IMA Journal of Mathematical Control and Information (2020) **37**, 143–158
doi:10.1093/imamci/dny043
Advance Access publication on 1 November 2018

A dissipative approach to the stability of multi-order fractional systems

JAVIER A. GALLEGOS*
Department of Electrical Engineering, University of Chile, Av. Tupper 2007, Santiago, Chile
*Corresponding author. Email: jgallego@ing.uchile.cl

AND

MANUEL A. DUARTE-MERMOUD
Department of Electrical Engineering, University of Chile, Av. Tupper 2007, Santiago, Chile and Advanced Mining Technology Center, University of Chile, Av. Tupper 2007, Santiago, Chile

[Received on 27 April 2018; revised on 21 August 2018; accepted on 30 September 2018]

Real-order generalization of dissipativeness and passivity concepts are presented in this paper. They are characterized as properties of a system; that is, they are independent of the system's internal representation and independent of the type of fractional derivative defining that representation. With the aid of these extended concepts, the stability analysis of linearly interconnected multi-order (mixed-order or multivariable) linear or nonlinear systems consisting of integer and fractional order subsystems becomes a well-defined problem and it is reduced to verify algebraic inequalities and/or the dissipativenes of each subsystem. In particular, small gain and passivity theorems for multi-order systems are obtained. Examples show the benefits in simplicity obtained with this approach when analysing the stability of large-scale multi-order nonlinear systems.

Keywords: dissipative systems; passivity; fractional order systems; large-scale systems; multi-order; multivariable.

1. Introduction

The stability study of large-scale input–output integer order systems relying on dissipative properties (Moylan & Hill, 1978) has been carried out in much the same way that energy concepts are used in statistical mechanic; that is, an abstraction of all internal subsystem dynamics and a focus on rather measures of their input–output behaviour. In this way, the analysis of complex nonlinear systems has been developed by considering the input–output properties of their subsystems and the interconnection among them, as shown in multi-agent (Chopra & Spong, 2006), network (Kottenstette , 2013), control (Ghanbari , 2016) or cooperative (Arcak, 2007) problems.

This approach will be extended to the cases where the subsystems are defined by different orders of differentiation. The motivation of our results are as follows: first, some fractional models of complex processes have been recently proposed (Podlubny, 1999; Caponetto *et al.*, 2010) using the same order of differentiation in each equation; and second, models of some real process are more precise if it used multi-order large-scale systems Baleanu *et al.* (2016); Diethelm (2013). Our contribution is detailed in the following paragraphs.

In Section 2 we adopt and generalize to real order, the operator approach in which the dissipative property can be asserted by input–output measures (Desoer & Vidyasagar, 1975; Hill & Moylan, 1980; van der Schaft, 2000), and therefore it is system property rather than of its internal representations. The alternative storage function approach proposed in Rakhshan *et al.* (2017) (which is a generalization to real order of the integer order approach; Willems, 1972a) is dependent on the representation. It must be noted that in integer order both approaches are equivalent (Hill & Moylan, 1980), but in fractional systems, the same proof fails because of its non-local behaviour (Gallegos & Duarte-Mermoud, 2016b). Moreover, the notions are established independent of the type of fractional derivative involved (c.f. Rakhshan *et al.*, 2017 where the results depend on the choice of Caputo derivative), a crucial fact since there is no preferred fractional derivative as a generalization of integer derivative (Kilbas *et al.*, 2006). We show this feature with several examples. Moreover, in Section 2, we provide elementary stability properties of fractional dissipative systems on which relies the large-scale stability. These results generalize to real order the ones contained in Hill & Moylan (1980) and van der Schaft (2000).

In Section 3 we prove that systems consisting of interconnection of subsystems defined with possibly different order derivation are well defined under mild Lipschitz-type assumptions. When particularized to the same derivation order, this result generalizes the one in van der Schaft (2000), which was done for feedback connection of two systems.

In Section 4 we prove the main results which exhibit conditions for the input–output stability of large-scale systems. In particular, we generalize the small gain theorem in van der Schaft (2000), both for a general connection and for real-order finite-gain subsystems. Next, we generalize the result in Moylan & Hill (1978) by considering real-order dissipative subsystems. Finally, we introduce the concept of passive interconnection of subsystems as a way to assure finite-gain stability.

In Section 5 we show through examples the benefits of this approach to analyse the stability of large-scale mixed-order nonlinear systems. On the one hand, the Lyapunov method to study stability (Gallegos & Duarte-Mermoud, 2016b; Tuan & Trinh, 2017) has been only extended to subsystems having the same order of differentiation and lesser than one—the difficulties lying in the non-group property of the fractional differentiation operators, which makes a hard problem to find a Lyapunov function of systems with different differentiation orders. On the other hand, large-scale systems can have involved internal (pseudo) state representations with unknown parameters. These difficulties are more simply handled with the proposed input–output approach, as shown in several examples.

2. Definitions

In this section, we present the definition of real-order dissipativeness based on a norm defined by the fractional integral and some stability properties of dissipativeness systems.

2.1. *Fractional inner product*

The Riemann–Liouville fractional integral of a function $f : [0, T] \to \mathbb{C}$ is given by (Diethelm, 2010),

$$ {}_aI^\alpha f(t) := \left[{}_aI^\alpha f(\cdot)\right](t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \qquad (1)$$

where $\alpha \in \mathbb{R}_{>0}$, the set of positive real numbers. It is well defined for locally integrable functions. We assume, w.l.o.g., $a = 0$ and omit the sub-index a. From this concept, definitions of commonly used fractional derivative are obtained. The Riemann–Liouville (RL) fractional derivative of order α is given by ${}^RD^\alpha f := D^m I^{m-\alpha} f$ where $m = \lceil \alpha \rceil$ and the Caputo derivative of order α is given by ${}^CD^\alpha f := I^{m-\alpha} D^m f$ (see Diethelm, 2010, §2, 3).

The semi-group property of the fractional integral states that for any $\alpha, \beta > 0$ and any $t \geq 0$, $[I^{\alpha+\beta}f](t) = [I^{\alpha}I^{\beta}f](t)$ whenever f is a continuous function (Diethelm, 2010, Theorem 2.2).

Using the same arguments in integer integrals, an inner product can be defined for the fractional integral, since it will be linear, symmetric and positive-definite, module functions of null measure (see also Gallegos *et al.*, in press, §2.1).

Definition 1 Let $x, y : [0, \infty] \to \mathbb{C}^n$, $A \in \mathbb{C}^n \times \mathbb{C}^n$. Considering $T > 0$ and $\alpha > 0$ are fixed real numbers, we define

- inner product: $\langle x, y\rangle_{\alpha,T} := [I^{\alpha}x^{*}y](t = T)$,
- norm: $||x||^2_{\alpha,T} := \langle x, x\rangle_{\alpha,T}$.
- $x \in \mathcal{L}^2_{\alpha,e}$ if for any $T > 0$, $||x||^2_{\alpha,T} < \infty$,

where $*$ denotes the conjugate transpose.

In the following results, we mainly use the linearity of the inner product and the Cauchy–Schwarz inequality obtained from the induced norm. The Euclidean and its induced matrix norm are denoted by $||\cdot||$. Note also that for any constant matrix F,

$$\begin{aligned} ||Fx||^2_{\alpha,T} &= [I^{\alpha}||Fx||^2](t = T) \\ &\leq ||F||^2[I^{\alpha}||x||^2](t = T) = ||F||^2||x||^2_{\alpha,T}. \end{aligned}$$

In the following T will be often omitted when it refers to any $T > 0$. Note also that if $||x||_{\alpha} < \infty$ and x is uniformly continuous, we obtain that x is bounded.

2.2. *Dissipative systems*

A large-scale system is an interconnection among many subsystems, which can be linear or nonlinear, fractional or integer order, finite or infinite dimensional and/or single or multiple input–output. For the sake of simplicity, it will be assumed a linear interconnection and that each subsystem has finite-dimensional input and output. Fractional systems would be examples of infinite and finite dimensional state and input/output, respectively (see Gallegos & Duarte-Mermoud, 2016b).

A desired feature in a large-scale system is finite gain since it is connected with standard control requirements such as robustness and stability (van der Schaft, 2000). Finite gain and passivity concepts can be enclosed in the dissipative one in an operator approach, which does not depend on the internal state representation. Therefore, they become system's properties determined only by its input and output. The concept of dissipative systems, which is defined through an integral inequality, has its intuition in mechanical systems, where that inequality is just an energetic balance verified for most of the physical systems.

Definition 2 Let a system or map from $\mathcal{U}$ to $\mathcal{Y}$ (function spaces) be denoted by (u, y), $y = y(u)$ or $y = G(u)$. Consider a null initial condition for the system, i.e. such that $y \equiv 0$ whenever $u \equiv 0$. Let I be the identity operator, $\alpha > 0$ and some real numbers $\epsilon, \delta > 0$.

- (u, y) is α-dissipative for (Q, S, R) if the operators Q, R are self-adjoint and such that $(\forall u \in \mathcal{L}^2_{\alpha,e})$, $(\forall T < \infty)$

$$\langle y, Qy\rangle_{\alpha,T} + 2\langle y, Su\rangle_{\alpha,T} + \langle u, Ru\rangle_{\alpha,T} \geq 0. \qquad (2)$$

- (u, y) has α-finite gain if it is α-dissipative for $(-I, 0, \gamma^2 I)$.
- (u, y) is α-passive if it is α-dissipative for $(0, I, 0)$.
- (u, y) is strictly input α-passive if it is α-dissipative for $(0, I, -\epsilon I)$.
- (u, y) is strictly output α-passive if it is α-dissipative for $(-\epsilon I, I, 0)$.
- (u, y) is strictly α-passive if it is α-dissipative for $(\epsilon I, I, \delta I)$.

REMARK 1 (i) More generally, it is possible to define an $(\alpha_1, \alpha_2, \alpha_3)$-dissipativenes. For instance, the (α_1, α_2)-finite gain will be equivalent to $||y||_{\alpha_1} \leq \gamma ||u||_{\alpha_2}$.

(ii) For a non-null initial condition, a suited constant term must be added in the definition, e.g. $\langle y, Qy\rangle_{\alpha,T} + 2\langle y, Su\rangle_{\alpha,T} + \langle u, Ru\rangle_{\alpha,T} \geq C \quad (\forall u \in \mathcal{L}^2_{\alpha,e}), (\forall T < \infty)$. This approach is independent of the internal representations of the system, in the sense that they differ only in that constant term. It follows that an α-passive system with feedback control $u = -ky$ have $||y||_\alpha$ bounded when $k \geq 0$. In particular, for $u = 0$, $\alpha = 1$ and detectability properties of the output with respect to its state, the asymptotic attractiveness of the internal state's equilibrium point can be asserted (see Section 5).

Next, we show some properties intended to establish conditions for asserting the finite-gain property.

PROPOSITION 1 Let (u, y) a system or map from $\mathcal{U}$ to $\mathcal{Y}$

(i) If it is α-dissipative with $Q < 0$ then it has α-finite gain.

(ii) If it has α-finite gain and strictly input α-passive then it is α-strictly output passive.

(iii) If it is α-dissipative then it is β-dissipative for any $\beta > \alpha$.

(iv) If it is strictly output α-passive then it has α-finite gain.

(v) If it α-passive then the system $S : v \to z$, $v = y + u$, $z = y - u$ has α-finite gain.

Proof.

(i) Since Q is self-adjoint, $Q^{1/2}$ is self-adjoint. Let $\hat{S} := Q^{1/2}S$ and η such that $R + \hat{S}^T\hat{S} \leq \eta^2 I$ (e.g. η^2 could be the largest eigenvalue of $R + \hat{S}^T\hat{S}$), then

$$\left|\left|Q^{1/2}y - \hat{S}u\right|\right|^2_{\alpha,T} \leq \eta^2 ||u||^2_{\alpha,T}.$$

Therefore,

$$||y||^2_{\alpha,T} \leq K||u||^2_{\alpha,T},$$

where $K = ||Q||^{-1}(\eta + ||Q^{1/2}S||)^2$.

(ii) Using the finite gain and next the strictly input α-passivity, we get

$$||y||^2_\alpha \leq \gamma^2||u||^2_\alpha \leq \frac{2\gamma^2}{\epsilon}\langle y, u\rangle_\alpha,$$

implying strictly output α-passivity.

(iii) It follows from Definition 2 and the $(\beta - \alpha)$-integration of (2).

(iv) By definition and adding a positive term

$$\epsilon||y||_\alpha^2 \leq \langle y, u\rangle_\alpha + 1/2||1/\sqrt{\epsilon}u - \sqrt{\epsilon}y||_\alpha^2.$$

Developing the square, i.e.

$$\epsilon||y||_\alpha^2 \leq 1/2\epsilon||u||_\alpha^2 + \epsilon/2||y||_\alpha^2,$$

the finite gain is obtained.

(v) By using the definition of passivity and noting that $2u = (v - z)$ and $2y = (v + z)$, the claim follows. □

REMARK 2 It is an easy problem to show that these properties remain true when one considers a non-null term associated to the initial condition as in Remark 1(ii), with the exception of Proposition 1(iii). For the latter, we observe that if the term to add is a non-positive constant, we still have that the dissipation inequality (2) holds.

Next, we show two examples of the existence of dissipative fractional systems in the nonlinear and linear cases, respectively. It must be noted that when the initial condition term in both cases is non-null, a constant term will be added to inequality (2). Otherwise, since for null initial condition all fractional systems have the same behaviour, it follows that the dissipative property can be verified for any kind of fractional systems starting from rest, using the requirements below.

EXAMPLE 1 Consider the input–output system

$$\begin{cases} {}^R D^\alpha x = Ax + Bu \\ y = B^T P x \\ \lim_{t\to 0^+} I^{1-\alpha}x(t) = b, \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $y(t), u(t) \in \mathbb{R}^m$ for all $t > 0$, $0 < \alpha < 1$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, T denotes the transpose and $P \in \mathbb{R}^{n\times n}$ is a constant positive-definite matrix such that $A^T P + PA = Q$ is a negative semi-definite constant matrix. The solutions of (3) are continuous on $(0, T]$ for every $T > 0$ Diethelm (2010), with a singularity at $t = 0$. Define $V(\xi(t)) := \xi(t) := [I^{1-\alpha}x^T Px](t)$. The function ξ is continuous on $[0, T]$ for every $T > 0$. By applying inequality of RL derivative (see Alsaedi *et al.*, 2015), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\xi) \leq x^T Qx + x^T PBu \leq y^T u,$$

and by integration, we get

$$\int_0^t [y^T u]\mathrm{d}\tau \geq V(t) - V(0) \geq -V(0) \quad \forall t > 0.$$

Then, the RL fractional system (3) is 1-passive. The same holds for any fractional system of the form (3), not necessarily RL, starting from a null initial condition.

The following example provides a linear matrix inequality (LMI) characterization of the dissipativeness property for fractional linear systems.

EXAMPLE 2 Consider the commensurate system

$$\begin{cases} {}^C D^\alpha x = Ax + Bu \\ y = Cx + Du, \end{cases}$$

with $\alpha \leq 1$, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and real constant matrices A, B, C, D of suited dimensions. If the following system

$$\begin{cases} \begin{pmatrix} A^T Q + QA & QB - C^T \\ B^T Q - C & -D - D^T \end{pmatrix} \leq 0 \\ Q = Q^T \geq 0 \end{cases} \tag{4}$$

has a solution for Q, then the linear system is α-passive. Indeed, by defining $V = x^T Qx$ and using the second condition of (4), we have

$${}^C D^\alpha V \leq 2x^T QAx + x^T QBu = x^T QAx + x^T A^T Qx + x^T QBu,$$

where we have employed that ${}^C D^\alpha V \leq 2x^T P\, {}^C D^\alpha x$ (see Tuan & Trinh, 2017). Using the first condition of (4), we obtain the inequality (details omitted)

$$D^\alpha V \leq y^T u.$$

By α-integration, we have $I^\alpha y^T u \geq V(x(t)) \geq 0$, for a null initial condition (and thus, for any derivative) or $I^\alpha y^T u \geq -V(x(0))$, otherwise.

Note that condition (4) is independent of α. Therefore, if an integer linear system is dissipative, the system defined by changing its order of derivation to $\alpha \leq 1$ is still dissipative. On the other hand, condition (4) is necessary for dissipativeness when $\alpha = 1$ as shown in Willems (1972b, Theorem 3), so that in principle the set of quadruplets (A, B, C, D) could be enlarged for $\alpha < 1$.

3. Well-posed large-scale systems

When subsystems of different orders of differentiation are considered, we must especially consider the following existence and uniqueness problem. Given an external input e, there must exist a unique u such that it is consistent with a feedback connection since otherwise the calculations will remain ill-defined. Mathematically, u must satisfy $u = e + Fy = e + FG(u)$, where F is the linear interconnection matrix and (y, u) collects the output–input pairs of all subsystems. This connection encompasses others like $u = A_1 e + A_2 y$, by redefining e.

Hence, the problem can be seen as a fixed point one and, intuitively, it will require a Lipschitz-like condition on the system. We begin generalizing to fractional order the feedback connection of two systems (van der Schaft, 2000),

$$\begin{cases} u_1 = e_1 - y_2(u_2) \\ u_2 = e_2 + y_1(u_1), \end{cases} \tag{5}$$

where (e_1, e_2) are external inputs to the total system with output (y_1, y_2) with internal inputs (u_1, u_2). Let X to be a Banach space (e.g. $\mathcal{C}^m[0,T]$ the space of continuous function from $[0,T]$ to $\mathbb{R}^m$ or $\mathcal{L}^\infty(0,T)$ the space of bounded function from $[0,T]$ to $\mathbb{R}^m$).

Proposition 2 Consider the connection (5) of two systems such that for any $u, v \in X$

$$||y_i(u) - y_i(v)||_\alpha \le L_i||u - v||_\alpha, \quad i \in \{1, 2\}, \tag{6}$$

where $L_1 L_2 < 1$. Then, the system (5) has a unique solution in X.

Proof. From (5), $u_2 = e_2 + G_1(e_1 - G_2(u_2)) =: f(u_2, e)$. Hence,

$$||f(u,e) - f(v,e)||_\alpha \le L_1 L_2 ||u - v||_\alpha < ||u - v||_\alpha.$$

Thus, for any fixed e, f is a contraction map in X and it has a unique fixed point called u_2. A similar argument for u_1 completes the proof. □

A finite-gain linear system holds assumption (6) since $||y(u) - y(v)||_\alpha = ||y(u - v)||_\alpha \le L||u - v||_\alpha$ (note that initial condition terms cancel). Assumption (6) implies causality (i.e. if $u \equiv v$, then their outputs are the same) and for systems with $y(u \equiv 0) \equiv 0$ (e.g. a linear system) (6) implies finite gain. This simple result give us the idea for the general case.

Theorem 1 Consider the connection $u = e + Fy$ where $F \in \mathbb{R}^{m \times m}$ and $||F|| = C$. Suppose that for any $u, v \in X$

$$||y(u) - y(v)||_\alpha \le L||u - v||_\alpha, \tag{7}$$

where $L < C^{-1}$. Then the system $u = e + Fy$ has a unique solution in X, for any $e \in X$. Moreover, the map $e \in X \longrightarrow u \in X$ is well defined and Lipschitz.

Proof. The first part of the proof follows the one of Proposition 2 by defining $f(u, e) := e + FG(u)$ and noting that from the assumptions we get

$$||f(u,e) - f(v,e)||_\alpha \le C||G(u) - G(v)||_\alpha < ||u - v||_\alpha.$$

Since for every $e \in X$ there exists a unique $u \in X$, a map can be (well) defined. Considering $e_1, e_2 \in X$, we have

$$||u_1 - u_2||_\alpha \le ||e_1 - e_2||_\alpha + CL||u_1 - u_2||_\alpha,$$

whereby

$$||u_1 - u_2||_\alpha \le (1 - CL)^{-1}||e_1 - e_2||_\alpha,$$

which shows that the map is Lipschitz. □

By taking the induced matrix norm for the feedback matrix (5), that is $F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we get $||F|| = 2$ yielding a weaker claim than that given by Proposition 2 since (7) implies $L_1, L_2 < 1/2$. The reason is that in Theorem 1 it is not need to know the internal structure of the feedback matrix. From the Lipschitz property of the map $e \longrightarrow u$, it follows that a system holding (12) has α-finite gain and is causal. From the uniqueness, and if $y(0) = 0$, it follows that $e \equiv 0$ implies $u \equiv 0$.

The following proposition, together with Theorem 1, allows us to have a well-defined feedback system composed of different order subsystems.

PROPOSITION 3 Consider that for system (u, y) there exist numbers L and α such that for any $u, v \in X$

$$||y(u) - y(v)||_\alpha \leq L||u - v||_\alpha. \tag{8}$$

Then, for any $\beta > \alpha$

$$||y(u) - y(v)||_\beta \leq L||u - v||_\beta. \tag{9}$$

Proof. It follows from the $(\beta - \alpha)$-integration of (8) and the semi-group property of fractional integrals. □

4. Finite gain of large-scale systems

In this section, the main results asserting the finite gain of large-scale systems are established by imposing conditions on their subsystems and their connections. Finite gain is a special case of input–output stability (see e.g. Khalil, 1996; Sastry, 1999) where a bounded output is obtained when a bounded input is applied—the boundedness being given in a (fractional) integral norm. It also encompasses robust stability (e.g. if e_2 is a bounded noise in the connection (5)).

These results are established by requiring the same dissipative order for each subsystem. Due to Proposition 1(iii), Example 1 and the examples of the next section, they also work for multi-order systems.

4.1. *Small-gain conditions*

The intuition comes from requiring small enough finite gains on some subsystems so that the large-scale feedback system is also finite gain. Again, we start generalizing the two subsystems Small-gain theorem (see e.g. van der Schaft, 2000).

PROPOSITION 4 Consider the feedback connection (5) of two finite gain systems (u_i, y_i) for $i = 1, 2$ such that their finite gains hold the following small-gain condition

$$\gamma_1 \cdot \gamma_2 < 1. \tag{10}$$

Then the system with inputs (e_1, e_2) and outputs (y_1, y_2) has α-finite gain.

Proof. From (5), finite gain and triangular inequality we have

$$\begin{aligned} ||u_1||_\alpha &\le ||e_1||_\alpha + \gamma_2||u_2||_\alpha \\ &\le ||e_1||_\alpha + \gamma_2||e_2||_\alpha + \gamma_2\gamma_1||u_1||_\alpha \\ &= (1-\gamma_2\gamma_1)^{-1}\left[||e_1||_\alpha + \gamma_2||e_2||_\alpha\right]. \end{aligned}$$

Similarly, we obtain

$$||u_2||_\alpha \le (1-\gamma_2\gamma_1)^{-1}\left[||e_2||_\alpha + \gamma_1||e_1||_\alpha\right].$$

Therefore, $||(e_1, e_2)||_\alpha < \infty \Rightarrow ||(u_1, u_2)||_\alpha < \infty \Rightarrow ||(y_1, y_2)||_\alpha < \infty$, where the last implication is due to the finite gain of the subsystems. □

Note that from the proof, it is indifferent in (5) the sign of its elements, due to the norm inequality. The next result also makes this abstraction and requires even less information from the interconnection.

THEOREM 2 Consider a collection of subsystems holding the finite-gain condition

$$||y||_\alpha \le \gamma||u||_\alpha, \tag{11}$$

linearly connected trough $u = e + Fy$, where $u, e \in \mathcal{C}^n(0,T)$, $y \in \mathcal{C}^m(0,T)$ and the matrix $F \in \mathbb{R}^{n\times m}$ holding

$$||F|| < \gamma^{-1}. \tag{12}$$

Then, the system with input e and outputs y has finite gain.

Proof. From triangular inequality and hypothesis we obtain

$$\begin{aligned} ||u||_\alpha &\le ||e||_\alpha + ||F||\cdot||y||_\alpha \\ &\le ||e||_\alpha + ||F||\cdot\gamma\cdot||u||_\alpha \\ &\le (1-||F||\cdot\gamma)^{-1}||e||_\alpha. \end{aligned}$$

Therefore, $||(e_1,\dots,e_n)||_\alpha < \infty \Rightarrow ||(u_1,\dots,u_n)||_\alpha < \infty \Rightarrow ||(y_1,\dots,y_n)||_\alpha < \infty$. □

A sufficient condition for (11) in the single input single output (SISO) case, is to have finite gain. Indeed, if $\gamma := \max_{i=1,\dots,n}\gamma_i$, then $||y||_\alpha^2 = \sum_i ||y_i||_\alpha^2 \le \sum_i \gamma_i||u_i||_\alpha^2 \le \gamma\sum_i ||u_i||_\alpha^2 = \gamma||u||_\alpha^2$.

When particularized to connection (5), condition (12) requiring $\gamma_1, \gamma_2 < 1$, is conservative in regard to condition (10), due to the fact that we use less information on the interconnection structure. In this way, robustness against uncertainties in the value of the interconnection parameters is obtained. If the matrix of interconnection is fixed, some subsystems could be adjusted to get condition (12). Conversely, condition (12) indicates that when the subsystems have an unknown finite gain, matrix F can be adjusted to get a stable system.

4.2. *Dissipativeness conditions*

According to Proposition 1(i), a dissipative system has finite gain if a linear matrix inequality is verified. In the large-scale case, this LMI could be easier to check than conditions (11–12) since on the one hand the latter assume known bounds on the finite gains of each subsystem, and on the other, an LMI is numerically tractable. However, in contrast with the previous theorem, the specific structure of the connection matrix will be relevant, as shown in the next result, which generalizes to fractional order the main result in Moylan & Hill (1978).

THEOREM 3 Let (Q_i, S_i, R_i) α-dissipative systems interconnected trough $u = e + Fy$, with y and u vectors whose components are the outputs and inputs of each subsystem, respectively, e is an external input and F is a constant matrix. Assume that the matrices $Q := diag(Q_i), S := diag(S_i)$ and $R := diag(R_i)$ satisfy

$$\hat{Q} := SF + F^T S^T + F^T RF + Q < 0, \tag{13}$$

where T denotes the transpose. Then, the system (e, y) has α-finite gain.

Proof. From the definition of the matrices in the hypotheses, the form of the interconnection and the dissipativeness of each subsystem, we obtain

$$2\big\langle y, S(e+Fy)\big\rangle_{\alpha,T} + \big\langle (e+Fy), R(e+Fy)\big\rangle_{\alpha,T} + \langle y, Qy\rangle_{\alpha,T} \geq 0,$$

which is developed as

$$2\langle y, Se\rangle_{\alpha,T} + \langle e, Re\rangle_{\alpha,T} + \langle e, RFy\rangle_{\alpha,T} + \langle Re, Fy\rangle_{\alpha,T} + \langle y, Qy\rangle_{\alpha,T} + 2\langle y, SFy\rangle_{\alpha,T} + \langle Fy, RFy\rangle_{\alpha,T} \geq 0.$$

Using the definition of $\hat{Q}$, this inequality is compactly written as

$$\langle y, \hat{Q}y\rangle_{\alpha,T} + 2\langle y, (S + F^T R)e\rangle_{\alpha,T} + \langle e, Re\rangle_{\alpha,T} \geq 0.$$

Therefore, the large-scale system (e, y) is dissipative (note that $\hat{Q}, R$ are self-adjoint since Q, R are). By Proposition 1(i) finite gain is obtained provided that $\hat{Q} < 0$ i.e. (13) is verified. □

REMARK 3 Note that (13) is an algebraic condition independent of the dissipative order α. Thus, classic integer order results to obtain simplified versions of (13) can be used. For instance, if some subsystems are passive, that is $Q_i = 0$ and/or $R_i = 0$ (see Moylan & Hill, 1978) or if the interconnection has symmetric structure (see Ghanbari *et al.*, 2016). Note also that (13) is sometimes obtained in the integer literature for the connection $u = e - Fy$, which explains the flip of signs.

More flexibility in the verification of the stability of a large-scale system than (13) is obtained through the following result.

COROLLARY 1 Let (Q_i, S_i, R_i) α-dissipative systems interconnected as in Theorem 3. If there exists a diagonal matrix $D > 0$ such that

$$\hat{Q} := DSF + F^T S^T D + F^T FRF + DQ < 0, \tag{14}$$

then the system (e, y) has α-finite gain.

Proof. It follows along the same lines of the proof of Theorem 3, by redefining the inner product as $\langle x, y\rangle_{T,D,\alpha} := [{}_0I^{\alpha} x^T D y](T)$. □

4.3. *Passivity conditions*

Since the passive property is stronger than dissipativeness, in the sense that a α-passive system is also α-dissipative but the converse is not necessarily true, a simplification of condition (13) or (14) to get stability can be expected if passivity instead of dissipativeness is assumed. We start generalizing a result the passivity theorem (van der Schaft, 2000), showing an alternative condition to (13).

PROPOSITION 5 Let $y_1 = G_1(u_1)$ be a passive system and $y_2 = G_1(u_2)$ strictly input passive (or G_1 strictly output passive and G_2 passive). Consider the interconnection (5) with $e_2 = 0$. If $e_1 \in \mathcal{L}^2_\alpha$, then $y_1 \in \mathcal{L}^2_\alpha$.

Proof. By the linearity of the inner product and (5), we get

$$\langle y_1, e_1\rangle_\alpha = \langle y_1, u_1\rangle_\alpha + \langle y_1, y_2\rangle_\alpha.$$

Using the passivity hypotheses, (5) and $e_2 = 0$, we get

$$\langle y_1, e_1\rangle_\alpha \geq \epsilon_2 ||u_2||^2_\alpha.$$

From Cauchy–Schwarz

$$||y_1||_\alpha ||e_1||_\alpha \geq \epsilon_2 ||u_2||^2_\alpha = \epsilon_2 ||y_1||^2_\alpha,$$

yielding the finite-gain result. The case G_1 strictly output passive and G_2 passive is similar. □

REMARK 4 If G_2 is a controller of the plant G_1, Proposition 5 can be seen as a robustness result, since the closed loop remains stable even if the plant is not exactly G_1, although it is still passive.

The proof of the above simple case motivates the next large-scale result, where passivity of each subsystem and a passive condition on the interconnection are required. The latter asserts that collecting all the inputs and outputs of each subsystem in vectors y, u and e the following inequality is verified

$$y^T e \geq y^T u \quad \forall y, u, e \in X, \tag{15}$$

where X is a vector space of dimension n. In the linear case studied in this paper, where $u = e + Fy$, this is equivalent to the condition $F \leq 0$. For instance, the feedback connection (5) is passive since its connection matrix has null eigenvalues. Neutral or power preserving connections (Willems, 1972a) are thus special cases of passive connections.

THEOREM 4 Consider a passive interconnection of n strictly output passive systems. If $e_i \in \mathcal{L}^2_\alpha$ for $i = \{1, \ldots, n\}$, then $y \in \mathcal{L}^2_\alpha$.

Proof. From the passive interconnection, we obtain

$$\sum_{i=1}^{n} \langle y_i, e_i\rangle_\alpha \geq \sum_{i=1}^{n} \langle y_i, u_i\rangle_\alpha,$$

and from the passivity of each subsystem, it follows that

$$\sum_{i=1}^{n} \langle y_i, e_i \rangle_\alpha \geq \sum_{i=1}^{n} \epsilon_i ||y_i||_\alpha^2. \tag{16}$$

Using Cauchy–Schwarz for the Fractional inner product, we obtain

$$\sum_{i=1}^{n} ||y_i||_\alpha ||e_i||_\alpha \geq \sum_{i=1}^{n} \epsilon_i ||y_i||_\alpha^2.$$

Therefore, if $e_i \in \mathcal{L}_\alpha^2$ for all i then $y_i \in \mathcal{L}_\alpha^2$ for all i. Note that this also follows from Proposition 1(iv) and the fact that from (16) the large-scale system is strictly output passive. □

5. Examples

In this section we show the benefits of the dissipative extension previously presented, to analyse the stability of mixed-order systems. These are systems that present subsystems of different differentiation orders and provide a more precise model of real process (Caponetto *et al.*, 2010; Podlubny, 1999). The Lyapunov method to study stability (Gallegos & Duarte-Mermoud, 2016b; Tuan & Trinh, 2017) has been only extended to subsystems having the same less than one order of differentiation. Existence and smoothness of solutions for this class of systems can be found in Gallegos *et al.* (accepted), where there is also a discussion of the difficulties involved in the Lyapunov approach. Moreover, since the fractional systems are started from rest, the following results hold for any fractional derivative.

EXAMPLE 3 Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - x_2 + u_1 \\ y_1 &= h(x_1), \end{aligned}$$

where h is an arbitrary smooth function taking values at $(0, \infty)$ and $x_1, x_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$. This system is strictly output 1-passive (e.g. see Khalil, 1996, p. 259). Consider the fractional system

$$\begin{aligned} D^\alpha x_3 &= x_4 \\ D^\alpha x_4 &= -x_3^3 - x_4 + u_2 \\ y_2 &= x_4, \end{aligned}$$

where $x_3, x_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\alpha \leq 1$. Define $V_2 = (1/4)x_1^4 + (1/2)x_2^2$, which is convex and differentiable. Applying Tuan & Trinh (2017, Theorem 3) (since the fractional system started from rest, we can use it for any fractional derivative) and considering the vector field $f(x)$ of components $x_4, -x_3^3 - x_4 + u_2$, we have

$$\begin{aligned} D^\alpha V &\leq \frac{\partial V}{\partial x}^T f(x) \\ &= -y_2^2 + y_2 u_2. \end{aligned}$$

We deduce, by α-integration, that the system is strictly output α-passive (see Example 2). Consider the interconnection given by

$$u_1 = e + y_2$$
$$u_2 = y_1.$$

The resulting mixed-order system is driven only by the external input e. In particular, the fractional system is also 1-passive according to Proposition 1(iii), since we are considering systems starting from rest. From Theorem 4, we deduce that the interconnected system is finite gain 1-stable.

We consider now a connection having star-shaped symmetry, i.e. one where subsystems do not have interconnections with each other and the base system has interconnections with all the subsystems. It can be seen as a model of hierarchical control or an internet protocol command. Although more generally formulated, the base system is intended to be an integer system and the non-local behaviour is displaced to the subordinate systems.

EXAMPLE 4 Consider a base system (Q, S, R) α-dissipative connected to n subsystems (q, s, r) β-dissipative systems with $0 < \beta \le \alpha \le 1$ through the connection

$$F = \begin{bmatrix} F_0 & F_{12} & \dots & \dots & F_{12} \\ F_{21} & F_1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ F_{21} & 0 & \dots & 0 & F_1 \end{bmatrix}.$$

We suppose that the subsystems are driven only by the external input e, but the base system can have a non-null state. It follows that the interconnected system is $(\hat{Q}, \hat{S}, \hat{R})$ α-dissipative (see the proof of Theorem 3 together with Proposition 1(iii), for details). Hence, according to Theorem 3, if $\hat{Q} < 0$, the total system has α-finite gain.

Given the specific form of the connection and since the algebraic condition $\hat{Q} < 0$ is the same if $\alpha = \beta = 1$, we can use Ghanbari *et al.* (2016, Theorem 2) to give a bound on n so that the stability is assured. Particularizing to passive systems, that is $(Q, S, R) = (0, (1/2)I, 0)$ and $(q, s, r) = (0, (1/2)I, 0)$, we have that a necessary condition for the total system to have α-finite gain is that

$$n < \frac{\bar{\sigma}(\hat{Q})}{\sigma(\beta\hat{q}\beta)},$$

where $\sigma, \bar{\sigma}$ are the minimum and maximum eigenvalue functions, respectively, $\hat{Q} = (1/2)(H_0 + H_0^T)$, $\hat{q} = (1/2)(H_1 + H_1^T)$, $\beta = (1/2)(H_{12} + H_{12}^T)$ and $H = -F$. For instance, consider SISO systems with

$$F = \begin{bmatrix} 2 & 0.5 & \dots & \dots & 0.5 \\ 0.5 & 0.4 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0.5 & 0 & \dots & 0 & 0.4 \end{bmatrix}.$$

Then, $n < 20$ is enough to guarantee stability of the interconnected system.

Finally, we show that, from this approach, the attractiveness of an equilibrium point for an internal representation of a system, can be asserted, resembling the role of Lyapunov functions.

EXAMPLEX 5 Consider a system (u, y). The Lyapunov functions for unforced systems (i.e. systems with $u \equiv 0$) allow to deduce asymptotic properties of the solutions with initial conditions $x(t_0)$ close to the equilibrium point x^*, where x is the internal variable (state or pseudo-state). Instead, by restating this in the input–output framework, we work with an input u driving the system from rest to $x(t_0)$, after that being identically zero. Since we start from rest, the type of fractional derivative being used is indifferent, when the system has an internal representation expressed by a fractional equation.

More specifically, we consider bounded compactly supported functions u (i.e bounded functions vanishing outside of a bounded interval). If the system is controllable, for any $x(t_0)$ there exists such a function driving the system from x^* to $x(t_0)$. Therefore, the set of $x(t_0)$ closes to the equilibrium can be translated in the input–output sense as functions u whose infinite norm is close to zero.

If the system has α-finite gain, by Definition 2, we have that for any $T > 0$

$$||y||_{\alpha,T} \leq \gamma ||u||_{\alpha,T}.$$

Since u is compactly supported and bounded, $||u||_{\alpha,T}$ converges to zero as $T \to \infty$ for $\alpha < 1$ (Gallegos *et al.*, 2015, Property 17) or it remains bounded if $\alpha = 1$. Therefore, $||y||_{\alpha,T}$ also converges to zero for $\alpha < 1$ or it remains bounded if $\alpha = 1$. Thus, if y is uniformly continuous, then y converges to zero (Gallegos *et al.*, 2015, Lemma 21). Moreover, if the system is asymptotically detectable, that is, if $\lim_{t\to\infty} y(t) = 0$ implies $\lim_{t\to\infty} x(t) = x^*$, then x^* is locally attractive. Therefore, if the system is (globally) controllable and asymptotically detectable, x^* is (globally) attractive.

For instance, consider the system

$$D^\alpha y = -a(t)y + u,$$

where $0 < \alpha < 1$, a is a continuous bounded function such that $a(t) > \epsilon > 0$ for any $t \geq 0$. Assume null IC, that is $y(t) \equiv 0$ for $t \leq 0$. It is clear that this system is asymptotically detectable (since $x \equiv y$). On the other hand, by choosing $u = a(t)y + \frac{y_0}{[I^\alpha p](t_0)}$ for $t \in [0, t_0]$ and zero otherwise, we obtain that the system is driven from $y(0) = 0$ to $y(t_0) = y_0$, with p a function taking the value 1 on $[0, t_0]$ and zero otherwise, and $[I^\alpha p](t_0)$ is its fractional integral evaluated at $t = t_0$. Hence, the system is controllable.

Note that with this input the system becomes $D^\alpha y = -a(t)y$ for $t \geq t_0$ and $y(t_0) = y_0$. Then, we have

$$yD^\alpha y = -a(t)y^2 + uy$$
$$\frac{1}{2}D^\alpha y^2 \leq -\epsilon y^2 + uy,$$

where we use for the last inequality that, since the *IC* is null, any fractional derivative can be used, in particular, the Caputo one, which has this property (Tuan & Trinh, 2017). By α-integration (taking from $t = 0$), it follows that

$$-I^\alpha \epsilon y^2 + I^\alpha uy \geq \frac{1}{2}y^2 \geq 0.$$

Thus, by Definition 2, the system is strictly output α-passive. By Proposition 1(iv) the system has α-finite gain. By setting $V = V(y) := \frac{1}{2}y^2$, we have that for any $t \geq t_0$, $D^\alpha V \leq 0$ and V is a bounded function at $[0, t_0]$. Using Gallegos & Duarte-Mermoud (2016a, Theorem 8), it follows that V (i.e. y) is a bounded function (where, again, we use the argument above since the IC is null). Since $D^\alpha y = -a(t)y^2 + yu$ and a is bounded, it follows that $D^\alpha y$ is also a bounded function. Using Gallegos & Duarte-Mermoud (2016a, Proposition 1), it follows that y is uniformly continuous. Therefore, $y = 0$ is globally attractive and the solutions are bounded.

6. Conclusion

Real-order notions of dissipativeness, passivity and finite gain are established to deal with large-scale mixed-order systems consisting of subsystems, possibly defined with different orders of differentiation. These notions become properties of the system in the sense that they are independent of its internal representation and they are formulated for any type of fractional system. The well-posedness of such interconnected systems is then proved by requiring Lipschitz conditions on the subsystems. Conditions for input–output stability are established for multi-order large-scale systems. Essentially, these conditions are a trade-off between the knowledge of the interconnection and the dissipativeness requirements of the subsystems. These results can be employed in the controller synthesis of large-scale multi-order systems, as shown in examples.

Acknowledgements

The first author would like to thank CONICYT-PCHA/National PhD Scholarship Program 2018.

Funding

CONICYT-Chile (grant no. FB0809 to M.D.M.); and FONDECYT (1150488 to M.D.M.).

References

Alsaedi, A., Ahmad, B. & Kirane, M. (2015) Maximum principle for certain generalized time and space fractional diffusion equations. *Quart. Appl. Math*, **73**, 163–175.

Arcak, M. (2007) Passivity as a design tool for group coordination. *IEEE Trans. Automat. Control.*, **52**, 1380–1390.

Baleanu, D., Diethelm, K., Scalas, E. & Trujillo, J. (2016) *Fractional Calculus: Models and Numerical Methods*, 2nd edition. Singapore: World Scientific.

Caponetto, R., Dongola, G., Fortuna, L. & Petras, I. (2010) *Fractional Order Systems: Modeling and Control Applications*. Singapore: World Scientific.

Chopra, N. & Spong, M. W. (2006) Passivity-based control of multi-agent systems. In *Advances in Robot Control*. Berlin Germany: Springer, pp. 107–134.

Desoer, C. A. & Vidyasagar, M. (1975) *Feedback Systems: Input-Output Properties*. New York: Academic Press.

Diethelm, K. (2010) *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*. Lecture Notes in Mathematics, 2004. Berlin: Springer.

Diethelm, K. (2013) A fractional calculus based model for the simulation of an outbreak of dengue fever. *Nonlinear Dynam.*, **71**, 613–619.

Gallegos, J. A., Duarte-Mermoud, M. A., Aguila-Camacho, N. & Castro, R. (2015) On fractional extensions of Barbalat Lemma. *Syst. Control Lett.*, **84**, 7–12.

Gallegos, J. A. & Duarte-Mermoud, M. A. (2016a) Boundedness and convergence on fractional order systems. *J. Comput. Appl. Math.*, **296**, 815–826.

Gallegos, J. A. & Duarte-Mermoud, M. A. (2016b) On the Lyapunov Theory for fractional order systems. *Appl. Math. Comput.*, **287**, 161–170,

Gallegos, J. A., Duarte-Mermoud, M. A. & Castro-Linares, R. Mixed order robust adaptive control for general linear time invariant systems. *J. Franklin Inst.* (in press).

Gallegos J. A., Aguila-Camacho, N. & Duarte-Mermoud, M. A. (2018) Smooth solutions to mixed-order fractional differential systems with applications to stability analysis. *J. Integral Equations Appl.*, JIE17050.

Ghanbari, V., Wu, P. & Antsaklis, P. (2016). Large-scale dissipative and passive control systems and the role of star and cyclic symmetries. *IEEE Trans. Automat. Contr.*, **61**, 3676–3680.

Hill, D. J. & Moylan, P. J. (1980) Connections between finite-gain and asymptotic stability. *IEEE Trans. Automat. Contr.*, **AC-25**, 931–936.

Khalil H. K. (1996) *Nonlinear Systems*, 2nd ed. Englewood Cliffs, NJ: Prentice Hall.

Kilbas, A. A., Srivastava, H. M. & Trujillo, J. J. (2006) Theory and applications of fractional differential equations, In *North-Holland Mathematics Studies*, **204**. Amsterdam: Elsevier Science B.V.

Kottenstette, N., Hall, J., Koutsoukos, X., Sztipanovits, J. & Antsaklis, P. (2013) Design of networked control systems using passivity. *IEEE Trans. Control Syst. Technol.*, **21**, 649–665.

Moylan, P. & Hill, D. (1978) Stability criteria for large-scale systems. *IEEE Trans. Automat. Control*, **AC-23**, 143–149.

Podlubny, I. (1999) *Fractional Differential Equations*. San Diego: Academic Press.

Rakhshan, M., Gupta, V. & Godwine, B. (2017) On passivity of fractional order systems. arXiv:1706.07551v1 [math.DS].

Sastry S. (1999) *Nonlinear Systems: Analysis, Stability, and Control*. New York: Springer.

van der Schaft, A. (2000) *L2-Gain and Passivity Techniques in Nonlinear Control*. Berlin, Germany: Springer.

Tuan, H. & Trinh H. (2017) Stability of fractional-order nonlinear systems by Lyapunov direct method, *IET Control Theory Appl.,* DOI: 10.1049/iet-cta.2018.5233.

Willems, J. C. (1972a) Dissipative dynamical systems. Part I: General theory *Arch. Ration. Mechan. Anal.*, **45**, 321–351.

Willems, J.C. (1972b) Dissipative dynamical system. Part II: Linear systems with quadratic supply rates. *Arch. Rational Mech. Anal*, **45**, 352–393.