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Graphs admitting antimagic labeling for arbitrary sets of positive numbers☆

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ABSTRACT

Hartsfield and Ringel in 1990 conjectured that any connected graph with $q \geq 2$ edges has an edge labeling f with labels in the set $\{1, \dots, q\}$, such that for every two distinct vertices u and v, $f^u \neq f^v$, where $f^v = \sum_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges of the graph incident to vertex v.

We say that a graph $G = (V, E)$, with q edges, is *universal antimagic*, if for every set B of q positive numbers there is a bijection $f : E \to B$ such that $f^u \neq f^v$, for any two distinct vertices u and v. It is *weighted universal antimagic* if for any vertex weight function w and every set B of q positive numbers there is a bijection $f : E \to B$ such that $w(u) + f^u \neq w(v) + f^v$, for any two distinct vertices u and v.

In this work we prove that paths, cycles, and graphs whose connected components are cycles or paths of odd lengths are universal antimagic. We also prove that a split graph and any graph containing a complete bipartite graph as a spanning subgraph is universal antimagic. Surprisingly, we are also able to prove that any graph containing a complete bipartite graph $K_{n,m}$ with $n, m \geq 3$ as a spanning subgraph is weighted universal antimagic. From all the results we can derive effective methods to construct the labelings.

1. Introduction

Hartsfield and Ringel in 1990 conjectured that any connected graph with $q \geq 2$ edges has an *antimagic labeling*, that is, an edge labeling f with labels in the set $\{1, \dots, q\} =: [q]$, such that for every two distinct vertices u and v, $f^u \neq f^v$, where $f^v = \sum_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges of the graph incident to vertex v, see [7].

In the past fifteen years there have been interesting advances and several classes of graphs have been shown to admit antimagic labelings (see [1–6,9–15]). Yet, Hartsfield and Ringel's conjecture is still open even for trees.

In [16], the stronger notion of weighted-k-antimagic graphs was introduced, based on previous concepts presented in [8]. A graph $G = (V, E)$ with q edges is *weighted-k-antimagic* if for any vertex weight function w there is an edge labeling $f : E \to [q + k]$ such that $w(u) + f^u \neq w(v) + f^v$, for any two distinct vertices u and v. A function f that satisfies this property is called a (w, k)-*antimagic labeling* of G.

In this work we consider the following related notions: *universal antimagic* and *weighted universal antimagic* labelings.

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Universal antimagic

A graph $G = (V, E)$, with q edges, is *universal antimagic*, if for every set B of q positive numbers there is a bijection $f : E \to B$ such that $f^u \neq f^v$, for any two distinct vertices u and v. A function f satisfying this condition is called a B-antimagic labeling of G. Universal antimagic graphs should be rare. So far we do not know any connected graph with at least three vertices which is not universal antimagic. However, if we allow the set B to contain non-positive numbers, then there are graphs with no B-antimagic labeling. By instance, in [8] it was noticed that no path of i vertices P_i, for each $i \in \{3, 4, 5\}$ has an antimagic labeling using numbers in $\{-1, 0, \ldots, i-3\}$. The same idea applies to the complete graph $K_{1,n}$.

In this work we prove in Proposition 1 that paths, cycles, and graphs whose connected components are cycles or path of odd lengths are universal antimagic. In Proposition 2 we prove that split graphs are universal antimagic. In Theorem 3 we prove that any graph containing a complete bipartite graph as a spanning subgraph is universal antimagic.

Weighted universal antimagic

In [16] it was shown that the complete bipartite graph $K_{1,n}$ is weighted-2-antimagic but it is not weighted-0-antimagic. By instance, if we assign the weight 4 to all vertices of degree 1 of $K_{1,3}$, and weight 0 to the vertex v of degree three, then any labeling of $K_{1,3}$ will fail to be antimagic. In [11] it was proved that each connected graph G on p vertices having a universal vertex (a vertex which is adjacent to all other vertices of the graph) is weighted-1-antimagic, unless $G = K_{1,p-1}$ and p is even. Moreover, for each complete partite graph H, with $H \neq K_{1,n}, K_{2,2}$, any connected graph G containing H as a spanning subgraph is weighted-0-antimagic. The result is tight in the sense that $K_{1,n}$ and $K_{2,2}$ are not weighted-0-antimagic. In fact, if we assign weight 2 to the vertices in one of the independent sets of the graph $K_{2,2}$, and weight 0 to the vertices in the other independent set, then no labeling of this graph is antimagic.

A graph $G = (V, E)$, with q edges, is *weighted universal antimagic* if for any vertex weight function w and every set B of q positive numbers there is a bijection $f : E \to B$ such that $w(u) + f^u \neq w(v) + f^v$, for any two distinct vertices u and v.

It is clear that every weighted universal antimagic is also universal antimagic and there are universal antimagic graphs which are not weighted universal antimagic. By instance, $K_{1,n}$ and $K_{2,2}$ are universal antimagic but they are not weighted universal antimagic. Surprisingly, in Theorem 5 we prove that any graph containing a complete bipartite graph $K_{n,m}$ with $n, m \geq 3$ as a spanning subgraph is weighted universal antimagic. We left open the case $K_{2,n}$ with $n \geq 3$.

2. Universal antimagic graphs

We first consider graphs with maximum degree two. On the one hand, when they are connected we can build explicit B-antimagic labelings. On the other hand, there are disconnected graphs with maximum degree two for which there is no antimagic labeling, hence they are not universal antimagic. By instance, any disconnected graph whose connected components are paths with three vertices. However, we can prove the following result.

Proposition 1. *Let G be a graph with maximum degree 2. If G is connected or any of its connected components is a cycle or a path of odd length at least three, then G is universal antimagic.*

Proof. Since G has maximum degree two, we know that its connected components are cycles or paths. We first consider the case when G is a path with vertices $v_1, \ldots, v_{q+1}$ and edges $e_1, \ldots, e_q$, where $e_i = v_i v_{i+1}$, for each $i = 1, \ldots, q$. Let $B = \{b_1, \ldots, b_q\}$ be a set of q positive numbers $0 < b_1 < \cdots < b_q$.

For $q = 2$, the edge labeling $f(e_1) = b_1$ and $f(e_2) = b_2$ is B-antimagic since $b_1 < b_2 < b_1 + b_2$. For $q = 3$ the edge labeling $f(e_1) = b_1, f(e_3) = b_2$ and $f(e_2) = b_3$ is also antimagic since $b_1 < b_2 < b_1 + b_2 < b_1 + b_3$. For $q \geq 4$ we define an edge labeling of G as follows. We define $f(e_1) = b_1, f(e_q) = b_2$, and we iterate this process with the set $\{b_3, \ldots, b_q\}$ on the path with $q - 2$ edges $e_2, \ldots, e_{q-1}$. This clearly defines an B-antimagic labeling since

$$b_1 < b_2 < b_1 + b_3 < \cdots < b_{q-3} + b_{q-1} < b_{q-2} + b_q < b_{q-1} + b_q.$$

Notice that all previous labelings f for paths satisfy the following property: for each edge e incident to a vertex of degree one and each edge e' with both ends of degree two, $f(e) < f(e')$. When this happens we say that f is *degree-monotone*.

When G is a cycle with q edges, we can split any vertex into two new vertices, obtaining a path P with the same set of edges as G. Given a set $B = \{b_1, \ldots, b_q\}$ of q positive numbers $0 < b_1 < \cdots < b_q$, let f be the B-antimagic labeling previously obtained for P. Then f is also a B-antimagic labeling for G as the value $b_1 + b_2$ is smaller than $b_i + b_j$ for any $\{i, j\} \neq \{1, 2\}$.

We now assume that G has q edges and has at least two connected components. Let G' be the subgraph of G containing precisely the cycles of G and $G'' = G \setminus G'$. Then, by hypothesis, each connected component of G'' is a path of odd length with at least three vertices.

Let B be a set of q positive numbers $0 < b_1 < \cdots < b_q$. Let r be the number of edges of G' and let B' be the subset of B containing the r largest number of B. We assign the values of B' to the edges in G' in the following way: the largest ones to one cycle, the next largest ones to the next cycles and so on. Inside each cycle we proceed as in the unique cycle case. It is clear that the labeling f defined in this way satisfies that $f^u \neq f^v$ for any two distinct vertices u and v in G' and that

$f^u > a + b$ for any two a and b not in B'. Moreover, any labeling g of G'' with labels in $B'' = B \setminus B'$ satisfies that $g^v < f^u$, for each $v \in G''$ and each $u \in G'$.

Then, in order to finish the proof, we can assume that G contains only paths of odd length at least three.

We prove by induction on the number of edges of G that given any set B of q positive numbers there is a B-antimagic labeling of G which is also degree-monotone: the label of any edge incident to a vertex of degree one is less than the label assigned to any edge whose two ends have degree two.

We already noticed that when G is a path, there is a B-labeling which is degree-monotone.

Let k be the number of vertices of G of degree one. Let B' be the set of labels obtained from B by deleting the smaller k values and let $B'' = B \setminus B'$ be its complement.

If G has at least two connected components and one of them is P_4, then let b' be the smallest label in B', and let $b_1 < b_2$ be the smallest labels in B''. We apply the induction to the graph G' obtained from G by removing P_4 and with the set of labels $C = B \setminus \{b', b_1, b_2\}$. By the induction hypothesis, G' has a degree-monotone C-antimagic labeling. By labeling P_4 with its only degree-monotone $\{b', b_1, b_2\}$-antimagic labeling, we extend the labeling of G' to a B-labeling of G which is degree-monotone and B-antimagic because,

$$b_1 < b_2 < b'' < b' + b_1 < b' + b_2 < c + d$$

for every $b'' \in B''$ and every $c, d \in B'$.

If no connected component of G is P_4, then its connected components are path of odd length at least five. We apply induction as follows. Let G' be obtained from G by deleting all its vertices of degree one. Then, each connected component of G' is a path of odd length at least three.

By the induction hypothesis, G' has a degree-monotone B'-antimagic labeling g. Hence, $g^v > a+b > a$, for each vertex v of degree two in G', for each $a \notin B'$ and each $b \in B'$ assigned by g to an edge incident to a vertex of degree one. Let $v_1, \ldots, v_k$ be the vertices of G' of degree one, where the ordering is such that $g^{v_1} < \cdots < g^{v_k}$. Let $b_1 < \cdots < b_k$ be the labels in B''. We extend g to a labeling f of G by defining $f(v_i'v_i) = b_i$, where v_i is the neighbor of v_i not in G', for each $i \in [k]$. It is clear that

$$f^{v_1'} = b_1 < \cdots < f^{v_k'} = b_k < f^{v_1} = b_1 + g^{v_1} < \cdots < b_k + g^{v_k} < g^v,$$

for all v of degree two in G'.

Hence, f is a B-antimagic labeling which is also degree monotone. □

A *split-partition* of a connected graph $G = (V, E)$ is a partition $\{S, K, R\}$ of the set V, where S is an independent set and the following properties are satisfied: (1) for each $x \in S$, $N_G(x) \subsetneq K$ and (2) for each $x \in K$, $R \subseteq N_G(x)$, where $N_G(x)$ denotes the set of neighbors of a vertex x in G.

In [2], Barrus proves that any connected graph with at least three vertices and admitting a split-partition $\{S, K, R\}$, with K a set of pairwise adjacent vertices, is antimagic. It is not hard to see that the proof of this result can be modified to show that graphs admitting such a split-partition are universal antimagic. Therefore, the following result holds.

Proposition 2. *Any connected graph with at least three vertices and admitting a split-partition $\{S, K, R\}$, with K a set of pairwise adjacent vertices, is universal antimagic. In particular, split graphs are universal antimagic.*

In [13], Barrus' result was extended to each graph $G = (V, E)$ admitting a split-partition $\{S, K, R\}$, where K induces in G a regular graph. The proof of this result heavily relies on arithmetic relations between the elements of the set $\{1, \ldots, |E|\}$, which does not hold for general sets of positive numbers.

In what follows, we consider a similar situation where a graph contains a spanning subgraph which is a complete bipartite graph.

Theorem 3. *For each $1 \leq m \leq n$, any graph $G = (V, E)$ containing the complete bipartite graph $K_{m,n}$ as a spanning subgraph is universal antimagic.*

Proof. Let B be a subset of positive numbers of size $|E|$. We prove that there is a B-antimagic labeling of G.

We start with the case $m = 1$. If $K_{1,n}$ is a spanning subgraph of G, then there exists a universal vertex v in G. Let g be any partial labeling of the edges of the graph $G - v$, with the smallest numbers of B. We denote the vertices of $G - v$ by $v_1, v_2, \ldots, v_n$ such that, for each $i \in [n-1]$, $g^{v_i} \leq g^{v_{i+1}}$.

Let $b_1 < b_2 < \cdots < b_n$ be the largest n numbers of B. We extend the labeling g to a labeling f of G by defining $f(vv_i) = b_i$, for each $i \in [n]$. We have that

$$f^{v_1} = g^{v_1} + b_1 < f^{v_2} = g^{v_2} + b_2 < \cdots < f^{v_n} = g^{v_n} + b_n.$$

Since v has degree n and we assigned the largest numbers of B to the edges incident to v, then $f^{v_n} < f^v$, which means that f is a B-antimagic labeling of G.

Now we prove the case $m = 2$. We can assume G has no universal vertex, as otherwise, we can apply previous case. The case $n \leq 2$ was already considered in Propositions 1 and 2.

We now consider $n \geq 3$. Let $V = \{x, w\} \cup Y$ be the set of vertices of G, where $\{x, w\}$ is the independent set of size two of the spanning $K_{2,n}$, and Y is the independent set of size n. Let B be any set of $|E|$ positive numbers. As before, we shall define a B-antimagic labeling of G.

We can assume that $xw \notin E$ because G has no universal vertex. Let g be any labeling of the edges of the graph induced by Y with the smallest numbers of B. Let $Y = \{v_1, \dots, v_n\}$ such that $g^{v_i} \leq g^{v_{i+1}}$, for each $i \in [n-1]$.

Let $b_1 < \cdots < b_{2n}$ be the elements of B not used by g. We extend the labeling g to a labeling f of G by defining $f(xv_i) = b_{2i-1}$ and $f(wv_i) = b_{2i}$, for each $i \in [n]$.

Then, $f^{v_i} = g^{v_i} + b_{2i-1} + b_{2i}$, for each $i \in \{1, \dots, n\}$, $f^x = \sum_{i=1}^{n} b_{2i-1}$ and $f^w = \sum_{i=1}^{n} b_{2i}$. Thus,

$$f^{v_1} < \cdots < f^{v_n}$$

and $f^x < f^w$. Moreover, for each $i \in [n]$, the vertex v_i has at most $n-2$ neighbors because is not a universal vertex. Thus, for each $i \in [n]$, g^{v_i} is the sum of at most $n-2$ numbers of $B \setminus \{b_1, \dots, b_{2n}\}$ which implies that $f^{v_n} < f^w$ and

$$\begin{aligned} f^{v_{n-1}} &= g^{v_{n-1}} + b_{2n-3} + b_{2n-2} \\ &< \sum_{i=1}^{n-2} b_{2i-1} + b_{2n-3} + b_{2n-2} \\ &< \sum_{i=1}^{n-2} b_{2i-1} + b_{2n-3} + b_{2n-1} \\ &= f^x. \end{aligned}$$

Hence, if f is not a B-antimagic labeling, it is because $f^{v_n} = f^x$.

We have that some of $b_{2n} - b_{2n-1}$ or $b_{2n-2} - b_{2n-3}$ is smaller than $(f(w) - f(x))/2$ because $n \geq 3$ and $f(w) - f(x) = \sum_{i=1}^{n}(b_{2i} - b_{2i-1})$. Let $j \in \{n-1, n\}$ such that $b_{2j} - b_{2j-1} < (f(w) - f(x))/2$. Define h as the labeling obtained from f by interchanging the values at edges xv_j and wv_j. Clearly the new sums at the vertices change only at x and w. Moreover,

$$h^{v_n} = f^{v_n} = f^x < h^x < h^w < f^w.$$

Therefore,

$$h^{v_1} < \cdots < h^{v_n} = f^{v_n} < h^x < h^w$$

which implies that h is a B-antimagic labeling. □

In the next section, in Theorem 5, we shall prove that $K_{m,n}$ is weighted universal antimagic, whenever $m, n \geq 3$. As this property is stronger than what we need here, we omit the proof for the case $m \geq 3$.

3. Weighted antimagic graphs

In [16] it was observed that if G has a spanning subgraph H which is weighted-k-antimagic, then G itself is weighted-k-antimagic. Here we can prove the analogous property for weighted universal antimagic graphs.

Proposition 4. *Let $G = (V, E)$ be a graph and let $H = (V, F)$ be a spanning subgraph of G which is weighted universal antimagic. Then G is weighted universal antimagic.*

Proof. Let $w : V \to \mathbb{R}$ be a vertex weight function. Let $q = |E|$ and let B be a set with q positive numbers. We must prove that there is a bijection $f : E \to B$ such that $w(x) + f^x \neq w(y) + f^y$, for every $x, y \in V$ with $x \neq y$.

Let $p = |F|$ and let $A \subseteq B$ with $|A| = p$. We define f in two steps. We first assign the values in $B \setminus A$ to the edges not in F, arbitrarily. This defines f in the set $E \setminus F$. Let $w'(x) = w(x) + \sum_{zx \in E \setminus F} f(zx)$, for each $x \in V$.

Since H is weighted universal antimagic, given w' there is a function $g : F \to A$ such that $w'(x) + \sum_{zx \in F} g(zx) \neq w'(y) + \sum_{zy \in F} g(zy)$, for every $x, y \in V$ with $x \neq y$.

We define f restricted to F as the function g. Then, f is a bijection and for each pair of distinct vertices x and y we have that $w(x) + f^x = w(x) + \sum_{zx \in E \setminus F} f(zx) + \sum_{zx \in F} g(zx) = w'(x) + \sum_{zx \in F} g(zx) \neq w'(y) + \sum_{zy \in F} g(zy) = w(y) + f^y$. Therefore, G is weighted universal antimagic. □

Theorem 5. *Let G be a graph and let $K_{m,n}$ be a complete bipartite spanning subgraph of G with $n, m \geq 3$. Then G is weighted universal antimagic.*

Proof. From Proposition 4 it is enough to prove that $K_{m,n} = (V, F)$ is weighted universal antimagic, when $m, n \geq 3$.

Let R and C be the independent sets of $K_{m,n}$, with $|R| = m$ and $|C| = n$. Let $w : V \to \mathbb{R}$ be a vertex weight function. We assume that $R = \{u_1, \dots, u_m\}$ and $C = \{v_1, \dots, v_n\}$ such that $w(u_1) \leq \cdots \leq w(u_m)$ and $w(v_1) \leq \cdots \leq w(v_n)$.

Let $B = \{b_{i,j} : (i,j) \in [m] \times [n]\}$ be a subset of positive numbers such that

$$b_{1,1} < \cdots < b_{m,1} < \cdots < b_{1,n} < \cdots < b_{m,n}.$$

For a function $f : E \to B$, let

$$r_i^f = w(u_i) + \sum_{j=1}^{n} f(u_i v_j),$$

for each $i \in [m]$ and let

$$c_j^f = w(v_j) + \sum_{i=1}^{m} f(u_i v_j),$$

for each $j \in [n]$. Let $C^f = \{c_j^f : j \in [n]\}$ and $R^f = \{r_j^f : j \in [m]\}$.

For the following definitions we denote by (x, y) and $[x, y]$ the open and closed intervals defined by $x, y \in \mathbb{R}$. We say that i is a r-crash of f if

$$(r_i^f, r_{i+1}^f) \cap C^f = \emptyset \neq [r_i^f, r_{i+1}^f] \cap C^f.$$

Similarly, we say that j is a c-crash of f if

$$(c_j^f, c_{j+1}^f) \cap R^f = \emptyset \neq [c_j^f, c_{j+1}^f] \cap R^f.$$

A crash of f is either a r-crash of f or a c-crash of f.

With these definitions we have that f has a crash if and only if the set $C^f \cap R^f$ is non-empty. Notice that if we find a function f with $C^f \cap R^f$ empty, then we get the result.

Let $\mathcal{F}$ be the set of all functions f from $[m] \times [n]$ to B, satisfying the following.

1. f is a bijection, $r_1^f < \cdots < r_m^f$ and $c_1^f < \cdots < c_n^f$.
2. If $i \in [m]$ is a r-crash of f, then for all $k \in [n]$, $f(i,k) < f(i+1,k)$.
3. If $j \in [n]$ is a c-crash of f, then for all $k, l \in [m]$, $f(k,j) < f(l,j+1)$.

The set $\mathcal{F}$ is not empty: it is easy to see that the function $f(i,j) = b_{i,j}$ belongs to $\mathcal{F}$.

Indeed, the sums $\sum_{j=1}^{n} f(u_i v_j)$ increase with i and the sums $\sum_{i=1}^{m} f(u_i v_j)$ increase with j. Hence, $r_1^f < \cdots < r_m^f$ and $c_1^f < \cdots < c_n^f$.

Let $f \in \mathcal{F}$ be a function with $|R^f \cap C^f|$ minimum. To get the result we shall prove that $R^f \cap C^f$ is empty.

Assume that $i \in [m]$ is a r-crash of f. As $f \in \mathcal{F}$, for all $k \in [n]$, $f(i,k) < f(i+1,k)$. Moreover, as $n \geq 3$,

$$\begin{aligned} r_{i+1}^f - r_i^f &= w(u_{i+1}) - w(u_i) + \sum_{l=1}^{n} (f(i+1,l) - f(i,l)) \\ &> f(i+1,1) - f(i,1) + f(i+1,2) - f(i,2). \end{aligned}$$

Then, there is $(x,y) \in \{(f(i,1), f(i+1,1)), (f(i,2), f(i+1,2))\}$ such that $2(y-x) < r_{i+1}^f - r_i^f$.

By interchanging the values x and y in f we obtain a function g such that $r_i^f < r_i^g < r_{i+1}^g < r_{i+1}^f$, $r_k^g = r_k^f$, for all $k \neq i, i+1$ and $c_j^g = c_j^f, j \in [n]$. Hence, i is not a r-crash of g, $C^g = C^f$ and $R^g = (R^f \setminus \{r_i^f, r_{i+1}^f\}) \cup \{r_i^g, r_{i+1}^g\}$. Therefore, $|C^g \cap R^g| = |C^f \cap R^g| = |C^f \cap (R^f \setminus \{r_i^f, r_{i+1}^f\})| < |C^f \cap R^f|$ which is a contradiction since g belongs to $\mathcal{F}$. In fact, on the one hand, if t is a r-crash of g and $t < i$, then $g(t,k) = f(t,k) < f(t+1,k) \leq g(t+1,k)$, for each $k \in [n]$. And, if $t > i$, then $g(t,k) \leq f(t,k) < f(t+1,k) = g(t+1,k)$, for each $k \in [n]$. On the other hand, if $s \in [n]$ is a c-crash of g, then $g(k,s) < g(l,s)$, for each $k, l \in [m]$ since x and y belong to $\{f(t,1) : t \in [n]\}$ or both belong to $\{f(t,2) : t \in [n]\}$.

Assume now that f has a c-crash at $j \in [n]$. As $f \in \mathcal{F}$, for all $k, l \in [m]$, $f(k,j) < f(l,j+1)$. As $m \geq 3$, we can apply the same argument as in the previous case to show that there is $(x,y) \in \{(f(1,j), f(1,j+1)), (f(2,j), f(2,j+1))\}$ such that $2(y-x) < c_{j+1}^f - c_j^f$. By interchanging x and y we obtain g such that $c_j^f < c_j^g < c_{j+1}^g < c_{j+1}^f$, $c_l^g = c_l^f$, for all $l \neq j, j+1$ and $r_i^g = r_i^f$, for each $i \in [m]$. Hence, j is not a c-crash of g and, as f has no r-crash, neither has g.

If g has a c-crash at $s \in [n]$, then when $s < j$, we have $g(k,s) = f(k,s) < f(l,s+1) \leq g(l,s+1)$, for each $k, l \in [m]$. And, if $s > j$, then $g(k,s) \leq f(k,s) < f(l,s+1) = g(l,s+1)$, for each $k, l \in [m]$. Therefore, g belongs to $\mathcal{F}$ and, as before, it can be proved that $|C^g \cap R^g| < |C^f \cap R^f|$ which is again a contradiction. We conclude that f has no crash at all and therefore, $R^f \cap C^g$ is empty. □

All the results we present in this work give rise to effective method to compute the labelings. In particular, we can turn the proof of Theorem 5 into an algorithm as follows. Initially, we assign to the edge $i_i v_j$ of $K_{m,n}$ the value $b_{i,j}$. This defines a function in $\mathcal{F}$. Iteratively, we reduce the crashes of f. First, we reduce all r-crashes. The proof of Theorem 5 shows that we can build a new function in $\mathcal{F}$ with less r-crashes than f. When we get a function without r-crashes we start reducing c-crashes until no one remains. One can see that the initial function has at most $\min\{m, n\}$ crashes. Moreover, the reduction of one crash takes constant time. Therefore, after the initial step the algorithm uses linear time. In order to carry out the initial step, we need to sort the set B and the weights given by the function w. This is the more time consuming part of the algorithm which is $O(mn\log(mn))$.

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