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A nonlinear spectral rate-dependent constitutive equation for electro-viscoelastic solids

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Abstract. In this communication a spectral constitutive equation for nonlinear viscoelastic-electroactive bodies with short-term memory response is developed, using the total stress formulation and the electric field as the electric independent variable. Spectral invariants, each one with a clear physical meaning and hence attractive for use in experiment, are used in the constitutive equation. A specific form for constitutive equation containing single-variable functions is presented, which are easy to analyze compared to multivariable functions. The effects of viscosity and an electric field are studied via the results of boundary value problems for cases considering homogeneous distributions for the strains and the electric field, and some these results are compared with experimental data.

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1. Introduction

In the last 2 decades there has been a renewed interest in the study of electromagnetic interactions with solid media (see, for example, [23] and the references therein). In the particular case of electric interactions, the interest comes mostly from the development of some composite materials, where electro (or magneto) active particles are added to a rubber-like matrix during curing [4,12]. In other cases thin plates made of rubber are coated with electrodes, which upon the application of an electric potential, are compressed due to the electric forces that appear between the electrodes [3]. In biomechanics some types of soft tissues such as muscles can react to electric fields (see [7] and the references mentioned therein). We note that most of the early mechanical models of electro-active materials are simplified by assuming that these materials are elastic, i.e., there is no dissipation of energy [10,11,23]. However, in reality, most materials are not purely elastic and they exhibit some form of dissipation. In view of this, we are particularly concerned with viscoelastic-electroactive bodies that represent a wide range of materials and physical systems sensitive to mechanical forces and electric fields. Applications where these materials are used, for example, include biomimetics, micro-robotics and actuators. This has created considerable interest during the last years and many publications have resulted from attempts to understand the influence of electric fields on the mechanical behaviour of viscoelastic solids (see, for example, [1,2,13,16,17]). For example, in reference [8], Chen proposed a very general model for electro-thermo-viscoelastic solids with memory, using a Gibbs' potential. In references [5,9,19,43,44] models have been presented based on the decomposition of the energy into an electro-elastic part plus a visco-electro-elastic part, with the use of an evolution equation to find the viscous part of the deformation [5,9] or the internal variables [19,43,44]. In the case of specific models for applications in biomechanics, for example, an orthotropic visco-electro-elastic model has been proposed for myocardium [7], where the energy of the body is split into an elastic passive part and an active visco-electro-elastic counterpart. A general rate-dependent dissipation model was presented, for example, by Saxena et al. [25], where not only the strains are decomposed into an elastic and a visco-elastic part, but the same happens with the electric field, which is decomposed into a non-dissipative part and a dissipative (called viscous) part, in order to account not only for the dissipation due to the mechanical interactions, but also the electrical interactions, where the viscous parts are found from some evolution equation.

In the past, most of these viscoelastic-electroactive models used classical invariants [41] (or their variants) to describe their constitutive equations. Since the 1940s classical invariants have played an important role in the development of constitutive laws in continuum mechanics. Rivlin and others developed trace based invariants, because they are convenient and easy to evaluate. However, in many theoretical works, where such invariants are used, there is no interest about fitting with experimental data, the issue of propagation of error, nor being consistent with physics and the infinitesimal theory. Problems arise because most of the classical invariants do not have an immediate physical meaning and, hence, they are not attractive in seeking to design a rational program of experiments. For example, it is not straightforward to design an experiment [14,27] (denoted by R-experiment), where to rigorously construct a specific functional form of the energy function it requires to capture the behavior of a body in terms of a single classical invariant while keeping the remaining (classical) invariants fixed. We note that an R-experiment requires the number of independent invariants in the set of invariants of the corresponding minimal integrity or irreducible basis. It is shown in references [30,38,39], that the number of independent invariants is generally less than the number of invariants in the corresponding minimal integrity or irreducible basis, and is far less if the number of classical invariants in a minimal integrity or irreducible basis is large. Because of the unclear physical meaning of the classical invariants it is not clear how to select the relevant independent classical invariants from the set of invariants in the corresponding minimal or irreducible basis. In addition to this, researchers are not sure which invariants are best needed for a given problem, and for simplicity a reduced number of invariants is commonly considered, which may create problems in order to capture the response of the material [21,31]. However, it is shown by Shariff [27,28] that spectral invariants, each one with a clear physical meaning, are easy to analyze and attractive for use in R-experiment. Furthermore, to evaluate the number of independent classical invariants in a minimal integrity basis is not straightforward due to the difficulty in constructing relations (syzygies) among classical invariants. However, relations among the spectral variables are easily constructed [30,38,39] and, hence, the number of independent spectral invariants can be easily obtained.

In view of the advantages of using spectral invariants, in this paper, based on the authors' previous work [6,33,36,37], we develop a spectral constitutive model to describe the mechanical behaviour of viscoelastic-electroactive bodies with short-term memory response.

This paper is organized as follows. In Sect. 2, the relevant kinematic variables and the material model are introduced. The construction of a visco-elastic potential in the present of an electric field is presented in Sect. 3 and its corresponding spectral formulation is given in Sect. 4. In Sect. 5, we propose a specific form of the constitutive equation and its performance is evaluated in Sect. 6 by comparing our theory with different experiments. Pure homogeneous boundary value problems are discussed in Sect. 7 and we conclude our paper in Sect. 8.

2. Preliminaries

2.1. Kinematics

Let $\boldsymbol{X}$ denotes the typical position vector of a material particle in the reference configuration $\mathcal{B}_r$ of the body, and let $\boldsymbol{x}$ denotes the corresponding position vector of the same particle in the deformed configuration $\mathcal{B}_t$ at time t. It is assumed that there exists a one-to-one mapping $\boldsymbol{\chi}$ such that it assigns to each point $\boldsymbol{X}$ just one point $\boldsymbol{x}$ at each instant t, i.e., $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ and $\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$. The deformation gradient tensor is $\boldsymbol{F} = \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$. The left and right Cauchy-Green stretch tensors, respectively $\boldsymbol{B}$ and $\boldsymbol{C}$, are given by $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ and $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$.

The particle velocity $\boldsymbol{v}$ is defined as $\boldsymbol{v} = \dfrac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t}$. The velocity gradient tensor, denoted by $\boldsymbol{L}$, is the gradient of the velocity. It follows immediately that $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, where the superimposed dot designates the time derivative. In this paper repeated indices do not mean sum in those indices. More details about the kinematics of continua can be found, for example, in [42].

2.2. Electrostatics

If there is no interaction with magnetic fields and there is no distribution of free charges and electric current, then the simplified forms of the Maxwell equations are

$$\mathrm{curl}(\boldsymbol{e}) = 0, \quad \mathrm{div}(\boldsymbol{d}) = 0, \tag{1}$$

where $\boldsymbol{e}$ is the electric field and $\boldsymbol{d}$ is the electric displacement in the current configuration, and div and curl are the divergence and curl operators with respect to $\boldsymbol{x}$, respectively. In vacuum $\boldsymbol{e}$ and $\boldsymbol{d}$ are related through

$$\boldsymbol{d} = \varepsilon_0 \boldsymbol{e}, \tag{2}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m is the electric permittivity in vacuum. For a condensed matter an extra field is needed, which is called the electric polarization $\boldsymbol{p}$, where

$$\boldsymbol{d} = \varepsilon_0 \boldsymbol{e} + \boldsymbol{p}. \tag{3}$$

In the present communication we use the concept of total the Cauchy stress $\boldsymbol{T}$ tensor defined in [10,11]. In the absence of surface electric charges $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{T}$ must satisfy the continuity equations

$$\boldsymbol{n} \cdot [\![\boldsymbol{d}]\!] = 0, \quad \boldsymbol{n} \times [\![\boldsymbol{e}]\!] = \boldsymbol{0}, \quad \boldsymbol{T}\boldsymbol{n} = \hat{\boldsymbol{t}} + \boldsymbol{T}_{\mathrm{M}}\boldsymbol{n}, \tag{4}$$

where $\cdot$ and $\times$ denote the dot product and cross product, respectively, between vectors, $\boldsymbol{n}$ is the unit outward normal vector to $\partial \mathcal{B}_t$, $\hat{\boldsymbol{t}}$ is the external mechanical traction, $[\![\ \ ]\!]$ denotes the difference of a quantity from outside and inside a body and $\boldsymbol{T}_{\mathrm{M}}$ is the Maxwell stress tensor outside the body in vacuum which is defined as

$$\boldsymbol{T}_{\mathrm{M}} = \boldsymbol{d} \otimes \boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \boldsymbol{e})\boldsymbol{I}. \tag{5}$$

More details about electrostatics and continuum mechanics can be obtained, from, for example, in [18,23].

2.3. Mechanical balance laws

The conservation of mass equation for a continuum may be written in the form

$$J\rho = \rho_0, \tag{6}$$

where ρ_0 and ρ are the mass densities in the reference and deformed configurations, respectively. The first law of movement, in the absence of the external body force per unit mass which is independent of the electrostatic field, is

$$\rho \dot{\boldsymbol{v}} = \mathrm{div}\, \boldsymbol{\sigma} + \boldsymbol{f}_e, \tag{7}$$

where $\boldsymbol{\sigma}$ is the (elastic) Cauchy stress tensor and

$$\boldsymbol{f}_e = (\mathrm{grad}\, \boldsymbol{e})^T \boldsymbol{p}, \tag{8}$$

is the electric body force [10]. The Lagrangian counter part of the electric and displacement fields have the forms [10]

$$\boldsymbol{e}_L = \boldsymbol{F}^T \boldsymbol{e}, \quad \boldsymbol{d}_L = J\boldsymbol{F}^{-1}\boldsymbol{d} \tag{9}$$

and satisfy

$$\mathrm{Div}\, \boldsymbol{d}_L = 0, \quad \mathrm{Curl}\, \boldsymbol{e}_L = \boldsymbol{0}, \tag{10}$$

where Div and Curl are the divergence and curl operators with respect to $\boldsymbol{X}$, respectively. In view of (3), the balance of angular momentum requires that [10]

$$\boldsymbol{\sigma} + \boldsymbol{d} \otimes \boldsymbol{e} \tag{11}$$

is symmetric. In view of of the identity

$$(\mathrm{grad}\, \boldsymbol{e})^T \boldsymbol{p} = \mathrm{div}\left[\boldsymbol{d} \otimes \boldsymbol{e} - \frac{1}{2}\varepsilon_0 (\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{I}\right] \tag{12}$$

we can define the symmetric **total Cauchy** stress tensor

$$\boldsymbol{T} = \boldsymbol{\sigma} + \boldsymbol{d} \otimes \boldsymbol{e} - \frac{1}{2}\varepsilon_0 (\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{I} \tag{13}$$

and the first law of movement becomes

$$\rho \dot{\boldsymbol{v}} = \mathrm{div}\, \boldsymbol{T}. \tag{14}$$

3. Electro-viscoelastic potential

In this communication, we only model electro-viscoelastic solids with *short-term memory response*; the model is only capable of modeling rate dependent deformations but not the common phenomenon such as stress-relaxation. From the principle of material objectivity [42] it follows that all constitutive relations are independent of $\boldsymbol{v}$. As a result, we assume the mechanical behaviour of electro-viscoelastic solids is governed by the variables

$$\boldsymbol{F}, \quad \dot{\boldsymbol{C}}, \quad \boldsymbol{e}_L = e\boldsymbol{f}, \quad e = \mid \boldsymbol{e}_L \mid . \tag{15}$$

We also assume there exist an electro-viscoelastic potential

$$W_v = W_{(v)}(\boldsymbol{C}, \dot{\boldsymbol{C}}, \boldsymbol{f}, e) \tag{16}$$

that is responsible for the internal dissipation due to the viscous effects in the sense that [24]

$$\mathrm{tr}\left(\frac{\partial W_v}{\partial \dot{\boldsymbol{C}}}\dot{\boldsymbol{C}}\right) \geq 0. \tag{17}$$

If $W_e = W_{(e)}(\boldsymbol{C}, \boldsymbol{e})$ is the elastic free energy function for an electro-elastic solid, then the Clausius–Duhem inequality takes the form

$$\begin{aligned} &\frac{\mathrm{tr}(\boldsymbol{T}^{(2)}\dot{\boldsymbol{C}})}{2} - \mathrm{tr}\left(\frac{\partial W_v}{\partial \dot{\boldsymbol{C}}}\dot{\boldsymbol{C}}\right) - \boldsymbol{p}\cdot\dot{\boldsymbol{e}} - \rho \dot{W}_e \\ &= \frac{1}{2}\mathrm{tr}\left[\left(\boldsymbol{T}^{(2)} - 2\frac{\partial W_v}{\partial \dot{\boldsymbol{C}}} - 2\rho\frac{\partial W_e}{\partial \boldsymbol{C}}\right)\dot{\boldsymbol{C}}\right] - \left(\boldsymbol{p} + \rho\frac{\partial W_e}{\partial \boldsymbol{e}}\right)\cdot\dot{\boldsymbol{e}} \geq 0, \end{aligned} \tag{18}$$

where $\boldsymbol{T}^{(2)}$ is the second Piola–Kirchhoff stress tensor. Since $\dot{\boldsymbol{C}}$ and $\dot{\boldsymbol{e}}$ are arbitrary in (18), we have that

$$\boldsymbol{T}^{(2)} = \boldsymbol{S}_{(e)} + \boldsymbol{S}_{(v)}, \quad \boldsymbol{p} = -\rho\frac{\partial W_e}{\partial \boldsymbol{e}}. \tag{19}$$

where

$$\boldsymbol{S}_{(e)} = 2\rho \frac{\partial W_e}{\partial \boldsymbol{C}}, \quad \boldsymbol{S}_{(v)} = 2\frac{\partial W_v}{\partial \dot{\boldsymbol{C}}}. \tag{20}$$

Since $\boldsymbol{\sigma}$ is not symmetric, $\boldsymbol{T}^{(2)}$ is not symmetric. If $\boldsymbol{S}_{(v)}$ is assumed symmetric, it is clear in (19) that $\boldsymbol{S}_{(e)}$ is not symmetric.

Consider

$$W_{(e)}(\boldsymbol{C}, \boldsymbol{e}) = W_{(e)}(\boldsymbol{C}, \boldsymbol{F}^{-T}\boldsymbol{e}_L) = \Phi_{(e)}(\boldsymbol{F}, \boldsymbol{e}_L). \tag{21}$$

Then

$$\boldsymbol{F}\frac{\partial \Phi_{(e)}}{\partial \boldsymbol{F}} = 2\boldsymbol{F}\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T - \frac{\partial W_{(e)}}{\partial \boldsymbol{e}} \otimes \boldsymbol{e} = 2\boldsymbol{F}\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T + \boldsymbol{p}\otimes\boldsymbol{e}. \tag{22}$$

Note that the Cauchy stress

$$\boldsymbol{\sigma} = \boldsymbol{F}\boldsymbol{T}^{(2)}\boldsymbol{F}^T = 2\rho\boldsymbol{F}\frac{\partial W_{(e)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T + 2\boldsymbol{F}\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}}\boldsymbol{F}^T. \tag{23}$$

Hence,

$$\rho\boldsymbol{F}\frac{\partial \Phi_{(e)}}{\partial \boldsymbol{F}} = \boldsymbol{\sigma} + \boldsymbol{p}\otimes\boldsymbol{e} - 2\boldsymbol{F}\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}}\boldsymbol{F}^T. \tag{24}$$

In view of (11) and (3) it is clear that $\boldsymbol{F}\dfrac{\partial \Phi_{(e)}}{\partial \boldsymbol{F}}$ is symmetric. Using (24), we define the total stress

$$\boldsymbol{T} = \rho\boldsymbol{F}\frac{\partial \Phi_{(e)}}{\partial \boldsymbol{F}} + \boldsymbol{F}\boldsymbol{S}_{(\boldsymbol{v})}\boldsymbol{F}^T + \varepsilon_0\left[(\boldsymbol{e}\otimes\boldsymbol{e}) - \frac{1}{2}(\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{I}\right]. \tag{25}$$

Using the relation

$$\frac{\partial(J\boldsymbol{e}_L\cdot\boldsymbol{C}^{-1}\boldsymbol{e}_L)}{\partial \boldsymbol{F}} = -2J\boldsymbol{F}^{-1}\left[\boldsymbol{e}\otimes\boldsymbol{e} - \frac{1}{2}(\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{I}\right] \tag{26}$$

we can write

$$J\boldsymbol{T} = \boldsymbol{F}\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{F}} + J\boldsymbol{F}\boldsymbol{S}_{(\boldsymbol{v})}\boldsymbol{F}^T = 2\boldsymbol{F}\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T + J\boldsymbol{F}\boldsymbol{S}_{(\boldsymbol{v})}\boldsymbol{F}^T, \tag{27}$$

where

$$\Omega_{(e)} = J\left[\rho\Phi_{(e)} - \frac{\varepsilon_0}{2}\boldsymbol{e}_L\cdot(\boldsymbol{C}^{-1}\boldsymbol{e}_L)\right]. \tag{28}$$

The Lagrangian form of the polarization $\boldsymbol{p}_L$ is related to $\boldsymbol{p}$ via

$$\boldsymbol{p} = -\rho\boldsymbol{F}\frac{\partial \Phi_{(e)}}{\partial \boldsymbol{e}_L} = J^{-1}\boldsymbol{F}\boldsymbol{p}_L, \tag{29}$$

where

$$\boldsymbol{p}_L = -\rho_0\frac{\partial \Phi_{(e)}}{\partial \boldsymbol{e}_L}. \tag{30}$$

Hence, the Lagrangian counterparts of $\boldsymbol{d}$ and $\boldsymbol{p}$ take the form [10]

$$\boldsymbol{d}_L = -\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{e}_L}, \quad \boldsymbol{p}_L = \boldsymbol{d}_L - \varepsilon_o\boldsymbol{C}^{-1}\boldsymbol{e}_L. \tag{31}$$

In the case when the material is incompressible, we have $J = 1$, and

$$\boldsymbol{T} = 2\boldsymbol{F}\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}}\boldsymbol{F}^T + \boldsymbol{F}\boldsymbol{S}_{(v)}\boldsymbol{F}^T - p\boldsymbol{I}, \tag{32}$$

where

$$\Omega_{(e)} = \rho\Phi_{(e)} - \frac{\varepsilon_0}{2}\boldsymbol{e}_L\cdot(\boldsymbol{C}^{-1}\boldsymbol{e}_L) \tag{33}$$

and p is the Lagrange multiplier associated with the incompressibility constraint $J = \det(\boldsymbol{F}) = 1$.

4. Spectral formulation

We assume that $\Omega_{(e)}$ is independent of the signs of $\boldsymbol{f}$, i.e., it depends on $\boldsymbol{f} \otimes \boldsymbol{f}$, and for simplicity we write

$$\Omega_{(e)} = \Omega_{(a)}(\boldsymbol{U}, \boldsymbol{f} \otimes \boldsymbol{f}, e) = \Omega_{(b)}(\boldsymbol{U}, \boldsymbol{f}, e) = \Omega_{(b)}(\boldsymbol{Q}\boldsymbol{U}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{f}, e) \tag{34}$$

for any rotation $\boldsymbol{Q}$ [41], where $\boldsymbol{U}$ is the right stretch tensor with the relations $\boldsymbol{C} = \boldsymbol{U}^2$. Hence, we can express $\Omega_{(e)}$ in terms of the isotropic invariants of the set $S = \{\boldsymbol{U}, \boldsymbol{f}\}$. To obtain these isotropic invariants, following the work of Shariff et al. [37], we simply express the components of the elements of S using the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where $\boldsymbol{u}_i$ is an eigenvector of $\boldsymbol{U}$ and

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i. \tag{35}$$

Hence, we can express $\Omega_{(e)}$ in terms of e and the spectral component invariants

$$\lambda_i = \boldsymbol{u}_i \cdot \boldsymbol{U}\boldsymbol{u}_i = \boldsymbol{Q}\boldsymbol{u}_i \cdot \boldsymbol{Q}\boldsymbol{U}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{u}_i, \quad f_i = \boldsymbol{u}_i \cdot \boldsymbol{f} = \boldsymbol{Q}\boldsymbol{u}_i \cdot \boldsymbol{Q}\boldsymbol{f}. \tag{36}$$

We must emphasize that the components of the vectors and tensors in the set S, with respect to an *arbitrary* basis, are not, in general, invariants. Hence, we can express $\Omega_{(e)}$ as (taking note that $\Omega_{(e)}$ depends on $\boldsymbol{f} \otimes \boldsymbol{f}$):

$$\Omega_{(e)} = \Omega(\lambda_{1,2,3}, \zeta_{1,2,3}, e), \tag{37}$$

where $\zeta_i = f_i^2$ and the term such as $\lambda_{1,2,3}$ represents the expression $\lambda_1, \lambda_2, \lambda_3$ and the function Ω must satisfy the P-property (see "Appendix") as described in [31]. Since $\boldsymbol{f}$ is a unit vector, we must have the constraint

$$\sum_{i=1}^{3} \zeta_i = 1. \tag{38}$$

In the case of the electro-viscoelastic potential, we first note that

$$\dot{\boldsymbol{C}} = \sum_{i=1}^{3} 2\lambda_i \dot{\lambda}_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i + \sum_{i \neq j}^{3} \Omega_{ij}(\lambda_j^2 - \lambda_i^2)\boldsymbol{u}_i \otimes \boldsymbol{u}_j, \tag{39}$$

where $\Omega_{ij} = -\Omega_{ji} = \boldsymbol{u}_i \bullet \dot{\boldsymbol{u}}_j$. The electro-viscoelastic potential W_v can be expressed in terms of the spectral invariants

$$\lambda_i, \quad g_{ij}, \quad \zeta_i, \quad e, \tag{40}$$

where

$$g_{ij} = g_{ji} = \boldsymbol{u}_i \cdot (\dot{\boldsymbol{C}}\boldsymbol{u}_j). \tag{41}$$

It is clear that, in view of (38), only 11 of the invariants

$$\lambda_i, \quad g_{ij}, \quad \zeta_i \tag{42}$$

are independent. We note that if use the classical invariants [41], the number of classical invariants in the minimal integrity basis for the set of tensors and vector

$$\boldsymbol{C}, \quad \dot{\boldsymbol{C}}, \quad \boldsymbol{f} \tag{43}$$

is 18 [41]. Since, the classical invariants can be expressed explicitly in terms of the spectral invariants (42) [39], hence, it is clear that only 11 of the 18 classical invariants are independent. In [39], 7 relations between the classical invariants were given which, alternatively, proves that only 11 classical invariants are independent.

To facilitate the construction of the P-property, we use the 6 independent invariants

$$\alpha_i = \boldsymbol{u}_i \cdot (\dot{\boldsymbol{C}}\boldsymbol{u}_i), \quad \beta_i = \boldsymbol{u}_i \cdot (\dot{\boldsymbol{C}}^2 \boldsymbol{u}_i) > 0 \tag{44}$$

instead of the invariants g_{ij}. Hence the electro-viscoelastic potential is of the form

$$W_{(v)} = W(\lambda_{1,2,3}, \alpha_{1,2,3}, \beta_{1,2,3}, \zeta_{1,2,3}, e), \tag{45}$$

where the function W must satisfy the P-property as described in [31]. In view of (42), the mechanical behaviour of an electro-viscoelastic material can be described using only 13 spectral invariants (including e), where only 12 of them are independent. Due to the P-property described in [31], the constitutive equations may be expressed in terms of a number of invariants (that depend on the spectral invariants and material constants) that is much less than 13 as exemplified in Sect. 5 below.

4.1. Spectral components of the tensor derivatives

The evaluation of the spectral components of the stress requires the spectral components of the tensor derivatives $\dfrac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}}$ and $\dfrac{\partial W_v}{\partial \dot{\boldsymbol{C}}}$. Following the work of Shariff [27], we have

$$\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}} = \sum_{i,j=1}^{3} \left(\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}} \right)_{ij} \boldsymbol{u}_i \otimes \boldsymbol{u}_j, \tag{46}$$

where

$$\left(\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}} \right)_{ii} = \frac{1}{2\lambda_i} \frac{\partial \Omega}{\partial \lambda_i} \quad \text{(there is no sum in } i), \tag{47}$$

$$\left(\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{C}} \right)_{ij} = \frac{1}{\lambda_i^2 - \lambda_j^2} \left(\frac{\partial \Omega}{\partial \zeta_i} - \frac{\partial \Omega}{\partial \zeta_j} \right) f_i f_j, \quad i \neq j. \tag{48}$$

It is assumed that W has sufficient regularity to ensure that, as λ_i approaches λ_j, (48) has a limit. For the electro-visco potential, we have,

$$\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} = \sum_{i,j=1}^{3} \left(\boldsymbol{u}_i \cdot \frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} \boldsymbol{u}_j \right)_{ij} \boldsymbol{u}_i \otimes \boldsymbol{u}_j. \tag{49}$$

We also require the relations

$$\frac{\partial \alpha_i}{\partial \dot{\boldsymbol{C}}} = \boldsymbol{u}_i \otimes \boldsymbol{u}_i, \quad \frac{\partial \beta_i}{\partial \dot{\boldsymbol{C}}} = \boldsymbol{u}_i \otimes \dot{\boldsymbol{C}}\boldsymbol{u}_i + \dot{\boldsymbol{C}}\boldsymbol{u}_i \otimes \boldsymbol{u}_i. \tag{50}$$

4.2. Electric and displacement fields

Using the relations

$$\frac{\partial e}{\partial \boldsymbol{e}_{(L)}} = \boldsymbol{f}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{e}_{(L)}} = \frac{1}{e}(\boldsymbol{I} - \boldsymbol{f} \otimes \boldsymbol{f}), \tag{51}$$

we obtain the Lagrangian electric displacement [10,11]

$$\boldsymbol{d}_{(\mathrm{L})} = -\frac{\partial \Omega_{(e)}}{\partial \boldsymbol{e}_{(\mathrm{L})}} = -\frac{\partial \Omega}{\partial e} \boldsymbol{f} + \frac{1}{e} (\boldsymbol{I} - \boldsymbol{H})^T \frac{\partial \Omega}{\partial \boldsymbol{f}}, \tag{52}$$

where $\boldsymbol{H} = \boldsymbol{f} \otimes \boldsymbol{f}$.

The Lagrangian spectral components for the electric displacement $\boldsymbol{d}$ are:

$$\boldsymbol{d}_{(\mathrm{L})} = -\sum_{k=1}^{3} d_{(\mathrm{L})_k} \boldsymbol{u}_k, \tag{53}$$

where

$$d_{(\mathrm{L})_k} = \frac{\partial \Omega}{\partial e}(\boldsymbol{f} \cdot \boldsymbol{u}_k) + \frac{1}{e}\left[(\boldsymbol{I} - \boldsymbol{H})^T \frac{\partial \Omega}{\partial \boldsymbol{f}}\right] \cdot \boldsymbol{u}_k. \tag{54}$$

The electric field in the deformed configuration can simply be expressed by

$$\boldsymbol{d} = -\sum_{k=1}^{3} \lambda_k d_{(\mathrm{L})_k} \boldsymbol{v}_k, \tag{55}$$

where $\boldsymbol{v}_k = \boldsymbol{R}\boldsymbol{u}_k$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is the spectral Eulerian basis and the rotation $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$.

5. Specific form

In order to give quantitative and qualitative results given in Sect. 7, we use simple specific forms for $W_{(v)}$ and $\Omega_{(e)}$ for incompressible bodies. So we propose

$$\Omega_{(e)} = \mu H_1 + c(e) H_2 - \varepsilon_0 e^2 H_3, \tag{56}$$

where the material dependent invariants H_1, H_2 and H_3 are defined as

$$H_1 = \sum_{i=1}^{3} r_1(\lambda_i), \quad H_2 = \sum_{i=1}^{3} \zeta_i r_2(\lambda_i), \quad H_3 = \sum_{i=1}^{3} \zeta_i \frac{1}{2\lambda_i^2}, \tag{57}$$

$\mu > 0$ is a material constant [26]. In the absence of electric field $\Omega_{(e)}$ should be independent of e and, hence, we impose the condition $c(0) = 0$. To be consistent with infinitesimal elasticity, the following conditions are required

$$r_1(1) = r_2(1) = r_1'(1) = r_2'(1) = 0, \quad r_1''(1) = r_2''(1) = 2. \tag{58}$$

For $\mid \lambda_i - 1 \mid << 1$, both $r_1(\lambda_i)$ and $r_2(\lambda_i)$ are approximately quadratic in $\lambda_i - 1$. Extending the quadratic behaviour to finite strain, we also impose the conditions

$$r_1(\lambda_i), r_2(\lambda_i) \geq 0 \tag{59}$$

and both derivative functions r_1' and r_2' are monotonically increasing. In the case of $W_{(v)}$, in view of (39), we simply have

$$\alpha_i = 2\lambda_i \dot{\lambda}_i \tag{60}$$

and propose the specific form

$$\frac{1}{\mu} W_{(v)} = \frac{\nu_1}{2} \sum_{i=1}^{3} \beta_i r_3(\lambda_i) + \nu_2 H_4 \mathrm{tr}(\dot{\boldsymbol{C}}^2), \tag{61}$$

where ν_1 and ν_2 are dimensionless material constants, and

$$\mathrm{tr}(\dot{\boldsymbol{C}}^2) = \sum_{i=1}^{3} \beta_i, \quad H_4 = \sum_{i=1}^{3} r_4(\lambda_i) \geq 0. \tag{62}$$

The functions r_3 and r_4 have the same properties as that of the functions r_1 and r_2, described above. We note that since r_α $(\alpha = 1, 2, 3, 4)$ and c are general single-variable functions, they are much easier to handle than multivariable functions; this is evident when we compare our model with different experiments as indicated in Sect. 6. It is clear from (61) that

$$\frac{1}{\mu}\frac{\partial W_{(v)}}{\partial \dot{\boldsymbol{C}}} = \frac{\nu_1}{2}\sum_{i=1}^{3}[r_3(\lambda_i)(\boldsymbol{u}_i \otimes \dot{\boldsymbol{C}}\boldsymbol{u}_i + \dot{\boldsymbol{C}}\boldsymbol{u}_i \otimes \boldsymbol{u}_i)] + 2\nu_2 H_4 \dot{\boldsymbol{C}}. \tag{63}$$

The internal dissipation (17) becomes

$$\frac{\nu_1}{2}\sum_{i=1}^{3}[r_3(\lambda_i)(\beta_i + \dot{\boldsymbol{C}}\boldsymbol{u}_i \cdot \dot{\boldsymbol{C}}\boldsymbol{u}_i)] + 2\nu_2 H_4 \mathrm{tr}(\dot{\boldsymbol{C}}^2) \geq 0. \tag{64}$$

Since $r_3, H_4, \beta_i \geq 0$, hence

$$\nu_1, \nu_2 \geq 0 \tag{65}$$

are necessary and sufficient for the inequality (64).

The restriction on the material constant $c(e)$ is evaluated at quasi-static deformation when $\dot{\boldsymbol{C}} = 0$ using the strong ellipticity condition at the reference configuration ($\boldsymbol{F} = \boldsymbol{I}$). Mathematically, the strong ellipticity condition for an incompressible bodies requires that (see [22])

$$\boldsymbol{m} \cdot [\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{m}] > 0, \quad \boldsymbol{m} \cdot \boldsymbol{n} = 0 \tag{66}$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are unit vectors, and where, in Cartesian components, we have

$$(\boldsymbol{Q}(\boldsymbol{n}))_{ij} = \sum_{p,q=1}^{3}\left(\frac{\partial^2 \Omega_{(e)}}{\partial \boldsymbol{F}^2}\right)_{piqj} n_p n_q, \tag{67}$$

where n_i is a Cartesian component of $\boldsymbol{n}$. The ellipticity condition in the reference configuration is obtained in a similar manner as in the work of Shariff et al. [32], where

$$\boldsymbol{Q}(\boldsymbol{n}) = \boldsymbol{Q}_1(\boldsymbol{n}) + \boldsymbol{Q}_2(\boldsymbol{n}) + \boldsymbol{Q}_3(\boldsymbol{n}) \tag{68}$$

where

$$\boldsymbol{Q}_1(\boldsymbol{n}) = \mu(\boldsymbol{I} + \boldsymbol{n} \otimes \boldsymbol{n}), \tag{69}$$

$$\boldsymbol{Q}_2(\boldsymbol{n}) = \frac{c(e)}{2}[\boldsymbol{H}\boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{H}\boldsymbol{n} + (\boldsymbol{n} \cdot \boldsymbol{H}\boldsymbol{n})\boldsymbol{I} + \boldsymbol{H}], \tag{70}$$

$$\boldsymbol{Q}_3(\boldsymbol{n}) = -\epsilon_0 e^2(\boldsymbol{n} \otimes \boldsymbol{H}\boldsymbol{n} + \boldsymbol{H}\boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{H}). \tag{71}$$

In this section, we deal with problems that can be considered as two dimensional, we only consider the case for $\boldsymbol{m}$ and $\boldsymbol{n}$ are in a plane. Take note that when $\boldsymbol{F} = \boldsymbol{I}$, the basis $[\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$ is arbitrary. the necessary and sufficient condition for (66) is

$$b_1 > 0 \quad \text{and} \quad 4b_1 b_2 > b_3, \tag{72}$$

where

$$b_1 = \mu + \frac{c(e)}{2}(f_1^2 + f_2^2) - \epsilon_0 e^2 f_2^2, \tag{73}$$

$$b_2 = \mu + \frac{c(e)}{2}(f_1^2 + f_2^2) - \epsilon_0 e^2 f_1^2, \tag{74}$$

$$b_3 = 2\epsilon_0 e^2 f_1 f_2. \tag{75}$$

Using $(9)_2$, (52) and (54), the Eulerian electric displacement and polarization then simply take the forms

$$\boldsymbol{d} = -\boldsymbol{F}\frac{\partial \aleph}{\partial \boldsymbol{e}_{(\mathrm{L})}} + \varepsilon_0 \boldsymbol{e}, \quad \boldsymbol{p} = -\boldsymbol{F}\frac{\partial \aleph}{\partial \boldsymbol{e}_{(\mathrm{L})}}, \tag{76}$$

where

$$\aleph = \sum_{i=1}^{3} c(e)\zeta_i r_2(\lambda_i). \tag{77}$$

The components $d_{(\mathrm{L})_k}$ (54) simply take the form

$$d_{(\mathrm{L})_k} = (\boldsymbol{f} \bullet \boldsymbol{u}_k)\left[\frac{\partial \aleph_a}{\partial e} + \frac{2}{e}\left(\frac{\partial \aleph_a}{\partial \zeta_k} - \sum_{i=1}^{3}\frac{\partial \aleph_a}{\partial \zeta_i}\zeta_i\right)\right], \tag{78}$$

where $\aleph_a = \aleph - \varepsilon_0 \sum_{i=1}^{3} \zeta_i \dfrac{e^2}{2\lambda_i^2}$.

6. Comparison with experimental data

In this section, we compare our theory with experiments on three different types of materials; the elastic experiment of Jones and Treloar biaxial data [15], the viscoelastic experiment of soft biological tissues [24] and the electro-viscoselastic experiment of Menhert et al. [20]. We note that values of the ground-state constants used to curve fit the experiments satisfy the inequalities (65) and (72).

6.1. Jones and Treloar biaxial data [15]

In the absence of an electric field and in quasi-static deformations ($\dot{\boldsymbol{C}} = \boldsymbol{0}$), we have $\boldsymbol{T} = \boldsymbol{\sigma}$, and for an incompressible biaxial deformation of a thin (in the 3-direction) purely elastic solid [26],

$$\boldsymbol{U} = \lambda_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \lambda_3 \boldsymbol{u}_3 \otimes \boldsymbol{u}_3, \tag{79}$$

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i, \tag{80}$$

and

$$\sigma_1 - \sigma_2 = \mu[\lambda_1 r_1'(\lambda_1) - \lambda_2 r_1'(\lambda_2)], \tag{81}$$

where σ_i is a principal component of the Cauchy stress $\boldsymbol{\sigma}$ and $\sigma_3 = 0$. Following the work of Shariff [26], we use the function

$$r_1(\lambda) = \ln(\lambda)^2 + \kappa_1\left[\int_1^{\lambda} \frac{e^s - 1}{s}\,\mathrm{d}s + \lambda - 2\ln(\lambda) - 1\right] + \kappa_2\left(\int_1^{\lambda}\frac{1 - e^s}{s}\,\mathrm{d}s - \lambda + 1\right), \tag{82}$$

for the rubberlike material, where κ_1 and κ_2 are dimensionless parameters. In Fig. 1 the theoretical curves $\sigma_1 - \sigma_2$ versus λ_1 at fixed λ_2 is compared with the biaxial experimental data of Jones and Treloar experiment data [15] using the values

$$\mu = 0.4\,\mathrm{MPa}, \quad \kappa_1 = 2.4669, \quad \kappa_2 = 0.3771. \tag{83}$$

It is clear in Fig. 1 the our theory fits the experimental data well.

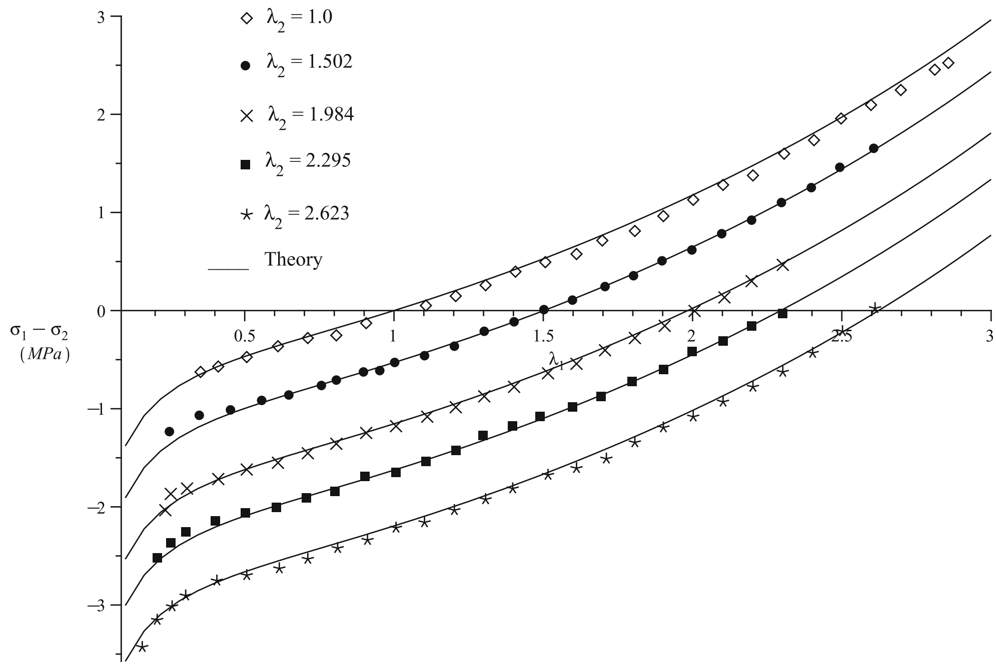

Fig. 1. Comparison of theory with the biaxial experimental data of Jones and Treloar [15]. Quasi-static deformation in the absence of electric field $e = 0$ and $\dot{\boldsymbol{C}} = \mathbf{0}$

6.2. Simple tension viscoelastic experiment data of soft biological tissues [24]

For a simple tension stretch in the 3-direction of a rectangular strip, we have,

$$\boldsymbol{F} = \frac{1}{\sqrt{\lambda}}(\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 + \boldsymbol{g}_2 \otimes \boldsymbol{g}_2) + \lambda \boldsymbol{g}_3 \otimes \boldsymbol{g}_3, \tag{84}$$

$$\dot{\boldsymbol{C}} = -\frac{\dot{\lambda}}{\lambda}(\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 + \boldsymbol{g}_2 \otimes \boldsymbol{g}_2) + 2\lambda\dot{\lambda} \boldsymbol{g}_3 \otimes \boldsymbol{g}_3, \tag{85}$$

where λ is the axial strain and $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ is a fixed orthonormal basis. The axial component of second-Piola Kirchhoff stress

$$\tau = \frac{\mu}{\lambda^2}[\lambda r_1'(\lambda) - \frac{1}{\sqrt{\lambda}} r_1'(\frac{1}{\sqrt{\lambda}}) + sv_1 - sv_2], \tag{86}$$

where

$$sv_1 = 4\lambda^3 \dot{\lambda}[\nu_1 r_3(\lambda) + 2\nu_2 H_4], \tag{87}$$

$$sv_2 = -2\frac{\dot{\lambda}}{\lambda^2}\left[\nu_1 r_3(\frac{1}{\sqrt{\lambda}}) + 2\nu_2 H_4\right]. \tag{88}$$

For this type of soft tissue biological materials, following the work of Shariff [29,34] on passive myocardium, we use, for simplicity, the functions

$$r_1 = r_3 = r_4 = s^2, \tag{89}$$

where

$$s(x) = \frac{2}{\rho_0\sqrt{\pi}}\operatorname{erf}^{-1}(\rho_0 \ln(x)) + \rho_1(e^{1-x} + x - 2), \tag{90}$$

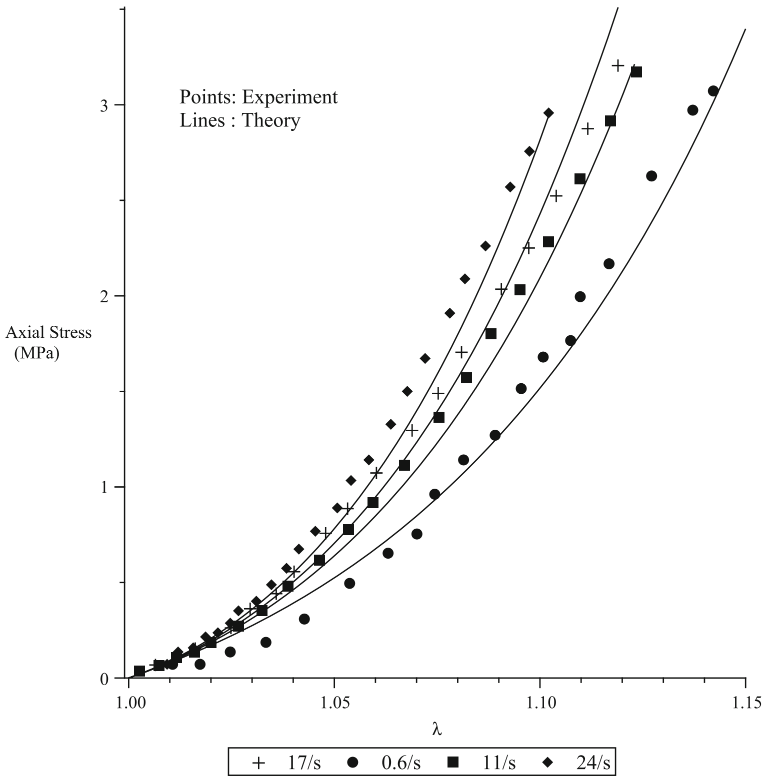

FIG. 2. Comparison of theory with the axial experimental biological soft tissues of Pioletti and Rakotomanana [24]. The axial stress is the second-Piola Kirchhoff stress. λ is the axial strain. The curves for strain rates $h = 0.6\%/s$ and $h = 24\%/s$ are fitted in an ad-hoc manner. The curves for strain rates $h = 11\%/s$ and $h = 17\%/s$ are predicted using fitted material constant values

$\text{erf}^{-1}(x)$ is the inverse error function and, ρ_o and ρ_1 are dimensionless material parameters.

In Fig. 2, we fit the experimental data of Pioletti and Rakotomanana [24] with the ad-hoc values

$$\rho_0 = 4.0, \quad \rho_1 = 10, \quad \mu = 2.0\,\text{MPa}, \quad \nu_1 = \nu_2 = 0.05, \tag{91}$$

for strain rates $h = 0.6\%/s$ and $h = 24\%/s$. Using the values given in (91), we predict the data for $h = 11\%/s$ and $h = 17\%/s$. From Fig. 2, it seems that our simple constitutive equation reasonably describes the mechanical behaviour of a short-term memory response visco-elastic material.

6.3. Axial tension of a thin electro-viscoelastic VHB 4905$^{\text{TM}}$ sheet [20].

The experiment of Menhert et al. [20] uses a thin rectangular strip of VHB 4905$^{\text{TM}}$ polymer with thickness 500 microns, width 70 mm and length 100 mm. In view of this, we consider the undeformed configuration (dimensions are in metres)

$$\begin{aligned} &-2.5 \times 10^{-4} \leq X \leq 2.5 \times 10^{-4}, \quad -3.5 \times 10^{-2} \leq Y \leq 3.5 \times 10^{-2}, \\ &-0.05 \leq Z \leq 0.05. \end{aligned} \tag{92}$$

The strip is clamped at $Z = -0.05$ and $Z = 0.05$ and stretch in the Z-direction. Although the deformation is not homogeneous, but it is mainly homogeneous near $Z = 0$. Hence, near $Z = 0$ we approximate the deformation as a homogeneous axial tension deformation and the deformation gradient is given by (84). Details of the experiment are given in [20] and hence, we will not discuss them here. The area of the undeformed surface, where the force acts, is 0.35×10^{-4} m^2 and the electric field is

$$\boldsymbol{e} = \eta\, \boldsymbol{g}_1, \tag{93}$$

where $\eta = \dfrac{V}{5 \times 10^{-4}}$ V/m and V is the applied voltage. For this material we use

$$r_1'(x) = \frac{ln(x) + \dfrac{e^{(a_1(x-1))} - 1}{a_1} + \dfrac{a_2}{100}\mathrm{erf}(100(x-1))}{1 + \dfrac{a_2}{\sqrt{\pi}}}, \tag{94}$$

$$r_2(x) = (x-1)^2, \quad c(e) = c_1 e^2, \quad r_3(x) = r_4(x) = \frac{\sqrt{\pi}\mathrm{erf}(1000(x-1)) r_1'(x)}{4000x^2}, \tag{95}$$

where a_1, a_2 are dimensionless parameters, erf is the error function and $e^2 = \boldsymbol{e}_L \cdot \boldsymbol{e}_L = \dfrac{\eta^2}{\lambda}$. The Cartesian components of the Maxwell stress outside the body is

$$(\boldsymbol{T}_m)_{11} = \frac{\varepsilon_0 \eta^2}{2} = -(\boldsymbol{T}_m)_{22} = -(\boldsymbol{T}_m)_{33} \tag{96}$$

and,in view of

$$\sigma_{11} = (\boldsymbol{T}_m)_{11}, \tag{97}$$

the axial total stress is given by

$$\sigma_{33} = \lambda r_1'(\lambda) - \lambda_1 r_1'(\lambda_1) - \eta^2 (\frac{c_1 \lambda_1 r_2'(\lambda_1)}{\lambda} + \frac{\varepsilon_0}{2}), \tag{98}$$

where σ_{11} and σ_{33} are Cartesian components of the total stress $\boldsymbol{T}$. The applied force TF acting on the material is

$$TF = \frac{3.5 \times 10^{-5} \sigma_{33}}{\lambda}. \tag{99}$$

We do an ad-hoc curve fitting of Menhert et al. [20] experiment with the constant values

$$\mu = 0.55\,\mathrm{MPa}, \quad a_1 = -4, \quad a_2 = 7, \quad \nu_1 = \frac{2000}{\sqrt{\pi}}, \quad \nu_2 = \frac{4}{\sqrt{\pi}}, \quad c_1 = -0.35 \times 10^{-9}\,\mathrm{F/m}. \tag{100}$$

In Fig. 3, we curve fit the *loading* data of Menhert et al. [20] in an ad-hoc manner. From the figure, the simple functions in (94) and (95), seem able to model the *loading* experiment results of Menhert et al. [20].

7. Boundary value problems

For obtaining the numerical results in this section, we use r_1 in (82) and the material-constant values in (83) and, for simplicity, we use

$$r_2(x) = r_3(x) = r_4(x) = (x-1)^2, \quad c(e) = c_1 e^2, \tag{101}$$

$$\nu_1 = \nu_2 = 0.001, \quad c_1 = 10^{-10}\,\mathrm{F/m}. \tag{102}$$

As well as this, for simplicity we only consider pure homogeneous deformations, and homogeneous distributions for the electric field. Two problems are studied, namely the simple tension of a cylinder and the simple shear of a slab, both at constant strain rates, using the proposed specific forms given in Sect. 5.

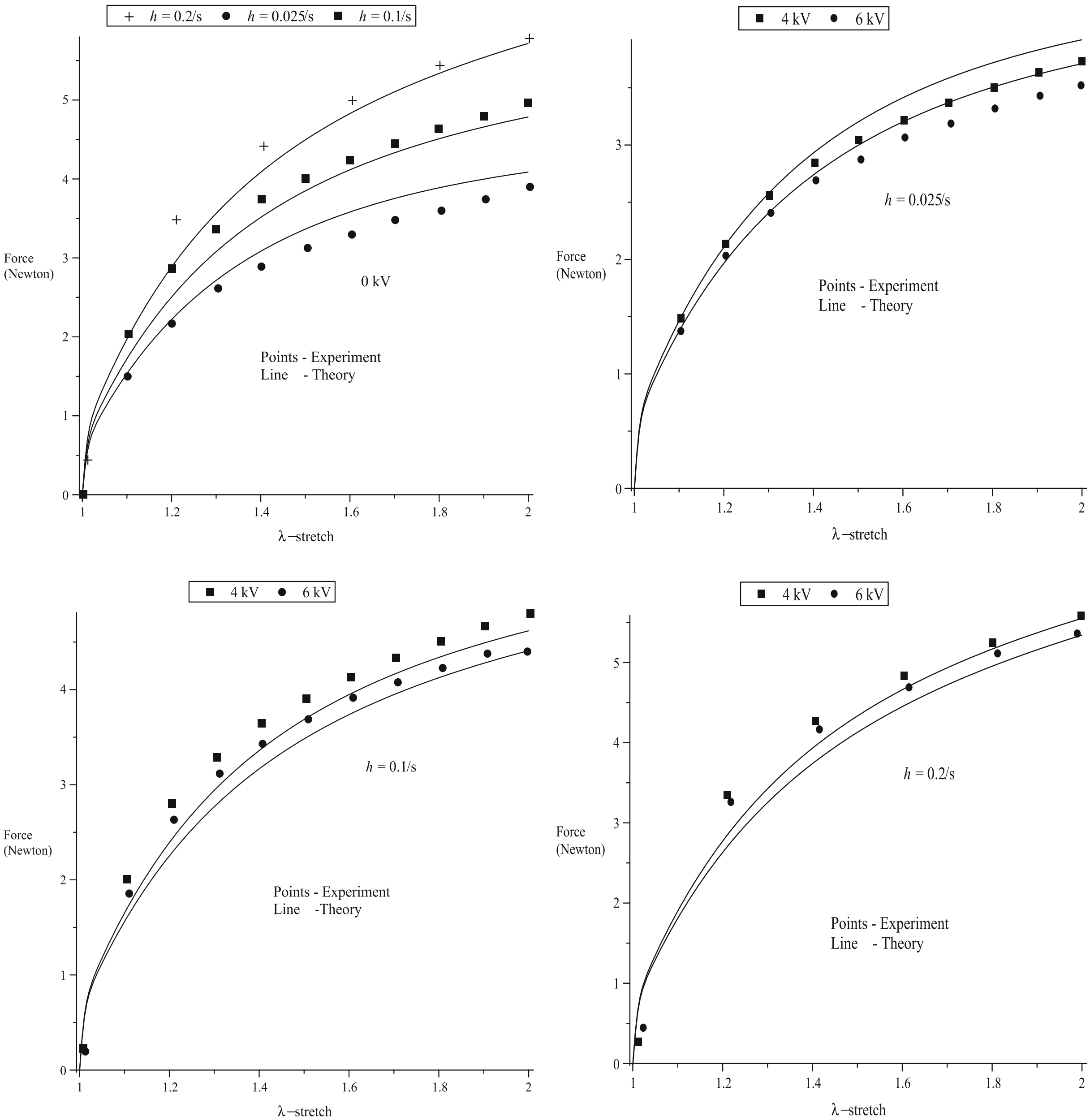

FIG. 3. Force versus λ for various values of strain rate $h = \dot{\lambda}$ and applied voltages

7.1. Simple tension of a cylinder

Here, we consider a solid cylinder and the deformation

$$r = \frac{1}{\sqrt{\lambda_z}} R, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{103}$$

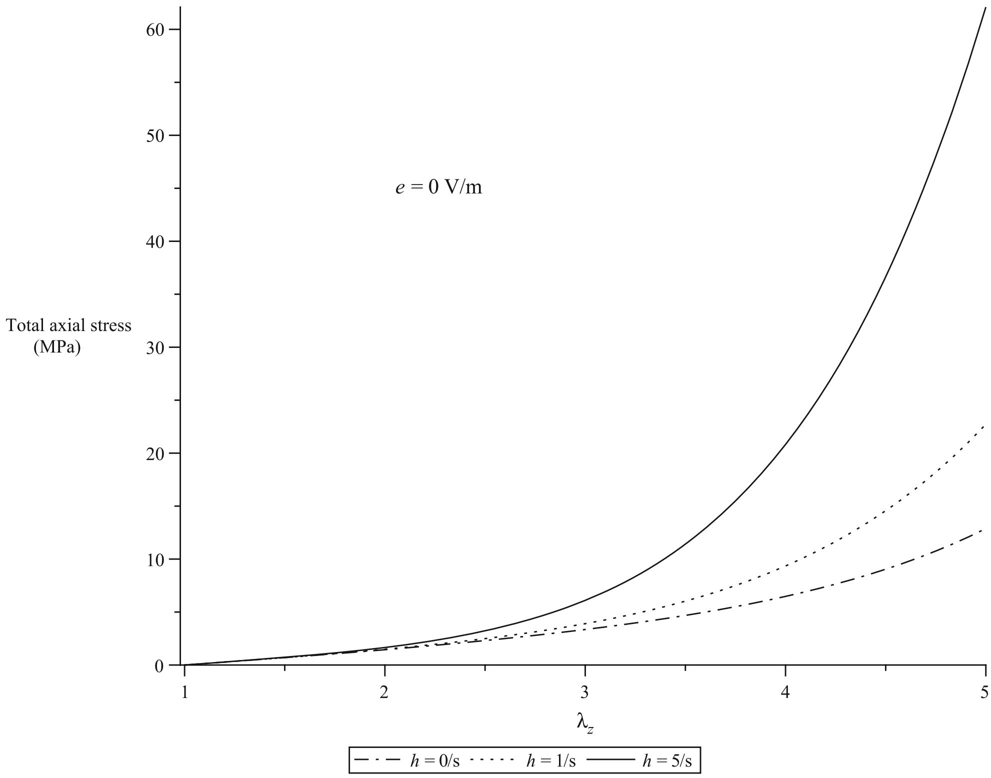

Fig. 4. Plot of the total axial stress (113) versus λ_z when the electric field $e = 0$ V/m for various values of strain rate $h = \dot{\lambda}_z$

where (r, θ, z) and (R, Θ, Z) are the polar coordinate in the deformed and undeformed configurations, respectively. All tensor and vector components in this section are defined with respect to the cylindrical polar basis $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$. We have

$$\boldsymbol{F} \equiv \begin{pmatrix} \frac{1}{\sqrt{\lambda_z}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_z}} & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{104}$$

where $\lambda_z > 0$ is the uniaxial stretch that depends on time t. The principal stretches are

$$\lambda_1 = \lambda_r = \frac{1}{\sqrt{\lambda_z}}, \quad \lambda_2 = \lambda_r = \frac{1}{\sqrt{\lambda_z}}, \quad \lambda_3 = \lambda_z, \tag{105}$$

and the Lagrangian spectral vectors are given as

$$\boldsymbol{u}_1 = \boldsymbol{e}_r, \quad \boldsymbol{u}_2 = \boldsymbol{e}_\theta, \quad \boldsymbol{u}_3 = \boldsymbol{e}_z. \tag{106}$$

Also

$$\boldsymbol{C} \equiv \begin{pmatrix} \frac{1}{\lambda_z} & 0 & 0 \\ 0 & \frac{1}{\lambda_z} & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix},$$

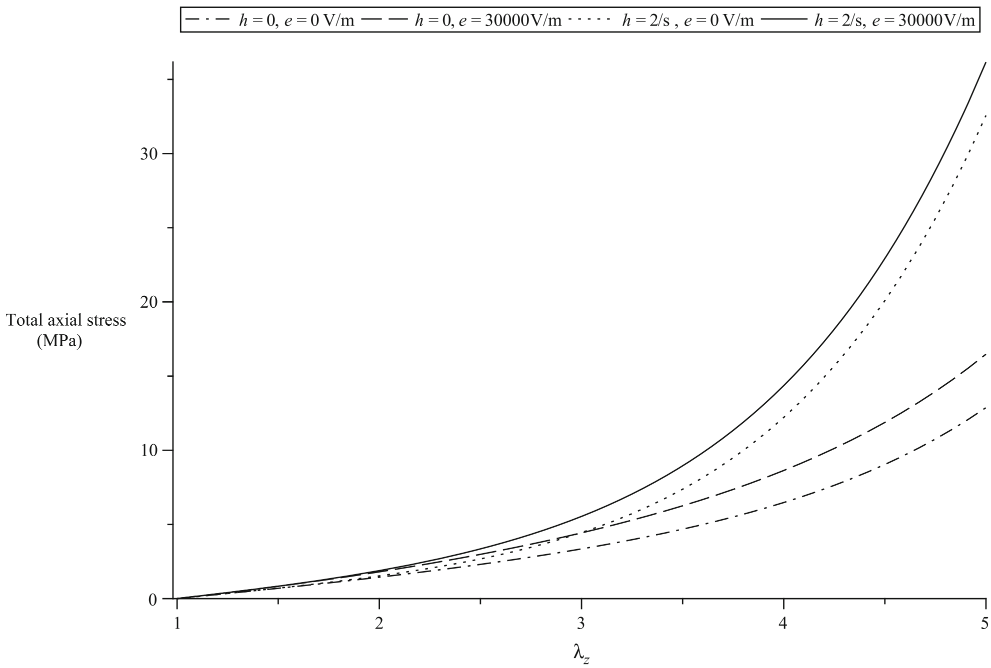

FIG. 5. Plot of the total axial stress (113) versus λ_z for various values of electric field e and strain rate $h = \dot{\lambda}_z$

$$\dot{\boldsymbol{C}} \equiv \begin{pmatrix} \alpha_1 = -\dfrac{\dot{\lambda}_z}{\lambda_z} & 0 & 0 \\ 0 & \alpha_2 = -\dfrac{\dot{\lambda}_z}{\lambda_z} & 0 \\ 0 & 0 & \alpha_3 = 2\lambda_z\dot{\lambda}_z \end{pmatrix}. \tag{107}$$

Here, we consider the case $\boldsymbol{e}_{(L)} = e\boldsymbol{e}_z$, where e is a constant. Hence, $\boldsymbol{f} = \boldsymbol{e}_z$, the condition $\mathrm{Curl}\,\boldsymbol{e}_{(L)} = 0$ is automatically satisfied and

$$f_1 = f_2 = 0, \quad f_3 = 1, \quad \beta_i = \alpha_i^2. \tag{108}$$

The Lagrangian components of the electric displacement are simplified to

$$d_{(\mathrm{L})_1} = d_{(\mathrm{L})_2} = 0, \quad d_{(\mathrm{L})_3} = 2e\left[\sum_{i=1}^{3} c_1\zeta_i r_2(\lambda_i) - \varepsilon_0 \sum_{i=1}^{3}\zeta_i \frac{1}{2\lambda_i^2}\right]. \tag{109}$$

It is clear that $d_{(\mathrm{L})_3}$ is independent of $\boldsymbol{X}$ and hence $\mathrm{Div}(\boldsymbol{d}_{(\mathrm{L})}) = 0$ is automatically satisfied. The electric field is

$$\boldsymbol{e} = \frac{e}{\lambda_z}\boldsymbol{e}_z. \tag{110}$$

The non-zero Maxwell stress components in vacuo are

$$(\boldsymbol{T}_\mathrm{M})_{zz} = \frac{\varepsilon_0 e^2}{2\lambda_z^2} = -(\boldsymbol{T}_\mathrm{M})_{rr} = -(\boldsymbol{T}_\mathrm{M})_{\theta\theta}. \tag{111}$$

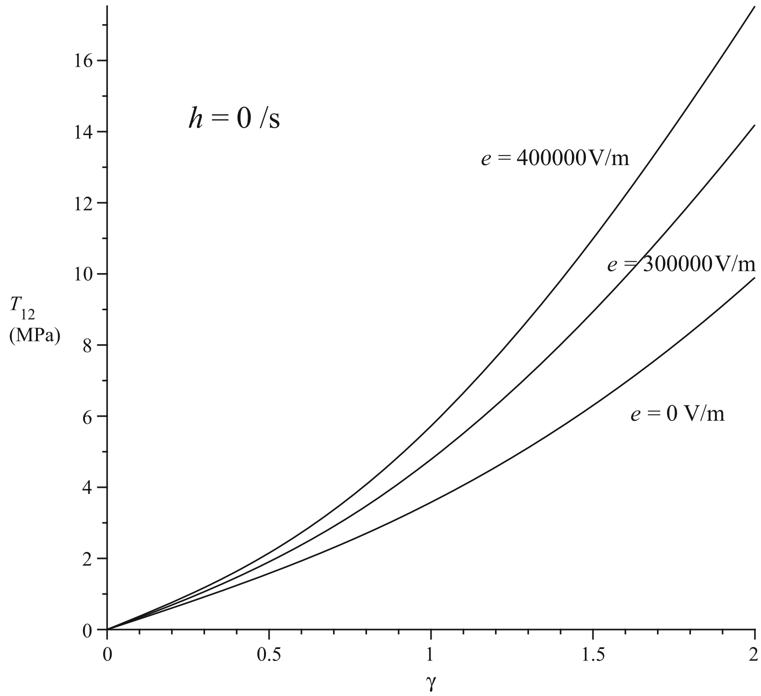

FIG. 6. Plot of total shear stress T_{12} versus the amount of shear γ when $h = \dot{\gamma} = 0/\mathrm{s}$

Let $\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}$ be the non-shear cylindrical polar components of the total stress $\boldsymbol{T}$. If there is no mechanical stress on the cylindrical free surface, we have for the total Cauchy stress at the free surface

$$\tau_{rr} = -\frac{\varepsilon_0 e^2}{2\lambda_z^2}. \tag{112}$$

We then have

$$\frac{\tau_{zz}}{\mu} = \lambda_z r_1'(\lambda_z) - \lambda_1 r_1'(\lambda_1) + c_1 e^2 \lambda_z r_2'(\lambda_z) + \frac{\varepsilon_0 e^2}{2\lambda_z^2} + sv_z - sv_r, \tag{113}$$

where

$$sv_z = 4\lambda_z^3 \dot{\lambda}_z [\nu_1 r_3(\lambda_z) + 2\nu_2 H_4], \tag{114}$$

$$sv_r = -2\frac{\dot{\lambda}_z}{\lambda_z^2}\left[\nu_1 r_3(\frac{1}{\sqrt{\lambda_z}}) + 2\nu_2 H_4\right]. \tag{115}$$

It is clear from the constitutive equation that, since $\lambda_1 = \lambda_2$ (and also from the equilibrium equation), that

$$\tau_{rr} = \tau_{\theta\theta} = -\frac{\varepsilon_0 e^2}{2\lambda_z^2}. \tag{116}$$

In Fig. 4 the plot of the total axial stress (113) versus λ_z when the electric field $e = 0$ V/m for various values of strain rate $\dot{\lambda}_z$ is depicted. It is clear from Fig. 4 that, as expected, the magnitude of the total axial stress increases as the strain rate $\dot{\lambda}_z$ increases. The behaviour of the total axial stress (113) for various values of electric field e and strain rate $\dot{\lambda}_z$ is depicted in Fig. 5. From Fig. 5, both the presence of an electric field and strain rate increase the magnitude of the total axial stress.

7.2. Simple shear of a slab

In this section, we give results for a simple shear deformation, where the principal directions $\boldsymbol{u}_i$ change continuously during deformation. Here, all the components of vectors and tensors are relative to a fixed Cartesian system. Consider a simple shear for a slab, where the deformation gradient is of the form

$$\boldsymbol{F} \equiv \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{117}$$

where $\gamma > 0$ is the amount of shear that depends on time t. We then have,

$$\boldsymbol{C} \equiv \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dot{\boldsymbol{C}} \equiv \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 2\gamma\dot{\gamma} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{118}$$

The spectral invariants of $\boldsymbol{C}$ are obtained using the following methodology. Let θ denote the orientation (in the anticlockwise sense relative to the 1-axis) of the in plane Lagrangean principal axes. The angle θ is restricted accordingly by the following (see [27])

$$\frac{\pi}{4} \leq \theta < \frac{\pi}{2}. \tag{119}$$

The principal directions are $\boldsymbol{u}_1 \equiv [c, s, 0]^T$, $\boldsymbol{u}_2 \equiv [-s, c, 0]^T$ and $\boldsymbol{u}_3 \equiv [0, 0, 1]^T$, where $c = \cos(\theta)$ and $s = \sin(\theta)$. It can be easily shown (see [27]) that the principal stretches take the values

$$\lambda_1 = \frac{\gamma + \sqrt{\gamma^2+4}}{2} \geq 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2+4} - \gamma}{2} \leq 1, \quad \lambda_3 = 1 \tag{120}$$

and

$$c = \frac{1}{\sqrt{1+\lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}, \quad c^2 - s^2 = -\gamma c s. \tag{121}$$

We consider the case when the electric field $\boldsymbol{e}_{(L)} \equiv [0, e, 0]^T$, where e is constant. Hence,

$$\zeta_1 = s^s, \quad \zeta_2 = c^2, \quad \zeta_3 = 0 \tag{122}$$

and $\operatorname{Curl} \boldsymbol{e}_{(L)} = 0$. The nonzero components of the Maxwell stress in vacuo are

$$(\boldsymbol{T}_M)_{22} = \frac{\varepsilon_0 e^2}{2} = -(\boldsymbol{T}_M)_{11} = -(\boldsymbol{T}_M)_{33}. \tag{123}$$

The total shear stress

$$T_{12} = \sigma_{12}^{(e)} + \sigma_{12}^{(v)}, \tag{124}$$

where

$$\sigma_{12}^{(e)} = 2[l_1(\gamma s^2 + cs) + l_2(\gamma c^2 - cs) + l_4 \gamma cs], \tag{125}$$

where

$$l_\alpha = \frac{1}{2\lambda_\alpha}\left\{\mu r_1'(\lambda_\alpha) + e^2\zeta_\alpha\left[c_1 r_2'(\lambda_\alpha) + \frac{\varepsilon_o}{\lambda_\alpha^3}\right]\right\}, \quad \alpha = 1, 2, \tag{126}$$

$$l_4 = e^2\left\{\frac{c_1[r_2(\lambda_1) - r_2(\lambda_2)]}{\lambda_1^2 - \lambda_2^2} + \frac{\varepsilon_0}{2\lambda_1^2\lambda_2^2}\right\} \tag{127}$$

and

$$\sigma_{12}^{(v)} = 2\mu[P_1 + 2\nu_2 H_2(\dot{\gamma} + 2\gamma^2\dot{\gamma})], \tag{128}$$

$$P_1 = \frac{\nu_1}{2}\left\{r_3(\lambda_1)[\dot{\gamma}(1 + 4cs\gamma + 4\gamma^2 s^2)] + r_3(\lambda_2)[\dot{\gamma}(1 - 4cs\gamma + 4\gamma^2 c^2)]\right\}. \tag{129}$$

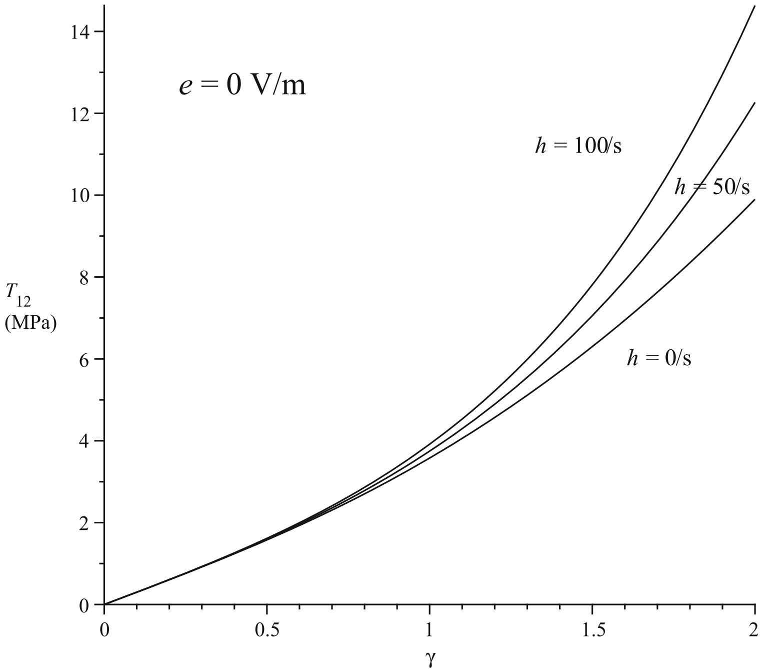

FIG. 7. Plot of total shear stress T_{12} versus the amount of shear γ when $e = 0$ V/m

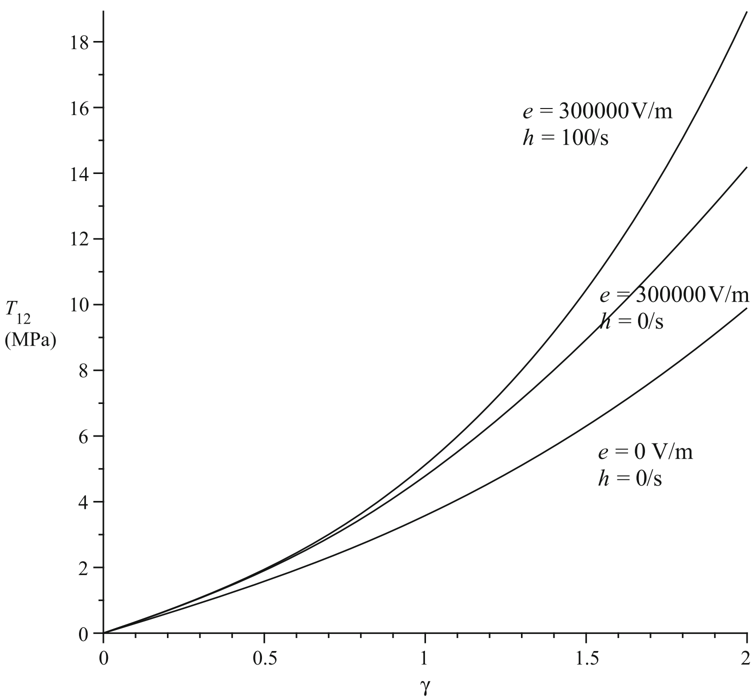

FIG. 8. Plot of total shear stress T_{12} versus the amount of shear γ for various values $h = \dot{\gamma}$ and e

In Fig. 6 we see that at $\dot{\gamma} = 0$/s, the magnitude of the shear stress increases as the electric field e increases. In the case when the electric field is absent, $e = 0$ V/m, it is indicated in Fig. 7 that the magnitude of the shear stress increases as the value $\dot{\gamma}$ increases. In Fig. 8 we see that both the presence of an electric field and shear rate $\dot{\gamma}$ have the effect of increasing the magnitude of the shear stress.

8. Conclusion

In this article a model for short-term memory response visco-electro-elastic solids has been presented, where the scalar potentials that are used to obtained the constitutive equations, are defined in terms of spectral invariants. The use of such invariants, which have clearer physical meanings in comparison with the classical invariants [41], permits to obtain simple but general expressions for the stresses and the dependent electrical variable. In future works we will consider additionally the presence of one and two preferred directions (transversely isotropic body and two directions elasticity), which are interesting from the point of view of applications in biomechanics, and also in the modelling of some electro-active polymers, where a rubber-like material is filled with electro-active particles that are aligned in a preferred direction (chains of particles) [4].

Appendix: $\boldsymbol{P}$-property

The description of the P-property uses the eigenvalues (λ_i) and eigenvectors ($\boldsymbol{u}_i$) of the symmetric tensor $\boldsymbol{U}$. A general anisotropic scalar function Φ, such as that given in (37) and (45), where its arguments are expressed in terms spectral invariants with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ can be written in the form

$$\Phi = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3), \tag{130}$$

with the symmetrical property

$$\tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \tilde{W}(\lambda_2, \lambda_1, \lambda_3, \boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_3) = \tilde{W}(\lambda_3, \lambda_2, \lambda_1, \boldsymbol{u}_3, \boldsymbol{u}_2, \boldsymbol{u}_1). \tag{131}$$

In view of the non-unique values of $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ when $\lambda_i = \lambda_j$, a function $\tilde{W}$ should be independent of $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ when $\lambda_i = \lambda_j$, and $\tilde{W}$ should be independent of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ when $\lambda_1 = \lambda_2 = \lambda_3$. Hence, when two or three of the principal stretches have equal values the scalar function Φ must have any of the following forms

$$\Phi = \begin{cases} W_{(a)}(\lambda, \lambda_k, \boldsymbol{u}_k), & \text{when } \lambda_i = \lambda_j = \lambda, i \neq j \neq k \neq i \\ W_{(b)}(\lambda), & \text{when } \lambda_1 = \lambda_2 = \lambda_3 = \lambda \end{cases} \tag{132}$$

As an example of (132), consider $\Phi = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} = \sum_{i=1}^{3} \lambda_i^2 (\boldsymbol{a} \bullet \boldsymbol{u}_i)^2$, where $\boldsymbol{a}$ is a fixed unit vector and $\sum_{i=1} (\boldsymbol{a} \bullet \boldsymbol{u}_i)^2 = 1$. If $\lambda_1 = \lambda_2 = \lambda$, we have $\Phi = W_{(a)}(\lambda, \lambda_3, \boldsymbol{u}_3) = \lambda^2 + (\lambda_3^2 - \lambda^2)(\boldsymbol{a} \bullet \boldsymbol{u}_3)^2$ and in the case of $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\Phi = W_{(b)}(\lambda) = \lambda^2$. Note that, for example, $\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{u}_i \otimes \boldsymbol{u}_i$ (or $\boldsymbol{U}$) and all the classical invariants described in Spencer [41] , satisfy the P-property. In Refs. [35] and [40], the P-property described here is extended to non-symmetric tensors such as the two-point deformation tensor $\boldsymbol{F}$.

References

[1] Ask, A., Menzel, A., Ristinmaa, M.: Electrostriction in electro-viscoelastic polymers. Mech. Mater. **50**, 9–21 (2012)
[2] Ask, A., Menzel, A., Ristinmaa, M.: Phenomenological modeling of viscous electrostrictive polymers. Int. J. Nonlinear Mech. **47**, 156–165 (2012)
[3] Bar-Cohen, Y.: Electrostictive polymers: current capabilities and challenges. In: Proceedings of the SPIE Smart Structures and Materials Symposium, EAPAD Conference, San Diego, CA, Paper 4695-02 (2002)
[4] Bossis, G., Abbo, C., Cutillas, S., Lacis, C., Métayer, C.: Electroactive and electrostructures elastomers. Int. J. Mod. Phys. B **15**, 564–573 (2001)
[5] Büschel, A., Klinkel, S., Wagner, W.: Dielectric elastomers. Numerical modeling of nonlinear visco-electroelasticity. Int. J. Numer. Methods Eng. **93**, 834–856 (2013)
[6] Bustamante, R., Shariff, M.H.B.M.: New sets of invariants for an electro-elastic body with one and two families of fibres. Eur. J. Mech. A-Solid **58**, 42–53 (2016)
[7] Canzız, B., Dal, H., Kaliske, M.: Computational cardiology: a modified Hill model to describe the electro-visco-elasticity of the myocardium. Comput. Methods Appl. Mech. Eng. **315**, 434–466 (2017)
[8] Chen, X.: Nonlinear electro-thermo-viscoelasticity. Acta Mech. **211**, 49–59 (2010)
[9] Denzer, R., Menzel, A.: Configurational forces for quasi-incompressible large strain electro-viscoelasticity. Applications to fracture mechanics. Eur. J. Mech. A-Solids **48**, 3–15 (2014)
[10] Dorfmann, A., Ogden, R.W.: Nonlinear electroelasticity. Acta Mech. **174**, 167–183 (2005)
[11] Dorfmann, A., Ogden, R.W.: Nonlinear electroelastic deformations. J. Elast. **82**, 99–127 (2006)
[12] Fang, D.N., Soh, A.K., Li, C.Q., Jiang, B.: Nonlinear behavior of 0–3 type ferroelectric composites with polymer matrices. J. Mater. Sci. **36**, 5281–5288 (2001)
[13] Gizzia, A., Anna Pandolf, A.: Visco-hyperelasticity of electro-active soft tissues. Procedia IUTAM **12**, 162–175 (2015)
[14] Holzapfel, G.A., Ogden, R.W.: On planar biaxial tests for anisotropic nonlinearly elastic solids: a continuum mechanical framework. Math. Mech. Solids **14**, 474–489 (2009)
[15] Jones, D.F., Treloar, L.R.G.: The properties of rubber in pure homogeneous strain. J. Phys. D Appl. Phys. **8**(11), 1285–1304 (1975)
[16] Hong, W.: Modeling viscoelastic dielectrics. J. Mech. Phys. Solids **59**, 637–650 (2011)
[17] Khan, K., Wufai, H., El Sayed, T.: A variational constitutive framework for the nonlinear viscoelastic response of a dielectric elastomer. Comput. Mech. **52**, 345–360 (2013)
[18] Kovetz, A.: Electromagnetic Theory. University Press, Oxford (2000)
[19] Mehnert, M., Hossain, M., Steinmann, P.: Numerical modeling of thermo-electro-viscoelasticity with field dependent material parameters. Int. J. Nonlinear Mech. **106**, 13–24 (2018)
[20] Mehnert, M., Hossain, M., Steinmann, P.: Experimental and numerical investigations of the electro-viscoelastic behavior of VHB 4905^{TM}. Eur. J. Mech./ A Solids **77**, 103797 (2019)
[21] Murphy, J.G., Biwa, S.: The counterintuitive mechanical response in simple tension of arterial models that are separable functions of the I_1, I_4, I_6 invariants. Int. J. Nonlinear Mech. **90**, 72–81 (2017)
[22] Ogden, R.W.: Non-Linear Elastic Deformations. Ellis Horwood, Chichester (1984)
[23] Ogden, R.W., Steigmann, D.J.: Mechanics and Electrodynamics of Magneto- and Electro-elastic Materials. CISM Courses and Lectures Series, vol. 527. Springer, Wien (2011)
[24] Pioletti, D.P., Rakotomanana, L.R.: Non-linear viscoelastic laws for soft biological tissues. Eur. J. Mech. A-Solid **19**(5), 749–759 (2000)
[25] Saxena, P., Vu, D.K., Steinmann, P.: On rate-dependent dissipation effects in electro-elasticity. Int. J. Nonlinear Mech. **62**, 1–11 (2014)
[26] Shariff, M.H.B.M.: Strain energy function for filled and unfilled rubberlike material. Rubber Chem. Technol. **73**, 1–21 (2000)
[27] Shariff, M.H.B.M.: Nonlinear transversely isotropic elastic solids: an alternative representation. Q. J. Mech. Appl. Math. **61**, 129–149 (2008)
[28] Shariff, M.H.B.M.: Physical invariants for nonlinear orthotropic solids. Int. J. Solids Struct. **48**, 1906–1914 (2011)
[29] Shariff, M.H.B.M.: Physical invariant strain energy function for passive myocardium. Biomech. Model. Mechanobiol. **12**(2), 215–223 (2013)
[30] Shariff, M.H.B.M.: The number of independent invariants of an n-preferred direction anisotropic solid. Math. Mech. Solids **22**(10), 1989–1996 (2016)
[31] Shariff, M.H.B.M.: Anisotropic separable free energy functions for elastic and non-elastic solids. Acta Mech. **227**(11), 3213–3237 (2016)
[32] Shariff, M.H.B.M., Bustamante, R., Merodio, J.: On the spectral analysis of residual stress in finite elasticity. IMA J. Appl. Math. **82**(3), 656–680 (2017)

[33] Shariff, M.H.B.M., Bustamante, R., Merodio, J.: Rate type constitutive equations for fiber reinforced nonlinearly viscoelastic solids using spectral invariants. Mech. Res. Commun. **84**, 60–64 (2017)
[34] Shariff, M.H.B.M.: On the spectral constitutive modelling of transversely isotropic soft tissue: physical invariants. Int. J. Eng. Sci. **120**, 199–219 (2017)
[35] Shariff, M.H.B.M.: Spectral derivatives in continuum mechanics. Q. J. Mech. Appl. Mech. **70**(4), 479–476 (2017)
[36] Shariff, M.H.B.M., Bustamante, R., Merodio, J.: A nonlinear electro-elastic model with residual stresses and a preferred direction. Math. Mech. Solids **25**(3), 838–865 (2020)
[37] Shariff, M.H.B.M., Bustamante, R., Merodio, J.: A nonlinear constitutive model for a two preferred direction electro-elastic body with residual stresses. Int. J. Nonlinear Mech. **119**, 103352 (2020)
[38] Shariff, M.H.B.M.: The number of independent invariants for n symmetric second order tensors. J. Elast. **134**(1), 119–126 (2019)
[39] Shariff, M.H.B.M.: The number of independent invariants for m unit vectors and n symmetric second order tensors is $2m + 6n - 3$. arXiv preprint arXiv:1907.09941 (2019)
[40] Shariff, M.H.B.M.: A general spectral nonlinear elastic consistent tangent modulus tensor formula for finite element software. Results Appl. Math. **7**, 100113 (2020)
[41] Spencer, A.J.M.: Theory of invariants. In: Eringen, A.C. (ed.) Continuum Physics I, pp. 239–253. Academic Press, New York (1971)
[42] Truesdell, C.A., Toupin, R.: The classical field theories. In: Flügge, S. (ed.) Handbuch der Physik, vol. III/1, pp. 226–902. Springer, Berlin (1960)
[43] Vogel, F., Göktepe, S., Steinmann, P., Kuhl, E.: Modeling and simulation of viscous electro-active polymer. Eur. J. Mech. A-Solids **48**, 112–128 (2014)
[44] Wang, S., Decker, M., Henann, D.L., Chester, S.A.: Modeling of dielectric viscoelastomers with applications to electromechanical instabilities. J. Mech. Phys. Solids **95**, 213–229 (2016)

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