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Boundary of Maximal Monotone Operators Values

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Abstract We characterize in Hilbert spaces the boundary of the values of maximal monotone operators, by means only of the values at nearby points, which are close enough to the reference point but distinct of it. This allows to write the values of such operators using finite convex combinations of the values at at most two nearby points. We also provide similar characterizations for the normal cone to prox-regular sets.

Keywords Maximal monotone operators · Prox-regular sets · Boundary points

Mathematics Subject Classification 52A41 · 90C25 · 90C34

1 Introduction

Given a continuous convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, according to [4, Theorem 3.1] the topological boundary of the Fenchel subdifferential of φ is completely characterized by means of the values of such a subdifferential mapping at points, which are close

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enough to the reference point but distinct of it. More specifically, for every $x \in \mathbb{R}^n$ we have that

$$\mathrm{bd}(\partial\varphi(x)) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} \partial\varphi(y). \tag{1.1}$$

The aim of this work is to extend this relation to general proper lower-semicontinuous functions defined in infinite-dimensional Hilbert spaces, that are not necessarily continuous. This question is motivated by the expected applications of this kind of formulas to the stability issues of semi-infinite linear programming problems as we explain below. Because our approach and proofs use techniques from the theory of monotone operators [3], we shall investigate the validity of (1.1) for general maximal monotone operators. It is well known, due to Rockafellar's Theorem (see [15]), that this family of operators strictly includes the family of proper lower-semicontinuous convex functions. Apart from its generality, the current characterization of the boundary of the values of maximal monotone operators could enable the study of the stability of more general semi-infinite linear programming problems whose restrictions systems are described by means of saddle-type functions (see [16]).

It is worth observing that the above relation (1.1) leads to the characterization of the whole set of the subdifferential, using only its values at close enough points different from the reference one. Namely, we obtain in Theorem 3.3 the following characterization for every maximal monotone operator A and every x such that $\mathrm{bd}(Ax) \neq \emptyset$,

$$Ax = \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, A)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} Ay \right\}.$$

In particular, when $A = \partial\varphi$ for a proper lower semi-continuous convex function φ, not necessarily continuous, we obtain that

$$\partial\varphi(x) = \mathrm{N}_{\mathrm{dom}\,\varphi}(x) + \overline{\mathrm{co}}\{\operatorname*{Limsup}_{y \longrightarrow_{\neq} x} \partial\varphi(y)\}.$$

The last formula has been obtained in [16, Theorem 25.6] for proper lower semi-continuous convex functions defined in $\mathbb{R}^n$ and having domains with a nonempty interior. The characterization of [16, Theorem 25.6] considers nearby points which belong to dense subsets of the domain of φ. This result was extended next in [18, Theorem 3.1] to Banach spaces.

The characterization in (1.1) has been shown useful for many stability purposes of parametrized semi-infinite linear programming problems, given in $\mathbb{R}^n$ as [8]

$$P(c, a, b) : \begin{cases} \text{minimize } c'x \\ \text{subject to } a_t'x \leq b_t, t \in T, \end{cases}$$

for a compact indexing set T and continuous functions a and b on T. Relation (1.1) was the main ingredient in [4–6] to derive point-based explicit expressions for the so-called *calmness moduli* of the associated feasible and optimal solutions set-valued mappings; we refer to [9–11] for more details on this calmness property. For instance, if $\mathcal{F}_a : C(T,\mathbb{R}) \to \mathbb{R}^n$ denotes the feasible set-valued mapping

$$\mathcal{F}_a(b) := \{x \in \mathbb{R}^n : a_t' x \leq b_t \forall t \in T\}, \;\; b \in C(T,\mathbb{R}),$$

then the calmness modulus of $\mathcal{F}_a$ at a point $(\bar{b}, \bar{x})$ belonging to its graph, defined as

$$\operatorname{clm} \mathcal{F}_a(\bar{b},\bar{x}) := \limsup_{\substack{x\to\bar{x}, b\to\bar{b} \\ x\in\mathcal{F}_a(b)}} \frac{d(x, \mathcal{F}_a(\bar{b}))}{d(b,\bar{b})},$$

is written in the more explicit following form (using the convention $\frac{1}{0} = +\infty$)

$$\operatorname{clm} \mathcal{F}_a(\bar{b},\bar{x}) = \left(\liminf_{x\to\bar{x}, s(x)>0} d_*(0, \partial s(x)) \right)^{-1},$$

where $s : \mathbb{R}^n \to \mathbb{R}$ is the continuous convex function given by

$$s(x) := \max_{t\in T} \{a_t' x - b_t\}.$$

In this way, from a qualitative point of view, the calmness of the mapping $\mathcal{F}_a$, say $\operatorname{clm} \mathcal{F}_a(\bar{b},\bar{x}) > 0$, is equivalent to the fact that the function s has an (global) error bound at $\bar{x}$ (see [12,13]). Moreover, if, in addition, the set $\mathcal{F}_a(\bar{b})$ turns out to be the singleton $\{\bar{x}\}$, in which case $s(x) > 0$ iff $x \neq \bar{x}$, then formula (1.1) entails the following point-based expression of the calmness modulus of the mapping $\mathcal{F}_a$,

$$\operatorname{clm} \mathcal{F}_a(\bar{b},\bar{x}) = \left(d_*(0, \operatorname{bd}(\partial s(\bar{x})))\right)^{-1}.$$

The advantage of this approach is that, due to the Valadier formula [19], the subdifferential mapping of the function s at $\bar{x}$ can be easily estimated by means only of the data vectors a and b. It is worth noting that in the framework of semi-infinite linear programming problems, this singleton's assumption is required for the solutions set-valued mapping and not for the feasible set-valued mapping (see [4–6] for more details).

For the aim of adapting in a further research the analysis above to more general semi-infinite linear programming problems, with not necessarily compact indexing sets T, so that the function s above lacks to be continuous, we extend in this paper formula (1.1) to the class of proper and lower semicontinuous convex functions. More generally, we establish similar characterizations for maximal monotone operators in the setting of Hilbert spaces. The first result stated in Theorem 3.1 asserts that, given a maximal monotone operator $A : H \rightrightarrows H$, for all $x \in H$ we have that

$$\operatorname{bd}(Ax) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} \operatorname{bd}(Ay) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} Ay,$$

where the Limsup is taken with respect to the norm topology. As a consequence, we prove that the value of A at x can be expressed using only different nearby points, in the sense that for every $x \in H$ such that $\mathrm{bd}(Ax) \neq \emptyset$ (Theorem 3.3)

$$Ax = \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, A)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} Ay \right\},$$

where co_2 is the set of all the segments generated by the elements of the underlying set, and $\mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, A)}(x)$ is the normal cone in the sense of convex analysis to the closure of the domain of the operator A. Characterizations of similar type are given for the faces of the values of A; see Theorem 3.2. Extensions to nonconvex objects, as prox-regular sets and functions, are also considered; see Theorems 4.1 and 4.2.

This paper is organized as follows: After Sect. 2, dedicated to present the necessary notations and the preliminary tools, we give the main result in Sect. 3: Theorem 3.1 characterizes the boundary of the values of maximal monotone operators, while Theorem 3.3 recovers the values of such operators using these boundary points. Theorem 3.2 specifies such characterizations to the faces of the values of maximal monotone operators. In Sect. 4 we extend this analysis to non-convex objects, covering the normal cone to prox-regular sets (Theorem 4.1) and the subdifferential of uniformly prox-regular functions (Theorem 4.2).

2 Notations and Preliminary Results

In this paper, H is a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|| \cdot ||$. The null vector in H is denoted by 0. The weak topology on H is denoted by ω, while the strong and the weak convergences in H are denoted by $\to$ and $\rightharpoonup$, resp. We denote by $\mathrm{B}(x, \rho)$ the closed ball with center $x \in H$ and radius $\rho > 0$; in particular, we write $\mathrm{B}_\rho := \mathrm{B}(0, \rho)$. Given a set $S \subset H$, $\mathrm{co}\{S\}$ and $\mathrm{co}_2\{S\}$ are respectively the *convex hull* of S and the set

$$\mathrm{co}_2\, S := \{\alpha s_1 + (1 - \alpha) s_2 : \alpha \in [0, 1], s_1, s_2 \in S\}.$$

Observe that $\mathrm{co}_2\, S$ coincides with $\mathrm{co}\, S$ when $H = \mathbb{R}$, but the two sets may be different from each other in general. By $\mathrm{int}(S)$, $\mathrm{bd}(S)$ and $\mathrm{cl}(S)$ (or, indistinctly, $\overline{S}$), we denote the *interior,* the *boundary* and the *closure* of S, respectively. The *indicator, the support* and the *distance functions* to the set S are respectively given by

$$\begin{aligned} \mathrm{I}_S(x) &:= 0 \ \text{ if } x \in S; \ +\infty \text{ if not}, \\ \sigma_S(x) &:= \sup\{\langle x, s \rangle : s \in S\}, \\ d_S(x) &:= \inf\{||x - y|| : \ y \in S\} \end{aligned}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). We shall write $\overset{S}{\rightharpoonup}$ for the weak convergence when restricted to the set S, and similarly for $\xrightarrow{S}$. We also write

$y \longrightarrow_{\neq} x$ when $y \to x$ with $y \neq x$. We denote by P_S the (*orthogonal*) *projection mapping* onto S defined as

$$P_S(x) := \{y \in S : \ ||x - y|| = d_S(x)\}.$$

Next, we review some classical facts about convex functions and monotone operators; we refer to [3,20] for more details. Given a function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, we say that φ is proper if its domain $\operatorname{dom} \varphi := \{x \in H : \varphi(x) < +\infty\}$ is nonempty, lower semicontinuous (lsc, for short) if its epigraph $\operatorname{epi} \varphi := \{(x, \lambda) \in H \times \mathbb{R} : \varphi(x) \leq \lambda\}$ is closed, and convex if its epigraph is convex. If φ is convex, the Fenchel sudifferential mapping of φ is defined as

$$\partial\varphi(x) := \{x^* \in H : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \, \forall y \in H\}, \text{ if } x \in \operatorname{dom} \varphi,$$

and $\partial\varphi(x) := \emptyset$ when $x \notin \operatorname{dom} \varphi$. The normal cone to a convex set $S \subset H$ is $\mathrm{N}_S(x) := \partial \mathrm{I}_S(x)$ for $x \in H$.

Given a set-valued operator $A : H \rightrightarrows H$, the *domain* and the *graph* of A are respectively given by

$$\operatorname{dom} A := \{x \in H : Ax \neq \emptyset\}, \ \operatorname{Gr} A := \{(x, x^*) : x^* \in Ax\}.$$

The operator A is said to be *monotone* if

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0 \text{ for all } (x_1, x_1^*), (x_2, x_2^*) \in \operatorname{Gr} A,$$

and *maximal monotone* if, in addition, A coincides with every monotone operator containing its graph. In such a case, it is known that $\operatorname{cl}(\operatorname{dom} A)$ is convex, and that Ax is convex and weakly closed for every $x \in H$. Hence, the *minimal norm* element of Ax; that is,

$$A^\circ x := \{x^* \in Ax : \|x^*\| = \min_{z^* \in Ax} \|z^*\|\},$$

is well-defined and unique, whenever $x \in \operatorname{dom} A$.

Finally, given a multifunction $F : H \rightrightarrows H$ we denote

$$\begin{aligned}
\operatorname*{Limsup}_{y \longrightarrow x} F(y) &:= \{x^* \in H : \exists y_n \longrightarrow x, \ y_n^* \to x^*, \text{ s.t. } y_n^* \in F(y_n) \ \forall n \geq 1\},\\
\operatorname*{Limsup}_{y \rightharpoonup x} F(y) &:= \{x^* \in H : \exists y_n \rightharpoonup x, \ y_n^* \to x^*, \text{ s.t. } y_n^* \in F(y_n) \ \forall n \geq 1\},\\
\omega - \operatorname*{Limsup}_{y \longrightarrow x} F(y) &:= \{x^* \in H : \exists y_n \longrightarrow x, \ y_n^* \rightharpoonup x^*, \text{ s.t. } y_n^* \in F(y_n) \ \forall n \geq 1\}.
\end{aligned}$$

3 Boundary of Maximal Monotone Operators

In this section, we give the desired property which expresses the value of a given maximal monotone operator $A : H \rightrightarrows H$, defined on a Hilbert space H, by means of its values at nearby points.

Definition 3.1 Given $x \in \operatorname{dom} A$ and $v \in H$, we define the set $A(x; v) \subset H$ as

$$A(x; v) := \left\{x^* \in Ax : \langle x^*, v\rangle = \sigma_{Ax}(v)\right\},$$

with the convention that $A(x, v) = \emptyset$ when $\sigma_{Ax}(v) = +\infty$.

Since Ax, $x \in \operatorname{dom} A$, is convex and weakly closed, $A(x; \cdot)$ coincides with the subdifferential mapping of the proper, convex and lsc support function σ_{Ax}. As a consequence, the following remark resumes some easy properties of the set $A(x; v)$.

Remark 3.1 Given $x \in \operatorname{dom} A$ and $v \in H$, we have:

(i) $A(x; v)$ is convex and closed (possibly empty), and nonempty whenever the set Ax is bounded.
(ii) $A(x; 0) = Ax$, and if $v \neq 0$ then $A(x; v)$ is a subset of $\operatorname{bd}(Ax)$. In the last case, we refer to $A(x; v)$ as the face of Ax with respect to the direction v.
(iii) $A(x; \alpha v) = A(x; v)$ for any $v \neq 0$ and $\alpha > 0$; thus, the face $A(x; v)$ depends only on the direction v.

We shall need the following lemma.

Lemma 3.1 (see, e.g., [7]) *For any nonempty closed convex set $S \subset H$, the set of points $s \in \operatorname{bd}(S)$ such that $\mathrm{N}_S(s) \neq \{0\}$ is dense in* $\operatorname{bd}(S)$.

Proposition 3.1 *Let $x \in \operatorname{dom} A$ and $v \neq 0$ be given. Then we have that*

$$\operatorname{bd}(Ax) = \operatorname{cl}\left(\bigcup_{v\neq 0} A(x; v)\right).$$

Proof The inclusion "$\supset$" being obvious, due to the definition of the set $A(x; v)$, we only need to prove the inclusion "$\subset$". Take an arbitrary vector $\xi \in \operatorname{bd}(Ax)$. According to Lemma 3.1, there exists a sequence $(\xi_n)_n \subset \operatorname{bd}(Ax)$ such that $\xi_n \to \xi$ and $\mathrm{N}_{Ax}(\xi_n) \neq \{0\}$. Hence, for each n there exists $v_n \neq 0$ such that $v_n \in \mathrm{N}_{Ax}(\xi_n) = \partial \mathrm{I}_{Ax}(\xi_n)$, or, equivalently, $\xi_n \in \partial\sigma_{Ax}(v_n) = A(x; v_n)$; that is, $\xi \in \operatorname{cl}\left(\bigcup_{v\neq 0} A(x; v)\right)$. □

Theorem 3.1 *For every $x \in H$ we have*

$$\operatorname{bd}(Ax) = \operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(Ay) = \operatorname*{Limsup}_{y \to_{\neq} x} Ay.$$

Proof To prove the first statement of the theorem we proceed by verifying the following inclusions, for every fixed $x \in H$,

$$\mathrm{bd}(Ax) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(Ay) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset \mathrm{bd}(Ax). \tag{3.1}$$

First, we observe that when $x \notin \mathrm{dom}\, A$, these inclusions follow since that, using the norm-weak (and a fortiori, the norm-norm) upper semicontinuity of the maximal monotone operator A,

$$\mathrm{bd}(Ax) = \emptyset \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(Ay) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset Ax = \emptyset.$$

So, we may assume that $x \in \mathrm{dom}\, A$. Also, if $\mathrm{bd}(Ax) = \emptyset$, then we would have that $Ax = H$, so that $\mathrm{dom}\, A = \{x\}$ and this leads to

$$\operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(Ay) = \operatorname*{Limsup}_{y \to_{\neq} x} Ay = \emptyset;$$

that is, the conclusion of the first statement is also true in this case.

From the observation above we assume now that $\mathrm{bd}(Ax) \neq \emptyset$. Take $x^* \in \mathrm{bd}(Ax)$ $(\subset Ax)$. According to Lemma 3.1, for each $n \geq 1$ there exists $x_n^* \in \mathrm{bd}(Ax)$ such that $\|x_n^* - x^*\| \leq \frac{1}{n}$ and $\mathrm{N}_{Ax}(x_n^*) \neq \{0\}$. We take $u_n \in \mathrm{N}_{Ax}(x_n^*)$ such that $\|u_n\| = 1$ and put $v_n := x_n^* + u_n$. It is clear that

$$v_n \neq x_n^*,\; v_n \notin Ax \;\text{ and } x_n^* = P_{Ax}(v_n). \tag{3.2}$$

We fix $n \geq 1$ and consider the following differential inclusion

$$\dot{z}(t) \in v_n - Az(t) \;\text{ for almost every } t \in [0, 1],\; z(0) = x, \tag{3.3}$$

which, according to [3, Proposition 3.3], possesses a (strong) unique solution $z_n(\cdot)$ such that $z_n(t) \in \mathrm{dom}\, A$ for all $t \in [0, 1]$, and that the function

$$t \mapsto \frac{d^+ z_n(t)}{dt} = (v_n - Az_n(t))^\circ = v_n - P_{Az_n(t)}(v_n) \tag{3.4}$$

is right-continuous on $[0, 1)$. In particular, we have that (recall (3.2))

$$\frac{d^+ z_n(0)}{dt} = (v_n - Az_n(0))^\circ = (v_n - Ax)^\circ = v_n - P_{Ax}(v_n) = v_n - x_n^*;$$

hence, since $v_n - x_n^* \neq 0$, by (3.2), it follows that $z_n(t) \neq x$ for all small $t \in [0, 1)$. Then, from the right-continuity of $\frac{d^+ z_n(\cdot)}{dt}$ and the expressions in (3.4), there exists a sequence $t_k \downarrow 0$ such that

$$z_{n,k}^* := P_{Az_n(t_k)}(v_n) \to x_n^* \text{ as } k \text{ goes to } +\infty, \tag{3.5}$$

and $z_n(t_k) \neq x$ for all $k \geq 1$. We observe that $z^*_{n,k} \in \mathrm{bd}(Az_n(t_k))$ for all $k \geq 1$ in a cofinite set, because for otherwise, since $z^*_{n,k} \in Az_n(t_k)$ we would have $z^*_{n,k} \in \mathrm{int}(Az_n(t_k))$ for infinitely many k, and due to (3.5) this would lead to $v_n \in Az_n(t_k)$ for all $k \in K$. Consequently, as $z_n(t_k) \to z_n(0) = x$ when k goes to $+\infty$, the maximal monotonicity of A would give us $v_n \in Ax$, which is a contradiction with (3.2). Now, we may choose a diagonal sequence $(z^*_{n,k_n})_n$ such that $z^*_{n,k_n} \to x^*$ as $n \to +\infty$, and this shows that $x^* \in \mathrm{Limsup}_{y \to_{\neq} x} \mathrm{bd}(Ay)$, which yields the first inclusion in (3.1).

We take now $x^* \in \mathrm{Limsup}_{y \to_{\neq} x} Ay$, so that $x^* = \lim\limits_{n\to\infty} x^*_n$ for some $x^*_n \in Ax_n$ with $x_n \to x$ and $x_n \neq x$. Then by the norm-weak upper semicontinuity of the operator A, we deduce that $x^* \in Ax$. Thus, it suffices to prove that $x^* \in H \setminus \mathrm{int}(Ax)$. Proceeding by contradiction, we assume that $x^* + r\mathbb{B} \subset Ax$ for some $r > 0$. Then, using the monotonicity of A, for every $n \geq 1$ one has that

$$\left\langle x^*_n - \left(x^* + r\frac{x_n - x}{\|x_n - x\|}\right), x_n - x \right\rangle \geq 0,$$

which gives

$$\|x^*_n - x^*\|\|x_n - x\| \geq \langle x^*_n - x^*, x_n - x\rangle \geq \left\langle r\frac{x_n - x}{\|x_n - x\|}, x_n - x \right\rangle = r\|x_n - x\|;$$

that is, $\|x^*_n - x^*\| \geq r$ for every $n \geq 1$, and this contradicts the convergence of (x^*_n) to x^*. Hence, $x^* \in \mathrm{bd}(Ax)$ and we conclude the proof of (3.1). □

It easily follows from Theorem 3.1 that

$$\mathrm{bd}(Ax) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset \omega - \operatorname*{Limsup}_{y \to_{\neq} x} Ay,$$

but the last inclusion may be strict, as the following example shows.

Example 3.1 Assume that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis for H, and consider the maximal monotone operator $A := \partial\|\cdot\|$. So,

$$A0 = \mathrm{B}(0, 1) \text{ and } Ax = \frac{x}{\|x\|} \text{ for all } x \neq 0.$$

We observe that the sequence $(\frac{e_n}{n})_{n\in\mathbb{N}}$ strongly converges to 0, and

$$A\frac{e_n}{n} = e_n \rightharpoonup 0 \in \mathrm{int}(\mathrm{B}(0, 1)) = \mathrm{int}(A0).$$

We give an interesting corollary of Theorem 3.1.

Corollary 3.1 *For every $x \in H$ we have*

$$d(0, \mathrm{bd}(Ax)) = \liminf_{y \to_{\neq} x} d(0, Ay).$$

Consequently, if x *is such that* $0 \notin \operatorname{int}(Ax)$, *then*

$$\|A^\circ x\| = \liminf_{y \longrightarrow_{\neq} x} \|A^\circ y\|.$$

Proof It suffices to consider the case when $x \in \operatorname{dom} A$, because otherwise both sides of the equality are equal to $+\infty$.

We may distinguish two cases: If $0 \notin Ax$, then $d(0, \operatorname{bd}(Ax)) = d(0, Ax) = \|A^\circ x\|$. Thus, according to Theorem 3.1 there are sequences (y_n), $(y_n^*) \subset H$ such that

$$y_n \to_{\neq} x, \;\; y_n^* \in Ay_n, \;\; \text{and } y_n^* \to A^\circ x \;\text{ as } n \to +\infty.$$

Hence,

$$\|A^\circ x\| = \lim_{n\to\infty} \|y_n^*\| \geq \liminf_{n\to\infty} d(0, Ay_n) \geq \liminf_{y \to_{\neq} x} d(0, Ay),$$

and so $d(0, \operatorname{bd}(Ax)) = \|A^\circ x\| \geq \liminf_{y \to_{\neq} x} d(0, Ay)$. Hence, if $\liminf_{y \to_{\neq} x} d(0, Ay) = +\infty$, then the first equality of the corollary obviously holds. Otherwise, we suppose that $\liminf_{y \to_{\neq} x} d(0, Ay) < \alpha$ for some $\alpha \in \mathbb{R}$, and let sequences (y_n), $(y_n^*) \subset H$ be such that

$$y_n \to_{\neq} x, \;\; y_n^* \in Ay_n, \;\; \text{and } \lim_{n\to\infty} \left\|y_n^*\right\| < \alpha.$$

Thus, taking into account Theorem 3.1, we may suppose that $y_n^* \to x^* \in \operatorname{bd}(Ax)$; that is,

$$d(0, \operatorname{bd}(Ax)) \leq \left\|x^*\right\| \leq \alpha.$$

We get the desired inequality "$\leq$" when α goes to $\liminf_{y \to_{\neq} x} d(0, Ay)$, and this completes the proof of the first statement.

To prove the last statement, we observe that under the current assumption, we have that $\|A^\circ x\| = d(0, Ax) = d(0, \operatorname{bd}(Ax))$, and so it suffices to use the first assertion of the theorem. □

We give the following corollary in which we use the notation

$$\|Az\| := \sup\{\|z^*\| : z^* \in Az\}.$$

Corollary 3.2 *For every* $x \in H$ *such that* Ax *is a nonempty bounded set, we have*

$$\|Ax\| \leq \limsup_{y \longrightarrow_{\neq} x} \|Ay\|,$$

and, provided that H is finite-dimensional,

$$\|Ax\| = \limsup_{y \longrightarrow_{\neq} x} \|Ay\|.$$

Proof Let $x \in H$ be as in the current corollary. Then for any $\varepsilon > 0$ there exists $x^* \in \mathrm{bd}(Ax)$ such that $\|x^*\| \geq \|Ax\| - \varepsilon$. According to Theorem 3.1, there exist sequences $y_n \to x$ and $y_n^* \in Ay_n$ such that $y_n \neq x$ and $y_n^* \to x^*$ as $n \to +\infty$. Thus,

$$\limsup_{y \longrightarrow_{\neq} x} \|Ay\| \geq \limsup_{n \to +\infty} \|Ay_n\| \geq \lim_{n \to \infty} \|y_n^*\| = \|x^*\| \geq \|Ax\| - \varepsilon,$$

and the desired inequality follows when ε goes to 0.

We assume now that H is finite-dimensional, so that according to the first statement we only need to prove that

$$\|Ax\| \geq \limsup_{y \longrightarrow_{\neq} x} \|Ay\|.$$

Indeed, we first observe that, due to the finite-dimensional assumption, according to [3, Remark 2.1] we have

$$\mathrm{int}(\mathrm{dom}\, A) = \mathrm{int}(\mathrm{cl}(\mathrm{dom}\, A)), \tag{3.6}$$

and, consequently, A is locally bounded in $\mathrm{int}(\mathrm{cl}(\mathrm{dom}\, A))$ whenever this last set is nonempty [3]. Thus, if $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| = +\infty$, then we would have $x \in \mathrm{bd}(\mathrm{cl}(\mathrm{dom}\, A))$ (otherwise, by (3.6), $x \in \mathrm{int}(\mathrm{dom}\, A)$ and $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| < +\infty$). Hence, the finite-dimensional framework ensures that $\mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, A)}(x) \neq \{0\}$. Now, because A is a maximal monotone operator we have $Ax = Ax + \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, A)}(x)$, which would entail the contradiction $\|Ax\| = +\infty$. Consequently, we may suppose that $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| < +\infty$. We let a sequence $(y_n, y_n^*)_n \subset \mathrm{Gr}\, A$ be such that $y_n \to x$, $y_n \neq x$ and $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| = \lim_{n \to \infty} \|y_n^*\|$. We may also assume that the sequence $(y_n^*)_n$ converges to some $x^* \in Ax$. Then

$$\|Ax\| \geq \|x^*\| = \lim_{n \to \infty} \|y_n^*\| = \limsup_{y \longrightarrow_{\neq} x} \|Ay\|,$$

as we wanted to prove. □

The following result concerns the faces of the values of maximal monotone operators.

Theorem 3.2 *For every $x \in \mathrm{dom}\, A$ and $v \neq 0$ we have*

$$A(x; v) = \operatorname*{Limsup}_{w \to v, t \downarrow 0} A(x + tw) = \operatorname*{Limsup}_{w \rightharpoonup v, t \downarrow 0} A(x + tw) = \omega\text{-} \operatorname*{Limsup}_{w \to v, t \downarrow 0} A(x + tw).$$

Proof We fix $x \in \operatorname{dom} A$ and $v \neq 0$, and take $x^* \in A(x; v)$. From Definition 3.1, we have that $v \in (\partial\sigma_{Ax})^{-1}(x^*) = \mathrm{N}_{Ax}(x^*)$, which ensures that $x^* = P_{Ax}(x^* + v)$. Let us consider the following differential inclusion

$$\dot{z}(t) \in x^* + v - Az(t) \;\; t \geq 0, \;\; z(0) = x.$$

According to [3, Proposition 3.3], this differential inclusion has a unique (strong) solution $z(\cdot)$ such that

$$\begin{aligned}\lim_{t\downarrow 0} \frac{d^+z(t)}{dt} &= \lim_{t\downarrow 0}(x^* + v - Az(t))^\circ \\ &= \frac{d^+z(0)}{dt} = (x^* + v - Ax)^\circ = (x^* + v) - x^* = v. \end{aligned} \tag{3.7}$$

We denote

$$x_n^* := P_{Az(\frac{1}{n})}(x^* + v), \;\; w_n := \frac{z(\frac{1}{n}) - x}{\frac{1}{n}};$$

hence, (3.7) ensures that $\frac{d^+z(\frac{1}{n})}{dt} = (x^* + v - Az(\frac{1}{n}))^\circ = x^* + v - x_n^* \to \frac{d^+z(0)}{dt} = v$. Therefore, as $n \to +\infty$ we obtain that

$$x_n^* \to x^*, w_n \to \frac{d^+z(0)}{dt} = v,$$

and so

$$x^* = \lim_{n\to\infty} x_n^* \subset \operatorname*{Limsup}_{n\to\infty} Az\left(\frac{1}{n}\right) = \operatorname*{Limsup}_{n\to\infty} A\left(x + \frac{1}{n}w_n\right) \subset \operatorname*{Limsup}_{w\to v, t\downarrow 0} A(x + tw),$$

showing that

$$A(x; v) \subset \operatorname*{Limsup}_{w\to v, t\downarrow 0} A(x + tw) \subset \operatorname*{Limsup}_{w\rightharpoonup v, t\downarrow 0} A(x + tw).$$

Thus, since $A(x; v) \subset \operatorname{Limsup}_{w\to v, t\downarrow 0} A(x+tw) \subset \omega - \operatorname{Limsup}_{w\to v, t\downarrow 0} A(x+tw)$, we only need to verify that

$$\operatorname*{Limsup}_{w\rightharpoonup v, t\downarrow 0} A(x + tw) \subset A(x; v) \text{ and } \omega - \operatorname*{Limsup}_{w\to v, t\downarrow 0} A(x + tw) \subset A(x; v). \tag{3.8}$$

To see the first inclusion, we take $x^* \in \operatorname{Limsup}_{w\rightharpoonup v, t\downarrow 0} A(x+tw)$, so that $x^* = \lim_n x_n^*$ for some sequences (x_n^*), $(w_n) \in H$, $(t_n) \subset \mathbb{R}_+$, such that $x_n^* \in A(x + t_n w_n)$, $w_n \rightharpoonup v$, and $t_n \downarrow 0$. It follows by the maximal monotonicity of A that $x^* \in Ax$, and for all $\xi \in Ax$

$$\langle x_n^* - \xi, w_n \rangle = \frac{1}{t_n} \langle x_n^* - \xi, x + t_n w_n - x \rangle \geq 0.$$

So, by taking the limit as $n \to +\infty$ we obtain that $\langle x^*, v \rangle \geq \sup_{\xi \in Ax} \langle \xi, v \rangle \geq \langle x^*, v \rangle$, which shows that $x^* \in A(x; v)$, and the first inclusion in (3.8) follows. We conclude the proof of the theorem because the second inclusion in (3.8) can be obtained using the same arguments as those used for the first inclusion. □

The following example shows the necessity of moving the vector v in the expression of Theorem 3.2.

Example 3.2 Consider the maximal monotone operator A defined on H as

$$Ax := x + \mathrm{N}_{\mathrm{B}(0,1)}(x),$$

and let $x, v \in H \setminus \{0\}$ be such that

$$\|x\| = 1 \text{ and } \langle v, x \rangle = 0.$$

Then one can easily check that $Ax = [1, +\infty[\ x$, and so

$$A(x; v) = \left\{ x^* \in Ax : \langle x^*, v \rangle = \sup_{\xi \in Ax} \langle \xi, v \rangle = \sup_{\alpha \in [1, +\infty[} \langle \alpha x, v \rangle = 0 \right\} = Ax.$$

But for any $t > 0$ we have that $A(x + tv) = \emptyset$, which shows that

$$\omega - \operatorname*{Limsup}_{t \downarrow 0} A(x + tv) = \operatorname*{Limsup}_{t \downarrow 0} A(x + tv) = \emptyset.$$

In Theorem 3.3 we give the expression of the values of maximal monotone operators by using the values at nearby points. We need first to check the following lemma.

Lemma 3.2 *Given* $x \in \operatorname{dom} A$, *for every* $x^* \in Ax$ *it holds*

$$\mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) = \{v \in H : x^* + tv \in Ax,\ \forall t \geq 0\} =: d_\infty(Ax). \tag{3.9}$$

Proof Since the operator $A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}$ is monotone and $\operatorname{Gr} A \subset \operatorname{Gr}(A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)})$, the maximality of A ensures that $Ax + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) = Ax$, which implies that $\mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) \subset d_\infty(Ax)$. Take now $v \in d_\infty(Ax)$, so that $x^* + tv \in Ax$ for all $t \geq 0$. Then, by the monotonicity of A we get

$$\langle y^* - (x^* + tv), y - x \rangle \geq 0\ \forall y^* \in Ay,\ \forall t \geq 0,$$

which in turn leads to

$$\langle y^* - x^*, y - x \rangle \geq t \langle v, y - x \rangle\ \forall y^* \in Ay,\ \forall t \geq 0.$$

Hence, $\langle v, y - x \rangle \leq 0$ for every $y \in \operatorname{dom} A$, and we deduce that $v \in \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x)$. □

Theorem 3.3 *For every* $x \in \operatorname{dom} A$ *such that* $\operatorname{bd}(Ax) \neq \emptyset$ *we have that*

$$Ax = \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} Ay \right\}.$$

Proof First, according to Theorem 3.1, ensuring that $\operatorname{bd}(Ax) = \operatorname{Limsup}_{y \to_{\neq} x} Ay$, and to the maximal monotonicity of the operator A, ensuring that $A = A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}$, we only need to prove the following inclusion when $\operatorname{int}(Ax) \neq \emptyset$,

$$\operatorname{int}(Ax) \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2 \left\{ \operatorname{bd}(Ax) \right\}. \tag{3.10}$$

Given $x^* \in \operatorname{int}(Ax)$, we fix $x_0^* \in \operatorname{bd}(Ax)$ and introduce the set

$$S := \{x_0^* + t(x^* - x_0^*) : t \geq 1\}.$$

On the one hand, if $S \cap \operatorname{bd}(Ax) = \emptyset$, then $S \subset Ax$ and, due to the convexity of Ax, we obtain $x_0^* + \mathbb{R}_+(x^* - x_0^*) \subset Ax$. Hence, thanks to Lemma 3.2 we deduce that $x^* - x_0^* \in \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x)$, and we get

$$x^* \in x_0^* + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2\{\operatorname{bd}(Ax)\},$$

which yields (3.10). On the other hand, if $S \cap \operatorname{bd}(Ax) \neq \emptyset$, then there exists some $t > 1$ such that $z^* = x_0^* + t(x^* - x_0^*) \in \operatorname{bd}(Ax)$. Thus,

$$x^* = \frac{1}{t} z^* + (1 - \frac{1}{t}) x_0^* \in \mathrm{co}_2\{\operatorname{bd}(Ax)\} \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2\{\operatorname{bd}(Ax)\},$$

and this completes the proof of the theorem. □

4 Prox-Regular Analysis

In this section, we extend the results of the previous section to two classes of operators of nonsmooth analysis, the normal cone to uniformly r -prox-regular sets, and the class of uniformly prox-regular extended-real-valued functions. As before, we work in the setting of a given Hilbert space H.

We start by giving the definition of the proximal normal cone.

Definition 4.1 [7] Given a set $C \subset H$ and $x \in C$, the proximal normal cone to C at x, denoted by $\mathrm{N}_C^P(x)$, is the set of vectors $x^* \in H$ for which there exists $m > 0$ such that

$$\langle x^*, y - x \rangle \leq m \left\| y - x \right\|^2 \text{ for all } y \in C.$$

Definition 4.2 [14] For positive numbers r and α, a closed set C is said to be (r, α)-prox-regular at $\overline{x} \in C$ provided that one has $x = P_C(x + v)$, for all $x \in C \cap \mathrm{B}(\overline{x}, \alpha)$

and all $v \in \mathrm{N}_C^P(x)$ such that $||v|| < r$. The set C is r-prox-regular (prox-regular, resp.) at $\overline{x}$ when it is (r, α)-prox-regular at $\overline{x}$ for some real $\alpha > 0$ (for some numbers $r, \alpha > 0$, resp.). The set C is said to be r -uniformly prox-regular when $\alpha = +\infty$.

The following theorem describes the boundary set of the normal cone of a uniformly r-prox-regular set, by means of its values at nearby points, which are different from the reference point. We also characterize such normal cones by means of their boundaries points. Recall that the Bouligand tangent cone of a prox-regular closed set C at $x \in C$ is given by

$$\mathrm{T}_C(x) := \left(\mathrm{N}_C^P(x)\right)^* := \{u^* \in H \mid \langle x^*, u\rangle \leq 0 \ \text{ for all } u \in \mathrm{N}_C^P(x)\}.$$

Theorem 4.1 *Let $C \subset H$ be a uniformly r-prox-regular closed set for some $r > 0$. Then for every $x \in C$ we have that*

$$\mathrm{bd}(\mathrm{N}_C^P(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(\mathrm{N}_C^P(y)) = \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y). \tag{4.1}$$

Consequently, if $\mathrm{int}(\mathrm{T}_C(x)) \neq \emptyset$ *and* $\dim H > 1$ *then*

$$\mathrm{N}_C^P(x) = \mathrm{co}_2\left\{\mathrm{bd}(\mathrm{N}_C^P(x))\right\} = \mathrm{co}_2\left\{\operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y)\right\}. \tag{4.2}$$

Proof First, we observe that the inclusions

$$\mathrm{bd}(\mathrm{N}_C^P(x)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(\mathrm{N}_C^P(y)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y), \tag{4.3}$$

follow as in the the proof of Theorem 3.1, since the following differential inclusion,

$$\dot{z}(t) \in f(z(t)) - \mathrm{N}_C^P(z(t)) \ \ t \in [0, 1], \ \ z(0) = x \in C,$$

for a given Lipschitz function $f : H \to H$, also possesses a unique solution $z(\cdot)$ such that the function $\frac{d^+ z(\cdot)}{dt}$ is right-continuous on $[0, 1[$ and $\frac{d^+ z(t)}{dt} = \left(f(z(t)) - \mathrm{N}_C^P(z(t))\right)^\circ$ for all $t \in [0, 1[$ (see [1, Theorem 4.6] for more details).

We are going to prove the converse inclusions of (4.3). We take $\xi \in \mathrm{Limsup}_{y \to_{\neq} x}$ $\mathrm{N}_C^P(y)$, and let the sequences (y_n) and (ξ_n) be such that

$$\xi_n \in \mathrm{N}_C^P(y_n), \ \ y_n \to x, \ \ \xi_n \to \xi \ \text{ as } n \to +\infty;$$

hence, we may suppose that for some $M > 0$ we have that $\xi_n \in \mathrm{N}_C^P(y_n) \cap \mathrm{B}_M$ for all $n \in \mathbb{N}$. Next, using the r-uniform prox-regularity of the set C, we obtain that $\xi \in \mathrm{N}_C^P(x)$ [14]. We claim that $\xi \in \mathrm{bd}(\mathrm{N}_C^P(x))$. Proceeding by contradiction, we assume that for some positive number ρ such that $\rho < M$ it holds $\xi + \mathrm{B}_\rho \subset \mathrm{N}_C^P(x)$; that is,

$$\xi + \rho \frac{y_n - x}{\|y_n - x\|} \in \mathrm{N}_C^P(x) \ \forall n \in \mathbb{N}.$$

Now, using the monotonicity of the mapping $x \to \mathrm{N}_C^P(x) \cap \mathrm{B}_{2M} + \frac{2M}{r} x$ (see [14]), we get

$$\left\langle \xi_n + \frac{2M}{r} y_n - \left(\xi + \rho \frac{y_n - x}{\|y_n - x\|} + \frac{2M}{r} x \right), y_n - x \right\rangle \geq 0 \text{ for all } n \geq 1,$$

which implies that

$$\begin{aligned} &\|\xi_n - \xi\| \|y_n - x\| + \frac{2M}{r} \|y_n - x\|^2 \\ &\geq \langle \xi_n - \xi, y_n - x \rangle + \frac{2M}{r} \|y_n - x\|^2 \geq \rho \|y_n - x\|, \end{aligned}$$

and, dividing by $\|y_n - x\|$,

$$\|\xi_n - \xi\| + \frac{2M}{r} \|y_n - x\| \geq \rho,$$

which is a contradiction. Hence, $\xi \in \mathrm{bd}(\mathrm{N}_C^P(x))$ and (4.3) holds as equalities.

In this last part of the proof, we assume that $\mathrm{int}(\mathrm{T}_C(x)) \neq \emptyset$; that is, there exist $v \in H$ and $\eta > 0$ such that $v + \mathrm{B}_\eta \subset \mathrm{int}(\mathrm{T}_C(x))$. According to the first statement of the theorem we only need to prove that

$$\mathrm{int}(\mathrm{N}_C^P(x)) \subset \mathrm{co}_2 \left\{ \mathrm{bd}(\mathrm{N}_C^P(x)) \right\}. \tag{4.4}$$

Indeed, If $\mathrm{int}(\mathrm{N}_C^P(x)) = \emptyset$, then (4.4) holds trivially. Otherwise, we suppose that $\mathrm{int}(\mathrm{N}_C^P(x)) \neq \emptyset$ and fix an arbitrary vector $\nu \in \mathrm{int}(\mathrm{N}_C^P(x)) \setminus \{0\}$. We also choose $v \in \mathrm{int}(\mathrm{T}_C(x)) \setminus \{0\}$ such that $v \notin \mathbb{R}\nu$; such a vector always exists (due to the current hypothesis $\dim H > 1$, there always exists $w \neq 0$ such that $v + w \in \mathrm{int}(\mathrm{T}_C(x))$ and $v + w \notin \mathbb{R}\nu$). Consequently, $v \notin \mathrm{N}_C(x)$, and so there exists some $t > 0$ such that $\nu + tv \notin \mathrm{N}_C(x)$. Since $\nu \in \mathrm{int}(\mathrm{N}_C^P(x))$ it follows that

$$z^* := \nu + \bar{t} v \in \mathrm{bd}(\mathrm{N}_C^P(x)) \setminus \{0\}, \tag{4.5}$$

for some $\bar{t} \in (0, t]$.

Now, we take $\xi \in \mathrm{int}(\mathrm{N}_C^P(x)) \setminus \{0\}$, so that $-\xi \notin \mathrm{N}_C^P(x)$, by [17, Exercise 9.42] (this exercise was given in finite dimensions but it can be easily extended to infinite dimensions). We are going to show that for some $t_0 > 0$ the vector z^* defined in (4.5) satisfies

$$\xi + t_0(\xi - t_0 z^*) \notin \mathrm{N}_C^P(x). \tag{4.6}$$

Proceeding by contradiction, we suppose that $\xi + t(\xi - tz^*) \in \mathrm{N}_C^P(x)$ for all $t \geq 0$, and we get

$$\frac{1+t}{t^2}\xi - z^* \in \mathrm{N}_C^P(x) \;\; \forall t > 0,$$

which as $t \to +\infty$ gives us $-z^* \in \mathrm{N}_C^P(x)$, which contradicts the nonemptyness of the set $\mathrm{int}(\mathrm{T}_C(x))$ (again by [17, Exercise 9.42]). Then, for t_0 being as in (4.6), there exists some $\beta \in (0, 1)$ such that $w^* := \xi + \beta t_0(\xi - t_0 z^*) \in \mathrm{bd}(\mathrm{N}_C^P(x))$, and hence $\xi = \frac{1}{1+\beta t_0} w^* + \frac{\beta t_0}{1+\beta t_0}(t_0 z^*) \in \mathrm{co}_2 \left\{\mathrm{bd}(\mathrm{N}_C^P(x))\right\}$. □

Remark 4.1 If $\dim H = 1$, the first equality in (4.2) is not true in general, but in this case, the geometry of $\mathrm{T}_C(x)$ and $\mathrm{N}_C(x)$ are easily determined.

In this last part of the paper, we extend the results of Sect. 3 to the proximal subdifferential mapping of lsc functions.

Definition 4.3 [2, *Definition* 3.1] Given a lsc function $f : H \to \mathbb{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, a vector $x^* \in H$ is called a proximal subgradient of f at x, written $x^* \in \partial_P f(x)$, if there are $\rho, \delta > 0$ such that

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - \delta||y - x||^2, \;\; \forall y \in \mathrm{B}(x, \rho).$$

A vector $x^* \in H$ is called a limiting subgradient of f at x, written $\xi \in \partial_L f(x)$, if there are sequence (x_k), $(x_k^*) \subset H$ such that

$$x^* = \omega - \lim_{k\to\infty} x_k^*, x_k \longrightarrow x, \;\; f(x_k) \longrightarrow f(x), x_k^* \in \partial_P f(x_k).$$

Definition 4.4 [2, *Definition* 3.1] A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be uniformly prox-regular on a set $E \subset H$ if there exist $\varepsilon, r > 0$ such that for any $\bar{x} \in E$ and $\bar{v} \in \partial_L f(\bar{x})$, one has, for all $(x, v) \in \mathrm{Gr}(\partial_L f)$ satisfying $\|x - \bar{x}\| < \varepsilon$, $|f(x) - f(\bar{x})| < \varepsilon$ and $\|v - \bar{v}\| < \varepsilon$,

$$f(x') \geq f(x) + \langle v, x' - x\rangle - \frac{r}{2}\|x' - x\|^2 \;\; \forall x' \in \mathrm{B}(\bar{x}, \varepsilon).$$

It is worth observing that for uniformly prox-regular functions f at $\{\bar{x}\} \subset \mathrm{dom}\, f$, we have that $\partial_P f(\bar{x}) = \partial_L f(\bar{x})$, and, in particular, if f is convex, then $\partial_P f(\bar{x}) = \partial_L f(\bar{x}) = \partial f(\bar{x})$. In the following result, we give the counterpart of Theorem 3.1 to the proximal subdifferential mapping of prox-regular functions.

Theorem 4.2 *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lsc function and let $x \in \operatorname{dom} f$. If f is uniformly prox-regular on a neighborhood of x, then*

$$\operatorname{bd}(\partial_P f(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \partial_P f(y),$$

and, provided that $\operatorname{bd}(\partial_P f(x)) \neq \emptyset$,

$$\partial_P f(x) = \mathrm{N}^P_{\operatorname{dom} f}(x) + \operatorname{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} \partial_P f(y) \right\}.$$

Proof According to [2, Proposition 3.6], the current property of f entails the existence of some $r > 0$, an open convex neighborhood U of x, and a lsc convex function g such that

$$f(y) = g(y) - \frac{r}{2}\|y\|^2 \quad \forall y \in U; \tag{4.7}$$

hence, $\partial_P f(y) = \partial g(y) - ry$ for all $y \in U$. Thus, since ∂g is a maximal monotone operator [15], by applying Theorem 3.1 we get

$$\begin{aligned} \operatorname{bd}(\partial_P f(x)) &= \operatorname{bd}(\partial g(x) - rx) \\ &= \operatorname{bd}(\partial g(x)) - rx \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} \partial g(y) - rx \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} (\partial g(y) - ry) \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} (\partial_P f(y)), \end{aligned}$$

which yields the first conclusion.

To prove the second statement we observe that $\operatorname{dom} f \cap U = \operatorname{dom} g \cap U$, which yields

$$\mathrm{N}^P_{\operatorname{dom} f}(x) = \mathrm{N}^P_{\operatorname{dom} g}(x) = \mathrm{N}^P_{\operatorname{cl}(\operatorname{dom} g)}(x) = \mathrm{N}_{\operatorname{cl}(\operatorname{dom} g)}(x)$$

(because the proximal normal cone coincides with the usual normal cone to convex sets). Thus, since $\operatorname{bd}(\partial g(x)) = \operatorname{bd}(\partial_P f(x)) + rx \neq \emptyset$, due to the current assumption, by applying Theorem 3.3 and taking into account (4.7) we get

$$
\begin{aligned}
\partial_P f(x) &= \partial g(x) - rx \\
&= \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\,\partial g)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} (\partial g(y) - ry) \right\} \\
&= \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\,g)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} (\partial g(y) - ry) \right\} \\
&= \mathrm{N}^P_{\mathrm{dom}\,f}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} (\partial_P f(y)) \right\},
\end{aligned}
$$

where we used the fact that $\mathrm{cl}(\mathrm{dom}\,\partial g) = \mathrm{cl}(\mathrm{dom}\,g)$ (see, e.g. [20]). □

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