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On some graph classes related to perfect graphs: A survey[☆]

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ABSTRACT

Perfect graphs form a well-known class of graphs introduced by Berge in the 1960s in terms of a min–max type equality involving two famous graph parameters. In this work, we survey certain variants and subclasses of perfect graphs defined by means of min–max relations of other graph parameters; namely: clique-perfect, coordinated, and neighborhood-perfect graphs. We show the connection between graph classes and both hypergraph theory, the clique graph operator, and some other graph classes. We review different partial characterizations of them by forbidden induced subgraphs, present the previous results, and the main open problems. Computational complexity problems are also discussed.

1. Introduction

Perfect graphs were defined by Berge in the 1960s in terms of a min–max type equality involving two important parameters: the *clique-number* and the *chromatic number*. Coloring a graph is the task of assigning colors to its vertices in such a way that no two adjacent vertices receive the same color. In many situations we are interested in knowing the minimum number of different colors needed to color a certain graph G. This minimum number is called the *chromatic number* of G and is denoted by $\chi(G)$. A *complete set* of a graph is a set of vertices that are pairwise adjacent and a *clique* (also known as a *maximal clique*) is a complete set that is not properly contained in any other. The maximum cardinality of a clique of a graph G is called the *clique number* of G and is denoted by $\omega(G)$. Clearly, in any coloring, the vertices of a clique must receive different colors. Thus, $\omega(G)$ is a trivial lower bound for $\chi(G)$, i.e., the min–max type inequality

$$\omega(G) \leq \chi(G) \quad \text{holds for any graph } G.$$

☆ Dedicated to Thomas Liebling and Jayme Luiz Szwarcfiter on the occasion of their 75th birthdays.

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Notice that the difference between $\chi(G)$ and $\omega(G)$ can be arbitrarily large. Moreover, Mycielski [70] presented a family of graphs $\{G_n\}_{n\geq 2}$ with $\omega(G_n) = 2$ and $\chi(G_n) = n$. In this context, Berge defined a graph G to be *perfect* if and only if the min–max type equality $\omega(G') = \chi(G')$ holds for each induced subgraph G' of G.

Min–max type relations play a remarkable role in the field of discrete mathematics. In the following pages, we will recall two famous min–max type theorems due to Kőnig for bipartite graphs. Other notable examples are Dilworth's theorem [36] that dictates that in any partial order the maximum size of an antichain equals the minimum number of chains needed to cover it, Menger's theorem [69] that states that the maximum number of disjoint paths joining two vertices s and t equals the minimum number of edges in an st-cut, and its generalization, the max-flow min-cut theorem [45] that ensures that the maximum amount of flow in a network equals the capacity of a minimum cut.

The complement of a graph G is the graph $\overline{G}$ whose vertex set is the same as the vertex set of G but such that any pair of different vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. In 1972, Lovász, and shortly after Fulkerson, proved a conjecture by Berge stating the following:

Theorem 1 (*Perfect Graph Theorem [64]*)**.** *A graph is perfect if and only if its complement is perfect.*

A stable set of a graph is a set of vertices that are pairwise nonadjacent. The *stability number* of a graph G is the maximum cardinality $\alpha(G)$ of a stable set of G. The *clique covering number* of a graph G is defined as the minimum number of cliques of G needed to cover the vertices of G, and it is denoted by $\theta(G)$. Clearly, $\alpha(G) \leq \theta(G)$. Moreover, $\alpha(G) = \omega(\overline{G})$ and $\theta(G) = \chi(\overline{G})$. Therefore, by the Perfect Graph Theorem, the notion of perfection can also be formulated in terms of a min–max type equality involving the stability number and the clique covering number: *a graph G is perfect if and only if $\alpha(G') = \theta(G')$ for each induced subgraph G' of G.*

A *hole* is a chordless cycle of length at least 5 (a chord is an edge joining two nonconsecutive vertices of the cycle). An *antihole* is the complement of a hole. A hole, or antihole, is said to be *odd* or *even* if it has an odd or an even number of vertices. We say that a graph *has an odd hole* (resp. *antihole*) if it contains an induced odd hole (resp. antihole).

It is not difficult to verify that odd holes and odd antiholes are *imperfect* (i.e., not perfect). Since the class of perfect graphs is hereditary, any perfect graph has no odd holes and no odd antiholes. Furthermore, Berge conjectured, and Chudnovsky, Robertson, Seymour, and Thomas proved the following *forbidden induced subgraph characterization* for perfect graphs.

Theorem 2 (*Strong Perfect Graph Theorem [27]*)**.** *A graph G is perfect if and only if G has no odd hole and no odd antihole.*

Shortly after, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković devised a polynomial-time algorithm for recognizing perfect graphs [25]. Recently, a polynomial-time algorithm for deciding whether any given graph has an odd hole was proposed [28]; this algorithm leads to a considerably simpler recognition algorithm for perfect graphs than that in [25].

Perfect graphs were widely studied and extensively surveyed. The books [7,51], and [74] are completely devoted to perfect graphs. The book [21] and the volume [78] contain chapters about perfect graphs. Some additional surveys on perfect graphs and/or the Perfect Graph Theorem are [26,66,76,79], and [86]. More than one hundred subclasses of perfect graphs were defined and surveyed in [54].

In this work, we survey certain variants and subclasses of perfect graphs defined by means of min–max relations of other graph parameters; namely: clique-perfect, coordinated, and neighborhood-perfect graphs. Our interest in clique-perfect and coordinated graphs is twofold: (i) their connection with two important properties in hypergraph theory: the Kőnig property and the edge coloring property (see Section 3), and (ii) their connection with perfectness of the clique graph (see Section 4). The class of neighborhood-perfect graphs is a class of perfect graphs introduced by Lehel and Tuza [61], which turns out to coincide with clique-perfect graphs when restricted to certain graph classes: chordal graphs, Helly circular-arc graphs and hereditary clique-Helly claw-free graphs (see Theorems 19 and 40 and Corollary 43).

The class of *clique-perfect graphs* is defined in a somewhat similar fashion to perfect graphs. A *clique-independent set* of a graph G is a subset of pairwise disjoint cliques of G. A *clique-transversal* of G is a subset of vertices intersecting all the cliques of G. Denote by $\alpha_c(G)$ and $\tau_c(G)$ the maximum cardinality of a clique-independent set and the minimum cardinality of a clique-transversal of G, respectively. Clearly, the min–max type inequality

$$\alpha_c(G) \leq \tau_c(G) \quad \text{holds for any graph } G.$$

Analogously to perfect graphs, a graph G is said to be *clique-perfect* if and only if $\alpha_c(G') = \tau_c(G')$ holds for each induced subgraph G' of G. A graph that is not clique-perfect is said to be *clique-imperfect*. It is important to mention that clique-perfect graphs do not need to be perfect since, for instance, odd antiholes of length $6n + 3$ are clique-perfect for each $n \geq 1$ (Reed, 2001, cf. [41]). The difference between $\alpha_c(G)$ and $\tau_c(G)$ can be arbitrarily large. Moreover, Durán, Lin, and Szwarcfiter presented in [41] a family of graphs $\{G_n\}_{n\geq 2}$ such that $\alpha_c(G_n) = 1$ and $\tau_c(G_n) = n$. The number of vertices of G_n grows exponentially. Later, Lakshmanan S. and Vijayakumar [58] found another family of graphs $\{H_n\}_{n\geq 1}$ such that $\alpha_c(H_n) = 2n + 1$ and $\tau_c(H_n) = 3n + 1$ but H_n has only $5n + 2$ vertices.

The equality between $\alpha_c(G)$ and $\tau_c(G)$ has been implicitly studied in the literature for long time, but the name 'clique-perfect' was first introduced by Guruswami and Pandu Rangan [52]. Some well-known graph classes that are clique-perfect are: balanced graphs [8], comparability graphs [2], dually chordal graphs [20], complements of acyclic graphs [10], and distance-hereditary graphs [59].

A *matching* of a graph G is a set of edges that pairwise do not share endpoints. A *vertex cover* of G is a set S of vertices of G such that every edge of G has at least one endpoint in S. The *matching number* $\nu(G)$ is the maximum cardinality of a matching of G, and the *vertex covering number* $\tau(G)$ is the minimum cardinality of a vertex cover. Kőnig's matching theorem [56] asserts that the min–max equality

$$\nu(G) = \tau(G) \quad \text{holds for any bipartite graph } G.$$

Notice that if G is bipartite and with no isolated vertices then the cliques of G coincide with its edges and, as a consequence, $\alpha_c(G) = \nu(G)$ and $\tau_c(G) = \tau(G)$. Then, by Kőnig's matching theorem, if G is bipartite and without isolated vertices then $\alpha_c(G) = \tau_c(G)$. It is easily seen that this equality holds even if the bipartite graph G is permitted to have isolated vertices. Since the induced subgraphs of a bipartite graph are also bipartite, it follows that bipartite graphs are clique-perfect. Thus, bipartite graphs can be regarded as a special type of clique-perfect graphs.

Coordinated graphs form a subclass of perfect graphs and are defined similarly. Let G be a graph, let $\gamma_c(G)$ be the minimum number of colors needed to color the cliques of G in such a way that two intersecting cliques receive different colors, and let $\Delta_c(G)$ be the maximum cardinality of a family of cliques all of which have at least one vertex of G in common. Clearly,

$$\Delta_c(G) \leq \gamma_c(G) \quad \text{holds for any graph } G.$$

Parameters Δ_c and γ_c are generally denoted in the literature by M and F, respectively.

A graph G is called *coordinated* if $\Delta_c(G') = \gamma_c(G')$ for each induced subgraph G' of G. Since the class of coordinated graphs is hereditary by definition, and since odd holes and odd antiholes are not coordinated [13] then, by Theorem 2, coordinated graphs are perfect. Also in this case, the difference between the parameters can be arbitrarily large. In fact, in [13] it is shown that, for antiholes G, the difference $\gamma_c(G) - \Delta_c(G)$ grows exponentially in the number of vertices of G.

If G is a triangle-free graph without isolated vertices, the cliques of G coincide with the edges, then $\gamma_c(G)$ coincides with $\gamma(G)$, the so called *chromatic index* (minimum number of colors to color the edges of a graph so that edges that share an endpoint receive different colors), and the parameter $\Delta_c(G)$ coincides with the *maximum degree* $\Delta(G)$ of the vertices of G. By Kőnig's edge coloring theorem [56],

$$\gamma(G) = \Delta(G) \quad \text{holds for any bipartite graph } G.$$

As in the case of clique-perfection, we can conclude that $\gamma_c(G) = \Delta_c(G)$ holds for any bipartite graph G and hence bipartite graphs are coordinated. Thus, coordinated graphs constitute another way of generalizing bipartite graphs.

Neighborhood-perfect graphs were defined in [61], also by the equality of two parameters for all induced subgraphs. Given a graph G, a set $C \subseteq V(G)$ is a *neighborhood-covering set* (or *neighborhood set*) if each edge and each vertex of G belongs to $G[v]$ for some $v \in C$, where $G[v]$ denotes the subgraph of G induced by the closed neighborhood of the vertex v. Two elements of $E(G) \cup V(G)$ are *neighborhood-independent* if there is no vertex $v \in V(G)$ such that both elements are in $G[v]$. A set $S \subseteq V(G) \cup E(G)$ is said to be a *neighborhood-independent set* if every pair of elements of S is neighborhood-independent. Let $\rho_n(G)$ be the size of a minimum neighborhood-covering set and $\alpha_n(G)$ of a maximum neighborhood-independent set. Clearly, $\alpha_n(G) \leq \rho_n(G)$ for every graph G. When $\rho_n(G') = \alpha_n(G')$ for every induced subgraph G' of G, G is called a *neighborhood-perfect* graph. It was proved in [61] that odd holes and odd antiholes are not neighborhood-perfect and hence, Theorem 2 implies that neighborhood-perfect graphs are also perfect.

In Section 2, we present the basic definitions and preliminary results. In Section 3, we discuss the connection between clique-perfect and coordinated graphs and hypergraph theory. In Section 4, we study how clique-perfection and coordination depend on properties of the clique graph. In Section 5, the classes of clique-perfect, coordinated, and neighborhood-perfect graphs are analyzed when restricted to different graph classes, detailing the previous results and the main open problems.

2. Definitions and preliminaries

All graphs in this paper are undirected, without loops and without multiple edges. We denote the vertex set of the graph G by $V(G)$, and the edge set by $E(G)$. For any set S, $|S|$ will denote its cardinality. C_n will denote the chordless cycle with n vertices, P_n the chordless path with n vertices, and K_n a complete graph with n vertices. Paths and cycles are assumed to be simple (i.e., with no repeated vertices aside from the starting and ending vertices in the case of cycles). A cycle of a graph is *Hamiltonian* if it visits every vertex of the graph. By the *edges of a cycle* we mean those edges joining two consecutive vertices of the cycle. A *triangle* is a complete graph with three vertices. A graph is *triangle-free* if it contains no triangle. Some small graphs to be referred in the sequel are depicted in Fig. 1.

A *universal vertex* in a graph is a vertex that is adjacent to all the other vertices of the graph. An *isolated vertex* is a vertex that is not adjacent to any other vertex of the graph. The *neighborhood* of a vertex v in a graph G is the set $N_G(v)$ consisting of all the vertices that are adjacent to v. The *degree* of v is $d_G(v) = |N_G(v)|$. The *common neighborhood* of an edge $e = vw$ is $N_G(e) = N_G(v) \cap N_G(w)$, and, in general, the *common neighborhood* of a nonempty subset W of vertices is $N_G(W) = \bigcap_{w \in W} N_G(w)$, while $N_G(\emptyset) = V(G)$. If H is a subgraph of G then $N_H(v) = N_G(v) \cap V(H)$, $N_H(e) = N_G(e) \cap V(H)$ and $N_H(W) = N_G(W) \cap V(H)$ for every vertex v, every edge e and every subset of vertices W. The *closed neighborhood* of v is

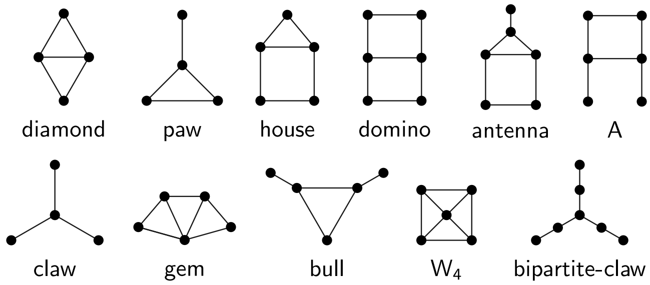

Fig. 1. Some small graphs.

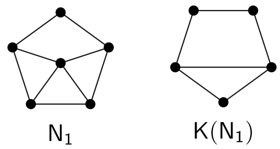

Fig. 2. The graph N_1 and its clique graph.

the set $N_G[v] = N_G(v) \cup \{v\}$. Two vertices v and w are *true twins* in G if $N_G[v] = N_G[w]$, and *false twins* if $N_G(v) = N_G(w)$. The subgraph of G *induced* by the vertex set $W \subseteq V(G)$ is denoted by $G[W]$, and $G - W$ denotes $G[V(G) \setminus W]$.

A graph G is *anticonnected* if $\overline{G}$ is connected. An *anticomponent* of G is the subgraph of G induced by the vertices of a connected component of $\overline{G}$.

A class $\mathcal{C}$ of graphs is called *hereditary* if, for every graph of $\mathcal{C}$, all its induced subgraphs belong to $\mathcal{C}$. Let G and H be graphs. We say that G is *H-free* to mean that G contains no induced H. If $\mathcal{H}$ is a collection of graphs we say that G is *$\mathcal{H}$-free* to mean that G contains no induced H for any $H \in \mathcal{H}$.

Let $\mathcal{F}$ be a family of sets. The *intersection graph* of $\mathcal{F}$ is a graph whose vertices are the members of $\mathcal{F}$, and such that two members of $\mathcal{F}$ are adjacent if and only if they intersect. For instance, the *line graph* $L(G)$ of a graph G is the intersection graph of the edges of G. Whitney [90] proved that if H and H' are connected graphs such that $L(H) = L(H') \neq K_3$ then $H = H'$.

Another example of an intersection graph is the clique graph. The *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G. The map $K : G \mapsto K(G)$ is known as the *clique graph operator* or simply the *clique operator*. A graph G is said to be *K-perfect* if $K(G)$ is perfect. If G is not K-perfect we say that it is *K-imperfect*. Notice that the class of K-perfect graphs is not hereditary. For instance, the graph N_1 of Fig. 2 is K-perfect but it contains an induced C_5 and $K(C_5) = C_5$ is imperfect. Because of this, the following terminology is introduced in [77]: a graph is *hereditary K-perfect* if all its induced subgraphs are K-perfect. It turns out that hereditary K-perfect graphs are perfect, as implied by Theorem 2 together with the following lemma.

Lemma 3 (*[77]*)**.** *A hereditary K-perfect graph has no odd holes and has no antiholes with more than* 6 *vertices.*

Interestingly, hereditary K-perfection has been implicitly characterized when restricted to several graph classes; many of these characterizations are presented in Section 5.

A family $\mathcal{F}$ of nonempty sets is said to *satisfy the Helly property* if every nonempty subfamily of $\mathcal{F}$ of pairwise intersecting members has nonempty intersection. A graph G is said to be *clique-Helly (CH)* if the family of its cliques satisfies the Helly property. The graphs of Fig. 3 are examples of graphs that are not clique-Helly. Clique-Helly graphs were characterized in [37] and independently in [84]. Notice that any graph with a universal vertex is clique-Helly and thus a clique-Helly graph may contain any prescribed induced subgraph. Instead, a graph is *hereditary clique-Helly (HCH)* [73] if all its induced subgraphs are clique-Helly. Prisner gave several characterizations of hereditary clique-Helly graphs, one by means of minimal forbidden induced subgraphs:

Theorem 4 (*[73]*)**.** *A graph is hereditary clique-Helly if and only if it contains none of the graphs of Fig. 3 as induced subgraph.*

In the sequel, we call any of the graphs in Fig. 3 a *pyramid*. The graph 0-pyramid is also called 3-*sun*. In [73,88], it is proved that a hereditary clique-Helly graph G has at most $|V(G)| + |E(G)|$ cliques.

Let G and H be two graphs. Assume that $V(G) \cap V(H) = \emptyset$. The *disjoint union* of G and H is a graph $G \cup H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. The disjoint union is clearly an associative operation, and for each nonnegative integer t we will denote by tG the disjoint union of t copies of G. The *join* of G and H is a graph $G + H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H) \cup \{vw : v \in V(G),\ w \in V(H)\}$.

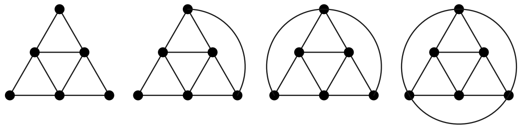

Fig. 3. From left to right: 0-, 1-, 2- and 3-pyramid.

A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) stable sets. A graph is *chordal* if every cycle of length at least 4 has at least one chord. A *comparability graph* is a graph that admits a transitive acyclic orientation of its edges. Bipartite and chordal graphs can be recognized in linear time and comparability graphs can be recognized in polynomial time [75,83]. Bipartite, chordal and comparability graphs are subclasses of perfect graphs.

A *cograph* [34] is a P_4-free graph, that is, a graph without chordless paths on 4 vertices. Equivalently, cographs are those graphs that can be obtained from isolated vertices by successively applying disjoint union and join operations. Cographs form a well-known class of perfect graphs.

We will study in this survey two superclasses of cographs: *P_4-tidy graphs* and *tree-cographs*. A graph $G = (V, E)$ is *P_4-tidy* if for every vertex set A inducing a P_4 in G there is at most one vertex $v \in V \setminus A$ such that $G[A \cup \{v\}]$ contains at least two induced P_4's. They were introduced in [50]. A *starfish* is a graph whose vertex set can be partitioned into three sets S, C and R, where each of the following conditions holds: (1) $S = \{s_1, \dots, s_t\}$ is a stable set and $C = \{c_1, \dots, c_t\}$ is a complete set, for some $t \geq 2$; (2) s_i is adjacent to c_j if and only if $i \neq j$; and (3) R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. An *urchin* is a graph whose vertex set can be partitioned into three sets S, C, and R satisfying the same conditions (1) and (3) but that instead of condition (2) satisfies: (2') s_i is adjacent to c_j if and only if $i = j$. Clearly, urchins are the complements of starfishes and vice versa. A *fat starfish* (resp. *fat urchin*) arises from a starfish (resp. urchin) with partition (S, C, R) by substituting exactly one vertex of $S \cup C$ by K_2 or $2K_1$.

Theorem 5 (*[50]*). *If G is a P_4-tidy graph, then exactly one of the following statements holds:*

1. *G or $\overline{G}$ is disconnected;*
2. *G is isomorphic to C_5, P_5, $\overline{P_5}$, a starfish, a fat starfish, an urchin, or a fat urchin.*

Tree-cographs were introduced in [85] by the following recursive definition:

1. Every tree is a tree-cograph.
2. If G is a tree-cograph, then $\overline{G}$ is a tree-cograph.
3. The disjoint union of tree-cographs is a tree-cograph.

This definition implies that if G is a tree-cograph, then either G or $\overline{G}$ is disconnected, or G is a tree or the complement of a tree. Tree-cographs are also a subclass of perfect graphs.

Distance-hereditary graphs form another superclass of cographs. A graph G is called *distance-hereditary* if and only if the distance between any two vertices of G is the same in every connected induced subgraph of G containing the two vertices. Equivalently, a graph is distance-hereditary if and only if it is {house,domino,gem}-free and has no holes [3].

A *circular-arc graph* [51] is the intersection graph of arcs of the unit circle. A *representation* of a circular-arc graph is a collection of arcs (of the unit circle), each corresponding to a unique vertex of the graph, such that two arcs intersect if and only if the corresponding vertices are adjacent. A *Helly circular-arc (HCA) graph* [49] is a circular-arc graph admitting a representation whose arcs satisfy the Helly property.

Let G be a graph, $Q_1, \dots, Q_k$ all its cliques and $v_1, \dots, v_n$ all its vertices. A *clique matrix* (or *clique-vertex incidence matrix*) of G is the $k \times n$ matrix $A = (a_{ij})$ where a_{ij} is 1 if $v_j \in Q_i$ and 0 otherwise. The clique matrix of a graph is unique up to permutations of rows and/or columns. Let A be a $m \times n$ zero–one matrix. We say that A is *perfect* if the set packing polytope

$$\{x \in \mathbb{R}^n \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$$

has all integer extreme points. Perfect graphs and perfect matrices are related by the following result [30,72]: *a graph is perfect if and only if its clique matrix is perfect.*

A zero–one matrix A is said to be *balanced* if and only if it contains no odd square submatrix with exactly two 1's in each row and in each column. Clearly, balancedness is preserved by row permutations, column permutations and transpositions. There is a forbidden submatrix characterization for balanced matrices in terms of perfect matrices: *Let A be a zero–one matrix. Then A is balanced if and only if all submatrices of A are perfect* [4,72]. In particular, balanced matrices are perfect. By analogy with the relation between perfect graphs and perfect matrices, Dahlhaus, Manuel and Miller proposed to call a graph *balanced* if its clique matrix is balanced [35]. There is a characterization of balanced graphs in terms of forbidden

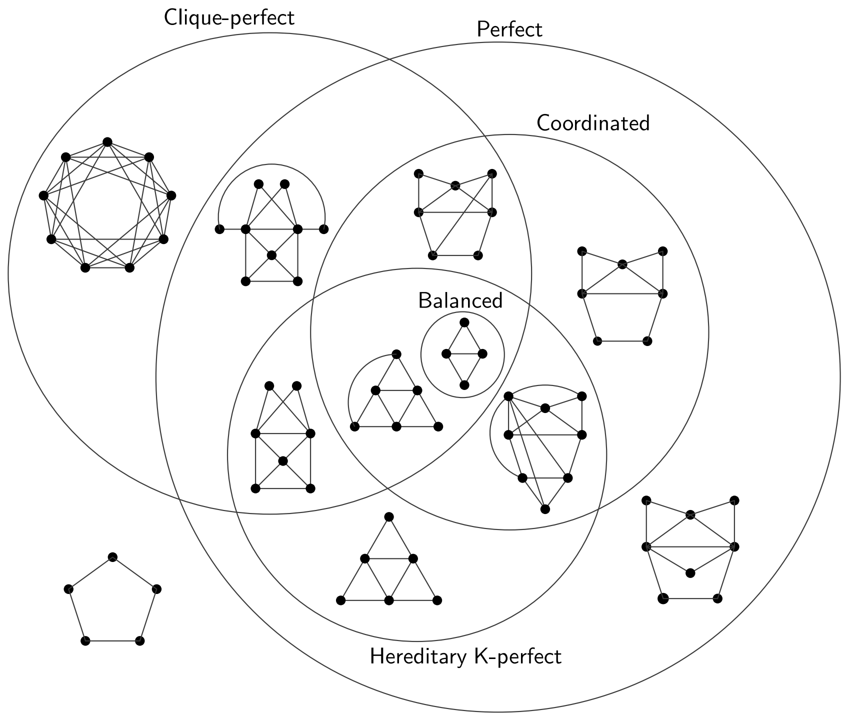

Fig. 4. Inclusions and intersections of the studied classes related to coordinated graphs, together with separating examples.

structures defined as follows. An *unbalanced cycle* of a graph G is an odd cycle C such that for each edge e of C there exists a (possibly empty) complete set W_e of G such that $W_e \subseteq N_G(e) \setminus C$ and $N_C(W_e) \cap N_C(e) = \emptyset$.

Theorem 6 ([4,15])**.** *A graph is balanced if and only if it contains no unbalanced cycle.*

In [73], Prisner proved that a graph is hereditary clique-Helly if and only if its clique matrix does not contain a 3×3 submatrix with exactly two 1's in each row and in each column. In particular, it turns out that balanced graphs are hereditary clique-Helly, thus the number of cliques of a balanced graph is bounded by its number of vertices plus its number of edges. Dahlhaus, Manuel and Miller observed that combining the polynomial-time algorithm in [87] that outputs the clique matrix of a hereditary clique-Helly graph, together with the polynomial-time algorithm in [32] that recognizes balanced matrices, a polynomial-time algorithm to recognize balanced graphs is obtained. The algorithm proposed in [32] by Conforti, Cornuéjols, Kapoor and Vušković was quite involved, but Zambelli [91] developed a simpler polynomial-time algorithm to test balancedness of a matrix.

Figs. 4 and 5 show the inclusion and intersection schemes of the graph classes that are the subject of this paper related to coordinated and neighborhood-perfect graphs, respectively.

3. Connection with hypergraph theory

A *hypergraph* H is an ordered pair $(X, \mathcal{E})$ where X is a finite set and $\mathcal{E}$ is a family of nonempty subsets of X. The elements of X are the *vertices* of H and the elements of $\mathcal{E}$ are the *hyperedges* of H. If $x_1, \ldots, x_n$ are the vertices of H and $E_1, \ldots, E_m$ are the hyperedges of H then a *hyperedge-vertex incidence matrix* of H is the $m \times n$ matrix $A = (a_{ij})$ where a_{ij} is 1 if $x_j \in E_i$ and 0 otherwise. A hypergraph has the *Helly property* if every nonempty family of pairwise intersecting hyperedges has a nonempty intersection. The *line graph* (or *representative graph*) of a hypergraph H, denoted by $L(H)$, is the intersection graph of the family $\mathcal{E}$ of hyperedges of H.

We will restrict ourselves to hypergraphs $(X, \mathcal{E})$ where $\bigcup \mathcal{E} = X$. A *partial hypergraph* of H is a hypergraph H' whose hyperedge set $\mathcal{E}'$ is a subset of the hyperedge set of H and whose vertex set is the union of the members of $\mathcal{E}'$.

We will be mostly interested in studying clique hypergraphs of graphs. Namely, the *clique hypergraph* of a graph G is the hypergraph $\mathcal{K}(G) = (X, \mathcal{E})$ where X is the set of vertices of G and $\mathcal{E}$ is the family of cliques of G. The hyperedge-vertex incidence matrix of $\mathcal{K}(G)$ is the clique matrix of G, and G is clique-Helly if and only if $\mathcal{K}(G)$ has the Helly property. Furthermore, clique graph and clique hypergraph are related in the following way: $K(G) = L(\mathcal{K}(G))$.

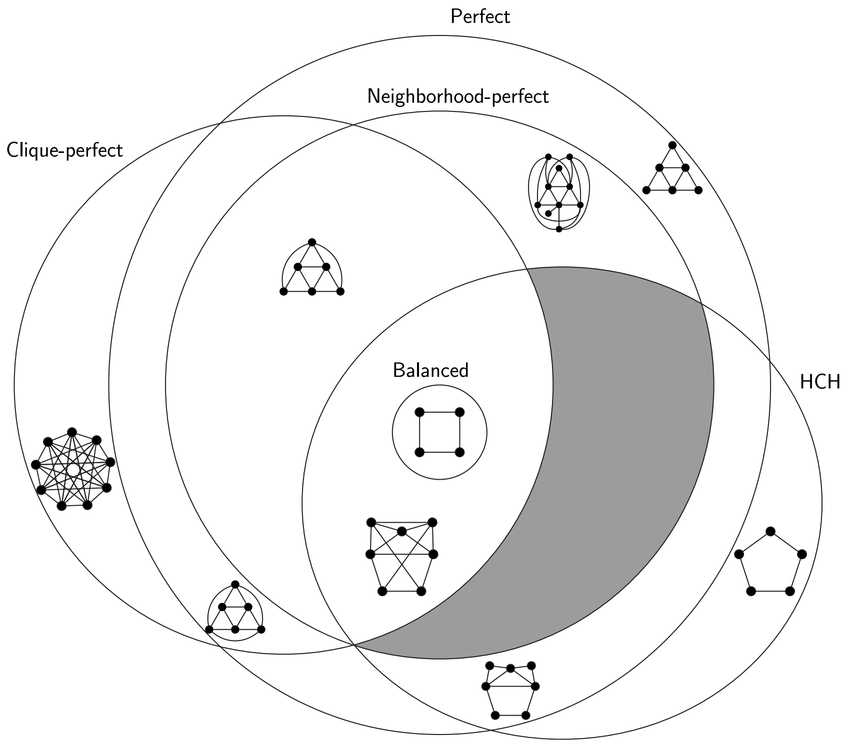

Fig. 5. Inclusions and intersections of the studied classes related to neighborhood-perfect graphs, with separating examples. The shaded region corresponds to an empty set.

3.1. The Kőnig property

A *matching* of H is a family of pairwise disjoint hyperedges and the *matching number* $\nu(H)$ is the maximum cardinality of a matching of H. A *transversal* of H is a set of vertices that meet all the hyperedges and the *transversal number* $\tau(H)$ is the minimum cardinality of a transversal of H. Clearly, $\nu(H) \leq \tau(H)$ for each hypergraph H. A hypergraph is said to satisfy the *Kőnig property* if $\nu(H) = \tau(H)$. Notice that $\alpha_c(G) = \nu(\mathcal{K}(G))$ and $\tau_c(G) = \tau(\mathcal{K}(G))$. Thus, we have by definition:

Remark 1. A graph G is clique-perfect if and only if the clique hypergraph $\mathcal{K}(G')$ has the Kőnig property for each induced subgraph G' of G.

3.2. The colored edge property

Another property of hypergraphs that are of our interest is the following. Let H be a hypergraph. The *chromatic index* $\gamma(H)$ of H is the least number of colors necessary to color the hyperedges of H such that any two intersecting hyperedges are colored with different colors. The *degree* $d_H(x)$ of a vertex x of H is the number of hyperedges of H containing x. The *maximum degree* of the hypergraph H is defined as $\Delta(H) = \max_{x\in X} d_H(x)$. Clearly, $\Delta(H) \leq \gamma(H)$ for any hypergraph H. Finally, a graph is said to have the *colored edge property* [6, p. 15] if and only if $\gamma(H) = \Delta(H)$. Since $\gamma_c(G) = \gamma(\mathcal{K}(G))$ and $\Delta_c(G) = \Delta(\mathcal{K}(G))$ hold, we obtain:

Remark 2. A graph G is coordinated if and only if the clique hypergraph $\mathcal{K}(G')$ has the colored edge property for each induced subgraph G' of G.

3.3. Normality

Lovász proved the following (we use the formulation of Berge [6, pp. 195–197]):

Theorem 7 ([64]). *Let H be a hypergraph, A_H the hyperedge-vertex incidence matrix of H and A_H^T its transpose. Then the following conditions are equivalent:*

1. *Every partial hypergraph of H has the Kőnig property.*
2. *Every partial hypergraph of H has the colored edge property.*
3. *The matrix A_H^T is perfect.*
4. *H satisfies the Helly property and $L(H)$ is perfect.*

Any hypergraph satisfying any of these conditions is said to be *normal*. Since $K(G) = L(\mathcal{K}(G))$, it follows as a corollary:

Corollary 8. *Let G be a graph, A_G the clique matrix of G and A_G^T its transpose. Then the following are equivalent:*

1. *Every partial hypergraph of $\mathcal{K}(G)$ has the Kőnig property.*
2. *Every partial hypergraph of $\mathcal{K}(G)$ has the colored edge property.*
3. *The matrix A_G^T is perfect.*
4. *G is clique-Helly and $K(G)$ is perfect.*

In light of this theorem we introduce the following terminology: a graph G is *clique-normal* if G is clique-Helly and $K(G)$ is perfect, or equivalently, if its clique hypergraph $\mathcal{K}(G)$ is normal. Notice that an induced subgraph of a clique-normal graph may not be clique-normal. For instance, the graph N_1 of Fig. 2 is clique-normal but contains an induced C_5 which is not even K-perfect.

So we introduce also the following definition: a graph G is said to be *hereditary clique-normal* if all the induced subgraphs of G are clique-normal. Equivalently, G is hereditary clique-normal if it is hereditary clique-Helly and hereditary K-perfect. Combining Corollary 8 with Remarks 1 and 2 it follows:

Corollary 9. *If G is hereditary clique-normal (i.e., hereditary clique-Helly and hereditary K-perfect) then G is clique-perfect and coordinated.*

The converse is not true because there are graphs that are clique-perfect and coordinated but not even hereditary K-perfect (see Fig. 4). Nevertheless, Corollaries 16 and 18 of Section 4 can be regarded as partial converses of Corollary 9.

A different characterization of clique-normal graphs arises from defining clique subgraphs, which are closer to partial hypergraphs of the clique hypergraph than induced subgraphs. Let G be a graph, $\mathcal{Q}$ the set of cliques of G and $\mathcal{Q}' \subseteq \mathcal{Q}$. Denote by $G_{\mathcal{Q}'}$ the subgraph of G formed exactly by the vertices and edges corresponding to the cliques in $\mathcal{Q}'$. If every clique of $G_{\mathcal{Q}'}$ is also a clique of G then $G_{\mathcal{Q}'}$ is called a *clique subgraph* of G [14]. A graph G is called *c–clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every clique subgraph H of G, and *c–coordinated* if $\gamma_c(H) = \Delta_c(H)$ for every clique subgraph H of G.

By definition, if H is a clique subgraph of a graph G then $K(H)$ is an induced subgraph of $K(G)$. This property allows to prove the following theorem:

Theorem 10 ([13,14])**.** *Let G be a clique-normal graph. Then G is c-clique-perfect and c-coordinated.*

Moreover, when G is hereditary clique-Helly, every induced subgraph of $K(G)$ is the clique graph of a clique subgraph of G [73]. So the following holds:

Theorem 11 ([13,14])**.** *Let G be a hereditary clique-Helly graph. Then the following statements are equivalent:*

1. *$K(G)$ is perfect.*
2. *G is c-clique-perfect.*
3. *G is c-coordinated.*
4. *G is clique-normal.*

It remains an open question whether the equivalence among assertions (1), (2), and (3) of the above theorem holds for (general) clique-Helly graphs G.

Berge defined in 1969 a hypergraph to be *balanced* (cf. [38, p. 397]) if its hyperedge-vertex incidence matrix is balanced. Recall that balanced graphs are those whose clique matrix is balanced, that is, those graphs whose clique hypergraph is balanced. We have also the following:

Theorem 12 ([8,64])**.** *If a hypergraph is balanced then it is also normal.*

Since the class of balanced graphs is hereditary [15,51] then:

Corollary 13. *Balanced graphs are hereditary clique-normal (i.e., hereditary clique-Helly and hereditary K-perfect). In particular, balanced graphs are clique-perfect and coordinated.*

The class of balanced graphs is a common subclass of clique-perfect and coordinated graphs that is interesting from a computational point of view. In fact, the problems of determining each of the parameters α_c, τ_c, Δ_c and γ_c are NP-complete [22], NP-hard [43], #P-complete [9] and {#P,NP}-hard [9], respectively. However, all these problems are known to be polynomially solvable when restricted to balanced graphs. Indeed, as we already mentioned, the size of the clique

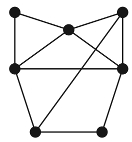

Fig. 6. An example of an $\alpha(K)$-perfect and $\chi(K)$-perfect but K-imperfect graph.

matrix of a balanced graph is bounded by a polynomial in the number of vertices and, consequently, can be computed in polynomial time. This, combined with the fact that the set packing and set covering polyhedra of balanced matrices are integral [46], implies the following:

Theorem 14 ([35,46]). *Each of the parameters α_c, τ_c, Δ_c and γ_c can be computed in polynomial time (in the number of vertices) for balanced graphs.*

4. Connection with the clique graph operator

The following result relates the parameters used to define clique-perfect graphs with the parameters used to define perfect graphs applied to the clique graph.

Theorem 15 ([14]). *Let G be a graph. Then:*

1. $\alpha_c(G) = \alpha(K(G))$.
2. $\tau_c(G) \geq \theta(K(G))$.
3. *If G is clique-Helly then* $\tau_c(G) = \theta(K(G))$.

The theorem above asserts, in particular, that

$$\alpha_c(G) = \alpha(K(G)) \leq \theta(K(G)) \leq \tau_c(G) \quad \text{holds for any graph } G \tag{$*$}$$

and that a graph is clique-perfect exactly when both inequalities are satisfied at equality for each of its induced subgraphs.

A graph G is *$\alpha(K)$-perfect* if $\alpha(K(G')) = \theta(K(G'))$ for each induced subgraph G' of G. The graph depicted in Fig. 6 is an example of an $\alpha(K)$-perfect graph that is not hereditary K-perfect. Notice that given a graph G, the clique graph $K(G)$ may contain some induced subgraphs which are not clique graphs of any induced subgraph of G and this is one reason why $\alpha(K)$-perfection turns out to be a strictly weaker property than hereditary K-perfection. With all this terminology, Theorem 15 implies the following variant of Corollary 9.

Corollary 16 ([14]). *If G is a clique-perfect graph then G is $\alpha(K)$-perfect. Furthermore, if G is hereditary clique-Helly then the converse also holds.*

Thus the class of $\alpha(K)$-perfect graphs is a superclass of both clique-perfect graphs and hereditary K-perfect graphs.

The analogue to Theorem 15 for coordinated graphs was found in [13].

Theorem 17 ([13]). *Let G be a graph. Then:*

1. $\gamma_c(G) = \chi(K(G))$.
2. $\Delta_c(G) \leq \omega(K(G))$.
3. *If G is clique-Helly then* $\Delta_c(G) = \omega(K(G))$.

As a consequence,

$$\Delta_c(G) \leq \omega(K(G)) \leq \chi(K(G)) = \gamma_c(G) \quad \text{holds for any graph } G,$$

and coordinated graphs are exactly those for which both inequalities are satisfied at equality for each induced subgraph. Again, as a corollary of Theorem 17 we will derive Corollary 18, which is another variant of Corollary 9 but in terms of *$\chi(K)$-perfectness*. A graph G is *$\chi(K)$-perfect* if $\chi(K(G')) = \omega(K(G'))$ for each induced subgraph G' of G. Again, the graph of Fig. 6 shows that $\chi(K)$-perfection is strictly weaker than hereditary K-perfection. Finally, the analogue to Corollary 16 is the following.

Corollary 18. *If G is a coordinated graph then G is $\chi(K)$-perfect. Furthermore, if G is hereditary clique-Helly then the converse also holds.*

Therefore $\chi(K)$-perfect graphs constitute a superclass of both coordinated graphs and hereditary K-perfect graphs.

Finally, note that the $\alpha(K)$-perfect and $\chi(K)$-perfect graphs define classes that are incomparable because $\overline{C_9}$ is $\alpha(K)$-perfect but not $\chi(K)$-perfect, and the viking with 7 vertices (cf. Fig. 11) is $\chi(K)$-perfect but not $\alpha(K)$-perfect. The graph of Fig. 6 is $\alpha(K)$-perfect and $\chi(K)$-perfect but not K-perfect, and proves the following proper inclusion:

hereditary K-perfect $\subset \alpha(K)$-perfect $\cap\ \chi(K)$-perfect

5. Partial characterizations

In this section we review characterizations of clique-perfect and coordinated graphs when restricted to different graph classes. We review the previous results and formulate the main open problems. Computational complexity issues are also discussed.

The class of clique-perfect graphs is hereditary and thus admits some forbidden induced subgraph characterization. Nevertheless, although some families of forbidden induced subgraphs were identified and some partial characterizations were formulated, a complete list of forbidden induced subgraphs for the class of clique-perfect graphs is not known. Furthermore, the problem of determining the complexity of the recognition of clique-perfect graphs is also open. These two questions are regarded as the main open problems related to clique-perfect graphs (see for instance [11]).

The coordinated graph recognition problem is NP-hard and it is NP-complete even restricted to {gem, C_4,odd hole}-free graphs with $\Delta = 4$, $\omega = 3$ and $\Delta_c = 3$ [80]. The main open problem regarding coordinated graphs is to find the complete list of minimal forbidden induced subgraphs. This problem seems to be difficult because in [82] several families of minimally non-coordinated graphs were described whose cardinality grows exponentially on the number of vertices and edges.

Although there are some partial results, the problem of given a characterization by forbidden induced subgraphs of neighborhood-perfect graphs is also open in general. Similarly, although some polynomial-time algorithms for recognizing neighborhood-perfectness when the input graph is known to belong to certain graph classes, the computational complexity of recognizing neighborhood-perfect graphs in general is not known. These are the main open problems regarding neighborhood-perfect graphs.

In this section we present the known partial results on clique-perfect, coordinated graphs, and/or neighborhood-perfect graphs regarding the problems of characterizing by minimal forbidden induced subgraphs and determining the computational complexity of their recognition when restricted to different graph classes. These graph classes are: chordal graphs, diamond-free graphs, P_4-tidy graphs, Helly circular-arc graphs, complements of forests, line graphs and complements of line graphs, some other subclasses of claw-free graphs and two superclasses of triangle-free graphs.

5.1. Chordal graphs

An r-*sun* or simply *sun* [44] is a chordal graph G on $2r$ vertices, $r \geq 3$, whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is a stable set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \mod r$. A sun is *odd* if r is odd. A sun is *complete* if U is a complete set.

Lehel and Tuza [61] proved that for chordal graphs, balancedness and neighborhood-perfection coincide with the absence of odd suns as induced subgraphs. This result, together with the fact that odd suns are clique-imperfect [14] implies the following.

Theorem 19 (*[14,61]*)**.** *For each chordal graph G, the following assertions are equivalent:*

1. *G is clique-perfect.*
2. *G is balanced.*
3. *G is neighborhood-perfect.*
4. *G contains no induced odd sun.*

Notice that odd suns may properly contain odd suns as induced subgraphs, thus unfortunately this characterization is not by *minimal* forbidden induced subgraphs. Indeed, it is an open problem to determine the minimal odd suns in the sense of induced subgraphs. As balanced graphs can be recognized in polynomial time, the same algorithm solves, in polynomial time, the problem of recognizing clique-perfect graphs (and thus neighborhood-perfect graphs) restricted to chordal graphs.

Since clique-perfect chordal graphs coincide with balanced chordal graphs, it follows that both are subclasses of coordinated chordal graphs. The inclusion is proper since, for instance, the graph displayed in Fig. 7 is an example of an odd sun that is coordinated. The fact that clique-perfect chordal graphs are balanced implies, by Theorem 14, that for clique-perfect chordal graphs both parameters α_c and τ_c (that coincide) can be computed in polynomial time.

Notice that also the 3-sun is chordal and hereditary K-perfect but not clique-perfect, or coordinated. So hereditary K-perfection does not coincide with coordination or clique-perfection or balancedness even when restricted to chordal graphs.

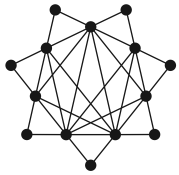

Fig. 7. A chordal coordinated odd sun.

5.2. Diamond-free graphs

The concept of suns was extended as follows. Let G be a graph and C a cycle of G not necessarily induced. An edge of C is *non-proper* (or *improper*) if it forms a triangle with some vertex of C. An *r-generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non-proper edges $\{e_j\}_{j\in J}$ (J is permitted to be an empty set) and a stable set $U = \{u_j\}_{j\in J}$, such that for each $j \in J$, u_j is adjacent exactly to the endpoints of e_j. An r-generalized sun is said to be *odd* if r is odd. Clearly odd holes and odd suns are odd generalized suns. We call a cycle *proper* if none of its edges is improper. By definition, proper odd cycles are odd generalized suns. It turns out that odd generalized suns are not clique-perfect. Indeed, the following holds.

Theorem 20 (*[14,41]*)**.** *Odd generalized suns and antiholes of a length not divisible by* 3 *are clique-imperfect.*

In fact, the characterization by forbidden induced subgraphs of those diamond-free graphs that are clique-perfect is the following.

Theorem 21 (*[12]*)**.** *Let G be a diamond-free graph. Then G is clique-perfect if and only if G contains no induced odd generalized sun.*

In fact, the authors prove that diamond-free graphs with no odd generalized suns are hereditary clique-Helly and hereditary K-perfect, and therefore clique-perfect.

The problem of deciding whether there exists a polynomial-time algorithm for the recognition of the clique-perfection of a diamond-free graph was left open in [12]. This problem was solved in [16], where it was proved that such an algorithm exists, based on the following result.

Theorem 22 (*[16]*)**.** *Let G be a diamond-free graph. Then G is clique-perfect if and only if G is balanced.*

Thus, the problems of recognizing balancedness and recognizing clique-perfection coincide when restricted to diamond-free graphs. Therefore the recognition of clique-perfection can also be solved in polynomial time for diamond-free graphs. Moreover, by Theorems 14 and 22, α_c (and therefore also τ_c) can be computed in polynomial time for clique-perfect diamond-free graphs.

Notice that if G is a diamond-free graph, the problem of deciding whether G is a minimal odd generalized sun can be solved in polynomial time (it suffices to check that G is not clique-perfect but $G-\{v\}$ is clique-perfect for every vertex v of G). Surprisingly, the problem of deciding whether G is an odd generalized sun (not necessarily minimal) is NP-complete even if G is a triangle-free graph [57]. Notice that an odd cycle in a triangle-free graph cannot have improper edges. Hence, if G is a triangle-free graph with an odd number of vertices, then G is an odd generalized sun if and only if G has a Hamiltonian cycle, and the Hamiltonian cycle problem on triangle-free graphs with an odd number of vertices is NP-complete [48, pp. 56–60].

Since

$$\text{balanced graphs} \subset \text{hereditary clique-Helly} \cap \text{hereditary } K\text{-perfect} \subset \text{clique-perfect}$$

holds and diamond-free graphs are hereditary clique-Helly we conclude, as a corollary of Theorem 22, the following.

Corollary 23. *Let G be a diamond-free graph. Then the following conditions are equivalent (and can be decided in polynomial time):*

1. *G is balanced.*
2. *G is hereditary K-perfect.*
3. *G is clique-perfect.*
4. *G contains no induced proper odd cycle.*

Since diamond-free clique-perfect graphs are balanced, they are also coordinated. Therefore the class of clique-perfect diamond-free graphs is a subclass of the class of coordinated diamond-free graphs. The inclusion is proper since the graph of Fig. 8 is coordinated but not clique-perfect.

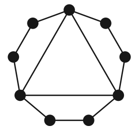

Fig. 8. A coordinated diamond-free odd generalized sun.

Odd holes and complete odd suns are minimally clique-imperfect. However, there are other odd generalized suns that contain proper induced odd generalized suns and consequently are not minimally clique-imperfect. The same is true even for proper odd cycles. Thus, the characterizations of Theorem 22 and Corollary 23 are not by *minimal* forbidden induced subgraphs. The minimal forbidden induced subgraph characterization of clique-perfect (or equivalently balanced) graphs restricted to diamond-free graphs was described in [1]. The corresponding minimal forbidden diamond-free induced subgraphs are the odd holes and the *sunoids* (which can be put in correspondence with Dyck-paths); the reader is referred to [1] for the details.

5.3. Superclasses of cographs

It is known that comparability graphs are clique-perfect [2]. Since cographs are comparability graphs [47] then cographs are also clique-perfect. In [58] the authors gave a simpler proof of the clique-perfection of cographs based on the following result.

Theorem 24 ([58]). *Let G be the join of the graphs G_1 and G_2. Then $\alpha_c(G) = \min\{\alpha_c(G_1), \alpha_c(G_2)\}$ and $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$. In particular, G is clique-perfect if and only if G_1 and G_2 are clique-perfect.*

As a corollary it is possible to compute $\alpha_c(G)$ and $\tau_c(G)$ of a given cograph G in linear time by first computing its cotree [34].

Distance-hereditary graphs and P_4-tidy graphs define two superclasses of cographs not included in the class of comparability graphs. In [59], distance-hereditary graphs were shown to be clique-perfect, relying on a decomposition tree of distance-hereditary graphs introduced in [23]. In [59], linear-time algorithms for computing $\alpha_c(G)$ and $\tau_c(G)$ for any distance-hereditary graph are also presented. In [16], those P_4-tidy graphs that are clique-perfect were characterized by minimal forbidden induced subgraphs. Moreover, it was shown that the problems of recognizing clique-perfect graphs can be solved in linear time for P_4-tidy graphs.

Theorem 25 ([16]). *Let G be a P_4-tidy graph. Then G is clique-perfect if and only if G contains neither C_5 nor 3-sun as an induced subgraph. Moreover, clique-perfectness of P_4-tidy graphs can be decided in linear time.*

Furthermore, it was shown that both parameters that define clique-perfectness can be computed in linear time for P_4-tidy graphs.

Theorem 26 ([16]). *There are linear-time algorithms that compute $\alpha_c(G)$ and $\tau_c(G)$ for any given P_4-tidy graph G.*

The proof of the above theorems relies on the structural characterization of P_4-tidy graphs given in Theorem 5.

Gyárfás et al. [53] characterized those cographs that are neighborhood-perfect by forbidden induced subgraphs.

Theorem 27 ([53]). *A cograph G is neighborhood-perfect if and only if G contains no induced $\overline{3K_2}$.*

The above characterization was generalized from cographs to P_4-tidy graphs in [42], where a linear-time recognition algorithm for neighborhood-perfectness of P_4-tidy graphs was also devised.

Theorem 28 ([42]). *A P_4-tidy graph G is neighborhood-perfect if and only if G has no induced $\overline{3K_2}$, 3-sun, or C_5. Moreover, neighborhood-perfectness of P_4-tidy graphs can be decided in linear time.*

Moreover, α_n and ρ_n can be computed in linear time for P_4-tidy graphs.

Theorem 29 ([42]). *There are linear-time algorithms that compute $\alpha_n(G)$ and $\rho_n(G)$ for any given P_4-tidy graph G.*

Since the 3-sun and C_5 are not coordinated, the class of coordinated P_4-tidy graphs is included in the class of clique-perfect P_4-tidy graphs. Moreover, the inclusion is proper since the graph $\overline{\text{tent} \cup K_2}$ is P_4-tidy and clique-perfect (having $\alpha_c = \tau_c = 2$) but not coordinated (since $\Delta_c = 3$ and $\gamma_c = 4$). The graph $\overline{3K_2}$ is an example of a coordinated P_4-tidy graph that is not neighborhood-perfect; thus, coordinated P_4-tidy graphs are not contained in neighborhood-perfect P_4-tidy graphs.

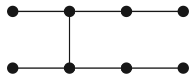

Fig. 9. The graph R.

It was proved in [10] that if F is a forest then $\overline{F}$ is clique-perfect. As the class of complements of forests is closed under taking induced subgraphs, it suffices to prove that $\alpha_c(\overline{F}) = \tau_c(\overline{F})$. We may assume that F has no isolated vertex, because if u were an isolated vertex of F then every clique of $\overline{F}$ would contain u implying $\alpha_c(\overline{F}) = \tau_c(\overline{F}) = 1$. Thus F has no universal vertex, which means that $\tau_c(\overline{F}) > 1$. Moreover F has some leaf u and let v be the only neighbor of u in F. Then $\{u, v\}$ is a clique-transversal of $\overline{F}$, which shows that $\tau_c(\overline{F}) = 2$. Furthermore, since every connected component of F is a tree with at least two vertices, $\alpha_c(\overline{F}) = \tau_c(\overline{F}) = 2$. This, together with Theorem 24 immediately implies the following.

Theorem 30 (*[10,58]*)**.** *Every tree-cograph is clique-perfect.*

In [42], the following characterization of neighborhood-perfect graphs by forbidden induced subgraphs was proved.

Theorem 31 (*[42]*)**.** *If G is a tree-cograph, then G is neighborhood-perfect if and only if G contains no induced $\overline{3K_2}$ or $P_6 + 3K_1$.*

Moreover, from this a linear-time recognition algorithm for neighborhood-perfectness of tree-cographs was also found in [42]. Furthermore, in the same work, linear-time algorithms for computing $\alpha_n(G)$ and $\rho_n(G)$ for any tree-cograph G were devised. In the following subsection, we will present the characterization of coordinated graphs for complements of forests, a subclass of tree-cographs.

5.4. Complements of forests

Recall from the previous subsection that all complements of forests are clique-perfect. Moreover, it follows from Theorem 31 that those complements of forests that are neighborhood-perfect are those that are 3-pyramid-free. In [19] a characterization by minimal forbidden induced subgraphs of those coordinated graphs within the class of complements of forests was found. Recall that $2P_4$ is the disjoint union of two P_4's and let R be the graph depicted in Fig. 9. Both $\overline{2P_4}$ and $\overline{R}$ are not coordinated, since $\Delta_c(\overline{2P_4}) = \Delta_c(\overline{R}) = 6$ and $\gamma_c(\overline{2P_4}) = \gamma_c(\overline{R}) = 7$. The characterization is as follows.

Theorem 32 (*[19]*)**.** *Let G be the complement of a forest. Then G is coordinated if and only if G contains no induced $\overline{2P_4}$ and no induced $\overline{R}$.*

A characterization of those forests obtained by identifying the false twins of $\overline{G}$ when G is coordinated, called *c-forest*, leads to a linear time recognition algorithm for coordinated graphs within the class of complements of forests [19].

5.5. Line graphs and complements of line graphs

The characterization by forbidden induced subgraphs of those line graphs that are clique-perfect is as follows.

Theorem 33 (*[11]*)**.** *Let G be a line graph. Then G is clique-perfect if and only if G has no odd holes and contains no induced 3-sun.*

The first part of the proof consists of proving that perfect line graphs are K-perfect, and relies on the characterization of perfect line graphs in [67] and on a property of clique-cutsets in perfect graphs [5]. Therefore, those perfect line graphs that are in addition hereditary clique-Helly are also clique-perfect. In the second part of the proof those line graphs that are not hereditary clique-Helly are treated separately. In fact, hereditary K-perfection and clique-perfection do not coincide for line graphs as one realizes by considering the 3-sun.

Those line graphs that are coordinated were characterized by forbidden induced subgraphs in [19].

Theorem 34 (*[19]*)**.** *Let G be a line graph. Then G is coordinated if and only if G has no odd hole and contains no induced 3-sun.*

Reasoning as in the first part of the proof of Theorem 33, line graphs without odd holes are K-perfect, and then those that in addition are hereditary clique-Helly, are coordinated. Therefore the proof consists of studying those line graphs that have no odd holes but have some family of cliques that does not satisfy the Helly property.

Notice that by Theorems 33 and 34, coordinated and clique-perfect graphs coincide when restricted to line graphs.

In [19] also a characterization of coordinated line graphs in terms of its root graph is given. To formulate it, the authors introduce the following definitions. Given a graph H and a set $S \subseteq V(H)$, denote by $E_H(S)$ the set of edges of H that have both endpoints in S. The set $S \subseteq V(H)$ is an *edge separator of H* if every vertex of S belongs to a different connected component of $G \setminus E_H(S)$. Let $t = \{v_1, v_2, v_3\}$ be a triangle of a graph H. The triple (v_1, v_2, v_3) is a *well ordering* of t if

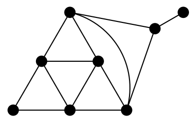

Fig. 10. A line graph that is not neighborhood-perfect.

$d_H(v_1) \leq d_H(v_2) \leq d_H(v_3)$ and at least one of the following conditions holds: (*i*) $d_H(v_1) < d_H(v_2)$, or (*ii*) $N_H[v_3]$ is equal to both $N_H[v_1]$ and $N_H[v_2]$, or (*iii*) $N_H[v_3]$ is equal to none of $N_H[v_1]$ and $N_H[v_2]$. Every triangle admits some well ordering permutation [19]. Let E_t be the set of edges $\{v_1v_2, v_2v_3, v_3v_1\}$, and if $\mathcal{T}$ is a family of triangles, let $E_{\mathcal{T}} = \bigcup_{t\in\mathcal{T}} E_t$. With this terminology, the characterization can now be formulated as follows:

Theorem 35 (*[19]*)**.** *Let H be a graph and $\mathcal{T}$ be the set of triangles of H. Then the following statements are equivalent:*

1. *$L(H)$ is coordinated.*
2. *$H \setminus E_{\mathcal{T}}$ is bipartite and every well ordered triangle (v_1, v_2, v_3) of H satisfies one of the following statements:*
   - *(a) $d_H(v_1) = 2$ and $N_H[v_2] \cap N_H[v_3]$ is an edge separator of H.*
   - *(b) $d_H(v_1) = 3$, v_1 and v_2 are true twins and $N_H[v_1]$ is an edge separator of G.*

This characterization leads to a linear-time recognition algorithm for coordinated graphs within the class of line graphs. As a corollary, a linear-time recognition algorithm for clique-perfect graphs within this class follows. In [19], a linear-time algorithm for determining $\Delta_c(G)$ and $\gamma_c(G)$ for any coordinated line graph $G = L(H)$ is also presented.

We would like to remark that the first part of the proof of Theorem 34 implies the following: a line graph is hereditary K-perfect if and only if it has no odd holes, if and only if it is perfect.

In [60], Lehel characterized those line graphs that are neighborhood-perfect by forbidden induced subgraphs.

Theorem 36 (*[60]*)**.** *Let G be a line graph. Then G is neighborhood-perfect if and only if G contains none of the following graphs as an induced subgraph:* 3*-sun,* $\overline{3K_2}$ *and the graph in Fig. 10.*

In [17], clique-perfectness of complements of line graphs was characterized by minimal forbidden induced subgraphs.

Theorem 37 (*[17]*)**.** *If G is the complement of a line graph, then G is clique-perfect if and only if G contains no induced* 3*-sun and has no antihole $\overline{C_k}$ for any $k \geq 5$ such that k is not a multiple of* 3*.*

Let G be the complement of the line graph of a graph H. In order to prove the above theorem, the parameters $\alpha_c(G)$ and $\tau_c(G)$ are expressed in terms of the graph H. Clearly, the cliques of G are precisely the maximal matchings of H. The *matching-transversal number of H*, denoted by $\tau_m(H)$, is defined as the minimum size of a set of edges of H that meets every maximal matching of H. Similarly, the *matching-independence number of H*, denoted by $\alpha_m(H)$, is defined as the maximum size of a set of pairwise-disjoint maximal matchings of H. The graph H is called *matching-perfect* [17] if $\tau_m(H') = \alpha_m(H')$ for every subgraph (induced or not) H' of H. It is not hard to see that G is clique-perfect if and only if H is matching-perfect. In fact, the proof of Theorem 37 follows as a corollary of the following characterization of matching-perfect graphs. The *bipartite claw* is the graph that arises from claw by subdividing each of its edges once.

Theorem 38 (*[17]*)**.** *A graph H is matching-perfect if and only if H contains no bipartite claw and the length of each cycle of H is at most* 4 *or a multiple of* 3*.*

In its turn, Theorem 38 is proved by means of a decomposition theorem describing in detail the linear and circular structure of graphs containing no bipartite claw proved in [17].

Notice that since the 3-sun and antiholes different from $\overline{C_6}$ are neither coordinated nor neighborhood-perfect, coordinated (resp. neighborhood-perfect) complements of line graphs are clique-perfect. Moreover, these inclusions are proper and coordination and neighborhood-perfectness differ for complements of line graphs. In fact, the 3-pyramid $= \overline{L(K_4)}$ is clique-perfect and coordinated but not neighborhood-perfect, whereas $\overline{2P_4} = \overline{L(2P_5)}$ is clique-perfect and neighborhood-perfect but not coordinated.

5.6. Helly circular-arc graphs

In [12] a characterization of those Helly circular-arc graphs that are clique-perfect was formulated in terms of forbidden induced subgraphs. The forbidden induced subgraphs are displayed in Fig. 11. The 3-sun, odd holes, vikings and 2-vikings are all odd generalized suns. The authors show that, within Helly circular-arc graphs, there are only two families of minimally clique-imperfect graphs which are not odd generalized suns or antiholes, namely, S_k and T_k for each $k \geq 2$. Since the graphs of Fig. 11 do not contain properly each other as induced subgraphs, it follows that the following is a characterization by *minimal* forbidden induced subgraphs:

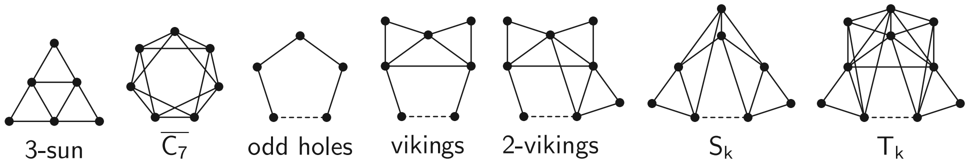

Fig. 11. Minimal forbidden induced subgraphs for clique-perfect graphs inside the class of Helly circular-arc graphs. Dashed lines represent induced paths of length $2k-3$ for each $k \geq 2$.

Theorem 39 ([12]). *Let G be a Helly circular-arc graph. Then G is clique-perfect if and only if it does not contain a 3-sun, an antihole of length 7, an odd hole, a viking, a 2-viking or one of the graphs S_k or T_k for each $k \geq 2$.*

Whether a graph is a Helly circular-arc graph can be decided in linear time and, if affirmative, both parameters $\alpha_c(G)$ and $\tau_c(G)$ can also be computed in linear time [39,40,63]. However, these facts do not immediately imply the existence of a polynomial-time recognition algorithm for clique-perfect Helly circular-arc graphs (because we need to verify the equality for every induced subgraph). The characterization given in [12] leads to such an algorithm, which is strongly based on the recognition of perfect graphs [25]. The idea of the algorithm is similar to the one used in [33] for recognizing balanceable matrices. It was proved in [89] that clique-perfectness coincides with neighborhood-perfectness for Helly circular-arc graphs. As in [12], the proof strategy consists in studying separately those Helly circular-arc graphs that are hereditary clique-Helly from those that are not so.

Theorem 40 ([89]). *If G is a Helly circular-arc graph, then G is clique-perfect if and only if G is neighborhood-perfect.*

Coordinated Helly circular-arc graphs and clique-perfect Helly circular-arc graphs are not comparable. In fact, on the one hand, for each $k \geq 2$, the viking with $2k+3$ vertices is coordinated but not clique-perfect (since $\alpha_c = k$ and $\tau_c = k+1$), and on the other hand, the graph $\bar{R}$ (cf. Fig. 9) is an example of a Helly circular-arc graph which is clique-perfect but not coordinated (since $\Delta_c = 6$ and $\gamma_c = 7$).

5.7. Some subclasses of claw-free graphs

Another class where clique-perfectness, coordination and neighborhood-perfectness coincide is the class of hereditary clique-Helly claw-free graphs.

Theorem 41 ([11]). *Let G be a hereditary clique-Helly claw-free graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd hole or an antihole of length 7.*

It is well-known that for hereditary clique-Helly graphs, each clique has some *proper edge*; i.e., an edge that belongs to that clique only. Moreover, it was proved in [61] that every graph G for which each clique has a proper edge satisfies $\alpha_c(G) = \alpha_n(G)$ and $\tau_c(G) = \rho_n(G)$. Thus, the following holds.

Theorem 42 ([61]). *A hereditary clique-Helly graph G is clique-perfect if and only if G is neighborhood-perfect.*

The proof of Theorem 41 relies on the decomposition theorem for claw-free graphs that appears in [29]. Indeed, in [11] it is proved more, that hereditary clique-Helly claw-free graphs with no odd holes and no induced $\overline{C_7}$ are K-perfect. Since, conversely, odd holes and $\overline{C_7}$ are not K-perfect, then, by Theorem 41, Corollary 9, Theorem 42, and the fact that coordinated graphs are perfect, the following set of equivalencies follow.

Corollary 43 ([11]). *Let G be a hereditary clique-Helly claw-free graph. Then the following assertions are equivalent:*

1. *G is clique-perfect.*
2. *G is hereditary K-perfect.*
3. *G is coordinated.*
4. *G is neighborhood-perfect.*
5. *G is perfect.*
6. *G has no odd holes and contains no induced $\overline{C_7}$.*

Notice that deciding whether a claw-free graph is perfect is solvable in polynomial time [31].

Relying on Theorem 41, the following characterization of those claw-free planar graph that are clique-perfect was found in [62].

Theorem 44 ([62]). *Let G be a claw-free graph such that G is planar or $\Delta(G) \leq 4$. Then, G is clique-perfect if and only if G contains no odd hole and neither a 3-sun nor the graph of Fig. 12 as an induced subgraph.*

As an immediate consequence, they observe that every cubic claw-free graph is clique-perfect.

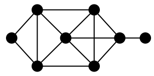

Fig. 12. A minimal forbidden induced subgraph for the class of clique-perfect graphs.

5.8. Superclasses of triangle-free graphs

Triangle-free graphs were extensively studied in the literature, usually in the context of graph coloring problems (see for instance [55,68,71]). It is easy to see that if a graph G is triangle-free, then G is perfect if and only if G is clique-perfect, if and only if G is coordinated, because bipartite graphs are perfect, clique-perfect, and coordinated.

In [18], these equivalencies were extended to two superclasses of triangle-free graphs: paw-free and {gem, W_4,bull}-free graphs.

Theorem 45 (*[18]*)**.** *Let G be a paw-free graph. The following statements are equivalent:*

1. *G is perfect.*
2. *G is clique-perfect.*
3. *G is coordinated.*
4. *G does not contain odd holes.*

For paw-free graphs, not every perfect graph is hereditary K-perfect. In [81], it is proved that if G is a paw-free graph having at least three anticomponents and each anticomponent of G has at least 3 vertices, then $K(G)$ contains an induced C_5. In particular, $K(\overline{3K_3})$ contains an induced C_5, so it is not perfect. Moreover, the following theorem is proved.

Theorem 46 (*[81]*)**.** *Let G be a perfect connected paw-free graph. Then the following statements are equivalent:*

1. *G is K-perfect.*
2. *$K(G)$ does not contain an induced C_5.*
3. *Either G is bipartite or at least one anticomponent of G has at most two vertices.*

As a corollary of Theorem 46 (and reasoning as in the proof of Theorem 45) the following characterizations of hereditary K-perfect paw-free graphs can be obtained:

Corollary 47. *Let G be a paw-free graph. Then the following conditions are equivalent:*

1. *G is hereditary K-perfect.*
2. *G has no odd holes and contains no induced $\overline{3K_3}$.*
3. *Each connected component of G is either bipartite or not anticonnected and, in the latter case, at most two of its anticomponents have more than two vertices.*

For {gem, W_4,bull}-free graphs instead, all the discussed main notions coincide. Notice that {gem, W_4,bull}-free graphs are hereditary clique-Helly by Theorem 4 and recall Theorem 42.

Theorem 48 (*[18]*)**.** *Let G be a {gem, W_4,bull}-free graph. Then the following statements are equivalent:*

1. *G is perfect.*
2. *G is clique-perfect.*
3. *G is coordinated.*
4. *G is neighborhood-perfect.*
5. *G is hereditary K-perfect.*
6. *G does not contain odd holes.*

Using this characterization and the fact that perfect graphs can be recognized in polynomial time, the clique-perfect recognition problem restricted to the class of {gem, W_4,bull}-free graphs can be solved in polynomial time.

The major step to verify Theorem 48 is to prove that {gem, W_4,bull}-free graphs without odd holes are K-perfect. Since the class of {gem, W_4,bull,odd hole}-free graphs is hereditary then it follows that they are hereditary K-perfect. Since {gem, W_4,bull}-free graphs are hereditary clique-Helly then the remaining implications follow by Corollary 9.

In [18] it is mentioned as an interesting open problem to determine the 'biggest' superclass of triangle-free graphs where clique-perfect and coordinated graphs are equivalent.

6. Final remarks

We would like to put forward some possibilities to study these classes further. Take for instance the case of clique-perfect graphs. We feel that there may be still many opportunities for finding more obstructions to clique-perfection.

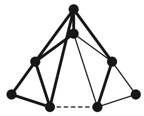

Fig. 13. A viking as a partial subgraph of S_k.

Indeed, the larger family of obstructions for clique-perfection that are clearly identified in the literature are those that correspond to odd generalized suns but we suspect that they might represent only a restricted portion of the total set of obstructions.

In most of the cases the approach taken to obtain the partial characterization of clique-perfection (and coordination) reviewed in Section 5 depends heavily on the equality between clique-perfection and hereditary K-perfection (at least when the graphs are further restricted to be hereditary clique-Helly). So one of the main parts of these characterizations consists on studying how the imperfection of $K(G)$ translates into a forbidden structure on G. This part of the proof differs radically when restricting to different graph classes. So it would represent a progress to find a way to handle this situation more generally. Notice that by Theorem 2, if $K(G)$ is imperfect then $K(G)$ or $\overline{K(G)}$ contains an odd hole. Consider the case when $K(G)$ contains a hole $Q_1, \ldots, Q_n$ of cliques of G. By taking a vertex in each consecutive pair of cliques a cycle C arises in G. It happens, as one may suspect, that many possibilities emerge by combining presence of some adjacencies and absence of others. At this point we could think about grouping the arising possibilities together by similarity and a tool that can help to achieve this is the concept of trigraph [24]. A trigraph is a generalization of a graph where some adjacencies are set to be present, some adjacencies are set to be not present, but the remaining adjacencies can be freely set to be present or not. That is, a trigraph represents the whole family of graphs resulting from deciding the presence or not of each of the 'undecided' edges. This could help to reduce the number of forbidden structures into consideration. Notice that there are still minimally clique-imperfect graphs that are hereditary K-perfect so that their clique graphs do not have odd holes or odd antiholes, this indicates that to describe the full set of obstructions we may need to study also some cycles in the clique graph that do not correspond to odd holes or antiholes.

Further possibilities of attack may arise by classifying the obstructions according to the underlying structure of the odd holes of $K(G)$. This structure may resemble the structure of hypomatchable graphs, which can be constructed from odd holes by attaching paths each of which creates a new hole (see [65] for details). A similitude is observed when we look at the clique graph as extension of the line graph, since for a hypomatchable graph, the graphs obtained by the removal of any vertex admit a perfect matching, that is exactly a stable set in the line graph that, in addition, is a vertex cover.

In the same spirit, there is an interesting fact that shows up if we take a closer look at the graphs of Fig. 11. For instance, we could consider S_k as a merging of two vikings. Fig. 13 shows a partial subgraph of S_k that is a viking, and other candidates can be found symmetrically. Contrary to what one could expect the clique graph of S_k contains just one odd hole and not two odd holes that share some edges. It would be interesting to generalize this or find new merging procedure as they may allow us to classify certain obstructions as nonbasic and concentrate on the more restricted basic ones that do not admit such decompositions.

Furthermore, it is interesting to find more general characterizations of hereditary K-perfect graphs, as well as characterizing $\alpha(K)$-perfect and $\chi(K)$-perfect graphs.

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