Characterization of Filippov representable maps and Clarke subdifferentials

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Abstract

The ordinary differential equation $\dot{x}(t) = f(x(t)),\ t \geq 0$, for f measurable, is not sufficiently regular to guarantee existence of solutions. To remedy this we may relax the problem by replacing the function f with its Filippov regularization F_f and consider the differential inclusion $\dot{x}(t) \in F_f(x(t))$ which always has a solution. It is interesting to know, inversely, when a set-valued map Φ can be obtained as the Filippov regularization of a (single-valued, measurable) function. In this work we give a full characterization of such set-valued maps, hereby called Filippov representable. This characterization also yields an elegant description of those maps that are Clarke subdifferentials of a Lipschitz function.

Keywords: Filippov regularization, Krasovskii regularization, Differential inclusion, cusco map, Clarke subdifferential.

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1 Introduction

We consider the differential equation

$$\dot{x}(s) = f(x(s)),\ s \geq 0\,,\ x(0) = x_0\,, \tag{1}$$

where $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a bounded measurable function and $x_0 \in \mathbb{R}^d$. The above Cauchy problem might have no solution due to the lack of regularity of f. A way to overcome this difficulty is to replace (1) by a "minimal" differential inclusion which is sufficiently regular to have a solution. A natural way to do this is to replace f by its Krasovskii regularization K_f given by

$$K_f(x) := \bigcap_{\delta>0} \overline{\text{co}}\, f(B_\delta(x))$$

and obtain, accordingly:

$$\dot{x}(s) \in K_f(x(s)),\ x(0) = x_0,\ s \geq 0. \tag{2}$$

Another possibility is to consider, instead of K_f, the Filippov regularization F_f of f given by

$$F_f(x) := \bigcap_{\mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, f((B_\delta(x)) \setminus N),$$

where the first intersection is taken over the sets $N \subset \mathbb{R}^d$ with Lebesgue measure $\mathcal{L}(N)$ equal to zero. In this way, we obtain the so-called Filippov solutions of (1), that is, solutions of the differential inclusion

$$\dot{x}(s) \in F_f(x(s)),\ x(0) = x_0,\ s \geq 0. \tag{3}$$

The Filippov regularization is based on the idea that sets of measure zero should play no role in the relaxed dynamics.

Inclusions (2) and (3) always have a solution, since the set-valued mappings K_f and F_f are upper semicontinuous, with nonempty convex compact values (*c.f.* [1], [14]). For simplicity, borrowing terminology from [5], [4], we shall refer to such set-valued mappings as *cusco* maps (see forthcoming Definition 2.1). If the function f is continuous, then both maps K_f and F_f are single-valued and equal to f.

The techniques of Krasovskii and Filippov regularizations were introduced for obtaining solutions of discontinuous differential equations. Both regularizations have further been widely used in optimal control and differential games, see [3], [9], [16], [19], [21], [24], [23] *e.g.*

The main goal of this paper is to consider the inverse problem: given a cusco set-valued mapping F from $\mathbb{R}^d$ to $\mathbb{R}^d$, does there exist a singe-valued function f, such that F is the Krasovskii / Filippov regularization of f? We shall refer to such maps as Krasovskii representable (respectively, Filippov representable). Notice that "being cusco" is clearly a necessary condition for being representable. We completely characterize Filippov representable maps, even in a slightly more general setting, namely, for maps defined in $\mathbb{R}^d$ with values in $\mathbb{R}^\ell$.

The other main contribution of this work is an equivalent characterization of the set-valued maps that are Clarke subdifferentials of a Lipschitz function in the finite-dimensional case. We show that these maps are exactly the Filippov regularizations of functions satisfying a so-called *nonsmooth Poincaré condition*. This condition is recently stated and used independently in [18] and [10] for a different purpose. We refer to [4] for another characterization of set-valued maps that are Clarke subdifferentials of a Lipschitz function in Banach spaces.

The manuscript is organized as follows: In Section 2 we introduce basic notation and background for Krasovskii and Filippov regularizations. In Section 3 we obtain several key results for both regularizations, while in Section 4 we provide the main result (characterization of Filippov representability) and use it to obtain an alternative characterization of those set-valued maps that are Clarke subdifferentials of Lipschitz functions (Section 5).

2 Preliminaries

Throughout the paper, we denote by B_X (respectively, $\bar{B}_X$) the open (respectively, closed) unit ball, centered at the origin of the normed space X. The index will often be omitted if there is no ambiguity about the space. In this case, we denote by $B_\delta(x) := x + \delta B_X$ the (open) ball centered at x with radius δ. We also denote by $\mathcal{L}_d$ the Lebesgue measure in $\mathbb{R}^d$ and by $\mathcal{N}_d$ the set of $\mathcal{L}_d$-null subsets of $\mathbb{R}^d$, that is,

$$\mathcal{N}_d = \{N \subset \mathbb{R}^d : \mathcal{L}_d(N) = 0\}.$$

We shall also omit the index d and simply write $\mathcal{L}$ for the Lebesgue measure and $\mathcal{N}$ for the family of null sets, whenever there is no ambiguity about the dimension.

For a set-valued mapping Φ from $\mathbb{R}^d$ to the subsets of $\mathbb{R}^\ell$, we will use the notation $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$, while a (single-valued) function will be denoted by $f : \mathbb{R}^d \longrightarrow \mathbb{R}^\ell$. The following definition provides a convenient abbreviation for several statements in the sequel.

Definition 2.1 (Cusco map)**.** An upper semi-continuous set-valued map $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ with nonempty compact convex values will be called cusco.

Under the above terminology, the Krasovskii regularization K_f is the smallest cusco map Φ satisfying $f(x) \in \Phi(x)$ for all $x \in \mathbb{R}^d$ and the Filippov regularization F_f is the smallest cusco map Ψ satisfying $f(x) \in \Psi(x)$ for *almost all* $x \in \mathbb{R}^d$. We refer the reader to [16], [17] and [7] for more information on Filippov's regularization and its applications. We also refer to [4], [5] for properties of cusco maps.

We shall also need the following classical notion of a point of approximate continuity of a measurable function.

Definition 2.2 (Points of approximate continuity)**.** Let $f : \mathbb{R}^d \to \mathbb{R}^\ell$ be a measurable function. A point $x \in \mathbb{R}^d$ is called a point of approximate continuity for f if for every $\varepsilon > 0$ it holds:

$$\lim_{\delta \to 0^+} \frac{\mathcal{L}\{x' \in B_\delta(x),\ |f(x') - f(x)| \geq \varepsilon\}}{\mathcal{L}(B_\delta(x))} = 0. \tag{4}$$

It is well-known that the complement $\mathbf{N}_f$ of the set of points of approximate continuity of a locally bounded measurable $f : \mathbb{R}^d \to \mathbb{R}^\ell$ is $\mathcal{L}_d$-null (*c.f.* [15] e.g.). Based on this result we can establish the following useful lemma.

Lemma 2.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^\ell$ be a (locally) bounded measurable function and $\mathbb{R}^d \diagdown \mathbf{N}_f$ be the set of points of approximate continuity. Then for every $\bar{x} \in \mathbb{R}^d$, $\delta > 0$ and $N \in \mathcal{N}$ we have:*

$$f(B_\delta(\bar{x}) \setminus \mathbf{N}_f) \subset \overline{f(B_\delta(\bar{x}) \setminus (\mathbf{N}_f \cup N))} \quad \text{and} \quad \overline{\text{co}}\, f(B_\delta(\bar{x}) \diagdown \mathbf{N}_f) = \overline{\text{co}}\, \left(f(B_\delta(\bar{x}) \setminus (\mathbf{N}_f \cup N))\right). \tag{5}$$

Consequently, for every $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$ it holds:

$$\overline{\text{co}}\ f(B_\delta(\bar{x}) \diagdown \mathbf{N}_f) = \bigcap_{N \in \mathcal{N}} \overline{\text{co}}\ f(B_\delta(\bar{x}) \diagdown N). \tag{6}$$

Proof. Let us prove (5). Fix $\varepsilon > 0$, $N \in \mathcal{N}$ and $x \in B_\delta(\bar{x}) \diagdown N_f$. Take $\delta_1 < \delta$ such that $B_{\delta_1}(x) \subset B_\delta(\bar{x})$. By (4), there exists $\delta_2 \in (0, \delta_1)$ such that

$$\frac{\mathcal{L}\{x' \in B_{\delta_2}(x),\ |f(x') - f(x)| \geq \varepsilon\}}{\mathcal{L}(B_{\delta_2}(x))} < 1,$$

which yields

$$\mathcal{L}\{x' \in B_{\delta_2}(x),\ |f(x') - f(x)| < \varepsilon\} > 0.$$

Thus

$$\mathcal{L}(\{x' \in B_{\delta_2}(x),\ |f(x') - f(x)| < \varepsilon\} \diagdown (N_f \cup N)) > 0.$$

Hence there exists $x' \in B_{\delta_2}(x) \diagdown (N_f \cup N)) \subset B_\delta(\bar{x}) \diagdown (N_f \cup N))$ such that $|f(x') - f(x)| < \varepsilon$. Since ε is arbitrary we deduce

$$f(x) \in \overline{f(B_\delta(\bar{x}) \diagdown (N_f \cup N))}.$$

The right-hand side of (5) follows from the fact that for every subset A of $\mathbb{R}^\ell$ we have

$$A \subset \mathrm{co}(A) \implies \overline{A} \subset \overline{\mathrm{co}}(A) \implies \overline{\mathrm{co}}(\overline{A}) = \overline{\mathrm{co}}(A).$$

Assertion (6) follows directly from (5). $\square$

We recall the following result due to Castaing (see [2, Theorem 8.1.4] e.g.)

Proposition 2.4. *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a measurable set-valued map. Then there exists a sequence of measurable selections $\{f_n\}_{n=1}^\infty$ of Φ such that*

$$\Phi(x) = \overline{\{f_n(x) \mid n \in \mathbb{N}\}}, \quad \textit{for all } x \in \mathbb{R}^d.$$

Combining above proposition with Lemma 2.3, we deduce the following useful result.

Corollary 2.5. *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be cusco. Then there exists $\mathbf{N}_\Phi \in \mathcal{N}_d$ (Lebesgue null set) such that for every $\bar{x} \in \mathbb{R}^d$, $\delta > 0$ and $N \in \mathcal{N}$ we have:*

$$\Phi(B_\delta(\bar{x})\backslash \mathbf{N_\Phi}) \subset \overline{\Phi(B_\delta(\bar{x}) \setminus (\mathbf{N}_\Phi \cup N))} \quad \textit{and} \quad \overline{\mathrm{co}}\,\Phi(B_\delta(\bar{x})\backslash N_\Phi) = \overline{\mathrm{co}}\,(\Phi(B_\delta(\bar{x})\backslash(\mathbf{N}_\Phi \cup N)). \quad (7)$$

Consequently, for every $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$ it holds:

$$\overline{co}\,\Phi(B_\delta(\bar{x})\diagdown \mathbf{N}_\Phi\,) = \bigcap_{N \in \mathcal{N}} \overline{co}\,\Phi(B_\delta(\bar{x})\diagdown N). \quad (8)$$

Proof. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable sets associated to Φ (*c.f.* Proposition 2.4). We set $\mathbf{N}_\Phi := \bigcup_{k\geq 1} N_k$, where $N_k = \mathbf{N}_{f_k}$ is the complement of the set of points of approximate continuity of f_k. We obviously have that $\mathbf{N}_\Phi$ is a null set. Let us show that (7) holds.
To this end, let $N \in \mathcal{N}$, $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$. Fix $x \in B_\delta(\bar{x})\diagdown N_\Phi$ and take $\delta_1 \in (0, 1)$ such that $B_{\delta_1}(x) \subset B_\delta(\bar{x})$. By Lemma 2.3 we have for any $k \geq 1$,

$$\begin{aligned} f_k(x) \in f_k(B_{\delta_1}(x)\diagdown N_k) &\subset \overline{f(B_{\delta_1}(\bar{x})\diagdown (N_k \cup N_\Phi \cup N))} = \overline{f(B_{\delta_1}(\bar{x})\diagdown (N_\Phi \cup N))} \\ &\subset \overline{\Phi(B_\delta(\bar{x})\diagdown (N_\Phi \cup N))}. \end{aligned}$$

So

$$\Phi(x) = \overline{\{f_k(x),\ k \geq 1\}} \subset \overline{\Phi(B_\delta(\bar{x})\diagdown (N_\Phi \cup N))}\,,$$

which established the left-hand side of (7). The remaining assertions are easily deduced in a similar manner as in Lemma 2.3. $\square$

Let us now recall (see [7, Proposition 2] *e.g.*) the following useful results. In [7], the results below have been stated and proved for the case $\ell = d$. The proofs for the general case (ℓ arbitrary) are identical. In what follows, $\mathcal{N}$ will always denote the class of Lebesgue null sets.

Proposition 2.6. *Let $f : \mathbb{R}^d \to \mathbb{R}^\ell$ be a measurable and (locally) bounded function. Then,*

(**i**). *there exists a set $\mathbf{N}_f \in \mathcal{N}$ such that*

$$F_f(x) := \bigcap_{\delta>0} \overline{co}\ f((B_\delta(x)) \setminus \mathbf{N}_f), \quad \textit{for all } x \in \mathbb{R}^d$$

and $f(x) \in F_f(x)$ for almost all $x \in \mathbb{R}^d$.

(ii). *F_f is the smallest cusco map Φ such that $f(x) \in \Phi(x)$, for almost all $x \in \mathbb{R}^d$.*

(iii). *F_f is single-valued if and only if there exists a continuous function g which coincides almost everywhere with f. In this case, $F_f(x) = \{g(x)\}$ for almost all $x \in \mathbb{R}^d$.*

(iv). *there exists a (necessarily measurable) function $\bar{f}$ which is equal almost everywhere to f and such that*

$$F_f(x) := \bigcap_{\delta>0} \overline{co}\ \bar{f}(B_\delta(x)), \quad \text{for all } x \in \mathbb{R}^d.$$

(v). *if a function $\widetilde{f}$ coincides with f for almost all $x \in \mathbb{R}^d$, then*

$$F_f(x) = F_{\widetilde{f}}(x), \quad \text{for all } x \in \mathbb{R}^d.$$

(vi). for all $x \in \mathbb{R}^d$

$$F(x) = \bigcap_{\widetilde{f}=f\, a.e.} \bigcap_{\delta>0} \overline{\mathrm{co}}\, \widetilde{f}(B_\delta(x))\,,$$

where the first intersection is taken over all functions $\widetilde{f}$ equal to f almost everywhere.

3 Cusco maps and Filippov representability

Before we proceed, we shall need the following classical result, whose proof is provided for completeness. According to the terminology of Kirk [20], the result asserts the existence, for every Euclidean space, of a countable partition that splits the family of open sets. For alternative proofs, or proofs of similar statements see [25], [12], [11].

Lemma 3.1 (Splitting partition)**.** *There exists a partition $\{A_n\}_{n=1}^{\infty}$ of $\mathbb{R}^d$, such that for every $n \in \mathbb{N}$ the set A_n has a positive measure in every open subset of $\mathbb{R}^d$.*

Proof. Consider the countable family $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open balls with rational centers and rational radii in $\mathbb{R}^d$. Let

$$b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

be a bijection such that $b(1,1) = 1$.

Using that each nonempty open set contains a closed nowhere dense set with positive measure (*e.g.* a Smith–Volterra–Cantor set, also called "fat" Cantor set), we can choose $T_1 \subset \mathcal{U}_1$ to be a nowhere dense closed set with positive measure. Then, we construct a sequence $\{T_m\}_{m=2}^{\infty}$ of disjoint closed nowhere dense sets with positive measure such that

$$\text{if } m = b(k,j), \text{ then } T_m \subset \mathcal{U}_k \setminus \cup_{l<m} T_l\,. \tag{9}$$

This can be done since the set $\mathcal{U}_k \setminus \bigcup_{l<m} T_l$ is open.

We now set

$$A_n := \bigcup_{k=1}^{\infty} T_{b(k,n)},\ n \geq 2$$

and

$$A_1 := \mathbb{R}^d \setminus \bigcup_{n=2}^{\infty} A_n\,.$$

It is clear that $\{A_n\}_{n=1}^{\infty}$ are measurable and disjoint. Moreover, if O be a nonempty open set, then there exists k such that $\mathcal{U}_k \subset O$. Using (9), we obtain that

$$A_n \cap O \supset A_n \cap \mathcal{U}_k \supset T_{b(k,n)}\,,\ n \geq 2$$

and

$$A_1 \cap O \supset (\mathbb{R}^d \setminus \bigcup_{n=2}^{\infty} A_n) \bigcap \mathcal{U}_k \supset T_{b(k,1)}\,.$$

Hence, $\mathcal{L}(A_n \bigcap O) \geq \mathcal{L}(T_{b(k,n)}) > 0$ and $\mathcal{L}(A_1 \cap O) \geq \mathcal{L}(T_{b(k,1)}) > 0$. This completes the proof of the lemma.

□

We are now ready to prove the following result.

Theorem 3.2. *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then there exists a measurable function $f : \mathbb{R}^d \to \mathbb{R}^\ell$ such that Φ is almost everywhere equal to F_f (the Filippov regularization of f), that is:*

$$\Phi(x) = F_f(x), \quad \textit{for almost every } x \in \mathbb{R}^d.$$

Proof. In view of Proposition 2.4, there exists a sequence of measurable selections $\{f_n\}_{n=1}^{\infty}$ of Φ such that

$$\Phi(x) = \overline{\{f_n(x) \mid n \in \mathbb{N}\}}, \quad \text{for every } x \in \mathbb{R}^d.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a splitting partition of $\mathbb{R}^d$ given in Lemma 3.1. We define the measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$f(x) := \sum_{n=1}^{\infty} f_n(x) \mathbf{1}_{A_n}(x),$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A (equal to 1 if $x \in A$ and to 0 if $x \notin A$). Let

$$F_f(x) := \bigcap_{N, \mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, f(B_\delta(x) \setminus N)$$

be the Filippov regularization of f. Since $\mathcal{L}(B_\delta(x) \cap A_n) > 0$ for all $n \in \mathbb{N}$ and for all $\delta > 0$, we obtain that

$$\begin{aligned} F_f(x) &\supset \bigcap_{N,\, \mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, f((B_\delta(x) \cap A_n) \setminus N) \\ &= \bigcap_{N,\, \mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, f_n((B_\delta(x) \cap A_n) \setminus N) \end{aligned} \tag{10}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$.

The next step in the proof consists in showing that the last expression in (10) contains $f_n(x)$ for almost all $x \in \mathbb{R}^d$. In order to do it, we will need the following assertion.

Claim. There exists a sequence of measurable sets $\{K_m\}_{m=1}^{\infty}$ such that:

1. $K_1 \subset K_2 \subset \ldots K_m \subset \ldots$
2. $\mathbb{R}^d = \bigcup_{m=1}^{\infty} K_m \cup N_0$, where $\mathcal{L}(N_0) = 0$

3. the restrictions $f_n|_{K_m}$ are continuous for all $m, n \in \mathbb{N}$.

We postpone the proof of the claim at the end of this proof. Assuming the above claim, we deduce from Lemma 3.1 that for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $\delta > 0$ it holds:

$$\begin{aligned} 0 <& \mathcal{L}(B_\delta(x) \cap A_n) = \mathcal{L}(B_\delta(x) \cap A_n \cap (\mathbb{R}^d \setminus N_0)) \\ =& \mathcal{L}\left(B_\delta(x) \cap A_n \cap \bigcup_{m=1}^{\infty} K_m\right) = \mathcal{L}\left(\bigcup_{m=1}^{\infty} (B_\delta(x) \cap A_n \cap K_m)\right) \\ =& \lim_{m\to\infty} \mathcal{L}(B_\delta(x) \cap A_n \cap K_m)\,, \end{aligned}$$

since $K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$. Therefore, for some $m_0 \in \mathbb{N}$ sufficiently large we have

$$\mathcal{L}(B_\delta(x) \cap A_n \cap K_m) > 0,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\delta > 0$ and $m \geq m_0$.

Let us fix an arbitrary $x \notin N_0$. Then, $x \in K_{m_1}$ for some $m_1 \in \mathbb{N}$. Let $\bar{m} := \max(m_0, m_1)$. Since $x \in K_m$ for all $m \geq m_1$, we can continue (10) in the following way

$$F_f(x) \supset \bigcap_{N,\, \mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, f_n(B_\delta(x) \cap A_n \cap K_{\bar{m}} \setminus N, t) \ni f_n(x)\,,$$

where the last inclusion is due to continuity of $f_n|_{K_{\bar{m}}}$.

We have obtained that for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^d \setminus N_0$

$$F_f(x) \ni f_n(x)\,.$$

Since the Filippov regularization F_f is closed-valued, we obtain

$$F_f(x) \supset \Phi(x) \ni f(x), \qquad \text{for all } x \in \mathbb{R}^d \setminus N_0.$$

We deduce from Proposition 2.6 (ii) that $F_f(x) = \Phi(x)$ for almost every $x \in \mathbb{R}^d$.

It remains to prove the claim about the existence of the sequence of sets $\{K_m\}_{m=1}^{\infty}$. Since the functions f_n are measurable, due to Lusin's theorem, for every $m, n \in \mathbb{N}$ we can find a set $K_{n,m} \subset \mathbb{R}^d$ such that $f_n|_{K_{n,m}}$ is continuous and

$$\mathcal{L}(\mathbb{R}^d \setminus K_{n,m}) < \frac{1}{2^{n+m}}\,.$$

Let us set $K'_m := \bigcap_{n=1}^{\infty} K_{n,m}$. We have that the restrictions $f_n|_{K'_m}$ are continuous for all $m, n \in \mathbb{N}$ and

$$\mathcal{L}(\mathbb{R}^d \setminus K'_m) = \mathcal{L}\left(\bigcup_{n=1}^{\infty} (\mathbb{R}^d \setminus K_{n,m})\right) \leq \sum_{n=1}^{\infty} \mathcal{L}(\mathbb{R}^d \setminus K_{n,m}) < \sum_{n=1}^{\infty} \frac{1}{2^{n+m}} = \frac{1}{2^m}\,.$$

The inclusions $K_1 \subset K_2 \subset \ldots K_m \subset \ldots$ are obtained by taking

$$K_m := \bigcap_{l \geq m} K'_l, \ m = 1, 2, \ldots\,.$$

We have that

$$\mathcal{L}(\mathbb{R}^d \setminus K_m) = \mathcal{L}\left(\bigcup_{l=m}^{\infty} (\mathbb{R}^d \setminus K_l')\right) \le \sum_{l=m}^{\infty} \mathcal{L}(\mathbb{R}^d \setminus K_l') < \sum_{l=m}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m-1}}\,.$$

Let us set $N_0 := \mathbb{R}^d \setminus \cup_{m=1}^{\infty} K_m$. Since $\mathbb{R}^d \setminus K_{m+1} \subset \mathbb{R}^d \setminus K_m$, we obtain that

$$\mathcal{L}(N_0) = \mathcal{L}\left(\bigcap_{m=1}^{\infty} (\mathbb{R}^d \setminus K_m)\right) = \lim_{m\to\infty} \frac{1}{2^{m-1}} = 0\,.$$

The proof is complete. $\square$

We also obtain the following

Proposition 3.3. *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then, there exists a measurable selection $f : \mathbb{R}^d \to \mathbb{R}^\ell$ of Φ (that is, $f(x) \in \Phi(x)$ for all $x \in \mathbb{R}^d$), such that*

(**i**). *Φ is equal almost everywhere to the Filippov regularization of f, that is,*

$$\Phi(x) = F_f(x), \quad \textit{for almost all } x \in \mathbb{R}^d.$$

(**ii**). *there exists some $\hat{f} : \mathbb{R}^d \to \mathbb{R}^\ell$ such that Φ is equal almost everywhere to the Krasovskii regularization of $\hat{f}$, that is,*

$$\Phi(x) = K_{\hat{f}}(x), \quad \textit{for almost all } x \in \mathbb{R}^d.$$

(**iii**). *Φ is equal almost everywhere to the intersection of all Filippov regularizations defined by functions $\widetilde{f}$ which are equal to f almost everywhere, that is,*

$$\Phi(x) = \bigcap_{\widetilde{f}=f\, a.e.} F_{\widetilde{f}}(x), \quad \textit{for almost all } x \in \mathbb{R}^d.$$

Proof. Using Theorem 3.2, we obtain a measurable function $\bar{f} : \mathbb{R}^d \to \mathbb{R}^\ell$ such that Φ is equal almost everywhere to the Filippov regularization F_f of $\bar{f}$, that is,

$$\Phi(x) = \bigcap_{N,\mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\text{co}}\, \bar{f}(B_\delta(x) \setminus N), \quad \text{for almost every } x \in \mathbb{R}^d.$$

Due to Proposition 2.6 (iv) there exists a function $\hat{f} : \mathbb{R}^d \to \mathbb{R}^\ell$ such that for all $x \in \mathbb{R}^d$

$$\Phi(x) := \bigcap_{\delta>0} \overline{\text{co}}\, \hat{f}\,(B_\delta(x))\,.$$

Clearly at every point $x \in \mathbb{R}^d \diagdown \hat{\mathbf{N}}_f$ of approximate continuity of $\hat{f}$ we have that $\hat{f}(x) \in \Phi(x)$. So setting $f(x) = \hat{f}(x)$, whenever $x \in \mathbb{R}^d \diagdown \mathbf{N}_{\hat{f}}$ and taking $f(x)$ to be any element of $\Phi(x)$ if $x \in \mathbf{N}_{\hat{f}}$, we obtain both claims (i) and (ii).

In order to establish (iii), we use (i) to obtain that for all $x \in \mathbb{R}^d \setminus \mathbf{N}_{\hat{f}}$

$$\Phi(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\, f\left(B_\delta(x)\right) \supset \bigcap_{\widetilde{f}=f\text{a.e.}} \bigcap_{\delta>0} \overline{\mathrm{co}}\, \widetilde{f}\left(B_\delta(x)\right) .$$

At the same time we also have:

$$\bigcap_{\widetilde{f}=f\text{a.e.}} \bigcap_{\delta>0} \overline{\mathrm{co}}\, f\left(B_\delta(x)\right) \supset \bigcap_{\widetilde{f}=f\text{a.e.}} \bigcap_{N,\mathcal{L}(N)=0} \bigcap_{\delta>0} \overline{\mathrm{co}}\, \widetilde{f}(B_\delta(x) \setminus N) .$$

The right-hand side is $\bigcap_{\widetilde{f}=f\text{a.e.}} F_{\widetilde{f}}(x)$, which by Proposition 2.6 (vi) is equal to $F_f(x)$, for all $x \in \mathbb{R}^d$. The proof is complete. $\square$

Remark 3.4. Notice that (completely) different functions may give rise to the same Filippov regularization: Indeed, let $A \subset \mathbb{R}$ be a splitting set, that is, A and $\mathbb{R}\diagdown A$ have positive measure on every nontrivial interval. Then both $f(x) := \mathbf{1}_A(x)$ and $\widetilde{f}(x) := \mathbf{1}_{\mathbb{R}\diagdown A}(x)$ satisfy $F_f(x) = F_{\widetilde{f}}(x) = [0,1]$ and at the same time $f(x) \neq \widetilde{f}(x)$ for all $x \in \mathbb{R}$.

Definition 3.5 (The map $m(\Phi)$)**.** Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. We define the following "minimal" map:

$$m(\Phi)(x) := \bigcap_{N\in\mathcal{N}} \bigcap_{\delta>0} \overline{\mathrm{co}}\ \Phi(B_\delta(x)\diagdown N), \quad \text{for all } x \in \mathbb{R}^d. \tag{11}$$

Thanks to Corollary 2.5, we have also

$$m(\Phi)(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\ \Phi(B_\delta(x)\diagdown \mathbf{N}_\Phi\ ). \tag{12}$$

Proposition 3.6. *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then the map $m(\Phi)$ is cusco and satisfies*

$$m(\Phi)(\bar{x}) \subset \bigcap_{\delta>0} \overline{co}\ \Phi(B_\delta(x)) \subset \Phi(\bar{x}), \quad \textit{for all } \bar{x} \in \mathbb{R}^d \tag{13}$$

$$m(\Phi)(\bar{x}) = \Phi(\bar{x}), \quad \textit{for almost all } \bar{x} \in \mathbb{R}^d. \tag{14}$$

Proof. Fix $N \in \mathcal{N}$, $x \in \mathbb{R}^d$ and set

$$G_N(x) := \bigcap_{\delta>0} \overline{\mathrm{co}}\, \Phi(B_\delta(x)\backslash N).$$

Being a decreasing intersection of nonempty compact convex sets, $G_N(x)$ is itself a nonempty compact convex set. Notice that the family $G_N(x\}_{N\in\mathcal{N}}$ has the finite intersection property. It follows from (11) that the map $m(\Phi)$ has nonempty convex compact values, while from its definition it follows easily that it is also upper semicontinuous, that is, $m(\Phi)$ is cusco.

We now fix $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$. Since Φ is upper semicontinuous there exists $\delta > 0$ such that

$$\forall x \in B_\delta(\bar{x}),\ \Phi(x) \in \Phi(\bar{x}) + \varepsilon B.$$

So $\Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + \varepsilon B$ and $\overline{\mathrm{co}}\, \Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + 2\varepsilon B$ because $\Phi(\bar{x})$ is convex closed. Therefore

$$\bigcap_{\delta>0} \overline{\mathrm{co}}\, \Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + 2\varepsilon B.$$

Taking the intersection over all $\varepsilon > 0$ we get

$$\bigcap_{\delta>0} \overline{\text{co}}\,\Phi(B_\delta(\bar{x})) \subset \bigcap_{\varepsilon>0} (\Phi(\bar{x}) + 2\varepsilon B) = \Phi(\bar{x}).$$

This proves (13). Let us prove (14). In view of Corollary 2.5 we get from (13)

$$\forall \bar{x} \in \mathbb{R}^d,\ m(\Phi)(\bar{x}) = \bigcap_{\delta>0} \overline{\text{co}}\,\Phi(B_\delta(x) \diagdown N_\Phi) \subset \Phi(\bar{x}). \tag{15}$$

If $\bar{x} \notin N_\Phi$ then

$$\Phi(\bar{x}) \subset \bigcap_{\delta>0} \Phi(B_\delta(x) \diagdown N_\Phi) \subset m(\Phi)(\bar{x}).$$

Consequently in view of (15) we obtain (14) for any $\bar{x} \notin N_\Phi$. □

4 Characterization of Filippov representable maps

Let $\hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$ be the set of all cusco maps $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$. We now define on $\hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$ the equivalence relation

$$\Phi_1 \sim \Phi_2 \quad \Longleftrightarrow \quad \Phi_1(x) = \Phi_2(x) \text{ for almost all } x \in \mathbb{R}^d$$

and the associated quotient set

$$\hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)/_\sim \ := \{\, [\Phi],\ \Phi \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)\, \}$$

where

$$[\Phi] := \{\Psi \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell),\ \Phi \sim \Psi\, \}.$$

We also define an order on $\hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$ by

$$\Phi_1 \preceq \Phi_2 \quad \Longleftrightarrow \quad \Phi_1(x) \subseteq \Phi_2(x), \text{ for all } x \in \mathbb{R}^d. \tag{16}$$

Lemma 4.1 (Equivalent elements in $\hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$)**.** *For all $\Phi_1, \Phi_2 \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$ we have:*

$$\Phi_1 \sim \Phi_2 \quad \Longleftrightarrow \quad m(\Phi_1) = m(\Phi_2).$$

Proof. Let $N \in \mathcal{N}$ be such that $\Phi_1(x) = \Phi_2(x)$ for all $x \in \mathbb{R}^d \setminus N$. Fix $\bar{x} \in \mathbb{R}^d$. In view of Corollary 2.5, we deduce that for every $\delta > 0$

$$\begin{aligned}\overline{\text{co}}\,\Phi_1(B_\delta(\bar{x}) \setminus N_{\Phi_1}) &= \overline{\text{co}}\,\Phi_1(B_\delta(\bar{x}) \setminus (\mathbf{N}_{\Phi_1} \cup \mathbf{N}_{\Phi_2} \cup N) \\ &= \overline{\text{co}}\,\Phi_2\,(B_\delta(\bar{x}) \setminus (\mathbf{N}_{\Phi_1} \cup \mathbf{N}_{\Phi_2} \cup N)) = \overline{\text{co}}\,\Phi_2(B_\delta(\bar{x}) \setminus \mathbf{N}_{\Phi_2})\end{aligned}$$

because $\Phi_1 = \Phi_2$ on the complement of N. By taking intersection over all $\delta > 0$ we obtain

$$m(\Phi_1)(\bar{x}) = \bigcap_{\delta>0} \overline{\text{co}}\,\Phi_1(B_\delta(\bar{x}) \setminus N_{\Phi_1}) = \bigcap_{\delta>0} \overline{\text{co}}\,\Phi_2(B_\delta(\bar{x}) \setminus N_{\Phi_2}) = m(\Phi_2)(\bar{x}).$$

The proof is complete. □

Corollary 4.2 (minimality of $m(\Phi)$)**.** *Let $\Phi \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$. Then $m(\Phi) \in [\Phi]$ and $m(\Phi)$ is the minimum element in $[\Phi]$ for the order $\preceq$ defined in* (16)*.*

The fact that every cusco map Φ is equivalent to $m(\Phi)$ and that the latter is the minimum element of $[\Phi]$ under set-inclusion, has an interesting consequence, see (17) in the following remark.

Remark 4.3. For every cusco map $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ we have:

$$m(\Phi)(x) = \bigcap_{\Phi' \sim \Phi} \Phi'(x), \quad \text{for all } x \in \mathbb{R}^d .$$

This yields the following relation (which is not completely obvious at a first glance):

$$\Phi(x) = \bigcap_{\Phi' \sim \Phi} \Phi'(x), \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{17}$$

We are now ready to establish our main result

Theorem 4.4 (Characterization of Filippov representable maps)**.** *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then Φ is Filippov representable if and only if $\Phi = m(\Phi)$ (that is, Φ is the $\preceq$-minimal element in its equivalent class).*

Proof. Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a Filippov representable cusco map. Then

$$\Phi(x) = F_f(x) = \bigcap_{\delta>0} \overline{\text{co}}\ f(B_\delta(x)) \setminus \mathbf{N}_f), \quad \text{for all } x \in \mathbb{R}^d,$$

where $f : \mathbb{R}^d \longrightarrow \mathbb{R}^\ell$ is some (bounded) measurable function. By Lemma 2.3 we deduce that

$$f(x) \in \Phi(x),\ \forall x \in \mathbb{R}^d \backslash N_f.$$

This together with (12) and Lemma 2.3 yields that for any $x \in \mathbb{R}^d$

$$\Phi(x) = \bigcap_{\delta>0} \overline{\text{co}}\ f(B_\delta(x) \setminus (\mathbf{N}_f \cup \mathbf{N}_\Phi)) \subset \bigcap_{\delta>0} \overline{\text{co}}\ \Phi(B_\delta(x) \setminus (\mathbf{N}_f \cup \mathbf{N}_\Phi)).$$

In view of Corollary 2.5, we get

$$\bigcap_{\delta>0} \overline{\text{co}}\ \Phi(B_\delta(x) \setminus (\mathbf{N}_f \cup \mathbf{N}_\Phi)) = \bigcap_{\delta>0} \overline{\text{co}}\ \Phi(B_\delta(x) \setminus (\mathbf{N}_\Phi))$$

which is equal to $m(\Phi)(x)$ by (12). This yields $\Phi = m(\Phi)$.

To prove the opposite direction, note that by Theorem 3.2 every cusco map Φ is equivalent to a Filippov regularization F_f, and consequently, $F_f = m(F_f) = m(\Phi)$. $\square$

The following corollary follows directly.

Corollary 4.5. *The following assertions are equivalent for every cusco map $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$:*

(**i**). *Φ is a Filippov representable map ;*

(**ii**). *$\Phi = m(\Phi)$;*

(**iii**). *for every $\bar{x} \in \mathbb{R}^d$ and $N \in \mathcal{N}$ we have:*

$$\overline{\text{co}} \left(\limsup_{x \notin N\, x \longrightarrow \bar{x}} \Phi(x) \right) = \Phi(\bar{x})\,.$$

Whenever Φ is cusco, the left-hand side of (iii) above is always contained in $\Phi(\bar{x})$. According to (ii) above, it is very easy to obtain explicit examples of cusco maps that are not Filippov representable. Indeed, take any measurable function f, consider its Filippov regularization F_f and modify it at some point $\bar{x}$ (or at all points of a discrete set) to get an equivalent cusco map Φ different from F_f. Indeed, it is sufficient to replace $F_f(\bar{x})$ by any convex compact strict superset $\Phi(\bar{x}) \supset F_f(\bar{x})$. Then Φ is not Filippov representable, since $\Phi \neq F_f = m(F_f) = m(\Phi)$, see forthcoming examples.

Example 4.6. (i). We deduce easily that the following cusco maps, based on a one-point modification of the minimal map $F_f(x) = \{0\}$, for all $x \in \mathbb{R}$ (trivial regularization of the constant function $f \equiv 0$), cannot be obtained as Filippov regularizations:

$$\Phi_1(x) = \begin{cases} [0,1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0 \end{cases} \qquad \text{and} \qquad \Phi_2(x) = \begin{cases} [-1,1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0. \end{cases}$$

It is worth noting that Φ_2 cannot even be a Krasovskii regularization of a function, while $\Phi_1 = K_g$, where $g(x) = 0$, for $x \neq 0$ and $g(0) = 1$.

(ii). A slightly more elaborated example of a function that can neither be obtained as Filippov nor as Krasovskii regularization is the following:

$$\Phi_3(x) = \begin{cases} [-\frac{1}{m}, \frac{1}{m}], & \text{if } x = p/m \in \mathbb{Q}\diagdown\{0\} \\ \{0\}, & \text{if } x \notin \mathbb{Q}\diagdown\{0\}. \end{cases}$$

where every nonzero rational number is given its irreducible form p/m, where p, m are relatively prime integers.

(iii). Let us define the following measurable function:

$$f(x) = \begin{cases} \frac{1}{m}, & \text{if } x = p/m \in \mathbb{Q}\diagdown\{0\} \\ \{0\}, & \text{if } x \notin \mathbb{Q}\diagdown\{0\}. \end{cases}$$

Then for every $x \in \mathbb{R}$ we have: $F_f(x) = \{0\}$ and $K_f(x) = [0, f(x)]$. In particular $F_f \sim K_f$ and consequently, the cusco map $\Phi = K_f$ cannot be represented as a Filippov regularization.

5 Characterization of Clarke subdifferentials

In this section we deal with the problem of determining whether a cusco map $\Phi \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^d)$ is the Clarke subdifferential of some locally Lipschitz function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$. A full characterization of such maps has been given in [4] and relevant results had been previously established in [5]. We shall complement the results of [4] by establishing, via our approach, another elegant characterization of Clarke subdifferentials. Our method is based on the characterization of Filippov representability (for the case $\ell = d$) together with a *nonsmooth Poincaré condition*. This latter has been recently stated and used independently in [18] and [10] for a different purpose (namely, to identify the free space of a finite-dimensional Euclidean space). Before we proceed, let us recall the relevant statement.

Theorem 5.1 (nonsmooth Poincaré condition (Proposition 3.2(ii) in [10])). *Let $\mathcal{U} \neq \emptyset$ be an open convex subset of $\mathbb{R}^d$. An essentially (locally) bounded measurable function $f : \mathcal{U} \longrightarrow \mathbb{R}^d$ is equal almost everywhere to the derivative of a (locally) Lipschitz function $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$ if and only if*

$$\partial_i f_j = \partial_j f_i \text{ for all } i, j \in \{1, \dots, d\}, \tag{18}$$

where $\partial_i f_j$ denotes the partial derivative (in the sense of distributions) of the j-th component of f with respect to x_i. That is, if $\mathcal{C}_0^\infty(\mathcal{U})$ is the space of compactly supported $\mathcal{C}^\infty$-functions on $\mathcal{U}$ (test functions), then (18) *becomes:*

$$\int_{\mathcal{U}} f_j(x) \frac{\partial \psi}{\partial x_i}(x) dx = \int_{\mathcal{U}} f_i(x) \frac{\partial \psi}{\partial x_j}(x) dx, \quad \text{for every } \psi \in \mathcal{C}_0^\infty(\mathcal{U}).$$

We now give an elegant characterization of Clarke subdifferentials in the spirit of this work.

Theorem 5.2 (Characterization of Clarke subdifferentials). *Let $\Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a cusco map. The following are equivalent:*

(**i**). $\Phi = \partial \varphi$ *for some locally Lipschitz function* $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *;*

(**ii**). $\Phi = F_f$ *for some measurable selection f of Φ that satisfies* (18).

Proof. (i)$\Longrightarrow$(ii). Assume that $\Phi = \partial \varphi$ for a locally Lipschitz function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$. Then by Rademacher's theorem, there exists $N_\varphi \in \mathcal{N}$ such that the derivative $\nabla \varphi(x)$ exists for all $x \in \mathbb{R}^d \diagdown N_\varphi$. For $x \in N_\varphi$, pick $s(x) \in \partial \varphi(x)$ and set

$$f(x) = \begin{cases} \nabla \varphi(x), & \text{if } x \in \mathbb{R}^d \diagdown N_\varphi \\ s(x), & \text{if } x \in N_\varphi. \end{cases}$$

Then $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a measurable selection of $\partial \varphi$ and (being a.e. equal to a gradient) it satisfies (18), see [10, Proposition 3.1 (ii)]. Moreover,

$$F_f(x) := \bigcap_{\delta > 0} \overline{\text{co}}\ f(B_\delta(x) \setminus N_\varphi) = \bigcap_{\delta > 0} \overline{\text{co}}\ \left\{ \nabla \varphi(x') : x' \in B_\delta(x) \diagdown N_\varphi \right\}. \tag{19}$$

Since φ is locally Lipschitz, we deduce ([8, Chapter 2.6])

$$\bigcap_{\delta > 0} \overline{\text{co}}\ \left\{ \nabla \varphi(x') : x' \in B_\delta(x) \diagdown N_\varphi \right\} = \overline{\text{co}} \left\{ \lim_{x_n \longrightarrow x} \nabla \varphi(x_n) :\ \{x_n\} \subset \mathbb{R}^d \diagdown N_\varphi \right\} = \partial \varphi(x), \tag{20}$$

which shows that (ii) holds for f being equal to $\nabla \varphi$ a.e.

(ii)$\Longrightarrow$(i). Assume that $\Phi = F_f$, where $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a measurable selection of Φ that satisfies (18). Then by Theorem 5.1, there exists a locally Lipschitz function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $f(x) = \nabla \varphi(x)$, for a.e. $x \in \mathbb{R}^d$. Then it follows from Proposition 2.6(v) and (19), (20) above that

$$\partial \varphi(x) = F_{\nabla \varphi}(x) = F_f(x) = \Phi(x) \text{ for all } x \in \mathbb{R}^d.$$

$\square$

Remark 5.3. (i) It is possible to have $\Phi = F_f$, without Φ being a subdifferential; consider for instance the function $f(x_1, x_2) = (x_2, -x_1)$, for all $(x_1, x_2) \in \mathbb{R}^2$ (which obviously fails (18)). Then $\Phi = f$ cannot be a subdifferential.

(ii) It is possible to have infinite many measurable selections $f(x) \in \Phi(x)$, for all $x \in \mathbb{R}^d$, each of which satisfies the nonsmooth Poincaré condition (18). Indeed, if we take Φ to be identically equal to the closed ball $\bar{B}$ for all $x \in \mathcal{U}$, then the set of all measurable selections that satisfy (18) contains isometrically the unit ball of the nonseparable Banach space $\ell^\infty(\mathbb{N})$, see [12].

(iii) If $\Phi = F_f$ and f is unique a.e. and satisfies (18), then by Theorem 5.2, $\Phi = \partial\varphi$ and $f = \nabla\varphi$ a.e. It follows that the locally Lipschitz function φ is unique up to a constant.

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