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Singularity formation for the two-dimensional harmonic map flow into S^2

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Abstract We construct finite time blow-up solutions to the 2-dimensional harmonic map flow into the sphere S^2,

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0, T) \\ u &= \varphi \quad \text{on } \partial\Omega \times (0, T) \\ u(\cdot, 0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^2$, $u : \Omega \times (0, T) \to S^2$, $u_0 : \bar{\Omega} \to S^2$ is smooth, and $\varphi = u_0\big|_{\partial\Omega}$. Given any k points $q_1, \dots, q_k$ in the domain, we find initial and boundary data so that the solution blows-up precisely at those points. The profile around each point is close to an asymptotically singular

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scaling of a 1-corotational harmonic map. We build a continuation after blow-up as a H^1-weak solution with a finite number of discontinuities in space–time by "reverse bubbling", which preserves the homotopy class of the solution after blow-up. Furthermore, we prove the codimension one stability of the one point blow-up phenomenon.

1 Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We denote by S^2 the standard 2-sphere. We consider the *harmonic map flow* for maps from Ω into S^2, given by the semilinear parabolic equation

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0, T) \tag{1.1}$$

$$u = \varphi \quad \text{on } \partial\Omega \times (0, T) \tag{1.2}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{1.3}$$

for a function $u : \Omega \times [0, T) \to S^2$. Here $u_0 : \bar{\Omega} \to S^2$ is a given smooth map and $\varphi = u_0\big|_{\partial\Omega}$. Local existence and uniqueness of a classical solution follows from the works [3,10,26]. Equation (1.1) formally corresponds to the negative L^2-gradient flow for the Dirichlet energy $\int_\Omega |\nabla u|^2 dx$. This energy is decreasing along smooth solutions $u(x, t)$:

$$\frac{\partial}{\partial t} \int_\Omega |\nabla u(\cdot, t)|^2 = -\int_\Omega |u_t(\cdot, t)|^2.$$

Struwe [26] established the existence of an H^1-weak solution, where just for a finite number of points in space–time loss of regularity occurs. This solution is unique within the class of weak solutions with decreasing energy, see Freire [11] and also Lin-Wang [15] for another proof.

If $T > 0$ designates the first instant at which smoothness is lost, we must have

$$\|\nabla u(\cdot, t)\|_\infty \to +\infty \quad \text{as} \quad t \uparrow T.$$

Several works have clarified the possible blow-up profiles as $t \uparrow T$. The following fact follows from results by Ding-Tian [9], Lin-Wang [13], Qing [19], Qing-Tian [20], Struwe [26], Topping [27] and Wang [32]:

Along a sequence $t_n \to T$ and points $q_1, \dots, q_k \in \Omega$, not necessarily distinct, $u(x, t_n)$ blow-up occurs at exactly those k points in the form of *bubbling*.

More precisely, under some technical assumptions we have

$$u(x,t_n) - u_*(x) - \sum_{i=1}^{k}\left[U_i\left(\frac{x-q_i^n}{\lambda_i^n}\right) - U_i(\infty)\right] \to 0 \quad \text{in } H^1(\Omega) \quad (1.4)$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.$$

The U_i's are entire, finite energy harmonic maps, namely solutions $U : \mathbb{R}^2 \to S^2$ of the equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

After stereographic projection, U lifts to a smooth map in S^2, so that its value $U(\infty)$ is well-defined. It is known that U is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N}, \quad (1.5)$$

see Topping [27].

In particular, $u(\cdot, t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and for some positive integers m_i, we have

$$|\nabla u(\cdot,t_n)|^2 \rightharpoonup |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi m_i\, \delta_{q_i} \quad (1.6)$$

in the measures sense, were δ_q denotes the unit Dirac mass at q.

Topping [28] estimated the blow-up rates as $\lambda_i^n = o((T-t_n)^{\frac{1}{2}})$ (also valid for more general targets), a fact that tells that the blow-up is of "type II", namely it does not occur at a self-similar rate.

A decomposition similar to (1.4) holds if blow-up occurs in infinite time, $T = +\infty$. In such a case one has the additional information that u_* is a harmonic map, and the convergence in (1.4) also holds uniformly in Ω (the latter is called the "no-neck property"), see Qing and Tian [20]. Finer properties of the bubble-decomposition have been found by Topping [27].

A *least energy* entire, non-trivial harmonic map is given by

$$W(x) = \frac{1}{1+|x|^2}\begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}, \quad x \in \mathbb{R}^2, \tag{1.7}$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla W|^2 = 4\pi, \quad W(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Very few examples are known of solutions, which exhibit the singularity formation phenomenon (1.6), and all of them concern single-point blow-up in radially symmetric *corotational* classes. When Ω is a disk or the entire space, a 1-corotational solution of (1.1) is one of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}. \tag{1.8}$$

Within this class, (1.1) reduces to the scalar, radially symmetric problem

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}. \tag{1.9}$$

We observe that the function

$$w(r) = \pi - 2\arctan(r)$$

is a steady state of (1.9) which corresponds precisely to the harmonic map W in (1.7). Indeed,

$$W(x) = \begin{pmatrix} e^{i\theta} \sin w(r) \\ \cos w(r) \end{pmatrix}.$$

Chang, Ding and Ye [4] found the first example of a blow-up solution of problem (1.1)–(1.3) (which was previously conjectured not to exist). They obtained the result in the 1-corotational class in a disk D by finding appropriate sub-super solutions to (1.9). Assuming that the initial energy satisfies $\int_D |\nabla u_0|^2 < 8\pi$, the decomposition (1.4) implies that

$$u(x,t) = W\left(\frac{x}{\lambda(t)}\right) + u_* + o(1), \tag{1.10}$$

with $u_* \in H^1$, $o(1) \to 0$ in H^1-norm, and $0 < \lambda(t) \to 0$ as $t \to T$. No information on the precise blow-up rate $\lambda(t)$ is obtained. Angenent, Hulshof and Matano [1] estimated the blow-up rate of 1-corotational maps as $\lambda(t) = o(T - t)$. Using matched asymptotics formal analysis for problem (1.9), van den Berg, Hulshof and King [30] demonstrated that this rate for 1-corotational maps should generically be given by

$$\lambda(t) \approx \kappa \frac{T - t}{|\log(T - t)|^2}, \tag{1.11}$$

for some $\kappa > 0$. Raphael and Schweyer [23] succeeded to rigorously construct an entire 1-corotational solution with this blow-up rate.

In this paper we deal with the general, nonsymmetric case in (1.1)–(1.3). Our first result asserts that for any given finite set of points of Ω and suitable initial and boundary values, a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation and rotation of that in (1.10) around each bubbling point.

To state our result, we observe that the functions

$$U_{\lambda,q,Q}(x) := QW\left(\frac{x-q}{\lambda}\right)$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and Q an orthogonal matrix in $\mathbb{R}^3$ do solve problem (1.5), and all share the least energy property:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,Q}|^2 = 4\pi.$$

Let us consider the α-rotation matrix around the third axis given by

$$e^{J\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In all what follows, we consider problem (1.1)–(1.3) with the boundary condition (1.2) given by the constant

$$\varphi(x) = \mathbf{e_3}. \tag{1.12}$$

Here and in what follows we denote

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.13}$$

The constant boundary value $\mathbf{e_3}$ precisely corresponds to $W(\infty)$ where W is the standard 1-corotational harmonic map (1.7). This choice is made for convenience, in fact any sufficiently small perturbation of it is also admissible. In the radial 1-corotational equation (1.9), this boundary condition in the disk $\Omega = D(0, R)$ simply corresponds to $v(R, t) = 0$. All results below do apply to a boundary condition which slightly perturbs (1.12), or in the case of entire space $\mathbb{R}^2$ where this value is set as a condition at infinity.

Theorem 1 *Given points $q = (q_1, \ldots, q_k) \in \Omega^k$ and any sufficiently small $T > 0$, there exist u_0 such the solution $u_q(x, t)$ of problem (1.1)–(1.3), for φ given by (1.12), blows-up at exactly those k points as $t \uparrow T$. More precisely, there exist numbers $\kappa_i^* > 0$, α_i^* and a function $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$ such that*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} e^{J\alpha_i^*}\left[\, W\left(\frac{x - q_i}{\lambda_i}\right) - W(\infty)\,\right] \to 0 \quad as \quad t \uparrow T, \tag{1.14}$$

in the H^1 and uniform senses in Ω where

$$\lambda_i(t) = \kappa_i^* \frac{T - t}{|\log(T - t)|^2}\big(1 + o(1)\big) \quad as \quad t \uparrow T. \tag{1.15}$$

In particular, we have

$$|\nabla u(\cdot, t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j} \quad as \quad t \uparrow T.$$

The blow-up solution we constructed in Theorem 1 has *no necks*. By the results of Qing-Tian [20], (see also Lin-Wang [13,14]), this follows from the directly checked fact that the L^2 norm of the tension field $\tau := u_t$ is bounded as $t \uparrow T$. Our construction suggests that no necks should be present in planar solutions with isolated least energy blow-up points.

In the next result we analyze the stability of the solutions constructed in Theorem 1. We recall that in the 1-corotational class in a disc, Chang-Ding-Ye [4] provided robust conditions on initial and boundary data that guarantee finite time blow-up. Raphael-Schweyer [23] established stability *within* the 1-corotational class in entire space for a solution blowing-up with the rate (1.11). Merle-Raphael-Rodnianski [18] and Raphael-Schweyer [23] conjectured instability outside the 1-corotational class. Van der Berg and Williams [31] provided formal and numerical evidence that blow-up may indeed be destroyed by small non-radial perturbations of a 1-corotational singularity.

Our proof of Theorem 1 yields *codimension-one stability* of the predicted blow-up phenomenon in the case of a single blow-up point when no symmetries are assumed. The meaning of this form of stability is as follows:

Theorem 2 *Let $u(x,t)$ be the solution predicted in Theorem 1 of the problem (1.1)–(1.3) that blows-up at a point $q \in \Omega$ and a time $T > 0$. Then there exists a C^1 manifold $\mathcal{M}$ in $C^1(\bar{\Omega}, S^2)$ with codimension one that contains u_0 such that for any $\tilde{u}_0 \in \mathcal{M}$ close to u_0, the solution $\tilde{u}(x,t)$ of problem (1.1)–(1.3) with initial datum $\tilde{u}_0$ blows-up at a point $\tilde{q} \in \Omega$ and a time $\tilde{T}$ which are close respectively to q and T.*

We discuss the general reason for the codimension-1 stability in Remark 2.1 in §2. The generalization of the previous theorem to the solution with k blow-up points of Theorem 1 is that there is a manifold in C^1 of codimension $2k-1$ of initial data that leads to k simultaneous blow-up points at a time T.

The solutions in Theorems 1 are classical in $[0, T)$. Our next result concerns the continuation of the solution after blow-up. As we have mentioned Struwe [26] defined a global H^1-weak solution of (1.1)–(1.3). Struwe's solution is obtained by just dropping the bubbles appearing at the blow-up time and then restarting the flow. The energy has jumps at each blow-up time generated by this procedure and it is decreasing. Decreasing energy suffices for uniqueness of the weak solution, as proven in [11,15]. On the other hand the bubble-dropping procedure modifies in time the topology of the image of the solution map. Topping [28] showed a different way to construct a continuation after blow up in the symmetric 1-corotational class. The solution in [4] is continued after blow-up by attaching a bubble with opposite orientation, which unfolds continuously the energy. The solution referred to is a *reverse bubbling solution*. As emphasized in [28], this continuation has the advantage that, unlike Struwe's solution, it preserves the homotopy class of the map after blow-up. Formal asymptotic rates for 1-corotational reverse bubbling were found in [30]. In [2] other forms of continuation of radial solutions were found.

We establish that Topping's continuation can be made without symmetry assumptions, with exact asymptotics, for the solution in Theorem 1. We define the bubble $\bar{w}$ with reverse orientation to that of W as

$$\bar{W}(x) = e^{J\pi} W(x) = \frac{1}{1+|x|^2}\begin{pmatrix} -2x \\ |x|^2 - 1\end{pmatrix} = \begin{pmatrix} -e^{i\theta}\sin w(r) \\ \cos w(r)\end{pmatrix}. \qquad (1.16)$$

Theorem 3 *Let $u_q(x,t)$ be the solution in Theorem 1. Then u_q can be continued as an H^1-weak solution in $\Omega \times (0, T+\delta)$, which is continuous except at the points (q_i, T), with the property that, besides expansion (1.14), we have $u_q(x,T) = u_*(x)$*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} e^{J\alpha_i^*} \left[\bar{W}\left(\frac{x-q_i}{\lambda_i(t)}\right) - \bar{W}(\infty) \right] \to 0 \quad as \quad t \downarrow T,$$

in the H^1 *and uniform senses in* Ω*, where*

$$\lambda_i(t) = \kappa_i^* \frac{t-T}{|\log(t-T)|^2}. \tag{1.17}$$

We observe that the energy in this continuation fails to be decreasing: it has a jump exactly at time T and it goes back to its previous level immediately after.

We consider a question related to Theorem 3 treated in the 1-corotational symmetric class in [28] and in [2]: the occurrence of perfectly smooth solutions which spontaneously develop a singularity in finite time by the addition of an infinitely concentrated bubble which instantaneously raises the energy in a multiple of 4π. We find that the typical rate for this backward bubbling is $\dot{\lambda}(t)$ of order $\frac{t-T}{|\log(t-T)|}$ rather than (1.17). This was formally derived in [30].

Theorem 4 *Given points* $q_1, \ldots, q_k$ *in* Ω *and any sufficiently small* $T > 0$ *there exists an* H^1*-weak solution* $u(x,t)$ *of problem* (1.1)–(1.3) *in* $\Omega \times (0, T+\delta)$ *which is continuous except at the points* (q_i, T)*, it is smooth in* $\Omega \times (0, T]$ *and has spontaneous reverse bubbling at the points* q_i *in the form*

$$u(x,t) - u(x,T) - \sum_{j=1}^{k} \left[W\left(\frac{x-q_i}{\lambda_i(t)}\right) - W(\infty) \right] \to 0 \quad as \quad t \downarrow T,$$

in the H^1 *and uniform senses in* Ω*, where for some positive numbers* κ_i

$$\lambda_i(t) = \kappa_i \frac{t-T}{|\log(t-T)|}. \tag{1.18}$$

Before proceeding into the proof we make some further comments. It is plausible that the solutions of the form described in Theorem 1 represent a form of "generic" bubbling phenomena for the two-dimensional harmonic map flow. For instance, it is reasonable to think that the limits along any sequence should have the same elements in the bubble decomposition. On the other hand, evidence in the literature suggests that typically only simple blow-up is present, having as a profile scalings of the 1-corotational maps W and $\bar{W}$. Higher degree maps are represented by the d-corotational symmetry class, $d \geq 1$,

$$u(x,t) = \begin{pmatrix} e^{di\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}.$$

Steady states in this class correspond to scalings of $v = w_d(r) = \pi - 2\arctan(r^d)$. It turns out that blow-up is not present in this class for $d \geq 4$. See Guan-Gustafson-Tsai [12]. It is conjectured that no blow-up exists also for $d = 2, 3$. This essentially discards higher degree blow-up. On the other hand, no multiple blow-up (bubble trees) in the 1-corotational class exists. See Van der Hout [29]. Infinite time multiple bubbling was found by Topping [27] in a target different from S^2. Bubbling rates faster than (1.15) do exist in the 1-corotational case, but they are not stable, see Raphaël and Schweyer [24]. Many other results on bubbling phenomena, and regularity for harmonic maps and the harmonic map flow are available in the literature, we refer the reader to the book by Lin and Wang [16].

In bubbling phenomena in this and related problems very little is known in nonradial situations. The method in [23,24], was successfully applied to very related blow-up phenomena in dispersive equations in symmetric classes. See for instance Rodnianski-Sterbenz [25], Merle-Raphaël-Rodnianski [18], Raphaël[21], Raphaël-Rodnianski [22]. Our results share a flavor with finite time multiple blow-up in the subcritical semilinear heat equation, as in the results by Merle and Zaag [17]. Bubbling associated to the critical exponent has been recently studied in [5,6]. Our approach is parabolic in nature. It is based on the construction of a good approximation and then linearizing inner and outer problems. An appropriate inverse for the inner equation is then found (which works well if the parameters of the problems are suitably adjusted) which makes it possible the application of fixed point arguments. The general approach, which we call inner-outer gluing, has already been applied to various singular perturbation elliptic problems, see for instance [7,8]. A major difficulty we have to overcome is the coupled nonlocal ODE satisfied by the scaling and rotation parameter. We now explain in more details below.

2 The 1-corotational harmonic maps and the ansatz for a blowing-up solution

The harmonic map equation for functions $U : \mathbb{R}^2 \to S^2$ is the elliptic problem

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad |U| = 1. \tag{2.1}$$

For $\xi \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, $\lambda > 0$, we consider the family of solutions of (2.1) given by the following 1-corotational harmonic maps

$$U_{\lambda,\xi,\omega}(x) := Q_\omega W(y), \quad y = \frac{x - \xi}{\lambda}$$

where $W(y)$ is the canonical 1-corotational harmonic map

$$W(y) = \frac{1}{1+|y|^2}\begin{pmatrix} 2y \\ |y|^2-1 \end{pmatrix}, \quad y \in \mathbb{R}^2,$$

and Q_ω is the ω-rotation matrix

$$Q_\omega := \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Our purpose is to build a smooth blowing-up solution $u : \bar{\Omega} \times [0, T) \to S^2$ of the problem

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0, T) \\ u = \mathbf{e_3} & \text{on } \partial\Omega \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

In order to keep the notation to a minimum, we shall do this in the case $k = 1$ of a single given bubbling point $q \in \Omega$. The changes needed for the general case of Theorem 1 are minor. More precisely for any sufficiently small number $T > 0$ we look for an initial datum u_0 such that the solution $u(x, t)$ of problem (2.2) looks at main order like

$$U(x, t) := U_{\lambda(t),\xi(t),\omega(t)}(x) = Q_{\omega(t)}\, W(y), \quad y = \frac{x - \xi(t)}{\lambda(t)}, \tag{2.3}$$

for certain functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1([0, T])$ such that

$$\xi(T) = q, \quad \lambda(T) = 0.$$

We shall find values for these functions so that for a small remainder $v(x, t)$ we have that $u = U + v$ solves (2.2). The condition $|U + v| = 1$ tells us that u can be written as

$$u(x, t) = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U, \tag{2.4}$$

where φ is a small function with values into $\mathbb{R}^3$ and we denote

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U, \quad a(\zeta) := \sqrt{1 - |\zeta|^2} - 1.$$

The term $a(\Pi_{U^\perp}\varphi)$ has a quadratic size in φ so it is of smaller order. We choose to decompose the remainder $\varphi(x, t)$ in (2.4) as the addition of an "outer" part, better expressed in the global variable x, and an "inner" part which supported near the singularity and it is naturally expressed as function of the slow variable y. More precisely, we let

$$\varphi(x,t) \;=\; \varphi^{out}(x,t)+\varphi^{in}(y,t), \quad y=\frac{x-\xi(t)}{\lambda(t)} \tag{2.5}$$

where

$$\varphi^{in}(y,t) \;=\eta_{R(t)}\,(y)\,Q_{\omega(t)}\phi(y,t), \quad \phi(y,t)\cdot W(y)\equiv 0$$

and $\eta_R(y) := \eta\left(\frac{|y|}{R}\right)$ with $\eta(s)$ a smooth cut-off function so that

$$\eta(s)=\begin{cases}1 & \text{for } s<1,\\ 0 & \text{for } s>2.\end{cases}$$

The function $\phi(y,t)$ is defined for $|y|<3R(t)$ where $R(t)\to+\infty$ and $\lambda(t)R(t)\to 0$ as $t\to T$. With these definitions we see that $\Pi_{U^\perp}\varphi^{in}=\varphi^{in}$.

We choose to the decompose the outer part $\varphi^{out}(x,t)$ in (2.5) as

$$\varphi^{out}(x,t) \;=\; \Phi^0[\omega,\lambda,\xi]+Z^*(x,t)\;+\;\psi(x,t), \tag{2.6}$$

where Φ^0 and + $Z^*(x,t)$ are explicit functions chosen as follows: $\Phi^0[\omega,\lambda,\xi]$ is a function (which will be precisely described in the next section) that at main order eliminates the largest slow-decaying part of the error of approximation U_t in (2.2). Writing $p(t):=\lambda(t)e^{i\omega(t)}$ and using polar coordinates $x=\xi(t)+re^{i\theta}$, we require

$$\partial_t\Phi^0-\Delta_x\Phi^0\approx\frac{2}{r}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix}\approx U_t.$$

On the other hand, we let $Z^*:\Omega\times(0,\infty)\to\mathbb{R}^3$ satisfy

$$\begin{cases} Z^*_t=\Delta Z^* & \text{in } \Omega\times(0,\infty),\\ Z^*(\cdot,t)=0 & \text{in } \partial\Omega\times(0,\infty),\\ Z^*(\cdot,0)=Z^*_0 & \text{in } \Omega,\end{cases} \tag{2.7}$$

where

$$Z^*_0(x)=\begin{bmatrix} z^*_0(x)\\ z^*_{03}(x)\end{bmatrix},\quad z^*_0(x)=z^*_{01}(x)+iz^*_{02}(x) \tag{2.8}$$

is a small, sufficiently regular function essentially satisfying

$$Z_0^*(q) = 0, \quad \operatorname{div} z_0^*(q) + i \operatorname{curl} z_0^*(q) \neq 0.$$

In summary, we make the ansatz

$$u = U + v, \quad v = \Pi_{U^\perp}\left(\Phi^0[\omega, \lambda, \xi] + Z^* + \psi\right) + \eta_R Q_\omega \phi + aU \tag{2.9}$$

for a blowing-up solution $u(x, t)$ of (2.2), where Φ and ψ are lower order corrections. Our task is to find functions $\omega(t)$, $\lambda(t)$, $\xi(t)$, $\psi(x, t)$ and $\phi(y, t)$ as described above, such that the remainder v remains uniformly small.

We will define a system of equations that we call the *inner-outer gluing system*, essentially of the form

$$\begin{cases} \lambda^2 \phi_t = L_W[\phi] + H[p, \xi, \psi, \phi], \quad \phi \cdot W = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi_t = \Delta_x \psi + G[p, \xi, \psi, \phi] & \text{in } \Omega \times (0, T) \end{cases} \tag{2.10}$$

where

$$L_W[\phi] = \Delta_y \phi + |\nabla_y W|^2 \phi + 2(\nabla_y \phi \cdot \nabla_y W) W, \quad \phi \cdot W = 0$$

is the linearized operator for equation (2.1) around $U = W$, so that if the pair of functions $(\phi(y, t), \psi(x, t))$ solves it then u given by (2.9) is a solution of (2.2). The point is to adjust the parameter functions ω, λ, ξ such that the inner problem can be solved for $\phi(y, t)$ which decays as $|y| \to \infty$. To fix the idea, let us consider the approximate elliptic equation, where time is regarded just as a parameter,

$$L_W[\phi] + H[p, \xi, 0, 0] = 0 \quad \text{in } \mathbb{R}^2$$

As we will discuss, a space-decaying solution $\phi(y, t)$ to this problem exists if a set of orthogonality conditions of the form

$$\int_{\mathbb{R}^2} H[p, \xi, 0, 0](y, t)\, Z(y)\, dy = 0 \quad \text{for all} \quad Z \in \mathcal{Z} \tag{2.11}$$

where $\mathcal{Z}$ is a 4-dimensional space constituted by decaying functions $Z(y)$ with $L_W[Z] = 0$. These solvability conditions lead to an essentially explicit system of equations for the parameter functions which will tell us in particular that for some small $\sigma > 0$

$$p(t) = -(\operatorname{div} z_0^*(q) + i\operatorname{curl} z_0^*(q))\frac{|\log T|(T-t)}{\log^2(T-t)}(1 + O(|\log T|^{-1+\sigma})),$$
$$\xi(t) = q + O((T-t)^{1+\sigma}), \tag{2.12}$$

and we recall that we are consistently asking $\operatorname{div} z_0^*(q) + i\operatorname{curl} z_0^*(q) \neq 0$.

Remark 2.1 In the case of blow-up at a single point, our codimension-1 stability result is directly connected to the solvability conditions (2.11). Indeed, the solution we construct depends at main order on four parameters functions: a scaling $\lambda(t) > 0$, a rotation angle $\omega(t) \in \mathbb{R}$, and the concentration point $\xi(t) \in \Omega$, see formula (2.3). The presence of decaying functions in the kernel of the operator L_W limits the decay of solutions to the inner linearized evolution. Too slow decay could make the contribution to the error in the remote regime too large. Sufficient decay in the linearized evolution can only be achieved if the right hand side satisfies four solvability conditions at all times $t \in [0, T)$. These are conditions (2.11), which translate into a system of integro-differential equations for $\lambda(t)$, $\omega(t)$, $\xi(t)$. For $\xi(t)$ the equation is almost a first order ODE, which imposes a constraint between $\xi(0)$ and $\xi(T)$. The equations for $\lambda(t)$ and $\omega(t)$ are better expressed for the combined quantity $p(t) = \lambda(t)e^{i\omega(t)}$. It is an integro-differential equation, whose solution has the expansion (2.12). This relation evaluated at time $t = 0$ says that $\lambda(0)e^{i\omega(0)} = -(\operatorname{div} z_0^*(q)+i\operatorname{curl} z_0^*(q))\frac{T}{|\log T|}(1+O(|\log T|^{-1+\sigma}))$. Considering z_0^* as fixed, this equation links $\lambda(0)$ with T and determines $\omega(0)$ uniquely in $[0, 2\pi)$. In other words in the initial condition we lose the freedom to choose $\omega(0)$. We also loose the freedom of choosing $\lambda(0)$ if T was fixed, but this is recovered by letting T vary. In the 1-corotational case, the symmetries imply that $\operatorname{curl} z_0^*(0) = 0$ and $\omega \equiv 0$, and therefore there is no loss of stability in this situation. The argument above considers z_0^* as fixed, but the analysis with all variables taken into consideration is detailed in Sect. 10.

Remark 2.2 Let us explain why the numbers $\operatorname{div} z_0^*(q)$ and $\operatorname{curl} z_0^*(q)$ appear in expression (2.12). Let us restrict the analysis to the 1-corotational ansatz (1.8) so that the harmonic map flow reduces to (1.9). We look for a solution that approximately looks like the superposition of a bubble (2.3) with $\xi(t) \equiv 0$, $\omega(t) \equiv 0$ perturbed by (2.6) consisting only of a term Z^* of the form

$$Z^*(r,t) = \begin{bmatrix} e^{i\theta} f(r,t) \\ 0 \end{bmatrix}$$

with f satisfying $\partial_t f = \partial_{rr} f + \frac{1}{r}\partial_r f - \frac{1}{r^2} f$ and $f(0,t) = 0$, namely we propose an approximate solution $v(r,t) = w(\frac{r}{\lambda}) + f(r,t)$ of (1.9). With the notation (2.8), we have that $\operatorname{div} z_0^*(0) = 2\partial_r f(0,0)$, $\operatorname{curl} z_0^*(0) = 0$. Expanding $f(r,t) \approx \partial_r f(0,0) r$ we get that

$$-\partial_t v + \Delta v - \frac{\sin(2v)}{2r^2} \approx \rho w_\rho \frac{\dot\lambda}{\lambda} - \frac{1}{\lambda}\frac{w_\rho}{\rho}\partial_r f(0,0), \quad \rho = \frac{r}{\lambda}.$$

Imposing that the right hand side above is L^2-orthogonal to the kernel of the linearized equation in a ball of radius $\sqrt{T-t}$ suggests that

$$|\log(T-t)|\dot\lambda(t) \approx c f_r(0,0),$$

for a positive universal constant c. This derivation is not correct because significant boundary terms appear in the integration. This issue is solved by the addition of the nonlocal term Φ^0. On the other hand, this suggests the role played by div $z_0^*(0)$ in the expression for λ. The term curl $z_0^*(0)$ appears when we introduce the rotation angle ω, which is needed outside the 1-corotational regime.

In the next sections we will carry out in detail the program for the construction sketched above. In Sect. 3 we will set up several facts about the elliptic linearized operator that will be needed in all subsequent computations. In Sect. 4 we will compute in precise way the error of approximation and define the function Φ^0 mentioned. We also introduce the precise terms appearing in the inner-outer gluing system (2.10). In Sect. 5 we will perform the computations of the orthogonality conditions which lead to expressions (2.12). In Sect. 6 we will carry out the full construction setting up the system as a fixed point problem. We make precise statements of the necessary (major) steps needed, in particular a subtle linear theory for the parabolic inner problem that mimics the Fredholm alternative for the elliptic equation mentioned above, which is developed in Sect. 7. Related Lipschitz estimates and linear bounds for the outer problem are performed in Sect. 7.6 and § A. The adjustment of the parameters to solve the full system is the purpose of Sect. 8. The stability statement is proved in Sect. 10. Finally, we discuss the continuation and reverse bubbling results in Sect. 11.

3 The linearized operator around the bubble

The linearized operator for (2.1) around $U = U_{\lambda,\xi,\omega}$ is the elliptic operator

$$L_U[\varphi] = \Delta\varphi + |\nabla U|^2\varphi + 2(\nabla\varphi\cdot\nabla U)U.$$

Differentiating U with respect to each of its parameters we obtain functions that annihilate this operator, namely solutions of $L_U[\varphi] = 0$. Setting $y = \frac{x-\xi}{\lambda}$, these functions are

$$\partial_\lambda U_{\lambda,\xi,\omega}(x) = \frac{1}{\lambda} Q_\omega \nabla W(y)\cdot y,$$

$$\partial_\omega U_{\lambda,\xi,\omega}(x) = (\partial_\omega Q_\omega) W(y),$$
$$\partial_{\xi_j} U_{\lambda,\xi,\omega}(x) = \frac{1}{\lambda} Q_\omega \partial_{y_j} W(y).$$

We observe that

$$(\partial_\omega Q_\omega) = Q_\omega J_0, \quad \text{where } J_0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can represent $W(y)$ in polar coordinates,

$$W(y) = \begin{pmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta}.$$

We notice that

$$w_\rho = -\frac{2}{1+\rho^2}, \quad \sin w = -\rho w_\rho = \frac{2\rho}{1+\rho^2}, \quad \cos w = \frac{\rho^2 - 1}{1+\rho^2},$$

and derive the alternative expressions

$$\begin{aligned} \partial_\lambda U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{01}(y), \quad Z_{01}(y) = \rho w_\rho(\rho)\, E_1(y), \\ \partial_\omega U_{\lambda,\xi,\omega}(x) &= \quad Q_\omega Z_{02}(y), \quad Z_{02}(y) = \rho w_\rho(\rho)\, E_2(y), \\ \partial_{\xi_j} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{11}(y), \quad Z_{11}(y) = w_\rho(\rho)\, [\cos\theta\, E_1(y) + \sin\theta\, E_2(y)], \\ \partial_{\xi_j} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} Q_\omega Z_{12}(y), \quad Z_{12}(y) = w_\rho(\rho)\, [\sin\theta\, E_1(y) - \cos\theta\, E_2(y)], \end{aligned} \tag{3.1}$$

where

$$E_1(y) = \begin{pmatrix} e^{i\theta} \cos w(\rho) \\ -\sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} i e^{i\theta} \\ 0 \end{pmatrix}.$$

The relation $|U_{\lambda,\xi,\omega}| = 1$ implies that all the functions Z_{ij} are pointwise orthogonal to $U_{\lambda,\xi,\omega}$. In fact the vectors $E_1(y)$, $E_2(y)$ constitute an orthonormal basis of the tangent space to S^2 at the point $W(y)$.

We have $L_W[Z_{lj}] = 0$ where for a function $\phi(y)$ we define

$$L_W[\phi] = \Delta_y \phi + |\nabla W(y)|^2 \phi + 2(\nabla W(y) \cdot \nabla \phi) W(y).$$

In addition to the elements (3.1) in the kernel of L_W there are also two other relevant functions in the kernel, namely

$$Z_{-1,1} = \rho^2 w_\rho(\rho)(\cos\theta E_1 - \sin\theta E_2),\ Z_{-1,2} = \rho^2 w_\rho(\rho)(\sin\theta E_1 + \cos\theta E_2). \tag{3.2}$$

It is worth noticing the connection between this operator and L_U which is given by

$$L_U[\varphi] = \frac{1}{\lambda^2} Q_\omega L_W[\phi], \quad \varphi(x) = \phi(y), \quad y = \frac{x-\xi}{\lambda}.$$

The linearized operator at functions orthogonal to *U*

It will be especially significant to compute the action of L_U on functions with values pointwise orthogonal to U. In what remains of this section we will derive various formulas that will be very useful later on.

For an arbitrary function $\Phi(x)$ with values in $\mathbb{R}^3$ we denote the projection

$$\Pi_{U^\perp}\Phi := \Phi - (\Phi\cdot U)U.$$

A direct computation shows the validity of the following:

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + \tilde{L}_U[\Phi]$$

where

$$\tilde{L}_U[\Phi] := |\nabla U|^2 \Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U,$$

and

$$\nabla(\Phi\cdot U)\nabla U = \partial_{x_j}(\Phi\cdot U)\,\partial_{x_j} U.$$

A very convenient expression for $\tilde{L}_U[\Phi]$ is obtained if we use polar coordinates. Writing in complex notation

$$\Phi(x) = \Phi(r,\theta), \quad x = \xi + re^{i\theta},$$

we find

$$\tilde{L}_U[\Phi] = -\frac{2}{\lambda} w_\rho(\rho)\left[(\Phi_r\cdot U)Q_\omega E_1 - \frac{1}{r}(\Phi_\theta\cdot U)Q_\omega E_2\right], \quad \rho = \frac{r}{\lambda}. \tag{3.3}$$

We single out two consequences of formula (3.3) which will be crucial for later purposes. Let us assume that $\Phi(x)$ is a C^1 function $\Phi : \Omega \to \mathbb{C} \times \mathbb{R}$, which we express in the form

$$\Phi(x) \;=\; \begin{pmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) \end{pmatrix}. \tag{3.4}$$

We also denote

$$\varphi = \varphi_1 + i\varphi_2, \quad \bar{\varphi} = \varphi_1 - i\varphi_2$$

and define the operators

$$\operatorname{div} \varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \operatorname{curl} \varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

We have the validity of the following formula

$$\tilde{L}_U[\Phi] = \tilde{L}_U[\Phi]_0 + \tilde{L}_U[\Phi]_1 + \tilde{L}_U[\Phi]_2\,, \tag{3.5}$$

where

$$\begin{cases} \tilde{L}_U[\Phi]_0 = \lambda^{-1}\rho w_\rho^2 \left[\operatorname{div}(e^{-i\omega}\varphi)\, Q_\omega E_1 + \operatorname{curl}(e^{-i\omega}\varphi)\, Q_\omega E_2 \right] \\ \tilde{L}_U[\Phi]_1 = -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1}\varphi_3)\cos\theta + (\partial_{x_2}\varphi_3)\sin\theta \right] Q_\omega E_1 \\ \qquad\qquad - 2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1}\varphi_3)\sin\theta - (\partial_{x_2}\varphi_3)\cos\theta \right] Q_\omega E_2\,, \\ \tilde{L}_U[\Phi]_2 = \lambda^{-1}\rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\varphi})\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar{\varphi})\, \sin 2\theta \right] Q_\omega E_1 \\ \qquad\qquad + \lambda^{-1}\rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\varphi})\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar{\varphi})\, \cos 2\theta \right] Q_\omega E_2. \end{cases} \tag{3.6}$$

Another corollary of formula (3.3) that we single out is the following: assume that

$$\Phi(x) = \begin{pmatrix} \phi(r)e^{i\theta} \\ 0 \end{pmatrix}, \quad x = \xi + re^{i\theta}, \quad \rho = \frac{r}{\lambda}$$

where $\phi(r)$ is complex valued. Then

$$\tilde{L}_U[\Phi] = \frac{2}{\lambda} w_\rho(\rho)^2 \left[\operatorname{Re}(e^{-i\omega}\partial_r\phi(r)) Q_\omega E_1 + \frac{1}{r}\operatorname{Im}(e^{-i\omega}\phi(r)) Q_\omega E_2 \right]. \tag{3.7}$$

A final result in this section is a computation (in polar coordinates) of the operator L_U acting on a function of the form

$$\Phi(x) = \varphi_1(\rho,\theta) Q_\omega E_1 + \varphi_2(\rho,\theta) Q_\omega E_2, \quad x = \xi + \lambda\rho e^{i\theta}.$$

We have:

$$L_U[\Phi] = \lambda^{-2}\left(\partial_\rho^2\varphi_1 + \frac{\partial_\rho\varphi_1}{\rho} + \frac{\partial_\theta^2\varphi_1}{\rho^2} + \left(2w_\rho^2 - \frac{1}{\rho^2}\right)\varphi_1 - \frac{2}{\rho^2}\partial_\theta\varphi_2\cos w\right) Q_\omega E_1$$
$$+\lambda^{-2}\left(\partial_\rho^2\varphi_2 + \frac{\partial_\rho\varphi_2}{\rho} + \frac{\partial_\theta^2\varphi_2}{\rho^2} + \left(2w_\rho^2 - \frac{1}{\rho^2}\right)\varphi_2 + \frac{2}{\rho^2}\partial_\theta\varphi_1\cos w\right) Q_\omega E_2.$$

4 The error and the inner-outer gluing system

The linearized operator for (2.1) around $U = U_{\lambda,\xi,\omega}$ is the elliptic operator

$$L_U[\varphi] = \Delta_x\varphi + |\nabla_x U|^2\varphi + 2(\nabla_x\varphi\cdot\nabla_x U)U,$$

where $\varphi = \varphi(x,t)$. Consistently we denote for a function $\phi = \phi(y,t)$. Let us denote

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u$$

A useful observation that we make is that as long as the constraint $|u| = 1$ is kept at all times and $u = U + v$ with $|v| \le \frac{1}{2}$ uniformly, then for u to solve equation (2.2) it suffices that

$$S(U+v) = b(x,t)U \tag{4.1}$$

for some scalar function b. Indeed, we observe that since $|u| \equiv 1$ we have

$$b\,(U\cdot u) = S(u)\cdot u = -\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\Delta|u|^2 = 0,$$

and since $U\cdot u \ge \frac{1}{2}$, we find that $b \equiv 0$.

Using that

$$\Delta U + |\nabla U|^2 U = 0$$

we find the following expansion for $S(U+v)$ with v given by (2.4):

$$S(U + \Pi_{U^\perp}\varphi + aU) = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi)$$
$$+c(\Pi_{U^\perp}\varphi)U$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a = a(\zeta)$,

$$L_U(\zeta) = \Delta\zeta + |\nabla U|^2\zeta + 2(\nabla U\cdot\zeta)U$$

$$N_U(\zeta) = \big[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U\cdot\nabla\zeta + |\nabla\zeta|^2 + |\nabla(aU)|^2\big]\zeta - aU_t$$
$$+ 2\nabla a\nabla U,$$
$$c(\zeta) = \Delta a - a_t + (|\nabla(U+\zeta+aU)|^2 - |\nabla U|^2)(1+a) - 2\nabla U\cdot\nabla\zeta$$

Since we just need to have an equation of the form (4.1) satisfied, we find that

$$u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U$$

solves (2.2) if and only if φ satisfies

$$0 = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + b(x,t)U, \tag{4.2}$$

for some scalar function b. The logic of the construction goes like this: We decompose φ into the sum of two functions $\varphi = \varphi^i + \varphi^o$, the "inner" and "outer" solutions and reduce equation (4.2) to solving a system of two equations in (φ^i, φ^o) that we call the inner and outer problems.

The inner function $\varphi^i(x,t)$ will be assumed supported only near $x = \xi(t)$ and better read as a function of the scaled space variable $y = \frac{x-\xi(t)}{\lambda(t)}$ with zero initial condition and such that $\varphi^i\cdot U = 0$, so that $\Pi_{U^\perp}\varphi^i = \varphi^i$. The outer function $\varphi^o(x,t)$ will be made out of several pieces and its role is essentially to satisfy (4.2) far away from the concentration point $x = \xi(t)$.

We write equation (4.2) in the following way:

$$0 = -\partial_t\varphi^i + L_U[\varphi^i] + \tilde L_U[\varphi^o] - \Pi_{U^\perp}[\partial_t\varphi^o - \Delta\varphi^o + U_t] \tag{4.3}$$
$$+ N_U(\varphi^i + \Pi_{U^\perp}\varphi^o) + (\varphi^o\cdot U)U_t + bU.$$

For the outer problem, we consider a function Φ^0 that depends explicitly on the parameter functions chosen in such a way that $\Pi_{U^\perp}[\partial_t\Phi^0 - \Delta\Phi^0 + U_t]$ gets concentrated near $x = \xi(t)$ by elimination of the terms in the first error U_t associated to dilation and rotation. Then we write

$$\varphi^o(x,t) = \Phi^0(x,t) + \Psi^*(x,t). \tag{4.4}$$

For the inner solution, we consider a smooth smooth cut-off function $\eta_0(s)$ with $\eta_0(s) = 1$ for $s < 1$ and $= 0$ for $s > \frac{3}{2}$. We also consider a positive, large smooth function $R(t) \to +\infty$ as $t \to T$ that we will later specify. We define

$$\eta(x,t) := \eta_0\left(R(t)^{-1}|y|\right), \quad y = \frac{x-\xi(t)}{\lambda(t)}$$

and let

$$\varphi^i(x,t) = \eta(x,t) Q_\omega \phi(y,t), \quad y = \frac{x-\xi(t)}{\lambda(t)}$$

for a function $\phi(y,t)$ with initial condition $\phi(\cdot,0) = 0$ that satisfies $\phi(\cdot,t)\cdot W \equiv 0$, defined for $|y| \le 2R(t)$ and that vanishes as $t \to T$. Then we have

$$\begin{aligned} Q_{-\omega} L_U[\varphi^i] &= \lambda^{-2}\eta L_W[\phi] + (\Delta_x\eta)\phi + 2\lambda^{-1}\nabla_x\eta\nabla_y\phi \\ Q_{-\omega}\varphi^i_t &= \eta\big(\phi_t - \lambda^{-1}\dot\lambda y\cdot\nabla_y\phi - \lambda^{-1}\dot\xi\cdot\nabla_y\phi + \dot\omega Q_{-\omega}\partial_\omega Q_\omega\phi\big) + \eta_t\phi. \end{aligned}$$

Equation (4.3) then becomes

$$\begin{aligned} 0 = {} & \lambda^{-2}\eta Q_\omega[-\lambda^2\phi_t + L_W[\phi] + \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]] \\ & + \eta Q_\omega(\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi - \dot\omega J\phi) \\ & + \tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t] \\ & - \partial_t\Psi^* + \Delta\Psi^* + (1-\eta)\tilde L_U[\Psi^*] + Q_\omega[(\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi] \\ & + N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) + ((\Psi^*+\Phi^0)\cdot U)U_t + bU. \end{aligned} \tag{4.5}$$

Next we will define precisely the operator Φ^0 and estimate the quantity

$$\tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t]. \tag{4.6}$$

The idea is to choose Φ^0 such that $\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t \approx 0$ whenever $|x-\xi| \gg \lambda$, so that in particular the last error term in the outer equation (4.4) is of smaller order.

Invoking formulas (3.1) to compute U_t we get

$$U_t = \dot\lambda\partial_\lambda U_{\lambda,\xi,\omega} + \dot\omega\partial_\omega U_{\lambda,\xi,\omega} + \partial_\xi U_{\lambda,\xi,\omega}\cdot\dot\xi = \mathcal{E}_0 + \mathcal{E}_1,$$

where, setting $y = \frac{x-\xi}{\lambda} = \rho e^{i\theta}$, we have

$$\begin{aligned} \mathcal{E}_0(x,t) &= -Q_\omega\left[\frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\, E_1(y) + \dot\omega\rho w_\rho(\rho)\, E_2(y)\right] \\ \mathcal{E}_1(x,t) &= -\frac{\dot\xi_1}{\lambda}\, w_\rho(\rho)\, Q_\omega[\cos\theta\, E_1(y) + \sin\theta\, E_2(y)] \\ &\quad - \frac{\dot\xi_2}{\lambda}\, w_\rho(\rho)\, Q_\omega[\sin\theta\, E_1(y) - \cos\theta\, E_2(y)]. \end{aligned}$$

Since $\mathcal{E}_1$ has faster space decay in ρ than $\mathcal{E}_0$ we will choose Φ^0 to be an approximate solution of

$$\Phi^0_t - \Delta_x \Phi^0 + \mathcal{E}_0 = 0. \tag{4.7}$$

For $x = \xi + re^{i\theta}$ and $r \gg \lambda$ we have

$$\begin{aligned}\mathcal{E}_0(x,t) &= -\frac{2r}{r^2+\lambda^2}\left[\dot\lambda Q_\omega E_1 + \lambda\dot\omega Q_\omega E_2\right]\\ &\approx -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix}.\end{aligned}$$

Here and in what follows we let

$$p(t) = \lambda(t)e^{i\omega(t)}.$$

Then

$$-\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix} = -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix} =: \tilde{\mathcal{E}}_0(x,t).$$

With the aid of Duhamel's formula for the standard heat equation, we find that the following function is a good approximate solution of $\Phi^0_t - \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = 0$ and hence of (4.7). We define

$$\begin{aligned}\Phi^0[\omega,\lambda,\xi] &:= \begin{bmatrix}\varphi^0(r,t)e^{i\theta}\\ 0\end{bmatrix}\\ \varphi^0(r,t) &= -\int_{-T}^{t}\dot p(s) r k(z(r), t-s)\, ds\\ z(r) &= \sqrt{r^2+\lambda^2}, \quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2},\end{aligned}$$

where for technical reasons that will be made clear later on, $p(t)$ is also assumed to be defined for negative values of t.

A direct computation yields

$$\Phi^0_t + \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = \tilde{\mathcal{R}}_0 + \tilde{\mathcal{R}}_1, \quad \tilde{\mathcal{R}}_0 = \begin{pmatrix}\mathcal{R}_0\\ 0\end{pmatrix}, \quad \tilde{\mathcal{R}}_1 = \begin{pmatrix}\mathcal{R}_1\\ 0\end{pmatrix}$$

where

$$\mathcal{R}_0 := -re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^{t}\dot p(s)(zk_z - z^2k_{zz})(z(r), t-s)\, ds$$

and

$$\mathcal{R}_1 := -e^{i\theta}\text{Re}\,(e^{-i\theta}\dot{\xi}(t))\int_{-T}^{t} \dot{p}(s)\, k(z(r), t-s)\, ds$$
$$+ \frac{r}{z^2}e^{i\theta}\,(\lambda\dot{\lambda}(t) - \text{Re}\,(re^{i\theta}\dot{\xi}(t)))\int_{-T}^{t} \dot{p}(s)\; zk_z(z(r), t-s)\, ds.$$

We observe that $\mathcal{R}_1$ is actually a term of smaller order. Using formulas (3.5), (3.7) and the facts

$$\frac{\lambda^2 r}{z^4} = \frac{1}{4\lambda}\rho w_\rho^2, \quad \frac{r}{z^2}(1-\cos w) = \frac{1}{2\lambda}\rho w_\rho^2,$$

we derive an expression for the quantity (4.6):

$$\begin{aligned}&\tilde{L}_U[\Phi^0] + \Pi_{U^\perp}[-U_t + \Delta\Phi^0 - \Phi^0_t]\\ &= \tilde{L}_U[\Phi^0] - \mathcal{E}_1 + \Pi_{U^\perp}[\tilde{\mathcal{E}}_0] - \mathcal{E}_0 + \Pi_{U^\perp}[\tilde{\mathcal{R}}_0] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\\ &= \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\end{aligned}$$

where

$$\mathcal{K}_0[p,\xi] = \mathcal{K}_{01}[p,\xi] + \mathcal{K}_{02}[p,\xi]$$

with

$$\mathcal{K}_{01}[p,\xi] := -\frac{2}{\lambda}\rho w_\rho^2 \int_{-T}^{t}\Big[\text{Re}\,(\dot{p}(s)e^{-i\omega(t)})Q_\omega E_1 + \text{Im}\,(\dot{p}(s)e^{-i\omega(t)})Q_\omega E_2\Big] \cdot k(z, t-s)\, ds \tag{4.8}$$

$$\begin{aligned}\mathcal{K}_{02}[p,\xi] := {}& \frac{1}{\lambda}\rho w_\rho^2 \left[\dot{\lambda} - \int_{-T}^{t} \text{Re}\,(\dot{p}(s)e^{-i\omega(t)}) r k_z(z, t-s) z_r\, ds\right] Q_\omega E_1\\ &- \frac{1}{4\lambda}\rho w_\rho^2 \cos w \left[\int_{-T}^{t} \text{Re}\,(\dot{p}(s)e^{-i\omega(t)})\,(zk_z - z^2 k_{zz})(z, t-s)\, ds\right] Q_\omega E_1\\ &- \frac{1}{4\lambda}\rho w_\rho^2 \left[\int_{-T}^{t} \text{Im}\,(\dot{p}(s)e^{-i\omega(t)})\,(zk_z - z^2 k_{zz})(z, t-s)\, ds\right] Q_\omega E_2,\end{aligned} \tag{4.9}$$

$$\mathcal{K}_1[p,\xi] := \frac{1}{\lambda} w_\rho \big[\text{Re}\big((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta}\big)Q_\omega E_1 + \text{Im}\big((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta}\big)Q_\omega E_2\big]. \tag{4.10}$$

We insert this decomposition in equation (4.5) and see that we will have a solution to the equation if the pair (ϕ, Ψ^*) solves the *inner-outer gluing system*

$$\begin{cases} \lambda^2\phi_t = L_W[\phi] + \lambda^2 Q_{-\omega}\Big[\tilde{L}_U[\Psi^*] + \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]\Big] & \text{in } \mathcal{D}_{2R}\\ \phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \phi(\cdot,0) = 0 = \phi(\cdot,T), \end{cases} \tag{4.11}$$

$$\Psi_t^* = \Delta_x \Psi^* + g[p, \xi, \Psi^*, \phi] \quad \text{in } \Omega \times (0, T) \tag{4.12}$$

where

$$\begin{aligned} g[p, \xi, \Psi^*, \phi] := {} & (1-\eta)\tilde{L}_U[\Psi^*] + (\Psi^* \cdot U)U_t \\ & + Q_\omega\big((\Delta_x \eta)\phi + 2\nabla_x \eta \nabla_x \phi - \eta_t \phi\big) \\ & + \eta Q_\omega\big(-\dot{\omega} J\phi + \lambda^{-1}\dot{\lambda} y \cdot \nabla_y \phi + \lambda^{-1}\dot{\xi} \cdot \nabla_y \phi\big) \\ & + (1-\eta)[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0 \cdot U)U_t \\ & + N_U(\eta Q_\omega \phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)), \end{aligned} \tag{4.13}$$

and we denote

$$\mathcal{D}_{\gamma R} = \{(y, t) \in \mathbb{R}^2 \times (0, T) \ / \ |y| < \gamma R(t)\}.$$

Indeed if (ϕ, Ψ^*) solves this system, then we have that

$$u(x, t) = U + \Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi])U \tag{4.14}$$

solves equation (2.2). The boundary condition $u = \mathbf{e}_3$ amounts to

$$\Pi_{U^\perp}[\Phi^0 + \Psi^*] + a(\Pi_{U^\perp}[U + \Phi^0 + \Psi^*])U = (\mathbf{e}_3 - U)$$

and then it suffices that we take the boundary condition for (4.12)

$$\Psi^*\big|_{\partial\Omega} = \mathbf{e}_3 - U - \Phi^0. \tag{4.15}$$

Since we want $u(x, t)$ to be a small perturbation of $U(x, t)$ when we stand close to (q, T), it is natural to require that Ψ^* satisfies the final condition

$$\Psi^*\big(q, T) = 0.$$

This constraint amounts to three Lagrange multipliers when we solve the problem, which we choose to put in the initial condition. Then we assume

$$\Psi^*\big(x, 0) = Z_0^*(x) + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3,$$

where c_1, c_2, c_3 are undetermined constants and $Z_0^*(x)$ is a small function for which specific assumptions will later be made.

5 The reduced equations

In this section we will informally discuss the procedure to achieve our purpose in particular deriving the order of vanishing of the scaling parameter $\lambda(t)$ as $t \to T$.

The main term that couples equations (4.11) and (4.12) inside the second equation is the linear expression

$$Q_\omega[(\Delta_x \eta)\phi + 2\nabla_x \eta \nabla_x \phi + \eta_t \phi],$$

which is supported in $|y| = O(R)$. This motivates the fact that we want ϕ to exhibit some type of space decay in $|y|$ since in that way Ψ^* will eventually be smaller and in turn that would make the two equations at main order *uncoupled*. Equation (4.11) has the form

$$\begin{aligned} \lambda^2 \phi_t &= L_W[\phi] + h[p, \xi, \Psi^*](y, t) \quad \text{in } \mathcal{D}_{2R} \\ \phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)}, \end{aligned}$$

where, for convenience we assume that $h(y, t)$ is defined for all $y \in \mathbb{R}^2$ extending outside $\mathcal{D}_{2R}$ as

$$\begin{aligned} h[p, \xi, \Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*]\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi] \\ + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi]\chi_{\mathcal{D}_{2R}}, \end{aligned} \tag{5.1}$$

where χ_A designates characteristic function of a set A, $\mathcal{K}_0$ is defined in (4.8), (4.9) and $\mathcal{K}_1$ in (4.10). If $\lambda(t)$ has a relatively smooth vanishing as $t \to T$ it seems natural that the term $\lambda^2 \phi_t$ be of smaller order and then the equation is approximately represented by the elliptic problem

$$L_W[\phi] + h[p, \xi, \Psi^*] = 0, \quad \phi \cdot W = 0 \quad \text{in } \mathbb{R}^2. \tag{5.2}$$

Let us consider the decaying functions $Z_{lj}(y)$ defined in formula (3.1), which satisfy $L_W[Z_{lj}] = 0$. If $\phi(y, t)$ is a solution of (5.2) with sufficient decay, then necessarily

$$\int_{\mathbb{R}^2} h[p, \xi, \Psi^*](y, t) \cdot Z_{lj}(y)\, dy = 0 \qquad \text{for all} \quad t \in (0, T), \tag{5.3}$$

for $l = 0, 1$, $j = 1, 2$. These relations amount to an integro-differential system of equations for $p(t)$, $\xi(t)$, which, as a matter of fact, *detemine* the correct values of the parameters so that the solution (ϕ, Ψ^*) with appropriate asymptotics exists.

We derive next useful expressions for relations (5.3). Let us first compute the quantities

$$\mathcal{B}_{0j}[p](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi]+\mathcal{K}_1[p,\xi]]\cdot Z_{0j}(y)\,dy. \tag{5.4}$$

Using (4.8), (4.9) the following expressions for $\mathcal{B}_{01}$, $\mathcal{B}_{02}$ are readily obtained:

$$\mathcal{B}_{01}[p](t) = \int_{-T}^{t} \mathrm{Re}\,(\dot p(s)e^{-i\omega(t)})\,\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s} - 2\dot\lambda(t)$$
$$\mathcal{B}_{02}[p](t) = \int_{-T}^{t} \mathrm{Im}\,(\dot p(s)e^{-i\omega(t)})\,\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s}$$

where $\Gamma_j(\tau)$, $j=1,2$ are the smooth functions defined as follows:

$$\Gamma_1(\tau) = -\int_0^\infty \rho^3 w_\rho^3 \Big[K(\zeta)+2\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2} -4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\Big]_{\zeta=\tau(1+\rho^2)}\,d\rho$$
$$\Gamma_2(\tau) = -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta)-\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)}\,d\rho$$

where

$$K(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta},$$

and we have used that $\int_0^\infty \rho^3 w_\rho^3 d\rho = -2$. Using these expressions we find that

$$|\Gamma_l(\tau)-1| \le C\tau(1+|\log\tau|) \quad \text{for } \tau<1, \tag{5.5}$$
$$|\Gamma_l(\tau)| \le \frac{C}{\tau} \qquad \text{for } \tau>1,\ l=1,2.$$

Let us define

$$\mathcal{B}_0[p] := \frac{1}{2}e^{i\omega(t)}\left(\mathcal{B}_{01}[p]+i\mathcal{B}_{02}[p]\right) \tag{5.6}$$

and

$$a_{0j}[p,\xi,\Psi^*] := -\frac{\lambda}{2\pi}\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi^*]\cdot Z_{0j}(y)\,dy,\ j=1,2,$$
$$a_0[p,\xi,\Psi^*] := \frac{1}{2}e^{i\omega(t)}\left(a_{01}[p,\xi,\Psi^*]+ia_{02}[p,\xi,\Psi^*]\right). \tag{5.7}$$

Similarly, we let

$$\mathcal{B}_{1j}[\xi](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] \cdot Z_{1j}(y)\, dy,\ j = 1, 2,$$
$$\mathcal{B}_1[\xi](t) := \mathcal{B}_{11}[\xi](t) + i\mathcal{B}_{12}[\xi](t).$$

Using (4.10), (3.1) and the fact that $\int_0^\infty \rho w_\rho^2 d\rho = 2$ we get

$$\mathcal{B}_1[\xi](t) = 2[\dot{\xi}_1(t) + i\dot{\xi}_2(t)].$$

At last, we set

$$a_{1j}[p,\xi,\Psi^*] := \frac{\lambda}{2\pi}\int_{B_{2R}} Q_{-\omega}\tilde{L}_U[\Psi^*] \cdot Z_{1j}(y)\, dy,\ j = 1, 2,$$
$$a_1[p,\xi,\Psi^*] := -e^{i\omega(t)}(a_{11}[p,\xi,\Psi^*] + ia_{12}[p,\xi,\Psi^*]).$$

We get that the four conditions (5.3) reduce to the system of two complex equations

$$\mathcal{B}_0[p] = a_0[p,\xi,\Psi^*], \tag{5.8}$$
$$\mathcal{B}_1[\xi] = a_1[p,\xi,\Psi^*]. \tag{5.9}$$

At this point we will make some preliminary considerations on this system that will allow us to find a first guess of the parameters $p(t)$ and $\xi(t)$. First, we observe that

$$\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds + O\big(\|\dot{p}\|_\infty\big).$$

To get an approximation for a_0, we analyze the operator $\tilde{L}_U$ in a_0. For this let us write

$$\Psi^* = \begin{bmatrix}\psi^* \\ \psi_3^*\end{bmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*.$$

From formula (3.5) we find that

$$\tilde{L}_U[\Psi^*](y) = [\tilde{L}_U]_0[\Psi^*] + [\tilde{L}_U]_1[\Psi^*] + [\tilde{L}_U]_2[\Psi^*],$$

where

$$\lambda Q_{-\omega}[\tilde{L}_U]_0[\Psi^*] = \rho w_\rho^2 \left[\operatorname{div}(e^{-i\omega}\psi^*)\, E_1 + \operatorname{curl}(e^{-i\omega}\psi^*)\, E_2\right]$$

$$\lambda Q_{-\omega}[\tilde{L}_U]_1[\Psi^*] = -2w_\rho \cos w \left[(\partial_{x_1}\psi_3^*)\cos\theta + (\partial_{x_2}\psi_3^*)\sin\theta \right] E_1 \\ - 2w_\rho \cos w \left[(\partial_{x_1}\psi_3^*)\sin\theta - (\partial_{x_2}\psi_3^*)\cos\theta \right] E_2 \,,$$

$$\lambda Q_{-\omega}[\tilde{L}_U]_2[\Psi^*] = \rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta \right] E_1 \\ + \rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta \right] E_2,$$

and the differential operators in Ψ^* on the right hand sides are evaluated at (x,t) with $x = \xi(t) + \lambda(t)y$, $y = \rho e^{i\theta}$ while $E_l = E_l(y)$, $l = 1, 2$.

From the above decomposition, assuming that Ψ^* is of class C^1 in space variable, we find that

$$a_0[p, \xi, \Psi^*] = [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi, t) + o(1),$$

where $o(1) \to 0$ as $t \to T$.

Similarly, we have that

$$a_1(p,\xi) = 2(\partial_{x_1}\psi_3^* + i\partial_{x_2}\psi_3^*)(\xi,t)\int_0^\infty \cos w\, w_\rho^2 \rho\, d\rho + o(1) \\ = o(1) \quad \text{as} \quad t \to T,$$

since $\int_0^\infty w_\rho^2 \cos w \rho\, d\rho = 0$.

Let us discuss informally how to handle (5.8)–(5.9). For this we simplify this system in the form

$$\int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds = [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi(t), t) + o(1) + O(\|\dot{p}\|_\infty) \\ \dot{\xi}(t) = o(1) \quad \text{as} \quad t \to T. \tag{5.10}$$

We assume for the moment that the function $\Psi^*(x,t)$ is fixed, sufficiently regular, and we regard T as a parameter that will always be taken smaller if necessary. We recall that we want $\xi(T) = q$ where $q \in \Omega$ is given, and $\lambda(T) = 0$. Equation (5.10) immediately suggests us to take $\xi(t) \equiv q$ as a first approximation. Neglecting lower order terms, we arrive at the "clean" equation for $p(t) = \lambda(t)e^{i\omega(t)}$,

$$\int_{-T}^{t-\lambda(t)^2} \frac{\dot{p}(s)}{t-s} ds = \operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0) =: a_0^* \tag{5.11}$$

At this point we make the following assumption:

$$\operatorname{div}\psi^*(q,0) < 0. \tag{5.12}$$

This implies that $a_0^* = -|a_0^*|e^{i\omega_0}$ for a unique $\omega_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let us take $\omega(t) \equiv \omega_0$. Then equation (5.11) becomes

$$\int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s} ds = -|a_0^*|. \tag{5.13}$$

We claim that a good approximate solution of (5.13) as $t \to T$ is given by

$$\dot\lambda(t) = -\frac{\kappa}{\log^2(T-t)}$$

for a suitable $\kappa > 0$. In fact, substituting, we have

$$\begin{aligned}\int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s}\, ds + \dot\lambda(t)\left[\log(T-t) - 2\log(\lambda(t))\right] \\ &\quad + \int_{t-(T-t)}^{t-\lambda(t)^2} \frac{\dot\lambda(s)-\dot\lambda(t)}{t-s} ds \\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s}\, ds \;-\; \dot\lambda(t)\log(T-t) \;=:\; \beta(t)\end{aligned} \tag{5.14}$$

as $t \to T$. We see that

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}(\log^2(T-t)\,\dot\lambda(t)) = 0$$

from the explicit form of $\dot\lambda(t)$. Hence $\beta(t)$ is constant. As a conclusion, equation (5.13) is approximately satisfied if κ is such that

$$\kappa \int_{-T}^{T} \frac{\dot\lambda(s)}{T-s} \;=\; -|a_0^*|.$$

And this finally gives us the approximate expression

$$\dot\lambda(t) = -|\mathrm{div}\,\psi^*(q,0) + i\,\mathrm{curl}\,\psi^*(q,0)|\,\dot\lambda_*(t),$$

where

$$\dot\lambda_*(t) = -\frac{|\log T|}{\log^2(T-t)}.$$

Naturally imposing $\lambda_*(T) = 0$ we then have

$$\lambda_*(t) = \frac{|\log T|}{\log^2(T-t)}(T-t)\,(1+o(1)) \quad \text{as} \quad t \to T.$$

6 Solving the inner-outer gluing system

Our purpose is to determine, for a given $q \in \Omega$ and a sufficiently small $T > 0$, a solution (ϕ, Ψ^*) of system (4.11)–(4.12) with a boundary condition of the form (4.15) such that $u(x,t)$ given by (4.14) blows up with $U(x,t)$ as its main order profile. This will only be possible for adequate choices of the parameter functions $\xi(t)$ and $p(t) = \lambda(t)e^{i\omega(t)}$. These functions will eventually be found by fixed point arguments, but a priori we need to make some assumptions regarding their behavior. For some positive numbers a_1, a_2, σ independent of T we will assume that

$$a_1|\dot{\lambda}_*(t)| \le |\dot{p}(t)| \le a_2|\dot{\lambda}_*(t)| \quad \text{for all} \quad t \in (0,T), \tag{6.1}$$

$$|\dot{\xi}(t)| \le \lambda_*(t)^{\sigma} \quad \text{for all} \quad t \in (0,T). \tag{6.2}$$

We also take

$$R(t) = \lambda_*(t)^{-\beta}, \tag{6.3}$$

where $\beta \in (0, \frac{1}{2})$.

To solve the outer equation (4.12) we will decompose Ψ^* in the form

$$\Psi^* = Z^* + \psi$$

where we let $Z^* : \Omega \times (0,\infty) \to \mathbb{R}^3$ satisfy (2.7) with $Z_0^*(x)$ a function satisfying certain conditions to be described below. Since we would like that $u(x,t)$ given by (4.14) has a blow-up behavior given at main order by that of $U(x,t)$, we will require

$$\Psi^*(q,T) = 0.$$

This constraint has three parameters. Therefore we need three "Lagrange multipliers" which we include in the initial datum.

6.1 Assumptions on Z_0^*

To describe the assumptions on Z_0^*, let us write

$$Z_0^*(x) = \begin{bmatrix} z_0^*(x) \\ z_{03}^*(x) \end{bmatrix}, \quad z_0^*(x) = z_{01}^*(x) + iz_{02}^*(x).$$

A first condition that we require, consistent with (5.12), is $\operatorname{div} z_0^*(q) < 0$. In addition we require that $Z_0^*(q) \approx 0$ in a non-degenerate way.

We want also Z^* to be sufficiently small, but independently of T, so that the heat equation (2.7) is a good approximation of the linearized harmonic map flow far from the singularity. In order to achieve later the desired stability property, it is convenient to split Z_0^* into two parts

$$Z_0^* = Z_0^{*0} + Z_0^{*1},$$

where Z_0^{*0} is sufficiently smooth and Z_0^{*1} allows more irregular perturbations. More precisely, for Z_0^{*0} we assume that for some $\alpha_0 > 0$ small and some $\alpha_1, \alpha_2 > 0$, all independent of T, we have

$$\begin{cases} \|Z_0^{*0}\|_{C^3(\overline{\Omega})} \leq \alpha_0, \\ |Z_0^{*0}(q)| \leq 5T, \\ |(Dz_0^{*0}(q))^{-1}| \leq \alpha_1, \\ -\alpha_1 \leq \operatorname{div} z_0^{*0}(q) \leq -\alpha_2. \end{cases} \tag{6.4}$$

(The notation here is analogous to (2.8)).

To describe Z_0^{*1} we introduce the following norm

$$\begin{aligned} \|Z_0^{*1}\|_* = \sup_\Omega |Z_0^{*1}(x)| + \frac{1}{|\log \varepsilon_*|} \sup_\Omega |\nabla_x Z_0^{*1}(x)| \\ + \frac{1}{|\log \varepsilon_*|^{1/2}} \sup_\Omega (|x - q_0| + \varepsilon_*) \, |D_x^2 Z_0^{*1}(x)|, \end{aligned} \tag{6.5}$$

where

$$\varepsilon_* = \lambda_*(0). \tag{6.6}$$

Then we assume that for some $\sigma > 0$ fixed we have

$$\|Z_0^{*1}\|_* \leq T^\sigma. \tag{6.7}$$

In summary, the conditions on Z_0^* are the following:

$$Z_0^* = Z_0^{*0} + Z_0^{*1} \text{ with } Z_0^{*0}, Z_0^{*1} \text{ satisfying (6.4) and (6.7).} \tag{6.8}$$

6.2 Linear theory for the inner problem

The inner problem (4.11) is written as

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h[p, \xi, \Psi^*] & \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) = 0 & \text{in } B_{2R(0)} \end{cases}$$

where $h[p, \xi, \Psi^*]$ is given by (5.1). To find a good solution to this problem we would like that $h[p, \xi, \Psi^*]$ satisfies the orthogonality conditions (5.3).

We split the right hand side $h[p, \xi, \Psi^*]$ and the inner solution into components with different roles regarding these orthogonality conditions.

Recall that

$$h[p, \xi, \Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*] \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi] + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi] \chi_{\mathcal{D}_{2R}},$$

the decomposition of $\tilde{L}_U$ given in (3.5):

$$\tilde{L}_U[\Psi^*] = \tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_1 + \tilde{L}_U[\Psi^*]_2 \,,$$

with $\tilde{L}_U[\Phi]_j$ defined in (3.6). Using the notation (3.4), we then define

$$\begin{aligned} \tilde{L}_U[\Phi]_1^{(0)} = & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t)) \cos\theta \right. \\ & \left. + (\partial_{x_2} \varphi_3(\xi(t), t))) \sin\theta \right] Q_\omega E_1 \\ & - 2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t))) \sin\theta \right. \\ & \left. - (\partial_{x_2} \varphi_3(\xi(t), t))) \cos\theta \right] Q_\omega E_2 \,. \end{aligned}$$

We then decompose

$$h = h_1 + h_2 + h_3$$

where

$$\begin{aligned} h_1[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} (\tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_2) \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi], \\ h_2[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*]_1^{(0)} \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi] \chi_{\mathcal{D}_{2R}}, \\ h_3[p, \xi, \Psi^*] &= \lambda^2 Q_{-\omega} (\tilde{L}_U[\Psi^*]_1 - \tilde{L}_U[\Psi^*]_1^{(0)}) \chi_{\mathcal{D}_{2R}}. \end{aligned}$$

Next we decompose $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. The function ϕ_1 will solve the inner problem with right hand side $h_1[p, \xi, \Psi^*]$ projected so that it satisfies essentially (5.3). The advantage of doing this is that h_1 has faster spatial decay, which gives better bounds for the solution. For this we let, for any function $h(y, t)$ defined in $\mathbb{R}^2 \times (0, T)$ with sufficient decay,

$$c_{lj}[h](t) := \frac{1}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{lj}|^2} \int_{\mathbb{R}^2} h(y, t) \cdot Z_{lj}(y) \, dy. \tag{6.9}$$

Note that $h[p, \xi, \Psi^*]$ is defined in $\mathbb{R}^2 \times (0, T)$, and for simplicity we will assume that the right hand sides appearing in the different linear equations are always defined in $\mathbb{R}^2 \times (0, T)$.

We would like that ϕ_1 solves

$$\lambda^2 \partial_t \phi_1 = L_W[\phi_1] + h_1[p, \xi, \Psi^*] - \sum_{l=-1}^{1} \sum_{j=1}^{2} c_{lj}[h_1(p, \xi, \Psi^*)] w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R},$$

but the estimates for ϕ_1 are better if the projections $c_{0j}[h(p, \xi, \Psi^*)]$ are modified slightly.

Here is the precise result that we will use later. We define the norms

$$\|h\|_{\nu,a} = \sup_{\mathbb{R}^2 \times (0,T)} \frac{|h(y,t)|}{\lambda_*^\nu (1+|y|)^{-a}}, \tag{6.10}$$

and

$$\|\phi\|_{*,\nu,a,\delta} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)|\nabla_y \phi(y,t)|}{\lambda_*^\nu \max\left(\frac{R^{\delta(5-a)}}{(1+|y|)^3}, \frac{1}{(1+|y|)^{a-2}} \right)}. \tag{6.11}$$

Proposition 6.1 *Let $a \in (2, 3)$, $\delta \in (0, 1)$, $\nu > 0$. Assume $\|h\|_{\nu,a} < \infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,1}[h]$, $\tilde{c}_{0j}[h]$ of*

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h - \sum_{j=1,2} \tilde{c}_{0j}[h] Z_{0j} \chi_{B_1} - \sum_{\substack{l=-1,1 \\ j=1,2}} c_{lj}[h] Z_{lj} \chi_{B_1} & \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) = 0 & \text{in } B_{2R(0)} \end{cases}$$

where c_{lj} is defined in (6.9), which is linear in h, such that

$$\|\phi\|_{*,\nu,a,\delta} \le C \|h\|_{\nu,a}$$

and such that

$$|c_{0j}[h] - \tilde{c}_{0j}[h]| \le C \lambda_*^\nu R^{-\frac{1}{2}\delta(a-2)} \|h\|_{\nu,a}.$$

The function ϕ_2 solves the equation with right hand side $h_2[p, \xi, \Psi^*]$, which is in *mode 1*, a notion that we define next (this is basically motivated by the analysis of Sect. 7, where we consider the linearized parabolic equation and use a Fourier decomposition of the right hand side and the solution).

Let $h(y,t) \in \mathbb{R}^3$, be defined in $\mathbb{R}^2 \times (0,T)$ or $\mathcal{D}_{2R}$ with $h \cdot W = 0$. We say that h is a mode $k \in \mathbb{Z}$ if h has the form

$$h(y,t) = \mathrm{Re}(\tilde{h}_k(|y|,t)e^{ik\theta})E_1 + \mathrm{Re}(\tilde{h}_k(|y|,t)e^{ik\theta})E_2,$$

for some complex valued function $\tilde{h}_k(\rho,t)$.

Consider then

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h - \sum_{j=1,2} c_{1j}[h] w_\rho^2 Z_{1j} & \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0) = 0 & \text{in } B_{2R(0)} \end{cases} \tag{6.12}$$

Proposition 6.2 *Let $a \in (2,3)$, $\delta \in (0,1)$, $\nu > 0$. Assume that h is in mode 1 and $\|h\|_{\nu,a} < \infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,2}[h]$ of (6.12), which is linear in h, such that*

$$\|\phi\|_{\nu,a-2} \le C\|h\|_{\nu,a}.$$

In the above statemen the norm $\|\phi\|_{\nu,a-2}$ analogous to the one in (6.10), but the supremum is taken in $\mathcal{D}_{2R}$.

Another piece of the inner solution, ϕ_3, will handle $h_3[p,\xi,\Psi^*]$, which does not satisfy orthogonality conditions in mode 0. We will still project it to satisfy the orthogonality condition in mode 1. Let us consider then (6.12) without any orthogonality conditions on h in mode 0. We define

$$\|\phi\|_{**,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\left|\nabla_y \phi(y,t)\right|}{\lambda_*(t)^\nu R(t)^2 (1+|y|)^{-1}}. \tag{6.13}$$

Proposition 6.3 *Let $1 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that if $\|h\|_{a,\nu} < +\infty$ there is a solution $\phi = \mathcal{T}_{\lambda,3}[h]$ of (6.12), which is linear in h and satisfies the estimate*

$$\|\phi\|_{**,\nu} \le C\|h\|_{a,\nu}.$$

Note that we allow a to be less than 2 in the previous proposition.

Next we have a variant of Proposition 6.3 when h is in mode -1.

Proposition 6.4 *Let $2 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that for any h in mode -1 with $\|h\|_{a,\nu} < +\infty$, there is a solution $\phi = \mathcal{T}_{\lambda,4}[h]$ of problem (6.12), which is linear in h and satisfies the estimate*

$$\|\phi\|_{***,\nu} \le C\|h\|_{a,\nu},$$

where

$$\|\phi\|_{***,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\left|\nabla_y \phi(y,t)\right|}{\lambda_*(t)^\nu \log(R(t))}.$$

All propositions stated here are corollaries of Proposition 7.1 and proved in Sect. 7.

6.3 The equations for $p = \lambda e^{i\omega}$

We need to choose the free parameters p, ξ so that $c_{lj}[h(p,\xi,\Psi^*)] = 0$ for $l = -1, 0, 1,\ j = 1, 2$. This will be easy to do for $l = 1$ (mode 1), but mode $l = 0$ is more complicated.

To handle c_{0j} we note that by definitions (5.1), (5.4), (5.7)

$$c_{0,j}[h(p,\xi,\Psi^*)] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \left(\mathcal{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*]\right)$$

where B_0, a_0 are defined in (5.6), (5.7) and we recall that $p = \lambda e^{i\omega}$.

So to achieve $c_{0j}[h(p,\xi,\Psi^*)] = 0$ we should solve

$$\mathcal{B}_0[p](t) = a_0[p,\xi,\Psi^*](t), \quad t \in [0,T], \tag{6.14}$$

adjusting the parameters $\lambda(t)$ and $\omega(t)$. This equation is delicate and we will instead impose a modified version of this condition. The modification of (6.14) consists in introducing another term in the equation, essentially modifying the operator $\mathcal{B}_0$.

To make this precise we define the following norms. Let I denote either the interval $[0,T]$ or $[-T,T]$. For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$ we let

$$\|g\|_{\Theta,l} = \sup_{t\in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|, \tag{6.15}$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, and $l \in \mathbb{R}$ we let

$$[g]_{\gamma,m,l} = \sup (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma}, \tag{6.16}$$

where the supremum is taken over $s \le t$ in I such that $t - s \le \frac{1}{10}(T-t)$.

We have then the following result, whose proof is in Sect. 8.

Proposition 6.5 *Let* $\alpha, \gamma \in (0, \frac{1}{2})$*,* $l \in \mathbb{R}$*,* $C_1 > 1$*. There is* $\alpha_0 > 0$ *such that if* $\Theta \in (0, \alpha_0)$ *and* $m \le \Theta - \gamma$*, then for* $a : [0, T] \to \mathbb{C}$ *is such that*

$$\begin{cases} \dfrac{1}{C_1} \le |a(T)| \le C_1, \\ T^{\Theta} |\log T|^{1+\sigma-l} \|a(\cdot) - a(T)\|_{\Theta, l-1} + [a]_{\gamma, m, l-1} \le C_1, \end{cases} \tag{6.17}$$

for some $\sigma > 0$*, then, for* $T > 0$ *small enough there are two operators* $\mathcal{P}$ *and* $\mathcal{R}_0$ *so that* $p = \mathcal{P}[a] : [-T, T] \to \mathbb{C}$ *satisfies*

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad t \in [0, T], \tag{6.18}$$

with

$$\begin{aligned} &|\mathcal{R}_0[a](t)| \\ &\quad \le C\Big(T^{\sigma} + T^{\Theta} \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta, l-1} + [a]_{\gamma, m, l-1}\Big) \\ &\quad \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^{l}}, \end{aligned} \tag{6.19}$$

for some $\sigma > 0$.

We have additional properties of the solution to this problem.

Proposition 6.6 *Let us make the same assumptions as in Proposition 6.5. Then* $\mathcal{P}[a]$ *can be written as*

$$\mathcal{P}[a] = p_{0,\kappa[a]} + \mathcal{P}_1[a] + \mathcal{P}_2[a]$$

where $p_{0,\kappa}$ *is defined in (8.2) and each term*

$$\kappa = \kappa[a], \quad p_1 = \mathcal{P}_1[a], \quad p_2 = \mathcal{P}_2[a],$$

has the following bounds:

$$\begin{aligned} \kappa &= |a(T)| \left(1 + O\left(\frac{1}{|\log T|}\right)\right), \\ |\dot{p}_1(t) - \dot{p}_{0,\kappa}(t)| &\le C \frac{|\log T|^{1-\sigma} \log(|\log T|)^2}{|\log(T-t)|^{3-\sigma}}, \\ |\ddot{p}_1(t)| &\le C \frac{|\log T|}{|\log(T-t)|^3 (T-t)}, \\ \|\dot{p}_2\|_{\Theta, l} &\le C\big(T^{\frac{1}{2}+\sigma-\Theta} + \|a(\cdot) - a(T)\|_{\Theta, l-1}\big), \end{aligned}$$

$$[\dot{p}_2]_{\gamma,m,l} \le C\Bigg(|\log T|^{l-3} T^{\alpha_0 - m - \gamma} + T^{\Theta} \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \Bigg),$$

where $\alpha_0 > 0$ *is some fixed some constant and* $\sigma > 0$ *is arbitrary (with* C *depending on* σ*).*

Roughly speaking, to obtain the modified equation (6.18) we notice that the main term in p in $\mathcal{B}_0[p]$ is the integral operator

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}(s)}{t-s} ds.$$

Thus we define

$$\tilde{\mathcal{B}}_0[p] = \mathcal{B}_0[p] - \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}(s)}{t-s} ds.$$

It will be sufficient to solve approximately equations (5.3) replacing in part this integral operator by a "regularized" version of it following the logic of the formal derivation of the rate (5.14). For $\alpha > 0$ let us write

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}(s)}{t-s} ds = S_\alpha[\dot{p}] + R_\alpha[\dot{p}]$$

where

$$S_\alpha[g] := g(t)[-2\log\lambda_*(t) + (1+\alpha)\log(T-t)] + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s} ds, \tag{6.20}$$

$$R_\alpha[g] := -\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*^2} \frac{g(t)-g(s)}{t-s} ds. \tag{6.21}$$

Thus equation (6.14) can be written in the form

$$S_\alpha[\dot{p}] + R_\alpha[\dot{p}] + \tilde{\mathcal{B}}_0[p] = a(t), \quad \text{in } [0,T],$$

for some function $a(t)$. The modified equation is

$$S_\alpha[\dot{p}] + \tilde{\mathcal{B}}_0[p] = a(t) \quad \text{in } [0,T],$$

and the remainder $\mathcal{R}_0$ is essentially $R_\alpha[\dot{p}]$. This is a sketch of how we obtain the modified equation and remainder. For more details see Sect. 8.

Another modification to equations (6.14) that we introduce is to replace $a_0[p, \xi, \Psi^*]$ by its main term. To do this we write

$$a_0[p, \xi, \Psi] = a_0^{(0)}[p, \xi, \Psi] + a_0^{(1)}[p, \xi, \Psi] + a_0^{(2)}[p, \xi, \Psi]$$

where

$$a_0^{(l)}[p, \xi, \Psi] = -\frac{\lambda}{4\pi} e^{i\omega} \int_{B_{2R}} \left(Q_{-\omega} \tilde{L}_U[\Psi]_l \cdot Z_{01} + i Q_{-\omega} \tilde{L}_U[\Psi]_l \cdot Z_{02} \right) dy$$

for $l = 0, 1, 2$.

We define

$$\begin{aligned} & c_0^*[p, \xi, \Psi^*](t) \\ & := \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{01}|^2} e^{-i\omega} \Big(\mathcal{R}_0 \left[a_0^{(0)}[p, \xi, \Psi^*] \right](t) + a_0^{(1)}[p, \xi, \Psi^*](t) \\ & \quad + a_0^{(2)}[p, \xi, \Psi^*](t) \Big) - (c_0[h[p, \xi, \Psi^*]] - \tilde{c}_0[h_1[p, \xi, \Psi^*]]), \end{aligned}$$

and

$$c_{01}^* := \mathrm{Re}(c_0^*), \quad c_{02}^* := \mathrm{Im}(c_0^*),$$

where $\mathcal{R}_0$ is the operator given Proposition 6.5 and $\tilde{c}_0 = \tilde{c}_{01} + i\tilde{c}_{02}$ are the operators defined in Proposition 6.1.

6.4 The system of equations

We transform the system (4.11)–(4.12) in the problem of finding functions $\psi(x, t), \phi_1, \ldots, \phi_4$, parameters $p(t) = \lambda(t)e^{i\omega(t)}, \xi(t)$ and constants c_1, c_2, c_3 such that the following system is satisfied:

$$\begin{cases} \psi_t = \Delta_x \psi + g(p, \xi, Z^* + \psi, \phi_1 + \phi_2 + \phi_3 + \phi_4) & \text{in } \Omega \times (0, T) \\ \psi = (\mathbf{e}_3 - U) - \Phi^0 & \text{on } \partial\Omega \times (0, T) \\ \psi(\cdot, 0) = (c_1 \mathbf{e_1} + c_2 \mathbf{e_2} + c_3 \mathbf{e_3})\chi + (1 - \chi)(\mathbf{e}_3 - U - \Phi^0) & \text{in } \Omega \\ \psi(q, T) = -Z^*(q, T) \end{cases} \tag{6.22}$$

$$\begin{cases} \lambda^2 \partial_t \phi_1 = L_W[\phi_1] + h_1[p,\xi,\Psi^*] - \sum_{j=1,2} \tilde c_{0j}[h_1[p,\xi,\Psi^*]] w_\rho^2 Z_{0j} \\ \qquad - \sum_{\substack{l=-1,1\\ j=1,2}} c_{lj}[h_1[p,\xi,\Psi^*]] w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R} \\ \phi_1 \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_1(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.23}$$

$$\begin{cases} \lambda^2 \partial_t \phi_2 = L_W[\phi_2] + h_2[p,\xi,\Psi^*] - \sum_{j=1,2} c_{1j}[h_2[p,\xi,\Psi^*]] w_\rho^2 Z_{1j} \quad \text{in } \mathcal{D}_{2R} \\ \phi_2 \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_2(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.24}$$

$$\begin{cases} \lambda^2 \partial_t \phi_3 = L_W[\phi_3] + h_3 - \sum_{j=1,2} c_{1j}[h_3[p,\xi,\Psi^*]] w_\rho^2 Z_{1j} \\ \qquad + \sum_{j=1,2} c^*_{0j}[p,\xi,\Psi^*] w_\rho^2 Z_{0j} \quad \text{in } \mathcal{D}_{2R} \\ \phi_3 \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_3(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.25}$$

$$\begin{cases} \lambda^2 \partial_t \phi_4 = L_W[\phi_4] + \sum_{j=1,2} c_{-1,j}[h_1[p,\xi,\Psi^*]] w_\rho^2 Z_{-1j} \\ \phi_4 \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi_4(\cdot,t) = 0 \quad \text{on } \partial B_{2R(t)} \\ \phi_4(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.26}$$

$$c_{0j}[h(p,\xi,\Psi^*)](t) - \tilde c_{0j}[p,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t \in (0,T), \quad j = 1,2, \tag{6.27}$$

$$c_{1j}[h(p,\xi,\Psi^*)](t) = 0 \quad \text{for all} \quad t \in (0,T), \quad j = 1,2. \tag{6.28}$$

In (6.22) χ is a smooth cut-off function with compact support in Ω which is identically 1 on a fixed neighborhood of q independent of T and the function $g(p,\xi,\Psi^*,\phi)$ is given by (4.13).

We see that if $(\phi_1,\phi_2,\phi_3,\phi_4,\psi,p,\xi)$ satisfies system (6.22)–(6.28) then the functions

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4, \quad \Psi^* = Z^* + \psi$$

solve the outer-inner gluing system (4.11)–(4.12).

The way in which we will proceed to solve the full problem (6.22)–(6.28) is the following. For given functions $\phi_1, \ldots, \phi_4$ and parameters p, ξ in a suitable class, we solve first the outer problem (6.22) in the form of an operator $\psi = \Psi[\phi_1 + \phi_2 + \phi_3 + \phi_4, p, \xi]$ and denote $\Psi^*[\phi_1 + \phi_2 + \phi_3, p, \xi] = Z^* + \Psi[\phi_1 + \phi_2 + \phi_3 + \phi_4, p, \xi]$. Then we substitute $\Psi^*[\phi_1 + \phi_2 + \phi_3 + \phi_4, p, \xi]$ in (6.23)–(6.26) and solve for ϕ_1, ϕ_2, ϕ_3, ϕ_4 as operators of the pair (p, ξ). Finally, we solve for p and ξ the remaining equations. All this will be done by suitable control on the linear parts of the equation and contraction mapping principle.

6.5 Choice of constants

We state here the constraints we impose in the parameters involved in the different norms. The values assumed will be sufficient for the inner-outer gluing scheme to work.

- $\beta \in (0, \frac{1}{2})$ is so that $R(t) = \lambda_*(t)^{-\beta}$.
- $\alpha \in (0, \frac{1}{2})$ appears in Proposition 6.5. It is the parameter used to define the remainder $\mathcal{R}_\alpha$ in (6.21).
- We use the norm $\| \ \|_{*,\nu_1,a_1,\delta}$ (6.11) to measure the solution ϕ_1 in (6.23). Here we will ask that $\nu_1 \in (0, 1)$, $a_1 \in (2, 3)$, and $\delta > 0$ small and fixed.
- We use the norm $\| \ \|_{\nu_2,a_2-2}$ (6.10) to measure the solution ϕ_2 in (6.24), with $\nu_2 \in (0, 1)$, $a_2 \in (2, 3)$.
- We use the norm $\| \ \|_{**,\nu_3}$ (6.13) for the solution ϕ_3 of (6.25), with $\nu_3 > 0$.
- We use the norm $\| \ \|_{***,\nu_4}$ for the solution ϕ_4 of (6.26), with $\nu_4 > 0$.
- We are going to use the norm $\| \ \|_{\sharp,\Theta,\gamma}$ with a parameters Θ, γ satisfying some restrictions given below.
- We have parameters m, l in Proposition 6.5. We work with m given by

$$m = \Theta - 2\gamma(1 - \beta).$$

and l satisfying $l < 1 + 2m$.

We will assume that

$$\alpha - 1 + 2\beta > 0$$

which ensures that $m + (1 + \alpha)\gamma > \Theta$.

To get the estimates for the outer problem (6.22), we need (A.1) and

$$\Theta < \min\Big(\beta, \frac{1}{2} - \beta, \nu_1 - 1 + \beta(a_1 - 1), \nu_2 - 1 + \beta(a_2 - 1), \nu_3 - 1, \nu_4 - 1 + \beta\Big)$$

$$\Theta < \min\Big(\nu_1 - \delta\beta(5-a_1) - \beta, \nu_2 - \beta, \nu_3 - 3\beta, \nu_4 - \beta\Big)$$

and

$$\Theta > 0.$$

Also to control the nonlinear terms in (6.22) we need $\delta > 0$ in $\| \ \|_{*,\nu_1,a_1,\delta}$ to be small.

To find Θ in the range above we need

$$\begin{aligned}
\nu_1 &> \max\Big(1 - \beta(a_1 - 1), \delta\beta(5-a_1) - \beta\Big)\\
\nu_2 &> \max\Big(1 - \beta(a_2 - 1), \beta\Big)\\
\nu_3 &> \max(1, 3\beta)\\
\nu_4 &> \max(1-\beta, \beta).
\end{aligned}$$

To solve the inner system given by equations (6.23), (6.24), (6.25), and (6.26) we will need

$$\begin{aligned}
&\nu_1 < 1,\\
&\nu_2 < 1 - \beta(a_2 - 2),\\
&\nu_3 < \min\Big(1 + \Theta + \sigma_1, 1 + \Theta + 2\gamma\beta, \nu_1 + \frac{1}{2}\delta\beta(a_1 - 2)\Big),\\
&\nu_4 < 1,
\end{aligned}$$

where $\sigma_1 \in (0, \gamma(\alpha - 1 + 2\beta))$.

6.6 The outer problem

Our main result for problem (6.22) is the existence of a small solution for all small T, with certain precise absolute and Lipschitz estimates satisfied. To obtain this result we need a suitable norm that we define next.

Given $\Theta > 0$, $\gamma \in (0, \frac{1}{2})$ we define

$$\begin{aligned}
\|\psi\|_{\sharp,\Theta,\gamma} :=\ & \lambda_*(0)^{-\Theta} \frac{1}{|\log T|\lambda_*(0)R(0)} \|\psi\|_{L^\infty(\Omega\times(0,T))}\\
&+ \lambda_*(0)^{-\Theta} \|\nabla_x \psi\|_{L^\infty(\Omega\times(0,T))}\\
&+ \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta-1} R(t)^{-1} \frac{1}{|\log(T-t)|} |\psi(x,t) - \psi(x,T)|
\end{aligned}$$

$$
+ \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta}|\nabla_x\psi(x,t) - \nabla_x\psi(x,T)|
$$
$$
+ \sup \lambda_*(t)^{-\Theta}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla_x\psi(x,t) - \nabla_x\psi(x',t')|}{(|x-x'|^2+|t-t'|)^\gamma}, \tag{6.29}
$$

where the last supremum in taken in the region

$$
x, x' \in \Omega, \quad t, t' \in (0,T), \quad |x-x'| \le 2\lambda_* R(t), \quad |t-t'| < \frac{1}{4}(T-t).
$$

We define the spaces

$$
\begin{aligned}
E_1 &= \{\phi_1 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y\phi_1 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_1\|_{*,\nu_1,a_1,\delta} < \infty\}\\
E_2 &= \{\phi_2 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y\phi_2 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_2\|_{\nu_2,a_2} < \infty\}\\
E_3 &= \{\phi_3 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y\phi_3 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_3\|_{**,\nu_3} < \infty\}\\
E_4 &= \{\phi_4 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y\phi_4 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_4\|_{***,\nu_4} < \infty\}
\end{aligned}
$$

and use the notation

$$
E = E_1 \times E_2 \times E_3 \times E_4,
$$

$$
\begin{aligned}
\Phi &= (\phi_1, \phi_2, \phi_3, \phi_4) \in E\\
\|\Phi\|_E &= \|\phi_1\|_{*,\nu_1,a_1,\delta} + \|\phi_2\|_{\nu_2,a_2-2} + \|\phi_3\|_{**,\nu_3} + \|\phi_4\|_{***,\nu_4}
\end{aligned}
$$

We define the closed ball

$$
\mathcal{B} = \{\Phi \in E : \|\Phi\|_E \le 1\}.
$$

Proposition 6.7 *Assume Z_0^* satisfies (6.8). Let $p(t) = \lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (6.1), (6.2), $\Phi \in \mathcal{B}$. Then there exists $C > 0$ such that if $T > 0$ is sufficiently small then there exists a solution $\psi = \Psi(p, \xi, \Phi, Z_0^*)$ to equation (6.22) such that*

$$
\begin{aligned}
&\|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}\\
&\le CT^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).
\end{aligned} \tag{6.30}
$$

Proof The proof consists in writing problem (6.22) in a fixed point form involving an inverse for the inhomogeneous linear heat equation

$$\begin{cases} \psi_t = \Delta_x \psi + f(x,t) \quad \text{in } \Omega \times (0,T) \\ \psi = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \psi(q,T) = 0 \\ \psi(x,0) = (c_1\,\mathbf{e_1} + c_2\,\mathbf{e_2} + c_3\,\mathbf{e_3})\eta_1 \quad \text{in } \Omega \end{cases} \tag{6.31}$$

for suitable constants c_1, c_2, c_3, where $\mathbf{e_1}$, $\mathbf{e_1}$, $\mathbf{e_1}$ are defined in (1.13), and $q \in \Omega$ and $T > 0$ is sufficiently small. The fixed smooth cut-off η_1 has compact support in Ω and is such that $\eta_1 \equiv 1$ in a neighborhood of q. The right hand side is assumed to satisfy $\|f\|_{**} < \infty$ where

$$\|f\|_{**} := \sup_{\Omega\times(0,T)} \Big(1 + \sum_{i=1}^{3} \varrho_i(x,t)\Big)^{-1} |f(x,t)|.$$

and the weights are defined by

$$\begin{cases} \varrho_1 := \lambda_*^{\Theta}(\lambda_* R)^{-1}\chi_{\{r\le 3R\lambda_*\}} \\ \varrho_2 := T^{-\sigma_0}\dfrac{\lambda_*^{1-\sigma_0}}{r^2}\chi_{\{r\ge R\lambda_*\}} \\ \varrho_3 := T^{-\sigma_0}, \end{cases}$$

where $r = |x-q|$, $\Theta > 0$ and $\sigma_0 > 0$ is small. (The factor T^{σ_0} in front of ϱ_2 and ϱ_3 is a simple way to have parts of the error small in the outer problem.) These weights naturally adapt to the form of the outer error g in (4.13). In Proposition A.1 a solution of Problem (6.31) is built as a linear operator of f with the estimate

$$\|\psi\|_{\sharp,\Theta,\gamma} + \frac{\lambda_*(0)^{-\Theta}(\lambda_*(0)R(0))^{-1}}{|\log T|}(|c_1| + |c_2| + |c_3|) \le C\|f\|_{**},$$

This fact and direct estimates for the outer error make the the contraction mapping principle applicable in a suitable region, producing an operator as in (6.30). To illustrate some of these estimates, let us write $g = g_1 + g_2 + g_3 + g_4$ where

$$\begin{aligned} g_1 &= Q_\omega\big((\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi\big) \\ &\quad + \eta Q_\omega\big(-\dot\omega J\phi + \lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi\big) \\ g_2 &= (1-\eta)\tilde L_U[\Psi^*] + (\Psi^*\cdot U)U_t \\ g_3 &= (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0\cdot U)U_t, \\ g_4 &= N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi)^*). \end{aligned}$$

We claim that

$$\|g_1\|_{**} \le CT^{\sigma}\|\Phi\|_E,$$

for some $\sigma > 0$. Indeed, we have

$$\begin{aligned}
|\Delta_x \eta \phi_1| &\le C\lambda_*^{\nu_1-2} R^{-a_1} \chi_{[|x-q|\le 3\lambda_* R]} \|\phi_1\|_{*,\nu_1,a_1,\delta}\\
|\Delta_x \eta \phi_2| &\le C\lambda_*^{\nu_2-2} R^{-a_2} \chi_{[|x-q|\le 3\lambda_* R]} \|\phi_2\|_{\nu_2,a_2-2}\\
|\Delta_x \eta \phi_3| &\le C\lambda_*^{\nu_3-2} R^{-1} \chi_{[|x-q|\le 3\lambda_* R]} \|\phi_3\|_{**,\nu_3}\\
|\Delta_x \eta \phi_4| &\le C\lambda_*^{\nu_4-2} R^{-2} \log R \chi_{[|x-q|\le 3\lambda_* R]} \|\phi_4\|_{***,\nu_4}.
\end{aligned}$$

The norm $\|\ \|_{**}$ is actually motivated by the weights appearing above. If

$$\Theta < \min(\nu_1 - 1 + \beta(a_1 - 1), \nu_2 - 1 + \beta(a_2 - 1), \nu_3 - 1, \nu_4 - 1 + \beta),$$

we find that for any $j = 1, 2, 3, 4$:

$$|\Delta_x \eta \phi_j| \le CT^{\sigma} \lambda_*^{\Theta-1+\beta} \chi_{[|x-q|\le 3\lambda_* R]} \|\Phi\|_E,$$

for some $\sigma > 0$. Then we have

$$\|Q_\omega (\Delta_x \eta)\phi\|_{**} \le CT^{\sigma}\|\Phi\|_E$$

and similarly

$$\|(\partial_t \eta) Q_\omega \phi\|_{**} + \|Q_\omega \lambda^{-1} \nabla_x \eta \nabla_y \phi\|_{**} \le CT^{\sigma}\|\Phi\|_E.$$

The other terms g_2, g_3, g_4 can be estimated in the same way. In the estimate for g_2 it is important to have the property that $\Psi^* = Z^* + \psi$ vanishes at (q, T). Lipschitz properties are proved using similar calculations. □

The operator $\Psi(p, \xi, \Phi, Z_0^*)$ satisfies Lipschitz properties with respect to its arguments, which are consequence of its construction. See Corollaries C.1 and C.2 in the appendix.

What we do next is to take $\Phi \in E$ with $\|\Phi\|_E \le 1$ and substitute $\Psi^*(p, \xi, \Phi, Z_0^*) = Z^* + \Psi(p, \xi, \Phi, Z_0^*)$ into (6.23)–(6.26). We can then write equations (6.22)–(6.26) as the fixed point problem

$$\Phi = \mathcal{F}(\Phi) \tag{6.32}$$

where

$$\mathcal{F}(\Phi) = (\mathcal{F}_1(\Phi), \mathcal{F}_2(\Phi), \mathcal{F}_3(\Phi), \mathcal{F}_4(\Phi)), \quad \mathcal{F} : \bar{\mathcal{B}}_1 \subset E \to E$$

wiith

$$
\begin{aligned}
\mathcal{F}_1(\Phi) &= \mathcal{T}_{\lambda,1}(h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)])\\
\mathcal{F}_2(\Phi) &= \mathcal{T}_{\lambda,2}(h_2[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)])\\
\mathcal{F}_3(\Phi) &= \mathcal{T}_{\lambda,3}\Big(h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\\
&\qquad + \sum_{j=1}^{2} c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]w_\rho^2 Z_{0j}\Big)\\
\mathcal{F}_4(\Phi) &= \mathcal{T}_{\lambda,4}\Big(\sum_{j=1}^{2} c_{-1,j}[h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]]w_\rho^2 Z_{-1,j}\Big).
\end{aligned}
$$

Although $\mathcal{F}$ also depends on p, ξ, Z_0^* we will omit this dependence from the notation for the moment.

Our next step is to solve problem (6.32).

6.7 The inner problem

Proposition 6.8 *Assume that p and ξ satisfy estimates* (6.1) *and that Z_0^* satisfies* (6.8)*. Then the system of equations* (6.32) *for $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ has a solution $\Phi(p, \xi, Z_0^*)$ in $\bar{\mathcal{B}}_1 \subset E$.*

Proof We estimate in detail the operator $\mathcal{F}_1$. The others are handled similarly. We recall that we have decomposed $Z_0^* = Z_0^{*0} + Z_0^{*1}$ (c.f. 6.8). We claim that for $\|\Phi\|_E \le 1$ we have

$$
\begin{aligned}
\|\mathcal{F}_1(\Phi)\|_{*,a,1,\nu_1} \le{}& C\lambda_*(0)^\Theta T^\sigma(\|\Phi\|_E + \|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)})\\
&+ CT^\sigma\|Z_0^{*0}\|_*,
\end{aligned}
\tag{6.33}
$$

and for $\|\Phi_1\|_E, \|\Phi_2\|_E \le 1$

$$
\|\mathcal{F}_1(\Phi_1) - \mathcal{F}_1(\Phi_2)\|_{*,a_1,\nu_1} \le CT^\sigma\lambda_*(0)^\Theta\|\Phi_1 - \Phi_2\|_E. \tag{6.34}
$$

To prove (6.33), we recall that by Proposition 6.1 we have

$$
\|\mathcal{F}_1(\Phi)\|_{*,\nu_1,a_1,\delta} \le C\|h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1}.
$$

From the definition of h_1 and recalling that $\Psi^*(p,\xi,\Phi,Z_0^*) = Z^* + \Psi(p,\xi,\Phi,Z_0^*)$ we get

$$
\begin{aligned}
&\|h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1} \\
&\quad \le \|\lambda^2 Q_{-\omega}(\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_0 + \tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_2)\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \\
&\qquad + \|\lambda^2 Q_{-\omega}(\tilde L_U[Z^*]_0 + \tilde L_U[Z^*]_2)\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} + \|\lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi]\|_{\nu_1,a_1}.
\end{aligned}
$$

We claim that for $j=0$ and $j=2$:

$$
\begin{aligned}
&\|\lambda^2 Q_{-\omega}\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_j\,\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \\
&\quad \le C T^\sigma \lambda_*(0)^\Theta (\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} \\
&\qquad + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).
\end{aligned} \tag{6.35}
$$

Indeed, let $\psi = \Psi(p,\xi,\Phi,Z_0^*)$. From (3.6) we get, for $j=0$ and $j=2$:

$$
|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j| \le C\frac{\lambda_*}{(1+|y|)^3}\|\nabla_x\psi\|_{L^\infty}.
$$

We use $\nu_1 < 1$ and $a_1 < 3$ to estimate for $|y|\le 2R$

$$
\frac{\lambda_*}{(1+|y|)^3} \le \frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}.
$$

Then for $|y|\le 2R$ and $j=0,2$:

$$
|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j| \le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}\|\nabla_x\psi\|_{L^\infty}.
$$

By the definition of the norm $\|\ \|_{\sharp,\Theta,\gamma}$ (c.f. (6.29)) and Proposition 6.7 we have

$$
\begin{aligned}
\|\nabla_x\psi\|_{L^\infty} &\le C\lambda_*(0)^\Theta \|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma} \\
&\le C\lambda_*(0)^\Theta T^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).
\end{aligned}
$$

Hence for $j=0,2$

$$
\begin{aligned}
&|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j| \\
&\quad \le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}} T^\sigma \lambda_*(0)^\Theta(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} \\
&\qquad + \|Z_0^*\|_*),
\end{aligned}
$$

and therefore we see that (6.35) is valid. Next we claim that

$$
\|\lambda^2 Q_{-\omega}\tilde L_U[Z^*]_j\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \le C T^\sigma \|Z_0\|_*, \tag{6.36}
$$

for $j = 0, 2$ and some $\sigma > 0$. Indeed, we use estimate (10.8) of Lemma 10.2 to obtain for $j = 0, 2$:

$$|\lambda^2 Q_{-\omega} \tilde{L}_U[Z^*]_j \, \chi_{\mathcal{D}_{2R}}| \le C \frac{\lambda_*}{(1+\rho)^3} |\log \varepsilon| \|Z_0\|_*,$$

where $\varepsilon > 0$ is given by (6.6). Since $\nu_1 < 1$, we get

$$\|\lambda^2 Q_{-\omega} \tilde{L}_U[Z^*]_j \, \chi_{\mathcal{D}_{2R}}\|_{\nu_1, a_1} \le C \lambda_*(0)^{1-\nu_1} |\log \lambda_*(0)| \|Z_0\|_*.$$

This implies (6.36). Next we estimate $\lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi]$. We claim that

$$\|\lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi]\|_{\nu_1, a_1} \le C T^\sigma \|\dot{p}\|_{L^\infty(-T,T)}. \tag{6.37}$$

Indeed, consider $\mathcal{K}_{01}$ given in (4.8). We have

$$|\lambda^2 Q_{-\omega} \mathcal{K}_{01}[p, \xi]| \le C \frac{\lambda_*}{(1+\rho)^3} \int_{-T}^{t} |\dot{p}(s) k(z, t-s)| \, ds.$$

A direct computation shows that

$$\begin{aligned} \|\lambda^2 Q_{-\omega} \tilde{L}_U[\mathcal{K}_{01}[p, \xi]] \chi_{\mathcal{D}_{2R}}\|_{\nu_1, a_1} &\le C \lambda_*(0)^{1-\nu_1} \|\dot{p}\|_{L^\infty(-T,T)} \\ &\le C T^\sigma \|\dot{p}\|_{L^\infty(-T,T)}, \end{aligned}$$

for some $\sigma > 0$. The estimate for $\mathcal{K}_{02}$ is similar, and we obtain (6.37). Combining (6.35), (6.36), and (6.37) we finally obtain

$$\|h_1[p, \xi, \Psi^*(p, \xi, \Phi, Z_0^*)]\|_{\nu_1, a_1} \le C T^\sigma (\|\Phi\|_E + \|\dot{p}\|_{L^\infty(-T,T)} + \|Z_0^*\|_*).$$

Then thanks to Proposition 6.1 we get (6.33). The proof of estimate (6.34) is similar. □

Let $\Phi(p, \xi, Z_0^*)$ be the solution of (6.32) constructed in Proposition 6.8. As a consequence of the construction above and the Lipschitz estimates for the inner problem in Sect. 7.6 Φ is Lipschitz in the parameters p, ξ, Z_0^* in the following sense.

Corollary 6.1 *Assume that p_1, p_2 and ξ_1, ξ_2 satisfy estimates (6.1) and that $Z_{0,1}^*$, $Z_{0,2}^*$ have the form*

$$Z_{0,l}^* = Z_0^{*0} + Z_{0,l}^{*1}, \quad l = 1, 2,$$

with Z_0^{*0} *satisfying* (6.4) *and* $\|Z_{0,l}^{*1}\|_* \le T^\sigma$. *Let us write* $p_j = \lambda_j e^{i\omega_j}$ *for* $j = 1, 2$. *for some* $\sigma > 0$. *Then*

$$\begin{aligned}
&\|\Phi(p_1, \xi_1, Z_{0,1}^*) - \Phi(p_2, \xi_2, Z_{0,2}^*)\|_E \\
&\quad \le \lambda_*(0)^\sigma \Big[\|\lambda_*(\dot\omega_1 - \dot\omega_2)\|_\infty + \Big\| \frac{\lambda_1 - \lambda_2}{\lambda_*} \Big\|_{L^\infty} \\
&\qquad + \|\dot\lambda_1 - \dot\lambda_2\|_{L^\infty} + \Big\| \frac{\xi_1 - \xi_2}{\lambda_* R} \Big\|_{L^\infty} + \Big\| \frac{\dot\xi_1 - \dot\xi_2}{R} \Big\|_{L^\infty} \\
&\qquad + \|Z_{0,1}^{*1} - Z_{0,2}^{*1}\|_* \Big],
\end{aligned}$$

for some possibly smaller $\sigma > 0$.

With this we can now state the following result. Let $\Phi(p, \xi, Z_0^*)$ denote the solution of (6.32) constructed in Proposition 6.8.

Proposition 6.9 *Given* Z_0^* *of the form* (6.8) *there exists* $p = \lambda e^{i\omega}$ *and* ξ *such that* (6.27) *and* (6.28) *are satisfied.*

The proposition above yields the existence of a blow-up solution. The proof is given in Sect. 9.

7 Linear theory for the inner problem

At the very heart of capturing the bubbling structure is the construction of an inverse for the linearized heat operator around the basic harmonic map. We consider the linear equation

$$\begin{aligned}
\lambda^2 \partial_t \phi &= L_W[\phi] + h(y, t) \quad \text{in } \mathcal{D}_{2R} \\
\phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)} \\
\phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}
\end{aligned} \tag{7.1}$$

where

$$\mathcal{D}_{2R} = \{(y, t) \ / \ t \in (0, T), \ \ y \in B_{2R(t)}(0)\}.$$

We assume that $h(y, t)$ is defined for all $(y, t) \in \mathbb{R}^2 \times (0, T)$ and satisfies

$$h \cdot W = 0, \quad |h(y, t)| \le C \frac{\lambda_*^\nu}{(1 + |y|)^a},$$

where $\nu > 0$ and $a \in (2, 3)$ [so that $\|h\|_{a,\nu} < \infty$ with the norm defined in (6.10)].

The parameter R is given by (6.3), that is $R(t) = \lambda_*(t)^{-\beta}$, $\beta \in (\frac{1}{4}, \frac{1}{2})$. Also, we assume that the parameter function $\lambda(t)$ satisfies we have that

$$a\lambda_*(t) \le \lambda(t) \le b\lambda_*(t) \quad \text{for all} \quad t \in (0, T)$$

for some positive numbers a, b, c independent of T.

We observe that a priori we are not imposing boundary conditions in problem (7.1). Our purpose is to construct a solution ϕ that defines a linear operator of h and satisfies uniform bounds in terms of suitable norms. In some sense this is an extension of "Fredholm" theory for linear parabolic problem (7.1).

All functions $h(y,t)$ with $h(y,t) \cdot W(y) \equiv 0$ can be expressed in polar form as

$$h(y,t) \;=\; h^1(\rho,\theta,t)E_1(y) + h^2(\rho,\theta,t)E_2(y), \quad y = \rho e^{i\theta}. \tag{7.2}$$

We can also expand in Fourier series

$$\tilde{h}(\rho,\theta,t) := h^1 + ih^2 \;=\; \sum_{k=-\infty}^{\infty} \tilde{h}_k(\rho,t)e^{ik\theta}, \quad \tilde{h}_k = \tilde{h}_{k1} + i\tilde{h}_{k2} \tag{7.3}$$

so that

$$h(y,t) = \sum_{k=-\infty}^{\infty} h_k(y,t) =: h_0(y,t) + h_1(y,t) + h_{-1}(y,t) + h^{\perp}(y,t), \tag{7.4}$$

where

$$h_k(y,t) = \mathrm{Re}\,(\tilde{h}_k(\rho,t)e^{ik\theta})\,E_1 + \mathrm{Im}\,(\tilde{h}_k(\rho,t)e^{ik\theta})\,E_2. \tag{7.5}$$

We consider the functions $Z_{kj}(y)$ defined in (3.1) and (3.2) and define for $k = -1, 0, 1$,

$$\bar{h}_k(y,t) := \sum_{j=1}^{2} \frac{\chi Z_{kj}(y)}{\int_{\mathbb{R}^2} \chi |Z_{kj}|^2} \int_{\mathbb{R}^2} h(x,t) \cdot Z_{kj}(z)\,dz,$$

where

$$\chi(y,t) = \begin{cases} w_\rho^2(|y|) & \text{if } |y| < 2R(t), \\ 0 & \text{if } |y| \ge 2R(t). \end{cases}$$

The main result in this section is the following, where we use the norm $\|h\|_{a,\nu}$ defined in (6.10).

Proposition 7.1 *Let* $2 < a < 3$, $\nu > 0$ *and let* h *with* $\|h\|_{a,\nu} < +\infty$. *Let us write* $h = h_0 + h_1 + h_{-1} + h^{\perp}$ *with* $h^{\perp} = \sum_{k\neq 0,\pm 1} h_k$. *Then there exists a solution* $\phi[h]$ *of problem* (7.1)*, which defines a linear operator of* h*, and satisfies the following estimate in* $\mathcal{D}_{2R}$*:*

$$
\begin{aligned}
&(1+|y|)\left|\nabla_y \phi(y,t)\right| + |\phi(y,t)| \\
&\lesssim \frac{\lambda_*(t)^{\nu} R(t)^{\frac{5-a}{2}}}{1+|y|} \min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\, \|h_0 - \bar{h}_0\|_{a,\nu} + \frac{\lambda_*(t)^{\nu} R(t)^2}{1+|y|} \|\bar{h}_0\|_{a,\nu} \\
&\quad + \frac{\lambda_*(t)^{\nu}}{1+|y|^{a-2}} \left\|h_1 - \bar{h}_1\right\|_{a,\nu} + \frac{\lambda_*(t)^{\nu} R(t)^4}{1+|y|^2} \left\|\bar{h}_1\right\|_{a,\nu} \\
&\quad + \frac{\lambda_*(t)^{\nu} R(t)^{\frac{5-a}{2}}}{1+|y|} \min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\, \|h_{-1} - \bar{h}_{-1}\|_{a,\nu} \\
&\quad + \lambda_*(t)^{\nu} \log R(t)\, \|\bar{h}_{-1}\|_{a,\nu} \\
&\quad + \frac{\lambda_*(t)^{\nu}}{1+|y|^{a-2}} \|h^{\perp}\|_{a,\nu}.
\end{aligned}
$$

The construction of the operator $\phi[h]$ as stated in the proposition will be carried out mode by mode in the Fourier series expansion. We shall use the convention that $h(y,t) = 0$ for $|y| > 2R(t)$. Let us write

$$
\phi = \sum_{k=-\infty}^{\infty} \phi_k, \qquad \phi_k(y,t) = \mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\, E_1 + \mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\, E_2.
$$

We shall build a solution of (7.1) by solving separately each of the equations

$$
\begin{aligned}
\lambda^2 \partial_t \phi_k &= L_W[\phi_k] + h_k(y,t) = 0 \quad \text{in } \mathcal{D}_{4R}, \\
\phi_k(y,0) &= 0 \quad \text{in } B_{4R(0)}(0),
\end{aligned} \tag{7.6}
$$

which, are equivalent to the problems

$$
\begin{aligned}
\lambda^2 \partial_t \varphi_k &= \mathcal{L}_k[\varphi_k] + \tilde{h}_k(\rho,t) \quad \text{in } \tilde{D}_{4R}, \\
\varphi_k(\rho,0) &= 0 \quad \text{in } (0, 4R(0))
\end{aligned}
$$

with

$$
\tilde{D}_{4R} = \{(\rho,t)\ /\ t \in (0,T),\ \rho \in (0, 4R(t))\}
$$

and we recall

$$
\mathcal{L}_k[\varphi_k] := \partial_\rho^2 \varphi_k + \frac{\partial_\rho \varphi_k}{\rho} - (k^2 + 2k\cos w + \cos(2w))\frac{\varphi_k}{\rho^2}.
$$

We have the validity of the following result.

Lemma 7.1 *Let* $\nu > 0$ *and* $0 < a < 3$, $a \neq 1, 2$. *Assume that*

$$\|h_k(y,t)\|_{a,\nu} < +\infty.$$

Then problem (7.6) *has a unique bounded solution* $\phi_k(y,t)$ *of the form*

$$\phi_k(y,t) \;=\; \mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_1 + \mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_2$$

which in addition satisfies the boundary condition

$$\phi_k(y,t) = 0 \quad \text{for all } t \in (0,T), \quad y \in \partial B_{R(t)}(0). \tag{7.7}$$

These solutions satisfy the estimates

$$|\phi_k(y,t)| \le C\|h\|_{a,\nu}\,\lambda_*^{\nu} k^{-2} \begin{cases} R^{2-a} & \text{if } \ a<2, \\ (1+\rho)^{2-a} & \text{if } \ a>2, \end{cases} \quad \text{if } k \ge 2.$$

$$|\phi_{-1}(y,t)| \le C\|h\|_{a,\nu}\,\lambda_*^{\nu} \begin{cases} R^{2-a} & \text{if } \ a<2, \\ \log R & \text{if } \ a>2, \end{cases}$$

$$|\phi_0(y,t)| \le C\|h\|_{a,\nu}\lambda_*^{\nu}(1+\rho)^{-1} \begin{cases} R^{2} & \text{if } \ a>1, \\ R^{3-a} & \text{if } \ a<1, \end{cases}$$

$$|\phi_1(y,t)| \le C\|h\|_{a,\nu}\lambda_*^{\nu}(1+\rho)^{-2}R^4$$

with C *independent of* R *and* k.

Proof Standard parabolic theory yields existence of a unique solution to equation (7.6) that satisfies the boundary condition (7.7), for each k. Equivalently, the problem

$$\begin{aligned} \lambda^2\partial_t\varphi_k &= \mathcal{L}_k[\varphi_k] + \tilde{h}_k(\rho,t) \quad \text{in } \tilde{D}_{4R}, \\ \varphi_k(t,4R) &= 0 \quad \text{for all} \quad t \in (0,T) \\ \varphi_k(0,\rho) &= 0 \quad \text{in } (0,4R(0)), \\ \mathcal{L}_k[\varphi_k] &= \partial_\rho^2\varphi_k + \frac{\partial_\rho\varphi_k}{\rho} - (k^2 + 2k\cos w + \cos(2w))\frac{\varphi_k}{\rho^2} \end{aligned} \tag{7.8}$$

has a unique solution $\varphi_k(\rho,t)$ which is bounded in ρ for each t.

We use barriers to derive the desired estimates. A first observation we make is that for mode $k=-1$ the elliptic equation $\mathcal{L}_{-1}[\varphi] + g(\rho) = 0$ in $(0,4R)$ with $\varphi(4R) = 0$ has a unique bounded solution given by the variation of parameters formula

$$\varphi(\rho) := Z_{-1}(\rho)\int_\rho^{4R} \frac{dr}{\rho Z_{-1}(r)^2}\int_0^r g(s)Z_{-1}(s)s\,ds, \tag{7.9}$$

$$Z_{-1}(\rho) = -\rho^2 w_\rho = \frac{2\rho^2}{\rho^2+1}.$$

Here we have used that $\mathcal{L}_{-1}[Z_{-1}] = 0$. Let us call $\varphi_0(\rho)$ the function in (7.9) with $g(\rho) := 2(1+\rho)^{-a}$. We readily estimate

$$|\varphi_0(\rho)| \le \begin{cases} R^{2-a} & \text{if } a < 2, \\ (1+\rho)^{2-a} & \text{if } a > 2. \end{cases}$$

Let us call $\bar\varphi(\rho,t) = \lambda_*(t)^\nu \varphi_0(\rho)$. Then we see that

$$\begin{aligned} &-\lambda^2\bar\varphi_t(\rho,t) + \mathcal{L}_{-1}[\bar\varphi(\rho,t)] + \frac{\lambda_*^\nu}{(1+\rho)^a} \\ &\le c\,\lambda_*^{\nu+1}|\dot\lambda_*|\varphi_0(\rho) - \frac{\lambda_*^\nu}{(1+\rho)^a} \\ &\le -\lambda_*^\nu(1+\rho)^{-a}\left[1 - C\lambda_* R^{2-a}(1+\rho)^a\right] \\ &< 0 \end{aligned}$$

in $\tilde D_{4R}$. Indeed, since $R(t) \ll \lambda_*^{-\frac{1}{2}}$, the inequality holds provided that T was chosen sufficiently small. Thus for $k = -1$ the barrier $\|h\|_{a,\nu}\,\bar\varphi(\rho,t)$ dominates both, real and imaginary parts of $\varphi_{-1}(\rho,t)$. As a conclusion, we find

$$|\phi_{-1}(y,t)| \le C\|h\|_{a,\nu}\lambda_*^\nu \begin{cases} R^{2-a} & \text{if } a < 2, \\ (1+\rho)^{2-a} & \text{if } a > 2, \end{cases} \quad \text{in } \mathcal{D}_{4R}.$$

The cases $k = 0, 1, -2$ can be dealt with in exactly the same manner, by replacing Z_{-1} in Formula (7.9) respectively by the functions

$$Z_0(\rho) = \frac{\rho}{\rho^2+1}, \quad Z_1(\rho) = \frac{1}{\rho^2+1}, \quad Z_{-2}(\rho) = \frac{\rho^3}{\rho^2+1}. \tag{7.10}$$

The estimates for ϕ_k predicted in the lemma then readily follow for $k = -2, -1, 0, 1$. Finally, let us now consider k with $|k| \ge 2$ and $k \ne -2$ and the function $\bar\varphi(\rho,t)$ as above. Now we find

$$
\begin{aligned}
-\lambda^2\bar{\varphi}_t(\rho,t)+\mathcal{L}_k[\bar{\varphi}(\rho,t)] &\le (\mathcal{L}_k-\mathcal{L}_{-1})[\bar{\varphi}(\rho,t)]\\
&\le -C\lambda_*^{\nu}(k^2-1+2(k-1))\frac{1}{\rho^2}(1+\rho)^{2-a}\\
&< -C(k^2-1+2(k-1))\frac{\lambda_*^{\nu}}{(1+\rho)^a}\quad \text{in } \tilde{\mathcal{D}}_{4R}.
\end{aligned}
$$

The latter quantity is negative provided that $|k|\ge 2$ and $k\neq -2$ and hence we get the estimate

$$
|\phi_k(y,t)| \;\le\; \frac{C}{k^2}\|h\|_{a,\nu}\lambda_*^{-\nu}\begin{cases} R^{2-a} & \text{if } a<2,\\ (1+\rho)^{2-a} & \text{if } a>2,\end{cases}\quad \text{in } \mathcal{D}_{4R}.
$$

The proof is concluded. □

We can get gradient estimates for the solutions built in the above lemma by means of the following result.

Lemma 7.2 *Let ϕ be a solution of the equation*

$$
\begin{aligned}
&\lambda^2\partial_t\phi = L_W[\phi]+h(y,t)\quad \textit{in } \mathcal{D}_{4\gamma R}\\
&\phi(\cdot,0)=0\quad \textit{in } B_{4\gamma R(0)}.
\end{aligned}
\tag{7.11}
$$

Given numbers a, b, γ, there exists a C such that if for some $M>0$ we have

$$
|\phi(y,t)|+(1+|y|)^2|h(y,t)|\le M\,\lambda_*(t)^b(1+|y|)^{-a}\quad \textit{in } \mathcal{D}_{4\gamma R},\tag{7.12}
$$

then

$$
(1+|y|)|\nabla_y\phi(y,t)| \;\le\; C\,M\lambda_*(t)^b(1+|y|)^{-a}\quad \textit{in } \mathcal{D}_{3\gamma R}\tag{7.13}
$$

and we recall

$$
\mathcal{D}_{\gamma R}=\{(y,t)\ /\ |y|<\gamma R(t),\quad t\in(0,T)\}.
$$

If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t)=0$ on $\partial B_{4\gamma R(t)}$ for all $t\in(0,T)$ then estimate (7.13) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Proof To prove the gradient estimates, we change the time variable, defining

$$
\tau(t)=\int_0^t\frac{ds}{\lambda(s)^2},\tag{7.14}
$$

so that (7.11) becomes in the variables (y, τ)

$$\begin{aligned} \partial_\tau \phi &= L_W[\phi] + h(y,\tau) \quad \text{in } \mathcal{D}_{4\gamma R} \\ \phi(\cdot, 0) &= 0 \quad \text{in } B_{4R(0)} \end{aligned}$$

Let $\tau_1 > 0$ and $y_1 \in B_{3\gamma R(\tau_1)}(0)$. Let $\rho = \frac{|y_1|}{5} + 1$ so that $B_\rho(y_1) \subset B_{4\gamma R(\tau_1)}(0)$. Let us define

$$\tilde{\phi}(z,t) := \phi(y_1 + \rho z, \tau_1 + \rho^2 s), \quad z \in B_1(0), \quad s > -\frac{\tau_1}{\rho^2}.$$

We distinguish two cases. First, when $\tau_1 \geq \rho^2$, we use interior estimates for parabolic equations, while for the case $\tau_1 < \rho^2$, we use estimates for a parabolic equation with initial condition.

Assume $\tau_1 \geq \rho^2$. Then $\tilde{\phi}(z, s)$ satisfies an equation of the form

$$\tilde{\phi}_s = \Delta_z \tilde{\phi} + A \nabla_z \tilde{\phi} + B\tilde{\phi} + \tilde{h}(z,s) \quad \text{in } B_1(0) \times (-1, 0]$$

with coefficients $A(z, s)$ and $B(z, s)$ uniformly bounded by $O((1 + \rho)^{-2})$ in $B_1(0) \times (-1, 0]$ and

$$\tilde{h}(z,s) = \rho^2 h(y_1 + \rho z, \tau_1 + \rho^2 s).$$

Since $\rho \leq C R(\tau_1)$ and $R(\tau_1)^2 \ll \tau_1$ for τ_1 large we get

$$\lambda_*(\tau_1)^b \lesssim \lambda_*(\tau_1 + \rho^2 s)^b \lesssim \lambda_*(\tau_1)^b, \quad s \in (-1, 0].$$

Standard parabolic estimates and assumption (7.12) yield

$$\begin{aligned} \|\nabla_z \tilde{\phi}\|_{L^\infty(B_{\frac14}(0)\times(1,2))} &\lesssim \|\tilde{\phi}\|_{L^\infty B_{\frac12}(0)\times(0,2)} + \|\tilde{h}\|_{L^\infty(B_{\frac12}(0)\times(0,2))} \\ &\lesssim M\, \lambda_*(\tau_1)^b \rho^{2-a}, \end{aligned}$$

so that in particular

$$\rho |\nabla_y \phi(y_1, \tau_1)| = |\nabla_z \tilde{\phi}(0,1)| \lesssim M\, \lambda_*(\tau_1)^b \rho^{2-a}.$$

In the case $\tau_1 \geq \rho^2$ the argument is similar, but the equation for $\tilde{\phi}$ holds in $B_1(0) \times (-\frac{\tau_1}{\rho^2}, 0]$ and has initial condition 0 at $s = -\frac{\tau_1}{\rho^2}$. Finally, for the last assertion we argue in similar way but using boundary rather that interior gradient estimates. □

In addition to estimate (7.13) we have a Hölder gradient estimate which is more natural to express using the variable τ defined in (7.14) as follows. We denote

$$\mathcal{B}_\ell(y,\tau) = \{(y',\tau') \;/\; |y-y'|^2 + |\tau'-\tau| < \ell^2\}.$$

For a function $g(y,\tau)$, a number $0<\alpha<1$, and a set A we let

$$[g]_{\alpha,A} := \sup\left\{ \frac{|f(y,\tau)-f(y',\tau')|}{(|y-y'|^2+|\tau'-\tau|)^{\frac{\alpha}{2}}} \;/\; (y,\tau),\ (y',\tau') \in A \right\}.$$

Corollary 7.1 *Let ϕ be a solution of the equation (7.11) with $h(y,\tau) = \operatorname{div} H(y,\tau)$. Given $\alpha \in (0,1)$ and constants a, b, γ there is C such that if*

$$\begin{aligned}&|\phi(y,\tau)| + (1+|y|)|H(y,\tau)| + (1+|y|)^{1+\alpha}[H]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}}\\ &\quad \le M\,\lambda_*(\tau)^b(1+|y|)^{-a}\end{aligned}$$

in $\mathcal{D}_{4\gamma R}$, where $\ell(y) = 1 + \frac{|y|}{4}$, then

$$\begin{aligned}(1+|y|)|\nabla_y\phi(y,\tau)| + (1+|y|)^{1+\alpha}[\nabla_y\phi]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}}&\\ \le C\,M\lambda_*(t)^b(1+|y|)^{-a}& \qquad (7.15)\end{aligned}$$

in $\mathcal{D}_{3\gamma R}$. If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t) = 0$ on $\partial B_{4\gamma R(t)}$ for all $t \in (0,T)$ then estimate (7.15) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Our next goal is to construct an inverse for modes $k=-1,0,1$ with a better control when subject to a certain solvability condition.

7.1 Mode $k = 0$

Let us consider again equation (7.6) for $k=0$ and the functions $Z_{0j}(y)$ defined in (3.1) . We have the following result.

Lemma 7.3 *Let assume that $2<a<3$, $k=0$ and*

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\,dy \;=\; 0 \quad \text{for all} \quad t\in[0,T) \qquad (7.16)$$

for $j=1,2$. Then there exist a solution ϕ_0 to equation (7.6) for $k=0$ that defines a linear operator of h_0 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}}\lambda_*^\nu(1+|y|)^{-1}\,\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\,. \qquad (7.17)$$

A central feature of estimate (7.17) is that it matches the size of the solutions obtained in Lemma 7.1 for $k \neq 0, 1$ when $|y| \sim R$.

Proof We observe that conditions (7.16) can be written as

$$\int_0^{2R} \tilde{h}_0(\rho, t)\, Z_0(\rho)\rho\, d\rho \;=\; 0 \quad \text{for all} \quad \tau \in (0, T). \tag{7.18}$$

Let us consider the complex valued functions

$$\tilde{H}_0(\rho, t) := -Z_0(\rho) \int_\rho^\infty \frac{1}{s Z_0(s)^2} \int_s^\infty \tilde{h}_0(\zeta, t) Z_0(\zeta)\zeta\, d\zeta, \quad k = 0, 1.$$

They are well-defined thanks to (7.18). Then the function

$$H_0(y, t) := \mathrm{Re}\,(\tilde{H}_0(\rho, \tau))E_1(y) + \mathrm{Re}\,(\tilde{H}_0(\rho, t))E_2(y)$$

solves

$$L_W[H_0(y, \tau)] = h_0(y, \tau) \quad \text{in } \mathcal{D}_{4R}$$

and satisfies

$$|H_0(y, t)| \;\lesssim\; \lambda_*(t)^\nu (1 + |y|)^{2-a} \|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{4R}.$$

Moreover, elliptic gradient estimates yield

$$|\nabla_y H_0(y, \tau)| \lesssim \lambda_*(t)^\nu (1 + |y|)^{1-a} \|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{3R}.$$

Let us consider the problem

$$\begin{aligned} \lambda^2 \Phi_t &= L_W[\Phi] + H_0(y, t) \quad \text{in } \mathcal{D}_{4R}, \\ \Phi(y, 0) &= 0 \quad \text{in } B_{4R}(0) \\ \Phi(y, t) &= 0 \quad \text{for all } t \in (0, T), \quad y \in \partial B_{4R(0)}(0) \end{aligned} \tag{7.19}$$

According to Lemma 7.1, this problem has unique solution $\Phi = \Phi_0$ that satisfies the estimates

$$|\Phi_0(y, t)| \;\leq C \|H_0\|_{a-2,\nu} \lambda_*(\tau)^\nu (1 + |y|)^{-1} \;\; R^{5-a} \quad \text{in } \mathcal{D}_{4R}.$$

Applying Lemma 7.2 we deduce that, also,

$$|\nabla_y \Phi_0(y, t)| \lesssim \|H_0\|_{a-2,\nu} \lambda_*(\tau)^\nu (1 + |y|)^{-2} \;\; R^{5-a} \quad \text{in } \mathcal{D}_{3R}$$

Let us write

$$\Phi_{0j} := \partial_{y_j}\Phi_0, \quad H_{0j} := \partial_{y_j} H_0$$

Then we have

$$\begin{aligned}\lambda^2\partial_t\Phi_{0j} &= L_W[\Phi_{0j}] + \partial_{y_j}|\nabla W|^2\Phi_0 + 2\nabla\partial_{y_j}W\nabla\Phi_0 + H_{0j}(y,\tau)\\ &\quad + 2(\nabla\Phi_0\partial_{y_j}\nabla W)W + 2(\nabla\Phi_0\nabla W)\partial_{y_j}W \quad \text{in } \mathcal{D}_{3R},\\ \Phi_{0j}(y,0) &= 0 \quad \text{for all } y\in B_{3R(0)}(0)\end{aligned}$$

According to Lemma 7.2 and the above estimates we obtain that

$$\begin{aligned}(1+|y|)|\nabla\Phi_{0j}(y,t)| &\lesssim \|h_0\|_{a,\nu}\lambda_*(t)^\nu(1+|y|)^{-2}R^{5-a}\\ &\quad + \|h_0\|_{a,\nu}\lambda_*(t)^\nu(1+|y|)^{4-a} \quad \text{in } \mathcal{D}_{3R}.\end{aligned}$$

Then we define

$$\phi_0 := L_W[\Phi_0]$$

so that $\phi = \phi_0$ solves

$$\begin{aligned}\lambda^2\phi_t &= L_W[\phi] + h_0(y,t) \quad \text{in } \mathcal{D}_{3R},\\ \phi(y,0) &= 0 \quad \text{for all} \quad y\in B_{3R(0)}(0)\end{aligned}$$

and defines a linear operator of the function h_0. Moreover, observing that

$$|L_W[\Phi_0]| \lesssim \left|D_y^2\Phi_0\right| + O(\rho^{-4})\,|\Phi_0| + O(\rho^{-2})\left|D_y\Phi_0\right|$$

we then get the estimate

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu}R^{5-a}\lambda_*(t)^\nu(1+|y|)^{-3}. \tag{7.20}$$

To complete the proof of estimate (7.17), we let φ_0 be the complex valued function defined as

$$\phi_0(y,t) \;=\; \mathrm{Re}\,(\varphi_0(\rho,t))\,E_1 + \mathrm{Im}\,(\varphi_0(\rho,t))\,E_2$$

so that letting $R' = R^{\frac{5-a}{4}} \ll R$, using the notation in (7.8), φ_0 satisfies the equation

$$\begin{aligned}\lambda^2\partial_t\varphi_0 &= \mathcal{L}_0[\varphi_0] + \tilde h_0(\rho,t) \quad \text{in } \tilde D_{R'},\\ \varphi_0(0,\rho) &= 0 \quad \text{in } (0,R'),\end{aligned} \tag{7.21}$$

and from (7.20), we can find an explicit supersolution for the real and imaginary parts of equation (7.21), which also dominates their boundary values at R', which yields

$$|\varphi_0(y,t)| \lesssim \|h_0\|_{a,\nu}\lambda_*^{\nu}\,|R'|^2(1+|y|)^{-1}\,, \quad |y|<R'.$$

Combining this estimate and (7.20) yields the validity of (7.17). □

We mention next a variant of Lemma 7.3, in which we weaken the hypothesis on the right hand side, allowing it to be a divergence of Hölder continuous function. This will be needed when analyzing estimates of the derivative with respect to λ of operator $\mathcal{T}_{\lambda,2}$ (Proposition 6.3).

Lemma 7.4 *Let assume that* $2<a<3$, $\nu>0$, *and* $k=0$. *Let* h_0 *have the form*

$$h_0(y,\tau) = \operatorname{div} H_0(y,\tau)$$

such that

$$(1+|y|)|H_0(y,\tau)| + (1+|y|)^{1+\alpha}[H_0]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4R}} \le \lambda_*(\tau)^{\nu}(1+|y|)^{-a},$$

in $\mathcal{D}_{4R}$, *where* $\alpha\in(0,1)$ *and* $\ell(y)=1+\frac{|y|}{4}$. *Assume also that*

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\,dy \;=\; 0 \quad \textit{for all} \quad t\in[0,T)$$

for $j=1,2$. *Then there exist a solution* ϕ_0 *to equation* (7.6) *for* $k=0$ *that defines a linear operator of* h_0 *and satisfies*

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}}\lambda_0^{\nu}(1+|y|)^{-1}\,\min\{1,\, R^{\frac{5-a}{2}}|y|^{-2}\},$$

in $\mathcal{D}_{3R}$.

7.2 Mode $k=-1$

Let us consider equation (7.6) for $k=-1$ and the functions $Z_{-1j}(y)$ defined in (3.2) . We have the following result.

Lemma 7.5 *Let assume that* $2<a<3$, $k=0$ *and*

$$\int_{\mathbb{R}^2} h_{-1}(y,t)\cdot Z_{-1j}(y)\,dy = 0 \quad \textit{for all} \quad t\in[0,T)$$

for $j = 1, 2$*. Then there exist a solution* ϕ_{-1} *to equation* (7.6) *for* $k = -1$ *that defines a linear operator of* h_0 *and satisfies the estimate in* $\mathcal{D}_{3R}$*,*

$$|\phi_{-1}(y,t)| \lesssim \|h_{-1}\|_{a,\nu}\lambda_*^\nu \min\{\log R,\, R^{4-a}|y|^{-2}\}.$$

Proof The proof is essentially the same as that of Lemma 7.3. □

7.3 Mode $k = 1$

Now we deal with (7.6) for $k = 1$. For convenience we give the result for a right hand side more general than strictly need for the proof of Proposition 7.1. Let us assume that h_1 is defined in entire $\mathbb{R}^2 \times (0, T)$ and that

$$h_1(y,t) = \mathrm{div}_y\, G(y,t) \tag{7.22}$$

where

$$|G(y,t)| \le \frac{\lambda_*(t)^\nu}{1 + |y|^{a-1}}, \quad y \in \mathbb{R}^2,\ t \in (0,T), \tag{7.23}$$

for some $\nu > 0$, $a \in (2, 3)$. Then the following result holds.

Lemma 7.6 *Let assume that* $2 < a < 3$*,* $k = 1$*,* h_1 *has the form* (7.22) *so that* (7.23) *holds and*

$$\int_{\mathbb{R}^2} h_1(y,t) \cdot Z_1^j(y)\, dy \;=\; 0 \quad \textit{for all} \quad t \in (0,T)$$

for $j = 1, 2$*. Then there exist a solution* ϕ_1 *to equation* (7.6) *for* $k = 1$ *that defines a linear operator of* h_1 *and satisfies the estimate in* $\mathcal{D}_{3R}$*,*

$$|\phi_1(y,t)| \lesssim \lambda_*(t)^\nu (1 + |y|)^{2-a}.$$

From this we get directly the next result.

Corollary 7.2 *Let assume that* $2 < a < 3$*,* $k = 1$ *and*

$$\int_{B_{2R}} h_1(y,t) \cdot Z_1^j(y)\, dy \;=\; 0 \quad \textit{for all} \quad t \in (0,T)$$

for $j = 1, 2$*. Then there exist a solution* ϕ_1 *to equation* (7.6) *for* $k = 1$ *that defines a linear operator of* h_1 *and satisfies the estimate in* $\mathcal{D}_{3R}$*,*

$$|\phi_1(y,t)| \;\lesssim\; \|h_1\|_{a,\nu}\lambda_*(t)^\nu (1 + |y|)^{2-a}\ .$$

Let us do the same change of the time variable as in (7.14) so that (7.6) for $k=1$ in entire $\mathbb{R}^2$ becomes in the variables (y,τ)

$$\begin{aligned} \partial_\tau \phi &= L_W[\phi] + h \quad \text{in } \mathbb{R}^2 \times (0,\infty), \\ \phi(\cdot,0) &= 0 \quad \text{in } \mathbb{R}^2. \end{aligned} \tag{7.24}$$

Thus, we consider a function $h(y,\tau)$ defined in entire $\mathbb{R}^2 \times (0,+\infty)$ of the form

$$h = \mathrm{Re}\,(\tilde h e^{i\theta})\, E_1 + \mathrm{Im}\,(\tilde h e^{i\theta})\, E_2, \tag{7.25}$$

that satisfies the orthogonality conditions for $j=1,2$

$$\int_{\mathbb{R}^2} h(\cdot,\tau)\cdot Z_{1j} \; = \; 0 \quad \text{for all} \quad \tau \in (0,\infty) \tag{7.26}$$

and such that $h(y,\tau)=0$ for $|y| \geq 2R(\tau)$.

By standard parabolic theory, this problem has a unique solution, which is therefore of the form

$$\phi = \mathrm{Re}\,(\varphi e^{i\theta})\, E_1 + \mathrm{Im}\,(\varphi e^{i\theta})\, E_2, \tag{7.27}$$

where the complex valued function $\varphi(\rho,\tau)$ solves the initial value problem

$$\begin{aligned} \partial_\tau \varphi &= \mathcal{L}_1[\varphi] + \tilde h(\rho,\tau) \quad \text{in } (0,\infty)\times(0,\infty), \\ \varphi(\rho,0) &= 0 \quad \text{in } (0,\infty), \\ \mathcal{L}_1[\varphi] &= \partial_\rho^2 \varphi + \frac{\partial_\rho \varphi}{\rho} - (1 + 2\cos w + \cos(2w))\frac{\varphi}{\rho^2}. \end{aligned} \tag{7.28}$$

We have the validity of the following result.

Lemma 7.7 *Let $0<\sigma<1$, $\nu>0$. Assume that h is mode 1, that is, has the form (7.25), satisfies the orthogonality conditions (7.26), and can be written as in (7.22) with g_j satisfying (7.23) where $b=1+\sigma$. Then there exists a constant $C>0$ such that the solution ϕ of problem (7.24) satisfies the estimate*

$$|\phi(y,t)| \leq C \frac{\lambda_*(t)^\nu}{1+|y|^\sigma}. \tag{7.29}$$

For the proof of this result we will use the following Liouville type result.

Lemma 7.8 *Let $0<\sigma<1$. Suppose $\tilde\phi$ satisfies*

$$\tilde\phi_\tau = L_W[\tilde\phi] \quad \textit{in } \mathbb{R}^2 \times (-\infty, 0],$$

$$\int_{\mathbb{R}^2} \tilde{\phi}(\cdot,\tau)\cdot Z_{1j} = 0 \quad \textit{for all} \quad \tau \in (-\infty, 0],$$
$$|\tilde{\phi}(y,\tau)| \le \frac{1}{1+|y|^\sigma} \quad \textit{in}\ \mathbb{R}^2\times(-\infty,0], \quad j=1,2,$$
$$\tilde{\phi}(y,\tau) = \mathrm{Re}\,(\tilde{\varphi}(\rho,\tau)e^{i\theta})\, E_1 + \mathrm{Im}\,(\tilde{\varphi}(\rho,\tau)e^{i\theta})\, E_2.$$

Then $\tilde{\phi} = 0$.

Proof By standard parabolic regularity $\tilde{\phi}(y,\tau)$ is a smooth function. A scaling argument shows that

$$(1+|y|)^{-1}|D_y\tilde{\phi}| + |\tilde{\phi}_\tau| + |D_y^2\tilde{\phi}| \le C(1+|y|)^{-2-\sigma}.$$

Differentiating the equation in τ, we also get $\partial_\tau\phi_\tau = L_W[\phi_\tau]$ and we find the estimates

$$(1+|y|)^{-1}|D_y\tilde{\phi}_\tau| \ + \ |\tilde{\phi}_{\tau\tau}| \ + \ |D_y^2\tilde{\phi}_\tau| \ \le \ C(1+|y|)^{-3-\sigma}.$$

Testing suitably the equations (taking into account the asymptotic behaviors in y in integrations by parts) we find

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(\tilde{\phi},\tilde{\phi}) = -\int_{\mathbb{R}^2} L_W[\tilde{\phi}]\cdot\tilde{\phi} = \int_{\mathbb{R}^2} |\nabla\tilde{\phi}|^2 - |\nabla W|^2|\tilde{\phi}|^2.$$

It is useful to observe the following: since

$$\tilde{\phi}(y,\tau) \ = \ \mathrm{Re}\,(\tilde{\varphi}(\rho,\tau)e^{i\theta})\, E_1 + \mathrm{Im}\,(\tilde{\varphi}(\rho,\tau)e^{i\theta})\, E_2$$

then we compute, using that $\mathcal{L}_1[w_\rho] = 0$,

$$B(\tilde{\phi},\tilde{\phi}) = -\int_0^\infty \mathcal{L}_1[\varphi]\bar{\varphi}\rho d\rho \ = \ \int_0^\infty |(w_\rho^{-1}\tilde{\varphi})_\rho|^2 w_\rho^2 \rho\, d\rho \ \ge \ 0.$$

We also get

$$\int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 = -\frac{1}{2}\partial_\tau B(\tilde{\phi},\tilde{\phi}).$$

From these relations we find

$$\partial_\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 \ < \ +\infty$$

and hence $\tilde{\phi}_\tau = 0$. Thus $\tilde{\phi}$ is independent of τ and therefore $L_W[\tilde{\phi}] = 0$. Since $\tilde{\phi}$ is at mode 1, this implies that $\tilde{\phi}$ is a linear combination of Z_{1j}, $j = 1, 2$. Since $\int_{\mathbb{R}^2} \tilde{\phi} \cdot Z_{1j} = 0$, $j = 1, 2$ we conclude that $\tilde{\phi} = 0$, a contradiction. □

Proof of Lemma 7.7 Let us write

$$\|\phi\|_{b,\tau_1} := \sup_{\tau\in(0,\tau_1)} \lambda_*(\tau)^{-\nu} \|(1 + |y|^b)\phi\|_{L^\infty(\mathbb{R}^2)}.$$

We claim that for any $\tau_1 > 0$ we have that

$$\|\phi\|_{2+\sigma,\tau_1} < +\infty. \tag{7.30}$$

Let us recall that with the transformations (7.27) we have that the complex valued function $\varphi(y, \tau)$ is radial in y and solves the initial value problem

$$\partial_\tau\varphi = \Delta_{\mathbb{R}^2}\varphi - (1 + 2\cos w + \cos(2w))\frac{\varphi}{\rho^2} + \tilde{h}(\rho, \tau) \quad \text{in } \mathbb{R}^2 \times (0, \infty),$$
$$\varphi(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2$$

where $\rho = |y|$, $y \in \mathbb{R}^2$ and $\tilde{h}$ is related to h by (7.25). Let us write $\varphi = \varphi_a + \varphi_b$ where φ_a is the unique solution to

$$\partial_\tau\varphi_a = \Delta_{\mathbb{R}^2}\varphi_a + \tilde{h}(\rho, \tau) \quad \text{in } \mathbb{R}^2 \times (0, \infty),$$
$$\varphi_a(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2$$

given by Duhamel's formula. Using the heat kernel in $\mathbb{R}^2$ one readily shows that $\|\varphi_a\|_{2+\sigma,\tau_1} < +\infty$. Let

$$\partial_\tau\varphi_b = \Delta_{\mathbb{R}^2}\varphi_b - (1 + 2\cos w + \cos(2w))\frac{1}{\rho^2}(\varphi_a + \varphi_b) \quad \text{in } \mathbb{R}^2 \times (0, \infty),$$
$$\varphi_b(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2.$$

By standard linear parabolic theory $\phi_b(y, \tau)$ is locally bounded in time and space. More precisely, given $R > 0$ there is a $K = K(R, \tau_1)$ such that

$$|\phi_b(y, \tau)| \leq K \quad \text{in } B_R(0) \times (0, \tau_1].$$

If we fix R large and take K_1 sufficiently large, we see that $K_1\rho^{-\sigma}$ is a supersolution for the real and imaginary parts of the equivalent complex valued equation (7.28) in the region $\rho > R$. As a conclusion, we find that $|\phi_b| \le 2K_1\rho^{-\sigma}$, and therefore $\|\phi_b\|_{\sigma,\tau_1} < +\infty$ for any $\tau_1 > 0$. This proves (7.30).

Next we claim that

$$\int_{\mathbb{R}^2} \phi(\cdot,\tau)\cdot Z_{1j} = 0 \quad \text{for all} \quad \tau \in (1,\tau_1), \quad j = 1,2. \tag{7.31}$$

Indeed, let us test the equation against

$$Z_{1j}\eta, \quad \eta(y) = \eta_0(R^{-1}|y|)$$

where η_0 is a smooth cut-off function with $\eta_0(r) = 1$ for $r < 1$ and $= 0$ for $r > 2$ and R is an arbitrary large constant. We find that

$$\int_{\mathbb{R}^2} \phi(\cdot,\tau)\cdot Z_{1j}\eta \;=\; \int_0^\tau ds \int_{\mathbb{R}^2} \phi(\cdot,s)\cdot (L_W[\eta Z_{1j}] + h\cdot Z_{1j}\eta). \tag{7.32}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \phi\cdot (L_W[\eta Z_{1j}] + h\cdot Z_{1j}\eta_R)\\
&= \int_{\mathbb{R}^2} \phi\cdot (Z_{1j}\Delta\eta + 2\nabla\eta\cdot\nabla Z_{1j}) - h\cdot Z_{1j}(1-\eta_R)\\
&= O(R^{-2-\sigma})
\end{aligned}$$

uniformly on $\tau \in (0,\tau_1)$. Letting $R \to +\infty$ in (7.32) we get that (7.31) holds.

Now we claim that there exists a constant C such that for all $\tau_1 > 0$ we have the validity of the estimate

$$\|\phi\|_{\sigma,\tau_1} \le C, \tag{7.33}$$

so that in particular estimate (7.29) holds.

To prove (7.33) we assume by contradiction the existence of sequences $\tau_1^n \to +\infty$ and ϕ_n, h_n of the form (7.25), (7.27) satisfying

$$\begin{aligned}
\partial_\tau\phi_n &= L_W[\phi_n] + h_n \quad \text{in } \mathbb{R}^2\times(1,\tau_1^n),\\
\int_{\mathbb{R}^2} \phi_n(\cdot,\tau)\cdot Z_{1j} &= 0 \quad \text{for all} \quad \tau\in(1,\tau_1^n),\\
\phi_n(\cdot,1) &= 0 \quad \text{in } \mathbb{R}^2,
\end{aligned}$$

so that

$$\|\phi_n\|_{\sigma,\tau_1^n} = 1 \tag{7.34}$$

but

$$h_n = \sum_{j=1}^{2} \partial_{y_j} g_{j,n}, \quad \|g_{j,n}\|_{1+\sigma,\tau_1^n} \to 0, \quad \text{as } n \to \infty.$$

We claim first that

$$\sup_{1<\tau<\tau_1^n} \tau^\nu |\phi_n(y,\tau)| \to 0 \tag{7.35}$$

uniformly on compact subsets of $y \in \mathbb{R}^2$. If not, for some $M > 0$ there are $|y_n| \le M$ and $1 < \tau_2^n < \tau_1^n$ so that

$$(\tau_2^n)^\nu (1 + |y_n|^\sigma)|\phi(y_n, \tau_2^n)| \ge \frac{1}{2}.$$

Clearly we must have $\tau_2^n \to +\infty$. Let us define

$$\tilde{\phi}_n(y,\tau) = (\tau_2^n)^\nu \phi_n(y, \tau_2^n + \tau).$$

Then

$$\partial_\tau \tilde{\phi}_n = L_W[\tilde{\phi}_n] + \tilde{h}_n \quad \text{in } \mathbb{R}^2 \times (1 - \tau_2^n, 0]$$

where $\tilde{h}_n \to 0$ has the form

$$\tilde{h}_n = \sum_{j=1}^{2} \partial_{y_j} \tilde{g}_{j,n}, \quad |\tilde{g}_{j,n}(y,\tau)| \le o(1) \frac{(\tau_2^n)^\nu}{(\tau_2^n + \tau)^\nu} \frac{1}{1 + |y|^{1+\sigma}}$$

and

$$|\tilde{\phi}_n(y,\tau)| \le \frac{1}{1 + |y|^\sigma} \quad \text{in } \mathbb{R}^2 \times (1 - \tau_2^n, 0].$$

From standard parabolic estimates, we find that passing to a subsequence, $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compact subsets of $\mathbb{R}^2 \times (-\infty, 0]$ where $\tilde{\phi} \ne 0$ and

$$\tilde{\phi}_\tau = L_W[\tilde{\phi}] \quad \text{in } \mathbb{R}^2 \times (-\infty, 0],$$

$$\int_{\mathbb{R}^2} \tilde{\phi}(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all} \quad \tau \in (-\infty, 0],$$
$$|\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^\sigma} \quad \text{in } \mathbb{R}^2 \times (-\infty, 0], \quad j = 1, 2,$$
$$\tilde{\phi}(y, \tau) = \mathrm{Re}\,(\tilde{\varphi}(\rho, \tau) e^{i\theta})\, E_1 + \mathrm{Im}\,(\tilde{\varphi}(\rho, \tau) e^{i\theta})\, E_2.$$

But then Lemma 7.8 implies that $\tilde{\phi} \equiv 0$, which is a contradiction, and we conclude that (7.35) indeed holds.

From (7.34), we have that for a certain y_n with $|y_n| \to \infty$ and $\tau_2^n > 0$,

$$(\tau_2^n)^\nu |y_n|^\sigma |\phi_n(y_n, \tau_2^n)| \geq \frac{1}{2}.$$

Now we let

$$\tilde{\phi}_n(z, \tau) := (\tau_2^n)^\nu |y_n|^\sigma\, \phi_n(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n)$$

so that

$$\partial_\tau \tilde{\phi}_n = \Delta_z \tilde{\phi}_n + a_n \cdot \nabla_z \tilde{\phi}_n + b_n \tilde{\phi}_n + \tilde{h}_n(z, \tau)$$

where

$$\tilde{h}_n(z, \tau) = (\tau_2^n)^\nu |y_n|^{2+\sigma}\, h_n(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n),$$

and $|a_n| + |b_n| \to 0$ uniformly on compact sets of $\mathbb{R}^2 \setminus \{0\}$.

Note that

$$\tilde{h}_n = \sum_{j=1}^{2} \partial_{z_j} \tilde{g}_{j,n}$$

where

$$\tilde{g}_{j,n}(z, \tau) = (\tau_2^n)^\nu |y_n|^{1+\sigma}\, g_{j,n}(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n),$$

By assumption on $g_{j,n}$ we find that $\tilde{g}_{j,n} \to 0$ uniformly on compact sets of $(\mathbb{R}^2 \setminus \{0\}) \times (-\infty, 0]$. Besides $|\tilde{\phi}_n(\frac{y_n}{|y_n|}, 0)| \geq \frac{1}{2}$ and

$$|\tilde{\phi}_n(z, \tau)| \leq |z|^{-\sigma} ((\tau_2^n)^{-1} |y_n|^{-2}\tau + 1)^{-\nu}.$$

As a conclusion, we may assume that $\tilde{\phi}_n \to \tilde{\phi} \neq 0$ uniformly over compact subsets of $\mathbb{R}^2 \setminus \{0\} \times (-\infty, 0]$ where

$$\tilde{\phi}_\tau = \Delta_z \tilde{\phi} \quad \text{in } \mathbb{R}^2 \setminus \{0\} \times (-\infty, 0].$$

and

$$|\tilde{\phi}(z,\tau)| \le |z|^{-\sigma} \quad \text{in } \mathbb{R}^2 \setminus \{0\} \times (-\infty, 0].$$

Moreover, the mode 1 assumption for ϕ_n translates for $\tilde{\phi}$ into

$$\tilde{\phi}(z,\tau) = \begin{bmatrix} \varphi(\rho,\tau)e^{2i\theta} \\ 0 \end{bmatrix}, \quad z = \rho e^{i\theta}$$

for a complex valued function φ that solves

$$\varphi_\tau = \varphi_{\rho\rho} + \frac{\varphi_\rho}{\rho} - \frac{4\varphi}{\rho^2} \quad \text{in } (0,\infty) \times (-\infty, 0], \tag{7.36}$$
$$|\varphi(\rho,\tau)| \le \rho^{-\sigma} \quad \text{in } (0,\infty) \times (-\infty, 0].$$

Let us set

$$u(\rho, t) = (\rho^2 + t)^{-\sigma/2} + \frac{\varepsilon}{\rho^2}$$

Then

$$-u_t + \Delta u - \frac{4u}{r^2} < (\rho^2 + t)^{-\sigma/2-1}\left[\sigma(\sigma+2) - 4 + \frac{\sigma}{2}\right] < 0.$$

It follows that the function $u(x, \tau + M)$ is a positive supersolution for the real and imaginary parts of equation (7.36) in $(0,\infty) \times [-M, 0]$. We find then that $|\varphi(\rho,\tau)| \le 2u(\rho, \tau + M)$. Letting $M \to +\infty$ we find

$$|\varphi(\rho,\tau)| \le \frac{2\varepsilon}{\rho^2}$$

and since ε is arbitrary we conclude $\varphi = 0$. Hence $\tilde{\phi} = 0$, a contradiction that concludes the proof of the lemma. □

Proof of Lemma 7.6 We take h to be the extension as zero of the function h_1 as in the statement of the lemma. Then we let ϕ be the unique solution of the initial value problem (7.24), which clearly defines a linear operator of h_1. From Lemma 7.7, expressing the resulting estimate in the variables (y, t), we have that for any $t_1 \in (0, T)$

$$|\phi(y,t)| \le C\lambda_*(t)^\nu (1 + |y|)^{-\sigma} \|h\|_{2+\sigma,t_1} \quad \text{for all} \quad t \in (0, t_1), \quad y \in \mathbb{R}^2.$$

Then letting $\phi_1 := \phi\big|_{\mathcal{D}_{3R}}$ and letting $t_1 \uparrow T$ the result follows. □

7.4 Proof of Proposition 7.1

We let h be defined in $\mathcal{D}_{2R}$ with $\|h\|_{a,\nu} < +\infty$, with $a \in (2,3)$, $\nu > 0$. We consider the problem

$$\lambda^2 \partial_t \phi = L_W[\phi] + h \quad \text{in } \mathcal{D}_{4R} \phi(\cdot, 0) \qquad\qquad \text{in } B_{4R(0)},$$

(recall that h is assumed to be defined in $\mathbb{R}^2 \times (0,T)$. Let ϕ_k be the solution estimated in Lemma 7.1 of

$$\begin{aligned} &\lambda^2 \partial_t \phi_k = L_W[\phi_k] + h_k \quad \text{in } \mathcal{D}_{4R} \\ &\phi(\cdot, t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t \in (0,T), \\ &\phi(\cdot, 0) = 0 \quad \text{in } B_{4R(0)}. \end{aligned}$$

In addition we let $\phi_{01}, \phi_{11}, \phi_{-11}$ solve

$$\begin{aligned} &\lambda^2 \partial_t \phi_{k1} = L_W[\phi_{k1}] + \bar{h}_k \quad \text{in } \mathcal{D}_{4R} \\ &\phi_{k1}(\cdot, t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t \in (0,T), \\ &\phi_{k1}(\cdot, 0) = 0 \quad \text{in } B_{4R(0)} \end{aligned}$$

for $k = 0, 1, -1$. Let us consider the functions ϕ_{02} constructed in Lemma 7.3, $\phi_{-1,2}$ constructed in Lemma 7.5, and ϕ_{12} constructed in Lemma 7.6, that solve for $k = 0, 1, -1$

$$\begin{aligned} &\lambda^2 \partial_t \phi_{k2} = L_W[\phi_{k2}] + h_k - \bar{h}_k \quad \text{in } \mathcal{D}_{3R} \\ &\phi_{k2}(\cdot, 0) = 0 \quad \text{in } B_{3R(0)}. \end{aligned}$$

We define

$$\phi := \sum_{k=0,1,-1} (\phi_{k1} + \phi_{k2}) + \sum_{k \neq 0,1,-1} \phi_k$$

which is a bounded solution of the equation

$$\lambda^2 \phi_t = L_W[\phi] + h(y,t) \quad \text{in } \mathcal{D}_{3R}$$

that defines a linear operator of h. Applying the estimates for the components in Lemmas 7.1, 7.3, 7.5, and 7.6 we obtain

$$\begin{aligned} |\phi(y,t)| \lesssim{}& \frac{\lambda_*(t)^\nu \log R(t)}{1 + |y|^{a-2}} \|h^\perp\|_{a,\nu} \\ &+ \frac{\lambda_*(t)^\nu}{1 + |y|^{a-2}} \left\|h_1 - \bar{h}_1\right\|_{a,\nu} + \frac{\lambda_*(t)^\nu R^4}{1 + |y|^2} \left\|\bar{h}_1\right\|_{\nu,a} \end{aligned}$$

$$+\frac{\lambda_*(t)^\nu R^{\frac{5-a}{2}}}{1+|y|}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_0-\bar h_0\|_{a,\nu}+\frac{\lambda_*(t)^\nu R^2}{1+|y|}\|\bar h_0\|_{a,\nu}$$
$$+\lambda_*^\nu\min\{\log R, R^{4-a}|y|^{-2}\}\,\|h_{-1}-\bar h_{-1}\|_{a,\nu}+\lambda_*(t)^\nu\log R\|\bar h_{-1}\|_{a,\nu},$$

in $\mathcal{D}_{3R}$. Finally, Lemma 7.2 yields that the same bound is valid for $(1+|y|)|\nabla_y\phi|$ in $\mathcal{D}_{2R}$. The function $\phi\big|_{\mathcal{D}_{2R}}$ solves (7.1), it defines a linear operator of h and satisfies the required estimates. □

7.5 Modified theory for mode 0

Let us consider the problem

$$\begin{cases}\lambda^2\varphi_t = L_W\varphi + h(y,t) + \sum_{j=1,2}\tilde c_{0j}Z_{0j}w_\rho^2 & \text{in } \mathcal{D}_{2R}\\ \varphi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \varphi = 0 \quad \text{on } \partial B_{2R}\times(0,T)\\ \varphi(\cdot,0)=0 \quad \text{in } B_{2R(0)},\end{cases} \tag{7.37}$$

in mode 0. The result here is the following.

Proposition 7.2 *Let $\sigma\in(0,1)$, $\delta\in(0,1)$, $\nu>0$. Assume $\|h\|_{\nu,2+\sigma}<\infty$. Then there is a solution ϕ, $\tilde c_{0j}$ of (7.37), which is linear in h, such that*

$$|\varphi(y,t)|+(1+|y|)|\nabla_y\varphi(y,t)|\le C\lambda_*^\nu\|h\|_{\nu,2+\sigma}\begin{cases}\frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y|\le 2R^\delta\\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta\le|y|\le R,\end{cases}$$

and such that

$$\tilde c_{0j}[h] = -\frac{\int_{B_{\mathbb{R}^2}} h\cdot Z_{0j}}{\int_{\mathbb{R}^2} w_\rho^2|Z_{0j}|^2} - G[h]$$

where G is a linear operator of h satisfying the estimate

$$|G[h]|\le C\lambda_*^\nu R^{-\delta\sigma'}\|h\|_{\nu,2+\sigma}, \tag{7.38}$$

with $0<\sigma'<\sigma$.

We are using the terminology *mode 0* from §7, which means that φ has the form

$$\varphi = \mathrm{Re}(\tilde\varphi e^{i\theta})E_1+\mathrm{Im}(\tilde\varphi e^{i\theta})E_2$$

where $\tilde{\varphi}$ is a complex valued function of ρ and t. The equation $\lambda^2\varphi_t = L_W\varphi + h(y,t)$ (wit h also in mode 0) becomes

$$\lambda^2\partial_t\tilde{\varphi} = \mathcal{L}_0\tilde{\varphi} + \tilde{h}, \quad \text{where} \quad \mathcal{L}_0[\tilde{\varphi}] := \partial^2_\rho\tilde{\varphi} + \frac{1}{\rho}\partial_\rho\tilde{\varphi} - \frac{\cos(2w)}{\rho^2}\tilde{\varphi},$$

and we have a similar definition for $\tilde{h}$. Note that the operator $\mathcal{L}_0$ at $\rho = 0$ and $\rho = \infty$ is given by $\partial^2_\rho\tilde{\varphi} + \frac{1}{\rho}\partial_\rho\tilde{\varphi} - \frac{1}{\rho^2}\tilde{\varphi}$. The last equation can be written as a regular parabolic PDE by setting $\hat{\varphi}(y,t) = \tilde{\varphi}(\rho,t)e^{-i\theta}$, $y = \rho e^{i\theta}$,

$$\lambda^2\partial_t\hat{\phi} = \Delta_y\hat{\varphi} + \frac{16\hat{\varphi}}{(1+|y|^2)^2} + \hat{h}(y,t).$$

Thus, instead of (7.37) we will construct a solution to (changing the notation to φ and h)

$$\begin{cases} \lambda^2\varphi_t = \Delta_y\varphi + \dfrac{16}{(1+|y|^2)^2}\varphi + h(y,t) + \tilde{c}_0\rho w^3_\rho & \text{in } \mathcal{D}_{2R} \\ \varphi = 0 \quad \text{on } \partial B_{2R}\times(0,T) \\ \varphi(\cdot,0) = 0 \quad \text{in } B_{2R(0)}, \end{cases} \tag{7.39}$$

with φ complex valued of the form $\varphi(y) = e^{i\theta}\tilde{\varphi}(\rho,t)$ (and the same for h). Here $\tilde{c}_0$ is complex and related to $\tilde{c}_{0j}$ in (7.37) by $\tilde{c}_0 = \tilde{c}_{01} + i\tilde{c}_{02}$.

We will construct φ solving (7.39) of the form

$$\varphi = \eta\phi + \psi$$

where $\eta(y,t) = \eta_1\left(\frac{|y|}{R_1}\right)$ and $\eta_1(r) = 1$ for $r \le 1$, $\eta_1(r) = 0$ for $r \ge 2$. Here $R_1 = R^\delta$. We find a solution to (7.39) if we get ϕ, ψ solving the system

$$\begin{cases} \lambda^2\partial_t\phi = \Delta\phi + B\phi + B\psi + h(y,t) + c_0\rho w^3_\rho & \text{in } \mathcal{D}_{2R_1} \\ \phi(\cdot,0) = 0 \quad \text{in } B_{2R_1(0)}, \end{cases} \tag{7.40}$$

$$\begin{cases} \lambda^2\partial_t\psi = \Delta\psi + (1-\eta)B\psi + A\phi + (1-\eta)h(y,t) & \text{in } \mathcal{D}_{2R} \\ \psi = 0 \quad \text{on } \partial B_{2R}\times(0,T) \\ \psi(\cdot,0) = 0 \quad \text{on } B_{2R(0)}, \end{cases} \tag{7.41}$$

where

$$B = \frac{16}{(1+|y|^2)^2}, \quad A\phi = \phi\Delta\eta + 2\nabla\phi\nabla\eta - \phi\eta_t.$$

Consider

$$\begin{cases} \lambda^2 \partial_t \psi = \Delta \psi + (1-\eta) B\psi + h(y,t) & \text{in } \mathcal{D}_{2R} \\ \psi = 0 & \text{on } \partial B_{2R} \times (0,T), \\ \psi(y,0) = 0 & \forall y \in B_{2R(0)}, \end{cases} \tag{7.42}$$

with ψ and h of the form $\psi = \tilde{\psi}(\rho,t)e^{i\theta}$. Let

$$\|\psi\|^{(1)}_{\nu,\sigma} = \sup_{\mathcal{D}_{2R}} \big\{ \lambda_*^{-\nu}(t)(1+|y|)^{\sigma} \left[|\psi(y,t)| + (1+|y|)|\nabla_y \psi(y,t)| \right] \big\}.$$

Lemma 7.9 *Let $\sigma \in (0,1)$, $\nu > 0$ and let ψ solve (7.42). If R_1 is sufficiently large, then*

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{7.43}$$

If in (7.42) h is replaced by $(1-\eta)h$ we get the additional estimate

$$|\psi(y,t)| + R_1|\nabla\psi(y,t)| \le C\lambda_*^{\nu}\frac{1}{R_1^{\sigma}}, \quad |y| \le 2R_1.$$

Proof To prove this lemma, we first claim that for the equation

$$\begin{cases} \lambda^2 \partial_t \psi = \Delta \psi + h(y,t) & \text{in } \mathcal{D}_{2R} \\ \psi = 0 & \text{on } \partial B_{2R} \times (0,T), \\ \psi(y,0) = 0 & \forall y \in B_{2R(0)}, \end{cases}$$

with ψ and h of the form $\psi = \tilde{\psi}(\rho,t)e^{i\theta}$. we have

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{7.44}$$

This is obtained using a barrier for the real and imaginary parts of $\tilde{\psi}$, which satisfies

$$\lambda^2 \partial_t \tilde{\psi} = \partial_{\rho\rho}\tilde{\psi} + \frac{1}{\rho}\partial_\rho \tilde{\psi} - \frac{1}{\rho^2}\tilde{\psi} + \tilde{h}.$$

To find the estimate for the solution of (7.42) we need to estimate $\|(1-\eta)B\psi\|_{\nu,2+\sigma}$. We have that

$$\begin{aligned} (1-\eta)B\,|\psi| &\le (1-\eta)\lambda_*^{\nu}(1+|y|)^{-4-\sigma}\,\|\psi\|^{(1)}_{\nu,\sigma} \\ &\le R_1(0)^{-2}\lambda_*^{\nu}(1+|y|)^{-2-\sigma}\,\|\psi\|^{(1)}_{\nu,\sigma}, \end{aligned}$$

and therefore

$$\|(1-\eta)B\psi\|_{\nu,2+\sigma} \le CR_1(0)^{-2}\|\psi\|^{(1)}_{\nu,\sigma}.$$

Then, if ψ satisfies (7.42), using (7.44) we get

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|(1-\eta)B\psi + h\|_{\nu,2+\sigma} \le CR_1(0)^{-1}\|\psi\|^{(1)}_{\nu,\sigma} + C\|h\|_{\nu,2+\sigma}.$$

If $R_1(0)$ is large enough, we obtain (7.43). □

Proof of Proposition 7.2 We use Lemma 7.9 to find a solution $\psi[\phi]$ of (7.42) with h replaced by $A\phi$, and a solution $\psi[h]$ of (7.42) with h replaced by $(1-\eta)h$, so that $\psi[\phi]+\psi[h]$ is the solution of (7.41).

Let $\sigma_1 \in (0,1)$. We also get the estimate

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \le C\|A\phi\|_{\nu,2+\sigma_1}. \tag{7.45}$$

We take $R_1 = R^\delta$ and construct a solution of the system (7.40), (7.41). For this it suffices to find ϕ such that

$$\begin{cases} \lambda^2\partial_t\phi = \Delta\phi + B\phi + B\psi[\phi] + B\psi[h] + h(y,t) + c_0\rho w_\rho^3 & \text{in } \mathcal{D}_{2R_1} \\ \phi(\cdot,0) = 0 \quad \text{in } B_{2R_1(0)}. \end{cases} \tag{7.46}$$

Let $\mathcal{T}$ denote the linear operator given by Lemma 7.3, Applied in $\mathcal{D}_{2R_1}$. Then to solve (7.46) we consider the fixed point problem

$$\phi = \mathcal{T}[B\psi[\phi] + B\psi[h] + h].$$

Let $\sigma \in (0,1)$. By Lemma 7.3,

$$\|\mathcal{T}[g]\|_{*,\nu,2+\sigma} \le \|g\|_{\nu,2+\sigma}, \tag{7.47}$$

where

$$\|\phi\|_{*,\nu,\sigma} = \sup \frac{\lambda_*^{-\nu}(1+|y|)^3}{R_1^{3-\sigma}}\left[|\phi(y,t)| + (1+|y|)|\nabla_y\phi(y,t)|\right].$$

We claim that if $\sigma_1 < \sigma$ then

$$\|A\phi\|_{\nu,2+\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{7.48}$$

Indeed, we have

$$|\phi\Delta\eta| \leq \frac{1}{R_1^2}\lambda_*^\nu \frac{R_1^{3-\sigma}}{(1+|y|)^3}|\Delta\eta_1|\|\phi\|_{*,\nu,\sigma} \leq C\lambda_*^\nu \frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}$$
$$\leq CR_1(0)^{\sigma_1-\sigma}\lambda_*^\nu \frac{1}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.$$

Similarly

$$|\nabla\phi\nabla\eta| \leq \frac{1}{R_1}\lambda_*^\nu \frac{R_1^{3-\sigma}}{(1+|y|)^4}|\nabla\eta_1|\|\phi\|_{*,\nu,\sigma} \leq C\lambda_*^\nu \frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.$$

Similar estimates for the remaining terms in A prove (7.48).

From (7.45) and (7.48) we find

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \leq CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{7.49}$$

Now we claim that

$$\|B\psi\|_{\nu,2+\sigma} \leq C\|\psi\|^{(1)}_{\nu,\sigma_1}. \tag{7.50}$$

Indeed,

$$B\,|\psi| \leq C\frac{\lambda_*^\nu}{(1+|y|)^{4+\sigma_1}}\|\psi\|^{(1)}_{\nu,\sigma_1} \leq C\frac{\lambda_*^\nu}{(1+|y|)^{2+\sigma}}\|\psi\|^{(1)}_{\nu,\sigma_1}$$

so (7.50) follows. Combining (7.50) and (7.49) we get

$$\|B\psi[\phi\|_{\nu,2+\sigma} \leq C\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \leq CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}.$$

From the above inequality and (7.47) we then get

$$\|\mathcal{T}[B\psi[\phi]]\|_{*,\nu,\sigma} \leq CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma},$$

which shows that the operator $\phi \mapsto \mathcal{T}[B\psi[\phi]+B\psi[h]+h]$ is a contraction if $R_1(0)$ is sufficiently large, and we find a unique fixed point, which satisfies the estimate

$$\|\phi\|_{*,\nu,\sigma} \leq C\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}.$$

Next we estimate $\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}$. We have by (7.47)

$$\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma} \leq C\|B\psi[h]+h\|_{\nu,2+\sigma}$$
$$\leq C\|\psi[h]\|^{(1)}_{\nu,\sigma} + \|h\|_{\nu,2+\sigma} \leq C\|h\|_{\nu,2+\sigma},$$

and hence

$$\|\phi\|_{*,\nu,\sigma} \leq C\|h\|_{\nu,2+\sigma}. \tag{7.51}$$

Similar to (7.49) we have

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma} \leq C\|\phi\|_{*,\nu,\sigma} \leq C\|h\|_{\nu,2+\sigma}$$

and

$$\|\psi[h]\|^{(1)}_{\nu,\sigma} \leq C\|h\|_{\nu,2+\sigma}.$$

Recalling that $\varphi = \eta\phi + \psi$ and $R_1 = R^\delta$, we get

$$|\varphi(y,t)| + (1+|y|)|\nabla_y\varphi(y,t)| \leq C\lambda_*^\nu\|h\|_{\nu,2+\sigma}\begin{cases}\frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y| \leq 2R^\delta\\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta \leq |y| \leq R.\end{cases}$$

Finally, thanks to Lemma 7.3, we have that

$$c_{0j}[h] = -\frac{1}{\int_{B_1}|Z_{0j}|^2}\left[\int_{B_{2R_1}} h\rho w_\rho + \int_{B_{2R_1}}(B\psi[\phi] + B\psi[h])\rho w_\rho\right]$$

The last term is a linear operator of h, which we estimate next. A similar computation as in (7.48) shows that

$$\|A\phi\|_{\nu+\delta(\sigma-\sigma_1),2+\sigma_1} \leq C\|\phi\|_{*,\nu,\sigma}.$$

This implies

$$\|\psi[\phi]\|_{\nu+\delta(\sigma-\sigma_1),\sigma_1} \leq C\|\phi\|_{*,\nu,\sigma}$$

and therefore

$$\left|\int_{B_{2R_1}} B\psi[\phi]\cdot Z_{0j}\right| \leq C\lambda_*^\nu R^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}$$

and using (7.51)

$$\left|\int_{B_{2R_1}} B\psi[\phi]\cdot Z_{0j}\right| C\lambda_*^\nu R^{\sigma_1-\sigma}\|h\|_{\nu,2+\sigma}.$$

We have for $|y| \leq 2R^\delta$

$$|\psi[h](y,t)| + (1+|y|)|\nabla_y \psi[h](y,t)| \leq C R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

Then for $|y| \leq 2R^\delta$ we have

$$|B\,|\psi[h]| \leq C\lambda_*^\nu (1+|y|)^{-4} R_1^{-\sigma} \|h\|_{\nu,2+\sigma},$$

and hence

$$\left| \int_{B_{2R_1}} B\psi[h]\rho w_\rho \right| \leq C\lambda_*^\nu R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

We would like to have the orthogonality condition defined as an integral in $\mathbb{R}^2$. Note that

$$\begin{aligned}\left| \int_{(B_{2R^\delta})^c} h\rho w_\rho \right| &\leq C\|h\|_{\nu,2+\sigma}\lambda_*^\nu \int_{(B_{2R^\delta})^c} \frac{1}{(1+|y|)^{3+\sigma}}\, dy \\ &\leq C\|h\|_{\nu,2+\sigma}\lambda_*^\nu R^{-\delta(1+\sigma)}.\end{aligned}$$

Then, going back to the original notation, we get

$$c_{0j}[h] = -\frac{\int_{\mathbb{R}^2} h\cdot Z_{0j}}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} - G[h]$$

where G satisfies (7.38). □

7.6 Lipschitz bounds with respect to λ

Let us consider the linear operator we constructed in Proposition 7.1 as a solution $\phi[h] = \mathcal{T}_{\lambda,1}[h]$ of problem (7.1),

$$\begin{aligned}&\lambda^2 \partial_t \phi = L_W[\phi] + h(y,t) \quad \text{in } \mathcal{D}_{2R}\\ &\phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)}\\ &\phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\end{aligned}$$

where $\mathcal{D}_{2R} = \{(y,t)\ /\ t \in (0,T),\ \ y \in B_{2R(t)}(0)\}$, and we assume $h \cdot W = 0$ in $\mathcal{D}_{2R}$. The purpose in this section is find estimates for directional derivatives of the operator $\mathcal{T}_{\lambda,1}[h]$ with respect to the parameter function λ. Examining the construction of $\mathcal{T}_{\lambda,1}[h]$ as the superposition of the unique solutions of different problems, it is not hard to see that the directional derivative

$$\phi_\lambda := (\partial_\lambda \mathcal{T}_{\lambda,1})[h][\lambda_1] = \frac{d}{ds}\mathcal{T}_{\lambda+s\lambda_1,1}[h]\Big|_{s=0}$$

satisfies the equation

$$\lambda^2\partial_t\phi_\lambda = L_W[\phi_\lambda] - 2\frac{\lambda_1}{\lambda}(L_W[\phi] + h(y,t)) \quad \text{in } \mathcal{D}_{2R}$$
$$\phi_\lambda(\cdot,0) = 0 \quad \text{in } B_{2R(0)}$$

with $\phi = \mathcal{T}_{\lambda,1}[h]$. We will find estimates for this quantity inherited from those we have already established for ϕ. We assume that for some positive numbers a, b, c independent of T we have that

$$a\lambda_*(t) \le \lambda(t) \le b\lambda_*(t), \quad |\lambda_1(t)| \le c\lambda_*(t) \quad \text{for all} \quad t \in (0,T).$$

The following estimate holds.

Proposition 7.3 *The function ϕ_λ is well defined and satisfies the estimate*

$$(1+|y|)\left|\nabla_y\phi_\lambda(y,t)\right| + |\phi_\lambda(y,t)|$$
$$\lesssim \lambda_*^\nu \frac{R^{1+\frac{5-a}{2}}\log R}{1+|y|}\min\left\{\frac{R^{1+\frac{5-a}{2}}}{|y|^2}, 1\right\}\|h\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty \quad \textit{in } \mathcal{D}_{2R}.$$

Proof of Proposition 7.3 We recall that $\phi[h] = \mathcal{T}_{\lambda,1}[h]$ was constructed mode by mode. According to the decomposition (7.2), (7.3), (7.4), (7.5), we can write

$$\phi = \phi_0 + \phi_1 + \phi_{-1} + \phi^\perp, \quad h = h_0 + h_1 + h_{-1} + h^\perp,$$

where we can assume for $k = 0, 1$, $j = 1, 2$,

$$\int_{B_{2R}} h_k(y,t)\cdot Z_{kj}(y)dy = 0.$$

We will give the estimates for ϕ_λ in each mode separately, writing

$$\phi_\lambda = \phi_{0\lambda} + \phi_{1\lambda} + \phi_{-1\lambda} + \phi_\lambda^\perp.$$

We will estimate each of the terms $\phi_{0\lambda}, \phi_{1\lambda}, \phi_{-1\lambda}, \phi_\lambda^\perp$ separately. □

First we give some estimates for the equation in entire space with some suitable right hand side.

Lemma 7.10 *Let ϕ be the solution of*

$$\partial_\tau \phi = \Delta_y \phi + g(y,\tau) \quad \text{in } \mathbb{R}^2 \times (0,\infty)$$
$$\phi(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2$$

given by Duhamel's formula. The following holds: let $\nu \in (0,1)$, $a \in (2,3)$. Assume that $g(y,\tau) = \mathrm{div}_y G(y,\tau)$ where $|G(y,\tau)| \le \frac{1}{(1+\tau^\nu)(1+|y|^{a-1})}$. Then

$$|\phi(y,\tau)| \le \frac{C(1+\log_+ \tau))}{(1+\tau^\nu)(1+|y|^{a-2})}$$

where $\log_+ \tau = \max(0, \log \tau)$. *If instead, g satisfies $|g(y,\tau)| \le \frac{1}{(1+\tau^\nu)(1+|y|^a)}$, then*

$$|\phi(y,\tau)| \le \frac{C(1+\log_+ \tau))}{1+\tau^\nu}.$$

Proof The proof of the first estimate directly follows from the representation formula

$$\begin{aligned}\phi(y,\tau) &= \frac{1}{4\pi}\int_0^\tau \frac{1}{\tau - s}\int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \mathrm{div}_z G(z,s)\, dz\, ds\\ &= C\int_0^\tau \frac{1}{\tau-s}\int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \frac{y-z}{\tau-s}\cdot G(z,s)\, dz\, ds\end{aligned}$$

The second estimate is treated similarly. □

7.7 Mode 0: estimate of $\phi_{0\lambda}$

We claim that

$$\begin{aligned}&(1+|y|)|\nabla \phi_{0\lambda}(y,t)| + |\phi_{0\lambda}(y,t)|\\ &\lesssim \lambda_*^\nu \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu} \frac{R^{1+\frac{5-a}{2}}\log R}{|y|+1}\begin{cases}1 & \text{if } |y| < R^{\frac{1}{2}}\\ \frac{R}{|y|^2} & \text{if } |y| > R^{\frac{1}{2}}.\end{cases}\end{aligned} \tag{7.52}$$

Proof We refer to the notation in the proof of Lemma 7.3 on the construction of ϕ_0. We recall that $\phi_0 = L_W[\Phi_0]$ where Φ_0 is the unique solution of the problem (7.19),

$$\lambda^2 \Phi_t = L_W[\Phi] + H_0(y,t) \quad \text{in } \mathcal{D}_{4R},$$
$$\Phi(y,0) = 0 \quad \text{in } B_{4R}(0)$$

$$\Phi(y,\tau) = 0 \quad \text{for all } t \in (0,T), \quad y \in \partial B_{4R(0)}(0).$$

Then $\phi_{0\lambda} = L_W[\Phi_{0\lambda}]$ where $\Phi_{0\lambda}$ solves

$$\lambda^2 \partial_t \Phi_{0\lambda} = L_W[\Phi_{0\lambda}] - 2\frac{\lambda_1}{\lambda}(\phi_0 + H_0(y,t)) \quad \text{in } \mathcal{D}_{4R}, \tag{7.53}$$
$$\Phi_{0\lambda}(y,0) = 0 \quad \text{in } B_{4R}(0)$$
$$\Phi_{0\lambda}(y,\tau) = 0 \quad \text{for all } t \in (0,T), \quad y \in \partial B_{4R(0)}(0).$$

We recall that we obtained

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{5-a} \lambda_*(t)^\nu (1+|y|)^{-3},$$

and a posteriori the better estimate

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} \frac{R^{\frac{5-a}{2}}\lambda_*^\nu}{1+|y|} \begin{cases} 1 & \text{if } |y| \le R^{\frac{5-a}{4}}, \\ \frac{R^{\frac{5-a}{2}}}{|y|^2} & \text{if } |y| > R^{\frac{5-a}{4}}. \end{cases}$$

The use of an explicit barrier in (7.53) then yields

$$|\Phi_{0\lambda}| \lesssim \lambda_*^\nu \|h_0\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \frac{R^{\frac{5-a}{2}+2}\log R}{1+|y|}$$

and then, arguing similarly as in the construction of ϕ_0 we obtain the estimate for $\phi_{0\lambda} = L_W[\Phi_{0\lambda}]$,

$$|\phi_{0\lambda}(y,t)| \lesssim \lambda_*^\nu \|h_0\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \frac{R^{\frac{5-a}{2}+2}\log R}{1+|y|^3}. \tag{7.54}$$

Next we want to improve this estimate, as was done in Lemma 7.3. We have that $\phi_{0\lambda}$ satisfies the equation

$$\lambda^2 \partial_t \phi_{0\lambda} = L_W[\phi_{0\lambda}] + g(y,t)$$

where

$$g = -2\frac{\lambda_1}{\lambda}(L_W[\phi_0] + h_0(y,t)). \tag{7.55}$$

We have that $g(y,t) = \mathrm{div}_y\, G_0(y,t) + G_1(y,t)$ in $\mathcal{D}_{4R}$, where

$$(1+|y|)|G_1(y,t)| + (1+|y|)^{\alpha}[G_0]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} + |G_0(y,t)| \\ \lesssim \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \|h_0\|_{a,\nu} \frac{R^{\frac{5-a}{2}}\lambda_*^{\nu}}{1+|y|^2} \begin{cases} 1 & \text{if } |y| \le R^{\frac{5-a}{4}}, \\ \frac{R^{\frac{5-a}{2}}}{|y|^2} & \text{if } |y| > R^{\frac{5-a}{4}}. \end{cases} \tag{7.56}$$

We write

$$\phi_{0\lambda} = \phi_b + \phi_c$$

where ϕ_b is given by the Duhamel formula

$$\phi_b(y,t) = \int_0^\tau \frac{1}{4\pi(\tau-s)} ds \int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} g(z, t_\lambda(s))\, dz$$

with g given by (7.55) and τ by (7.14), and let ϕ_c solve

$$\begin{cases} \lambda^2 \partial_t \phi_c = L_W[\phi_c] + |\nabla W|^2 \phi_b + 2(\nabla W \cdot \nabla \phi_b) W & \text{in } \mathcal{D}_{4R} \\ \phi_c(\cdot, t) = -\phi_b \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t \in (0,T), \\ \phi_c(\cdot, 0) = 0 \quad \text{in } B_{4R(0)}. \end{cases}$$

Using Lemma 7.10 we find that

$$|\phi_b(y,t)| + (1+|y|)|\nabla \phi_b(y,t)| \lesssim \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^{\nu} R^{\frac{5-a}{2}} \log R \tag{7.57}$$

for $|y| \le 5R$. The above estimate implies that

$$|\nabla W|^2 |\phi_b| + 2\,|(\nabla W \cdot \nabla \phi_b) W| \lesssim \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^{\nu} R^{\frac{5-a}{2}} \log R (1+|y|)^{-3} \tag{7.58}$$

Let φ_c be the complex valued function defined by

$$\phi_c(y,t) \;=\; \mathrm{Re}\,(\varphi_c(\rho,t))\, E_1 + \mathrm{Im}\,(\varphi_c(\rho,t))\, E_2$$

so that using the notation in (7.8), φ_c satisfies the equation

$$\begin{cases} \lambda^2 \partial_t \varphi_c = \mathcal{L}_0[\varphi_c] + \tilde{g}_c(\rho,t) & \text{in } \tilde{D}_{4R}, \\ \varphi_c(0,\rho) = 0 \quad \text{in } (0, 4R), \end{cases} \tag{7.59}$$

where by (7.58) $\tilde{g}_c$ satisfies

$$|\tilde{g}_c| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^\nu R^{\frac{5-a}{2}} \log R (1+|y|)^{-3}.$$

We can find an explicit supersolution for the real and imaginary parts of equation (7.59) in $\tilde{D}_{R^{1/2}}$ of the form

$$\bar{\varphi}_c = d(t) Z_0(\rho) \int_\rho^{2R^{1/2}} \frac{1}{Z_0(r)^2 r} \int_0^r Z_0(s)(1+s)^{-3} s\, ds dr$$

where $d(t) = \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^\nu R^{\frac{5-a}{2}} \log R$ and Z_0 is defined in (7.10). We note that at $\rho = R^{1/2}$ the value of ϕ_c satisfies, by (7.54) and (7.57)

$$\begin{aligned} |\phi_c(R^{1/2}, t)| &\le |\phi_{0\lambda}(R^{1/2}, t)| + |\phi_b(R^{1/2}, t)| \\ &\lesssim \lambda_*^\nu \|h_0\|_{a,\nu} \left\|\frac{\lambda_1}{\lambda}\right\|_\infty R^{\frac{6-a}{2}} \log R \end{aligned}$$

and on the other hand

$$|\bar{\phi}_c(R^{1/2}, t)| \ge c \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^\nu R^{\frac{5-a}{2}} \log R R^{1/2}$$

for some $c > 0$. This yields

$$|\phi_c(y, t)| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^\nu R^{\frac{5-a}{2}+1} \log R (1+|y|)^{-1} \quad |y| < R^{1/2}.$$

and combining with (7.57) we get

$$|\phi_{0\lambda}(y, t)| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\, \lambda_*^\nu R^{\frac{5-a}{2}+1} \log R (1+|y|)^{-1} \quad |y| < R^{1/2}.$$

Using Schauder estimates together with (7.56) we obtain (7.52). □

7.8 Mode 1: estimate of $\phi_{1\lambda}$

From a similar argument we obtain the following estimate.

$$\begin{aligned} &(1+|y|)|\nabla_y \phi_{1\lambda}(y, t)| + |\phi_{1\lambda}(y, t)| \\ &\quad \le C \lambda_*(t)^\nu (1+|y|)^{2-a} \|h\|_{a,\nu} \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \quad \text{in } \mathcal{D}_R. \end{aligned}$$

7.9 Estimate of $\phi_\lambda^\perp$ and $\phi_{-1\lambda}$

We claim that for any $\sigma \in (0, 1)$ we have

$$
\begin{aligned}
&(1+|y|)|\nabla \phi_\lambda^\perp(y,t)| + |\phi_\lambda^\perp(y,t)| \\
&\quad \lesssim \lambda_*(t)^\nu R^{a-2} \log R (1+|y|)^{2-a} \|h^\perp\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \\
&(1+|y|)|\nabla \phi_{-1\lambda}(y,t)| + |\phi_{-1\lambda}^\perp(y,t)| \\
&\quad \lesssim \lambda_*(t)^\nu R^{a-2+\sigma} (1+|y|)^{2-a} \|h_{-1}\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty .
\end{aligned}
$$

8 The λ-ω system

In this section we prove Proposition 6.5, on approximate solvability of the equation

$$\mathcal{B}_0[p](t) = a(t), \quad t \in [0, T),$$

where $\mathcal{B}_0$ is the operator defined in (5.6) and $a : [0, T] \to \mathbb{C}$ is a given continuous function. We will also derive Lipschitz estimates that will be crucial in solving for the final adjustment of parameters p, ξ by a fixed point argument in the next section.

Consistently with the discussion in Sect. 5, we assume that $\frac{1}{C_1} \le |a(T)| \le C_1$ for some C_1 independent of T. We will construct an operator $\mathcal{P}$ that to a function a in a suitable class assigns $p = \mathcal{P}[a]$ such that

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad \text{in } [0, T). \tag{8.1}$$

so that $\mathcal{R}_0[a](t)$ is a suitably small.

We construct the function p in Proposition 6.5 by linearization, and the first approximation is a function p_κ that deals with the case of constant a.

First we introduce some notation. We work with $\kappa \in \mathbb{C}$ and let $p_{0,\kappa}$ be the function

$$p_{0,\kappa}(t) = \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2} \, ds, \quad t \le T, \tag{8.2}$$

so that

$$\dot{p}_{0,\kappa}(t) = -\frac{\kappa |\log T|}{|\log(T-t)|^2}. \tag{8.3}$$

We will always assume that for a large, fixed constant C_1 we have

$$\frac{1}{C_1} \le |\kappa| \le C_1, \tag{8.4}$$

so that we also have $\tilde{C}^{-1}\lambda_* \le |p_{0,\kappa}| \le \tilde{C}\lambda_*$. The first term in the function p constructed in Proposition 6.5 is a function close to $p_{0,\kappa}$ that actually more or less solves (8.1) in the case that a is constant.

Lemma 8.1 *Given $\kappa \in \mathbb{C}$ satisfying (8.4), there is a function $p_\kappa : [-T, T] \to \mathbb{C}$, a constant $c(\kappa) \in \mathbb{C}$, and $\mathcal{R}_1(\kappa)(t)$ such that*

$$\mathcal{B}_0[p_\kappa](t) = c(\kappa) + \mathcal{R}_1(\kappa)(t) \tag{8.5}$$

for $t \in [0, T]$, where $\mathcal{R}_1(\kappa)(t)$ satisfies

$$|\mathcal{R}_1(\kappa)(t)| \le C\lambda_*^{\alpha_0} \tag{8.6}$$

for some $\alpha_0 > 0$.

We have additional estimates for p_κ and the remainder $\mathcal{R}_1(\kappa)$ constructed above. The function p_κ can be decomposed as

$$p_\kappa = p_{0,\kappa} + p_{1,\kappa}.$$

Here $p_{0,\kappa}$ is defined in (8.2). The function $p_{1,\kappa}$ satisfies: given $k \in (1, 2)$ there is C such that

$$\|p_{1,\kappa}\|_{*,k+1} \le C|\log T|^{k-1}\log^2(|\log T|) \tag{8.7}$$

and

$$\|p_{1,\kappa_1} - p_{1,\kappa_2}\|_{*,k+1} \le C|\log T|^{k-1}\log^2(|\log T|)|\kappa_1 - \kappa_2| \tag{8.8}$$

for κ_1, κ_2 satisfying (8.4), where the norm $\| \ \|_{*,k}$ is defined for $g \in C([-T, T]; \mathbb{C}) \cap C^1([-T, T); \mathbb{C})$ with

$$g(T) = 0$$

and $k > 0$ by

$$\|g\|_{*,k} = \sup_{t\in[-T,T]} |\log(T - t)|^k |\dot{g}(t)|, \tag{8.9}$$

(here $\dot{g} = \frac{d}{dt}g$).

The remainder, satisfies together with (8.6) the estimate for the derivative in t:

$$\left|\frac{d}{dt}\mathcal{R}_1(\kappa)(t)\right| \le C\lambda_*^{\alpha_0-1} \tag{8.10}$$

and Lipschitz estimates

$$|\mathcal{R}_1(\kappa_1)(t) - \mathcal{R}_1(\kappa_2)(t)| \le C\lambda_*^{\alpha_0}|\kappa_1 - \kappa_2| \tag{8.11}$$

$$\left|\frac{d}{dt}\mathcal{R}_1(\kappa_1)(t) - \frac{d}{dt}\mathcal{R}_1(\kappa_2)(t)\right| \le C\lambda_*^{\alpha_0-1}|\kappa_1 - \kappa_2|, \tag{8.12}$$

for κ_1, κ_2 satisfying (8.4). The proof of Lemma 8.1 and estimates (8.7), (8.8), (8.10), (8.11), and (8.12) are in Sect. 8.3.

For the proof of Proposition 6.5 and Lemma 8.1 it will be useful to isolate the main part of the operator $\mathcal{B}_0$, defined in (5.6). Given the asymptotic expansion of Γ_l in (5.5) we write

$$\mathcal{B}_0[p] = \mathcal{I}[p] + \tilde{\mathcal{B}}[p],$$

where

$$\mathcal{I}[p] := \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}(s)}{t-s}\,ds, \qquad \tilde{\mathcal{B}}[p] := \tilde{\mathcal{B}}_1[p] + \tilde{\mathcal{B}}_2[p] - \operatorname{Re}(\dot{p}(t)), \tag{8.13}$$

where

$$\begin{aligned}
\tilde{B}_1[p](t) = e^{i\omega(t)}\Bigg[&\int_{-T}^{t-\lambda_*(t)^2} \frac{\operatorname{Re}(\dot{p}(s)e^{-i\omega(t)})}{t-s}\left(\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right) - 1\right) ds \\
&+ \int_{t-\lambda_*(t)^2}^{t} \frac{\operatorname{Re}(\dot{p}(s)e^{-i\omega(t)})}{t-s}\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right) ds\Bigg] \\
\tilde{B}_2[p](t) = ie^{i\omega(t)}\Bigg[&\int_{-T}^{t-\lambda_*(t)^2} \frac{\operatorname{Im}(\dot{p}(s)e^{-i\omega(t)})}{t-s}\left(\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right) - 1\right) ds \\
&+ \int_{t-\lambda_*(t)^2}^{t} \frac{\operatorname{Im}(\dot{p}(s)e^{-i\omega(t)})}{t-s}\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right) ds\Bigg]
\end{aligned}$$

and we use the notation $p(t) = \lambda(t)e^{i\omega(t)}$. To prove Proposition 6.5, we take p of the form

$$p = p_\kappa + p_2,$$

where p_κ is the function constructed in Lemma 8.1, for some $\kappa \in \mathbb{C}$ to be determined. The function $p_2(t)$ will have the property $p_2(t) = o(p_\kappa(t))$, as $t \to T$. We would like that

$$\mathcal{I}[p_\kappa](t) + \mathcal{I}[p_2](t) + \tilde{\mathcal{B}}[p_\kappa + p_2](t) \approx a(t). \tag{8.14}$$

Given $\alpha > 0$, let us decompose $\mathcal{I}[p] = S_\alpha[\dot{p}] + R_\alpha[\dot{p}]$ where S_α, R_α are defined as in (6.20), (6.21), that is The idea is to replace $\mathcal{I}[p_2]$ by $S_\alpha[\dot{p}_2]$ in (8.14) to make this equation more manageable, that is, we consider $S_\alpha[\dot{p}_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] + \mathcal{R}_1(\kappa) = a(t),\ t \in [0, T]$, where we have used (8.5). We introduce one more modification, so as to have a more convenient problem to treat. Let us split $S_\alpha[g] = L_0[g] + L_1[g]$ where

$$\begin{aligned} L_0[g] &= (1-\alpha)|\log(T-t)|g(t) \\ L_1[g] &= (4\log(|\log(T-t)|) - 2\log(\kappa) - 2\log(|\log(T)|))g(t) \\ &\quad + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s}\, ds. \end{aligned}$$

We actually introduce one more modification to (8.14). For this, it is convenient that a is defined in $[-T, T]$. So, given a function $a : [0, T] \to \mathbb{C}$ satisfying the hypotheses of Proposition 6.5, we extend a continuously by constant for $t \le 0$.

Let η be a smooth cut-off function such that $\eta(s) = 1 \quad \text{for } s \ge 0$, $\eta(s) = 0 \quad \text{for } s \le -\frac{1}{4}$. The equation that we are going to solve is the following one:

$$\begin{aligned} L_0[\dot{p}_2] + \eta\left(\frac{t}{T}\right) L_1[\dot{p}_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] \\ = a(t) - \mathcal{R}_1(\kappa) + c \quad \text{in } [-T, T] \end{aligned} \tag{8.15}$$

for some constant c. Later on we shall show that it is possible to adjust κ so that $c = 0$.

8.1 Construction of a solution to (8.15)

Since in (8.15) the terms $a(t)$ and $\mathcal{R}_1(\kappa)$ have similar behavior, we will consider just

$$L_0[\dot{p}_2] + \eta\left(\frac{t}{T}\right) L_1[\dot{p}_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] = a(t) + c \quad \text{in } [-T, T] \tag{8.16}$$

Consider the norm $\|\ \|_{\mu,l}$ defined in (6.15).

Lemma 8.2 *Let $\mu, \alpha \in (0, \frac{1}{2})$ and $l \in \mathbb{R}$. Assume that $\frac{1}{C_1} \le |a(T)| \le C_1$ and*

$$T^{\mu}|\log T|^{1+\sigma-l}\|a(\cdot)-a(T)\|_{\mu,l-1} \le C_1, \tag{8.17}$$

for some $\sigma > 0$ fixed. Then if $T > 0$ is small there is a solution p_2 to (8.16) for some $c \in \mathbb{C}$. Moreover this solution satisfies

$$\|\dot{p}_2\|_{\mu,l} \le C\|a(\cdot)-a(T)\|_{\mu,l-1}. \tag{8.18}$$

For the proof of this lemma we consider the linear equation

$$L_0[g] + \eta\left(\frac{t}{T}\right)L_1[g] = f + c \quad \text{in } [-T, T]. \tag{8.19}$$

We will assume that $f(T) = 0$, and hence $c = L_1[g](T)$ because all other terms in the equation vanish at T. Thanks to the cut-off function $\eta(\frac{t}{T})$, we need only to consider the values of $L_1[g](t)$ for $t \ge -\frac{T}{4}$. Then in the definition of $L_1[g]$, $t-(T-t)^{1+\alpha} \ge t - \frac{1}{2}(T-t) \ge -T$ of $T > 0$ is small.

For the right hand side of (8.19) we take the space $C([-T, T]; \mathbb{C})$ with $f(T) = 0$ and the norm $\|f\|_{\mu,l-1}$.

The next lemma asserts the solvability of (8.19) in the weighted spaces introduced above.

Lemma 8.3 *Let $\alpha \in (0, \frac{1}{2})$ and $T > 0$ be sufficiently small. Assume $\|f\|_{\mu,l-1} < \infty$ where $\mu \in (0, 1)$, $l \in \mathbb{R}$. Then for $T > 0$ small there is a solution $S[f]$ of (8.19) that defines a linear operator of f and such that*

$$\|S[f]\|_{\mu,l} \le C\|f\|_{\mu,l-1}. \tag{8.20}$$

Proof We consider (8.19) as a fixed point problem of the form

$$g = L_0^{-1}\left[f - \eta\left(\frac{t}{T}\right)(L_1[g](t) - L_1[g](T))\right],$$

where L_0^{-1} is defined the formula

$$L_0^{-1}[f](t) = \frac{1}{(1-\alpha)}\frac{f(t)}{|\log(T-t)|}.$$

It is clear that

$$\|L_0^{-1}[f]\|_{\mu,l} \le \frac{1}{1-\alpha}\|f\|_{\mu,l-1}. \tag{8.21}$$

and a calculation shows that

$$\|L_1[g](\cdot) - L_1[g](T)\|_{\mu,l-1} \le \Big(\alpha + \frac{C\log|\log T|}{|\log T|}\Big)\|g\|_{\mu,l}. \tag{8.22}$$

To estimate the integral term we decompose

$$\int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s}\,ds - \int_{-T}^{T} \frac{g(s)}{T-s}\,ds = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s}\,ds, \quad I_2 = \int_{-T}^{t-(T-t)/2} g(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right)ds,$$
$$I_3 = \int_{t-(T-t)/2}^{T} \frac{g(s)}{T-s}\,ds.$$

Then

$$\begin{aligned}|I_1| &\le \|g\|_{\mu,l} \int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha}} \frac{(T-s)^{\mu}}{|\log(T-s)|^{l}(t-s)}\,ds \\ &\le \|g\|_{\mu,l} \frac{(T-t)^{\mu}}{|\log(T-t)|^{l}}(\alpha|\log(T-t)| + C).\end{aligned}$$

and similarly

$$|I_2| \le C\|g\|_{\mu,l}\frac{(T-t)^{\mu}}{|\log(T-t)|^{l}}, \qquad |I_3| \le C\|g\|_{\mu,l}\frac{(T-t)^{\mu}}{|\log(T-t)|^{l}}.$$

These estimates imply (8.22). Then this inequality combined with (8.21) shows that

$$\begin{aligned}&\left\| L_0^{-1}\left[\eta\left(\frac{t}{T}\right)(L_1[g](t) - L_1[g](T))\right]\right\|_{\mu,l} \\ &\quad \le \frac{1}{1-\alpha}\Big(\alpha + \frac{C\log|\log T|}{|\log T|}\Big)\|g\|_{\mu,l}.\end{aligned}$$

Then for $\alpha \in (0, \frac{1}{2})$ and $T > 0$ sufficiently small this operator is a contraction and we obtain the conclusion of the lemma. □

Proof of Lemma 8.2 Let S denote the linear operator constructed in Lemma 8.3.

Then to find a solution to (8.16) it is sufficient to find a solution p_2 of the fixed point problem

$$p_2 = \mathcal{A}[p_2] \tag{8.23}$$

where $\tilde{p} = \mathcal{A}[p_2]$ is defined by $\tilde{p}(T) = 0$ and

$$\frac{d\tilde{p}}{dt} = S\Big[-\big(\tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa]\big) + a(t) - a(T)\Big].$$

Let $M_1 = C_0\|a(\cdot) - a(T)\|_{\mu,l-1}$, where C_0 is a sufficiently large fixed constant. We claim that if $T > 0$ is sufficiently small then $\mathcal{A}$ is a contraction in ball $\overline{B}_{M_1}$ of the space of complex valued functions $p_2 \in C^1([-T, T])$ with $p_2(T) = 0$ and with the norm $\|\dot{p}_2\|_{\mu,l}$. Note that with this norm we have

$$|p_2(t)| \le C\|\dot{p}_2\|_{\mu,l}\frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}.$$

In particular, thanks to (8.17), if $\|\dot{p}_2\|_{\mu,l} \le M_1$, then

$$\left|\frac{p_2}{\lambda_*}\right| + \left|\frac{\dot{p}_2}{\dot{\lambda}_*}\right| \ll 1$$

for $T > 0$ small.

Let us verify that $\mathcal{A}$ maps $\overline{B}_{M_1}$ into itself. Let $p_2 \in \overline{B}_{M_1}$. By (8.20) we have

$$\|\mathcal{A}[p_2]\|_{\mu,l} \le C\Big(\|\tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa]\|_{\mu,l-1} + \|a(\cdot) - a(T)\|_{\mu,l-1}\Big). \tag{8.24}$$

After some computations, we can check the validity of the following estimate: for $p_1, p_2 \in \overline{B}_{M_1}$ we have

$$\|\tilde{\mathcal{B}}[p_\kappa + p_1] - \tilde{\mathcal{B}}[p_\kappa + p_2]\|_{\mu,l-1} \le C\frac{1}{|\log T|}\|\dot{p}_1 - \dot{p}_2\|_{\mu,l}. \tag{8.25}$$

Assuming for now this estimate let us continue with proving that $\mathcal{A}$ maps $\overline{B}_{M_1}$ into itself. Let $p_1 \in \overline{B}_{M_1}$. By (8.24) and (8.25)

$$\|\mathcal{A}[p_2]\|_{\mu,l} \le C\frac{M_1}{|\log T|} + C\|a(\cdot) - a(T)\|_{\mu,l-1} \le M_1,$$

if $T > 0$ is small. Also thanks to (8.20) and (8.25) we see that $\mathcal{A}$ is a contraction in $\overline{B}_{M_1}$. This finishes the proof of the lemma. □

We also have a Lipschitz property of the solution constructed in Lemma 8.2.

Lemma 8.4 *Let* $\mu, \alpha \in (0, \frac{1}{2})$ *and* $l \in \mathbb{R}$. *Assume that for* $j = 1, 2$, a_j *satisfies* $\frac{1}{C_1} \leq |a_j(T)| \leq C_1$ *and* (8.17), *and let* κ_1, κ_2 *satisfy* (8.4). *Then for* $T > 0$ *is small the solution* $p_2[a, \kappa]$ *to* (8.16) *constructed in Lemma* 8.2 *satisfies*

$$\begin{aligned} \|\dot{p}_2[a_1, \kappa_1] - \dot{p}_2[a_2, \kappa_1]\|_{\mu,l} &\leq C\|a_1(\cdot) - a_1(T) - (a_2(\cdot) - a_2(T))\|_{\mu,l-1} \\ \|\dot{p}_2[a_1, \kappa_1] - \dot{p}_2[a_1, \kappa_2]\|_{\mu,l} &\leq C\|a_1(\cdot) - a_1(T)\|_{\mu,l-1}|\kappa_1 - \kappa_2|. \end{aligned} \tag{8.26}$$

8.2 Hölder estimate of the solution

We will show in this section that the solution constructed in Lemma 8.2 has some Hölder regularity inherited from the one of a.

We then have the following result, where the Hölder semi norm $[\ ]_{\gamma,m,l}$ is defined in (6.16).

Lemma 8.5 *Let* $\alpha \in (0, \frac{1}{2})$, $\mu, \gamma \in (0, 1)$, $m \leq \mu - \gamma$, $l \in \mathbb{R}$. *Assume that* $\frac{1}{C_1} \leq |a(T)| \leq C_1$ *and*

$$T^{\mu}|\log T|^{1+\sigma-l}\|a(\cdot) - a(T)\|_{\mu,l-1} + [a]_{\gamma,m,l-1} \leq C_1,$$

for some $\sigma > 0$. *Then the solution* p_2 *constructed in Lemma* 8.2 *satisfies*

$$\begin{aligned} [\dot{p}_2]_{\gamma,m,l} \lesssim\ & \frac{T^{\mu}}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\mu,l-1} \\ & + [a(\cdot) - a(T)]_{\gamma,m,l-1}. \end{aligned}$$

The proof follows from the fixed point representation (8.23) and estimates in the weighted Hölder norms for the operators involved there.

We will also need a Lipschitz estimate of p_2 as a function of κ and $a(t)$ in the semi norm $[\ ]_{\gamma,m,l}$.

Lemma 8.6 *Let* $\alpha \in (0, \frac{1}{2})$, $\mu, \gamma \in (0, 1)$, $m \leq \mu - \gamma$, $l \in \mathbb{R}$. *Assume that for* $j = 1, 2$, *we have* $\frac{1}{C_1} \leq |a_j(T)| \leq C_1$ *and*

$$T^{\mu}|\log T|^{1+\sigma-l}\|a_j(\cdot) - a_j(T)\|_{\mu,l-1} + [a_j]_{\gamma,m,l-1} \leq C_1,$$

for some $\sigma > 0$, *and that* κ_1, κ_2 *satisfy* (8.4). *Then the solution* $p_2 = p_2[a, \kappa]$ *constructed in Lemma* 8.2 *satisfies*

$$\begin{aligned}
&[\dot p_2[a_1,\kappa_1]-\dot p_2[a_2,\kappa_1]]_{\gamma,m,l}\\
&\quad\lesssim [a_1-a_2]_{\gamma,m,l-1}\\
&\qquad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1},
\end{aligned}$$

and

$$[\dot p_2[a_1,\kappa_1]-\dot p_2[a_1,\kappa_2]]_{\gamma,m,l}\le C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|\kappa_1-\kappa_2|.$$

Proof of Proposition 6.5 By Lemma 8.2 there is p_2 satisfying (8.15), where we have used this lemma with a replaced by $a-\mathcal{R}_1(\kappa)$, with $\mathcal{R}_1(\kappa)$ being the remainder appearing in (8.5).

Note that by (8.6) and using the assumption $\Theta<\alpha_0$, we have

$$\|\mathcal{R}_1(\kappa)\|_{\Theta,l-1}\le T^{\alpha_0-\Theta}|\log T|^{l-1}. \tag{8.27}$$

Therefore from (8.18) we find

$$\|\dot p_2\|_{\Theta,l}\le C\big(T^{\alpha_0-\Theta}|\log T|^{l-1}+\|a(\cdot)-a(T)\|_{\Theta,l-1}\big).$$

In equation (8.15) the constant c depends on κ and we claim that it is possible to choose κ satisfying (8.4) such that $c=0$. Evaluating (8.15) at $t=T$ we find

$$\int_{-T}^{T}\frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds=a(T)+c.$$

We consider then the equation $c=0$ with κ as an un known, that is, we look for κ satisfying

$$\int_{-T}^{T}\frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds=a(T). \tag{8.28}$$

Using (8.2), (8.8) and (8.26) we see that

$$\int_{-T}^{T}\frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds=\kappa+\tilde f(\kappa)$$

where $\tilde f$ satisfies

$$|\tilde f(\kappa_1)-\tilde f(\kappa_2)|\le\frac{C}{|\log T|}|\kappa_1-\kappa_2|$$

for κ_1, κ_2 satisfying (8.4). It follows that there exists a unique κ so that (8.28) holds. Moreover

$$\kappa = a(T)\left(1 + O\left(\frac{1}{|\log T|}\right)\right)$$

as $T \to 0$.

Now let us prove the estimate (6.19). For this we note that what we left out in (8.15) is $R_\alpha[\dot{p}_2]$. In other words, the remainder $\mathcal{R}_0[a]$ is just $R_\alpha[\dot{p}_2]$. By Lemma 8.5 we have

$$\begin{aligned}[\dot{p}_2]_{\gamma,m,l} &\le C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\Theta,l-1} \\ &\quad + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|\mathcal{R}_1(\kappa)\|_{\Theta,l-1} \\ &\quad + C[a(\cdot) - a(T)]_{\gamma,m,l-1} + C[\mathcal{R}_1(\kappa)]_{\gamma,m,l-1}.\end{aligned}$$

Using (8.10) we see that for $s \le t$ in $[0, T]$ such that $t - s \le \frac{1}{10}(T - t)$ we have

$$\frac{|\mathcal{R}_1(t) - \mathcal{R}_1(s)|}{(t-s)^\gamma} \le \lambda_*(t)^{\alpha_0-\gamma}$$

and since $m \le \Theta - \gamma$, $\Theta < \alpha_0$ by hypothesis we get

$$[\mathcal{R}_1(\kappa)]_{\gamma,m,l-1} \le C\lambda_*(0)^\sigma$$

for some $\sigma > 0$. From this and (8.27) we obtain

$$\begin{aligned}[\dot{p}_2]_{\gamma,m,l} &\lesssim T^\sigma + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\Theta,l-1} \\ &\quad + [a]_{\gamma,m,l-1},\end{aligned}$$

for some $\sigma > 0$. Then

$$\begin{aligned}|R_\alpha[\dot{p}_2]| &\le \int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*(t)^2} \frac{|\dot{p}_2(t) - \dot{p}_2(s)|}{t-s}\,ds \\ &\le C\Big(T^\sigma + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\Theta,l-1} \\ &\quad + [a]_{\gamma,m,l-1}\Big)\cdot\frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}.\end{aligned}$$

□

8.3 Proof of Lemma 8.1

To do this we look for p_κ of the form

$$p_\kappa = p_{0,\kappa} + p_1,$$

where $p_{0,\kappa}$ is defined in (8.2), and we would like

$$\mathcal{I}[p_{0,\kappa}] + \mathcal{I}[p_1] + \tilde{\mathcal{B}}[p_{0,\kappa} + p_1](t) - c(\kappa) = O((T-t)^{\alpha_0}) \quad \text{for } t \in [0, T]. \tag{8.29}$$

The idea is to replace in (8.29) the operator $\mathcal{I}[p_1]$ by $S_{\alpha_0}[\dot{p}_1]$ defined in (6.20) and try to solve the corresponding equation. We claim that if $\alpha_0 > 0$ is small, then we can find p_1 such that

$$\mathcal{I}[p_{0,\kappa}] + S_{\alpha_0}[\dot{p}_1] + \tilde{\mathcal{B}}[p_{0,\kappa} + p_1](t) - c(\kappa) = 0 \quad \text{in } [0, T], \tag{8.30}$$

for some $c(\kappa)$. This means that instead of (8.29) we have obtained

$$\mathcal{B}_0[p_{0,,\kappa} + p_1] - c(\kappa) = R_{\alpha_0}[\dot{p}_1] \quad \text{in } [0, T].$$

The second step is to prove that there is κ such that $c(\kappa) = A$. The final step is to show that

$$|R_{\alpha_0}[\dot{p}_1]| \le C(T-t)^{\alpha_0},$$

and this implies (8.29).

Construction of a solution to (8.30)

To obtain a function p satisfying (8.30) we formulate a fixed point problem as follows.

We decompose

$$S_{\alpha_0}[g] = \tilde{L}_0[g] + \tilde{L}_1[g]$$

where

$$\tilde{L}_0[g](t) = (1-\alpha_0)|\log(T-t)|g(t) + \int_{-T}^{t} \frac{g(s)}{T-s}\, ds$$

and $\tilde{L}_1$ contains all other terms, that is,

$$\begin{aligned}\tilde{L}_1[g](t) &= \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_0}} \frac{g(s)}{t-s}\,ds - \int_{t-(T-t)}^{t} \frac{g}{T-s}\,ds \\ &+ \int_{-T}^{t-(T-t)} g(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds \\ &+ (4\log(|\log(T-t)|) - 2\log(|\log(T)|))g(t).\end{aligned}$$

Given a continuous function f in $[-T, T]$ with a certain modulus of continuity at T, we would like to find g such that

$$S_{\alpha_0}[\dot{g}] = f \quad \text{in } [-T, T].$$

We will not quite obtain this, but we will solve a modified version of this equation. Let η be a smooth cut-off function such that

$$\eta(s) = 1 \quad \text{for } s \geq 0, \quad \eta(s) = 0 \quad \text{for } s \leq -\frac{1}{4}. \tag{8.31}$$

We will be able to find a function g such that

$$\tilde{L}_0[\dot{g}] + \eta\left(\frac{t}{T}\right)\tilde{L}_1[\dot{g}] = f + c \quad \text{in } [-T, T]. \tag{8.32}$$

We use the norm $\| \ \|_{*,k}$ defined in (8.9) for the solution g of the above equation. For the right hand side of (8.32) we take the space $C([-T, T]; \mathbb{C})$ with $f(T) = 0$ and the norm

$$\|f\|_{**,k} = \sup_{t\in[-T,T]} |\log(T-t)|^k |f(t)|. \tag{8.33}$$

Note that in (8.32) the expression $\eta(\frac{t}{T})\tilde{L}_1[\dot{g}](t)$ is well defined for g of class C^1 in $[-T, T)$. Indeed, because of the cut-off function, $\tilde{L}_1[\dot{g}](t)$ needs to be computed only for $t \geq -\frac{T}{4}$, and for $t \geq -\frac{T}{4}$ the integrals appearing in $L_1[\dot{g}]$ are well defined, since they start at either at $-T$ or $t - \frac{1}{2}(T-t) = \frac{3}{2}t - \frac{1}{2}T \geq -T$.

The next lemma gives the solvability of (8.32) in the weighted spaces introduced above. Let

$$\Upsilon = \frac{2-\alpha_0}{1-\alpha_0}$$

Lemma 8.7 *Let $C_2 > 1$ be fixed, κ satisfying (8.4), and assume that $k > \Upsilon - 1$. Then, there is $\bar{\alpha}_0 > 0$, so that for $0 < \alpha_0 \le \bar{\alpha}_0$, and $T > 0$ small, there is a linear operator T_1 such that $g = T_1[f]$ satisfies (8.32) for some constant c and*

$$\|g\|_{*,k+1} + |c| \le \frac{C}{k+1-\Upsilon} \|f\|_{**,k}. \tag{8.34}$$

The constant C is independent of T, α_0.

Let

$$\begin{aligned} E(t) &:= \mathcal{I}[p_{0,\kappa}](t), \\ \tilde{E}(t) &= E(t) - E(T), \end{aligned} \tag{8.35}$$

where $\mathcal{I}$ is given by (8.13), and consider the fixed point problem

$$p_1 = \mathcal{A}[p_1] \tag{8.36}$$

where

$$\mathcal{A}[p_1] = T_1\big[-\eta\tilde{E} - \tilde{\mathcal{B}}[p_{0,\kappa} + p_1]\big], \tag{8.37}$$

where η is the cut-off function defined in (8.31).

Note that if p_1 is a solution of (8.36) then p_1 satisfies

$$\tilde{L}_0[\dot{p}_1] + \eta\left(\frac{t}{T}\right)\tilde{L}_1[\dot{p}_1] = \eta\tilde{E} - \tilde{\mathcal{B}}[p_{0,\kappa} + p_1](t) + c$$

in $[-T, T]$ for some constant c. This implies that p_1 satisfies

$$S_{\alpha_0}[\dot{p}_1] + \tilde{\mathcal{B}}[p_{0,\kappa} + p_1] - E = c$$

in $[0, T]$ for some possibly different constant c. This is precisely the equation (8.30).

Proposition 8.1 *Let $k > 0$, $k < 2$ close to 2 and $\alpha_0 > 0$ small. Then for $T > 0$ small there is a function p_1 satisfying (8.36) and moreover*

$$\|p_1\|_{*,k+1} \le M$$

where

$$M = C_0 |\log(T)|^{k-1} \log(|\log(T)|)^2, \tag{8.38}$$

with C_0 a fixed large constant.

Moreover, if we denote by $p_1(\kappa)$ the solution just constructed, we have, for κ_1, κ_2 satisfying (8.4)

$$\|p_1(\kappa_1) - p_1(\kappa_2)\|_{*,k+1} \le C|\log T|^{k-1}\log(|\log T|)^2\,|\kappa_1 - \kappa_2|. \tag{8.39}$$

The rest of the subsection is devoted to the proof of Proposition 8.1.

We start with the construction of the linear operator T_1 in Lemma 8.7. We want to find an inverse for $\tilde{L}_0$, namely given f find g such that $\tilde{L}_0[\dot{g}] = f$. To do this, we differentiate this equation and we get

$$\ddot{g}(t) + \frac{2-\alpha_0}{1-\alpha_0}\frac{\dot{g}(t)}{(T-t)|\log(T-t)|} = \frac{1}{1-\alpha_0}\frac{\dot{f}(t)}{|\log(T-t)|}. \tag{8.40}$$

Then we can write a particular solution for $\dot{g}$ to (8.40) as

$$\begin{aligned}\dot{g}(t) = {} & \frac{f(t)}{(1-\alpha_0)|\log(T-t)|} \\ & + \frac{\Upsilon - 1}{1-\alpha_0}|\log(T-t)|^{-\Upsilon}\int_t^T \frac{|\log(T-s)|^{\Upsilon-2}}{T-s} f(s)\,ds,\end{aligned} \tag{8.41}$$

where $\Upsilon = \frac{2-\alpha_0}{1-\alpha_0}$ and where we have assumed that $\frac{|\log(T-s)|^{\Upsilon-2}}{T-s} f(s)$ is integrable near T (for example $f(s) = O(|\log(T-s)|^{-k})$ with $k > \Upsilon - 1$ suffices).

Define the operator

$$T_0[f] = g, \tag{8.42}$$

where g is such that $\dot{g}$ is given by (8.41) and $g(T) = 0$. Note that $g = T_0[f]$ solves (8.40) and therefore

$$\tilde{L}_0[\dot{g}] = f + c,$$

for some constant c.

Lemma 8.8 *Assume $k > \Upsilon - 1$. Then for $f \in C([-T, T]; \mathbb{C})$ with $f(T) = 0$*

$$\|T_0[f]\|_{*,k+1} \le \frac{C}{k+1-\Upsilon}\|f\|_{**,k}.$$

The constant is independent of Υ (if Υ is bounded), k, T.

Proof This is direct from (8.41). □

Proof of Lemma 8.7 We construct g as a solution of the fixed point problem

$$g = T_0\left[f - \eta\left(\frac{t}{T}\right)\tilde{L}_1[g]\right].$$

where T_0 is the operator constructed in (8.42) and η is the cut-off function (8.31).

By Lemma 8.8

$$\|T_0[\tilde{L}_1[g]]\|_{*,k+1} \le \frac{C}{k+1-\Upsilon}\|\tilde{L}_1[g]\|_{**,k}.$$

A computations shows that

$$\|T_0[\tilde{L}_1[g]]\|_{*,k+1} \le \frac{C}{k+1-\Upsilon}\left(\alpha_0 + \frac{1}{|\log T|} + \frac{\log|\log T|}{|\log T|}\right)\|g\|_{*,k+1}.$$

we get a contraction if $\alpha_0 > 0$ is fixed small and then $T > 0$ is sufficiently small. □

Next we need an estimate for the error E defined in (8.35).

Lemma 8.9 *Let $p_{0,\kappa}$ be given by (8.2) and assume $\kappa \in \mathbb{C}$ satisfies (8.4). Then*

$$|E(t) - E(T)| \le C\frac{|\log T|\log|\log(T-t)|}{|\log(T-t)|^2}, \quad -\frac{T}{4} \le t \le T. \tag{8.43}$$

Proof By definition we have

$$E(t) = \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}_{0,\kappa}(s)}{t-s}\,ds.$$

Let $t \in [-\frac{T}{4}, T]$ and let us write

$$\begin{aligned} E(t) &= \int_{-T}^{t} \frac{\dot{p}_{0,\kappa}(s)}{T-s}\,ds - \int_{t-(T-t)/5}^{t} \frac{\dot{p}_{0,\kappa}(s)}{T-s}\,ds \\ &\quad + \int_{-T}^{t-(T-t)/5} \dot{p}_{0,\kappa}(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right)ds \\ &\quad + \int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot{p}_{0,\kappa}(s)}{t-s}\,ds. \end{aligned}$$

We estimate

$$\left|\int_{t-(T-t)/5}^{t} \frac{\dot p_{0,\kappa}(s)}{T-s}\,ds\right| \le \frac{C\kappa|\log T|}{|\log(T-t)|^2},$$

and

$$\left|\int_{-T}^{t-(T-t)/5} \dot p_{0,\kappa}(s)\left(\frac{1}{t-s}-\frac{1}{T-s}\right)ds\right| \le \frac{C\kappa|\log(T)|}{|\log(T-t)|^2}.$$

With the fourth term in E we proceed as follows

$$\begin{aligned}\int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(s)}{t-s}\,ds &= \dot p_{0,\kappa}(t)(\log(T-t)-2\log(\lambda_*))\\ &\quad -\int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(t)-\dot p_{0,\kappa}(s)}{t-s}\,ds.\end{aligned}$$

But

$$\left|\int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(t)-\dot p_{0,\kappa}(s)}{t-s}\,ds\right| \le \frac{C\kappa|\log(T)|}{|\log(T-t)|^3},$$

and therefore

$$\begin{aligned}E &= \int_{-T}^{t} \frac{\dot p_{0,\kappa}(s)}{T-s}\,ds + \dot p_{0,\kappa}(t)(\log(T-t)-2\log(\lambda_*))\\ &\quad + O\left(\frac{\kappa|\log(T)|}{|\log(T-t)|^2}\right).\end{aligned}$$

We note that

$$\dot p_{0,\kappa}(t)|\log(T-t)| + \int_0^t \frac{\dot p_{0,\kappa}(s)}{T-s}\,ds = c$$

for some constant c. Indeed, by (8.3)

$$\begin{aligned}\frac{d}{dt}\left(\dot p_{0,\kappa}(t)|\log(T-t)| + \int_0^t \frac{\dot p_{0,\kappa}(s)}{T-s}\,ds\right) &= \frac{\frac{d}{dt}(\dot p_{0,\kappa}(t)|\log(T-t)|^2)}{|\log(T-t)|}\\ &= 0.\end{aligned}$$

This shows that

$$E(t) = E(T) + O\left(\frac{|\log T|[\log(|\log T|) + \log(|\log(T-t)|)]}{|\log(T-t)|^2}\right),$$

which implies the estimate (8.43). □

Proof of Proposition 8.1 Let T_1 be the operator constructed in Lemma 8.7 for $T > 0$, $\alpha_0 > 0$ small and $\mathcal{A}$ defined in (8.37).

We will apply inequality (8.34) with $k < 2$ close to 2. The constant in this inequality remains bounded as $\alpha_0 \to 0^+$, because $\Upsilon = \frac{2-\alpha_0}{1-\alpha_0} \to 2$ as $\alpha_0 \to 0^+$.

For the poof we use the norm (8.33) with $k < 2$, k close to 2 so $k + 1 < 3$ is close to 3. We work with p_1 in the space $X = C([-T,T];\mathbb{C}) \cap C^1([-T,T);\mathbb{C})$ with the norm $\|\cdot\|_{*,k+1}$ defined in (8.9). By Lemma 8.7

$$\|\mathcal{A}[p_1]\|_{*,k+1} \le C\Big(\|\eta\tilde{E}\|_{**,k} + \|\tilde{\mathcal{B}}[p_{0,\kappa} + p_1](t) - \tilde{\mathcal{B}}[p_{0,\kappa} + p_1](T)\|_{**,k}\Big), \tag{8.44}$$

and by Lemma 8.9

$$\|\eta\tilde{E}\|_{**,k} \le C_E|\log T|^{k-1}\log(|\log T|), \tag{8.45}$$

for some $C_E > 0$. We take in X the closed ball $\overline{B}_M(0)$ of center 0 and radius M given by (8.38) with $C_0 > 0$ suitably large. The proof of Proposition 8.1 consists in showing that $\mathcal{A} : \overline{B}_M(0) \to \overline{B}_M(0)$ is a contraction. The estimates required for this are the following: for $\|p_1\|_{*,k+1} \le M$ we have

$$\|\tilde{\mathcal{B}}[p_{0,\kappa} + p_1]\|_{**,k} \le C|\log(T)|^{k-1}, \tag{8.46}$$

and for $\|p_i\|_{*,k+1} \le M$, $i = 1, 2$ we have

$$\|\tilde{\mathcal{B}}[p_{0,\kappa} + p_1] - \tilde{\mathcal{B}}[p_{0,\kappa} + p_2]\|_{**,k} \le \frac{C}{|\log T|}\|p_1 - p_2\|_{*,k+1}. \tag{8.47}$$

These inequalities are proved in a straightforward way. We omit the details

Form these estimates we see that $\mathcal{A}$ is a contraction in the ball $\overline{B}_M$. Indeed, from (8.44), (8.45) and (8.46) we have

$$\begin{aligned}\|\mathcal{A}[p_1]\|_{*,k+1} &\le C \cdot C_E|\log T|^{k-1}\log(|\log T|) + C|\log(T)|^{k-1}\\ &\le C_0|\log T|^{k-1}\log(|\log T|)^2\end{aligned}$$

by fixing C_0 large. Therefore $\mathcal{A} : \overline{B}_M(0) \to \overline{B}_M(0)$.

Next, for $\|p_i\|_{*,k+1} \leq M$, $i = 1, 2$, by Lemma 8.7 and (8.47) we get

$$\begin{aligned}\|\mathcal{A}[p_1] - \mathcal{A}[p_2]\|_{*,k+1} &\leq C\|\tilde{\mathcal{B}}[p_{0,\kappa} + p_1] - \tilde{\mathcal{B}}[p_{0,\kappa} + p_2]\|_{**,k} \\ &\leq \frac{C}{|\log T|}\|p_1 - p_2\|_{*,k+1}.\end{aligned}$$

The proof of (8.39) will be given in Corollary 8.1 below. □

We also have the following estimates

Lemma 8.10 *Let p_1 be the solution constructed in Proposition 8.1. Then*

$$\begin{aligned}|\ddot{p}_1(t)| &\leq C\frac{|\log T|}{|\log(T-t)|^3(T-t)} \\ \left|\frac{d^3}{dt^3}p_1(t)\right| &\leq C\frac{|\log T|}{|\log(T-t)|^3(T-t)^2}.\end{aligned}$$

The proof is done by formally differentiating the equation and using suitable estimates on the operators involved. We omit the details.

Proposition 8.1 defines a function that to κ satisfying (8.4) associates $p_1(\kappa)$, which is the unique fixed point of $\mathcal{A}$ in the ball $\{\|p_1\|_{*,k+1} \leq M\}$, $M = C_0|\log(T)|^{k-1}\log(|\log(T)|)^2$.

The next result gives several Lipschitz estimates of this map.

Corollary 8.1 *Let $k \in (0, 2)$. For κ_1, κ_2 satisfying (8.4) we have*

$$\|p_1(\kappa_1) - p_1(\kappa_2)\|_{*,k+1} \leq C|\log T|^{k-1}\log(|\log T|)^2\,|\kappa_1 - \kappa_2|.$$

We will also need a Lipschitz estimate for $\ddot{p}_1$ in the norm $\| \ \|_{-1,3}$ and $\frac{d^3}{dt^3}p_1$ in the norm $\| \ \|_{-2,3}$.

Lemma 8.11 *For κ_1, κ_2 satisfying (8.4) we have*

$$\begin{aligned}\|\ddot{p}_1(\kappa_1) - \ddot{p}_1(\kappa_2)\|_{-1,3} &\leq C|\log T|\,|\kappa_1 - \kappa_2| \\ \left\|\frac{d^3}{dt^3}p_1(\kappa_1) - \frac{d^3}{dt^3}p_1(\kappa_2)\right\|_{-2,3} &\leq C|\log T|\,|\kappa_1 - \kappa_2|\end{aligned}$$

Next we use the previous results on p_1 to obtain an estimate of $R_{\alpha_0}[\dot{p}_1]$.

Lemma 8.12 *Let p_1 be the solution constructed in Proposition 8.1. Then*

$$|R_{\alpha_0}[\dot{p}_1](t)| \leq C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0},$$

and for κ_1, κ_2 satisfying (8.4) *we have*

$$|R_{\alpha_0}[\dot p_1(\kappa_1)] - R_{\alpha_0}[\dot p_1(\kappa_2)]| \le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0}|\kappa_1-\kappa_2|.$$

Lemma 8.13 *Let p_1 be the solution constructed in Proposition* 8.1. *Then*

$$\begin{aligned}
&\left|\frac{d}{dt}R_{\alpha_0}[\dot p_1](t)\right| \le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0-1},\\
&\left|\frac{d}{dt}R_{\alpha_0}[\dot p_1(\kappa_1)](t) - \frac{d}{dt}R_{\alpha_0}[\dot p_1(\kappa_2)](t)\right|\\
&\quad \le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0-1}|\kappa_1-\kappa_2|.
\end{aligned}$$

9 Final adjustment of the parameters p and ξ

In this section we prove that the last equations of the gluing system (6.22)–(6.28) can be solved, by adjusting the parameter functions $p=\lambda e^{i\omega}$ and ξ, as stated in Proposition 6.9, thus concluding the proof of Theorem 1.

Proof of Proposition 6.9 Let $\Psi(p,\xi,\Phi,Z_0^*)$ be the solution to equation (6.22) constructed in Proposition 6.7. Let $\Phi(p,\xi,Z_0^*)$ denote the solution of (6.32) constructed in Proposition 6.8. In (6.27)–(6.28) we replace Ψ^* by $\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)$. Then to find a solution of the full system (6.22)–(6.28) it is sufficient to find p, ξ such that

$$\begin{aligned}
&c_{0j}[h(p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*))](t)\\
&\quad - c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)](t) = 0 \qquad (9.1)\\
&c_{1j}[h(p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*))](t) = 0 \qquad (9.2)
\end{aligned}$$

for all $t\in(0,T)$, $j=1,2$.

We recall from Sect. 6 that (9.1) is equivalent to

$$\mathcal{B}_0[p] = a_0^{(0)}[p,\xi,\Psi^*] + \mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*]\right], \quad t\in[0,T] \qquad (9.3)$$

where $\Psi^* = \Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)$. We recall that $\mathcal{B}_0$ is the integral operator defined in (5.6) which has the approximate form

$$\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2}\frac{\dot p(s)}{t-s}ds + O\big(\|\dot p\|_\infty\big).$$

In Proposition 6.5 we constructed an approximate inverse $\mathcal{P}$ of the operator $\mathcal{B}_0$, so that given a satisfying (6.17), $p := \mathcal{P}[a]$, satisfies the equation

$$\mathcal{B}_0[p] = a + \mathcal{R}_0[a], \quad \text{in } [0, T],$$

for a small remainder $\mathcal{R}_0[a]$. The proof of that proposition gives the decomposition

$$\mathcal{P}[a] = p_{0,\kappa} + \mathcal{P}_1[a],$$

where $p_{0,\kappa}$ is defined in (8.2), $\kappa = \kappa[a] \in \mathbb{C}$ and the function $p_1 = \mathcal{P}_1[a]$ has the estimate

$$\|p_1\|_{*,3-\sigma} \le C|\log T|^{1-\sigma} \log^2(|\log T|),$$

where $\| \ \|_{*,3-\sigma}$ is defined in (8.9) and $\sigma \in (0, 1)$. This leads us to define the space $X_1 := \mathbb{C} \times \tilde{X}_1$ where

$$\tilde{X}_1 := \{p_1 \in C([-T, T; \mathbb{C}]) \cap C^1([-T, T; \mathbb{C}]) \mid p_1(T) = 0, \\ \|p_1\|_{*,3-\sigma} < \infty\}.$$

Let us rewrite equation (9.2) as follows. By (6.9), (9.2) is equivalent to

$$\int_{\mathbb{R}^2} h[p, \xi, \Psi^*] \cdot Z_{1j}(y)\, dy = 0, \quad t \in (0, T), \ \ j = 1, 2,$$

and recalling (5.1), this is equivalent to

$$\lambda \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j} + \lambda \int_{B_{2R}} \mathcal{K}_1[p, \xi] \cdot Z_{1j} = 0,$$

which yields the following equation

$$\dot{\xi}_j = \frac{1}{4\pi}(1 + (2R)^{-2}) \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j}, \quad j = 1, 2. \tag{9.4}$$

We reformulate (9.3)–(9.4) as the fixed point problem

$$[p, \xi] = \mathcal{A}[p, \xi] \quad \text{in } \mathcal{B} \tag{9.5}$$

where the space $\mathcal{B}$ will be introduced below and the operator $\mathcal{A} = [\mathcal{A}_1, \mathcal{A}_2]$ is defined by

$$\mathcal{A}_1[p,\xi] = \mathcal{P}\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)]\right]$$
$$\mathcal{A}_2[p,\xi] = q - \int_t^T b[p,\xi](s)\,ds$$

with

$$b_{1j}[p,\xi](t) = \frac{1}{4\pi}(1+(2R)^{-2})\int_{B_{2R}} Q_{-\omega}\tilde{L}_U[\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)]\cdot Z_{1j}.$$

To define $\mathcal{B}$ consider the closed ball

$$\mathcal{B}_1 = \overline{B}_{l_1}(\kappa_0)\times\overline{B}_{l_2}(0)\subset X_1,$$

where $\kappa_0 = \operatorname{div} z_0^{*0}(q) + i\operatorname{curl} z_0^{*0}(q)$ with z_0^{*0} so that

$$Z_0^{*0}(x) = \begin{bmatrix} z_0^{*0}(x)\\ z_{03}^{*0}(x)\end{bmatrix},\quad z_0^{*0}(x) = z_{01}^{*0}(x) + i z_{02}^{*0}(x),$$

and $Z_0^* = Z_0^{*0} + Z_0^{*1}$ is the initial condition as described in (6.8). Here the numbers l_1, l_2 are given by

$$l_1 = T^\sigma,\quad l_2 = C_0|\log T|^{1-\sigma}\log^2(|\log T|),$$

with $\sigma > 0$ small and and $C_0 > 0$ is a fixed large constant. We consider ξ in the space

$$X_2 = \{\xi \in C^1([0,T];\mathbb{R}^2)\ :\ \dot{\xi}(T) = 0\}$$

endowed with the norm

$$\|\xi\|_{X_2} = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*(t)^{-\sigma}|\dot{\xi}(t)|$$

where $\sigma \in (0,1)$ is fixed. In X_2 we consider the closed ball $\mathcal{B}_2 := \overline{B}_1(\xi^*)$, where $\xi^* \equiv q \in \Omega$. We consider the Banach space $X := X_1 \times X_2$ and its closed ball $\mathcal{B} := \mathcal{B}_1 \times \mathcal{B}_2$. We formulate the fixed point problem (9.5) in $\mathcal{B}$. We claim that $\mathcal{A}(\mathcal{B}) \subset \mathcal{B}$ and that $\mathcal{A}$ is a contraction mapping on $\mathcal{B}$ for the norm $\|\ \|_X$. This is consequence of the various bounds and Lipschitz estimates derived in Sect. 8 for the operator $\mathcal{P}$ and in Sect. 6 for the operators Ψ^* and Φ. □

10 Stability of blow-up

In this section we discuss the stability of the blow-up phenomenon predicted in Theorem 1 and prove Theorem 2. We consider the class of initial conditions that lead to blow-up at a given point as described in Sect. 6.1. The solution has the form

$$u(x,t) = U_{\lambda(t),\omega(t),\xi(t)} + \varphi + a(|\varphi|^2)U_{\lambda(t),\omega(t),\xi(t)}$$

where $a(s) = \sqrt{1-s} - 1$ and

$$\varphi(x,t) = \Pi_{U^{\perp}_{\lambda(t),\omega(t),\xi(t)}}\left[\tilde{Z}^*(x,t) + \Phi(\lambda,\omega,\xi)(x,t) + \psi(x,t) + \eta\phi(x,t)\right],$$

where the point $\xi(T) \in \Omega$ is prescribed. Changing slightly the proof we can achieve that the value $\xi(0) = q$ be prescribed. Let us denote $\varepsilon = \lambda(0)$. A simple application of implicit function theorem to the system of equations determining (λ, ω, ξ) leads to the fact that the blow-up time T and the final point $\xi(T)$ can be regarded as functions of arbitrary small values $\varepsilon > 0$ and points $q \in \Omega$.

The functions (λ, ω, ξ) as well as ψ and ϕ have Lipschitz dependence in $p := (\varepsilon, q)$ and Z^* in suitable topologies. We relabel

$$\omega(p) := \omega(0), \quad U_p := U_{\varepsilon,\omega(p),q}, \quad \tilde{\Phi}(p)(x) = \Phi(\lambda,\omega,\xi)(x,0) + \psi(x,0)$$

so that the initial condition of the solution above becomes

$$u_0(p) = U_p + \Pi_{U_p^{\perp}}[Z^* + \tilde{\Phi}(p)] + a(|\Pi_{U_p^{\perp}}[Z^* + \tilde{\Phi}(p)]|^2)U_p.$$

A generic initial condition close to

$$U_{p_0} + \Pi_{U_{p_0}^{\perp}}[Z_0^* + \tilde{\Phi}(p_0)] + a(|\Pi_{U_{p_0}^{\perp}}[Z_0^* + \tilde{\Phi}(p_0)]|^2)U_{p_0}$$

with values in S^2 can be written in the form

$$\begin{aligned} v(x;\varphi_1) := {} & U_{p_0} + \Pi_{U_{p_0}^{\perp}}[Z_0^* + \tilde{\Phi}(p_0) + \varphi_1] \\ & + a(|\Pi_{U_{p_0}^{\perp}}[Z_0^* + \tilde{\Phi}(p_0) + \varphi_1]|^2)U_{p_0} \end{aligned}$$

where φ_1 is a small function, otherwise arbitrary. We shall show that if φ_1 is sufficiently small in C^2-topology and it lies on a certain codimension-1 manifold, then problem (2.2) with initial condition $u_0(x) = v(x;\varphi_1)$ has blow-up as predicted. Thus what we need is that for suitable

$$\zeta = (\varepsilon, q, Z^*) = \zeta_0 + \zeta_1, \quad \zeta_1 = (\varepsilon_1, q_1, Z_1^*)$$

we have that

$$v(\cdot; \varphi_1) = u_0(p). \tag{10.1}$$

It is convenient to measure the size of ζ_1 with respect to the norm (see 6.5),

$$\|p_1\| := |q_1| + |\varepsilon_1| + \|Z_1^*\|_*.$$

We expand $u_0(p)$ around $p = p_0$ and get

$$u_0(\zeta) = U_{\zeta_0} + \varphi(\zeta) + a(|\varphi(\zeta)|^2))\, U_{\zeta_0},$$

where

$$\begin{aligned}
\varphi(\zeta) &= \Pi_{U_{\zeta_0}^{\perp}}[Z^* + \tilde{\Phi}(\zeta) + (U_\zeta - U_{\zeta_0})(1 - \gamma(\zeta) + a(p))],\\
\gamma(\zeta) &= U_p \cdot (Z^* + \tilde{\Phi}(\zeta))\\
a(p) &= a(|\Pi_{U_\zeta^{\perp}}[Z^* + \tilde{\Phi}(\zeta)]|^2).
\end{aligned}$$

Therefore, equation (10.1) becomes

$$\Pi_{U_{\zeta_0}^{\perp}}[Z_0^* + \tilde{\Phi}(\zeta_0) + \varphi_1] = \Pi_{U_{\zeta_0}^{\perp}}[Z^* + \tilde{\Phi}(\zeta) + (U_\zeta - U_{p_0})(1 - \gamma + a)]$$

or, equivalently

$$\Pi_{U_{\zeta_0}^{\perp}}[Z_1^* + \tilde{\Phi}(\zeta) - \tilde{\Phi}(\zeta_0) + (U_\zeta - U_{\zeta_0})(1 - \gamma(\zeta) + a(\zeta)) - \varphi_1] \;=\; 0.$$

We will get a solution to this equation if we find a constant c_0 such that

$$Z_1^* + \tilde{\Phi}(\zeta_0 + \zeta_1) - \tilde{\Phi}(\zeta_0) + (U_\zeta - U_{\zeta_0})(1 - \gamma(\zeta) + a(\zeta)) = \varphi_1 + c_0 U_{\zeta_0}$$

Let us consider the functions $Z_{lj}(y)$ defined in (3.1), $l = 0, 1,\ j = 1, 2$, with $y = \frac{x-q}{\varepsilon}$. We introduce the following intermediate problem: we want to find a function Z_1^* and five constants c_0, c_{lj} such that

$$Z_1^* + \tilde{\Phi}(\zeta_0 + p_1) - \tilde{\Phi}(\zeta_0) + (U_\zeta - U_{\zeta_0})(1 - \gamma(\zeta) + a(\zeta)) = \varphi_1 + c_0 U_{\zeta_0} + c_{lj} Z_{lj} \tag{10.2}$$

and the following five real constraints hold for the function $Z_1^*(x)$:

$$\operatorname{div} z_1^*(\zeta_0) = 0, \quad \operatorname{curl} Z_1^*(q_0) = 0, \quad Z_1^*(q_0) = 0. \tag{10.3}$$

Summation convention is used in (10.2).

To make the argument more transparent, we consider a simplified linearized version of (10.2)–(10.3), in which lower order terms are neglected, and only the constants associated to mode 0 (associated to dilations and rotations) are considered. Thus we consider the model equation for Z_1^*,

$$\begin{cases} Z_1^* + \Phi_0[Z_1^*] = \varphi_1 + \sum_{j=1}^{2} c_{0j} Z_{0j}, \\ \operatorname{div} z_1^*(q_0, 0) = 0, \quad \operatorname{curl} z_1^*(q_0, 0) = 0. \end{cases} \tag{10.4}$$

where

$$\Phi_0[Z_1^*](r) = \begin{pmatrix} \phi_0[Z_1^*](r, t) \\ 0 \end{pmatrix}$$

with

$$\phi_0[Z_1^*](r) = re^{i\theta} \int_{-T}^{0} \dot{p}(s) k(r^2 + \varepsilon^2, -s)\, ds, \quad k(\zeta, t) = 2\frac{1 - e^{-\frac{\zeta}{4t}}}{\zeta}, \tag{10.5}$$

where $p(t) = \lambda(t)e^{i\omega(t)}$, $r = |x - q_0|$, $\varepsilon = \lambda(0)$, and $p = p[Z_1^*]$ is such that the following equation is satisfied

$$\begin{cases} \dot{p}(t)|\log(T - t)| + \int_{-T}^{t} \frac{\dot{p}(s)}{T - s}\, ds = \operatorname{div} \tilde{z}_1(q, t) + i \operatorname{curl} \tilde{z}_1(q, t), \quad t \in [0, T]. \\ p(T) = 0, \end{cases} \tag{10.6}$$

where

$$\begin{cases} \partial_t \tilde{Z}_1(x, t) = \Delta \tilde{Z}_1(x, t) \quad \text{in } \Omega \times (0, T) \\ \tilde{Z}_1(x, 0) = Z_1^*(x) \quad x \in \Omega \\ \tilde{Z}_1(x, t) = 0 \quad (x, t) \in \partial\Omega \times (0, T), \end{cases} \tag{10.7}$$

and we use the notation

$$\tilde{Z}_1 = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_{1,3} \end{pmatrix}, \quad Z_1^* = \begin{pmatrix} z_1^* \\ z_{1,3}^* \end{pmatrix}.$$

The main result here is the solvability of (10.4).

Proposition 10.1 *Assume $\|\varphi_1\|_*$ is finite. Then for $T > 0$ sufficiently small equation (10.4) has a unique solution Z_1^*, c_{01}, $c_{0,2}$ and moreover*

$$\|Z_1^*\|_* + |c_{01}| + |c_{02}| \le C\|\varphi_1\|_*.$$

We can obtain a similar result if all constraints and constants are considered, with essentially the same proof as that below. On the other hand, to derive the corresponding result to the full problem (10.2)–(10.3), we need to use the linearized version and contraction mapping principle. For that we need to use the precise Lipschitz estimates of the solution of the inner-outer gluing system on the parameters involved as done in Sect. 6 and Sect. 8. The C^1 character of the manifold predicted in Theorem 2 follows from the fixed point characterization and the implicit function theorem.

We devote the rest of this section to the proof of the proposition, whose main step is the following estimate.

Lemma 10.1 *Assume that*

$$\operatorname{div} z_1^*(q_0) = 0, \quad \operatorname{curl} z_1^*(q_0) = 0.$$

Then

$$\|\Phi_0[Z_1^*]\|_* \le \frac{C}{|\log T|}\|Z_1^*\|_*.$$

To prove this we need a corollary of Lemma B.1 adapted to the norm $\|\ \|_*$ defined in (6.5) is the following.

Lemma 10.2 *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|\nabla_x Z_1^*(x)| \le |\log\varepsilon|, \quad x \in \Omega$$
$$|D_x^2 Z_1^*(x)| \le \frac{|\log\varepsilon|^{\frac12}}{|x-q_0|+\varepsilon} \quad x \in \Omega.$$

Then the solution $\tilde Z_1$ of (10.7) satisfies

$$|\nabla_x \tilde Z_1(x,t)| \le |\log\varepsilon|, \quad t \ge 0, \tag{10.8}$$

and

$$|\nabla_x \tilde Z_1(x,t) - \nabla_x \tilde Z_1(x,T)| \le C \begin{cases} |\log\varepsilon| & \text{if } 0 \le t \le \varepsilon^2 \\ |\log\varepsilon|^{\frac12}\frac{T-t}{T}(1+\log(\frac{T}{t})) & \text{if } \varepsilon^2 \le t \le T. \end{cases}$$

Proof As in Lemma B.1 we consider the function given by Duhamel's formula in $\mathbb{R}^2$ and then decompose the solution as a sum of the one in $\mathbb{R}^2$ and a smooth one in Ω with zero initial condition.

From (B.2) and $|\nabla_x Z_1^*(x)| \le |\log\varepsilon|$ we get (10.8).

For $0 \le t \le \varepsilon^2$ we get

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C|\log\varepsilon|$$

from (10.8). For $\varepsilon^2 \le t \le T$ from Lemma B.1 we obtain

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C|\log\varepsilon|^{1/2}\frac{\sqrt{T}-\sqrt{t}}{\sqrt{T}}\left(1+\log\left(\frac{T}{t}\right)\right).$$

□

Proof of Lemma 10.1 Let $f(t) = \operatorname{div}\tilde{z}_1(q,t) + i\operatorname{curl}\tilde{z}_1(q,t)$. Differentiating (10.6) we find

$$\frac{d}{dt}\left(\dot{p}(t)|\log(T-t)|^2\right) = |\log(T-t)|\dot{f}(t).$$

This can be integrated explicitly and we get

$$\dot{p}(t) = -\frac{1}{|\log(T-t)|^2}\int_t^T |\log(T-s)|\dot{f}(s)\,ds + \frac{c|\log T|}{|\log(T-t)|^2}$$

for some constant c to be determined. Integrating by parts we find that

$$\dot{p}(t) = \frac{f(t)-f(T)}{|\log(T-t)|} + \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds + \frac{c|\log T|}{|\log(T-t)|^2}.$$

This function is defined for $t \in [0,T]$ and we need to extend it to $[-T,T]$ to make sense of (10.6). A possible extension is $\dot{p}(t) = \dot{p}(0)$ for $t \in [-T,0]$ but this makes this lemma too simple and not useful to adapt to the real situation. For this reason we make the analysis with the following extension. Define

$$\dot{p}_1(t) = \frac{f(t)-f(T)}{|\log(T-t)|} + \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds \tag{10.9}$$

so that

$$\dot{p}(t) = \dot{p}_1(t) + \frac{c|\log T|}{|\log(T-t)|^2} \quad \text{for } t \in [0,T]$$

Then define

$$\dot{p}(t) = \dot{p}_1(0) + \frac{c|\log T|}{|\log(T-t)|^2}, \quad t \in [-T,0]. \tag{10.10}$$

We want to estimate

$$\phi_0[Z_1^*](r) = re^{i\theta} \int_{-T}^{0} \dot p(s) k(r^2+\varepsilon^2, -s)\, ds, \quad k(\zeta, t) = 2\frac{1-e^{-\frac{\zeta}{4t}}}{\zeta},$$

which, thanks to (10.10) depends only on $\dot p_1(0)$ and c. Therefore we need to estimate these quantities. We claim that

$$\dot p_1(0) = \frac{f(0) - f(T) + O\left(\frac{\log(|\log T|)}{|\log T|^{1/2}}\right) \|Z_1^*\|_*}{|\log T|} \left(1 + O\left(\frac{1}{|\log T|}\right)\right) \tag{10.11}$$

$$c = f(T)\left(1 + O\left(\frac{1}{|\log T|}\right)\right) + O\left(\frac{1}{|\log T|}\right) f(0) + O\left(\frac{\log(|\log T|)}{|\log T|^{1/2}}\right) \|Z_1^*\|_*. \tag{10.12}$$

To obtain these estimates we note that evaluating equation (10.6) at $t = 0$ we get

$$\dot p_1(0)(|\log T| + \log 2) + c\left(1 + O\left(\frac{1}{|\log T|}\right)\right) = f(0) \tag{10.13}$$

and evaluating equation (10.6) at $t = T$ we get

$$\int_{-T}^{T} \frac{\dot p_1(s)}{T-s}\, ds + c\left(1 + O\left(\frac{1}{|\log T|}\right)\right) = f(T). \tag{10.14}$$

Thus we need to estimate $\int_{-T}^{T} \frac{\dot p_1(s)}{T-s}\, ds$ where p_1 is given (10.9). We have

$$\begin{aligned}\int_{-T}^{T} \frac{\dot p_1(s)}{T-s}\, ds &= \int_{-T}^{0} \frac{\dot p_1(s)}{T-s}\, ds + \int_{0}^{T} \frac{\dot p_1(s)}{T-s}\, ds \\ &= \dot p_1(0)\log 2 + \int_{0}^{T} \frac{\dot p_1(s)}{T-s}\, ds.\end{aligned}$$

To estimate $\int_0^T \frac{\dot p_1(s)}{T-s}\, ds$ we write

$$\dot p_1 = \dot p_{1a} + \dot p_{1b}$$

with

$$\dot p_{1a}(t) = \frac{f(t)-f(T)}{|\log(T-t)|}$$
$$\dot p_{1b}(t) = \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds.$$

We compute

$$\int_0^T \frac{\dot p_{1a}(s)}{T-s}\,ds = \int_0^{\frac{T}{|\log T|}} \ldots + \int_{\frac{T}{|\log T|}}^T \ldots.$$

By Lemma 10.2 we have that

$$|f(t)-f(T)| \le C\|Z_1^*\|_* \begin{cases} \log(|\log T|)|\log T|^{1/2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \le t \le T \\ |\log T|, & 0 \le t \le \dfrac{T}{|\log T|}. \end{cases} \tag{10.15}$$

which in particular implies

$$|\dot p_{1a}(t)| \le C\frac{\|Z_1^*\|_*}{|\log(T-t)|} \begin{cases} \log(|\log T|)|\log T|^{1/2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \le t \le T \\ |\log T|, & 0 \le t \le \dfrac{T}{|\log T|}. \end{cases}$$

Therefore

$$\int_0^{\frac{T}{|\log T|}} \frac{|\dot p_{1a}(s)|}{T-s}\,ds \le \frac{C}{|\log T|}\|Z_1^*\|_*,$$

and

$$\int_{\frac{T}{|\log T|}}^T \frac{|\dot p_{1a}(s)|}{T-s}\,ds \le C\frac{\log(|\log T|)|\log T|^{1/2}}{|\log T|}\|Z_1^*\|_*.$$

It follows that

$$\int_0^T \frac{|\dot p_{1a}(s)|}{T-s}\,ds \le C\frac{\log(|\log T|)}{|\log T|^{1/2}}\|Z_1^*\|_*. \tag{10.16}$$

By (10.15), we find that

$$|\dot{p}_{1b}(t)| \leq C\|Z_1^*\|_* \begin{cases} \dfrac{\log(|\log T|)|\log T|^{1/2}}{|\log(T-t)|^2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \leq t \leq T \\ \dfrac{\log(|\log T|)}{|\log T|^{3/2}}, & 0 \leq t \leq \dfrac{T}{|\log T|}. \end{cases}$$

This implies that

$$\int_0^T \frac{|\dot{p}_{1b}(s)|}{T-s}\, ds \leq \frac{\log(|\log T|)}{|\log T|^{3/2}} \|Z_1^*\|_*. \tag{10.17}$$

From (10.16) and (10.17) we find that

$$\left|\int_0^T \frac{\dot{p}_1(s)}{T-s}\, ds\right| \leq \frac{\log(|\log T|)}{|\log T|^{1/2}} \|Z_1^*\|_*.$$

Therefore (10.14) gives

$$\dot{p}_1(0)\log 2 + O\left(\frac{\log(|\log T|)}{|\log T|^{1/2}}\right)\|Z_1^*\|_* + c\left(1 + O(\frac{1}{|\log T|})\right) = f(T). \tag{10.18}$$

Equations (10.13) and (10.18) form a system

$$\begin{bmatrix} |\log T| + \log 2 & 1 + O\left(\frac{1}{|\log T|}\right) \\ \log 2 & 1 + O\left(\frac{1}{|\log T|}\right) \end{bmatrix} \begin{bmatrix} \dot{p}_1(0) \\ c \end{bmatrix} = \begin{bmatrix} f(0) \\ f(T) + O\left(\frac{\log(|\log T|)}{|\log T|^{1/2}}\right)\|Z_1^*\|_* \end{bmatrix}$$

for $\dot{p}_1(0)$ and c, and solving we get (10.11), (10.12).

We use (10.11), (10.12) to estimate ϕ_0 given by (10.5): and obtain

$$\|\Phi_0[Z_1^*]\|_* \leq \frac{C}{|\log T|}\|Z_1^*\|_*.$$

□

Proof of Proposition 10.1 We look for a solution of (10.4) in the space of functions

$$\mathcal{Z} = \{Z_1^* \in C^2(\overline{\Omega}) : \|Z_1^*\|_* < \infty,\ \operatorname{div} z_1^*(q_0) = 0,\ \operatorname{curl} z_1^*(q_0) = 0\}.$$

To determinte c_{0j} we apply divergence and curl (10.4) at q_0 to obtain

$$c_{01} = \varepsilon\left(\operatorname{div}\phi_0[Z_1^*](q_0,0) - \operatorname{div}\varphi_1(q_0)\right),$$
$$c_{02} = \varepsilon\left(\operatorname{curl}\phi_0[Z_1^*](q_0,0) - \operatorname{curl}\varphi_1(q_0)\right).$$

With this equation (10.4) becomes the fixed point problem

$$Z_1^* = \mathcal{F}[Z_1^*] + \varphi_1 + \operatorname{div}\varphi_1(q_0)\varepsilon Z_{01} + \operatorname{curl}\varphi_1(q_0)\varepsilon Z_{02}. \tag{10.19}$$

where

$$\mathcal{F}[Z_1^*] = -\Phi_0[Z_1^*] - \operatorname{div}\phi_0[Z_1^*](q_0,0)\varepsilon Z_{01} - \operatorname{curl}\phi_0[Z_1^*](q_0,0)\varepsilon Z_{02}$$

By Lemma 10.1 we get

$$\left|\operatorname{div}\phi_0[Z_1^*](q_0,0) + i\operatorname{curl}\phi_0[Z_1^*](q_0,0)\right| \le C|\log\varepsilon|\|\Phi_0[Z_1^*]\|_* \le C\|Z_1^*\|_*.$$

But

$$\|\varepsilon Z_{0j}\|_* \le \frac{C}{|\log T|^{1/2}}.$$

This and Lemma 10.1 shows that

$$\|\mathcal{F}[Z_1^*]\|_* \le \frac{C}{|\log T|^{1/2}}\|Z_1^*\|_*.$$

By the contraction mapping principle, equation (10.19) has a unique fixed point in $\mathcal{Z}$. □

11 Reverse bubbling

The proofs of Theorems 3 and 4 follow very similar lines to those of Theorem 1, with a "backwards" construction. In Theorem 3 we consider the exact ansatz as in (4.14) for $u(x,t)$ in $(0,2T)$, extended for $T < t < 2T$ in the form

$$u(x,t) = \bar U + \Pi_{\bar U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega\phi] + a(\Pi_{\bar U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega\phi])\bar U$$

where $\lambda(t)$ is defined in the interval $(-T,2T)$ and satisfies $\lambda(T)=0$, while

$$\bar U(x,t) = Q_{\omega(t)}(t)\bar W\left(\frac{x-\xi(t)}{\lambda(t)}\right)$$

and $\bar{W}(y)$ is the reverse bubble as in (1.16). A main point is that the linear theory for the inner problem, corresponding to $\phi(y,t)$ has to be performed for $t > T$ for "ancient solutions", which exactly mirror the forward theory of Sect. 7. More precisely, we need to consider a problem of the form We consider the linear equation

$$\begin{aligned} \lambda^2 \partial_t \phi &= L_W[\phi] + h(y,t) \quad \text{in } \tilde{\mathcal{D}}_{2R} \\ \phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)} \\ \phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R} \end{aligned}$$

where

$$\tilde{\mathcal{D}}_{2R} = \{(y,t) \;/\; t \in (T, 2T),\; y \in B_{2R(t)}(0)\}.$$

We assume that $h(y,t)$ is defined for all $(y,t) \in \mathbb{R}^2 \times (0,T)$ and satisfies

$$h \cdot W = 0, \quad |h(y,t)| \le C \frac{\tilde{\lambda}_*(t)^\nu}{(1+|y|)^a},$$

where we extend the definition of $\lambda_*(t)$ for $t > T$ as $\lambda_*(t) = \frac{|\log T||t-T|)}{\log^2|t-T|}$. Inverses for the linear problem with right bounds (which vanish as $t \downarrow T$) are found as before.

In the full construction a major ingredient is the adjustment of the parameter $\lambda(t)$ for times $t > T$.

The main term in the error (the one due to the effect of dilations) has now the extended form

$$\begin{cases} \frac{\dot{\lambda}}{\lambda} \rho w_\rho \sim -2\frac{\dot{\lambda}}{r} & \text{if } 0 \le t < T \\ \frac{\dot{\lambda}}{\lambda} \rho \tilde{w}_\rho \sim 2\frac{\dot{\lambda}}{r} & \text{if } T < t \le 2T. \end{cases}$$

Therefore we extend Φ^0 by considering the function $\Phi^0[\omega, b\lambda, \xi]$, where

$$b(s) = \begin{cases} 1 & s \le T \\ -1 & s > T. \end{cases}$$

$\Psi^*(x,t)$ has $Z^*(x,t)$ as its main term. Testing the error as before by the generator of dilations, we get the approximate equation

$$\int_0^{t-\lambda(t)^2} b(s) \frac{\dot{\lambda}(s)}{t-s}\, ds = -|\,[\operatorname{div} \Psi^* + i \operatorname{curl} \Psi^*](q,t)|. \tag{11.1}$$

We would like to find a solution such that $\lambda(T) = 0, \dot\lambda(t) < 0$ if $t < T$, $\dot\lambda(t) > 0$ if $t > T$. The solution for $t < T$ is the one of the forward bubbling of Theorem 1, which we recall is given at main order by

$$\lambda(t) = \kappa_* \frac{|\log T|(T-t)}{|\log(T-t)|^2}, \quad t < T,$$

For $t > T$ the approximate equation reads

$$\int_{-T}^{t-\lambda^2} b(s) \frac{\dot\lambda(s)}{t-s}\, ds = -|[\operatorname{div} \psi^* + i \operatorname{curl} \psi^*](q,T)| + \int_0^T \dot\lambda(s) \left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds - \int_T^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds.$$

Equation (11.1) then approximately reads

$$\int_T^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds = \int_0^T \dot\lambda(s) \left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds \quad \text{for } t \geq T,$$

The integral in the left hand side is approximately $\dot\lambda(t)|\log(t-T)|$, while

$$\int_0^T \dot\lambda(s) \left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds = (t-T)\kappa_* |\log T| \int_0^T \frac{1}{(t-s)(T-s)|\log(T-s)|^2}\, ds.$$

Arguing as before we get

$$\int_0^T \frac{1}{(t-s)(T-s)|\log(T-s)|^2}\, ds = \frac{1}{t-T} \frac{1}{|\log(t-T)|} + O\left(\frac{1}{|\log(t-T)|^2}\right).$$

Hence, for $t > T$ we get the apprximate equation

$$\dot\lambda(t)|\log(t-T)| = \kappa_* \frac{1}{|\log(t-T)|}$$

which gives

$$\lambda(t) = \kappa_* \frac{(t-T)|\log T|}{|\log(t-T)|^2}, \quad \text{for } t > T$$

as desired. This computation can be made fully rigorous with the same arguments already employed in the forward bubbling construction, leading to the proof of Theorem 3.

For the proof of Theorem 4 we proceed in exactly the same way however, now with an ansatz that does not include a bubble for $t < T$. In that case the approximate equation for λ takes the form

$$\int_T^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds = -|[\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](q,T)|$$

for $t > T$. From this equation we get

$$\lambda(t) = \kappa_* \frac{T-t}{|\log(t-T)|},$$

as desired.

It is interesting to notice that this continuation, even at the level of the parameter $\lambda(t)$, does not seem to exhibit analyticity near $t = T$, even one-sided, in terms of $(T-t)$ or the natural parameter $s = \frac{1}{\log(T-t)}$. It is not hard to check for instance that, even though formal improvements of approximation in powers of s are possible for $\lambda(t)$, they so not lead to a power series with positive convergence radius.

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Appendix A: The heat equation with right hand side

We are going to measure the solution to (6.31) in the norm $\|\ \|_{\sharp,\Theta,\gamma}$, (c.f. 6.29) with Θ and β (recall that $R = \lambda_*^{-\beta}$) satisfying:

$$\beta \in \left(0, \frac{1}{2}\right), \quad \Theta \in (0, \beta) \tag{A.1}$$

Our main result in this section is the following, where we use the norm $\|\ \|_{\sharp,\Theta,\gamma}$ defined in (6.29).

Proposition A.1 *Assume* (A.1). *For* T, $\varepsilon > 0$ *small there is a linear operator that maps a function* $f : \Omega \times (0,T) \to \mathbb{R}^3$ *with* $\|f\|_{**} < \infty$ *into* ψ, c_1, c_2, c_3 *so that* (6.31) *is satisfied. Moreover the following estimate holds*

$$\|\psi\|_{\sharp,\Theta,\gamma} + \frac{\lambda_*(0)^{-\Theta}(\lambda_*(0)R(0))^{-1}}{|\log T|}(|c_1| + |c_2| + |c_3|) \le C\|f\|_{**}, \quad \text{(A.2)}$$

where $\gamma \in (0, \frac{1}{2})$.

Remark A.1 The condition $\beta \in (0, \frac{1}{2})$ is a basic assumption to have the singularity appear inside the self-similar region. The condition $\Theta > 0$ is needed for Lemma A.1. The assumption $\Theta < \beta$ is so that the estimates provided by Lemma A.2 are stronger than the ones of Lemma A.1.

To prove Proposition A.1 we consider

$$\begin{cases} \psi_t = \Delta\psi + f \quad \text{in } \Omega \times (0, T) \\ \psi(x, 0) = 0, \quad x \in \Omega \\ \psi(x, t) = 0, \quad x \in \partial\Omega, t \in (0, T), \end{cases} \quad \text{(A.3)}$$

and let q be a point in Ω.

We always assume that R is given by (6.3).

Lemma A.1 *Assume* $\beta \in (0, \frac{1}{2})$ *and* $\Theta > 0$. *Let* ψ *solve* (A.3) *with* f *such that*

$$|f(x, t)| \le \lambda_*(t)^{\Theta}(\lambda_*(t)R(t))^{-1}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}.$$

Then

$$|\psi(x, t)| \le C\lambda_*(0)^{\Theta}\lambda_*(0)R(0)|\log T|, \quad \text{(A.4)}$$

$$|\psi(x, t) - \psi(x, T)| \le \lambda_*(t)^{\Theta}\lambda_*(t)R(t)|\log(T - t)|, \quad \text{(A.5)}$$

$$|\nabla\psi(x, t)| \le C\lambda_*(0)^{\Theta}, \quad \text{(A.6)}$$

$$|\nabla\psi(x, t) - \psi(x, T)| \le C\lambda_*(t)^{\Theta}, \quad \text{(A.7)}$$

and for any $\gamma \in (0, \frac{1}{2})$,

$$\frac{|\nabla\psi(x, t) - \nabla\psi(x, t')|}{|t - t'|^{\gamma}} \le C\frac{\lambda_*(t)^{\Theta}}{(\lambda_*(t)R(t))^{2\gamma}} \quad \text{(A.8)}$$

for any x, *and* $0 \le t' \le t \le T$ *such that* $t - t' \le \frac{1}{10}(T - t)$, *and*

$$\frac{|\nabla\psi(x, t) - \nabla\psi(x', t')|}{|x - x'|^{2\gamma}} \le C\frac{\lambda_*(t)^{\Theta}}{(\lambda_*(t)R(t))^{2\gamma}} \quad \text{(A.9)}$$

for any $|x - x'| \le 2\lambda_*(t)R(t)$ *and* $0 \le t \le T$.

The proof is in Sect. A.1.

Lemma A.2 *Assume* $\beta \in (0, \frac{1}{2})$ *and* $m \in (\frac{1}{2}, 1)$*. Let* ψ *solve* (A.3) *with* f *such that*

$$|f(x,t)| \le \frac{\lambda_*(t)^m}{|z-q|^2} \chi_{\{|x-q| \ge \lambda_*(t) R(t)\}}.$$

Then

$$\begin{aligned} |\psi(x,t)| &\le C T^m |\log T|^{2-m}, \\ |\psi(x,t) - \psi(x,T)| &\le C |\log T|^m (T-t)^m |\log(T-t)|^{2-2m}, \\ |\nabla \psi(x,t)| &\le C \frac{T^{m-1} |\log T|^{2-m}}{R(T)}, \\ |\nabla \psi(x,t) - \nabla \psi(x,T)| &\le C \frac{\lambda_*(t)^{m-1} |\log(T-t)|}{R(t)} \end{aligned}$$

and for any $\gamma \in (0, \frac{1}{2})$*:*

$$\frac{|\nabla \psi(x,t) - \nabla \psi(x',t')|}{(|x-x'|^2 + |t-t'|)^\gamma} \le C \frac{1}{(\lambda_*(t) R(t))^{2\gamma}} \frac{\lambda_*(t)^{m-1} |\log(T-t)|}{R(t)}$$

for any $|x - x'| \le 2\lambda_*(t) R(t)$ *and* $0 \le t' \le t \le T$ *such that* $t - t' \le \frac{1}{10}(T-t)$*.*

The proof is in Sect. A.1.

Lemma A.3 *Let* ψ *solve* (A.3) *with* f *such that*

$$|f(x,t)| \le 1,$$

Then

$$\begin{aligned} |\psi(x,t)| &\le Ct, \\ |\psi(x,t) - \psi(x,T)| &\le C(T-t)|\log(T-t)|, \\ |\nabla \psi(x,t)| &\le T^{1/2} |\nabla \psi(x,t) - \nabla \psi(x,T)| \quad \le C(T-t)^{1/2} \\ |\nabla \psi(x,t_2) - \nabla \psi(x,t_1)| &\le C|t_2 - t_1|^{1/2}. \\ |\nabla \psi(x_1,t) - \nabla \psi(x_2,t)| &\le C|x_1 - x_2| |\log(|x_1 - x_2|). \end{aligned}$$

The proof is in Sect. A.1.

Proof of Proposition A.1 Let $\psi_0[f]$ denote the solution of (A.3) where f satisfies $\|f\|_{**} < \infty$.

We claim that $\|\psi_0[f]\|_* \le C\|f\|_{**}$. Indeed, given f with $\|f\|_{**} < \infty$ we decompose $f = \sum_{i=1}^3 f_i$ with $|f_i| \le C\|f\|_{**}\varrho_i$. By linearity it is sufficient to prove that when f is each of the ϱ_i, the corresponding ψ has finite $\| \ \|_{**}$ norm.

The case $f = \varrho_1$ is direct from Lemma A.1. Using the hypothesis $\Theta < \beta$ we can find σ_0 small so that the case $f = \varrho_2$ follows from Lemma A.2. The case $f = \varrho_3$ follows from Lemma A.3.

Finally, let us show that in problem (6.31) we can choose c_i so that that $\psi(q, T) = 0$. To do this we let ψ_i the solution

$$\begin{cases} \partial_t \psi_i = \Delta_x \psi_i \quad \text{in } \Omega \times (0, T) \\ \psi_i = 0 \quad \text{on } \partial\Omega \times (0, T) \\ \psi_i(x, 0) = \mathbf{e_i}\eta_1 \quad \text{in } \Omega \end{cases}$$

Let

$$\psi = \psi_0 + \sum_{i=1}^{3} c_i \psi_i.$$

Then for $T > 0$ small there is unique choice of c_i such that $\psi(q, T) = 0$. Moreover $|c_i| \le C\lambda_*(0)^\nu R(0)^{2-a} |\log T| \|f\|_{**}$ and hence ψ satisfies (A.2). □

A.1 Proof of Lemmas A.1, A.2, and A.3

The proof of the estimates is done by analyzing the solution ψ of

$$\begin{cases} \partial_t \psi_0 = \Delta \psi_0 + f \quad \text{in } \mathbb{R}^2 \times (0, T), \\ \psi_0(x, 0) = 0 \quad x \in \mathbb{R}^2, \end{cases} \tag{A.10}$$

defined by Duhamel's formula

$$\psi_0(x, t) = \int_0^t \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{4\pi(t-s)} f(y, s)\, dy ds$$

assuming

$$|f(x, t)| \le \chi_{\{|y| \le 2\lambda_*(s)R(s)\}} \lambda_*(t)^{\nu-2} R(t)^{-a}.$$

The solution to (A.3) is then given by $\psi = \psi_0 + \psi_1$ where ψ_1 solves the homogeneous heat equation in $\Omega \times (0, T)$ with boundary condition given by

$-\psi_0$. In the sequel we prove that the estimates (A.4)–(A.5) are valid for ψ_0. Then the conclusion for ψ_1 follows from standard parabolic estimates. In what follows we denote by ψ the solution to (A.10) given by Duhamel's formula.

Proof of (A.4) We have, using the heat kernel,

$$\psi(x,t) = C\int_0^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{t-s}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}}\,dyds$$
$$= C\int_0^t \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2}\,dzds$$

where $\tilde{x} = x(t-s)^{-1/2}$. First we estimate

$$\int_0^{t-(T-t)} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2}\,dzds$$
$$\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{t-s}ds \le C\lambda_*(0)^{\nu}R(0)^{2-a}. \tag{A.11}$$

Consider the integrals $\int_{t-(T-t)}^{t-\lambda_*(t)^2}$ and $\int_{t-\lambda_*(t)^2}^{t}$. We have

$$\int_{t-(T-t)}^{t-\lambda_*(t)^2} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds$$
$$\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|. \tag{A.12}$$

For the second part we have

$$\int_{t-\lambda_*(t)^2}^{t} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds$$
$$\le C\int_{t-\lambda_*(t)^2}^{t} \lambda_*(s)^{\nu-2}R(s)^{-a}\,ds \le C\lambda_*(t)^{\nu}R(t)^{-a}. \tag{A.13}$$

From (A.11), (A.12), (A.13),we deduce

$$|\psi(x,t)| \le C\lambda_*(0)^{\nu}R(0)|\log T|.$$

which is the desired estimate. Estimates (A.4), (A.5), (A.6), (A.7), (A.8), (A.9) follow in similar manner. □

The proofs of Lemmas A.2 and A.3 follow similar lines to those above, and we omit them.

Appendix B: The heat equation with initial condition

In this section we consider the heat equation

$$\begin{cases} \partial_t \tilde{Z}_1(x,t) = \Delta \tilde{Z}_1(x,t) & \text{in } \Omega \times (0,T) \\ \tilde{Z}_1(x,0) = Z_1^*(x) & x \in \Omega \\ \tilde{Z}_1(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{B.1}$$

and derive estimates assuming, roughly speaking, that Z_1^* behaves like $(r+\varepsilon)|\log(r+\varepsilon)|$.

Lemma B.1 *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x-q_0|+\varepsilon} \quad x \in \Omega.$$

Then the solution $\tilde{Z}_1$ of (B.1) *satisfies*

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C\frac{T-t}{T}\left(1+\log\left(\frac{T}{t}\right)\right) \quad \textit{if } \varepsilon^2 \le t \le T.$$

Proof We do the computation when Ω is $\mathbb{R}^2$ and we deal with the solution given by Duhamel's formula. The general case follows by the decomposing the solution as a sum of the one in $\mathbb{R}^2$ and a smooth one in Ω. Then

$$\nabla_x \tilde{Z}_1(x,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4t}} \nabla_x Z_1^*(x-y)\,dy. \tag{B.2}$$

Assume $\varepsilon^2 \le t \le T$. Then, using (B.2), we have

$$\begin{aligned} &|\nabla_x \tilde{Z}_1(0,t) - \nabla_x \tilde{Z}_1(0,T)| \\ &\quad = \frac{1}{4\pi}\left|\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 \nabla_x \tilde{Z}_1^*\big(-s\sqrt{T}y + (1-s)\sqrt{t}y\big)\,ds dy\right| \\ &\quad \le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 |D^2 Z_1^*\big(-s\sqrt{T}y + (1-s)\sqrt{t}y\big)|(\sqrt{T}-\sqrt{t})|y|\,ds dy \\ &\quad \le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 \frac{(\sqrt{T}-\sqrt{t})|y|}{s(\sqrt{T}-\sqrt{t})|y| + \sqrt{t}|y| + \varepsilon}\,ds dy \\ &\quad \le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \log\Big(\frac{\sqrt{T}|y|+\varepsilon}{\sqrt{t}|y|+\varepsilon}\Big)\,dy \end{aligned}$$

$$= C \int_0^\infty e^{-\frac{\rho^2}{4}} \log\Big(\frac{\sqrt{T}\rho + \varepsilon}{\sqrt{t}\rho + \varepsilon}\Big)\rho \, d\rho$$
$$= C(\sqrt{T} - \sqrt{t})\varepsilon \int_0^\infty e^{-\frac{\rho^2}{4}} \frac{1}{(\sqrt{T}\rho + \varepsilon)(\sqrt{t}\rho + \varepsilon)} \, d\rho.$$

Using this representation, after some computation the desired result follows. □

Similar computations leads us to the following estimates.

Lemma B.2 *Suppose* $Z_1^* \in C^2(\overline{\Omega})$ *satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x - q_0| + \varepsilon} \quad x \in \Omega.$$

Then the solution $\tilde{Z}_1$ *of* (B.1) *satisfies*

$$|D_x^2 \tilde{Z}_1(x, t) \le \frac{C}{\varepsilon + \sqrt{t}}$$

Lemma B.3 *Suppose* $Z_1^* \in C^2(\overline{\Omega})$ *satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x - q_0| + \varepsilon} \quad x \in \Omega.$$

Then the solution $\tilde{Z}_1$ *of* (B.1) *satisfies for* $0 \le t_0 \le t_1$*:*

$$|\nabla_x \tilde{Z}_1(x, t_1) - \nabla_x \tilde{Z}_1(x, t_0)| \le C \begin{cases} \frac{\sqrt{t_1} - \sqrt{t_0}}{\sqrt{t_1}} \log\left(2\frac{t_1}{t_0}\right) & \text{if } t_0 \ge \varepsilon^2 \\ \frac{\sqrt{t_1} - \sqrt{t_0}}{\sqrt{t_1}} \log\left(2\frac{t_1}{\varepsilon^2}\right) & \text{if } t_0 \le \varepsilon^2, \ t_1 \ge \varepsilon^2 \\ \frac{\sqrt{t_1} - \sqrt{t_0}}{\varepsilon} & \text{if } t_1 \le \varepsilon^2 \end{cases}$$

Let us recall the norm $\| \ \|_*$ defined in (6.5). As a corollary of the previous estimates we have.

Lemma B.4 *Suppose* $Z_0^* \in C^2(\overline{\Omega})$*. Then the solution* $\tilde{Z}^*$ *of* (B.1) *satisfies*

$$|\nabla_x \tilde{Z}^*(x, t)| \le |\log \varepsilon| \, \|Z_0^*\|_*, \quad t \ge 0,$$
$$|Z^*(x, t) - Z^*(x, T)| \le C |\log T| \frac{T - t}{\sqrt{T}} \|Z_0^*\|_*,$$
$$|\nabla_x \tilde{Z}_1(x, t) - \nabla_x \tilde{Z}_1(x, T)| \le C \|Z_0^*\|_* \begin{cases} |\log \varepsilon| & \text{if } 0 \le t \le \varepsilon^2 \\ |\log \varepsilon|^{1/2} \frac{T-t}{T} \left(1 + \log\left(\frac{T}{t}\right)\right) & \text{if } \varepsilon^2 \le t \le T. \end{cases}$$

Appendix C: Derivatives for the exterior problem

Corollary C.1 *Let $\Psi(p, \xi, \Phi, Z_0^*)$ be the solution to equation* (6.22) *constructed in Proposition* 6.7. *Let p_l, ξ satisfy* (6.1), (6.2) *and $p_l = \lambda e^{i\omega_l}$, $\|\Phi_l\|_E \le 1$, and $\|Z_{0l}^*\|_* < \infty$, $l = 1, 2$. Then*

$$\|\Psi(p_1, \xi, \Phi_1, Z_{01}^*) - \Psi(p_2, \xi, \Phi_2, Z_{02}^*)\|_{\sharp,\Theta,\gamma} \le CT^\sigma(\|\Phi_1 - \Phi_2\|_E + \|\lambda_*(\dot\omega_1 - \dot\omega_2)\|_\infty + \|Z_{01}^* - Z_{02}^*\|_*).$$

Corollary C.1 gives a partial Lipschitz property of the exterior solution $\Psi(p, \xi, \phi)$ of (6.22) with respect to p, namely it only considers variations of $p = \lambda e^{i\omega}$ with respect to ω. We will need Lipschitz estimates for variations of $p = \lambda e^{i\omega}$ in λ and also variations with respect to ξ. These estimates are obtained for $\Psi(p, \xi, \phi)$ when considered as a function of the inner variable $(y, t) \in \mathcal{D}_{2R}$.

For this let us introduce some notation. Suppose that $\psi(x, t)$ is defined in $\Omega \times (0, T)$. We let

$$\tilde\psi(y, t) = \psi(\xi(t) + \lambda(t)y, t), \quad (y, t) \in \mathcal{D}_{2R}.$$

The following expression is $\|\psi\|_{\sharp,\Theta,\gamma}$ expressed in terms of $\tilde\psi$ (and restricted to $\mathcal{D}_{2R}$):

$$\begin{aligned}
\|\tilde\psi\|_{\sharp\sharp,\Theta,\gamma} := {} & \lambda_*(0)^{-\Theta}\frac{1}{|\log T|\lambda_*(0)R(0)}\|\tilde\psi\|_{L^\infty(\mathcal{D}_{2R})} \\
& + \lambda_*(0)^{-\Theta-1}\|\nabla_y\psi\|_{L^\infty(\mathcal{D}_{2R})} \\
& + \sup_{\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{-1}\frac{1}{|\log(T-t)|}|\tilde\psi(y,t) - \tilde\psi(y,T)| \\
& + \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(y,T)| \\
& + \sup_{(y,t),(y',t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(y',t)|}{|y-y'|^{2\gamma}} \\
& + \sup \lambda_*(t)^{-\Theta-1}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t) - \nabla_y\tilde\psi(x',t')|}{|t-t'|^\gamma},
\end{aligned}$$

where the last supremum is taken in the region

$$(y, t), (y, t'), \in \mathcal{D}_{2R}, \quad |t - t'| \le \frac{1}{10}(T - t).$$

Corollary C.2 *Let $\Psi(p, \xi, \phi)$ be the solution to equation (6.22) in Proposition 6.7. Let $p_l = \lambda_l e^{i\omega}$, ξ_l satisfy (6.1), (6.2) and $\|\phi\|_{*,a,\nu} \le 1$. Then for $\tilde\Theta \in (0, \Theta)$ we have*

$$\|\tilde\Psi(p_1, \xi_1, \phi) - \tilde\Psi(p_2, \xi_2, \phi)\|_{\sharp\sharp,\tilde\Theta,\gamma} \le C\Big[\Big\|\frac{\lambda_1 - \lambda_2}{\lambda_*}\Big\|_{L^\infty} + \|\dot\lambda_1 - \dot\lambda_2\|_{L^\infty} + \Big\|\frac{\xi_1 - \xi_2}{\lambda_* R}\Big\|_{L^\infty} + \Big\|\frac{\dot\xi_1 - \dot\xi_2}{R}\Big\|_{L^\infty}\Big].$$

Let $f(y, t)$ be a function satisfying

$$|f(y, t)| \le \lambda_*(t)^\nu R(t)^{-a} \chi_{B_{R(t)}},$$

and let $\psi[\lambda, \xi]$ be the solution of

$$\begin{cases} \psi_t = \Delta_x \psi + \dfrac{1}{\lambda(t)^2} f\left(\dfrac{x - \xi}{\lambda}, t\right) & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi(x, 0) = 0 \quad x \in \mathbb{R}^2, \end{cases}$$

given by Duhamel's formula.

Let

$$\tilde\psi[\lambda, \xi](y, t) = \psi[\lambda, \xi](\xi(t) + \lambda(t) y, t).$$

We consider the directional derivative with respect to λ of $\tilde\psi$ in the direction of λ_1, defined by

$$D_\lambda \tilde\psi[\lambda, \xi][\lambda_1] = \lim_{s\to 0} \frac{1}{s}\left(\tilde\psi[\lambda + s\lambda_1, \xi] - \tilde\psi[\lambda, \xi]\right)$$

and also the directional derivative with respect to ξ of $\tilde\psi$ in the direction of ξ_1, defined by

$$D_\xi \tilde\psi[\lambda, \xi][\xi_1] = \lim_{s\to 0} \frac{1}{s}\left(\tilde\psi[\lambda, \xi + s\xi_1] - \tilde\psi[\lambda, \xi]\right).$$

C.1 Derivative with respect to λ

The proofs of the estimates below are based on Duhamel's formula for the solution:

$$\psi(x, t) = \int_0^t \int_{\mathbb{R}^2} \frac{\exp\left(-\frac{|x - x'|^2}{4(t-s)}\right)}{t - s} \frac{1}{\lambda(s)^2} f\left(\frac{x' - \xi(s)}{\lambda(s)}, s\right) dx' ds.$$

We change variables writing $x = \xi(t) + \lambda(t)y$ and $x' = \xi(s) + \lambda(s)y'$. Then

$$\tilde{\psi}(y,t) = \int_0^t \int_{\mathbb{R}^2} \frac{\exp\left(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)}\right)}{t-s} f(y',s)\,dy'ds,$$

and we obtain the following formula for the directional derivative:

$$\begin{aligned} &D_\lambda \tilde{\psi}[\lambda_1](y,t) \\ &\quad = -\frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{\exp\left(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)}\right)}{(t-s)^2} \\ &\qquad \cdot (\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y') \\ &\qquad \cdot (\lambda_1(t)y-\lambda_1(s)y') f(y',s)\,dy'ds. \end{aligned}$$

Lengthy but direct computations show the validity of the following estimates.

Lemma C.1 *We have*

$$|D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,t)| \le C\left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty}\right)\lambda_*(0)^\nu R(0)^{2-a},$$

and

$$\begin{aligned} &|D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,t) - D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,T)| \\ &\quad \le C\left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty}\right)\lambda_*(t)^\nu R(t)^{2-a}, \end{aligned}$$

for $|y| \le R(t)$, $t \in (0,T)$. *On the other hand, for any* $\sigma > 0$, $\gamma \in (0,\frac{1}{2})$ *there is a* C *such that*

$$\begin{aligned} &|\nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,t) - \nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,T)| \\ &\quad \le C\left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty}\right)\lambda_*(t)^{\nu-1-\sigma} R(t)^{1-a}, \\ &|\nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y_1,t) - \nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y_2,t)| \\ &\quad \le C\left(\frac{|y_1-y_2|}{R(t)}\right)^\gamma \left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty}\right)\lambda_*(t)^{\nu-1-\sigma} R(t)^{1-a}, \\ &|\nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,t_2) - \nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y,t_1)| \\ &\quad \le C\frac{(t_2-t_1)^\gamma}{(\lambda_*(t_2)R(t_2))^{2\gamma}}\left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty}\right)\lambda_*(t_2)^{\nu-1-\sigma} R(t_2)^{1-a} \end{aligned}$$

for t_1, t_2 *in* $[0,T]$ *with* $0 \le t_2 - t_1 \le \frac{1}{10}(T-t_2)$ *and* $|y| \le R(t_2)$.

We can derive a similar expression for the derivative with respect to ξ and obtain the following estimates.

Lemma C.2 *Assume that* $|\dot{\xi}(t)| \le C$, $|\dot{\lambda}(t)| \le C$, $C_1\lambda_*(t) \le \lambda(t) \le C_2\lambda_*(t)$, *in* $(0, T)$ *and let* $R(t) = \lambda_*(t)^{-\beta}$, $\beta < \frac{1}{2}$, $C, C_1, C_2 > 0$. *Then there is C such that*

$$|\nabla_x D_\xi \tilde{\psi}[\lambda, \xi](\xi_1)(y, t) - \nabla_x D_\xi \tilde{\psi}[\lambda, \xi](\xi_1)(y, T)| \le C\Big(\Big\|\frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R}\Big\|_{L^\infty} + \Big\|\frac{\dot{\xi}_1}{R}\Big\|_{L^\infty}\Big)\lambda_*(t)^{\nu-1}R(t)^{1-a},$$

for $|y| \le R(t)$, $t \in (0, T)$.

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