Taylor & Francis
Taylor & Francis Group

Optimization
A Journal of Mathematical Programming and Operations Research

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gopt20

Existence of quasi-equilibria on unbounded constraint sets

John Cotrina , Abderrahim Hantoute & Anton Svensson

To cite this article: John Cotrina , Abderrahim Hantoute & Anton Svensson (2020): Existence of quasi-equilibria on unbounded constraint sets, Optimization, DOI: 10.1080/02331934.2020.1778690

To link to this article: https://doi.org/10.1080/02331934.2020.1778690

Published online: 16 Jun 2020.

Submit your article to this journal

Article views: 49

View related articles

View Crossmark data

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=gopt20

OPTIMIZATION
https://doi.org/10.1080/02331934.2020.1778690

Taylor & Francis
Taylor & Francis Group

Check for updates

Existence of quasi-equilibria on unbounded constraint sets

John Cotrina [a], Abderrahim Hantoute[b,c] and Anton Svensson[d,e]

[a]Universidad del Pacífico, Lima, Perú; [b]Center for Mathematical Modeling (CMM), Santiago, Chile; [c]Mathematics Department, Universidad de Alicante, Alicante, Spain; [d]Mathematics Department, Universidad de Chile, Santiago, Chile; [e]Lab. PROMES UPR CNRS 8521, University of Perpignan, Perpignan, France

ABSTRACT

A quasi-equilibrium problem is an equilibrium problem where the constraint set does depend on the reference point. It generalizes important problems such as quasi-variational inequalities and generalized Nash equilibrium problems. We study the existence of equilibria on unbounded sets under a coerciveness condition. We discuss the relation of our results with others from the literature.

ARTICLE HISTORY
Received 10 February 2020
Accepted 24 April 2020

KEYWORDS
Coerciveness condition; generalized monotonicity; generalized convexity; quasi-equilibrium problems

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
49J40; 49J45; 54C60; 90C37

1. Introduction

Given a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a constraint set $C \subset \mathbb{R}^n$, the standard *equilibrium problem*, (EP) for short, as introduced by Blum and Oettli [1], consists of finding a point $x_0 \in C$ such that

$$f(x_0, y) \geq 0 \quad \text{for all } y \in C. \tag{1}$$

We consider next the *quasi-equilibrium problem*, (QEP) for short, which is an equilibrium problem but where the constraint set depends on the currently analysed point. More precisely, given a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a set-valued map $K : C \rightrightarrows C$, where C is a non-empty subset of $\mathbb{R}^n$, the QEP consists of finding

$$x_0 \in C \text{ such that } x_0 \in K(x_0), \quad \text{and} \quad f(x_0, y) \geq 0 \quad \text{for all } \; y \in K(x_0). \tag{2}$$

By QEP(f, K), we denote the solution set of problem (2).

Quasi-equilibrium problems have captured the attention of many researchers recently, since these problems summarize in a unified manner several particular classes of problems such as quasi-variational inequalities, generalized Nash

CONTACT John Cotrina cotrina_je@up.edu.pe

© 2020 Informa UK Limited, trading as Taylor & Francis Group

equilibrium problems, among others (see e.g. [2,3]). In particular, the difficult analysis of the existence of solutions can be carried out in the general setting of QEP and then, one can directly deduce the corresponding results for particular classes of problems.

In the case of a bounded (compact) set C, the first results in this setting require the continuity of the constraint set-valued map K, see [2,4]. The lower semi-continuity of K is relaxed in [5], while a version relaxing instead of the upper semi-continuity of K is provided in [6]. Ideas of the later reference were next used in [7,8] in order to generalize the famous Ky Fan minimax inequality. It is also worth recalling that [8] deals with a non-compact set C, though the constraint maps have compact values.

In the case of unbounded (hence, non-compact) sets, classical existence results for EP usually involve a coerciveness condition, see [9–11]. The QEP case with unbounded constraint sets was first studied in [12–14] under somewhat restrictive continuity conditions on the map K. Indeed, in [13,14], the continuity of both the bifunction and the constraints map were assumed, while in [12] the lower semi-continuity of K is used together with the upper semi-continuity of f.

Our aim in this work is to provide some general existence results for QEP on an unbounded constraint set, under a coerciveness-like condition [15]. In Section 2, we give the basic and classical notions on generalized convexity, generalized monotonicity, continuity for set-valued maps, among others, that are used in the sequel. The main result is given in Section 3, providing the existence result of quasi-equilibria under possibly unbounded constraint sets. Finally, in Section 4, we consider applications to quasi-variational inequality and generalized Nash equilibrium problems.

2. Preliminaries and basic results

Let S be a subset of $\mathbb{R}^n$. The convex hull, the closure and the interior of S are denoted by $\mathrm{co}(S)$, $\overline{S}$ and $\mathrm{int}(S)$, respectively. We denote the open and the closed balls in $\mathbb{R}^n$ with centre 0 and radius $\varepsilon > 0$ by B_ε and $\overline{B}_\varepsilon$, respectively.

We recall some classical definitions of generalized convexity. A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be

- *convex* if, for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, we have
$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y);$$
- *quasi-convex* if, for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, we have
$$h(tx + (1-t)y) \leq \max\{h(x), h(y)\};$$
- *semi-strictly quasi-convex at level* $\alpha \in \mathbb{R}$ if, for any $x, y \in \mathbb{R}^n$ such that $h(x) \leq \alpha$ and $h(y) < \alpha$, the following holds:
$$h(tx + (1-t)y) < \alpha \quad \text{for all } t \in]0, 1[.$$

- *semi-strictly quasi-convex* if it is semi-strictly quasi-convex at every level $\alpha \in \mathbb{R}$.

The definition above of semi-strictly quasi-convexity is equivalent to the usual one; that is, h is quasi-convex and, for any $x, y \in \mathbb{R}^n$ such that $h(x) \neq h(y)$, the following holds:

$$h(tx + (1-t)y) < \max\{h(x), h(y)\} \quad \text{for all } \ t \in]0,1[.$$

It is clear that every convex function is semi-strictly quasi-convex. A relevant and useful characterization of quasi-convexity is that a function is quasi-convex if and only if its sub-level sets are convex. A reference for quasi-convex functions and quasi-convex optimization is [16].

Let $K : X \rightrightarrows Y$ be a set-valued map with X and Y topological spaces. The map K is called:

- *closed* if its graph is a closed subset of $X \times Y$,
- *lower semi-continuous* (lsc, for short) at x_0, if for each open set V satisfying $K(x_0) \cap V \neq \emptyset$ there exists a neighbourhood U of x_0 such that $K(x) \cap V \neq \emptyset$ for all $x \in U$,
- *upper semi-continuous* (usc, for short) at x_0, if for each open set $V \supset K(x_0)$ there exists a neighbourhood U of x_0 such that $K(x) \subset V$ for all $x \in U$,
- lsc (usc), if it is lsc (usc) at every point of X,
- continuous, if it is lsc and usc.

The usual definition of lower semi-continuity of a set-valued map using sequences/nets is equivalent to the one given here using open sets (see, for instance, [17, Proposition 2.5.6]).

We present now some basic results on the lower semi-continuity of certain operations on set-valued maps.

Lemma 2.1: *Let X, Y be topological spaces, $T : X \rightrightarrows Y$ a set-valued map, and V an open subset of Y. If T is lsc at $x_0 \in X$, then the set-valued map $T_V : X \rightrightarrows Y$ defined by*

$$T_V(x) := T(x) \cap V \tag{3}$$

is also lsc at x_0.

Proof: This is an immediate consequence of [18, Lemma 2.2.5]. ■

Lemma 2.2: *Let X, Y and T be as in Lemma 2.1. Assume that T is lsc at $x_0 \in X$, and let a set-valued map $S : X \rightrightarrows Y$ be such that $S(x_0) \subset \overline{T(x_0)}$ and*

$$T(x) \subset S(x) \quad \forall\, x \in X.$$

Then, S is lsc at x_0.

Proof: Let V be an open subset of Y such that $S(x_0) \cap V \neq \emptyset$. Clearly, $\overline{T(x_0)} \cap V \neq \emptyset$, and we deduce that $T(x_0) \cap V \neq \emptyset$. Thus, by the lower semi-continuity of T there exists a neighbourhood U of x_0 such that $\emptyset \neq T(x) \cap V \subset S(x) \cap V$, for all $x \in U$. ■

Lemma 2.3: *Let X and T be as in Lemma 2.1, Y a topological vector space, and V an open convex subset of Y. Let $x_0 \in X$ such that $T(x_0) \cap V \neq \emptyset$. If T is lsc at x_0 and $T(x_0)$ is convex, then the set-valued map $T_{\overline{V}}$, defined similarly as in (3), is lsc at x_0.*

Proof: The set-valued map T_V, which is lsc at x_0 by Lemma 2.1, satisfies $T_V(x) \subset T_{\overline{V}}(x)$ for all $x \in X$ and, due to [19, Proposition 1.1 of §2] the following holds:

$$T_{\overline{V}}(x_0) \subset \overline{T(x_0)} \cap \overline{V} = \overline{T_V(x_0)}.$$

Thus, $T_{\overline{V}}$ is lsc at x_0 thanks to Lemma 2.2. ■

Remark 2.4: Lemma 2.3 is a slight refinement of [20, Lemma 1], since we do not need T to have closed values, nor $T_{\overline{V}}$ to have values with non-empty interior. Note that the last requirement excludes, for instance, single-valued maps.

The following lemma can be easily proved (see [21, Lemma 2.3]).

Lemma 2.5: *Let X, Y and T be as in Lemma 2.1, A a closed subset of X, and $S : A \rightrightarrows Y$ a set-valued map. We define the set-valued map $M : X \rightrightarrows Y$ as*

$$M(x) := \begin{cases} T(x) & \text{if } x \in X \setminus A, \\ S(x) & \text{if } x \in A. \end{cases}$$

If S, T are lsc and $S(x) \subset T(x)$, for all $x \in A$, then M is lsc.

The following result is [22, Theorem 5.9(c)].

Lemma 2.6: *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lsc at $x_0 \in \mathbb{R}^n$, then so is the set-valued map $\operatorname{co}(T) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined as*

$$\operatorname{co}(T)(x) := \operatorname{co}(T(x)).$$

The following is a consequence of Himmelberg's fixed point and Michael's selection theorems. Recall that for a set-valued map $T : C \subset \mathbb{R}^n \rightrightarrows C$, $\operatorname{Fix}(T)$ is the set of fixed points of T; that is, $x \in C$ with $x \in T(x)$.

Proposition 2.7 ([8, Corollary 1]): *Given a non-empty, convex and closed subset C of $\mathbb{R}^n$, if $T : C \rightrightarrows C$ is lsc with non-empty and convex values and $T(C)$ is bounded, then $\operatorname{Fix}(T) \neq \emptyset$.*

Given a set-valued map $T : X \rightrightarrows Y$, between sets X, Y, the *fibre* of T at $y \in Y$ is the set

$$T^{-1}(y) := \{x \in X : \ y \in T(x)\}.$$

The following result is a particular case of [23, Theorem 5] (see, also [5, Theorem 4 of §5]).

Proposition 2.8: *Let $C \subset \mathbb{R}^n$ be a compact, convex and non-empty set, and let $S, T : C \rightrightarrows C$ be set-valued maps such that*

(1) *S is usc with convex, compact and non-empty values,*
(2) *T is convex-valued with open fibres and* $\mathrm{Fix}(T) = \emptyset$,
(3) *the set $V := \{x \in C : \ S(x) \cap T(x) \neq \emptyset\}$ is open in C.*

Then there exists $x \in \mathrm{Fix}(S)$ such that $S(x) \cap T(x) = \emptyset$.

We recall now some notions of *generalized monotonicity* that will be used in the sequel. A bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said:

- to be *pseudo-monotone* on a subset C of $\mathbb{R}^n$ if the following implication holds, for all $x, y \in C$,

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

- to be *properly quasi-monotone* on a convex subset C of $\mathbb{R}^n$ if, for all $m \geq 1$, $x_1, \ldots, x_m \in C$ and $x \in \mathrm{co}\{x_1, \ldots, x_m\}$, we have

$$\min_{i=1,\ldots,m} f(x_i, x) \leq 0;$$

- to have the *upper sign property* on a convex subset C of $\mathbb{R}^n$ if, for all $x_0, x_1 \in C$, the following implication holds (see [24]):

$$\big(f(x_t, x_0) \leq 0, \ \forall\, t \in]0,1[\big) \Rightarrow f(x_0, x_1) \geq 0, \tag{4}$$

where $x_t := tx_1 + (1-t)x_0$.

The above notions, namely the first two, are inspired from similar properties for set-valued maps (see [25]), but they are not comparable in general (see the examples in [9]). However, in [8], the authors showed that the upper sign property of f is equivalent to the pseudo-monotonicity of $-f$ under suitable assumptions.

3. Main results

In this section, given a bifunction $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, a non-empty subset C of $\mathbb{R}^n$, and a set-valued map $K: C\rightrightarrows C$, we consider the following standard hypothesis:

$$(\mathcal{H})\begin{cases} C & \text{is closed and convex,}\\ K & \text{has non-empty convex values.}\end{cases}$$

We introduce below a coercivity-like condition, which has been used in [15, Theorem 3.7] and recently in [26]. First coercivity conditions for (EP) problems on unbounded constraint sets were given in [1]. We also refer to [27] for the so-called Karamardian condition, which is frequently used in the study of complementarity and variational inequality problems. For a detailed study of these conditions and many others, we refer to [28] (and references therein). See, also, [29], for conditions involving the position of the recession cones of the constraint set and appropriate sub-levels of the bifunction f.

Definition 3.1: We say that f and K satisfy the *uniform coercivity condition* (UCC, for short) at $\rho>0$ if:

(1) $K(x)\cap B_\rho\neq\emptyset$, for all $x\in C\cap\overline{B}_\rho$,
(2) for each $x\in\mathrm{Fix}(K)$ with $\|x\|=\rho$, there exists $y\in K(x)$ such that $\|y\|<\rho$ and $f(x,y)\leq 0$.

Remark 3.2: If $\mathrm{Fix}(K)$ is countable and part (1) of UCC is satisfied at some $\rho>0$, then there exists $\rho'>\rho$ such that the full statement of UCC is satisfied at ρ' (see [26, Theorem 4]).

Given $\rho>0$, we define the set-valued map $K_\rho: C\cap\overline{B}_\rho\rightrightarrows C\cap\overline{B}_\rho$ as

$$K_\rho(x):=K(x)\cap\overline{B}_\rho. \tag{5}$$

The following proposition, which is an extension of [10, Lemma 2.2], provides conditions under which we have $\mathrm{QEP}(f,K_\rho)\subset\mathrm{QEP}(f,K)$ for an appropriate $\rho>0$.

Proposition 3.3: *We assume that f and K satisfy $(\mathcal{H})$ and UCC at some $\rho>0$. If $x_0\in\mathrm{Fix}(K_\rho)$ is such that $f(x_0,x_0)\leq 0$, $f(x_0,\cdot)$ is semi-strictly quasi-convex at level* 0, *and*

$$f(x_0,y)\geq 0 \quad \textit{for all } y\in K(x_0)\cap B_\rho,$$

then $x_0\in\mathrm{QEP}(f,K)$.

Proof: If $x_0 \notin \mathrm{QEP}(f, K)$, then there would exists $y_0 \in K(x_0)$ such that $f(x_0, y_0) < 0$. Since $f(x_0, x_0) \le 0$, by the semi-strictly quasi-convexity of $f(x_0, \cdot)$ at level 0 we have that

$$f(x_0, y_t) < 0 \quad \text{for all } t \in]0, 1[,$$

where $y_t := (1-t)x_0 + ty_0$. If $\|x_0\| < \rho$, then for t close enough to 0, we would have that $y_t \in K(x_0) \cap B_\rho$ and $f(x_0, y_t) < 0$, which is a contradiction with our assumption. Otherwise, if $\|x_0\| = \rho$, then by UCC there exists $y_1 \in K(x_0) \cap B_\rho$ such that $f(x_0, y_1) \le 0$. Then, by proceeding as above we find an element $z_t := (1-t)y_1 + ty_0$, for small $t \in]0, 1[$, which yields the contradiction $f(x_0, z_t) < 0$. ■

Theorem 3.4: *We assume that f and K satisfy* $(\mathcal{H})$ *and UCC at some large* $\rho > 0$, *K is lsc,* $\mathrm{Fix}(K)$ *is closed, and* $f(x, \cdot)$ *is semi-strictly quasi-convex at level* 0, *for every* $x \in \mathrm{Fix}(K)$. *If f is properly quasi-monotone, has the upper sign property on C, and the set-valued map* $G : \mathrm{Fix}(K) \rightrightarrows C$ *defined as*

$$G(x) := \{y \in K(x) \cap \overline{B}_\rho : f(y, x) > 0\}$$

is lsc, then $\mathrm{QEP}(f, K)$ *is non-empty.*

Proof: We may assume that ρ is sufficiently large so that $C_\rho := C \cap \overline{B}_\rho \neq \emptyset$. Then, by UCC,

$$K(x) \cap B_\rho \neq \emptyset \quad \text{for all } x \in C_\rho,$$

and so the map K_ρ defined in (5) has non-empty and convex values. Moreover, due to Lemma 2.3, the relation above also ensures that K_ρ is lsc. Next, we define the set-valued map $M : C_\rho \rightrightarrows C_\rho$ by

$$M(x) := \begin{cases} K_\rho(x), & x \in C_\rho \setminus \mathrm{Fix}(K_\rho), \\ \mathrm{co}(G(x)), & x \in \mathrm{Fix}(K_\rho), \end{cases}$$

which is lsc due to Lemmas 2.5 and 2.6. The map M does not have any fixed point. In fact, every fixed point x of M is also a fixed point of K_ρ, and hence a fixed point of $\mathrm{co}(G)$; that is, $x \in \mathrm{co}\{x_i,\ i = 1, \ldots, k\}$ for some $x_i \in G(x)$. Hence, $\min_{i=1\cdots k} f(x_i, x) > 0$ and this contradicts the proper quasi-monotonicity of f.

Now, since the lsc map M has convex values and $M(C_\rho) \subset \overline{B}_\rho$, by Proposition 2.7 there exists $x_0 \in C_\rho$ such that $M(x_0) = \emptyset$. Thus, $x_0 \in \mathrm{Fix}(K_\rho)$ and $G(x_0) = \emptyset$. To show that $x_0 \in \mathrm{QEP}(f, K_\rho)$, we suppose by contradiction that $f(x_0, y) < 0$ for some $y \in K_\rho(x_0)$. Then, the upper sign property yields some $t \in]0, 1[$ such that

$$f(ty + (1-t)x_0, x_0) > 0;$$

that is, $ty + (1-t)x_0 \in G(x_0)$, a contradiction. Finally, since $f(x_0, x_0) \le 0$ by the proper quasi-monotonicity of f, by Proposition 3.3 we infer that $x_0 \in \mathrm{QEP}(f, K)$. ■

Since UCC holds at a sufficiently large ρ when C is compact, we obtain the following result.

Corollary 3.5: *Let C be a non-empty, compact and convex subset of $\mathbb{R}^n$ and assume that f is properly quasi-monotone, semi-strictly quasi-convex at level* 0 *in the second argument, and has the upper sign property. If the set*

$$\{y \in C : f(x,y) \leq 0\}$$

is closed, for each $x \in C$, then problem (1) *has a solution.*

Proof: First, the constant set-valued map $K(x) := C$, $x \in C$, is obviously lsc and has convex and non-empty values. Also, we have that $\mathrm{Fix}(K) = C$, which is obviously closed. Then condition UCC trivially holds, as well as hypothesis $(\mathcal{H})$. According to Theorem 3.4, it suffices to show that the map G, defined in Theorem 3.4, is lsc. Indeed, by the current assumption, for each $y \in C$, the fibre

$$G^{-1}(y) = \{x \in C : f(y,x) > 0\}$$

is open, and this easily implies the lower semi-continuity of G. ■

Corollary 3.5 is given in [10, Proposition 2.1], where instead of the upper sign property of f, the authors assume the quasi-convexity in the second argument of f together with the condition

$$f(x,x) = 0 \quad \forall\, x \in C,$$

as well as the *upper sign continuity* (see [10]) of f; that is,

$$\inf_{t \in]0,1[} f(tx + (1-t)y, y) \geq 0 \;\Rightarrow\; f(x,y) \geq 0 \quad \forall\, x,\, y \in C.$$

It is known that the last three conditions ensure the upper sign property of f (see [30, Lemma 3]).

Remark 3.6: It is worth recalling that, instead of the semi-strict quasi-convexity at level 0 of the bifunction f in Corollary 3.5, [10, Proposition 2.1] uses the so-called *sign preserving property*; that is, for all $x, y, z \in C$,

$$(f(x,y) = 0 \,\wedge\, f(x,z) < 0) \;\Rightarrow\; f(x, ty + (1-t)z) < 0 \quad \text{for all } t \in]0,1[\,.$$

We observe that, under the quasi-convexity of the functions $f(x,\cdot)$, $x \in C$, both the sign preserving property and the semi-strict quasi-convexity at level 0 are equivalent.

Corollary 3.7 ([4, Theorem 4.5]): *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction, C be a convex, compact and non-empty subset of $\mathbb{R}^n$, and $K : C \rightrightarrows C$ be a set-valued map. Suppose that the following properties hold:*

(1) *K is closed and lsc with convex values, and* $\operatorname{int}(K(x)) \neq \emptyset$, *for all* $x \in C$;
(2) *f is properly quasi-monotone*;
(3) *f is semi-strictly quasi-convex and lsc with respect to its second argument*;
(4) *for all $x, y \in \mathbb{R}^n$ and all sequence $(y_k)_k \subset \mathbb{R}^n$ converging to y, the following implication holds:*

$$\liminf_{k \to +\infty} f(y_k, x) \leq 0 \;\Rightarrow\; f(y, x) \leq 0;$$

(5) *f has the upper sign property.*

Then $\mathrm{QEP}(f, K)$ *is non-empty.*

Proof: Since C is compact, the set-valued map G in Theorem 3.4 can be described by $G(x) = \{y \in K(x) : f(y, x) > 0\}$ for every $x \in \mathrm{Fix}(K)$. We can prove the lower semi-continuity of G following the same steps of the proof of [8, Corollary 7], and thus the conclusion follows by applying Theorem 3.4. ■

Now, we present another existence result without generalized monotonicity.

Theorem 3.8: *We assume that f and K satisfy $(\mathcal{H})$ and UCC at some large $\rho > 0$, K is lsc,* $\mathrm{Fix}(K)$ *is closed, and $f(x, \cdot)$ is semi-strictly quasi-convex at level* 0 *for every* $x \in \mathrm{Fix}(K)$. *If* $f(x, x) = 0$, *for all* $x \in \mathrm{Fix}(K)$, *and the set-valued map* $R : \mathrm{Fix}(K) \rightrightarrows C$ *defined as*

$$R(x) := \{y \in K(x) : f(x, y) < 0\}$$

is lsc, then $\mathrm{QEP}(f, K)$ *is non-empty.*

Proof: We consider $R_\rho : \mathrm{Fix}(K_\rho) \rightrightarrows C$ defined as $R_\rho(x) := R(x) \cap B_\rho$, which is lsc (Lemma 2.1) with convex values. Thus, the set-valued map $M : C \rightrightarrows C$ defined as

$$M(x) := \begin{cases} K_\rho(x) & x \in C \setminus \mathrm{Fix}(K_\rho), \\ R_\rho(x) & x \in \mathrm{Fix}(K_\rho) \end{cases}$$

is lsc with convex values. If M is non-empty valued, then by Proposition 2.7 there exists $x_0 \in M(x_0)$; that is $x_0 \in \mathrm{Fix}(K_\rho)$ and $x_0 \in R_\rho(x_0)$, and this implies the contradiction $f(x_0, x_0) < 0$. Hence, there exists $x_0 \in C$ such that $M(x_0) = \emptyset$. Thus, $x_0 \in \mathrm{Fix}(K_\rho)$ and $R_\rho(x_0) = \emptyset$, i.e.

$$f(x_0, y) \geq 0 \quad \text{for all } \; y \in K(x_0) \cap B_\rho.$$

The last inequality can be extended to $K_\rho(x_0)$, we deduce that $x_0 \in \mathrm{QEP}(f, K_\rho)$. Finally, we conclude by applying Proposition 3.3. ■

The previous result has some similarities with [26, Theorem 3]. However, Theorem 3.8 does not require the upper semi-continuity of the bifunction.

The following simple example shows that Theorem 3.8 is independent of [26, Theorem 3].

Example 3.9: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the bifunction defined as

$$f(x,y) := \begin{cases} 0, & x = 1/2, \\ -2, & x = 0 \ \vee \ x = 1, \\ -1, & \text{otherwise.} \end{cases}$$

It is clear that f is semi-strictly quasi-convex in its second argument and is not upper semi-continuous in its first one. Consider the set $C = [0,1]$ and the constant set-valued map $K(x) := C$, $x \in C$. Thus, the set-valued map $R : C \rightrightarrows C$, defined in Theorem 3.8, is

$$R(x) = \begin{cases} [0,1], & x \neq 1/2, \\ \emptyset, & x = 1/2, \end{cases}$$

which is lsc. Since C is compact, we have that UCC holds at a sufficiently large ρ. Hence, by Theorem 3.8, the problem (1) admits at least one solution.

The next theorem can be considered a non-compact version of [5, Theorem 5 of §5], in the finite-dimensional setting.

Theorem 3.10: *Let C be a convex, closed and non-empty subset of $\mathbb{R}^n$, $K : C \rightrightarrows C$ be a set-valued map and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. Assume that f and K satisfy UCC at sufficiently large $\rho > 0$, and the following properties hold:*

(1) *K is closed with convex and non-empty values,*
(2) *$f(\cdot, y)$ is usc, for all $y \in C$,*
(3) *$f(x, \cdot)$ is quasi-convex, for all $x \in C$,*
(4) *the set $D = \{x \in C \cap \overline{B}_\rho : \inf_{y \in K(x) \cap \overline{B}_\rho} f(x,y) < 0\}$ is open in $C \cap \overline{B}_\rho$,*
(5) *$f(x,x) = 0$ for all $x \in C$,*
(6) *for each $x \in \mathrm{Fix}(K)$, $f(x, \cdot)$ is semi-strictly quasi-convex at level 0.*

Then $\mathrm{QEP}(f,K)$ is non-empty.

Proof: Consider the set $C_\rho := C \cap \overline{B}_\rho$ and the set-valued maps $K_\rho, T : C_\rho \rightrightarrows C_\rho$ defined as

$$K_\rho(x) := K(x) \cap \overline{B}_\rho \quad \text{and} \quad T(x) := \{y \in K_\rho(x) : f(x,y) < 0\}.$$

Clearly, $\mathrm{gph}(K_\rho) = \mathrm{gph}(K) \cap (C_\rho \times C_\rho)$ and $D = \{x \in C_\rho : \ T(x) \cap K_\rho(x) \neq \emptyset\}$. Condition UCC at ρ and assumption (1) imply that K_ρ is usc with convex, compact and non-empty values. Assumption (3) implies that T is convex-valued while assumption (2) implies that T has open fibres. Since f vanishes on the diagonal on $C \times C$, we deduce that $\mathrm{Fix}(T) = \emptyset$. Hence, by Proposition 2.8 there exists $x \in \mathrm{Fix}(K_\rho)$ such that $K_\rho(x) \cap T(x) = \emptyset$, which means that $x \in \mathrm{QEP}(f, K_\rho)$. The conclusion follows by applying Proposition 3.3. ∎

Our Theorem 3.10 has some similarities with [12, Theorem 3], but the set of assumptions in both results differ in two important aspects. Firstly, in [12] it was assumed that f is *0-diagonally convex* on the second variable, while in our case we assume that f is quasi-convex in its second argument and that f vanishes on the diagonal of $C \times C$. Examples in [31] show that these assumptions are not comparable in general. Secondly, there is a difference on the coerciveness conditions. In [12], the authors considered a quite restrictive coerciveness condition, which in particular implies that in a non-empty set, the images of K are compact.

The following corollary is related to [13, Theorem 3], where a slightly less general kind of 'quasi-equilibrium problem' was considered. Our condition 3 in the corollary is a consequence of this restriction.

Corollary 3.11: *Let C be a compact, convex and non-empty subset of $\mathbb{R}^n$, $K, K_C : C \rightrightarrows \mathbb{R}^n$ be set-valued maps such that $K_C(x) = K(x) \cap C$, and $f : C \times C \to \mathbb{R}$ be a bifunction. If the following assumptions hold*:

(1) *K_C is continuous with convex, compact and non-empty values,*
(2) *f is continuous and $f(x, \cdot)$ is convex, for all $x \in C$,*
(3) *$f(x, x) = 0$, for all $x \in C$,*
(4) *for each $x \in \mathrm{Fix}(K_C)$, there exists $y \in K_C(x)$ such that $f(x, y) \leq 0$ and $]y, z] \cap K_C(x) \neq \emptyset$, for all $z \in K(x) \setminus K_C(x)$,*

then $\mathrm{QEP}(f, K)$ *is non-empty.*

Proof: The set $\mathrm{QEP}(f, K_C)$ is non-empty, due to Theorem 3.10. Indeed, UCC as well as conditions (1)–(3) in Theorem 3.10 are obviously satisfied. To check condition (4) of Theorem 3.10 we first observe that the set D defined there is now given as, by choosing a sufficiently large number $\rho > 0$ such that $C \subset B_\rho$,

$$D = \left\{ x \in C : \inf_{y \in K_C(x)} f(x, y) < 0 \right\}.$$

To verify that D is open in C we take a sequence $(x_n)_n \subset C \setminus D$ that converges to some $x \in C$. Next, for each $y \in K(x) \cap C$, the continuity assumption on K_C gives rise to some sequence $(y_n)_n \subset C$ converging to y and such that $y_n \in K(x_n)$ for all n. Hence, $f(x_n, y_n) \geq 0$, for all n, and the continuity of f yields $f(x, y) \geq 0$. Since y is arbitrary in $K(x) \cap C$, we deduce that $\inf_{y \in K_C(x)} f(x, y) \geq 0$; that is, $x \in C \setminus D$, and the set D is open in C.

Finally, the conclusion of the corollary follows since assumption (4) implies that $\mathrm{QEP}(f, K_C) \subset \mathrm{QEP}(f, K)$. ∎

4. Applications

In this section, we consider applications on the study of existence of solutions for two well-known problems: (i) the quasi-variational inequality problem, and (ii) the generalized Nash equilibrium problem.

4.1. Quasi-variational inequality problem

Given a subset C of $\mathbb{R}^n$ and set-valued maps $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows C$, the set $\mathrm{QVI}(T, K)$ denotes the solution set of the *quasi-variational inequality problem* associated to T and K,

$$\{x \in C : \ x \in K(x) \ \text{and exists} \ x^* \in T(x) \ \text{such that} \ \langle x^*, y - x\rangle \geq 0, \forall\, y \in K(x)\}.$$

We say that T and K satisfy the *uniform coerciveness condition* at ρ if the following two conditions hold:

(1) $K(x) \cap B_\rho \neq \emptyset$, for all $x \in C \cap \overline{B}_\rho$,
(2) for each $x \in \mathrm{Fix}(K)$ such that $\|x\| = \rho$ there exists $y \in K(x)$ with $\|y\| < \rho$ such that $\left\langle x^*, y - x \right\rangle \leq 0$ for every $x^* \in T(x)$.

Now, we consider the bifunction $f_T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined as

$$f_T(x, y) := \sup_{x^* \in T(x)} \langle x^*, y - x\rangle. \tag{6}$$

Lemma 4.1: *Given any $\rho > 0$, T and K satisfy the uniform coerciveness condition at ρ if and only if f_T and K satisfies the UCC at ρ. Moreover, if T has non-empty and compact values, then* $\mathrm{QEP}(f_T, K) = \mathrm{QVI}(T, K)$.

Proof: Direct from the definition of f_T. ■

As a direct consequence of Lemma 4.1 and Theorem 3.4, we obtain the following existence result for quasi-variational inequality problems.

Theorem 4.2: *Let C be a closed, convex and non-empty subset of $\mathbb{R}^n$, and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $K : C \rightrightarrows C$ be set-valued maps. Assume that T and K satisfy the uniform coerciveness condition at a sufficiently large $\rho > 0$, and that the following conditions are satisfied*:

(1) *T has compact and non-empty values,*
(2) *T is properly quasi-monotone on C, i.e. for all $x_1, \ldots, x_m \in C$ and any $x \in \mathrm{co}(\{x_1, \ldots, x_m\})$, there exists i such that*

$$\langle x_i^*, x - x_i\rangle \leq 0, \quad \textit{for all } x_i^* \in T(x_i),$$

(3) *T is upper sign-continuous on C; that is, for all* $x, y \in C$

$$\left(\forall t \in]0,1[,\ \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0\right) \ \Rightarrow \ \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0,$$

where $x_t = tx + (1-t)y$,

(4) *K is lsc with convex and non-empty values,*

(5) *the set* $\mathrm{Fix}(K)$ *is closed and the set-valued map* $G : \mathrm{Fix}(K) \rightrightarrows C$ *defined as*

$$G(x) := \left\{ y \in K(x) \cap \overline{B}_\rho : \sup_{x^* \in T(x)} \langle x^*, y - x\rangle > 0 \right\} \tag{7}$$

is lsc.

Then, $\mathrm{QVI}(T, K)$ *is non-empty.*

Proof: Clearly, f_T is properly quasi-monotone and has the upper sign property. Therefore, the result follows from the fact $\mathrm{QVI}(T, K) = \mathrm{QEP}(f_T, K)$ and Theorem 3.4. ■

Remark 4.3: A few remarks about Theorem 4.2:

(1) The previous result is not a consequence of [20, Theorem 1], because T here is properly quasi-monotone (not pseudo-monotone) and the closedness of K is relaxed to the closedness of $\mathrm{Fix}(K)$. Aussel and Sultana [20, Theorem 3] proposes an existence result under quasi-monotonicity, which means that for all (x, x^*) and (y, y^*) in the graph of T the following implication holds:

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*,\ y - x\rangle \geq 0.$$

Observe that in this case one needs more regularity assumptions on the constraint map.

(2) Condition (5) in Theorem 4.2 holds, for instance, when the map K is continuous and the set

$$\left\{ (x, y) \in C \times C : \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \leq 0 \right\} \tag{8}$$

is closed. Indeed, the closedness of K leads easily to the closedness of the set $\mathrm{Fix}(K)$. To check that the set-valued map G defined in (7) is lsc, say at some $x \in \mathrm{Fix}(K)$, we proceed by contradiction and assume that for some open set $V \subset \mathbb{R}^n$ and a sequence $(x_n) \subset C$ converging to x we have that $G(x) \cap V \neq \emptyset$ and $G(x_n) \cap V = \emptyset$ for all n. Take $y \in G(x) \cap V \subset K(x) \cap \overline{B}_\rho \cap V$.

Since K is assumed continuous, we choose a sequence $(y_n)_n \subset C$ such that $y_n \in K(x_n) \cap \overline{B}_\rho \cap V$ and $y_n \to y$. But $G(x_n) \cap V = \emptyset$, and so

$$\sup_{x^* \in T(x_n)} \langle x^*, y_n - x_n \rangle \leq 0.$$

Consequently, the closedness of the set defined in (8) ensures that

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0,$$

which contradicts the fact that $y \in G(x)$.

4.2. Generalized Nash equilibrium problem

A generalized Nash equilibrium problem (GNEP) consists of p players. Each player ν controls the decision variable $x^\nu \in C_\nu$, where C_ν is a non-empty convex and closed subset of $\mathbb{R}^{n_\nu}$. We denote by $x = (x^1, \dots, x^p) \in \prod_{\nu=1}^p C_\nu = C$ the vector formed by all these decision variables and by $x^{-\nu}$, the strategy vector of all the players different from player ν. The set of all such vectors will be denoted by $C^{-\nu}$. We sometimes write $(x^\nu, x^{-\nu})$ instead of x in order to emphasize the νth player's variables within x. Note that this is still the vector $x = (x^1, \dots, x^\nu, \dots, x^p)$, and the notation $(x^\nu, x^{-\nu})$ does not mean that the block components of x are reordered in such a way that x^ν becomes the first block. Each player ν has an objective function $\theta_\nu : C \to \mathbb{R}$ that depends on all player's strategies. Each player's strategy must belong to a set identified by the set-valued map $K_\nu : C^{-\nu} \rightrightarrows C_\nu$ in the sense that the strategy space of player ν is $K_\nu(x^{-\nu})$, which depends on the rival player's strategies $x^{-\nu}$. Given the strategy $x^{-\nu}$, player ν chooses a strategy x^ν such that it solves the following optimization problem:

$$\min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), \quad \text{subject to } x^\nu \in K_\nu(x^{-\nu}), \tag{9}$$

for any given strategy vector $x^{-\nu}$ of the rival players. The solution set of problem (9) is denoted by $\mathrm{Sol}_\nu(x^{-\nu})$. Thus, a *generalized Nash equilibrium* is a vector $\hat{x}$ such that $\hat{x}^\nu \in \mathrm{Sol}_\nu(\hat{x}^{-\nu})$, for any ν.

Associated to a GNEP, there is a bifunction $f^{NI} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f^{NI}(x, y) := \sum_{\nu=1}^{p} \left\{ \theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu}) \right\},$$

which is called Nikaidô–Isoda function and was introduced in [32]. Additionally, we consider the set-valued map $K : C \rightrightarrows C$ defined as

$$K(x) := \prod_{\nu=1}^{p} K_\nu(x^{-\nu}).$$

Lemma 4.4: *A vector $\hat{x}$ is a solution of the GNEP if and only if $\hat{x} \in \mathrm{QEP}(f^{NI}, K)$.*

A GNEP satisfies the *coerciveness condition* at $\rho > 0$ if

(1) $K(x) \cap B_\rho \neq \emptyset$, for all $x \in C \cap \overline{B}_\rho$;
(2) for each $x \in \mathrm{Fix}(K)$, such that $\|x\| = \rho$ there exists $y \in K(x)$ with $\|y\| < \rho$ such that $\theta_\nu(y^\nu, x^{-\nu}) \leq \theta_\nu(x)$ for each ν.

If we consider in $\mathbb{R}^n$ the product norm given by the maximum of the norms of all the $\mathbb{R}^{n_\nu}$, then the above condition is equivalent to the following one, for each ν

(1) $K_\nu(x^{-\nu}) \cap B_{\mathbb{R}^{n_\nu},\rho} \neq \emptyset$, for all $x \in C \cap \overline{B}_\rho$;
(2) for each $x \in \mathrm{Fix}(K_\rho)$, if $\|x^\nu\|_{\mathbb{R}^{n_\nu}} = \rho$, then there exists $y^\nu \in K(x^{-\nu})$ with $\|y^\nu\|_{\mathbb{R}^{n_\nu}} < \rho$ such that $\theta_\nu(y^\nu, x^{-\nu}) \leq \theta_\nu(x)$.

Lemma 4.5: *If the GNEP satisfies the coerciveness condition at $\rho > 0$, then the pair f^{NI} and K satisfies the UCC at ρ.*

Proof: It is enough to see that if for each ν we have $\theta_\nu(y^\nu, x^{-\nu}) \leq \theta_\nu(x)$, then

$$f^{NI}(x, y) = \sum_{\nu=1}^{p} \theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x) \leq 0.$$

■

Thanks to Lemmas 4.4 and 4.5, we have the following result on the existence of solutions of a GNEP, which is a direct consequence of Theorems 3.4 and 3.10.

Theorem 4.6: *For any $\nu \in \{1, 2, \ldots, p\}$, let C_ν be a non-empty, closed and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $K_\nu : C^{-\nu} \rightrightarrows C_\nu$ be a set-valued map. Assume that the GNEP satisfies the coerciveness condition at ρ, that for each ν, θ_ν is convex with respect to the x^ν variable, and at least one of the following pair of Assumptions A1, A2 hold:*

A1 (a) *The set* $\mathrm{Fix}(K)$ *is closed.*
(b) *for each ν, the map K_ν is lower semi-continuous with non-empty and convex values.*

A2 (a) *For each ν, the map K_ν is closed with convex and non-empty values .*
(b) *The set*

$$N = \left\{ x \in C \cap \overline{B}_\rho : \inf_{y \in K(x) \cap \overline{B}_\rho} \sum_\nu \theta_\nu(y^\nu, x^{-\nu}) < \sum_\nu \theta_\nu(x) \right\}$$

is open in $C \cap \overline{B}_\rho$.

Then the GNEP admits a solution.

Proof: It is clear that f^{NI} is continuous and convex in its second argument and the map K is closed with convex and non-empty values. By Lemma 4.5, we have that f^{NI} and K satisfy the UCC at ρ. In case $A1$, K is lsc and has convex and non-empty values. Hence, the set-valued map R defined in the second case of Theorem 3.4 is also lsc and has convex values. So, the result follows from Theorem 3.4 and Lemma 4.4.

Finally, in case $A2$, the map K is closed with convex and non-empty values. Moreover, we have that

$$N = \left\{ x \in C \cap \overline{B}_\rho : \inf_{y \in K(x) \cap \overline{B}_\rho} f^{NI}(x, y) < 0 \right\}.$$

Hence, the result follows from Theorem 3.10 and Lemma 4.4. ■

The previous result is related to [20, Theorem 5]. However, we notice that in Assumption A1 the constraint set-valued maps K_ν are not necessarily closed, while for A2 K_ν are not necessarily lsc. Moreover, none of the cases assumes any differentiability, and the images of the constraint maps K_ν are allowed to have an empty interior. Finally, their 'coerciveness condition' is somehow weaker than ours. In fact, $\theta_\nu(y^\nu, x^{-\nu}) \leq \theta_\nu(x^\nu, x^{-\nu})$ clearly implies their condition $\langle \nabla_{x_\nu} \theta_\nu(x), x^\nu - y^\nu \rangle \geq 0$, due to the convexity assumption, while the converse implication is not true in general.

Acknowledgments

This work was partially supported by Cienciactiva-Concytec EE020, MATH Amsud No. 003-2017, CONICYT No. 1190012, Proyecto/Grant PIA AFB-170001, MICIU of Spain and Universidad de Alicante (Grant Beatriz Galindo BEA- GAL 18/00205), CONICYTPFCHA/Doctorado Nacional/2018-21180645, and PGMO project of FMJH. We would like to thank the reviewers for their valuable comments and suggestions which have improved the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

John Cotrina http://orcid.org/0000-0001-5034-6286

References

[1] Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems. Math Stud. 1993;63:1–23.

[2] Castellani M, Giuli M. An existence result for quasiequilibrium problems in separable Banach spaces. J Math Anal Appl. 2015;425:85–95.

[3] Cotrina J, Zúñiga J. A note on quasi-equilibrium problems. Oper Res Lett. 2018;46(1):138–140. Available from: https://www.sciencedirect.com/science/article/pii/S0167637717305138.
[4] Aussel D, Cotrina J, Iusem A. An existence result for quasi-equilibrium problems. J Convex Anal. 2017;24:55–66.
[5] Lassonde M. Minimax et point fixes. 2004. Unpublished. Available from: https://www.genconv.org/files/Kaohsiung_Lassonde_minimax.pdf.
[6] Cubiotti P. Existence of Nash equilibria for generalized games without upper semicontinuity. Int J Game Theory. 1997;26:267–273. Available from: https://doi.org/10.1007/BF01295855.
[7] Castellani M, Giuli M, Pappalardo M. A Ky Fan minimax inequality for quasiequilibria on finite-dimensional spaces. J Optim Theory Appl. 2018;179:53–64.
[8] Cotrina J, Zúñiga J. Quasi-equilibrium problems with non-self-constraint map. J Global Optim. 2019;75:177–197. Available from: https://doi.org/10.1007/s10898-019-00762-5.
[9] Bianchi M, Pini R. A note on equilibrium problems with properly quasimonotone bifunctions. J Global Optim. 2001;20:67–76.
[10] Bianchi M, Pini R. Coercivity conditions for equilibrium problems. J Optim Theory Appl. 2005 Jan;124(1):79–92. Availble from: https://doi.org/10.1007/s10957-004-6466-9.
[11] Cotrina J, García Y. Equilibrium problems: existence results and applications. Set-Valued Var Anal. 2018;26:159–177.
[12] Tian G, Zhou J. Quasi-variational inequalities without the concavity assumption. J Math Anal Appl. 1993;172:289–299.
[13] Noor MA, Oettli W. On general nonlinear complementary problems and quasi-equilibria. Le Mathematiche. 1994;49(2):313–331.
[14] Ding XP. Existence of solutions for quasi-equilibrium problems in noncompact topological spaces. Comput Math Appl. 2000;39:13–21.
[15] Cubiotti P. Existence of solutions for lower semicontinuous quasi-equilibrium problems. Comput Math Appl. 1995;30(12):11–22.
[16] Aussel D. New developments in quasiconvex optimization. Fixed point theory, variational analysis, and optimization. New York: Taylor & Francis; 2014. p. 173–208.
[17] Göpfert A, Riahi H, Tammer C, et al. Variational methods in partially ordered spaces. New York (NY): Springer; 2003. (CMS Books in Mathematics).
[18] Bank B, Guddat J, Klatte D, Kummer B, Tammer K. Non-linear parametric optimization. Birkhäuser Basel, 1982.
[19] Schaefer H. Topological vector spaces. Berlin: Springer-Verlag; 1971.
[20] Aussel D, Sultana A. Quasi-variational inequality problems with non-compact valued constraint maps. J Math Anal Appl. 2017;456(2):1482–1494.
[21] Naselli Ricceri O. On the covering dimension of the fixed point set of certain multifunctions. Comment Math Univ Carolin. 1991;32:281–286.
[22] Rockafellar R, Wets R. Variational analysis. Berlin: Springer; 1997.
[23] Ionescu CT. On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games. J Math Anal Appl. 1988;136:267–289.
[24] Oettli W. A remark on vector-valued equilibria and generalized monotonicity. Acta Math Vietnam. 1997;22:213–221.
[25] Daniilidis A, Hadjisavvas N. On generalized cyclically monotone operators and proper quasimonotonicty. Optimization. 2000;47(12):123–135.
[26] Castellani M, Giuli M. A coercivity condition for nonmonotone quasiequilibria on finite-dimensional spaces. J Global Optim. 2019;75:163–176. Available from: https://doi.org/10.1007/s10898-019-00811-z.

[27] Karamardian S. Generalized complementarity problems. J Optim Theory Appl. 1971;8:161–168.
[28] Flores-Bazán F. Existence theory for finite-dimensional pseudomonotone equilibrium problems. Acta Appl Math. 2003;77:249–297. Available from: https://doi.org/10.1023/A:1024971128483.
[29] Flores-Bazán F. Existence theorems for generalized noncoercive equilibrium problems: the Quasi-Convex Case. SIAM J Optim. 2001;11(3):675–690. Available from: https://doi.org/10.1137/S1052623499364134.
[30] Castellani M, Giuli M. Refinements of existence results for relaxed quasimonotone equilibrium problems. J Global Optim. 2013;57:1213–1227.
[31] Zhou JX, Chen G. Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities. J Math Anal Appl. 1988;132:213–225.
[32] Nikaidô H, Isoda K. Note on noncooperative convex games. Pacific J Math. 1955;52:807–815.