Nonlinearity

PAPER

Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension

To cite this article: María E Martínez 2020 *Nonlinearity* **33** 1156

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **33** (2020) 1156–1182 https://doi.org/10.1088/1361-6544/ab591c

Decay of small odd solutions for long range Schrödinger and Hartree equations in one dimension

María E Martínez[1]

Departamento de Ingenería Matemática DIM, FCFM Universidad de Chile, Santiago, Chile

E-mail: maria.martinez.m@uchile.cl

Received 8 July 2019, revised 14 October 2019
Accepted for publication 19 November 2019
Published 5 February 2020

Recommended by Dr Jean-Claude Saut

Abstract

We consider the long time asymptotics of (not necessarily small) odd solutions to the nonlinear Schrödinger equation with semi-linear and nonlocal Hartree nonlinearities, in one dimension of space. We assume data in the energy space $H^1(\mathbb{R})$ only, and we prove decay to zero in compact regions of space as time tends to infinity. We give three different results where decay holds: semilinear NLS, NLS with a suitable potential, and defocusing Hartree. The proof is based on the use of suitable virial identities, in the spirit of nonlinear Klein–Gordon models (Kowalczyk *et al* 2017 *Lett. Math. Phys.* **107** 921–31), and covers scattering sub, critical and supercritical (long range) nonlinearities. No spectral assumptions on the NLS with potential are needed.

Keywords: long-range, scattering, Schrödinger, Hartree, Coulomb potential, decay
Mathematics Subject Classification numbers: 35

1. Introduction

In this paper our goal is to study the long time behavoir of *small odd* global solutions of the one-dimensional nonlinear Schrödinger (NLS) and Hartree equations

$$\mathrm{i}u_t + u_{xx} = g(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

[1] Partially funded by Conicyt Beca Doctorado Nacional no. 21192076, CMM Conicyt PIA AFB170001 and Fondecyt Regular 1150202 and 1191412. Part of this work was done while the author was visiting Université Paris-Saclay (Sud), Institut Mittag-Leffler and Universidad de Granada FisyMat, whose support is greatly acknowledged.

In the Schrödinger case (see Ginibre–Velo [22], Cazenave–Weissler [8] and Cazenave [6]), we shall assume that the nonlinearity takes the form

$$g(u) = \mu V(x)u + f\left(|u|^2\right)u, \tag{1.2}$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ is a *Schwartz even function* and $f : \mathbb{R} \to \mathbb{R}$ is a function such that for $1 < p < 5$ (L^2 subcritical case),

$$|f(s)| \lesssim s^{\frac{p-1}{2}}, \tag{1.3}$$

and that satisfies that $f \circ s^2$ is locally Lipschitz continuous. In this context, we denote $F(s) = \int_0^s f(v)\mathrm{d}v$, for all $s > 0$, and

$$G(u) = \frac{\mu}{2}\int_{\mathbb{R}} V(x)|u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} F(|u|^2)\mathrm{d}x.$$

In the Hartree case, we have

$$g(u) = \sigma\left(W * |u|^2\right)u, \quad G(u) = \frac{\sigma}{4}\int_{\mathbb{R}}\left(W * |u|^2\right)|u|^2\mathrm{d}x, \tag{1.4}$$

where $\sigma = \pm 1$ and the potential W is given by

$$W(x) = \frac{1}{|x|^a}, \quad \text{with} \quad 0 < a < 1. \tag{1.5}$$

The equation (1.1) is Hamiltonian, and it is characterized by having at least the following conservation laws:

- Mass:

$$M(u(t)) := \int_{\mathbb{R}} |u(t)|^2\mathrm{d}x = M(u(0)). \tag{1.6}$$

- Energy:

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}} |\nabla u(t)|^2\mathrm{d}x + G(u(t)) = E(u(0)). \tag{1.7}$$

- Momentum:

$$P(u(t)) := \mathrm{Im}\int_{\mathbb{R}} u(t)\bar{u}_x(t)\mathrm{d}x = P(u(0)). \tag{1.8}$$

The NLS equations (1.1) and (1.2) with nonlinearity $f(s) = \pm s^{\frac{p-1}{2}}$ is commonly known as the semilinear Schrödinger equation [6]. In particular, if $f(s) = -s^{\frac{p-1}{2}}$, we say that the equation is focusing, while the defocusing case takes place when $f(s) = s^{\frac{p-1}{2}}$. It is well-known that this one-dimensional semilinear Schödinger equation is globally well-posed for initial data in $H^1(\mathbb{R})$ when $1 < p < 5$, and blow up may occur if $p \geqslant 5$, see e.g. [23, 35] and subsequent works.

On the other hand, the Hartree equation (1.1) with (1.4) is also locally well-posed in $H^1(\mathbb{R})$, and globally well-posed for small data, see [6, corollary 6.1.5] for instance. This comes from the fact that the potential W in (1.5) is an even function that satisfies the following properties:

- $W \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$,
- The function $(W * |u|^2)|u|^2$ is integrable. For the case (1.5), one has the estimate

$$\int_{\mathbb{R}} \left(|x|^{-a} * |u|^2\right) |u|^2 \mathrm{d}x < \infty,$$

(we prove this using the Hardy–Littlewood–Sobolev inequality [32, theorem 4.3, p 106] with $p = r = \frac{2}{2-a}$).

This means that we are in the case of [6, example 3.2.11] and [6, corollary 4.3.3], which implies the local well-posedness of the Hartree equation.

In this paper we are interested in the asymptotic behavior of small solutions to (1.1), both in the NLS case (with and without potential), and in the nonlocal Hartree case, at least in the defocusing case. The literature on this subject is huge; we present now a (far from complete) account of the most relevant results.

It is known that for subcritical (in the sense of GWP and scattering) semilinear NLS equation ($f(s) = \pm s^{\frac{p-1}{2}}$, $3 < p < 5$), scattering to a free solution exists (see, for instance, Ginibre and Velo [22], Tsutsumi [54] and Nakanishi–Ozawa [40]). Nevertheless, in Strauss [51] and Barab [3] it was proven that one cannot expect the same scattering for the critical ($p = 3$) and super critical case ($p < 3$), and modified scattering is believed to occur. This was generalized recently by Murphy and Nakanishi [38] for the semilinear NLS equation with potential and Hartree-type nonlinearities as (1.5).

Precisely, *modified scattering* for d dimensional critical NLS equation with nonlinearities

$$g(u) = \sigma|u|^{p-1}u, \quad p = 1 + \frac{2}{d}, \quad d = 1, 2, 3;$$

and the Hartree equation with Coulomb potential

$$g(u) = \sigma \left(|x|^{-1} * |u|^2\right) u, \quad d \geqslant 2,$$

and small initial condition, was first proved by Ozawa [43] and by Ginibre and Ozawa [21]. Moreover, it was shown that solutions u of such equations present the decay

$$\|u(t)\|_{L^\infty} \lesssim (1 + |t|)^{-d/2}, \tag{1.9}$$

when the initial data is sufficiently small in weighted Sobolev spaces (see also Hayashi–Naumkin [16], and Kato–Pusateri [27], for instance). Through a thorough analysis of the solution profile, a simplified proof of scattering in the critical defocusing NLS and Hartree equations has been exhibited in [27].

Similar recent results hold for the NLS case *with a potential*, as was shown by Cuccagna *et al* [14] for $p > 3$, and Naumkin [41] and Germain–Pusateri–Rousset [20] for the critical case $p = 3$ (see also [19]). Nakanishi [39] considered 3D NLS with a potential having a single negative eigenvalue, and proved asymptotics for large time. Indeed, assuming that the potential V is such that $-\frac{1}{2}\Delta + V$ does not have negative eigenvalues nor resonances at zero, they were able to prove the decay (1.9) for solutions of subcritical ($p > 3$) and critical ($p = 3$) NLS equation in one dimension. However different the methods to prove this decay are from each other, it is not clear to us if their technics still hold by assuming less restrictive spectral conditions.

Finally, following idea introduced in [29], about considering odd data only, Delort [17] proved modified scattering for small (smaller than a parameter ϵ) odd solutions u to (1.9) with data in $H^{0,1} \cap H^N$, N large, and showed (among other things) the precise decomposition for large time

$$u(t,x) = \frac{\epsilon}{\sqrt{t}} A_\epsilon\left(\frac{x}{t}\right) \exp\left[-\mathrm{i}\frac{x^2}{2t} + \mathrm{i}\epsilon^2 \log t \left|A_\epsilon\left(\frac{x}{t}\right)\right|^2\right] + r(t,x),$$

where the continuous function A_ϵ is bounded in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\theta \in (0, \frac{1}{4})$ and

$$\|r(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{3}{4}+\theta}), \quad \|A_\epsilon(x)\langle tx\rangle^{-2}\|_{L^\infty} = O(\epsilon t^{-\frac{1}{4}+\theta}),$$

and

$$\|r(t,\cdot)\|_{L^2} = O(\epsilon t^{-\frac{1}{4}+\theta}), \quad \|A_\epsilon(x)\langle tx\rangle^{-2}\|_{L^2} = O(\epsilon t^{-\frac{5}{8}+\frac{\theta}{2}}).$$

Notice that all positive decay/scattering results above mentioned cannot deal with the one dimensional NLS (for $p < 3$) and Hartree equations. This is in part explained by the lack of precise nonlinear estimates in the case of long range nonlinearities.

Our main goal in this paper is to extend in some sense the recently mentioned results [17, 20, 27, 41] and show decay of small solutions to the above equations, regardless the (supercritical with respect to scattering) power of the nonlinearity. In particular, we consider nonlinearities NLS with $1 < p < 5$ and Hartree long range supercritical in one dimension.

Our first result covers the NLS case without potential ($1 < p < 5$).

Theorem 1.1. *Suppose $u(t) \in H^1(\mathbb{R})$ is a global odd solution of the equation (1.1) and (1.2) and $\mu = 0$ such that, for some $\varepsilon > 0$ small,*

$$\|u(t=0)\|_{H^1(\mathbb{R})} \leqslant \varepsilon. \tag{1.10}$$

Then,

$$\lim_{t\to\infty} \left(\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)}\right) = 0, \tag{1.11}$$

for any bounded interval $I \subset \mathbb{R}$. Moreover, if the equation is defocusing, the smallness condition (1.10) is not needed.

Remark 1.1. NLS (1.1) preserves the oddness of the initial data along the flow.

Remark 1.2. As far as we could understand, theorem 1.1 is the first decay result for small data NLS in the long range supercritical nonlinearities $1 < p < 3$. Although we do not give a precise description of a possible limiting profile as in the previous literature, our results show dispersion after all.

Remark 1.3. Theorem 1.1 is sharp. Indeed, it is not true for $u(t) \in H^1$ even. A simple counterexample in this case is the non decaying soliton itself:

$$u(t,x) = Q_c(x)\mathrm{e}^{\mathrm{i}ct}, \qquad 0 < c \ll 1, \tag{1.12}$$

and $Q_c > 0$ solving $Q_c'' - cQ_c + Q_c^p = 0$, $Q_c \in H^1$. Note that this solution is even in space and small in H^1 provided $c \ll 1$. Also, the Satsuma–Yajima breather solution (see [46] and [1, equation (1.16)]) is an arbitrarily small non decaying even solution to NLS (1.2) in the integrable [56] case $p = 3$.

Remark 1.4. For an interval $I = I(t)$ growing in time, theorem 1.1 is also sharp. Indeed, see the works [33, 42] for the construction of odd solutions composed of two solitary waves with non zero speeds for finite time. These asymptotic two-soliton solutions can be arbitrarily small in the energy space, but they separate each other as time evolves, leaving any compact region in space for sufficiently large time. In this sense, these solutions do not contradict theorem 1.1.

Remark 1.5. From the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} x|u(t,x)|^2\mathrm{d}x = -2\,\mathrm{Im}\int_{\mathbb{R}} u(t)\bar{u}_x(t)\mathrm{d}x = -2P(u(t)),$$

valid if $xu(t=0)\in L^2$, we can see that nontrivial, non decaying periodic-in-time solutions (i.e. breathers) of NLS may exist only if their momentum vanishes. See [37] for more details on these properties of breather solutions.

Remark 1.6. Sometimes, instead of assuming odd data, the additional assumption $\|xu(t=0)\|_{L^2}\ll 1$ is considered. This condition works with even data, and rules out the existence of small solitary waves as in (1.12), since solitary waves that are small in terms of the seminorm $\dot{H}^1$ satisfy $\|xQ_c\|_{L^2}\gg 1$.

Remark 1.7. Note that (1.11) does not contain the $\dot{H}^1$ norm of the solution. This is a standard open issue in the field, see e.g. [17] for similar results. In our case, the lack of control on the decay of this semi-norm is due to the emergence of uncontrolled H^2 terms in the dynamics of the energy norm.

Remark 1.8. In the defocusing case, we expect better results. For instance, we can prove that $\liminf_{t\to+\infty}\|u_x(t)\|_{L^2(|x|\lesssim|t|\log^{-1}|t|)}=0$, but a better decay property is out of reach for the moment.

The proof of theorem 1.1 is based on the introduction of a virial identity adapted to the NLS dynamics. Following the ideas presented in [29, 30], which considered the nonlinear Klein–Gordon case, we use here a functional adapted to the momentum (1.8). Once this virial identity is established, decay is proved in a standard form.

Compared with the available results for Klein–Gordon [30], where H^1 decay is proven, the main novelty here is that we avoid the lack of H^1 decay in time for NLS (remark 1.7) by proving time decay in L^∞_x instead; also, we consider the cases of NLS with a nontrivial potential and with Hartree nonlinearities (see below), both of important physical interest, and not treated in [30].

Using inverse scattering techniques, Deift and Zhou [15] described the asymptotic behavior of solutions of the defocusing, nearly integrable quintic perturbation of cubic NLS

$$\mathrm{i}u_t+u_{xx}=|u|^2u+\epsilon|u|^4u,\quad \epsilon>0. \tag{1.13}$$

Using the techniques of this paper, we are able to give a partially complementary result to the one stated in [15]:

Corollary 1.2. *Let $\epsilon\neq 0$, and let $u\in C(\mathbb{R};H^1(\mathbb{R}))$ be a global small odd solution of (1.13). Then (1.11) is satisfied.*

The proof of this result immediately follows from theorem 1.1.

Our second result deals with NLS (1.1) with nonzero potential in (1.2). In this case, we also provide time decay results in the case μV small and spatially decaying fast enough, complementing [14, 17, 20, 41].

Theorem 1.3 (NLS with potential). *Assume $V \neq 0$ even as in (1.2). Under the assumptions of theorem 1.1, suppose additionally that V satisfies*

$$\int_{\mathbb{R}} (|V(x)| + |V'(x)|) \cosh(2x)\mathrm{d}x < +\infty. \tag{1.14}$$

Then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, (1.11) holds for any bounded interval $I \subset \mathbb{R}$.

Remark 1.9. Note that theorem 1.3 does not require that the operator $-\partial_x^2 \pm \mu V$ satisfies specific spectral properties as in [14, 20, 41]; only the decay hypothesis (1.14) is needed. In particular, no *non resonance* condition is needed for having (1.11). This fact reveals that the non resonance condition is essentially linked to the evenness of the involved data.

Remark 1.10. We can ask for V decaying slower than in (1.14), but proofs are probably more complicated; we hope to consider this problem elsewhere.

Finally, we deal with the Hartree case.

Theorem 1.4 (Defocusing Hartree equation). *Suppose that $u \in H^1(\mathbb{R})$ is a global odd solution of equation (1.1) with (1.5) and $\sigma = 1$. Then*

$$\lim_{t\to\infty} (\|u(t)\|_{L^2(I)} + \|u(t)\|_{L^\infty(I)}) = 0, \tag{1.15}$$

for any bounded interval $I \subset \mathbb{R}$.

Remark 1.11. Theorem 1.4 proves the non-existence of odd standing waves solutions for the equation (1.1) with defocusing Hartree type nonlinearities.

Remark 1.12. Theorem 1.4 does not include the focusing case, which is an open problem of independent interest. In that sense, the scattering problem for the $d \geqslant 2$ generalized Hartree equation was recently treated in Arora–Roudenko [2].

Remark 1.13. Focusing Hartree equation (1.1) with (1.5) ($\sigma = -1$) admits solitary waves solutions (or solitons)

$$u(t,x) = \mathrm{e}^{\mathrm{i}ct} Q_c(x) \in H^1$$

where $Q_c : \mathbb{R} \to \mathbb{R}$ is an H^1-solution of the Choquard equation

$$\Delta Q + \left(\frac{1}{|x|^a} * |Q|^p\right) Q - \lambda Q = 0, \quad c \in \mathbb{R}. \tag{1.16}$$

These solutions are, up to translation and inversion of the sign, positive and radially symmetric functions [9, 36]. Moreover, solitary waves for the focusing Hartree equation are stable, as was proven by Cazenave and Lions in [7]. See also Ruiz [45] for more details on solitary waves for Hartree.

Remark 1.14 (NLS around solitary waves). Solitary waves in mass subcritical NLS exist and they are stable. The first results on stability were provided by Cazenave and Lions in [7], where orbital stability of solitary waves for the NLS equations (1.1) and (1.2) without potential was proven (see also [25, 55]). Stability of several NLS solitons well-decoupled was proved in [34], and in [26] for the integrable case. The asymptotic stability for the same equation was studied by Buslaev and Perelman in [4] in the supercritical regime; this result was later generalized by Cuccagna in [10, 11, 13] for dimensions $d \geqslant 3$, and under special spectral conditions on the linearized operator around the solitary wave. The one dimensional case, under similar spectral assumptions and even data perturbations of the standing wave, was studied by Buslaev and Sulem [5]. For the NLS equation with potential (1.1) and (1.2), results for asymptotic stability of ground states (also, under spectral conditions) were provided by Soffer and Weinstein in [49, 50], see also [18, 48, 52, 53], and [44] for the case of multi-solitons in general dimensions. We believe that some of the ideas in this paper can be generalized to the case of asymptotic stability for solitary waves, but with harder proofs. See e.g. the recent paper by Cuccagna and Maeda [12], and the NLKG paper by Kowalczyk *et al* [31].

1.1. Notation

To simplify the notation we will denote $u_1 = \mathrm{Re}\ u$, $u_2 = \mathrm{Im}\ u$. Let $\alpha(x) \geqslant 0$ be a weight. We also denote by

$$\begin{aligned} \|u(t)\|^2_{L^2_\alpha(\mathbb{R})} &:= \int_{\mathbb{R}} \alpha(x)|u(t,x)|^2 \mathrm{d}x, \\ \|u(t)\|^2_{H^1_\alpha(\mathbb{R})} &:= \int_{\mathbb{R}} \alpha(x)\left(|u_x(t,x)|^2 + |u(t,x)|^2\right)\mathrm{d}x, \end{aligned} \tag{1.17}$$

the weighted L^2-norm and H^1-norm with weight α.

1.2. Organization of this paper

This paper is written as follows. In section 2 we prove theorem 1.1, NLS without potential. Section 3 is devoted to the proof of theorem 1.3, namely NLS with potential. Finally, section 4 deals with the Hartree case (theorem 1.4).

2. Schrödinger equation without potential

In this section we prove theorem 1.1. Consider the equation (1.1) with (1.2) and $V \equiv 0$. That is,

$$\mathrm{i}u_t + u_{xx} = f\left(|u|^2\right)u, \qquad u \in H^1 \text{ odd}. \tag{2.1}$$

As claimed in the introduction, the proof here follows the ideas in [30], with some minor differences.

2.1. A virial identity

We shall introduce a standard virial identity adapted to (2.1). Let $\varphi \in C^\infty(\mathbb{R})$ be bounded and to be chosen later, $u(t) \in H^1(\mathbb{R})$ a solution of equation (2.1) and define

$$I(u(t)) := \mathrm{Im} \int_{\mathbb{R}} \varphi(x)u(t,x)\bar{u}_x(t,x)\mathrm{d}x. \tag{2.2}$$

Then we have the following:

Lemma 2.1. *For $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ one has $I(u(t))$ well-defined and bounded in time. Moreover, we have the virial identity*

$$-\frac{\mathrm{d}}{\mathrm{d}t} I(t) = 2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}|u|^2 \mathrm{d}x - \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right] \mathrm{d}x. \tag{2.3}$$

Proof. Let $u(t) \in H^1(\mathbb{R})$ such that it satisfies equation (2.1). Then, we integrate by parts

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= \operatorname{Im} \int_{\mathbb{R}} \varphi u_t \bar{u}_x \mathrm{d}x + \operatorname{Im} \int_{\mathbb{R}} \varphi u \bar{u}_{xt} \mathrm{d}x \\ &= \operatorname{Im} \int_{\mathbb{R}} \varphi u_t \bar{u}_x \mathrm{d}x - \operatorname{Im} \int_{\mathbb{R}} (\varphi u)_x \bar{u}_t \mathrm{d}x. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= -\operatorname{Im} \int_{\mathbb{R}} \mathrm{i}\varphi\left(\mathrm{i}u_t\right) \bar{u}_x \mathrm{d}x - \operatorname{Im} \int_{\mathbb{R}} \mathrm{i}\left(\varphi u\right)_x \overline{\mathrm{i}u_t} \mathrm{d}x \\ &= -\operatorname{Re} \int_{\mathbb{R}} \varphi \overline{\mathrm{i}u_t} u_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} (\varphi u)_x \overline{\mathrm{i}u_t} \mathrm{d}x. \end{aligned}$$

Computing the derivative on the second term above

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= -2\operatorname{Re} \int_{\mathbb{R}} \varphi \overline{\mathrm{i}u_t} u_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x u \overline{\mathrm{i}u_t} \mathrm{d}x \\ &= -2\operatorname{Re} \int_{\mathbb{R}} \varphi(\mathrm{i}u_t) \bar{u}_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x \left(\mathrm{i}u_t\right) \bar{u} \mathrm{d}x. \end{aligned} \tag{2.4}$$

Thus, using (2.1), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= 2\operatorname{Re} \int_{\mathbb{R}} \varphi u_{xx} \bar{u}_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x u_{xx} \bar{u} \mathrm{d}x \\ &\quad - 2\operatorname{Re} \int_{\mathbb{R}} \varphi f\left(|u|^2\right) u \bar{u}_x \mathrm{d}x - \operatorname{Re} \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) u \bar{u} \mathrm{d}x. \end{aligned}$$

We notice that $2\operatorname{Re}\,(u_x \bar{u}) = 2\operatorname{Re}\,(u \bar{u}_x) = \left(|u|^2\right)_x$, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= \int_{\mathbb{R}} \varphi \left(|u_x|^2\right)_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x u_{xx} \bar{u} \mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \varphi f\left(|u|^2\right) \left(|u|^2\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) |u|^2 \mathrm{d}x. \end{aligned}$$

Recall the definition of $F(s) = \int_0^s f(v)\mathrm{d}v$, which implies that $(F(s))_x = f(s)s_x$. Furthermore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= \int_{\mathbb{R}} \varphi \left(|u_x|^2\right)_x \mathrm{d}x + \operatorname{Re} \int_{\mathbb{R}} \varphi_x \bar{u} u_{xx} \mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \varphi \left(F\left(|u|^2\right)\right)_x \mathrm{d}x - \int_{\mathbb{R}} \varphi_x f\left(|u|^2\right) |u|^2 \mathrm{d}x. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}}\varphi_{xx}\bar{u}u_x\mathrm{d}x + \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right]\mathrm{d}x\\ &= -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xx}\left(|u|^2\right)_x\mathrm{d}x + \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right]\mathrm{d}x.\end{aligned}$$

We integrate by parts again on the second term to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x + \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f\left(|u|^2\right)|u|^2\right]\mathrm{d}x.$$

□

2.2. Analysis of a bilinear form

With the identity (2.3) in mind, we define the bilinear form

$$B(w) = 2\int_{\mathbb{R}}\varphi_x w_x^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}w^2\mathrm{d}x, \quad w = u_i, \quad i = 1, 2. \tag{2.5}$$

Here, $u = u_1 + \mathrm{i}u_2$, with u_1, u_2 real-valued.

Let $\lambda \in (1, \infty)$. As we explained before, our intention is to prove some estimation of B using the weighted H^1_α-norm introduced in (1.17). To obtain this, we will consider $\varphi(x) = \lambda\tanh\left(\frac{x}{\lambda}\right)$ on the virial identity (2.3) and define the auxiliar function $\alpha(x) = \sqrt{\varphi_x(x)}$. Now, we estimate each term of the bilinear form B:

$$\begin{aligned}\int_{\mathbb{R}}(\alpha w)_x^2\,\mathrm{d}x &= \int_{\mathbb{R}}\alpha^2\left(w_x\right)^2\mathrm{d}x + 2\int_{\mathbb{R}}\alpha\alpha_x w w_x\mathrm{d}x + \int_{\mathbb{R}}\left(\alpha_x\right)^2 w^2\mathrm{d}x\\ &= \int_{\mathbb{R}}\varphi_x\left(w_x\right)^2\mathrm{d}x + \int_{\mathbb{R}}\alpha\alpha_x\left(w^2\right)_x\mathrm{d}x + \int_{\mathbb{R}}\left(\alpha_x\right)^2 w^2\mathrm{d}x\\ &= \int_{\mathbb{R}}\varphi_x\left(w_x\right)^2\mathrm{d}x - \int_{\mathbb{R}}\alpha\alpha_{xx}w^2\mathrm{d}x,\end{aligned}$$

using integration by parts in the last equality. Thus

$$\int_{\mathbb{R}}\varphi_x\left(w_x\right)^2\mathrm{d}x = \int_{\mathbb{R}}(\alpha w)_x^2\,\mathrm{d}x + \int_{\mathbb{R}}\frac{\alpha_{xx}}{\alpha}(\alpha w)^2\,\mathrm{d}x. \tag{2.6}$$

Furthermore, noticing that $\varphi_{xxx} = \left(\alpha^2\right)_{xx} = 2\left(\alpha\alpha_{xx} + \alpha_x^2\right)$, we get

$$\int_{\mathbb{R}}\varphi_{xxx}w^2\mathrm{d}x = 2\int_{\mathbb{R}}\left(\frac{\alpha_{xx}}{\alpha} + \frac{\alpha_x^2}{\alpha^2}\right)(\alpha w)^2\,\mathrm{d}x. \tag{2.7}$$

Hence, from (2.6) and (2.7),

$$B(w) = 2\int_{\mathbb{R}}(\alpha w)_x^2\,\mathrm{d}x - \int_{\mathbb{R}}\left(\frac{\alpha_x^2}{\alpha^2} - \frac{\alpha_{xx}}{\alpha}\right)(\alpha w)^2\,\mathrm{d}x.$$

Since $\alpha(x) = \mathrm{sech}\left(\frac{x}{\lambda}\right)$, then

$$\begin{aligned}\alpha_x(x) &= -\frac{1}{\lambda}\mathrm{sech}\left(\frac{x}{\lambda}\right)\tanh\left(\frac{x}{\lambda}\right)\\ \alpha_{xx}(x) &= \frac{1}{\lambda^2}\left(\mathrm{sech}\left(\frac{x}{\lambda}\right)\tanh\left(\frac{x}{\lambda}\right) - \mathrm{sech}^3\left(\frac{x}{\lambda}\right)\right)\end{aligned}$$

which implies that

$$B(w) = 2\int_{\mathbb{R}} (\alpha w)_x^2 \, dx - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right)(\alpha w)^2 \, dx.$$

In order to prove theorem 1.1 we need to prove that the bilineal part of (2.3) is coercive in some way. To be more precise, we would like the following

$$B(w) \geqslant \int_{\mathbb{R}} (\alpha w)_x^2 \, dx. \tag{2.8}$$

We introduce the auxiliar function $v = \alpha w$. Then we can set

$$\mathcal{B}(v) = 2\int_{\mathbb{R}} v_x^2 dx - \frac{1}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 dx$$

so that

$$\mathcal{B}(v) = B(w).$$

This way, coercivity of the operator $\mathcal{B}$ implies (2.8). We recall now

Proposition 2.2 (See [30]). *Let $v \in H^1(\mathbb{R})$ be odd, $\lambda > 0$. Then*

$$\mathcal{B}(v) \geqslant \frac{3}{2}\int_{\mathbb{R}} v_x^2 dx. \tag{2.9}$$

Sketch of proof. We write

$$\mathcal{B}(v) = \frac{3}{2}\int_{\mathbb{R}} v_x^2 dx + \frac{1}{2}\left(\int_{\mathbb{R}} v_x^2 dx - \frac{2}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 dx\right).$$

Notice that

$$-\frac{d^2}{dx^2} - \frac{2}{\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda}\right)$$

has only one negative eigenvalue corresponding to an even eigenfunction. This comes from the fact that (see [24, exercise 12]) the index of the operator

$$-\frac{h^2}{\nu}\frac{d^2}{dx^2} - \gamma \operatorname{sech}^2\left(\frac{x}{a}\right)$$

is equal to the largest integer N such that

$$N < \frac{1}{2}\sqrt{8\gamma\nu a^2 h^{-2} + 1} - \frac{1}{2}.$$

Since v is odd,

$$\int_{\mathbb{R}} v_x^2 dx - \frac{2}{\lambda^2}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) v^2 dx \geqslant 0, \tag{2.10}$$

and then (2.9) holds. □

2.3. Estimates of the terms on (2.3)

Lemma 2.3. *Let $u \in H^1(\mathbb{R})$ be odd. Then for some $C > 0$, $u = u_1 + \mathrm{i}u_2$,*

$$\|u\|^2_{H^1_\alpha(\mathbb{R})} \leqslant C\left(B(u_1) + B(u_2)\right). \tag{2.11}$$

Proof. We take $\lambda = 100$. First, notice that from (2.10), we have

$$\int_{\mathbb{R}} (\alpha w)_x^2 \geqslant \frac{2}{100^2} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{100}\right)(\alpha w)^2\,\mathrm{d}x.$$

Using proposition 2.2, this implies that

$$B(w) \geqslant \frac{3}{2}\int_{\mathbb{R}} (\alpha w)_x\,\mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right)u^2\mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}(x)w^2\mathrm{d}x. \tag{2.12}$$

Thus,

$$\int_{\mathbb{R}} \operatorname{sech}(x)u_i^2\mathrm{d}x \lesssim B(u_i), \quad i = 1, 2. \tag{2.13}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} (\alpha w)_x^2\,\mathrm{d}x &\gtrsim \int_{\mathbb{R}} \alpha^2 (\alpha w)_x^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \alpha^4 w_x^2\mathrm{d}x + \int_{\mathbb{R}} \alpha^3\alpha_x\left(w^2\right)_x\mathrm{d}x + \int_{\mathbb{R}} \alpha^2\alpha_x^2 w^2\mathrm{d}x.\end{aligned}$$

We integrate by parts,

$$\begin{aligned}\int_{\mathbb{R}} (\alpha w)_x^2\,\mathrm{d}x &\gtrsim \int_{\mathbb{R}} \alpha^4 w_x^2\mathrm{d}x - \int_{\mathbb{R}} \left(\alpha^3\alpha_x\right)_x w^2\mathrm{d}x + \int_{\mathbb{R}} \alpha^2\alpha_x^2 w^2\mathrm{d}x \\ &= \int_{\mathbb{R}} \alpha^4 w_x^2\mathrm{d}x - \int_{\mathbb{R}} \left(2\alpha^2\alpha_x^2 + \alpha^3\alpha_{xx}\right) w^2\mathrm{d}x.\end{aligned}$$

Then, from the definition of α,

$$\int_{\mathbb{R}} (\alpha w)_x^2\,\mathrm{d}x \gtrsim \int_{\mathbb{R}} \operatorname{sech}(x)w_x^2\mathrm{d}x - \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right)w^2\mathrm{d}x.$$

In other words,

$$\int_{\mathbb{R}} \operatorname{sech}(x)w_x^2\mathrm{d}x \lesssim \int_{\mathbb{R}} (\alpha w)_x^2\,\mathrm{d}x + \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{x}{100}\right)w^2\mathrm{d}x.$$

Then, from (2.12), we have that

$$\int_{\mathbb{R}} \operatorname{sech}(x)w_x^2\mathrm{d}x \lesssim \int_{\mathbb{R}} (\alpha w)_x^2\,\mathrm{d}x. \tag{2.14}$$

Hence, using proposition 2.2,

$$\int_{\mathbb{R}} \operatorname{sech}(x) u_{ix}^2 \mathrm{d}x \lesssim B(u_i), \quad i = 1, 2. \tag{2.15}$$

Finally, from (2.13) and (2.15), we get

$$\|u(t)\|^2_{H^1_\alpha(\mathbb{R})} \lesssim B(u_1) + B(u_2).$$

□

Lemma 2.4. *There exists $\varepsilon > 0$ such that:*
If u is an odd solution of (2.1) satisfying (1.10), then

$$-\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) \geqslant C \|u(t)\|^2_{H^1_\alpha(\mathbb{R})} \tag{2.16}$$

where $C > 0$.

Proof. Recall from (2.3) and the analysis of the previous section that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) &= 2\int_{\mathbb{R}} \varphi_x |u_x|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}|u|^2 \mathrm{d}x - \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right] \mathrm{d}x \\ &= B(u_1) + B(u_2) - \int_{\mathbb{R}} \varphi_x \left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right] \mathrm{d}x. \end{aligned}$$

Consequently, in order to complete the proof, we need to control the remaining terms of (2.3), since the terms involving the bilinear form B have already been estimated by lemma 2.3.

Note that

$$\left|F\left(|u|^2\right) - f(|u|^2)|u|^2\right| \lesssim |u|^{p+1}.$$

Since u is odd,

$$\begin{aligned} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{\lambda}\right) |u|^{p+1} \mathrm{d}x &= 2\int_0^\infty \operatorname{sech}^2\left(\frac{x}{\lambda}\right) |u|^{p+1} \mathrm{d}x \\ &= 2\int_0^\infty \operatorname{sech}^{-(p-1)}\left(\frac{x}{\lambda}\right) \operatorname{sech}^{p+1}\left(\frac{x}{\lambda}\right) |u|^{p+1} \\ &\simeq \int_0^\infty \mathrm{e}^{(p-1)x/\lambda} \operatorname{sech}^{p+1}\left(\frac{x}{\lambda}\right) |u|^{p+1} \mathrm{d}x. \end{aligned}$$

With a slight abuse of notation, set $v(t,x) := \operatorname{sech}\left(\frac{x}{\lambda}\right) u(t,x)$. Note that $v(t,0) = 0$ and vanishes at infinity $\forall t \in \mathbb{R}$. Then, integrating by parts,

$$\begin{aligned} \int_0^\infty \mathrm{e}^{(p-1)x/\lambda} |v|^{p+1} \mathrm{d}x &= -\frac{\lambda}{p-1}\int_0^\infty \mathrm{e}^{(p-1)x/\lambda} \left(|v|^{p+1}\right)_x \mathrm{d}x \\ &= -\frac{\lambda(p+1)}{p-1} \mathrm{Re} \int_0^\infty \mathrm{e}^{(p-1)x/\lambda} |v|^{p-1} \bar{v} v_x \mathrm{d}x. \end{aligned}$$

Hence,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x &= -\frac{\lambda(p+1)}{p-1}\mathrm{Re}\int_0^\infty \mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\overline{v}v_x\left(\mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\right)\mathrm{d}x\\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\mathrm{Re}\int_0^\infty \mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}\overline{v}v_x\mathrm{d}x\\
&\lesssim \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\int_0^\infty \mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p-1}{2}}|v||v_x|\mathrm{d}x\\
&= \|u\|_{L^\infty(\mathbb{R})}^{(p-1)/2}\int_0^\infty \mathrm{e}^{(p-1)x/2\lambda}|v|^{\frac{p+1}{2}}|v_x|\mathrm{d}x.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x &\lesssim \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^\infty |v_x|^2\mathrm{d}x + \int_0^\infty \mathrm{e}^{(p-1)x/\lambda}|v|^{p+1}\mathrm{d}x\\
&\simeq \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^\infty |v_x|^2\mathrm{d}x + \int_0^\infty \mathrm{sech}^{p-1}\left(\frac{x}{\lambda}\right)\mathrm{sech}^{p+1}\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x\\
&= \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^\infty |v_x|^2\mathrm{d}x + \int_0^\infty \mathrm{sech}^{2}\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x,\end{aligned}$$

which actually means that

$$\int_{\mathbb{R}} \mathrm{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \|u\|_{L^\infty(\mathbb{R})}^{p-1}\int_0^\infty |(\alpha u)_x|^2\mathrm{d}x.$$

By Sobolev's embedding,

$$\int_{\mathbb{R}} \mathrm{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^\infty |(\alpha u)_x|^2\mathrm{d}x.$$

Now, it is a fact that for every $0<\varepsilon<1$, there exists $\delta(\varepsilon)$ such that $\|u(0)\|_{H^1(\mathbb{R})}\leqslant\delta(\epsilon)$ implies that $\sup_{t\in\mathbb{R}}\|u\|_{H^1(\mathbb{R})}<\varepsilon$ (see [6, corollary 6.1.4] or the conservation of energy and mass (1.6) and (1.7)). This way, from proposition 2.2, we get

$$\int_{\mathbb{R}} \mathrm{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \lesssim \varepsilon^{p-1}\left(B(u_1)+B(u_2)\right).$$

So, choosing ε sufficiently small, (2.16) is proved. □

Remark 2.1 (Defocusing case). Note that in the semilinear defocusing case $f(|u|^2)=|u|^{p-1}$,

$$F\left(|u|^2\right)-f(|u|^2)|u|^2=\left(\frac{2}{p+1}-1\right)|u|^{p+1}.$$

Since $p>1$, $\frac{2}{p+1}|u|^{p+1}-1<0$ which means that the remaining term on (2.3) involving the nonlinearity is positive:

$$-\int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right)-f(|u|^2)|u|^2\right]\mathrm{d}x\geqslant 0$$

and then lemma 2.3 is enough to conclude lemma 2.4.

With this estimation, we can now prove the key to get theorem 1.1.

Proposition 2.5. *There exists a constant $C > 0$ such that*

$$\int_0^\infty \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t \leqslant C\varepsilon^2. \tag{2.17}$$

Proof. Let $\tau > 0$. We integrate (2.16) over $[0, \tau]$

$$\int_0^\tau \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t \leqslant C\left(I(u(0)) - I(u(\tau))\right) \leqslant CI(u(0)).$$

From Hölder inequality and (1.10) we get that

$$I(u(0)) \leqslant \|u(0)\|_{L^2(\mathbb{R})}\|u_x(0)\|_{L^2(\mathbb{R})} \leqslant \varepsilon^2.$$

This last fact implies that

$$\int_0^\tau \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t \leqslant C\varepsilon^2.$$

Now, taking $\tau \to \infty$, we conclude the proof. □

2.4. End of proof of theorem 1.1

Now theorem 1.1 is ready to be proved:

Step 1: The L^2 norm tends to zero: Let $\varphi \in C^\infty(\mathbb{R})$ be bounded. Then we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right) &= \mathrm{Re}\int_{\mathbb{R}}\varphi\bar{u}u_t\mathrm{d}x = -\mathrm{Re}\int_{\mathbb{R}} i\varphi\bar{u}\left(\mathrm{i}u_t\right)\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi\bar{u}\left(\mathrm{i}u_t\right)\mathrm{d}x.\end{aligned}$$

Hence, using equation (2.1) and integrating by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right) &= -\mathrm{Im}\int_{\mathbb{R}}\varphi\bar{u}u_{xx}\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)u\bar{u}\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi\bar{u}_xu_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi_x\bar{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)|u|^2\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\varphi|u_x|^2\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi_x\bar{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\varphi f(|u^2|)|u|^2\mathrm{d}x.\end{aligned}$$

Since the integrals on the first and third term are real, we get the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\right) = \mathrm{Im}\int_{\mathbb{R}}\varphi_x\bar{u}u_x\mathrm{d}x. \tag{2.18}$$

Thus

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right)\right| \leqslant \int_{\mathbb{R}}|\varphi_x||\bar{u}(t)||u_x(t)|\mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}}|\varphi_x||\bar{u}(t)|^2\mathrm{d}x + \int_{\mathbb{R}}|\varphi_x||u_x(t)|^2\mathrm{d}x.$$

We take $\varphi(x) = \mathrm{sech}(x)$ and get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\mathrm{sech}(x)|u(t,x)|^2\mathrm{d}x\right)\right|$$
$$\lesssim \int_{\mathbb{R}}\mathrm{sech}(x)|\bar{u}(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}}\mathrm{sech}(x)|u_x(t,x)|^2\mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.$$

From (2.17), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_w(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over $[t, t_n]$, and take $t_n \to \infty$. Then

$$\|u(t)\|^2_{L^2_\alpha(\mathbb{R})} \lesssim \int_t^\infty \|u(s)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}s.$$

In consequence

$$\lim_{t\to\infty}\|u(t)\|_{L^2_w(\mathbb{R})} = 0. \tag{2.19}$$

Step 2: The L^∞ norm tends to zero: We state the following:

Claim 2.6. For every interval I there exists $\tilde{x}(t) \in I$ such that, as t tends to infinity,

$$|u(t,\tilde{x}(t))|^2 \to 0.$$

Proof. Let $I \in \mathbb{R}$ be an interval. By contradiction, Suppose that there exists $\varepsilon_0 > 0$ such that $\forall n > 0$, $\exists t_n > n$

$$|u(t_n,x)|^2 > \varepsilon_0 \quad \forall x \in I.$$

Integrating over I, we get

$$\int_I |u(t_n,x)|^2\mathrm{d}x > |I|\varepsilon_0,$$

which contradicts (2.19). □

Let $x \in I$. By fundamental theorem of calculus and Hölder's inequality

$$|u(t,x)|^2 - |u(t,\tilde{x}(t))|^2 = \int_{\tilde{x}(t)}^x \left(|u|^2\right)_x \mathrm{d}x \leqslant 2\int_{\tilde{x}(t)}^x |u||u_x|\mathrm{d}x$$
$$\leqslant 2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)}.$$

Then we get

$$|u(t,x)|^2 \lesssim |u(t,\tilde{x}(t))|^2 + 2\|u(t)\|_{L^2(I)}\|u_x(t)\|_{L^2(I)}, \quad \forall x \in I. \tag{2.20}$$

Now, since (1.10) holds for $\varepsilon > 0$ as small as needed,

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} < \infty.$$

Also, this smallness condition is not needed if the nonlinearity is defocusing. Hence, taking $t \to \infty$ in (2.20), from claim 2.6 and (2.19), we get that

$$|u(t,x)|^2 \to 0, \quad \forall x \in I.$$

Which implies (1.11). The proof of theorem 1.1 is complete.

3. NLS with potential

This section is devoted to the proof of theorem 1.3. We consider now the NLS equation with a nontrivial potential V:

$$\mathrm{i}u_t + u_{xx} = \mu V(x)u + f\left(|u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{3.1}$$

As done in the previous section, we introduce a virial identity that will be used to estimate the H^1_α-norm of a solution of equation (3.1). However, because of the potential term V, new estimates must be proved in order to get theorem 1.3.

3.1. Virial identity

Suppose again $\varphi \in C^\infty(\mathbb{R})$ bounded and recall from section 2.1 the definition

$$I(u(t)) = \mathrm{Im}\int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_x(t,x)\mathrm{d}x.$$

Following the proof of lemma 2.1, we have now

Lemma 3.1. *Let $u(t) \in H^1(\mathbb{R})$ be a bounded in time solution of equation (3.1). Then*

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}I(t) &= 2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x - \mu\int_{\mathbb{R}}\varphi V_x|u|^2\mathrm{d}x\\ &\quad -\int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x.\end{aligned} \tag{3.2}$$

Sketch of proof. From the proof of lemma 2.1 (equation (2.4)) we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) = -2\mathrm{Re}\int_{\mathbb{R}}\varphi(\mathrm{i}u_t)\overline{u}_x\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}}\varphi_x(\mathrm{i}u_t)\overline{u}\mathrm{d}x.$$

We use (3.1) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(u(t)) &= 2\mathrm{Re}\int_{\mathbb{R}}\varphi u_{xx}\overline{u}_x\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}}\varphi_x u_{xx}\overline{u}\mathrm{d}x - 2\mu\mathrm{Re}\int_{\mathbb{R}}\varphi Vu\overline{u}_x\mathrm{d}x\\ &\quad - \mu\mathrm{Re}\int_{\mathbb{R}}\varphi_x Vu\overline{u}\mathrm{d}x - 2\mathrm{Re}\int_{\mathbb{R}}\varphi f\left(|u|^2\right)u\overline{u}_x\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}}\varphi_x f\left(|u|^2\right)u\overline{u}\mathrm{d}x.\end{aligned}$$

From the last equation, we are only interested in the terms involving the potential V, since the rest of them were analyzed in the proof of lemma 2.1. Then we compute

$$2\mathrm{Re}\int_{\mathbb{R}}\varphi Vu\bar{u}_x\mathrm{d}x+\mathrm{Re}\int_{\mathbb{R}}\varphi_x Vu\bar{u}\mathrm{d}x=\int_{\mathbb{R}}\varphi V\left(|u|^2\right)_x\mathrm{d}x+\int_{\mathbb{R}}\varphi_x V|u|^2\mathrm{d}x$$
$$=-\int_{\mathbb{R}}\varphi V_x|u|^2\mathrm{d}x.$$

Combining this with lemma 2.1, we conclude (3.2). □

3.2. Analysis of a modified bilinear form

In the following analysis, we will see more clearly the difference between the cases with and without potential. In this occasion, we define a new bilinear form ($u=u_1+\mathrm{i}u_2$, $u_i\in\mathbb{R}$)

$$B(w)=2\int_{\mathbb{R}}\varphi_x w_x^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}w^2\mathrm{d}x-\mu\int_{\mathbb{R}}\varphi V_x w^2\mathrm{d}x,\quad w=u_i,\quad i=1,2.$$

Consider $\lambda\in(1,\infty)$, $\varphi(x)=\lambda\tanh\left(\frac{x}{\lambda}\right)$ and $\alpha(x)=\sqrt{\varphi_x(x)}$. Since $\alpha^2=\varphi_x$, we can write

$$\int_{\mathbb{R}}\varphi V_x w^2\mathrm{d}x=\int_{\mathbb{R}}V_x\frac{\varphi}{\varphi_x}(\alpha w)^2\,\mathrm{d}x. \tag{3.3}$$

Thus, from (2.6), (3.3) and (2.7),

$$B(w)=2\int_{\mathbb{R}}(\alpha w)_x^2\,\mathrm{d}x-\int_{\mathbb{R}}\left(\frac{\alpha_x^2}{\alpha^2}-\frac{\alpha_{xx}}{\alpha}\right)(\alpha w)^2\,\mathrm{d}x-\mu\int_{\mathbb{R}}V_x\frac{\varphi}{\varphi_x}(\alpha w)^2\,\mathrm{d}x.$$

Then, from computations of section 2.2 we have that

$$B(w)=2\int_{\mathbb{R}}(\alpha w)_x^2\,\mathrm{d}x-\frac{1}{\lambda^2}\int_{\mathbb{R}}\mathrm{sech}^2\left(\frac{x}{\lambda}\right)(\alpha w)^2\,\mathrm{d}x-\mu\int_{\mathbb{R}}V_x\frac{\varphi}{\varphi_x}(\alpha w)^2\,\mathrm{d}x.$$

We set

$$\mathcal{B}(v)=2\int_{\mathbb{R}}v_x^2\mathrm{d}x-\frac{1}{\lambda^2}\int_{\mathbb{R}}\mathrm{sech}^2\left(\frac{x}{\lambda}\right)v^2\mathrm{d}x-\mu\int_{\mathbb{R}}V_x\frac{\varphi}{\varphi_x}v^2\mathrm{d}x,$$

where $v=\alpha w$. Then

$$\mathcal{B}(v)=B(w).$$

Now we prove a modified version of proposition 2.2.

Proposition 3.2. *Let $v\in H^1(\mathbb{R})$ be odd. Then, for $\lambda>0$ sufficiently small,*

$$\mathcal{B}(v)\geqslant\frac{1}{2}\int_{\mathbb{R}}v_x^2\mathrm{d}x. \tag{3.4}$$

Proof. We introduce

$$\mathcal{L}(v)=\int_{\mathbb{R}}v_x^2\mathrm{d}x-\frac{1}{\lambda^2}\int_{\mathbb{R}}\mathrm{sech}^2\left(\frac{x}{\lambda}\right)v^2\mathrm{d}x$$

and

$$\mathcal{K}(v) = \int_{\mathbb{R}} v_x^2 \mathrm{d}x + \mu \int_{\mathbb{R}} V_0 v^2 \mathrm{d}x,$$

where $V_0 = -V_x \frac{\varphi}{\varphi_x}$. Then,

$$\mathcal{B}(v) = \mathcal{L}(v) + \mathcal{K}(v).$$

Arguing as in the proof of proposition 2.2, we write

$$\mathcal{L}(v) = \frac{1}{2} \int_{\mathbb{R}} v_x^2 \mathrm{d}x + \frac{1}{2} \left(\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2} \int_{\mathbb{R}} \mathrm{sech}^2 \left(\frac{x}{\lambda} \right) v^2 \mathrm{d}x \right).$$

Since v is odd,

$$\int_{\mathbb{R}} v_x^2 \mathrm{d}x - \frac{2}{\lambda^2} \int_{\mathbb{R}} \mathrm{sech}^2 \left(\frac{x}{\lambda} \right) v^2 \mathrm{d}x \geqslant 0,$$

because the index N of such an operator is the integer that satisfies $N < \frac{1}{2}\sqrt{17} - \frac{1}{2} < 2$. Hence, we get that

$$\mathcal{L}(v) \geqslant \frac{1}{2} \int_{\mathbb{R}} v_x^2 \mathrm{d}x. \tag{3.5}$$

Then, in order to get the (3.4) it will be sufficient to demonstrate that $\mathcal{K}(v) \geqslant 0$.

To prove the positiveness of $\mathcal{K}$, we make use of the following result by Simon [47, theorem 2.5] (see also [28] for improved results):

Lemma 3.3. *Let V_0 be a non-identically zero potential that obeys*

$$\int_{\mathbb{R}} \left(1 + x^2\right) |V_0(x)| \mathrm{d}x < \infty.$$

Then

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$$

has a unique negative eigenvalue for all positive μ sufficiently small if and only if

$$\int_{\mathbb{R}} V_0(x) \mathrm{d}x \leqslant 0. \tag{3.6}$$

Moreover, since V_0 is even, such an eigenvalue is associated to an even eigenfunction.

Remark 3.1. We remark that in the case $\int_{\mathbb{R}} V_0 > 0$ there is no negative eigenvalue $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$, $\mu > 0$ sufficiently small.

Notice that, from the definition of φ and (1.14), we have

$$\int_{\mathbb{R}} V_0 \mathrm{d}x = -\int_{\mathbb{R}} V_x \frac{\varphi}{\varphi_x} \mathrm{d}x = -\lambda \int_{\mathbb{R}} V_x \sinh \left(\frac{x}{\lambda} \right) \cosh \left(\frac{x}{\lambda} \right) \mathrm{d}x.$$

We integrate by parts and get

$$\int_{\mathbb{R}} V_0 \mathrm{d}x = \int_{\mathbb{R}} \cosh\left(\frac{2x}{\lambda}\right) V \mathrm{d}x.$$

Since $\lambda > 1$, (1.14) tells us that V_0 integrates in space. Besides, since V is a Schwartz function,

$$\int_{\mathbb{R}} (1+x^2) \left| V_x \frac{\varphi}{\varphi_x} \right| \mathrm{d}x \leqslant \int_{\mathbb{R}} (1+x^2) |V_x| \cosh\left(\frac{2x}{\lambda}\right) \mathrm{d}x < \infty.$$

Then, lemma 3.3 implies that there exists $\mu_0 > 0$ such that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu V_0$$

has a unique negative eigenvalue for all $\mu < \mu_0$ and $\lambda > 1$. Since the corresponding eigenfunction is even, we have $\mathcal{K}(v) \geqslant 0$ for v odd. □

The conclusion that we obtain from proposition 3.4 is that for $i = 1, 2$,

$$B(u_i) \geqslant \frac{1}{2} \int_{\mathbb{R}} (\alpha u_i)_x^2 \, \mathrm{d}x.$$

This property of the bilinear form B will allow us to get an estimation of the operator $\frac{\mathrm{d}}{\mathrm{d}t} I(u(t))$ that will lead us to conclude the proof of theorem 1.3.

3.3. Estimates of the terms on (3.2)

Lemma 3.4. *Let u be an odd solution of (3.1). Then,*

$$\|u\|^2_{H^1_\alpha(\mathbb{R})} \leqslant C\,(B(u_1) + B(u_2)) \tag{3.7}$$

for some $C > 0$.

Proof. Direct from lemma 2.3. □

Lemma 3.5. *There exists $\varepsilon > 0$ such that for every odd solution u of (3.1) satisfying*

$$\|u(t)\|_{H^1(\mathbb{R})} \leqslant \varepsilon \quad \forall t \in \mathbb{R}, \tag{3.8}$$

then

$$-\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) \geqslant C \|u(t)\|^2_{H^1_\alpha(\mathbb{R})} \tag{3.9}$$

where $C > 0$.

Proof. The virial identity we have is

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}I(t) &= 2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x - \mu\int_{\mathbb{R}}\varphi V_x|u|^2\mathrm{d}x \\
&\quad - \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x \\
&= B(u_1) + B(u_2) - \int_{\mathbb{R}}\varphi_x\left[F\left(|u|^2\right) - f(|u|^2)|u|^2\right]\mathrm{d}x.
\end{aligned}
$$

As we already have an estimation for $B(u_1) + B(u_2)$ given by lemma 3.4, we need to check that the remaining terms can be controled. Replicating the proof of lemma 2.4, we get that

$$
\begin{aligned}
\int_{\mathbb{R}}\varphi_x\left[F(|u|^2) - f(|u|^2)|u|^2\right]\mathrm{d}x &\lesssim \int_{\mathbb{R}}\mathrm{sech}^2\left(\frac{x}{\lambda}\right)|u|^{p+1}\mathrm{d}x \\
&\lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^{\infty}\left|\left(\mathrm{sech}\left(\frac{x}{\lambda}\right)u_1\right)_x\right|^2\mathrm{d}x \\
&\quad + \|u\|_{H^1(\mathbb{R})}^{p-1}\int_0^{\infty}\left|\left(\mathrm{sech}\left(\frac{x}{\lambda}\right)u_2\right)_x\right|^2\mathrm{d}x.
\end{aligned}
$$

Thus, proposition 3.2 implies that

$$
\int_{\mathbb{R}}\varphi_x\left[F(|u|^2) - f(|u|^2)|u|^2\right]\mathrm{d}x \lesssim \|u\|_{H^1(\mathbb{R})}^{p-1}\big(B(u_1) + B(u_2)\big).
$$

Now, since $\|u\|_{H^1(\mathbb{R})}$ is small enough, we conclude. (In the defocusing case, this condition is not needed.) □

We can modify the proof of proposition 2.17, using lemma 3.5 instead of lemma 2.4 to obtain the following:

Proposition 3.6. *There exists a constant $C > 0$ such that*

$$
\int_0^{\infty}\|u(t)\|_{H_\alpha^1(\mathbb{R})}^2\mathrm{d}t \leqslant C\epsilon^2. \tag{3.10}
$$

3.4. Proof main result

Step 1: The L^2 norm tends to zero:

Let $\varphi \in C^{\infty}(\mathbb{R})$. Since $\mathrm{Im}\int_{\mathbb{R}}\varphi V|u|^2\mathrm{d}x = 0$, computing as in section 2.4, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\right) = \mathrm{Im}\int_{\mathbb{R}}\varphi_x\bar{u}u_x\mathrm{d}x. \tag{3.11}
$$

This identity implies that

$$
\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\varphi|u(t)|^2\mathrm{d}x\right)\right| \lesssim \int_{\mathbb{R}}|\varphi_x||\bar{u}(t)|^2\mathrm{d}x + \int_{\mathbb{R}}|\varphi_x||u_x(t)|^2\mathrm{d}x.
$$

Taking $\varphi(x) = \mathrm{sech}(x)$ we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \mathrm{sech}(x)|u(t)|^2\right)\right|$$
$$\lesssim \int_{\mathbb{R}} \mathrm{sech}(x)|\overline{u}(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}} \mathrm{sech}(x)|u_x(t,x)|^2\mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.$$

From (3.10), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_\alpha(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over *[t_n,t]*, and take $t_n \to \infty$. Then

$$\|u(t)\|^2_{L^2\alpha(\mathbb{R})} \lesssim \int_t^{\infty} \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t.$$

Passing to the limit

$$\lim_{t\to\infty} \|u(t_n)\|_{L^2_\alpha(\mathbb{R})} = 0. \tag{3.12}$$

Step 2: The L^∞ norm tends to zero:

One uses the same arguments as in section 2.4. We skip the proof.

4. The Hartree equation. Proof of theorem 1.4

Our goal in this section is to extend theorem 1.1 to the Hartree equation,

$$\mathrm{i}u_t + u_{xx} = \sigma\left(|x|^{-a} * |u|^2\right)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R} \tag{4.1}$$

where $\sigma = \pm 1$ and $0 < a < 1$. We start out with a virial identity.

4.1. Virial identity

As before (see (2.2)), let us consider $\varphi \in C^\infty(\mathbb{R})$ bounded and let

$$J(u(t)) := \mathrm{Im}\int_{\mathbb{R}} \varphi(x)u(t,x)\overline{u}_x(t,x)\mathrm{d}x, \tag{4.2}$$

then we state the following result.

Lemma 4.1. *Let $u \in H^1(\mathbb{R})$ be a solution of (4.1), then*

$$-\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = 2\int_{\mathbb{R}} \varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}|u|^2\mathrm{d}x + \sigma a\int_{\mathbb{R}} \varphi\left(\frac{x}{|x|^{a+2}} * |u|^2\right)|u|^2\mathrm{d}x. \tag{4.3}$$

Proof. Recall (2.4) from the proof of lemma 3.2,

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = -2\mathrm{Re}\int_{\mathbb{R}} \varphi(\mathrm{i}u_t)\overline{u}_x\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}} \varphi_x\overline{u}(\mathrm{i}u_t)\mathrm{d}x.$$

We use (4.1) to get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = {} & 2\mathrm{Re}\int_{\mathbb{R}}\varphi u_{xx}\overline{u}_x\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}}\varphi_x\overline{u}u_{xx}\mathrm{d}x \\
& - \sigma 2\mathrm{Re}\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)u\overline{u}_x\mathrm{d}x - \sigma\mathrm{Re}\int_{\mathbb{R}}\varphi_x\left(|x|^{-a}*|u|^2\right)u\overline{u}\mathrm{d}x \\
= {} & \int_{\mathbb{R}}\varphi\left(|u_x|^2\right)_x\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}}\varphi_x\overline{u}u_{xx}\mathrm{d}x \\
& - \sigma\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)\left(|u|^2\right)_x\mathrm{d}x - \sigma\int_{\mathbb{R}}\varphi_x\left(|x|^{-a}*|u|^2\right)|u|^2\mathrm{d}x.
\end{aligned}
$$

We integrate by parts once on the last term and twice on the second term to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) &= -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \mathrm{Re}\int_{\mathbb{R}}\varphi_{xx}\overline{u}u_x\mathrm{d}x + \sigma\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)_x|u|^2\mathrm{d}x \\
&= -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}}\varphi_{xx}\left(|u|^2\right)_x\mathrm{d}x + \sigma\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)_x|u|^2\mathrm{d}x \\
&= -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x + \sigma\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)_x|u|^2\mathrm{d}x.
\end{aligned}
$$

Computing the derivative on the last term,

$$
\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) = -2\int_{\mathbb{R}}\varphi_x|u_x|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}|u|^2\mathrm{d}x - \sigma a\int_{\mathbb{R}}\varphi\left(\frac{x}{|x|^{a+2}}*|u|^2\right)_x|u|^2\mathrm{d}x.
$$

□

Let us analyze the RHS of (4.3). Notice that if φ is a non-decreasing weight function, the integral on the last term in (4.3) is positive:

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x \geqslant 0. \tag{4.4}
$$

Indeed, we compute (all the computations below are justified by choosing suitably compactly supported functions, and taking the standard limit procedure)

$$
\begin{aligned}
\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)_x|u|^2\mathrm{d}x &= -a\int_{\mathbb{R}}\varphi\left(\frac{x}{|x|^{a+2}}*|u|^2\right)|u|^2\mathrm{d}x \\
&= -a\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x = {} & \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\varphi(x)-\varphi(y)\right)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x \\
& + \int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(y)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

After a change of variables on the second integral, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x=\int_{\mathbb{R}}\int_{\mathbb{R}}(\varphi(x)-\varphi(y))\,\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x$$
$$-\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x.$$

Then, we obtain that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}(\varphi(x)-\varphi(y))\,\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x.$$

If φ is non-decreasing, then $(\varphi(x)-\varphi(y))\,(x-y)\geqslant 0$. Moreover,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(\varphi(x)-\varphi(y))\,\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x\geqslant 0.$$

This implies that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x\geqslant 0,$$

as claimed.

4.2. Proof of theorem 1.4

Assume $\sigma=1$ in (4.1) and let $u=u_1+\mathrm{i}u_2\in H^1(\mathbb{R})$ be an odd solution of this equation. As done in section 2, we define the bilinear form

$$B(u_i)=2\int_{\mathbb{R}}\varphi_x u_{i_x}^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}}\varphi_{xxx}u_i^2\mathrm{d}x,\quad i=1,2.$$

This means that we can re-write the virial identity (4.2) as

$$-\frac{\mathrm{d}}{\mathrm{d}t}J(u(t))=B(u_1)+B(u_2)-\sigma\int_{\mathbb{R}}\varphi\left(|x|^{-a}*|u|^2\right)_x|u|^2\mathrm{d}x. \tag{4.5}$$

Now, as usual, take $\lambda>1$, $\varphi=\lambda\tanh\left(\frac{x}{\lambda}\right)$ and $\alpha=\sqrt{\varphi_x}$. From (2.6) and (2.7) and reasoning as before, we have that

$$B(u_i)=2\int_{\mathbb{R}}(\alpha u_i)_x^2\,\mathrm{d}x-\frac{1}{\lambda^2}\int_{\mathbb{R}}\mathrm{sech}^2\left(\frac{x}{\lambda}\right)(\alpha u_i)^2\,\mathrm{d}x,\quad i=1,2.$$

Thus, proposition 2.2 implies that

$$B(u_i)\geqslant\frac{3}{2}\int_{\mathbb{R}}(\alpha u_i)_x^2\,\mathrm{d}x,\quad \text{for } i=1,2. \tag{4.6}$$

Moreover, if we consider

$$\|u(t)\|_{H^1_\alpha(\mathbb{R})}=\int_{\mathbb{R}}\mathrm{sech}(x)u^2(t,x)\mathrm{d}x+\int_{\mathbb{R}}\mathrm{sech}(x)u_x^2(t,x)\mathrm{d}x,$$

then, from proposition 2.3 we obtain

$$\|u\|^2_{H^1_\alpha(\mathbb{R})}\lesssim B(u_1)+B(u_2). \tag{4.7}$$

Since

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\frac{x-y}{|x-y|^{a+2}}|u(y)|^2|u(x)|^2\mathrm{d}y\mathrm{d}x \geqslant 0,$$

it follows that

$$-\frac{\mathrm{d}}{\mathrm{d}t}J(u(t)) \geqslant \|u\|^2_{H^1_\alpha(\mathbb{R})}.$$

Replicating the proof of proposition 2.5, we use the last inequality to obtain

$$\int_0^\infty \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}\mathrm{d}t \leqslant C\varepsilon^2. \tag{4.8}$$

Step 1: The L^2 norm tends to zero: Let $\phi \in C^\infty(\mathbb{R})$ bounded. Then we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\mathrm{d}x\right) &= \mathrm{Re}\int_{\mathbb{R}}\phi\bar{u}u_t\mathrm{d}x\\ &= -\mathrm{Re}\int_{\mathbb{R}}\mathrm{i}\phi\bar{u}\left(\mathrm{i}u_t\right)\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\phi\bar{u}\left(\mathrm{i}u_t\right)\mathrm{d}x.\end{aligned}$$

Hence, using equation (4.1) with $\sigma = 1$ and integrating by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\mathrm{d}x\right) &= -\mathrm{Im}\int_{\mathbb{R}}\phi\bar{u}u_{xx}\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\phi\left(|x|^{-a}*|u|^2\right)u\bar{u}\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\phi\bar{u}_xu_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\phi\left(|x|^{-a}*|u|^2\right)|u|^2\mathrm{d}x\\ &= \mathrm{Im}\int_{\mathbb{R}}\phi|u_x|^2\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\phi_x\bar{u}u_x\mathrm{d}x + \mathrm{Im}\int_{\mathbb{R}}\phi\left(|x|^{-a}*|u|^2\right)|u|^2\mathrm{d}x.\end{aligned}$$

Since the only integral that can have an imaginary part is the second one, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\right) = \mathrm{Im}\int_{\mathbb{R}}\phi_x\bar{u}u_x\mathrm{d}x. \tag{4.9}$$

Thus

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}}\phi|u(t)|^2\mathrm{d}x\right)\right| &\leqslant \int_{\mathbb{R}}|\phi_x||\bar{u}(t)||u_x(t)|\mathrm{d}x\\ &\lesssim \int_{\mathbb{R}}|\phi_x||\bar{u}(t)|^2\mathrm{d}x + \int_{\mathbb{R}}|\phi_x||u_x(t)|^2\mathrm{d}x.\end{aligned}$$

We take $\phi(x) = \mathrm{sech}(x)$ and get

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2_\alpha(\mathbb{R})}\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\mathrm{sech}(x)|u(t)|^2\right)\right|\\ &\lesssim \int_{\mathbb{R}}\mathrm{sech}(x)|\bar{u}(t,x)|^2\mathrm{d}x + \int_{\mathbb{R}}\mathrm{sech}(x)|u_x(t,x)|^2\mathrm{d}x = \|u(t)\|^2_{H^1_\alpha(\mathbb{R})}.\end{aligned}$$

From (4.8), there exists a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$ such that $\|u(t_n)\|^2_{L^2_\alpha(\mathbb{R})} \to 0$. Consider $t \in \mathbb{R}$, integrate over $[t,t_n]$, and take $t_n \to \infty$. Then

$$\|u(t)\|_{L^2_\alpha(\mathbb{R})}^2 \lesssim \int_t^\infty \|u(s)\|_{H^1_\alpha(\mathbb{R})}^2 \mathrm{d}s.$$

In consequence

$$\lim_{t\to\infty} \|u(t)\|_{L^2_\alpha(\mathbb{R})} = 0. \tag{4.10}$$

The rest of the proof is exactly the same as in the proofs of theorems 1.1 and 1.3.

References

[1] Alejo M A, Fanelli L and Muñoz C 2019 Stability and instability of breathers in the $U(1)$ Sasa–Satsuma and nonlinear Schrödinger models (arXiv:1901.10381)
[2] Arora A K and Roudenko S 2019 Global behavior of solutions to the focusing generalized Hartree equation (arXiv:1904.05339v1)
[3] Barab J E 1984 Nonexistence of asymptotically free solutions for a nonlinear Schroödinger equation *J. Math. Phys.* **25** 3270
[4] Buslaev V S and Perelman G 1995 Nonlinear scattering: the states that are closed to a soliton *J. Math. Sci.* **77** 3161–9
[5] Buslaev V S and Sulem C 2003 On asymptotic stability of solitary waves for the nonlinear Schrödinger equations *Ann. l'Inst. Henri Poincaré* C **20** 419–75
[6] Cazenave T 2003 *Semilinear Schrödinger Equations* (*Courant Lecture Notes in Mathematics*) (New York: New York University)
[7] Cazenave T and Lions P-L 1982 Orbital stability of standing waves for some nonlinear Schrödinger equations *Commun. Math. Phys.* **85** 549–61
[8] Cazenave T and Weissler F 1990 The Cauchy problem for the critical nonlinear Schrödinger equation in H^s *Nonlinear Anal.* **14** 807–36
[9] Cingolani S, Secchi S and Squassina M 2017 Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities *Proc. R. Soc. Edinburgh* A **140** 973–1009
[10] Cuccagna S 2001 Stabilization of solutions to nonlinear Schrödinger equations *Commun. Pure Appl. Math.* **54** 1110–45
[11] Cuccagna S 2011 The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states *Commun. Math. Phys.* **305** 279–331
[12] Cuccagna S and Maeda M 2019 On stability of small solitons of the 1D NLS with a trapping delta potential (arXiv:1904.11869)
[13] Cuccagna S and Mizumachi T 2008 On asymptotic stability in energy space of ground states for nonlinear Schroödinger equations *Commun. Math. Phys.* **284** 51–77
[14] Cuccagna S, Visciglia N and Georgiev V 2014 Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential *Commun. Pure Appl. Math.* **67** 957–81
[15] Deift P and Zhou X 2002 Perturbation theory for the infinite-dimensional integrable system on the line. A case study *Acta Math.* **188** 163–262
[16] Hayashi N and Naumkin P I 1998 Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations *Am. J. Math.* **120** 369–89
[17] Delort J-M 2016 Modified scattering for odd solutions of cubic nonlinear Schrödinger equations with potential in dimension one (arXiv:hal-01396705)
[18] Gang Z and Sigal I M 2007 Relaxation of solitons in nonlinear Schrödinger equations with potential *Adv. Math.* **216** 443–90
[19] Germain P, Pusateri F and Rousset F 2016 Asymptotic stability of solitons for mKdV *Adv. Math.* **299** 272–330
[20] Germain P, Pusateri F and Rousset F 2018 The nonlinear Schrödinger equation with a potential *Ann. l'Inst. Henri Poincaré* C **35** 1477–530
[21] Ginibre J and Ozawa T 1993 Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n > 2$ *Commun. Math. Phys.* **151** 619–45
[22] Ginibre J and Velo G 1979 On a class of nonlinear Schrödinger equations. I: the Cauchy problem *J. Funct. Anal.* **32** 1–32

[23] Glassey R 1977 On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations *J. Math. Phys.* **18** 1794–7

[24] Gol'dman I I, Krivchenkov V D, Geilikman B T, Marquit E and Lepa E 1961 *Problems in Quantum Mechanics* authorised revised ed B T Geilikman (Oxford: Pergamon) (transl. Russian E Marquit and E Lepa)

[25] Grillakis M, Shatah J and Strauss W 1987 Stability theory of solitary waves in the presence of symmetry. I *J. Funct. Anal.* **74** 160–97

[26] Kapitula T 2007 On the stability of N-solitons in integrable systems *Nonlinearity* **20** 879

[27] Kato J and Pusateri F 2011 A new proof of long range scattering for critical nonlinear Schrödinger equations *Differ. Integral Equ.* **24** 923–40

[28] Klaus M 1977 On the Bound state of Schrödinger operators in one dimension *Ann. Phys., NY* **108** 288–300

[29] Kowalczyk M, Martel Y and Muñoz C 2017 Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space *J. Am. Math. Soc.* **30** 769–98

[30] Kowalczyk M, Martel Y and Muñoz C 2017 Nonexistence of small, odd breathers for a class of nonlinear wave equations *Lett. Math. Phys.* **107** 921–31

[31] Kowalczyk M, Martel Y and Muñoz C 2019 Soliton dynamic for the 1D NLKG equation with symmetry and in the absence of internal modes (arXiv:1903.12460)

[32] Lieb E and Loss M 2001 *Analysis* (*Graduate Studies in Mathematics* vol 14) 2nd edn (Providence, RI: American Mathematical Society)

[33] Martel Y and Merle F 2006 Multi-solitary waves for nonlinear Schrödinger equations *Ann. l'Inst. Henri Poincaré* C **23** 849–64

[34] Martel Y, Merle F and Tsai T-P 2006 Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations *Duke Math. J.* **133** 405–66

[35] Merle F and Raphaël P 2005 The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation *Ann. Math.* **161** 157–222

[36] Moroz V and Van Schaftingen J 2017 A guide to the Choquard equation *J. Fixed Point Theory Appl.* **19** 773–813

[37] Muñoz C and Ponce G 2019 Breathers and the dynamics of solutions to the KdV type equations *Commun. Math. Phys.* **367** 581–98

[38] Murphy J and Nakanishi K 2019 Failure of scattering to solitary waves for the long-range nonlinear Schrödinger equations (arXiv:1906.01802)

[39] Nakanishi K 2017 Global dynamics below excited solitons for the nonlinear Schrödinger equation with a potential *J. Math. Soc. Japan* **69** 1353–401

[40] Nakanishi K and Ozawa T 2002 Remarks on scattering for nonlinear Schrödinger equations *NoDEA* **9** 45–68

[41] Naumkin I P 2016 Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential *J. Math. Phys.* **57** 051501

[42] Nguyen T V 2019 Existence of multi-solitary waves with logarithmic relative distances for the NLS equation *C. R. Math.* **357** 13–58

[43] Ozawa T 1991 Long range scattering for nonlinear Schrödinger equations in one space dimension *Commun. Math. Phys.* **139** 479–93

[44] Rodnianski I, Schlag W and Soffer A 2003 Asymptotic stability of N-soliton states of NLS (arXiv:0309114)

[45] Ruiz D 2010 On the Schrödinger–Poisson–Slater system: behavior of minimizers, radial and nonradial cases *Arch. Ration. Mech. Anal.* **198** 349–68

[46] Satsuma J and Yajima N 1974 Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media *Suppl. Prog. Theor. Phys.* **55** 284–306

[47] Simon B 1976 The Bound state of weakly coupled Schrödinger operators in one and two dimensions *Ann. Phys., NY* **97** 279–88

[48] Soffer A and Weinstein M I 1999 Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations *Inventory Math.* **136** 9–74

[49] Soffer A and Weinstein M I 1990 Multichannel nonlinear scattering for nonintegrable equations *Commun. Math. Phys.* **133** 119–46

[50] Soffer A and Weinstein M I 1992 Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data *J. Differ. Equ.* **98** 376–90

[51] Strauss W 1974 Nonlinear scattering theory *Scattering Theory in Mathematical Physics* ed J A Lavita and J-P Marchand (Dordrecht: Reidel) pp 53–78

[52] Tsai T-P and Yau H-T 2002 Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions *Commun. Pure Appl. Math.* **55** 153–216
[53] Tsai T-P and Yau H-T 2002 Relaxation of excited states in nonlinear Schrödinger equations *Int. Math. Res. Not.* **31** 1629–73
[54] Tsutsumi Y 1985 Scattering problem for nonlinear Schrödinger equations *Ann. l'Inst. Henri Poincaré* A **43** 321–47
[55] Weinstein M I 1986 Lyapunov stability of ground states of nonlinear dispersive evolution equations *Commun. Pure Appl. Math.* **39** 51–68
[56] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *J. Exp. Theor. Phys.* **34** 62–9