentiable functions), the proximal subdifferential provides the most sharp characterization for Lyapunov pairs. However, for differentiable functions, and despite that the inclusion $\partial_P V(x) \subset \{\nabla V(x)\}$ could be strict, it can be deduced from Corollary 1 that property (7) alone suffices.

The consideration of nonsmooth Lyapunov functions would permit more flexibility in the choice of Lyapunov functions, as the following example shows.

Example 1 In (1), we put $A \equiv 0$ and let F be any multivalued mapping such that (1) admits solutions. We are going to show that for any Lipschitz continuous function $W : \mathbb{R}^n \to \mathbb{R}_+$ and any $a \ge 0$, there exists a function V such that (V, W) is a weak a-Lyapunov pair; that is, for every $x \in \mathbb{R}^n$ there is a solution $x(\cdot; x)$ such that

$$e^{at} V(x(t; x)) + \int_0^t W(x(s; x))ds \le V(x), \quad \forall t \ge 0; \tag{8}$$

consequently, the function V is a (weak) Lyapunov function.

For this aim, we define the value function V on $\mathbb{R}^n$ as

$$V(x) =: \inf \left\{ \int_0^{+\infty} e^{at} W(x(t))dt \ : \ \dot{x}(t) \in F(x(t)), \ x(0) = x \right\}.$$

Observe that for $F(x) = -2x$, $a = 1$ and $W(x) = \|x\|$, the solution of (1) is given by $x(t; x) = e^{-2t}x$, so that

$$V(x) = \int_0^{+\infty} e^t e^{-2t} dt \|x\| = \|x\|, \tag{9}$$

and V is not differentiable at 0 in the case of the Euclidean norm (or at every point having some zero components in the case of l_1 norm).

Next, we assume that the infimum above is always attained. Given $x \in \mathbb{R}^n$ and $t \geq 0$, we choose a solution $x(\cdot; x)$ for $V(x)$. Then, since $x(\cdot + t; x)$ is feasible for $V(x(t; x))$,

$$V(x(t; x)) \leq \int_0^{+\infty} e^{as} W(x(s + t; x)) ds = e^{-at} \int_t^{+\infty} e^{as} W(x(s; x)) ds$$

$$\leq e^{-at} V(x) - e^{-at} \int_0^t e^{as} W(x(s; x)) ds,$$

and (8) follows.

Example 2 (Nonholonomic integrator; see [15]) The following system, given in $\mathbb{R}^3$ as

$$\dot{x}(t) := (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_2(t)) \in F(x(t)) := \{(u, v, x_1(t)v - x_2(t)u) : u^2 + v^2 \leq 1\},$$

does not admit a smooth Lyapunov function.

The paper is organized as follows: After Section 2, reserved to give the necessary notation and present the main tools, we make in Section 3 a review of the existence theorems of differential inclusion (1), and establish some first properties of the solutions. In Section 4 we characterize weak and strong invariant closed sets with respect to (1), while in Section 5, criteria for strong Lyapunov pairs are provided.

2 Notation and Main Tools

In this paper, the notations $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the inner product and the norm in $\mathbb{R}^n$, respectively. For each $x \in \mathbb{R}^n$ and $\rho \geq 0$, $\mathrm{B}(x, \rho)$ is the closed ball with center x and radius ρ; in particular, we denote $\mathrm{B}_r := \mathrm{B}(\theta, r)$ where θ is the origin vector in $\mathbb{R}^n$. Given a nonempty set $S \subset \mathbb{R}^n$, we denote by $\mathrm{cl}(S)$, $\mathrm{int}(S)$, $\mathrm{co}\, S$, $\mathrm{conv}\, S$ and $\mathrm{cl(conv)}S$ the closure, the interior, the convex hull and the closed convex hull of S, respectively. We denote by $\|S\|$ the nonnegative real number define by

$$\|S\| := \sup\{\|v\| : v \in S\}.$$

The *projection mapping* onto S is defined as

$$\Pi_S(x) := \{s \in S : \|x - s\| = d_S(x)\},$$

where $d_S(x) := \inf\{\|x - s\| : s \in S\}$ is the *distance function* to S. If S is a closed set, then $\Pi_S(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$. We denote by $S^\circ := \Pi_S(\theta)$ the minimal norm vector in S. The *indicator function* of S is defined as

$$\mathrm{I}_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S, \end{cases}$$

and the *support function* of S is defined as

$$\sigma_S(x) := \sup\{\langle x, s \rangle : s \in S\},$$

with the convention that $\sigma_\emptyset = -\infty$. Given a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its *domain* and *epigraph* are defined by

$$\mathrm{dom}\, \varphi := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\};$$

$$\mathrm{epi}\, \varphi := \{(x, \alpha) \in \mathbb{R}^{n+1} : \varphi(x) \leq \alpha\}.$$

We say φ is proper if $\mathrm{dom}\, \varphi \neq \emptyset$; lower semicontinuous (lsc for short) if $\mathrm{epi}\, \varphi$ is closed. We denote by $\mathcal{F}(\mathbb{R}^n)$ the set all proper and lsc functions.

Next, we introduce some basic concepts of nonsmooth and variational analysis. Let $\varphi \in \mathcal{F}(\mathbb{R}^n)$ and $x \in \operatorname{dom} \varphi$. We call $\xi \in \mathbb{R}^n$ a *proximal subgradient* of φ at x, written $\xi \in \partial_P \varphi(x)$, if

$$\liminf_{y \to x, y \neq x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|^2} > -\infty.$$

A vector $\xi \in \mathbb{R}^n$ is said to be a *Fréchet subgradient* of φ at x, written $\xi \in \partial_F \varphi(x)$ if

$$\liminf_{y \to x, y \neq x} \frac{\varphi(y) - \varphi(x) - \langle \xi, y - x \rangle}{\|y - x\|} \geq 0.$$

The *Mordukhovich* (or *limiting*) *subdifferential* of φ at x is defined as

$$\partial_L \varphi(x) := \left\{ \lim_{n \to \infty} \xi_n : \xi_n \in \partial_P \varphi(x_n), x_n \to x, \varphi(x_n) \to \varphi(x) \right\},$$

and the singular subdifferential of φ at x as

$$\partial_\infty \varphi(x) := \left\{ \lim_{n \to \infty} \alpha_n \xi_n : \xi_n \in \partial_P \varphi(x_n), x_n \to x, f(x_n) \to f(x), \alpha_n \downarrow 0 \right\}.$$

The *Clarke* (or *generalized*) *subdifferential* of φ at x is

$$\partial_C \varphi(x) := \operatorname{cl}\{\operatorname{co}\{\partial_L \varphi(x) + \partial_\infty \varphi(x)\}\}.$$

In the case $x \notin \operatorname{dom} \varphi$, by convention we set $\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \emptyset$. We have the classical inclusions $\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x)$. If φ is locally Lipschitz continuous around x, then $\partial_\infty \varphi(x) = \{\theta\}$ and

$$\partial_C \varphi(x) = \operatorname{conv} \partial_L \varphi(x).$$

The *generalized directional derivative* of φ at x in the direction v is defined by

$$\varphi^0(x; v) := \limsup_{y \to x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

We have that

$$\varphi^0(x; v) = \sup_{\xi \in \partial_C \varphi(x)} \langle \xi, v \rangle, \quad \forall v \in \mathbb{R}^n.$$

We also remind the *lower Dini* (or *contingent*) *directional derivative* of φ at $x \in \operatorname{dom} \varphi$ in the direction $v \in \mathbb{R}^n$, which is given by

$$\varphi'(x; v) := \liminf_{t \to 0^+, w \to v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

From the definition of the proximal and the Fréchet subdifferentials, it is easy to prove that

$$\sigma_{\partial_P \varphi(x)}(\cdot) \leq \sigma_{\partial_F \varphi(x)}(\cdot) \leq \varphi'(x; \cdot), \quad \forall x \in \operatorname{dom} \varphi. \tag{10}$$

The *proximal*, the *Fréchet*, and the *Mordukhovich normal cones* are defined, respectively, by

$$N_S^P(x) := \partial_P I_S(x), \ N_S^F(x) := \partial_F I_S(x), \ N_S^L(x) := \partial_L I_S(x). \tag{11}$$

We also define the *prox-singular subdifferential* $\partial_{P,\infty} \varphi(x)$ of φ at x as those elements $\xi \in \mathbb{R}^n$ such that

$$(\xi, 0) \in N_{\operatorname{epi} \varphi}^P(x, \varphi(x)).$$

The *Bouligand tangent* (or *contingent*) *cone* to S at x is defined as

$$T_S^B(x) := \left\{ v \in H : \exists \ x_k \in S, \exists \ t_k \to 0, \ \text{s.t.} \ t_k^{-1}(x_k - x) \to v \text{ as } k \to +\infty \right\}.$$

Next we recall some basic concepts and properties of maximal monotone operators. For a multivalued operator $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the *domain* and the *graph* are given, respectively, by

$$\text{dom } A := \{x \in \mathbb{R}^n : A(x) \neq \emptyset\}, \ \text{graph } A := \{(x, y) : y \in A(x)\};$$

to simplify, we may identify A to its graph. The operator A is said to be *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \text{ for all } (x_i, y_i) \in \text{graph } A, i = 1, 2.$$

If, in addition, A is not properly included in any other monotone operator, then A is said to be *maximal monotone*. In this case, for any $x \in \text{dom } A$, $A(x)$ is closed and convex; hence, $(A(x))^\circ$ is singleton. By the maximal property, if a sequence $(x_n, y_n)_n \subset A$ is such that $(x_n, y_n) \to (x, y)$ as $n \to +\infty$, then $(x, y) \in A$.

Take $f \in L^1(0, T; \mathbb{R}^n)$ for $T > 0$. The differential inclusion given in $\mathbb{R}^n$ as

$$\dot{x}(t) \in f(t) - A(x(t)), \text{ a.e. } t \in [0, T], x(0) = x_0 \in \text{cl}(\text{dom } A),$$

always has a unique solution $x(\cdot) := x(\cdot; x_0)$ (see [11]), that satisfies for a.e. $t \in [0, T]$

$$\frac{d^+x(t)}{dt} := \lim_{t' \downarrow t} \frac{x(t') - x(t)}{t' - t} = f(t^+) - \Pi_{A(x(t))}(f(t+0)),$$

where $f(t^+) := \lim_{h \to 0, h \neq 0} \frac{1}{h} \int_t^{t+h} f(\tau)d\tau$.

Finally, we recall Gronwall's Lemma

Lemma 1 *(Gronwall's Lemma [2]) Let $T > 0$ and $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ such that $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. If, for some $0 \leq \alpha < 1$, an absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfies*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \text{ a.e. } t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau}b(s)ds, \ \forall t \in [t_0, t_0 + T].$$

3 Solutions of the System

In this section, we investigate and review some properties of the solutions of differential inclusion (1), that is given by

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \text{ a.e. } t \geq 0, x(0) = x_0 \in \text{cl}(\text{dom } A),$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping.

Definition 1 A continuous function $x : [0, \infty) \to \mathbb{R}^n$ is said to be a solution of (1) if it is absolutely continuous on every compact subset of $(0, +\infty)$ and satisfies

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \text{ a.e. } t \geq 0, x(0) = x_0 \in \text{cl}(\text{dom } A).$$

The following characterization will be useful in the sequel.

Proposition 1 *A continuous function $x : [0, \infty) \to \mathbb{R}^n$ is a solution of (1) iff $x(\cdot)$ is absolutely continuous on every compact subset of $(0, +\infty)$, and for every $T > 0$*

there exists a function $f \in L^\infty(0, T; \mathbb{R}^n)$ *with* $f(t) \in F(x(t))$ *a.e.* $t \in [0, T]$, *such that*

$$\dot{x}(t) \in f(t) - A(x(t)), \quad \text{a.e. } t \in [0, T], \; x(0) = x_0 \in \text{cl}(\text{dom } A). \tag{12}$$

Proof The sufficient condition is clear and, so, we only need to justify the necessary part. Suppose that $x(\cdot)$ is any solution of (1) and fix $T > 0$. Since F is Lipschitz and $x(\cdot)$ is continuous, there exists $m > 0$ such that $F(x(t)) \subset B_m$ for all $t \in [0, T]$. We define the set-valued mapping $G : [0, T] \rightrightarrows \mathbb{R}^n$ as

$$G(t) := [\dot{x}(t) + A(x(t))] \cap F(x(t)) = \Big[[\dot{x}(t) + A(x(t))] \cap B_m \Big] \cap F(x(t)).$$

We are going to check that G is measurable. Since operator A is maximal monotone, the mappings

$$x \mapsto A_n(x) := A(x) \cap B_n, \; n \geq 1,$$

are upper semi-continuous, and so are the mappings

$$t \mapsto A_n(x(t)) := A(x(t)) \cap B_n, \; n \geq 1,$$

due to the continuity of the solution $x(\cdot)$. Then, due to the relation $A(x(t)) = \cup_{n \in \mathbb{N}} A_n(x(t))$, we deduce that the multifunction $t \mapsto A(x(t))$ is measurable. Since $\dot{x}(t) = \lim_{n \to +\infty} n(x(t + \frac{1}{n}) - x(t))$ for a.e. $t \in [0, T]$, $\dot{x}(\cdot)$ is measurable, and we deduce that the multifunction $t \mapsto [\dot{x}(t) + A(x(t))] \cap B_m$ is measurable. Similarly, the multifunction $t \mapsto F(x(t))$ is measurable. Consequently, according to [14, Proposition III.4], the mapping G is measurable, and we conclude from [14, Theorem III.6] that G admits a measurable selection; i.e., a measurable function $f : [0, T] \to \mathbb{R}^n$ such that

$$f(t) \in G(t) = [\dot{x}(t) + A(x(t))] \cap B_m \cap F(x(t)) \subset F(x(t)), \quad \text{a.e. } t \in [0, T].$$

Hence, $\dot{x}(t) \in f(t) - A(x(t))$ and $\|f(t)\| \leq \|F(x(t))\| \leq m$, so that $f \in L^\infty(0, T; \mathbb{R}^n)$. $\qquad\square$

The next theorem shows that differential inclusion (1) has at least one solution whenever $x_0 \in \text{cl}(\text{dom } A)$. We use the following lemma, which is a particular case of [7, Theorem A].

Lemma 2 *Let* $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a Lipschitz multifunction with nonempty, convex and compact values, and let* $x \in \mathbb{R}^n$, $v \in G(x)$. *Then there exists a Lipschitz continuous selection* f *of* G *such that* $f(x) = v$.

Theorem 1 *Differential inclusion* (1) *has at least one solution.*

Proof Fix $x_0 \in \text{cl}(\text{dom } A)$ and, according to Lemma 2 , let f be a Lipschitz continuous selection of F. Then the differential inclusion

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \; x(0) = x_0,$$

admits a unique solution $x(\cdot)$, which is absolutely continuous on every compact subset of $(0, +\infty)$ (see e.g. [9, 11]). It follows that $x(\cdot)$ is also a solution of differential inclusion (1). $\qquad\square$

We also give some further properties of the solutions of differential inclusion (1), which will be used in the sequel. Given a set $S \subset H$ and $x \in \text{dom } A$ we denote

$$(S - A(x))^\circ := \bigcup_{s \in S} (s - A(x))^\circ = \{s - \Pi_{A(x)}(s) : s \in S\}.$$

Proposition 2 *Fix $x_0 \in \text{cl}(\text{dom } A)$ and let $x(\cdot) := x(\cdot; x_0)$ be any solution of (1). Then the following assertions hold:*

(i)   *$x(t) \in \text{dom } A$, for every $t > 0$, and for a.e. $t \geq 0$*

$$\frac{d^+ x(t)}{dt} := \lim_{h \downarrow 0} \frac{x(t + h) - x(t)}{h} \in (F(x(t)) - A(x(t)))^\circ .$$

  Conversely, if $x_0 \in \text{dom } A$, then for any $v \in [F(x_0) - A(x_0)]^\circ$ there exists a solution $y(\cdot)$ of (1) such that

$$y(0) = x_0, \quad \frac{d^+ y(0)}{dt} = v.$$

(ii)   *There exists a real number $c > 0$ such that for any $x_0 \in \text{dom } A$ and any solutions $x(\cdot) := x(\cdot; x_0)$ and $y(\cdot) := y(\cdot; x_0)$ of (1), one has for all $t \geq 0$*

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct},$$

$$\|x(t) - y(t)\| \leq 4(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct}.$$

Consequently, for every $T > 0$ there exists $\rho > 0$ such that

$$x(t) \in \text{B}(x_0, \rho), \quad \forall t \in [0, T].$$

Proof (i) According to Proposition 1, for each $T > 0$ there exists some $f \in L^\infty(0, T; \mathbb{R}^n)$ with $f(t) \in F(x(t))$ a.e. $t \in [0, T]$, such that $x(\cdot)$ is the unique solution of (12); hence, by [11] we deduce that $x(\cdot)$ satisfies $x(t) \in \text{dom } A$ for all $t \in (0, T)$, and

$$\frac{d^+ x(t)}{dt} = \left(f(t^+) - A(x(t)) \right)^\circ, \quad \text{a.e. } t \in (0, T), \tag{13}$$

where $f(t^+) := \lim_{h \to 0} h^{-1} \int_0^h f(t + \tau) d\tau$. Moreover, given $\varepsilon > 0$ there exists some $h > 0$ such that for a.e. $\tau \in (0, h)$ we have

$$f(t + \tau) \in F(x(t + \tau)) \subset F(x(t)) + L \|x(t + \tau) - x(t)\| \mathbb{B} \subset F(x(t)) + \varepsilon L \mathbb{B},$$

and so $\lim_{h \to 0^+} \frac{1}{h} \int_0^h f(t + \tau) d\tau \in F(x(t)) + \varepsilon L \mathbb{B}$ (this last set is convex and closed). Hence, as ε goes to 0 we get $f(t^+) \in F(x(t))$, and (i) follows from (13).

Conversely, we assume that $x_0 \in \text{dom } A$ and take $v \in [F(x_0) - A(x_0)]^\circ$. We choose $w \in F(x_0)$ such that $v = w - \Pi_{A(x_0)}(w)$. According to Lemma 2, there exists a Lipschitz continuous selection f of F such that $f(x_0) = w$. Then the unique solution $y(\cdot)$ of the following differential inclusion

$$\dot{y}(t) \in f(y(t)) - A(y(t)), \quad y(0) = x_0,$$

satisfies

$$\frac{d^+ y(0)}{dt} = f(x_0) - \Pi_{A(x_0)}(f(x_0)) = w - \Pi_{A(x_0)}(w),$$

and the proof of (i) is complete.

(ii) Let $x(\cdot)$ be a solution of differential inclusion (1), with $x(0) = x_0$, and fix $T > 0$. Then

by Proposition 1 there exist functions $k, g \in L^1(0, T; \mathbb{R}^n)$ such that $k(t) \in F(x(t))$, $g(t) \in A(x(t))$, and

$$\dot{x}(t) = k(t) - g(t), \quad \text{a.e. } t \in [0, T].$$

We also choose by Lemma 2 a Lipschitz continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, with Lipschitz constant c $(c \geq L)$, and consider the unique solution $z(\cdot)$ of the differential inclusion

$$\dot{z}(t) \in f(z(t)) - A(z(t)), \quad \text{a.e. } t \geq 0, \ z(0) = x_0.$$

So, for any $t \geq 0$ one has $\left\| \frac{d^+ z(t)}{dt} \right\| \leq e^{ct} \left\| \frac{d^+ z(0)}{dt} \right\|$ and

$$\left\| \frac{d^+ z(0)}{dt} \right\| = \left\| (f(x_0) - A(x_0))^\circ \right\| \leq \|F(x_0)\| + \left\| A^\circ(x_0) \right\|,$$

so that

$$\|z(t) - x_0\| \leq \int_0^t e^{c\tau} \left\| \frac{d^+ z(0)}{dt} \right\| d\tau = \frac{e^{ct} - 1}{c} \left\| \frac{d^+ z(0)}{dt} \right\|$$

$$\leq \frac{e^{ct} - 1}{c} (\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) \tag{14}$$

$$\leq t e^{ct} (\|F(x_0)\| + \left\| A^\circ(x_0) \right\|). \tag{15}$$

By the Lipschitz continuity of F we choose a function $w(\cdot) : [0, T] \to \mathbb{R}^n$ such that

$$w(t) \in F(z(t)), \quad \|k(t) - w(t)\| \leq L \|x(t) - z(t)\|, \ \forall t \in [0, T]. \tag{16}$$

Then we obtain

$$\langle \dot{x}(t) - \dot{z}(t), x(t) - z(t) \rangle$$
$$= \left\langle k(t) - g(t) - f(z(t)) + \Pi_{A(z(t))}(f(z(t))), x(t) - z(t) \right\rangle$$
$$= \langle k(t) - f(z(t)), x(t) - z(t) \rangle + \underbrace{\left\langle -g(t) + \Pi_{A(z(t))}(f(z(t))), x(t) - z(t) \right\rangle}_{\leq 0, \text{ by the monotonicity of } A}$$
$$\leq \langle k(t) - w(t), x(t) - z(t) \rangle + \langle w(t) - f(z(t)), x(t) - z(t) \rangle$$
$$\leq L \|x(t) - z(t)\|^2 + 2 \|F(z(t))\| \|x(t) - z(t)\| \quad \text{(by (16))}$$
$$\leq L \|x(t) - z(t)\|^2 + 2 \big(\|F(x_0)\| + L \|z(t) - x_0\| \big) \|x(t) - z(t)\|$$
$$\leq L \|x(t) - z(t)\|^2 + 2 \Big(\|F(x_0)\| + (e^{ct} - 1)(\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) \Big) \|x(t) - z(t)\|$$
$$\leq L \|x(t) - z(t)\|^2 + 2(\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) e^{ct} \|x(t) - z(t)\|.$$

Consequential, from the Gronwall Lemma we get, for every $t \geq 0$,

$$\|x(t) - z(t)\| \leq 2(\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) t e^{ct},$$

which together with (15) give us

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) t e^{ct},$$

and, for every other solution $y = y(\cdot; x_0)$,

$$\|x(t) - y(t)\| \leq \|x(t) - z(t)\| + \|y(t) - z(t)\| \leq 4(\|F(x_0)\| + \left\| A^\circ(x_0) \right\|) t e^{ct};$$

that is the conclusion of (ii) follows. $\qquad\square$

4 Strong and Weak Invariant Sets

In this section, we give explicit characterizations for a closed set $S \subset \mathbb{R}^n$ to be strong or weak invariant for differential inclusion (1),

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \ x(0) = x_0 \in \text{cl}(\text{dom } A),$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping. Invariance criteria are written exclusively by means of the data; that is, multifunction F and operator A, and involve the geometry of the set S, using the associated proximal and Fréchet normal cones.

Definition 2 Let S be a closed subset of $\mathbb{R}^n$.

(i) S is said to be strong invariant if for any $x_0 \in S \cap \text{cl}(\text{dom } A)$ and any solution $x(\cdot; x_0)$ of (1), we have

$$x(t; x_0) \in S, \quad \forall t \geq 0.$$

(ii) S is said to be weak invariant if for any $x_0 \in S \cap \text{cl}(\text{dom } A)$, there exists at least one solution $x(\cdot; x_0)$ of (1) such that

$$x(t; x_0) \in S, \quad \forall t \geq 0.$$

Since any solution of differential inclusion (1) lives in $\text{cl}(\text{dom } A)$ (Proposition 2), we may assume without loss of generality that S is a closed subset of $\text{cl}(\text{dom } A)$. We shall need the following two lemmas.

Lemma 3 *(e.g. [4, Lemma A.1]) Let $S \subset \mathbb{R}^n$ be closed. Then for every $x \in \mathbb{R}^n \setminus S$ we have*

$$\partial_L d_S(\cdot)(x) \in \left\{ \frac{x - \Pi_S(x)}{d_S(x)} \right\} \text{ and } \partial_C d_S(\cdot)(x) \in \text{conv} \left\{ \frac{x - \Pi_S(x)}{d_S(x)} \right\}.$$

Lemma 4 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an l-Lipschitz continuous function. Then for every $x \in \mathbb{R}^n$ we have*

$$\varphi(x + v) \leq \varphi(x) + \varphi^0(x; v) + o(\|v\|), \ v \in \mathbb{R}^n.$$

Proof We proceed by contradiction and suppose that for some $\alpha > 0$ and sequence $(v_n)_n \subset \mathbb{R}^n \setminus \{\theta\}$ converging to θ it holds

$$\varphi(x + v_n) - \varphi(x) > \varphi^0(x; v_n) + \alpha \|v_n\| \quad \text{for all } n \geq 1. \tag{17}$$

Without loss of generality, we can assume that $\frac{v_n}{\|v_n\|} \to v \neq \theta$. Then

$$\begin{aligned} \varphi(x + v_n) - \varphi(x) &= \varphi\big(x + v_n - \|v_n\| v + \|v_n\| v\big) - \varphi(x + v_n - \|v_n\| v) \\ &\quad + \varphi(x + v_n - \|v_n\| v) - \varphi(x) \\ &\leq \varphi\big(x + v_n - \|v_n\| v + \|v_n\| v\big) - \varphi(x + v_n - \|v_n\| v) \\ &\quad + l \left\| (v_n - \|v_n\| v) \right\|. \end{aligned}$$

Hence, from inequality (17) one gets

$$\frac{\varphi\big(x + v_n\big) - \varphi(x + v_n - \|v_n\| v)}{\|v_n\|} + l \left\| \frac{v_n}{\|v_n\|} - v \right\| \geq \varphi^0 \left(x; \frac{v_n}{\|v_n\|} \right) + \alpha,$$

which as $n \to \infty$ leads us to the contradiction $\varphi^0(x; v) \geq \varphi^0(x; v) + \alpha > \varphi^0(x; v)$.   □

Before we state the main strong invariance theorem we give the following result:

Proposition 3 *Let $S \subset \mathrm{cl}(\mathrm{dom}\, A)$ satisfy condition (2), and take $x_0 \in S$. If there is some $\rho > 0$ such that for any $x \in \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}\, A$,*

$$\sup_{\xi \in N_S^P(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0, \tag{18}$$

then given any solution $x(\cdot; x_0)$ of (1), there exists $T > 0$ such that $x(t; x_0) \in S$ for every $t \in [0, T]$.

Proof Let $x(\cdot) := x(\cdot; x_0)$ be any solution of differential inclusion (1), so that for some $T_1 > 0$ we have

$$x(t) \in \mathrm{B}\left(x_0, \frac{\rho}{3}\right) \cap \mathrm{dom}\, A, \quad \text{a.e. } t \in [0, T_1], \tag{19}$$

where $\rho > 0$ is as in the current assumption, and so (by condition (2))

$$\Pi_S(x(t)) \subset \mathrm{B}\left(x_0, \frac{2}{3}\rho\right) \cap S \cap \mathrm{dom}\, A \subset \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}\, A \quad \text{for a.e. } t \in (0, T_1]. \tag{20}$$

We denote the function $\eta : [0, T_1] \to \mathbb{R}$ as

$$\eta(t) := d_S^2(x(t)).$$

Fix $\varepsilon > 0$. Since the function $d_S^2(\cdot)$ is Lipschitz continuous on each bounded set and $x(\cdot)$ is absolutely continuous on $[\varepsilon, T_1]$, function η is also absolutely continuous on $[\varepsilon, T_1]$; hence, differentiable on a set $T_0 \subset [\varepsilon, T_1]$ of full measure (we may also suppose that (20) holds for all $t \in T_0$). We pick $t \in T_0$ so that, according to Lemma 4, for all $s > 0$

$$\begin{aligned}
d_S^2(x(t + s)) &= d_S^2(x(t) + \dot{x}(t)s + o(s)) \\
&\leq d_S^2(x(t) + \dot{x}(t)s) + o(s) \\
&\leq \left(d_S(x(t)) + s d_S^0(x(t); \dot{x}(t)) + o(s)\right)^2 + o(s) \\
&\leq d_S^2(x(t)) + 2 d_S(x(t)) d_S^0(x(t); \dot{x}(t)s) + o(s), \tag{21}
\end{aligned}$$

While by Lemma 3 we have

$$\begin{aligned}
d_S(x(t)) d_S^0(x(t); \dot{x}(t)) &= d_S(x(t)) \max_{\xi \in \partial_C d(x(t))} \langle \xi, \dot{x}(t) \rangle \tag{22} \\
&\leq \max_{u \in \Pi_S(x(t))} \langle x(t) - u, \dot{x}(t) \rangle.
\end{aligned}$$

Let us write $\dot{x}(t)$ as $\dot{x}(t) = v - w$ for some $v \in F(x(t))$ and $w \in A(x(t))$, and fix $u \in \Pi_S(x(t))$ ($\subset \mathrm{B}(x_0, \rho) \cap S \cap \mathrm{dom}\, A$ by (20)). By the Lipschitz continuity of F we choose some $v' \in F(u)$ such that

$$\left\| v - v' \right\| \leq L \, \|x(t) - u\| = L d_S(x(t)).$$

Since $x(t) - u \in N_S^P(u)$, by the current hypothesis of the theorem there exist $w' \in A(u)$ such that

$$\langle x(t) - u, v' - w' \rangle \leq 0,$$

which in turn yields, due to the monotonicity of A,

$$
\begin{aligned}
\langle x(t) - u, \dot{x}(t) \rangle &= \langle x(t) - u, v - w \rangle \\
&= \langle x(t) - u, v - v' \rangle + \langle x(t) - u, v' - w' \rangle \\
&\quad + \langle x(t) - u, w' - w \rangle \\
&\leq L \, \| x(t) - u \|^2 = L d_S^2(x(t)).
\end{aligned}
$$

Thus, continuing with (21) and (22) we arrive at

$$
\eta(t + s) \leq \eta(t) + 2L\eta(t)s + o(\|s\|),
$$

which implies that $\dot{\eta}(t) \leq 2L\eta(t)$. Hence, by the Gronwall Lemma, we obtain that $\eta(t) \leq \eta(\varepsilon)e^{2L(t-\varepsilon)}$ for all $t \in T_0$, or, equivalently, $\eta(t) \leq \eta(\varepsilon)e^{2L(t-\varepsilon)}$ for all $t \in [\varepsilon, T_1]$. Then, as ε goes to 0 we conclude that $\eta(t) = 0$ for all $t \in [0, T_1]$, which proves that $x(t) \in S$ for all $t \in [0, T_1]$. $\qquad\square$

We give the required characterization of strong invariant closed sets with respect to differential inclusion (1).

Theorem 2 *Let S be a closed subset of* $\mathrm{cl}(\mathrm{dom}\, A)$ *satisfying relation* (2). *Then the following statements are equivalent, provided that* $\mathrm{N}_S = \mathrm{N}_S^P$ *or* N_S^F *and* $\mathrm{T}_S = \mathrm{T}_S^B$, *or* $\mathrm{T}_S = \mathrm{conv}\, \mathrm{T}_S^B$.

(i) *S is strong invariant for differential inclusion* (1).

(ii) *For every* $x \in S \cap \mathrm{dom}\, A$, *one has*

$$
v - \Pi_{A(x)}(v) \in \mathrm{T}_S(x), \quad \forall v \in F(x). \tag{23}
$$

(iii) *For every* $x \in S \cap \mathrm{dom}\, A$, *one has*

$$
[v - A(x)] \cap \mathrm{T}_S(x) \neq \emptyset, \quad \forall v \in F(x). \tag{24}
$$

(iv) *For every* $x \in S \cap \mathrm{dom}\, A$, *one has*

$$
\sup_{\xi \in \mathrm{N}_S(x)} \sup_{v \in F(x)} \langle \xi, v - \Pi_{A(x)}(v) \rangle \leq 0. \tag{25}
$$

(v) *For every* $x \in S \cap \mathrm{dom}\, A$, *one has*

$$
\sup_{\xi \in \mathrm{N}_S(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{26}
$$

(vi) *For every* $x \in S \cap \mathrm{dom}\, A$, *one has*

$$
\sup_{\xi \in \mathrm{N}_S(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{\|F(x)\| + \|A^\circ(x)\|}} \langle \xi, v - x^* \rangle \leq 0. \tag{27}
$$

Proof The implication $(ii) \Rightarrow (iii)$ and $(vi) \Rightarrow (v)$ are trivial, while the implications $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (v)$ come from the relation $\mathrm{T}_S(x) \subset (\mathrm{N}_S^F(x))^*$ for all $x \in S$. The implications (v) (with $\mathrm{N}_S = \mathrm{N}_S^P) \Rightarrow (i)$ is a direct consequence of Proposition 3.

$(i) \Rightarrow (ii)$. To prove this implication we suppose that S is strong invariant and take $x_0 \in S \cap \mathrm{dom}\, A$ and $v \in F(x_0)$. According to Lemma 2, there exists a Lipschitz continuous selection f of F such that $f(x_0) = v$, and so there is a unique solution $x(\cdot)$ of the following differential inclusion,

$$
\dot{x}(t) \in f(x(t)) - A(x(t)), \quad \text{a.e. } t \geq 0, \ x(0) = x_0.
$$

It follows that $x(\cdot)$ is also a solution of differential inclusion (1), so that $x(t) \in S$ for any $t \geq 0$. Then we get

$$v - \Pi_{A(x_0)}(v) = (f(x_0) - A(x_0))^\circ = \frac{d^+ x(0)}{dt}$$

$$= \lim_{t\downarrow 0} \frac{x(t) - x_0}{t} \in \mathrm{T}_S^B(x_0) \subset \mathrm{T}_S(x_0).$$

$(iv) \Rightarrow (vi)$. This implication holds since for any $x \in \mathrm{dom}\, A$ and $v \in F(x)$ we have that

$$\begin{aligned}
\left\| \Pi_{A(x)}(v) \right\| &\leq \left\| \Pi_{A(x)}(v) - A^\circ(x) \right\| + \| A^\circ(x) \| \\
&= \left\| \Pi_{A(x)}(v) - \Pi_{A(x)}(\theta) \right\| + \| A^\circ(x) \| \\
&\leq \| v \| + \| A^\circ(x) \| \leq \| F(x) \| + \| A^\circ(x) \|.
\end{aligned}$$

The proof of the theorem is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following proposition, which provides the counterpart of Proposition 3 for the weak invariance, is essentially given in [24, Theorem 1]. The specification of the interval on which the solution remains in S also comes from the proof given in that paper.

Proposition 4 *Let $S \subset \mathrm{dom}\, A$ be closed and take $x_0 \in S$ such that, for some $r, m > 0$,*

$$\| A^\circ(x) \| \leq m, \ \forall x \in S \cap \mathrm{B}(x_0, r). \tag{28}$$

Assume that for all $x \in S \cap \mathrm{B}(x_0, r)$,

$$\sup_{\xi \in N_S(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{m + \| F(x) \|}} \langle \xi, v - x^* \rangle \leq 0. \tag{29}$$

Then there exists a solution $x(\cdot; x_0)$ of (1) such that $x(t; x_0) \in S$ for every $t \in [0, T]$ with $T = \frac{r}{3} \left(m + \sup_{x \in \mathrm{B}(x_0, r) \cap S} \| F(x) \| \right)^{-1}$.

Consequently, we obtain the desired characterization of weak invariant sets with respect to differential inclusion (1). Recall that A° is said to be locally bounded on S if for every $x \in S$ we have

$$\mathfrak{m}(x) := \limsup_{y \to x, y \in S} \| A^\circ(y) \| < +\infty. \tag{30}$$

Theorem 3 *Let $S \subset \mathrm{dom}\, A$ be a closed set such that A° is locally bounded on S. Then the following statements are equivalent provided that T_S and N_S are the same as the ones in Theorem 2 :*

(i)   *S is weak invariant for differential inclusion (1).*
(ii) *For every $x \in S$, one has*

$$\cup_{v \in F(x)} \left[v - A(x) \cap \mathrm{B}_{\mathfrak{m}(x) + \| F(x) \|} \right] \cap \mathrm{T}_S(x) \neq \emptyset. \tag{31}$$

(iii) *For every $x \in S$, one has*

$$\sup_{\xi \in N_S(x)} \inf_{v \in F(x)} \inf_{x^* \in A(x) \cap \mathrm{B}_{\mathfrak{m}(x) + \| F(x) \|}} \langle \xi, v - x^* \rangle \leq 0. \tag{32}$$

Proof $(i) \Rightarrow (ii)$. Given an $x_0 \in S$ we choose a solution $x(\cdot) := x(\cdot; x_0)$ of (1) that belongs to S. Fix $\varepsilon > 0$. By (30) and the current assumption we also choose $\rho > 0$ such that

$$\left\| A^\circ(x) \right\| \leq \mathfrak{m}(x_0) + \varepsilon \ \text{ for all } \ x \in \mathrm{B}(x_0, \rho) \cap S.$$

Then for any $x \in \mathrm{B}(x_0, \rho) \cap S$ and any $v \in F(x)$ we get

$$\left\| \Pi_{A(x)}(v) \right\| \leq \left\| \Pi_{A(x)}(v) - A^\circ(x) \right\| + \left\| A^\circ(x) \right\| \leq \|F(x)\| + \mathfrak{m}(x_0) + \varepsilon.$$

Let $T > 0$ be such that $x(t) \in \mathrm{B}(x_0, \rho) \cap S$ for all $t \in [0, T]$, so that for all $v \in F(x(t))$ and $t \in [0, T]$ we have

$$\left\| \Pi_{A(x(t))}(v) \right\| \leq \|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon;$$

hence, by Proposition 2(i),

$$\dot{x}(t) \in F(x(t)) - A(x(t)) \cap \mathrm{B}_{\|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon} \quad a.e. \ t \in [0, T], \tag{33}$$

and $x(\cdot)$ is Lipschitz continuous on $[0, T]$ (observing that $\mathrm{B}_{\|F(x(t))\| + \mathfrak{m}(x_0) + \varepsilon} \subset \mathrm{B}_{\|F(x_0)\| + L\rho + \mathfrak{m}(x_0) + \varepsilon}$). Take an element $w \in \mathrm{Limsup}_{t \downarrow 0} t^{-1}(x(t) - x_0)$ (this Painleve-Kuratowski upper limit is nonempty, due to the Lipschitz continuity of $x(\cdot)$). Then, since the mappings $x \mapsto A(x) \cap \mathrm{B}_{\|F(x)\| + \mathfrak{m}(x_0) + \varepsilon}$ and $x \mapsto F(x)$ are upper semicontinuous, by using (33) we get

$$
\begin{aligned}
w \in{}& \mathrm{Limsup}_{t \downarrow 0} \frac{1}{t} \int_0^t \dot{x}(\tau) d\tau \\
\subset{}& \mathrm{Limsup}_{t \downarrow 0} \left(\overline{\mathrm{co}} \left(\bigcup_{\tau \in [0,t]} F(x(\tau)) - A(x(\tau)) \cap \mathrm{B}_{\|F(x(\tau))\| + \mathfrak{m}(x_0) + \varepsilon} \right) \right) \\
\subset{}& F(x_0) - A(x_0) \cap \mathrm{B}_{\|F(x_0)\| + \mathfrak{m}(x_0) + \varepsilon},
\end{aligned}
\tag{34}
$$

and we conclude that, as ε goes to 0 (observe that v is independent of ε),

$$w \in F(x_0) - A(x_0) \cap \mathrm{B}_{\|F(x_0)\| + \mathfrak{m}(x_0)}.$$

Thus, (ii) follows, due to the obvious fact that $\mathrm{Limsup}_{t \downarrow 0} t^{-1}(x(t) - x_0) \subset \mathrm{T}_S(x_0)$.

$(iii) \Rightarrow (i)$. Fix $x_0 \in S$. By (30) we choose $r, m > 0$ such that $\mathfrak{m}(x) \leq m$ for every $x \in S \cap \mathrm{B}(x_0, r)$. It suffices to prove that the following quantity is equal to $+\infty$,

$$\overline{T} := \sup\{T : \exists \, x(\cdot; x_0) \text{ a solution of (1) such that } x(t; x_0) \in S, \ \forall t \in [0, T]\}.$$

According to Proposition 4, there exist some $T_1 > 0$ and a solution $x_1(\cdot; x_0)$ of differential inclusion (1) such that $x_1(t; x_0) \in S$ for all $t \in [0, T_1]$; hence, $\overline{T} \geq T_1 > 0$.

We proceed by contradiction and assume that $\overline{T} < +\infty$. By Proposition 2, we let $r_1 > 0$ be such that for every solution $x(\cdot; x_0)$ of (1) we have

$$x(t; x_0) \in \mathrm{B}(x_0, r_1), \ \forall t \in [0, \overline{T}].$$

We set

$$k := \sup_{x \in \mathrm{B}(x_0, r_1 + 1)} \|F(x)\| + \sup_{x \in \mathrm{B}(x_0, r_1 + 1) \cap S} \left\| A^\circ(x) \right\|,$$

so that $k < +\infty$, due to (30) and the compactness of the set $\mathrm{B}(x_0, r_1 + 1) \cap S$. By definition of $\overline{T}$, for $0 < \varepsilon < \min\left\{ \frac{1}{3k}, \overline{T} \right\}$ we choose a solution $x_\varepsilon(\cdot; x_0)$ of (1) such that $x_\varepsilon(t; x_0) \in S$ for all $t \in [0, \overline{T} - \varepsilon]$. We put $y_0 := x_\varepsilon(\overline{T} - \varepsilon; x_0) \in \mathrm{B}(x_0, r_1) \cap S$, so that $\mathrm{B}(y_0, 1) \subset \mathrm{B}(x_0, r_1 + 1)$ and the following relations follows easily

$$\|A^\circ(y)\| \leq \sup_{u \in \mathrm{B}(x_0, r_1 + 1) \cap S} \|A^\circ(u)\| =: m_1, \ \forall y \in S \cap \mathrm{B}(y_0, 1),$$

$$\sup_{\xi \in \mathrm{N}_S(y)} \inf_{v \in F(y)} \inf_{x^* \in A(y) \cap \mathrm{B}_{m_1 + \|F(y)\|}} \langle \xi, v - x^* \rangle \leq 0 \text{ for all } y \in S \cap \mathrm{B}(y_0, 1).$$

Then, according to Proposition 4, there exists a solution $x_2(\cdot; y_0)$ of (1) such that $x_2(t; y_0) \in S$ for all $t \in \left[0, \frac{1}{3k}\right]$. Consequently, the function $z(\cdot; x_0)$ defined as

$$z(t; x_0) := \begin{cases} x_\varepsilon(t; x_0) & \text{if } s \in [0, \overline{T} - \varepsilon], \\ x_2(t - \overline{T} + \varepsilon; y_0) & \text{if } s \in [\overline{T} - \varepsilon, +\infty[, \end{cases}$$

is a solution of (1) and satisfies $z(t; x_0) \in S$ for all $t \in [0, \tilde{T}]$ with $\tilde{T} := \overline{T} + \frac{1}{3k} - \varepsilon > \overline{T}$, which contradicts the definition of $\overline{T}$. Hence $\overline{T} = \infty$, and S is weak invariant. $\qquad\square$

We close this section by an example to illustrating the use of the previous invariance results, namely Theorem 4, in getting the existence of solutions for a system governed by the sum of a Cusco mapping and the normal cone to a prox-regular set. This new idea is exploited with further details in [3].

Example 3 We consider the following differential inclusion, given in $\mathbb{R}^n$,

$$\dot{x}(t) \in F(x(t)) - N_C(x(t)), \text{ a.e. } t \geq 0, x(0) = x_0 \in C, \tag{35}$$

where F is as before; i.e., an L-Lipschitz Cusco mapping, C is a closed r-uniformly prox-regular set of $\mathbb{R}^n$; that is, due to the finite-dimensional setting, the projection mapping onto C is single-valued in the region of points that are of at most distance r from C, and $N_C(x)$, $x \in C$, is the proximal normal cone to C at x (see (11)); that is,

$$N_C(x) := N_C^P(x) = \{\xi \in \mathbb{R}^n : \exists \sigma \geq 0 \text{ s. t. } \langle \xi, y - x \rangle \leq \sigma \|y - x\|^2, \ \forall y \in C\}$$

(we removed the superscript P from N^P, because $N_C^P = N_C^F = N_C^L$ in the current case.). We refer to [32] for a detailed analysis of these concepts. When the set C also depends on the time, the system above always admits a solution (see, for instance, [28]), and is referred to as a sweeping process (see [29]).

Most of the works on sweeping processes use approximate schemes based on the Moreau-Yoshida regularization of the indicator function of the set C ([28]), or discrete schemes as in [18, 29]. Here, we are going to prove this existence result using Theorem 4.

Towards this aim, we suppose for the sake of simplifying the presentation that C is bounded; hence, for being L-Lipschitz and Cusco, the mapping F is uniformly bounded on the set C.

In the first step, we recall an easy fact that comes from the prox-regularity of the set C (see [3, Lemma 4.1]):

Step 1: for each $m > 0$ large enough, there exists a maximal monotone operator A_C such that $C \subset \text{dom} A_C \subset \text{cl}(\text{conv } C)$; hence, C and A_C satisfy condition (2), and

$$N_C(x) \cap B_m + \frac{m}{r}x \subset A_C(x) \subset N_C(x) + \frac{m}{r}x, \ \forall x \in C. \tag{36}$$

Step 2: we consider the following differential inclusion, given in $\mathbb{R}^n$,

$$\dot{x}(t) \in F(x(t)) + \frac{m}{r}x(t) - A_C(x(t)), \text{ a.e. } t \geq 0, x(0) = x_0 \in C. \tag{37}$$

Then, due to Theorem 1, differential inclusion (37) has at least one solution.

Step 3: we use Theorem 2(v) to prove that the set C is invariant for (37). Indeed, we choose m large enough such that $\|F(y_0)\| + L\|y - y_0\| \leq m$ for all $y, y_0 \in C$ (C is assumed to be bounded). Next, we take $x \in C$, $\xi \in N_C(x)$, $v \in F(x)$, and, using the L-Lipschitzianity of the Cusco mapping F, pick an element $v_0 \in F(x_0)$ such

that $\|v - v_0\| \leq L\|x - x_0\|$. Hence, the projection point $z := \Pi_{N_C(x)}(v) \subset N_C(x)$ satisfies

$$\|z\| \leq \|v\| \leq \|v_0\| + \|v - v_0\| \leq \|F(x_0)\| + L\|x - x_0\| \leq m.$$

So, by (36), there exists some $x^* \in A_C(x)$ such that $z + \frac{m}{r}x = x^*$. Moreover, since $v - z$ is in the negative dual cone of $N_C(x)$, we obtain that

$$\left\langle \xi, v + \frac{m}{r}x - x^* \right\rangle = \langle \xi, v - z \rangle \leq 0.$$

Consequently, according to Theorem 2(v), the set C is invariant for (37).

Step 4: we conclude that differential inclusion (35) has a solution. Absolutely, this follows by combining Steps 1, 2 and 3 as follows. By Step 2, there exists a solution $x(\cdot) := x(\cdot; x_0)$ of (37). Since C is invariant for (37), thanks to Step 3, we have that $x(t) \in C$ for all $t \geq 0$, and so, using (36), for a.e $t \geq 0$

$$\dot{x}(t) \in F(x(t)) + \frac{m}{r}x(t) - A_C(x(t)) \subset F(x(t)) - N_C(x(t));$$

that is, $x(\cdot)$ is a solution of (35).

5 Strong a-Lyapunov Pairs

In this section, we use the invariance results of the previous section to characterize strong a-Lyapunov pairs with respect to differential inclusion (1),

$$\dot{x}(t) \in F(x(t)) - A(x(t)), \quad \text{a.e.} \, t \geq 0, \quad x(0) = x_0 \in \text{cl}(\text{dom} \, A),$$

where $A : H \rightrightarrows H$ is a maximal monotone operator and F is an L-Lipschitz Cusco mapping.

Definition 3 Let $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc functions such that $W \geq 0$ and let $a \geq 0$. We say that (V, W) is a strong a-Lyapunov pair for (1) if for any $x_0 \in \text{cl}(\text{dom} \, A)$ we have

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0), \quad \forall t \geq 0, \tag{38}$$

for every solution $x(\cdot; x_0)$ of (1).

The following lemma shows that the non-regularity of the functions V, W candidates to form a-Lyapunov pairs is mainly carried by the function V. For $k \geq 1$ we denote

$$W_k(x) := \inf_{z \in \mathbb{R}^n} \{W(z) + k\|x - z\|\}. \tag{39}$$

Lemma 5 Given a function $W : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$, W_k defined in (39) is k-Lipschitz continuous, and we have $W_k(x) \nearrow W(x)$ for all $x \in \mathbb{R}^n$. Moreover, if $x(\cdot; x_0)$ is a solution of differential inclusion (1), then W satisfies inequality (38) iff W_k does for all $k \geq 1$.

Proof The first statement of the lemma is known (see, e.g., [16]), and the second statement of the lemma follows easily from Fatou's lemma. □

Lemma 6 Consider the operator $\hat{A} : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n+3}$ and the function $\tilde{V} : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\hat{A}(x, \alpha, \beta, \gamma) := (A(x), \theta_{\mathbb{R}^3}), \quad \tilde{V}(x, \alpha, \beta) := e^{a\beta}V(x) + \alpha, \tag{40}$$

together with the mappings $\hat{F}_k : \mathbb{R}^{n+3} \to \mathbb{R}^{n+3}$, $k \geq 1$, *given by (recall* (39))

$$\hat{F}_k(x, \alpha, \beta, \gamma) := (F(x), W_k(x), 1, 0).$$

Then $\hat{A}$ *is maximal monotone with* $\text{dom }\hat{A} = \text{dom } A \times \mathbb{R}^3$, $\hat{F}_k$ *is Lipschitz continuous with constant* $\left(L^2 + k^2\right)^{\frac{1}{2}}$, *and consequently, the following differential inclusion possesses solutions,*

$$\begin{cases} \dot{z}(t) \in \hat{F}_k(z(t)) - \hat{A}(z(t)), & a.e. \ t \geq 0, \\ z(0) = Z_0 = (x_0, y_0, z_0, w_0) \in \text{cl}(\text{dom } A) \times \mathbb{R}^3, \end{cases} \tag{41}$$

and every solutions is written as

$$z(t; Z_0) = (x(t; x_0), y_0 + \int_0^t W_k(x(\tau; x_0))d\tau, z_0 + t, w_0),$$

for a solution $x(\cdot; x_0)$ *of* (1).

We need the following result which provides us with a local criterion for strong a-Lyapunov pairs.

Proposition 5 *Let* $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be two proper lsc functions such that* $\text{dom } V \subset \text{dom } A$, $W \geq 0$ *and let* $a \geq 0$. *Fix* $x_0 \in \text{dom } V$ *and assume that for some* $\rho > 0$ *we have, for all* $x \in B(x_0, \rho)$,

$$\sup_{\xi \in \partial_P V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0, \tag{42}$$

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{43}$$

Then there exists some $T > 0$ *such that for every solution* $x(\cdot; x_0)$ *of differential inclusion* (1) *one has*

$$e^{at} V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0), \ \forall t \in [0, T].$$

Proof First, by Proposition 2(ii) we let $c > 0$ be such that for any solutions $x(\cdot) := x(\cdot; x_0)$ of (1) it holds

$$\|x(t) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)te^{ct} \text{ for all } t \geq 0,$$

and choose $T > 0$ such that

$$3(\|F(x_0)\| + \|A^\circ(x_0)\|)Te^{cT} \leq \rho. \tag{44}$$

As in Lemma 6, we define the proper and lsc function $\tilde{V} : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ as $\tilde{V}(x, \alpha, \beta) := e^{a\beta} V(x) + \alpha$, so that $\text{epi }\tilde{V}$ is closed and satisfies

$$\text{epi } \tilde{V} \subset \text{dom } V \times \mathbb{R}^3 \subset \text{dom } A \times \mathbb{R}^3 = \text{dom } \hat{A},$$

where $\hat{A}$ is also defined as in Lemma 6; hence, condition (2) is obviously satisfied for epi $\tilde{V}$.

Claim We claim that for any given $\tilde{z} := (x_1, y_1, z_1, w_1) \in \text{epi }\tilde{V}$ with $\|x_1 - x_0\| < \rho$, there exists small enough $\varepsilon > 0$ such that for each $(x, y, z, w) \in B(\tilde{z}, \varepsilon) \cap \text{epi }\tilde{V}$, $(\tilde{\xi}, -\kappa) \in \mathrm{N}^P_{\text{epi }\tilde{V}}(x, y, z, w)$, and $(v, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w)$ there exists $x^* \in A(x)$ such that

$$\langle (\tilde{\xi}, -\kappa), (v - x^*, W_k(x), 1, 0) \rangle \leq 0. \tag{45}$$

Indeed, with $\tilde{z}$ as in the claim let us choose $\varepsilon > 0$ such that

$$(x, y, z, w) \in B(\tilde{z}, \varepsilon) \cap \text{epi}\, \tilde{V} \Rightarrow x \in B(x_0, \rho).$$

Let (x, y, z, w), $(\tilde{\xi}, -\kappa)$, and $(v, W_k(x), 1, 0)$ be as in the claim, so that $x \in B(x_0, \rho) \cap \text{dom}\, V$ and $v \in F(x)$, as well as $\kappa \geq 0$ (see [16, Exercise 2.1]). We may distinguish two cases:

(i) If $\kappa > 0$, then $w = \tilde{V}(x, y, z)$ and, without loss of generality, we may suppose that $\kappa = 1$. Hence, $\tilde{\xi} = (e^{az}\xi, 1, ae^{az}V(x)) \in \partial_P \tilde{V}(x, y, z)$ for some $\xi \in \partial_P V(x)$. Consequently, by the current hypothesis there exists $x^* \in A(x)$ such that

$$\langle \xi, v - x^* \rangle + aV(x) + W_k(x) \leq \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

In other words, we have $(v - x^*, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w) - \hat{A}(x, y, z, w)$ and

$$\begin{aligned}
\langle (\tilde{\xi}, -1), (v - x^*, W_k(x), 1, 0) \rangle &= \langle (e^{az}\xi, 1, ae^{az}V(x), -1), (v - x^*, W_k(x), 1, 0) \rangle \\
&= e^{az}\langle \xi, v - x^* \rangle + W_k(x) + ae^{az}V(x) \\
&= e^{az}(\langle \xi, v - x^* \rangle + aV(x) + W_k(x)) \\
&\quad + (1 - e^{az})W_k(x) \leq 0,
\end{aligned}$$

and (45) follows.

(ii) If $\kappa = 0$, then $\tilde{\xi} \in \partial_{P,\infty} \tilde{V}(x, y, z)$ and, so, $(\tilde{\xi}, -\kappa) = (\xi, \theta_{\mathbb{R}^3})$ for some $\xi \in \partial_{P,\infty} V(x)$. Then, by arguing as in the paragraph above, the current hypothesis yields some $x^* \in A(x)$ such that $\langle \xi, v - x^* \rangle \leq 0$. Hence, $(v - x^*, W_k(x), 1, 0) \in \hat{F}_k(x, y, z, w) - \hat{A}(x, y, z, w)$ and

$$\langle (\tilde{\xi}, 0), (v - x^*, W_k(x), 1, 0) \rangle = \langle \xi, v - x^* \rangle \leq 0; \tag{46}$$

that is, (45) follows in this case too. The claim is proved.

Now, we take a solution $x(\cdot; x_0)$ of (1), so that

$$z(\cdot; z_0) := \left(x(\cdot; x_0), \int_0^{\cdot} W_k(x(\tau; x_0))d\tau, \cdot, V(x_0)\right),$$

with $z_0 := (x_0, 0, 0, V(x_0))$, becomes a solution of (41). Then, from the claim (with $\tilde{z} := z_0$) above and Proposition 3, there exists some $\bar{t} > 0$ such that

$$z(t; z_0) \in \text{epi}\,\tilde{V}, \quad \forall t \in [0, \bar{t}];$$

that is,

$$\overline{T} := \sup\{t \geq 0 : \text{ such that } z(s; z_0) \in \text{epi}\,\tilde{V} \; \forall s \in [0, t]\} > 0. \tag{47}$$

Let us show that $\overline{T} \geq T$, where T is defined in (44). We proceed by contradiction and assume that $\overline{T} < T$. Then, because (by Proposition 2(ii))

$$\|x(\overline{T}; x_0) - x_0\| \leq 3(\|F(x_0)\| + \|A^\circ(x_0)\|)\overline{T}e^{c\overline{T}} < \rho,$$

and $z(\overline{T}; z_0) = (x(\overline{T}; x_0), \int_0^{\overline{T}} W_k(x(\tau; x_0))d\tau, \overline{T}, V(x_0)) \in \text{epi}\,\tilde{V}$, from the claim above (with $\tilde{z} := z(\overline{T}; z_0)$) and Proposition 3, there exists some $t_1 > 0$ such that $z(t; z(\overline{T}; z_0)) \in \text{epi}\,\tilde{V}$ for all $t \in [0, t_1]$. Thus, $z(t + \overline{T}; z_0) = z(t; z(\overline{T}; z_0)) \in \text{epi}\,\tilde{V}$ for every $t \in [0, t_1]$, and we get a contradiction to the definition of $\overline{T}$.

Finally, from (47) we get

$$e^{at}V(x(t; x_0)) + \int_0^t W_k(x(\tau; x_0))d\tau \leq V(x_0), \quad \forall t \in [0, T].$$

Moreover, because T is independent of k, by taking the limit as $k \to \infty$ we arrive at (as $W_k(x) \nearrow W(x)$, by Lemma 5)

$$e^{at} V(x(t; x_0)) + \int_0^t W(x(\tau; x_0)) d\tau \leq V(x_0), \quad \forall t \in [0, T],$$

which is the desired inequality.   □

We give now the desired characterization of strong a-Lyapunov pairs.

Theorem 4 *Let V, W, and a be as in Proposition 5, and let ∂ stand for either ∂_P or ∂_F. Then the pair (V, W) is a strong a-Lyapunov pair for* (1) *iff for all $x \in \text{dom } V$*

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0, \tag{48}$$

$$\sup_{\xi \in \partial_{P,\infty} V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x)} \langle \xi, v - x^* \rangle \leq 0. \tag{49}$$

Proof To prove the sufficiency part, we take $x_0 \in \text{dom } V$ and a solution $x(\cdot; x_0)$ of differential inclusion (1). By Proposition 5 there exists some $T > 0$ such that

$$e^{at} V(x(t; x_0)) + \int_0^t W(x(\tau; x_0)) d\tau \leq V(x_0), \quad \forall t \in [0, T]. \tag{50}$$

It suffices to prove that the following quantity is $+\infty$,

$$T := \sup\{s \geq 0 : (50) \text{ holds } \forall t \in [0, s]\}.$$

Otherwise, if T is finite, then $x(T; x_0) \in \text{dom } V$ (because V is lsc), and again from Proposition 5 we find $\eta > 0$ such that for all $t \in [0, \eta]$, using the semi-group property of $x(\cdot; x_0)$,

$$e^{a(t+T)} V(x(t + T; x_0)) + \int_0^{t+T} W(x(\tau; x_0)) d\tau$$

$$\leq e^{aT} \left(e^{at} V(x(t + T; x_0)) + \int_T^{t+T} W(x(\tau; x_0)) d\tau \right) + \int_0^T W(x(\tau; x_0)) d\tau$$

$$\leq e^{aT} V(x(T; x_0)) + \int_0^T W(x(\tau; x_0)) d\tau \leq V(x_0),$$

and we get the contradiction $T \geq T + \eta$. Hence, $T = +\infty$ and (50) holds for all $t \geq 0$, showing that (V, W) forms a strong Lyapunov pair for differential inclusion (1).

To prove the necessity of the current conditions, we start by verifying (48) with $\partial = \partial_F$. We fix $x_0 \in \text{dom } V$ ($\subset \text{dom } A$) and $v \in F(x_0)$, and, according to Proposition 2, we choose a solution $x(\cdot; x_0)$ of differential inclusion (1) such that $\frac{d^+ x(0; x_0)}{dt} = v - \Pi_{A(x_0)}(v)$. Thus, since (V, W) is assumed to be a strong a-Lyapunov pair for (1), we obtain for every $t > 0$

$$\frac{V(x(t; x_0)) - V(x_0)}{t} + \frac{e^{at} - 1}{t} V(x(t; x_0)) + \frac{1}{t} \int_0^t W(x(\tau; x_0)) d\tau \leq 0,$$

which give us, as $t \downarrow 0$,

$$\sigma_{\partial_F V(x_0)}(v - \Pi_{A(x_0)}(v)) \leq V'(x_0; v - \Pi_{A(x_0)}(v))$$

$$\leq \liminf_{t \downarrow 0} \frac{V(x(t; x_0)) - V(x_0)}{t} \leq -aV(x_0) - W(x_0).$$

Hence, (48) follows with either $\partial = \partial_F$ or $\partial = \partial_P$. To verify (49) we fix $x_0 \in \operatorname{dom} V$, $v \in F(x_0)$ and $\xi \in \partial_{P,\infty} V(x_0)$; that is, $(\xi, 0) \in N^P_{\operatorname{epi} V}(x_0, V(x_0))$. According to Proposition 2, we choose a solution $x(\cdot; x_0)$ of differential inclusion (1) such that $\frac{d^+ x(0; x_0)}{dt} = v - \Pi_{A(x_0)}(v)$. Since (V, W) is strong a-Lyapunov for differential inclusion (1), one has that $(x(t; x_0), e^{-at} V(x_0)) \in \operatorname{epi} V$ for all $t \geq 0$. Then, by the definition of the proximal normal cone, there exists $\eta > 0$ such that for all small $t \geq 0$

$$\langle (\xi, 0), (x(t; x_0), e^{-at} V(x_0)) - (x_0, V(x_0)) \rangle$$
$$\leq \eta \big(\|x(t; x_0) - x_0\|^2 + (e^{-at} - 1)^2 |V(x_0)|^2 \big),$$

and so

$$\langle \xi, x(t; x_0) - x_0 \rangle \leq \eta \big(\|x(t; x_0) - x_0\|^2 + (e^{-at} - 1)^2 |V(x_0)|^2 \big).$$

Hence, by dividing on $t > 0$ and taking limits as $t \downarrow 0$, we obtain that

$$\langle \xi, v - \Pi_{A(x_0)}(v) \rangle \leq 0,$$

as we wanted to prove. $\qquad \square$

We give in the following corollary other criteria for strong a-Lyapunov pairs for (1). Recall that A° is said to be locally bounded on $\operatorname{dom} V$ if condition (30) holds for all $x \in \operatorname{dom} V$; that is, for every $x \in \operatorname{dom} V$ we have

$$\mathfrak{m}(x) = \limsup_{y \to x, \, y \in \operatorname{dom} V} \|A^\circ(y)\| < +\infty.$$

We also observe that the function $\mathfrak{m}$ is upper semicontinuous at every $x \in \mathbb{R}^n$ such that $\mathfrak{m}(x) < +\infty$; that is,

$$\limsup_{y \to x, \, y \in \operatorname{dom} V} \mathfrak{m}(y) = \mathfrak{m}(x).$$

Corollary 1 *Let V, W, and a be as in Proposition 5, and let ∂ stand for either ∂_P, ∂_F, or ∂_L. If A° is locally bounded on $\operatorname{dom} V$, then (V, W) is a strong a-Lyapunov pair for (1) iff one of the following statements holds.*

(i) *For any $x \in \operatorname{dom} V$,*

$$\sup_{\xi \in \partial V(x)} \sup_{v \in F(x)} \inf_{x^* \in A(x) \cap B_{\|F(x)\| + \mathfrak{m}(x)}} \langle \xi, v - x^* \rangle + aV(x) + W(x) \leq 0.$$

(ii) *For any $x \in \operatorname{dom} V$,*

$$\sup_{v \in F(x)} V'(x; v - \Pi_{A(x)}(v)) + aV(x) + W(x) \leq 0.$$

(iii) *For any $x \in \operatorname{dom} V$,*

$$\sup_{v \in F(x)} \inf_{x^* \in A(x) \cap B_{\|F(x)\| + \mathfrak{m}(x)}} V'(x; v - x^*) + aV(x) + W(x) \leq 0.$$

Proof $(ii) \Rightarrow (iii)$. This implication follows since that for any $x \in \operatorname{dom} V$ ($\subset \operatorname{dom} A$) any $v \in F(x)$

$$\|\Pi_{A(x)}(v)\| \leq \|A^\circ(x)\| + \|\Pi_{A(x)}(v) - A^\circ(x)\| \leq \mathfrak{m}(x) + \|F(x)\|.$$

$(iii) \Rightarrow (i)$. When ∂ stands for either ∂_P or ∂_F this implication follows from the relation $\sigma_{\partial_P V(x)}(\cdot) \leq \sigma_{\partial_F V(x)}(\cdot) \leq V'(x; \cdot)$. If $\partial = \partial_L$, we take $\xi \in \partial_L V(x)$ and $v \in F(x)$, and choose sequences (x_i) and (ξ_i) such that

$$x_i \overset{V}{\to} x, \ \xi_i \in \partial_P V(x_i), \ \xi_i \to \xi \ \text{as } i \to \infty;$$

moreover, due to the upper semi-continuity of $\mathfrak{m}$ at x and $\mathfrak{m}(x) < +\infty$, by assumption, we may assume up to a subsequence that

$$\mathfrak{m}(x_i) \leq \mathfrak{m}(x) + \frac{1}{i}, \ \forall i \in \mathbb{N}.$$

By the Lipschitz continuity of F we also choose a sequence $(v_i)_{i \geq 1}$ such that $v_i \in F(x_i)$ and $v_i \to v$. Since (i) holds with $\partial = \partial_P$, for each i there exists $x_i^* \in A(x_i) \cap B_{\|F(x_i)\| + \mathfrak{m}(x_i)}$ such that

$$\langle \xi_i, v_i - x_i^* \rangle + a V(x_i) + W(x_i) \leq 0. \tag{51}$$

Then, since the maximal monotone operator A has a closed graph, and $(x_i^*)_i$ is bounded, we assume w.l.o.g. that

$$x_i^* \to x^* \in A(x) \cap B_{\mathfrak{m}(x)} \ \text{as } i \to \infty.$$

So, by passing to the limit in (51) as $i \to \infty$, and using the lower semicontinuity of W, we obtain that

$$\langle \xi, v - x^* \rangle + a V(x) + W(x) \leq 0,$$

which shows that (i) holds when $\partial = \partial_L$.

$(i) \Rightarrow (V, W)$ is a strong a-Lyapunov pair for (1). According to Theorem 4 we only need to show that (49) holds. We fix $x \in \operatorname{dom} V$, $\xi \in \partial_{P,\infty} V(x)$ and $v \in F(x)$. There exist sequences $(x_i)_i$, $(\xi_i)_i$, and $(\alpha_i)_i$ such that

$$x_i \overset{V}{\to} x, \ \xi_i \in \partial_P V(x_i), \ \alpha_i \downarrow 0, \ \alpha_i \xi_i \to \xi \ \text{as } i \to \infty.$$

By arguing as in the last paragraph above there also exists a sequence $(v_i)_i$ such that $v_i \in F(x_i)$ and $v_i \to v$ as $i \to \infty$. Moreover, using the current assumption on A°, there exists $m > 0$ such that $\sup_i \mathfrak{m}(x_i) \leq m$. Now, by assumption (ii), for each $i \in \mathbb{N}$ there exists a sequences $x_i^* \in A(x_i) \cap B_{\|F(x_i)\| + \mathfrak{m}(x_i)} \subset A(x_i) \cap B_{\|F(x_i)\| + m}$ and

$$\langle \xi_i, v_i - x_i^* \rangle + a V(x_i) + W(x_i) \leq 0. \tag{52}$$

By using again that A has a closed graph, and that $x_i^* \to x^* \in A(x)$, by multiplying the last inequality above (52) by α_i and next taking limits as $i \to \infty$, we arrive at (49). The proof of the corollary is finished since (ii) is a necessary condition for strong a-Lyapunov pairs, as we have shown in the proof of Theorem 4. □

We revisit Example 3 to show how can Theorem 4 (or, more precisely, its Corollary 1) be applied to study the stability of sweeping processes (more details can be found in [3]). Let us then retake system (35), which is given as

$$\dot{x}(t) \in F(x(t)) - N_C(x(t)), \ \text{a.e. } t \geq 0, x(0) = x_0 \in C,$$

with $C \subset \mathbb{R}^n$ being a bounded closed r-uniformly prox-regular set. Then, due to Corollary 1, the statement of the following example follows by arguing as in Example 3 (observing that (35) is equivalent to (37) and that A_C° is bounded on C, as comes from (36)).

Example 4 (Example 3 continued) A lsc function $V : \mathbb{R}^n :\to \mathbb{R} \cup \{+\infty\}$ such that dom $V \subset C$ is a strong Lyapunov function for (35) iff there exists some $k \geq 0$ such that, for all $x \in \text{dom } V$,

$$\sup_{\xi \in \partial_P V(x)} \sup_{v \in F(x)} \inf_{x^* \in N_C(x) \cap B_k} \langle \xi, v - x^* \rangle \leq 0.$$

6 Conclusion

The main contribution of this paper consists in providing primal and dual criteria for strong and weak invariant closed sets (Theorems 2 and 3), and for strong Lyapunov functions (Theorem 4 and Corollary 1), associated to differential inclusions (1). As in the classical invariance results, as one can consult in [16], the presented criteria are expressed in terms of the associated Hamiltonians. The novelty of this work lies in the consideration of differential inclusions that are governed by the sum of a Lipschitz Cusco mapping and a maximal monotone operator (yielding a one-sided Lipschitz multifunction, as in [24]); thus, allowing the right-hand side to have empty or unbounded values. We have confined ourselves to the finite-dimensional setting, because for the need of a Lipschitz continuous selection theorem, which is valid only in such a setting. Hence, our objective for a close future will be to go beyond this difficulty and extend the current results, for instance, to the Gelfand triple, where we expect that the Lyapunov stability approach could give some satisfactory results, especially for the study of the regularity of the solutions of differential inclusions (and partial differential equations). We also plan to extend this study to similar differential inclusions but with time-depending right-hand sides.

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