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Series A

Determination of convex functions via subgradients of minimal norm

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Abstract

We show, in Hilbert space setting, that any two convex proper lower semicontinuous functions bounded from below, for which the norm of their minimal subgradients coincide, they coincide up to a constant. Moreover, under classic boundary conditions, we provide the same results when the functions are continuous and defined over an open convex domain. These results show that for convex functions bounded from below, the slopes provide sufficient first-order information to determine the function up to a constant, giving a positive answer to the conjecture posed in Boulmezaoud et al. (SIAM J Optim 28(3):2049–2066, 2018) .

Keywords Subdifferential determination · Subgradient flows · Moreau–Yosida approximation · Dirichlet boundary condition · Slope

Mathematics Subject Classification Primary: 37C10 · 49J52; Secondary: 26B25 · 34A60 · 34C12 · 34G20 · 49J40

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1 Introduction

In 1966, Rockafellar [27] showed a fundamental result of determination of convex functions in terms of their subdifferentials: Every two convex proper lower semicontinuous functions φ_1 and φ_2 defined over a Banach space X with values in $\mathbb{R} \cup \{+\infty\}$ satisfying that

$$\partial\varphi_1(x) = \partial\varphi_2(x), \quad \forall x \in X, \tag{1}$$

are in fact equal up to a constant. This problem was first solved by Moreau [24] in the Hilbert space setting and finally extended to Banach spaces by Rockafellar, one year after. Since then, many authors have work in the problem of determination of convex and nonconvex functions, trying to weaken or adapt condition (1) (see, e.g., [5,8,13–16,22,26,29–31] and the references therein). Nowadays, in the convex setting, we know a more general result: Every two maximal monotone operators A and B over a Hilbert space $\mathcal{H}$ satisfying that

$$A(x)^\circ = B(x)^\circ, \quad \forall x \in \mathcal{H},$$

must coincide (see, e.g., [11]), where $A(x)^\circ$ is the element of minimal norm of the set $A(x)$.

Recently, Boulmezaoud et al. [10] obtained a very surprising improvement of the result above, in the case of convex twice continuously differentiable functions: They showed that if two convex functions $\varphi_1, \varphi_2 \in \mathcal{C}^2(\mathcal{H})$ satisfy that

(a) $\|\nabla\varphi_1(x)\| = \|\nabla\varphi_2(x)\|$ for all $x \in \mathcal{H}$, and
(b) $\inf_{x\in\mathcal{H}} \|\nabla\varphi_1(x)\| = 0$,

then they must coincide up to a constant. Their proof is based on the study of a second order dynamical system, for which the $\mathcal{C}^2$ condition is used. This is the very first result in the literature that reveals that slope information determines, up to a constant, a convex and bounded from below function.

Since hypothesis (b) holds whenever φ_1 is bounded from below, they asked in the same article if the result could be extended for $\varphi_1, \varphi_2 \in \mathcal{C}^1(\mathcal{H})$, assuming (a) and both functions to be bounded from below. Furthermore, they also conjectured the result for nonsmooth convex functions bounded from below [10, Conjecture 3.13] providing a proof for the one-dimensional case.

Up to our knowledge, the proof of the conjecture above for $\mathcal{C}^1$ convex functions was provided originally by J.-B. Baillon, but it was never published. His strategy (that we rediscover while working in this problem), was to study the solution of the steepest descent dynamical system for the function $\Phi = \varphi_1 + \varphi_2$, and then observe that the difference function $\varphi_1 - \varphi_2$ must be constant along the trajectories of that dynamical system.

In this work, we address Boulmezaoud–Cieutat–Daniilidis conjecture [10, Conjecture 3.13] under the same setting of Moreau–Rockafellar seminal result, that is, the case when φ_1 and φ_2 are convex proper lower semicontinuous and bounded from below, but not necessarily differentiable. Furthermore, we consider the case when the hypotheses over the functions are only verified in an open convex domain Ω. In this case, hypothesis (a) is replaced by

$$\|\partial\varphi_1(x)^\circ\| = \|\partial\varphi_2(x)^\circ\|, \quad \forall x \in \Omega,$$

since the element of minimal norm that determines the trajectories of steepest descent dynamics. As we will comment latter, in this setting, the technique used by Baillon cannot be directly applied.

The reader can note that the relation between the norm of the gradient $\|\nabla\varphi(x)\|$ for a continuously differentiable function and the size of the subgradient of minimal norm $\|\partial\varphi(x)^\circ\|$ for general convex functions is quite deep: In fact, both notions coincide with what is known as the *strong slope* of φ at x, usually denoted by $|\nabla\varphi|(x)$. The strong slope can be interpreted as the *fastest instantaneous rate of decrease* and it recently has been used to study steepest descent systems in metric spaces (see, e.g., [2,17–20,23,28] and the references therein). Moreover, it has strong links with the theory of error bounds and metric regularity (see, e.g., [4]). For a nice survey on recent developments on slope descent methods and tame optimization we refer the reader to [21].

Our main results (Corollary 3.1 and Theorem 4.1) can be interpreted as follows: *for two convex proper lower semicontinuous functions bounded from below, if they have the same slope, then they must coincide up to a constant*. In comparison with Moreau–Rockafellar condition (1), we may say that knowing the subdifferential of a convex function is more first-order information than what we really need in order to know the function. Thus, to completely determine a convex function (up to a constant) it is enough to know its strong slope. Our approach is based on dynamical systems, following the original ideas of [10] and J.-B. Baillon.

The article is organized as follows: In Sect. 2, we provide some preliminary notions of convex analysis and steepest descent dynamics with convex potentials. We also provide, for the sake of completeness, the simple proof of J.-B. Baillon in the continuously differentiable case. In Sect. 3 we show a comparison principle for convex functions and we prove Boulmezaoud–Cieutat–Daniilidis conjecture in the general case. In Sect. 4, we provide the same determination result for convex continuous functions over an open convex domain, under Dirichlet-type boundary conditions, which is not a direct consequence of the results presented above.

2 Preliminaries and problem formulation

In the rest of the article, we will assume that the reader has a basic knowledge of convex analysis and we will use the standard notation of this field, following [3,7,9,25]. In this section, we summarize the elements that we will use the most. For further details, we refer the reader to the aforementioned books.

From now on, $\mathcal{H}$ will denote a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Through Riesz representation theorem, we identify the dual of $\mathcal{H}$ with $\mathcal{H}$. For notational convenience, we denote $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For a (generalized) sequence (u_λ) in $\mathcal{H}$ and $u \in \mathcal{H}$, we write $u_\lambda \to u$ to denote strong convergence and $u_\lambda \rightharpoonup u$ to denote the weak convergence.

For a nonempty convex closed set C of $\mathcal{H}$ we denote by $d_C : \mathcal{H} \to \mathbb{R}_+$ the distance function to C, and by $\operatorname{proj}(\cdot\,; C) : \mathcal{H} \to \mathcal{H}$ its metric projection mapping. We write $\delta_C : \mathcal{H} \to \mathbb{R}_\infty$ to denote the indicator function of C in the sense of convex analysis, that is,

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

For a function $f : \mathcal{H} \to \mathbb{R}_\infty$ and a set-valued map $S : \mathcal{H} \rightrightarrows \mathcal{H}$, we denote

$$\operatorname{dom} f = \{x \in \mathcal{H} \;:\; f(x) \in \mathbb{R}\} \quad \text{and} \quad \operatorname{dom} S = \{x \in \mathcal{H} \;:\; S(x) \neq \emptyset\}$$

as the domains of f and S, respectively. We write $\operatorname{id}_{\mathcal{H}}$ to denote the identity operator on $\mathcal{H}$.

We denote by $\Gamma_0(\mathcal{H})$ the set of all convex lower semicontinuous proper functions from $\mathcal{H}$ with values in $\mathbb{R}_\infty$. For $f \in \Gamma_0(\mathcal{H})$, its Legendre–Fenchel conjugate function $f^* : \mathcal{H} \to \mathbb{R}_\infty$ is given by

$$f^*(x^*) = \sup_{v \in \mathcal{H}} \{\langle x^*, v\rangle - f(v)\}.$$

It is known that $f^* \in \Gamma_0(\mathcal{H})$ and that for every $(x, x^*) \in \mathcal{H} \times \mathcal{H}$, the Legendre–Fenchel inequality holds, that is

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*).$$

For $x \in \mathcal{H}$, the subdifferential of f at x is given by

$$\partial f(x) = \{x^* \in \mathcal{H} \;:\; f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in \mathcal{H}\}.$$

Recall that for $f \in \Gamma_0(\mathcal{H})$, the set-valued map $\partial f : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximal monotone operator (see [25] for the definition), $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$, and that ∂f can be characterized by the Legendre–Fenchel extremal condition, that is

$$x^* \in \partial f(x) \iff \langle x^*, x\rangle = f(x) + f^*(x^*).$$

For a convex, lower semicontinuous and proper function $\Phi : \mathcal{H} \to \mathbb{R}_\infty$ and any point $x \in \operatorname{dom} \partial\Phi$, we denote by $\partial\Phi(x)^\circ$ the element of $\partial\Phi(x)$ of minimal norm, that is,

$$\partial\Phi(x)^\circ := \operatorname{proj}\left(0; \partial\Phi(x)\right),$$

where $\operatorname{proj}(x; K)$ denotes the metric projection of x onto the closed convex set K. For $\lambda > 0$, the λ-Moreau–Yosida regularization of Φ is the function $\Phi_\lambda : \mathcal{H} \to \mathbb{R}$ given by

$$\Phi_\lambda(x) := \inf_{y\in\mathcal{H}} \left\{ \Phi(y) + \frac{1}{2\lambda}\|x-y\|^2 \right\}.$$

The above infimum is attained at a unique point, $J_\lambda x$. The Lipschitz mapping $J_\lambda : \mathcal{H} \to \mathcal{H}$ is called the *resolvent* of $\partial\Phi$ of index λ.

It is known that Φ_λ is a continuously differentiable convex function, and its derivative is given by

$$\nabla\Phi_\lambda(x) = \frac{1}{\lambda}(x - J_\lambda x).$$

The operator $A_\lambda = \lambda^{-1}(\mathrm{id}_{\mathcal{H}} - J_\lambda)$ is known as the *Yosida approximation* of the monotone operator $A := \partial\Phi$. One of the properties of the operator A_λ that will be used in the article is that

$$\|A_\lambda x\| \le \|\partial\Phi(x)^\circ\|, \quad \forall x \in \operatorname{dom}\partial\Phi. \tag{2}$$

For more properties on Yosida approximations, and the detailed definitions of the operators J_λ and A_λ, we refer the reader to [3].

The following theorem surveys the main known properties of the subgradient flow dynamical system for convex potentials, that we will use.

Theorem 2.1 ([3, Theorem 17.2.2 and Proposition 17.2.8]) *Let $\Phi : \mathcal{H} \to \mathbb{R}_\infty$ be a convex, lower semicontinuous, and proper function. Suppose that Φ is bounded from below, that is, $\inf_{\mathcal{H}} \Phi > -\infty$. Then, for any $u_0 \in \operatorname{dom}\Phi$ there exists a unique (strong) global solution $u : [0, +\infty[\to \mathcal{H}$ of the Cauchy problem*

$$\begin{cases} \dot{u}(t) \in -\partial\Phi(u(t)), & \forall t \ge 0 \\ u(0) = u_0. \end{cases}$$

Moreover, the following properties hold:

(S.i) $u(t) \in \operatorname{dom}\partial\Phi$ *for all* $t > 0$.

(S.ii) $\dot{u} \in L^2([0, +\infty[, \mathcal{H}) \cap L^\infty([0, +\infty[, \mathcal{H})$. *In particular, u is Lipschitz continuous on* $[0, +\infty[$.

(S.iii) *For each $t \ge 0$, u has a right derivative and*

$$\frac{d^+u}{dt}(t) = -\partial\Phi(u(t))^\circ.$$

(S.iv) *The map* $t \mapsto \left\| \frac{d^+u}{dt}(t) \right\|$ *is nonincreasing and*

$$\lim_{t\to+\infty} \left\| \frac{d^+u}{dt}(t) \right\| = 0.$$

(S.v) *The map* $t \mapsto \Phi(u(t))$ *is nonincreasing, absolutely continuous on each bounded interval of* $[0, T]$, *and*

$$\frac{d}{dt}\Phi(u(t)) = -\|\dot{u}(t)\|^2, \text{ a.e. } t > 0.$$

(S.vi) *The limit* $\lim_{t\to+\infty}\Phi(u(t))$ *exists and coincides with* $\inf_{\mathcal{H}}\Phi$.

(S.vii) *If* $\operatorname{argmin}\Phi \neq \emptyset$, *then there exists* $u_\infty \in \operatorname{argmin}\Phi$ *such that* $u(t) \rightharpoonup u_\infty$.

To finish this introductory part and for the sake of completeness, we present the rediscovered proof of Boulmezaoud, Cieutat and Daniilidis's conjecture in the everywhere differentiable case when the infimum of the function φ_1 is attained.

Theorem 2.2 *Let* $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}$ *be two convex continuous functions everywhere Gâteaux differentiable, such that*

(i) $\|\nabla\varphi_1(x)\| = \|\nabla\varphi_2(x)\|$, *for all* $x \in \mathcal{H}$.

(ii) φ_1 *attains its minimum, that is,* $\operatorname{argmin}\varphi_1 \neq \emptyset$.

Then, there exists $a \in \mathbb{R}$ *such that* $\varphi_1 = \varphi_2 + a$.

Proof Since $S := \operatorname{argmin}\varphi_1 \neq \emptyset$, hypothesis (i) entails that $\operatorname{argmin}\varphi_2 = S$. Now, consider the convex functional $\Phi = \varphi_1 + \varphi_2$ and fix $u_0 \in \mathcal{H}$. It is not hard to realize that $\inf\Phi = \inf\varphi_1 + \inf\varphi_2$ and that $\operatorname{argmin}\Phi$ coincides with S as well.

Let $u : [0, +\infty[\to \mathcal{H}$ be the unique global solution of

$$\begin{cases} \dot{u}(t) = -\nabla\Phi(u(t)), \\ u(0) = u_0. \end{cases}$$

Then, by Theorem 2.1.(S.vii), there exists $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$. By lower semicontinuity of φ_1, upper semicontinuity of $-\varphi_2$ and Theorem 2.1.(S.vi) we can write

$$\begin{aligned} \varphi_1(u_\infty) &\le \liminf \varphi_1(u(t)) \\ &\le \limsup \varphi_1(u(t)) \\ &\le \lim \Phi(u(t)) + \limsup -\varphi_2(u(t)) \\ &\le \inf\Phi - \inf\varphi_2 = \varphi_1(u_\infty). \end{aligned}$$

Thus, $\varphi_1(u(t)) \to \inf\varphi_1$. By symmetry, $\varphi_2(u(t)) \to \inf\varphi_2$ as well. Now, let us define $\Psi : [0, +\infty[\to \mathbb{R}$ given by

$$\Psi(t) = \varphi_1(u(t)) - \varphi_2(u(t)).$$

Clearly, Ψ is absolutely continuous and for almost all $t > 0$ we can write

$$\frac{d}{dt}\Psi(t) = \langle\nabla\varphi_1(u(t)) - \nabla\varphi_2(u(t)), \dot{u}(t)\rangle$$

$$= \langle \nabla\varphi_1(u(t)) - \nabla\varphi_2(u(t)), -(\nabla\varphi_1(u(t)) + \nabla\varphi_2(u(t)))\rangle$$
$$= \|\nabla\varphi_2(u(t))\|^2 - \|\nabla\varphi_1(u(t))\|^2 = 0.$$

Thus, Ψ is a constant function, and so

$$\varphi_1(u_0) - \varphi_2(u_0) = \Psi(0) = \lim_{t\to+\infty} \Psi(t) = \inf\varphi_1 - \inf\varphi_2.$$

Since, $u_0 \in \mathcal{H}$ is arbitrary, the conclusion follows by setting $a = \inf\varphi_1 - \inf\varphi_2$. □

3 Comparison principle and determination when $\Omega = \mathcal{H}$

Let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ two convex proper lower semicontinuous functionals and let Ω be a nonempty open convex subset of $\mathcal{H}$. For an arbitrary $x_0 \in \Omega \cap \operatorname{dom}\partial\varphi_1$, we consider the following system:

$$\begin{cases} \dot{x}(t) \in -\partial\varphi_1(x(t)), \\ x(0) = x_0. \end{cases} \tag{3}$$

We consider the following two hypotheses:

(H.1) $\|\partial\varphi_1(x)^\circ\| \geq \|\partial\varphi_2(x)^\circ\|$, for all $x \in \Omega$.
(H.2) $\inf_{\overline{\Omega}} \varphi_1 > -\infty$.

Let $x_1 : [0, +\infty[\to \mathcal{H}$ be the unique solution of system (3). Theorem 2.1 and hypothesis (H.2) ensure that φ_1 is nonincreasing along x_1 and $\varphi_1(x_1)$ converges to the infimum of φ_1. However, one first obstruction to compare φ_1 and φ_2 is that, when following x_1, we cannot guarantee that φ_2 is nonincreasing along this trajectory. The next lemma shows that if the trajectory x_1 never meets $\operatorname{bd}\Omega$, hypothesis (H.1) forces $\varphi_2(x_1)$ to converge to the infimum of φ_2.

Lemma 3.1 *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, and let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions verifying hypotheses (H.1) and (H.2). For $x_0 \in \Omega \cap \operatorname{dom}\partial\varphi_1$, let $x_1 : [0, +\infty[\to \mathcal{H}$ be the unique solution of system (3). If the trajectory $x_1(\cdot)$ remains entirely in Ω, then $\inf_{\overline{\Omega}} \varphi_2 = \inf_{\mathcal{H}} \varphi_2$ and $\varphi_2(x_1(t)) \to \inf_{\mathcal{H}} \varphi_2$ (which may be $-\infty$).*

Proof Let $v \in \Omega \cap \operatorname{dom}\varphi_2$. By convexity and the fact that $\Omega \cap \operatorname{dom}\partial\varphi_1 = \Omega \cap \operatorname{dom}\partial\varphi_2$, for any $t \in [0, +\infty[$ we can write

$$\varphi_2(x_1(t)) + \langle \partial\varphi_2(x_1(t))^\circ, v - x_1(t)\rangle \leq \varphi_2(v).$$

By hypothesis (H.1) and Theorem 2.1.(S.iv), we know that $\partial\varphi_2(x_1(t))^\circ$ converges strongly to 0, and therefore $\langle \partial\varphi_2(x_1(t))^\circ, v\rangle \to 0$. Let us prove that

$\langle \partial\varphi_2(x_1(t))^\circ, x_1(t)\rangle$ also converges to 0. Indeed, fix $\varepsilon > 0$ and, applying Theorem 2.1.(S.ii), let $t_\varepsilon > 0$ large enough such that

$$\int_{t_\varepsilon}^{+\infty} \|\dot{x}_1(s)\|^2 ds \leq \varepsilon.$$

Then, for $t \geq t_\varepsilon$ we can write

$$\begin{aligned}
\left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t)\rangle\right| &\leq \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t_\varepsilon)\rangle\right| \\
&\quad + \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t) - x_1(t_\varepsilon)\rangle\right| \\
&= \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t_\varepsilon)\rangle\right| + \left|\int_{t_\varepsilon}^{t} \langle \partial\varphi_2(x_1(t))^\circ, \dot{x}_1(s)\rangle ds\right| \\
&\leq \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t_\varepsilon)\rangle\right| + \int_{t_\varepsilon}^{t} \|\partial\varphi_2(x_1(t))^\circ\| \|\dot{x}_1(s)\| ds \\
&\leq \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t_\varepsilon)\rangle\right| + \int_{t_\varepsilon}^{t} \|\dot{x}_1(s)\|^2 ds,
\end{aligned}$$

where the last inequality follows by noting that

$$\begin{aligned}
\|\partial\varphi_2(x_1(t))^\circ\| &\leq \|\partial\varphi_1(x_1(t))^\circ\| \\
&= \left\|\frac{d^+x_1}{dt}(t)\right\| \leq \left\|\frac{d^+x_1}{dt}(s)\right\| = \|\dot{x}_1(s)\|, \text{ for a.e. } s \leq t.
\end{aligned}$$

Then, we get that

$$\left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t)\rangle\right| \leq \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t_\varepsilon)\rangle\right| + \varepsilon \xrightarrow{t\to+\infty} \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\limsup \left|\langle \partial\varphi_2(x_1(t))^\circ, x_1(t)\rangle\right| = 0,$$

proving the desired convergence. Then, for all $v \in \operatorname{dom} \varphi_2$ one has that

$$\begin{aligned}
\inf_{\overline{\Omega}} \varphi_2 &\leq \liminf \varphi_2(x_1(t)) \\
&\leq \limsup \varphi_2(x_1(t)) \\
&\leq \limsup(\varphi_2(v) + \langle \partial\varphi_2(x_1(t))^\circ, x_1(t) - v\rangle) = \varphi_2(v).
\end{aligned}$$

By taking infimum in v over $\operatorname{dom}\varphi_2$, we conclude that $\inf_{\overline{\Omega}}\varphi_2 = \inf_{\mathcal{H}}\varphi_2$, that $\lim \varphi_2(x_1(t))$ exists in $\overline{\mathbb{R}}$ (could be $-\infty$), and that it coincides with $\inf_{\mathcal{H}}\varphi_2$, finishing the proof. □

The following theorem allows us to compare two convex functions when the minimal norm of their respective subgradients are comparable. As in classic theory of viscosity solutions [6], this is the cornerstone to provide the uniqueness result we search. Here, we consider two types of boundary conditions, that we called Dirichlet and Neumann conditions. While the name of the first one is self-explained, the second one received its name due to its relation with normal derivatives in the continuously differentiable setting (see Remark 3.1).

Theorem 3.1 (Comparison principle) *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, and let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions satisfying hypotheses (H.1) and (H.2).*

Assume also that one of the following boundary conditions hold:

(C.1) ***(Dirichlet condition)*** *There exists $c \in \mathbb{R}$ such that $\varphi_1(x) \geq \varphi_2(x) + c$, for all $x \in \operatorname{bd}\Omega$.*

(C.2) ***(Neumann condition) For all $x \in \operatorname{bd}\Omega$, the following inequality holds:*** $\|\partial(\varphi_1 + \delta_{\overline{\Omega}})(x)^\circ\| \geq \|\partial(\varphi_2 + \delta_{\overline{\Omega}})(x)^\circ\|$.

Then, there exists $a \in \mathbb{R}$ such that $\varphi_1 \geq \varphi_2 + a$ in $\overline{\Omega}$. Moreover,

- *if (C.1) holds, then we can set $a = \min\left\{c, \inf_{\overline{\Omega}}\varphi_1 - \inf_{\overline{\Omega}}\varphi_2\right\}$; and*
- *if (C.2) holds, then we can set $a = \inf_{\overline{\Omega}}\varphi_1 - \inf_{\overline{\Omega}}\varphi_2$.*

Proof Assume first condition (C.1). Let $x_0 \in D := \Omega \cap \operatorname{dom}\partial\varphi_1$. Clearly, (H.1) yields that $x_0 \in \operatorname{dom}\partial\varphi_2$ as well. Let us consider then the subgradient flow problem (3) with initial condition x_0, and let $x_1 : [0, +\infty[\to \mathcal{H}$ be its solution, given by Theorem 2.1. Define the function $\Psi : [0, +\infty[\to \mathbb{R}$ given by

$$\Psi(t) := \varphi_1(x_1(t)) - \varphi_2(x_1(t)).$$

and consider the exit time

$$\tau := \inf\{t \geq 0 \ : \ x_1(t) \in \operatorname{bd}\Omega\}.$$

Clearly $\tau > 0$ and, since x_1 is Lipschitz-continuous with some constant $L > 0$ by Theorem 2.1.(S.ii), we know that almost every $t \in [0, \tau[$,

$$\|\partial\varphi_2(x_1(t))^\circ\| \leq \|\partial\varphi_1(x_1(t))^\circ\| = \|\dot{x}_1(t)\| \leq L.$$

Then, by [3, Proposition 17.2.5] we know that $t \mapsto \varphi_2(x_1(t))$ is absolutely continuous on $[0, \tau[$, and so is Ψ. Therefore, for almost all $t \in [0, \tau[$ we can write

$$\frac{d}{dt}\Psi(t) = \langle \partial\varphi_1(x_1(t))^\circ, \dot{x}_1(t)\rangle - \langle \partial\varphi_2(x_1(t))^\circ, \dot{x}_1(t)\rangle$$

$$\leq -\|\partial\varphi_1(x_1(t))^\circ\|^2 + \|\partial\varphi_2(x_1(t))^\circ\|\|\partial\varphi_1(x_1(t))^\circ\| \leq 0,$$

which yields that Ψ is nonincreasing on $[0, \tau[$. Now, on the one hand, if $\tau < +\infty$, then $x_1(\tau) \in \operatorname{bd} \Omega$ and

$$\varphi_1(x_0) - \varphi_2(x_0) = \Psi(0) \geq \Psi(\tau) = \varphi_1(x_1(\tau)) - \varphi_2(x_1(\tau)) \geq c.$$

On the other hand, if $\tau = +\infty$, we can apply Theorem 2.1.(S.v) and Lemma 3.1 to write

$$\varphi_1(x_0) - \varphi_2(x_0) \geq \Psi(t) \xrightarrow{t\to+\infty} \inf_{\Omega} \varphi_1 - \inf_{\Omega} \varphi_2 =: b.$$

Noting that $\varphi_1(x_0) - \varphi_2(x_0) < +\infty$, hypothesis (H.2) ensures that $\inf_{\overline{\Omega}} \varphi_2 > -\infty$ and so $b \in \mathbb{R}$. Then, defining $a = \min\{b, c\}$, we conclude that

$$\varphi_1(x) \geq \varphi_2(x) + a, \quad \forall x \in \Omega \cap \operatorname{dom} \partial\varphi_1.$$

Consider now $x \in \overline{\Omega} \cap \operatorname{dom} \varphi_1$. By Brønsted–Rockafellar theorem applied to $\varphi_1 + \delta_\Omega$ (see, e.g., [12, Theorem 2]), we have that there exist a sequence (x_n) in D converging to x such that $\varphi_1(x_n) \to \varphi_1(x)$. Then, by lower semicontinuity of φ_2,

$$\varphi_2(x) + a \leq \liminf(\varphi_2(x_n) + a) \leq \liminf \varphi_1(x_n) = \varphi_1(x).$$

The conclusion follows.

To finish, if we assume condition (C.2), we can replace φ_i by $\tilde{\varphi}_i = \varphi_i + \delta_{\overline{\Omega}}$, $i = 1, 2$ and apply the above development directly to these functions over the whole space $\mathcal{H}$, where condition (C.1) is trivially satisfied. This can be done due to the fact that Ω has nonempty interior and therefore, the sum rule of subdifferentials hold, that is,

$$\partial\tilde{\varphi}_i(x) = \partial\varphi_i(x) + \partial\delta_{\overline{\Omega}}(x), \quad \forall x \in \mathcal{H},\ i = 1, 2.$$

Then, condition (C.2) yields that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ verify hypothesis (H.1) over $\mathcal{H}$. Since in this case the exit time τ defined above will always be $+\infty$, we can set $a = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2$. The proof is now finished. □

Observe that, under Dirichlet boundary condition (C.1), the constant $a = \min\{c, \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2\}$ cannot be improved without extra hypotheses. For example:

1. On the one hand, if we consider $\Omega =]0, +\infty[$, $\varphi_1(x) = \exp(-x)$ and $\varphi_2(x) = 0$, then in this case we can choose $c = 1$ and we have that $\inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2 = 0$. Here, it is clear that $\varphi_1 \geq \varphi_2 + 0$. Thus, the infimum difference is the active constant. See Fig. 1, left figure.
2. On the other hand, if we consider $\Omega =]-1, +\infty[$, $\varphi_1(x) = x - 1$ and $\varphi_2(x) = |x|$, then in this case we can choose $c = -3$ and $\inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2 = -2$. Here, it is clear that $\varphi_1 \geq \varphi_2 - 3$. Thus, the boundary constant is the active one. See Fig. 1, right figure.

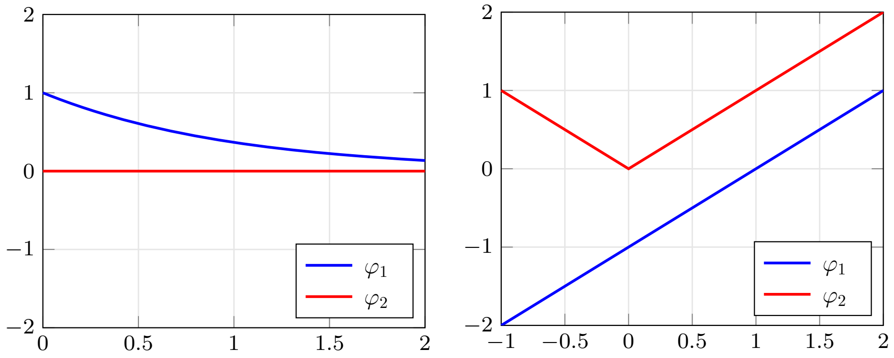

Fig. 1 Left: $a = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2$. Right: $a = c$

From the proof of Theorem 3.1, if $\operatorname{bd} \Omega = \emptyset$, then $\inf_{\mathcal{H}} \varphi_2 > -\infty$ and the constant $a \in \mathbb{R}$ can be set as $a = \inf_{\mathcal{H}} \varphi_1 - \inf_{\mathcal{H}} \varphi_2$. Thus, we get the following direct corollary:

Corollary 3.1 *Let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions satisfying that:*

(i) $\|\partial \varphi_1(x)^\circ\| = \|\partial \varphi_2(x)^\circ\|$, *for all $x \in \mathcal{H}$.*
(ii) $\inf_{\mathcal{H}} \varphi_1 > -\infty$.

Then, there exists $a \in \mathbb{R}$ such that $\varphi_1 = \varphi_2 + a$.

Also, the reader can appreciate (following the same argument in the end of the proof of Theorem 3.1) that the Neumann condition (C.2) allows to omit the set Ω as a constraint, by considering directly $\varphi_1 + \delta_{\overline{\Omega}}$ and $\varphi_2 + \delta_{\overline{\Omega}}$. Therefore, we also can write the following corollary:

Corollary 3.2 *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, and let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions satisfying that:*

(i) $\|\partial \varphi_1(x)^\circ\| = \|\partial \varphi_2(x)^\circ\|$, *for all $x \in \Omega$.*
(ii) $\inf_{\overline{\Omega}} \varphi_1 > -\infty$.

Assume also that the Neumann boundary condition holds, that is,

$$\|\partial(\varphi_1 + \delta_{\overline{\Omega}})(x)^\circ\| = \|\partial(\varphi_2 + \delta_{\overline{\Omega}})(x)^\circ\|, \quad \textit{for all } x \in \operatorname{bd} \Omega.$$

Then, there exists $a \in \mathbb{R}$ such that $\varphi_1 = \varphi_2 + a$ in $\overline{\Omega}$.

Remark 3.1 Observe that when the functions φ_1, φ_2 are continuously differentiable, we have that

$$\partial(\varphi_i + \delta_{\overline{\Omega}})(x) = \nabla \varphi_i(x) + N_{\overline{\Omega}}(x), \quad \forall x \in \operatorname{bd} \Omega \text{ and } i = 1, 2,$$

where $N_{\overline{\Omega}}(x)$ stands for the normal cone of $\overline{\Omega}$ at x. Then, if we choose a normal vector $\xi \in N_{\overline{\Omega}}(x)$, we will have, provided hypothesis (H.1) and continuity of the gradients, that

$$\begin{aligned}\|\nabla\varphi_1(x)+\xi\|^2 &= \|\nabla\varphi_1(x)\|^2+2\langle\nabla\varphi_1(x),\xi\rangle+\|\xi\|^2\\ &\geq \|\nabla\varphi_2(x)\|^2+2\langle\nabla\varphi_1(x),\xi\rangle+\|\xi\|^2.\end{aligned}$$

Then, it is easy to verify that the classic Neumann condition of normal derivatives, that is,

$$\langle\nabla\varphi_1(x),\xi\rangle \geq \langle\nabla\varphi_2(x),\xi\rangle, \quad \forall x\in \operatorname{bd}\Omega, \forall\xi\in N_{\overline{\Omega}}(x),$$

entails condition (C.2). This is why we refer to Neumann conditions in Theorem 3.1 and in Corollary 3.2.

4 Determination under Dirichlet boundary condition

In this section, we will show that the conclusion of Corollary 3.2 also holds under Dirichlet boundary conditions. However, in order to prove it, the comparison principle shown in Theorem 3.1 is not enough, mainly because it doesn't allow us to differentiate between the cases when trajectories remain in Ω and when they reach bd Ω.

Here, we will go back to the idea of using the sum and the difference of both functions, mentioned in Sect. 1 for the smooth case. The main obstruction to directly apply the technique is that in the nonsmooth case one cannot guarantee that $\partial(\varphi_1+\varphi_2)(x)^\circ$ coincide with $\partial\varphi_1(x)^\circ+\partial\varphi_2(x)^\circ$. Therefore, an adaptation is needed.

A first natural idea is to consider the Moreau–Yosida regularization of the involved functionals, to obtain approximate smooth versions of the problem, and then apply known convergence results. However, this is not possible since the norms of the gradients of the regularized approximations of φ_1 and φ_2 are not comparable between them, nor comparable with the gradient of the approximations of the sum $\varphi_1+\varphi_2$.

Here, we propose a new method consisting on partially regularize the sum, by regularizing only one of the functions. Formally, let $\Omega\subseteq\mathcal{H}$ be a nonempty open convex set, and let $\varphi_1,\varphi_2:\mathcal{H}\to\mathbb{R}_\infty$ be two convex proper lower semicontinuous functions. We will study the behavior of

$$\Phi := \varphi_1+\varphi_2+\delta_{\overline{\Omega}} \quad \text{and} \quad \Phi_\lambda := \varphi_{1,\lambda}+\varphi_2+\delta_{\overline{\Omega}},$$

where, for $\lambda>0$, $\varphi_{1,\lambda}$ is the λ-Moreau–Yosida approximation of $\varphi_1+\delta_{\overline{\Omega}}$, that is

$$\varphi_{1,\lambda}(x) = \inf_{y\in\mathcal{H}}\left\{(\varphi_1+\delta_{\overline{\Omega}})(y)+\frac{1}{2\lambda}\|x-y\|^2\right\}. \tag{4}$$

Let $u_0 \in \operatorname{dom} \partial \Phi$. We consider $u : [0, +\infty[\to \mathcal{H}$ and $u_\lambda : [0, +\infty[\to \mathcal{H}$ to be the global solutions of the systems

$$\begin{cases} \dot{u}(t) \in -\partial \Phi(u(t)), & \text{a.e. } t \geq 0 \\ u(0) = u_0, \end{cases} \\ \text{and} \\ \begin{cases} \dot{u}_\lambda(t) \in -\partial \Phi_\lambda(u_\lambda(t)), & \text{a.e. } t \geq 0 \\ u_\lambda(0) = u_0, \end{cases} \tag{5}$$

respectively. A priori, there is no guarantee that the solutions of the second system of (5) converge to the solution of the first one.

The following lemma establishes that, under some uniform boundedness condition, the desired convergences hold in the spirit of classic Moreau–Yosida approximations. It is mainly based on [3, Proposition 17.2.6] and its proof is an adaptation of the classic proofs of [3, Theorem 17.2.2 and Proposition 17.2.6].

Lemma 4.1 *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, and let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions such that the mappings $x \mapsto \|\partial \varphi_1(x)^\circ\|$ and $x \mapsto \|\partial \varphi_2(x)^\circ\|$ are uniformly bounded in $\overline{\Omega}$. Then,*

1. *$u_\lambda \to u$ uniformly in every interval* $[0, T]$.
2. *$\dot{u}_\lambda \to \dot{u}$ strongly in $L^2([0, T], \mathcal{H})$.*
3. *$\Phi_\lambda(u_\lambda(t)) \to \Phi(u(t))$ uniformly in every interval* $[0, T]$.
4. *$\varphi_{1,\lambda}(u_\lambda(t)) \to \varphi_1(u(t))$ and $\varphi_2(u_\lambda(t)) \to \varphi_2(u(t))$ uniformly in every interval* $[0, T]$.

Proof Without loss of generality, let us assume that $\varphi_1 = \varphi_1 + \delta_{\overline{\Omega}}$ and $\varphi_2 = \varphi_2 + \delta_{\overline{\Omega}}$. First, fix $T > 0$ and let $M_1, M_2 > 0$ be the global upper bounds on $\|\partial \varphi_1(x)^\circ\|$ and $\|\partial \varphi_2(x)^\circ\|$, respectively.

Let us show that $(u_\lambda)_{\lambda > 0}$ is a Cauchy net in $\mathcal{C}([0, T], \mathcal{H})$ with the uniform norm. Fix $\lambda, \mu > 0$ and define $h : [0, T] \to [0, +\infty[$ by

$$h(t) := \frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|^2.$$

Then, we can write

$$\begin{aligned} \dot{h}(t) &= \langle u_\lambda(t) - u_\mu(t), \dot{u}_\lambda(t) - \dot{u}_\mu(t)\rangle \\ &= \langle u_\lambda(t) - u_\mu(t), -\xi_\lambda(t) + \xi_\mu(t)\rangle \\ &\quad + \langle u_\lambda(t) - u_\mu(t), -\nabla \varphi_{1,\lambda}(u_\lambda(t)) + \nabla \varphi_{1,\mu}(u_\mu(t))\rangle, \end{aligned}$$

where $\xi_\lambda(t) \in \partial \varphi_2(u_\lambda(t))$ is such that $\xi_\lambda(t) + \nabla \varphi_{1,\lambda}(u_\lambda(t)) = \partial \Phi_\lambda(u_\lambda(t))^\circ$, and $\xi_\mu(t) \in \partial \varphi_2(u_\mu(t))$ is such that $\xi_\mu(t) + \nabla \varphi_{1,\mu}(u_\mu(t)) = \partial \Phi_\mu(u_\mu(t))^\circ$. The existence of the elements $\xi_\lambda(t)$ and $\xi_\mu(t)$ is given by the continuity of $\varphi_{1,\lambda}$ and $\varphi_{1,\mu}$, which entail the qualification conditions of the sum rule of convex subdifferentials (see, e.g., [9, Theorem 4.1.19]).

By monotonicity of $\partial\varphi_2$, we get that

$$\langle u_\lambda(t) - u_\mu(t), -\xi_\lambda(t) + \xi_\mu(t)\rangle \leq 0,$$

and so,

$$\dot{h}(t) \leq \langle u_\lambda(t) - u_\mu(t), -\nabla\varphi_{1,\lambda}(u_\lambda(t)) + \nabla\varphi_{1,\mu}(u_\mu(t))\rangle.$$

From now on, we will omit the variable t to simplify notation. Recalling that $u_\lambda = J_\lambda u_\lambda + \lambda A_\lambda u_\lambda$ and $u_\mu = J_\mu u_\mu + \mu A_\mu u_\mu$, the above inequality allows us to write

$$\dot{h} + \langle J_\lambda u_\lambda + \lambda A_\lambda u_\lambda - J_\mu u_\mu - \mu A_\mu u_\mu, A_\lambda u_\lambda - A_\mu u_\mu\rangle \leq 0.$$

Since $A_\lambda u_\lambda \in \partial\varphi_1(J_\lambda u_\lambda)$ and $A_\mu u_\mu \in \partial\varphi_1(J_\mu u_\mu)$, we get that $\langle J_\lambda u_\lambda - J_\mu u_\mu, A_\lambda u_\lambda - A_\mu u_\mu\rangle$ is nonnegative, which entails

$$\dot{h} + \langle \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu, A_\lambda u_\lambda - A_\mu u_\mu\rangle \leq 0.$$

Noting that $\langle A_\lambda u_\lambda, A_\mu u_\mu\rangle \leq \frac{1}{2}(\|A_\lambda u_\lambda\|^2 + \|A_\mu u_\mu\|^2)$, it is not hard to prove that

$$\langle \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu, A_\lambda u_\lambda - A_\mu u_\mu\rangle \geq -\mu\|A_\lambda u_\lambda\|^2 - \lambda\|A_\mu u_\mu\|^2,$$

and recalling that $\|A_\lambda u_\lambda\| \leq \|\partial\varphi_1(u_\lambda)^\circ\|$ and $\|A_\mu u_\mu\| \leq \|\partial\varphi_1(u_\mu)^\circ\|$, we conclude that

$$\dot{h} \leq +\mu\|A_\lambda u_\lambda\|^2 + \lambda\|A_\mu u_\mu\|^2 \leq (\lambda + \mu)M_1^2.$$

By integrating $\dot{h}$, we conclude that

$$\|u_\lambda - u_\mu\|_\infty \leq \sqrt{\lambda + \mu}M_1\sqrt{T},$$

which proves that (u_λ) is a Cauchy net. This yields that $u_\lambda \to v \in \mathcal{C}([0, T], \mathcal{H})$ uniformly. Since $(\nabla\varphi_{1,\lambda}(u_\lambda))$ is uniformly bounded above by M_1 we have that

$$\begin{aligned}\|\xi_\lambda + \nabla\varphi_{1,\lambda}(u_\lambda)\| = \|\Phi_\lambda(u_\lambda)^\circ\| &= \inf\{\|\zeta + \nabla\varphi_{1,\lambda}(u_\lambda)\| \,:\, \zeta \in \partial\varphi_2(u_\lambda)\}\\ &\leq \|\partial\varphi_2(u_\lambda)^\circ\| + M_1 \leq M_1 + M_2.\end{aligned}$$

Thus, for all $t \in [0, T]$,

$$\|\xi_\lambda\| \leq \|\xi_\lambda + \nabla\varphi_{1,\lambda}(u_\lambda)\| + \|\nabla\varphi_{1,\lambda}(u_\lambda)\| \leq 2M_1 + M_2,$$

and so (ξ_λ) is also uniformly bounded. We get then that v is absolutely continuous in $[0, T]$ and $\dot{u}_\lambda \rightharpoonup \dot{v} \in L^2([0, T], \mathcal{H})$. Indeed, it is clear that any sequence $(\dot{u}_{\lambda_n})$ with

$\lambda_n \to 0$ has a weakly convergent subsequence $(\dot{u}_{\lambda_{n_k}})$, and then by a mild application of Lebesgue Dominated Convergence theorem, we get that

$$\int_0^t \lim_k \dot{u}_{\lambda_{n_k}}(s)ds = v(t) - v(0), \quad \forall t \in [0, T],$$

proving that v is absolutely continuous and $\dot{u}_{\lambda_{n_k}} \rightharpoonup \dot{v}$. The convergence of the original net $(\dot{u}_\lambda)$ follows by noting that any sequence $(\dot{u}_{\lambda_n})$ with $\lambda_n \to 0$ has $\dot{v}$ as cluster point.

Moreover, by similar arguments, we get that

$$\xi_\lambda \rightharpoonup \xi \in L^2([0, T], \mathcal{H}) \quad \text{and} \quad \nabla\varphi_{1,\lambda}(u_\lambda) \rightharpoonup \eta \in L^2([0, T], \mathcal{H}),$$

with $-\dot{v} = \xi + \eta$. Finally, since $\|J_\lambda u_\lambda(t) - u_\lambda(t)\| \leq \lambda\|A_\lambda u_\lambda(t)\|$ for all $t \in [0, T]$, we conclude that $J_\lambda u_\lambda \to v$ uniformly as well.

Let us prove now that $v = u$. Recall that for every $\lambda > 0$, $\dot{u}_\lambda = -\xi_\lambda - \nabla\varphi_{1,\lambda}(u_\lambda)$, where $\xi_\lambda \in \partial\varphi_2(u_\lambda)$ and $\nabla\varphi_{1,\lambda}(u_\lambda) \in \partial\varphi_1(J_\lambda u_\lambda)$. Then, we get that

$$\begin{aligned}
&\varphi_1(J_\lambda u_\lambda) + \varphi_1^*(\nabla\varphi_{1,\lambda}(u_\lambda)) - \langle J_\lambda u_\lambda, \nabla\varphi_{1,\lambda}(u_\lambda)\rangle = 0,\\
&\varphi_2(u_\lambda) + \varphi_2^*(\xi_\lambda) - \langle u_\lambda, \xi_\lambda\rangle = 0.
\end{aligned}$$

Thus, by integrating and then taking inferior limit, we get that

$$\begin{aligned}
&\int_0^T \varphi_1(v) + \varphi_1^*(\eta) - \langle v, \eta\rangle \leq 0,\\
&\int_0^T \varphi_2(v) + \varphi_2^*(\xi) - \langle v, \xi\rangle \leq 0.
\end{aligned}$$

Noting that (see, e.g., [9, Proposition 4.4.1, Lemma 4.4.15])

$$\Phi^*(-\dot{v}) \leq \inf\{\varphi_1^*(a) + \varphi_2^*(b) \ : \ a + b = -\dot{v}\} \leq \varphi_1^*(\eta) + \varphi_2^*(\xi),$$

we can add the latter inequalities to get that

$$\int_0^T \Phi(v(t)) + \Phi^*(-\dot{v}(t)) - \langle v(t), \dot{v}(t)\rangle dt \leq 0.$$

Since the above integrand is always nonnegative due to the Legendre–Fenchel inequality, we conclude that

$$\Phi(v(t)) + \Phi^*(-\dot{v}(t)) - \langle v(t), \dot{v}(t)\rangle = 0, \quad \text{for almost all } t \in [0, T],$$

which yields that $v : [0, T] \to \mathcal{H}$ is a solution in $[0, T]$ of the first dynamical system in (5), entailing by uniqueness of solutions that $v = u\big|_{[0,T]}$.

Now, let us prove that $\dot{u}_\lambda$ converges strongly to $\dot{u}$. First, since u_λ is the solution of the second system in (5), we know that

$$\int_0^T \|\dot{u}_\lambda(t)\|^2 dt + \Phi_\lambda(u_\lambda(T)) - \Phi_\lambda(u_0) = 0.$$

Since the map $v \mapsto \int_0^T \|v(t)\|^2 dt$ is lower semicontinuous in $L^2([0, T], \mathcal{H})$, the weak convergence of $(\dot{u}_\lambda)$ to $\dot{u}$ yields that

$$\int_0^T \|\dot{u}(t)\|^2 dt \leq \liminf_\lambda \int_0^T \|\dot{u}_\lambda(t)\|^2 dt.$$

Also, since $\varphi_{1,\lambda}(u_0) \to \varphi_1(u_0)$, we get that

$$\Phi(u_0) = \lim_\lambda \Phi_\lambda(u_0).$$

Finally, noting that $\Phi_\lambda(u_\lambda(T)) \geq \varphi_1(J_\lambda u_\lambda(T)) + \varphi_2(u_\lambda(T))$, the convergences $u_\lambda(T) \to u(T)$ and $J_\lambda u_\lambda(T) \to u(T)$ yield that

$$\begin{aligned} \Phi(u(T)) &= \varphi_1(u(T)) + \varphi_2(u(T)) \\ &\leq \liminf_\lambda \varphi_1(J_\lambda u_\lambda(T)) + \liminf_\lambda \varphi_2(u_\lambda) \leq \liminf_\lambda \Phi_\lambda(u_\lambda(T)). \end{aligned}$$

Finally, since u is the solution of the first system in (5), we have

$$\int_0^T \|\dot{u}(t)\|^2 dt + \Phi(u(T)) - \Phi(u_0) = 0.$$

We conclude then, by applying [3, Lemma 17.2.1], that

$$\int_0^T \|\dot{u}_\lambda(t)\|^2 dt \to \int_0^T \|\dot{u}(t)\|^2 dt \quad \text{and} \quad \Phi_\lambda(u_\lambda(T)) \to \Phi(u(T)).$$

The weak convergence $\dot{u}_\lambda \rightharpoonup \dot{u}$ and the convergence of the norms imply that $(\dot{u}_\lambda)$ converges strongly to $\dot{u}$ in $L^2([0, T], \mathcal{H})$. Finally, since

$$\begin{aligned} \left| \frac{d}{dt} \Phi_\lambda(u_\lambda(t)) \right| &= \|\dot{u}_\lambda(t)\|^2 \\ &\leq \|\nabla \varphi_{1,\lambda}(u_\lambda(t)) + \partial \varphi_2(u_\lambda(t))^\circ\|^2 \leq (M_1 + M_2)^2, \text{ a.e. } t \in [0, T], \end{aligned}$$

we get that $\|\Phi_\lambda(u_\lambda)\|_\infty \leq (M_1 + M_2)^2 T$. By Arzelà–Ascoli Theorem in the space $\mathcal{C}([0, T], \mathbb{R})$, we conclude that

$$\Phi_\lambda(u_\lambda) \to \Phi(u), \quad \text{uniformly in } [0, T].$$

Now, due to the upper semicontinuity of $-\varphi_1$, for every $t \in [0, T]$ we have that

$$\begin{aligned}
\varphi_2(u(t)) &\leq \liminf_\lambda \varphi_2(u_\lambda(t)) \\
&\leq \limsup_\lambda (\Phi_\lambda(u_\lambda(t)) - \varphi_{1,\lambda}(u_\lambda(t))) \\
&\leq \limsup_\lambda (\Phi_\lambda(u_\lambda(t)) - \varphi_1(J_\lambda u_\lambda(t))) \\
&\leq \lim_\lambda \Phi_\lambda(u_\lambda(t)) + \limsup_\lambda -\varphi_1(J_\lambda u_\lambda(t)) \\
&\leq \Phi(u(t)) - \varphi_1(u(t)) = \varphi_2(u(t)).
\end{aligned}$$

Thus, $\varphi_2(u_\lambda(t)) \to \varphi_2(u(t))$, and therefore, $\varphi_{1,\lambda}(u_\lambda(t)) \to \varphi_1(u(t))$. Furthermore, both mappings $\varphi_{1,\lambda}(u_\lambda)$ and $\varphi_2(u_\lambda)$ are absolutely continuous (by, e.g., [3, Proposition 17.2.5]) and

$$\begin{aligned}
\left|\frac{d}{dt}\varphi_{1,\lambda}(u_\lambda(t))\right| &\leq \|\nabla\varphi_{1,\lambda}(u_\lambda(t))\|\|\dot{u}_\lambda\| \leq M_1(M_1 + M_2), \quad \text{a.e. } t \in [0, T], \\
\left|\frac{d}{dt}\varphi_2(u_\lambda(t))\right| &\leq \|\partial\varphi_2(u_\lambda(t))^\circ\|\|\dot{u}_\lambda\| \leq M_2(M_1 + M_2), \quad \text{a.e. } t \in [0, T].
\end{aligned}$$

We deduce again by Arzelà–Ascoli Theorem in $\mathcal{C}([0, T], \mathbb{R})$, that $\varphi_{1,\lambda}(u_\lambda)$ converges uniformly to $\varphi_1(u)$ in $[0, T]$ and $\varphi_2(u_\lambda)$ converges uniformly to $\varphi_2(u)$ in $[0, T]$. The proof is now finished. □

While uniform boundedness is usually a too strong hypothesis, the following basic lemma will help us to localize this property. We include the proof for the sake of completeness.

Lemma 4.2 *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, $f : \Omega \to \mathbb{R}$ be a convex continuous function, and let $u : [a, b] \subset \mathbb{R} \to \mathcal{H}$ (with $a < b$) be a continuous trajectory such that $u([a, b]) \subset \Omega$. Then, there exist two open convex sets U and V, and $\varepsilon > 0$ small enough such that*

1. *$u([a, b]) \subset \overline{V} \subset U \subset \Omega$;*
2. *$u([a, b]) + \varepsilon\mathbb{B} \subset V$; and*
3. *f is Lipschitz-continuous on U.*

Proof Let $K = \overline{\text{co}}(u([a, b]))$. Let us first show that $K \subset \Omega$. Since $u([a, b])$ is compact, there exists a finite sequence $(x_i)_{i=1}^p \subseteq u([a, b])$ and $\eta > 0$ small enough such that

$$u([a, b]) \subseteq \bigcup_{i=1}^p x_i + \eta\mathbb{B} \subseteq \Omega.$$

Then, since Ω is convex

$$\operatorname{co}\left(\bigcup_{i=1}^{p} x_i + \eta\mathbb{B}\right) = \operatorname{co}(\{x_i\}_{i=1}^{p}) + \eta\mathbb{B} \subseteq \Omega,$$

and so, since $\operatorname{co}(\{x_i\}_{i=1}^{p}) + \eta\mathbb{B}$ is closed, we deduce that $\overline{\operatorname{co}}(u[a,b]) \subset \Omega$, as claimed.

Now, we claim that there exists $\gamma > 0$ and $\kappa > 0$ such that f is κ-Lipschitz on $K + \gamma\mathbb{B} \subseteq \Omega$.

Since K is compact (see, e.g., [1, Theorem 5.35]), by using a finite cover argument, we can find $L > 0$ such that f is L-Lipschitz (locally) at every point $x \in K$. Furthermore, since K is convex, we get that f is L-Lipschitz over K. This yields that there exists $\gamma > 0$ small enough such that f is κ-Lipschitz on $K + \gamma\mathbb{B}$ with $\kappa = L + 1$.

Indeed, if this is not the case, then there exist two sequences (y_k) and (z_k) such that

- $y_k, z_k \in K + \frac{1}{k}\mathbb{B}$; and
- $|f(y_k) - f(z_k)| > (L+1)\|y_k - z_k\|$.

By studying the convergence of the sequence $(\operatorname{proj}(y_k, K))_k$ and the sequence $(\operatorname{proj}(z_k, K))_k$, it is not hard to realize that, up to subsequences, there exist $y, z \in K$ such that $y_k \to y$ and $z_k \to z$. Then, on the one hand, if $y \neq z$, we would have

$$(L+1)\|y - z\| \leq |f(y) - f(z)| \leq L\|y - z\|,$$

which is a contradiction. On the other hand, if $y = z$, there exists a neighborhood O of y such that f is L-Lipschitz on O. Then, for $k \in \mathbb{N}$ large enough, we will get that $y_k, z_k \in O$ and so

$$(L+1)\|y_k - z_k\| \leq |f(y_k) - f(z_k)| \leq L\|y_k - z_k\|,$$

which is also a contradiction. The claim is then proved.

The proof is finished by taking $U = \operatorname{int}(K + \gamma\mathbb{B})$, $V = \operatorname{int}(K + \frac{\gamma}{2}\mathbb{B})$ and $\varepsilon = \frac{\gamma}{3}$. $\square$

Theorem 4.1 *Let $\Omega \subseteq \mathcal{H}$ be a nonempty open convex set, and let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathbb{R}_\infty$ be two convex proper lower semicontinuous functions satisfying the following conditions:*

(i) φ_1 *is finite in* Ω.
(ii) $\|\partial\varphi_1(x)^\circ\| = \|\partial\varphi_2(x)^\circ\|$, *for all* $x \in \Omega$.
(iii) $\inf_{\overline{\Omega}} \varphi_1 > -\infty$.

Assume also that the Dirichlet border condition holds, that is, there exists $c \in \mathbb{R}$ such that $\varphi_1(x) = \varphi_2(x) + c$, for all $x \in \operatorname{bd}\Omega$ (Dirichlet condition). Then, there exists $a \in \mathbb{R}$ such that $\varphi_1 = \varphi_2 + a$ in $\overline{\Omega}$.

Proof Without loss of generality, assume that $\operatorname{bd}\Omega \cap \operatorname{dom}\varphi_1 \neq \emptyset$, otherwise, the result becomes just an application of Corollary 3.1.

Since φ_1 is finite in Ω, convexity and lower semicontinuity yield that φ_1 is continuous in Ω and that $\Omega \subseteq \operatorname{dom}\partial\varphi_1$. Hypothesis (ii) entails that the same properties (finiteness and continuity in Ω) hold for φ_2 as well.

Applying twice Theorem 3.1, there exist two constants $a_1, a_2 \in \mathbb{R}$ such that

$$\varphi_2(x) + a_1 \leq \varphi_1(x) \leq \varphi_2(x) + a_2, \quad \forall x \in \overline{\Omega}.$$

Let us define

$$a_1 := \sup\left\{a \in \mathbb{R} \,:\, a \leq \varphi_1(x) - \varphi_2(x), \quad \forall x \in \overline{\Omega}\right\},$$
$$a_2 := \inf\left\{a \in \mathbb{R} \,:\, a \geq \varphi_1(x) - \varphi_2(x), \quad \forall x \in \overline{\Omega}\right\}.$$

By following the proof of Theorem 3.1, we know that

$$\min\{c, \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2\} \leq a_1 \leq a_2 \leq \max\{c, \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2\}$$

We will show that $c = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2$. Proceeding by contradiction, suppose that this is not true and consider first that $\inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2 < c$.

Take $u_0 \in \operatorname{dom} \varphi_1 \cap \operatorname{bd} \Omega$ and let $x_1 : [0, +\infty[\to \mathcal{H}$ and $u : [0, +\infty[\to \mathcal{H}$ be the global solutions of

$$\begin{cases} \dot{x}_1(t) \in -\partial \varphi_1(x_1(t)), & \text{a.e. } t \geq 0 \\ x_1(0) = u_0, \end{cases} \quad \text{and} \quad \begin{cases} \dot{u}(t) \in -\partial \Phi(u(t)), & \text{a.e. } t \geq 0 \\ u(0) = u_0, \end{cases}$$

respectively. Since for all $x \in \operatorname{bd} \Omega$ we can write

$$\inf_{\overline{\Omega}} \varphi_1 < c + \inf_{\overline{\Omega}} \varphi_2 \leq \varphi_1(x),$$

we know that there must exist a time t_0, such that the trajectory x_1 remains in Ω from t_0 onwards, that is $x_1(]t_0, +\infty[) \subset \Omega$. This yields that we can apply Lemma 3.1 to conclude that

$$\varphi_2(x_1(t)) \to \inf_{\overline{\Omega}} \varphi_2,$$

and so $\Phi(x_1(t)) \to \inf_{\overline{\Omega}} \varphi_1 + \inf_{\overline{\Omega}} \varphi_2$. This proves that

$$\inf_{\overline{\Omega}} \Phi = \inf_{\overline{\Omega}} \varphi_1 + \inf_{\overline{\Omega}} \varphi_2 < 2 \inf_{\overline{\Omega}} \varphi_2 + c \leq \inf_{\operatorname{bd} \Omega} \Phi.$$

Thus, since $\Phi(u(t)) \to \inf \Phi$, we deduce that there must exist a time $\bar{t}$, such that the trajectory u remains in Ω from $\bar{t}$ onwards. Setting $\bar{t} = \sup\{t \geq 0 \,:\, u(t) \in \operatorname{bd} \Omega\}$ and replacing u_0 by $u(\bar{t})$ if necessary, we may assume that $\bar{t} = 0$, that is, we may assume that $u(]0, +\infty[) \subset \Omega$.

Note that, by [3, Proposition 17.2.5], $\varphi_1(u)$ and $\varphi_2(u)$ are absolutely continuous. Moreover, $\varphi_1(u(t)) \to \inf_{\overline{\Omega}} \varphi_1$ and $\varphi_2(u(t)) \to \inf_{\overline{\Omega}} \varphi_2$. Indeed, for the first limit we can write

$$\inf_{\overline{\Omega}} \varphi_1 \leq \liminf_t \varphi_1(u(t))$$

$$\begin{aligned}
&\leq \limsup_t \varphi_1(u(t)) \\
&= \limsup_t (\Phi(u(t)) - \varphi_2(u(t))) \\
&\leq \limsup_t \Phi(u(t)) - \inf_{\overline{\Omega}} \varphi_2 \\
&= \inf_{\overline{\Omega}} \Phi - \inf_{\overline{\Omega}} \varphi_2 = \inf_{\overline{\Omega}} \varphi_1.
\end{aligned}$$

For the second limit, we can do the same by exchanging the roles of φ_1 and φ_2. Now, let $\Psi : [0, +\infty[\to \mathbb{R}$ be the function given by

$$\Psi(t) := \varphi_1(u(t)) - \varphi_2(u(t)).$$

By the previous development, we know that Ψ is absolutely continuous and

$$\lim_{t \to +\infty} \Psi(t) = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2.$$

Let us prove that Ψ is nondecreasing.

Fix $\delta > 0$ and $T > \delta$. Let V and U the neighborhoods given by Lemma 4.2 associated to the function φ_1, the interval $[\delta, T]$ and the curve $u\big|_{[\delta,T]}$. Let $\Phi_V = \varphi_1 + \varphi_2 + \delta_{\overline{V}}$, and let $\Phi_{V,\lambda} = \psi_\lambda + \varphi_2 + \delta_{\overline{V}}$ where

$$\psi_\lambda(x) = \inf_{y \in \mathcal{H}} \left\{ (\varphi_1 + \delta_{\overline{V}})(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}.$$

Let $u_V : [\delta, T] \to \mathcal{H}$ and $u_{V,\lambda} : [\delta, T] \to \mathcal{H}$ be the solutions of

$$\begin{cases} \dot{u}_V(t) \in -\partial \Phi_V(u_V(t)), & \text{a.e. } t \in [\delta, T] \\ u_V(\delta) = u(\delta), \end{cases}$$
$$\text{and} \tag{6}$$
$$\begin{cases} \dot{u}_{V,\lambda}(t) \in -\partial \Phi_{V,\lambda}(u_{V,\lambda}(t)), & \text{a.e. } t \in [\delta, T] \\ u_{V,\lambda}(\delta) = u(\delta), \end{cases}$$

respectively. Since φ_1 is Lipschitz-continuous on U, the mapping $x \mapsto \|\partial \varphi_1(x)^\circ\|$ is uniformly bounded on $\overline{V}$. Then, since for every $x \in \overline{V}$

$$\|\partial(\varphi_1 + \delta_{\overline{V}})(x)^\circ\| \leq \|\partial \varphi_1(x)^\circ\| \text{ and } \|\partial(\varphi_2 + \delta_{\overline{V}})(x)^\circ\| \leq \|\partial \varphi_2(x)^\circ\| = \|\partial \varphi_1(x)^\circ\|,$$

we can apply Lemma 4.1 to conclude that

- $u_{V,\lambda} \to u_V$ uniformly in $[\delta, T]$.
- $\psi_\lambda(u_{V,\lambda}) \to (\varphi_1 + \delta_{\overline{V}})(u_V)$ uniformly in $[\delta, T]$.
- $(\varphi_2 + \delta_{\overline{V}})(u_{V,\lambda}) \to (\varphi_2 + \delta_{\overline{V}})(u_V)$ uniformly in $[\delta, T]$.

Let $\varepsilon > 0$ small enough such that $u_k([\delta, T]) + \varepsilon\mathbb{B} \subset V$. Since $u_{V,\lambda} \to u_V$ uniformly in $[\delta, T]$, there exists $\lambda_0 > 0$ such that

$$\forall\lambda > \lambda_0, u_{V,\lambda}([\delta, T]) \subseteq u_V([\delta, T]) + \varepsilon\mathbb{B} \subset V.$$

Then, since $\partial\delta_{\overline{V}}(v) = \{0\}$ for all $v \in V$, for each $\lambda > \lambda_0$ and for almost all $t \in [\delta, T]$ there exists $\xi_\lambda(t) \in \partial\varphi_2(u_{V,\lambda}(t))$ such that

$$\dot{u}_{V,\lambda}(t) = -(\nabla\psi_\lambda(u_{V,\lambda}(t)) + \xi_\lambda(t)).$$

Let $\Psi_{V,\lambda}(t) : [\delta, T] \to \mathbb{R}$ be given by $\Psi_{V,\lambda}(t) = \psi_\lambda(u_{V,\lambda}(t)) - \varphi_2(u_{V,\lambda}(t))$. For almost all $t \in [\delta, T]$, we can write

$$\begin{aligned}
\frac{d}{dt}\Psi_{V,\lambda}(t) &= \langle\nabla\psi_\lambda(u_{V,\lambda}(t)) - \xi_\lambda(t), -(\nabla\psi_\lambda(u_{V,\lambda}(t)) + \xi_\lambda(t))\rangle \\
&= \|\xi_\lambda(t)\|^2 - \|(\nabla\psi_\lambda(u_{V,\lambda}(t))\|^2 \\
&\geq \|\partial\varphi_2(u_{V,\lambda}(t))^\circ\|^2 - \|\partial\varphi_1(u_{V,\lambda}(t))^\circ\|^2 = 0.
\end{aligned}$$

Thus, $\Psi_{V,\lambda}$ is nondecreasing, which yields that $\Psi_{V,\lambda}(\delta) \leq \Psi_{V,\lambda}(T)$. It is not hard to realize, thanks to the convergences established by Lemma 4.1 and by noting that $u_V = u\big|_{[\delta,T]}$, that

$$\Psi_{V,\lambda}(\delta) \xrightarrow{\lambda} \Psi(\delta) \quad \text{and} \quad \Psi_{V,\lambda}(T) \xrightarrow{\lambda} \Psi(T).$$

This yields that

$$\Psi(\delta) \leq \Psi(T).$$

Since $0 < \delta < T$ were arbitrary, Ψ is nondecreasing, as we wanted to prove. Now, we can write

$$c = \Psi(0) = \lim_{\delta\to 0} \Psi(\delta) \leq \lim_{T\to+\infty} \Psi(T) = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2,$$

which is a contradiction. We have shown then that

$$c \leq \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2.$$

Finally, if we assume that $\inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2 > c$, we can replicate the same reasoning by exchanging the roles of φ_1 and φ_2 to prove that $\inf_{\overline{\Omega}} \varphi_2 - \inf_{\overline{\Omega}} \varphi_1 \leq -c$. Therefore, $c = \inf_{\overline{\Omega}} \varphi_1 - \inf_{\overline{\Omega}} \varphi_2$, which finishes the proof. □

It would be interesting to know whether hypothesis (i) in Theorem 4.1 can be dropped or not, even in finite dimension. The fact that the function φ_1 is finite in Ω

is used for two steeps: (1) to apply Lemma 4.1 in Ω and; (2) to apply the Dirichlet boundary condition whenever a trajectory hits the boundary bd Ω.

If we drop the hypothesis of finiteness in Ω we will deal with new difficulties that are not obvious to overcome. At least in finite dimensional spaces, Lemma 4.1 could possibly still be applied in the relative interior of the new set $\Theta := \Omega \cap \operatorname{dom} \varphi_1$. However, the Dirichlet boundary condition that holds in bd Ω doesn't necessarily holds in the new boundary bd Θ, and therefore, we would have two kinds of boundary points $x \in \operatorname{bd} \Theta$: Either x is also a boundary point of Ω and therefore it inherits the Dirichlet boundary condition $\varphi_1(x) = \varphi_2(x) + c$; or x is an interior point of Ω (and a boundary point of dom φ_1), and thus it verifies the Neumann boundary condition $\|\partial\varphi_1(x)^\circ\| = \|\partial\varphi_2(x)^\circ\|$. Up to now, we don't know how to deal with this situation. This problem remains open.

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