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Faces and Support Functions for the Values of Maximal Monotone Operators

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Abstract

Representation formulas for faces and support functions of the values of maximal monotone operators are established in two cases: either the operators are defined on reflexive and locally uniformly convex real Banach spaces with locally uniformly convex duals, or their domains have nonempty interiors on real Banach spaces. Faces and support functions are characterized by the limit values of the minimal-norm selections of maximal monotone operators in the first case while in the second case they are represented by the limit values of any selection of maximal monotone operators. These obtained formulas are applied to study the structure of maximal monotone operators: the local unique determination from their minimal-norm selections, the local and global decompositions, and the unique determination on dense subsets of their domains.

Keywords Maximal monotone operator · Face · Support function · Minimal-norm selection · Yosida approximation · Strong convergence · Weak convergence

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1 Introduction

Faces and support functions are important tools in representation and analysis of closed convex sets (see [1, Chapter V]). For a closed convex set, a face is the set of points on the given set which maximizes some (nonzero) linear form while the support function is the signed distance from the origin point to the supporting planes of that set. The face associated with a given direction can be defined via the value of the support function at this direction [2, Definition 3.1.3, p. 220]. Recently, this notion has been defined and studied for the values of maximal monotone operators in [2, Sect. 3]. In this paper, the authors provided some characterizations for the boundary and faces of the values of maximal monotones operators in Hilbert spaces. Their work is motivated by the applications of these characterizations to the stability issues of semi-infinite linear programming problems.

Motivated by the study of the structure of maximal monotone operators, our paper will investigate the faces and support functions for the values of maximal monotone operators in real Banach spaces. We aim to establish some representation formulas for the faces and support functions in two cases regarding the reflexivity and local uniform convexity of the given spaces and theirs duals, and the nonemptiness of the interiors of the domains of maximal monotone operators. For the first case, we will extend the characterizations of faces associated with directions in [2, Theorem 3.2] from Hilbert spaces to reflexive and locally uniformly convex ones with locally uniformly convex duals. In comparison with previous work, where the authors used the properties of solutions of differential inclusions governed by maximal monotone operators, the proof here is new, simpler and more directed since we only use some basic properties of the Yosida approximation of maximal monotone operators. We formulate in the context of reflexive and locally uniformly convex spaces since our proof strongly depends on the single valuedness the duality mapping and its inverse, and the strong convergence of the trajectories generated by Yosida approximation. The obtained characterizations and the graphical density of points of subdifferentiability of convex functions allow us to get the representation formulas for support functions in reflexive and locally uniformly convex spaces with locally uniformly convex duals. For the second case, we will work with maximal monotone operators whose domains have nonempty interiors in real Banach spaces. Under this assumption, we could refine the formulas obtained in the first case. We show that the faces and support functions can be represented by the limit values of any selection of maximal monotone operators.

Characterizations for faces and support functions allow us to investigate the structure of maximal monotone operators. On reflexive and locally uniformly convex spaces with locally uniformly convex duals, we show the local unique determination of maximal monotone operators from their minimal-norm selections, and their local decompositions when their minimal-norm selections are locally bounded. On real Banach spaces, we get some global decompositions of maximal monotone operators when their domains have nonempty interiors. The global decompositions allow us to prove the unique determination of maximal monotone operators on dense subset of their domains.

The rest of this paper is structured as follows. In Sect. 2, we recall some basic notations of geometry of reflexive real Banach spaces and monotone operator theory. We also collect preliminary results in this section for the reader's convenience. In Sect. 3, representation formulas for faces and support functions are established in reflexive and locally uniformly convex spaces with locally uniformly convex duals. These formulas help us to show the local unique determination and to get the local decomposition of a maximal monotone operator provided that its minimal-norm selection is locally bounded. In Sect. 4, we will work with maximal monotone operators whose domains have nonempty interiors in real Banach spaces. Under our assumptions, we could refine the formulas for faces and support functions obtained in Sect. 3. The refined formulas allow us to find some global decompositions of maximal monotone operators and to show their unique determination on dense subsets of their domains.

2 Basic Definitions and Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and X^* be its continuous dual. The value of a functional $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle$. The open unit balls on X and X^* are denoted, respectively, by $\mathbb{B}$ and $\mathbb{B}^*$. For $x \in X$ and $r > 0$, the open ball centered at x with radius r is denoted by $B(x;r)$. We use the symbol lim or $\to$ to indicate the strong convergence in X, $\rightharpoonup$ for the weak convergence in X and $\overset{*}{\rightharpoonup}$ for the weak star convergence in X^*. Denote on X the set-valued mapping $J : X \rightrightarrows X^*$

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

The mapping J is called the *duality mapping* of the space X. The inverse mapping $J^{-1} : X^* \rightrightarrows X$ defined by $J^{-1}(x^*) := \{x \in X : x^* \in J(x)\}$ also satisfies

$$J^{-1}(x^*) = \{x \in X : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

If X is reflexive, i.e., $X = X^{**}$ then J^{-1} is the duality mapping of X^*.

Let us recall some geometric properties of real Banach spaces.

Definition 2.1 Let X be a real Banach space.

(i) X is called *uniformly convex* if and only if, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that

$$\frac{\|x+y\|}{2} \le 1 - \delta(\varepsilon) \quad \text{whenever} \quad \|x-y\| \ge \varepsilon, \quad \text{and} \quad \|x\| = \|y\| = 1;$$

(ii) X is called *locally uniformly convex* if and only if, given $\varepsilon > 0$ and $x \in X$ with $\|x\| = 1$, there exists $\delta(\varepsilon, x) > 0$, such that

$$\frac{\|x+y\|}{2} \le 1 - \delta(\varepsilon, x) \quad \text{whenever} \quad \|x-y\| \ge \varepsilon, \quad \text{and} \quad \|y\| = 1;$$

(iii) X is called *strictly convex* if and only if for any two distinct vectors with $\|x\| = 1$ and $\|y\| = 1$ we have $\|x+y\| < 2$.

It is clear from the definitions that uniform convexity implies local uniform convexity, and local uniform convexity implies strict convexity. The continuity of a duality mapping is closely related to geometric properties of real Banach spaces.

Proposition 2.1 (see [3, Proposition 2.7.31]) *If X is a real reflexive Banach space with locally uniformly convex dual X^*, then the duality mapping $J : X \rightrightarrows X^*$ is single-valued and continuous with respect to strong topologies on X and X^*.*

The *effective domain* $\operatorname{dom} f$ of an extended real-valued function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is the set of points $x \in X$ where $f(x) \in \mathbb{R}$. The function f is *proper* if $\operatorname{dom} f \neq \emptyset$. It is *lower semicontinuous* if

$$f(x) \le \liminf_{y \to x} f(y), \quad \forall x \in X.$$

The *epigraph* of f is defined by

$$\operatorname{epi} f := \{(x, r) : x \in \operatorname{dom} f, r \ge f(x)\}.$$

Suppose now that f is a lower semicontinuous convex function, i.e., $\operatorname{epi} f$ is convex and closed in $X \times \mathbb{R}$. A functional $x^* \in X^*$ is said to be a *subgradient* of f at $x \in X$, if $f(x)$ is finite and

$$f(y) - f(x) \ge \langle x^*, y - x \rangle, \quad \forall y \in X.$$

The collection of all subgradients of f at x is called the *subdifferential* of f at x, that is,

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x \rangle, \ \forall y \in X\}.$$

If $f(x) = +\infty$ we denote $\partial f(x) = \emptyset$. The function f is said to be *subdifferentiable* at x if $\partial f(x) \neq \emptyset$. Clearly, $\partial f(x)$ is a weak star closed convex in X^*. The following result represents the graphical density of points of subdifferentiability of f (see [4] and [5]).

Proposition 2.2 *Let f be a proper lower semicontinuous convex function from X into $\overline{\mathbb{R}}$. Then for any $\bar{x} \in \operatorname{dom} f$ and any $\varepsilon > 0$ there exists $x \in X$ such that $\partial f(x) \neq \emptyset$ and*

$$\|x - \bar{x}\| + |f(x) - f(\bar{x})| < \varepsilon.$$

Given a nonempty set $S \subset X$, $\operatorname{conv} S$ is the *convex hull* of S, $\operatorname{int} S$ is the *interior* of S, $\overline{S}$ is the *closure* of S and $\operatorname{bd}(S)$ is the *boundary* of S with respect to strong topology on X. Suppose now that S is nonempty closed and convex. For every $x \in S$, the *tangent cone* and the *normal cone* of S at x (see [6, Section 2.2.4] or [7, Section 4.2]) are defined respectively as

$$T(x; S) := \overline{\bigcup_{t>0} t^{-1}(S - x)}, \quad N(x; S) := \{x^* \in X^* : \sup_{y \in S} \langle x^*, y - x \rangle \le 0\}. \quad (1)$$

The tangent cone can be expressed in terms of sequences [7, Proposition 4.2.1] as

$$T(x; S) = \{v \in X : \exists \text{ sequences } t_n \downarrow 0, v_n \to v \text{ with } x + t_n v_n \in S \text{ for all } n \in \mathbb{N}\}. \tag{2}$$

By the bipolar theorem [6, Proposition 2.40], we have the following dual relationships

$$T(x; S) = \{v \in X : \sup_{x^* \in N(x;S)} \langle x^*, v\rangle \le 0\},$$
$$N(x; S) = \{x^* \in X^* : \sup_{v \in T(x;S)} \langle x^*, v\rangle \le 0\}.$$

The function $I_S : X \to \overline{\mathbb{R}}$ defined by

$$I_S(x) := \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3}$$

is called the *indicator function* of S and its Fenchel conjugate function $\sigma_S : X^* \to \overline{\mathbb{R}}$,

$$\sigma_S(x^*) := \sup\{\langle x^*, s\rangle : s \in S\}, \quad \forall x^* \in X^*, \tag{4}$$

is called the *support function* of S (see [8, p. 79]).

Similarly, for a nonempty closed and convex set $K \subset X^*$ and $x^* \in K$, we define the normal cone $N(x^*; K) \subset X^{**}$ and the tangent cone $T(x^*; K) \subset X^*$ of K at x^* as (1). The indicator function $I_K : X^* \to \overline{\mathbb{R}}$ and the support function $\sigma_K : X \to \overline{\mathbb{R}}$ are also defined similarly as (3) and (4) respectively. Observe that, σ_K is weakly star lower semicontinuous and convex.

For the set-valued operator $A : X \rightrightarrows X^*$, the *domain* of A is $D(A) := \{x \in X : Ax \neq \emptyset\}$ and $G(A) := \{(x, x^*) \in X \times X^* : x^* \in Ax\}$ is the *graph* of A. Recall that A is *monotone*, iff for all $(x, x^*), (y, y^*) \in G(A)$, one has $\langle x^* - y^*, x - y\rangle \ge 0$, and *maximally monotone* iff A is monotone and A has no proper monotone extension (in the sense of graph inclusion). The duality mapping, the subdifferential of a lower semicontinuous proper convex function, the normal cone to a closed convex set are examples of maximal monotone operators (see [9, Theorem A]). The maximal monotone operator A has closed convex values and is *demiclosed* [10, Proposition 2.1], i.e., A satisfies

$$\left[x_n^* \in Ax_n (\forall n \in \mathbb{N}), x_n^* \to x^*, x_n \rightharpoonup x\right] \Longrightarrow \left[x^* \in Ax\right],$$
$$\left[x_n^* \in Ax_n (\forall n \in \mathbb{N}), x_n^* \rightharpoonup x^*, x_n \to x\right] \Longrightarrow \left[x^* \in Ax\right].$$

If $\operatorname{int} D(A) \neq \emptyset$ then $\operatorname{int} D(A) = \operatorname{int} \overline{D(A)}$ (see [11, Theorem 27.1 and Theorem 27.3]) and A is locally bounded at every $x \in \operatorname{int} D(A)$ (see [12, Theorem 2.28] or [13, Theorem 1]), i.e., there exist $r > 0$ and $M > 0$ such that $x + r\mathbb{B} \subset D(A)$ and

$$\sup_{y^* \in Ay} \|y^*\| \le M, \quad \forall y \in x + r\mathbb{B}.$$

Conversely, if $x \in \overline{D(A)}$ and A is locally bounded at x, then $x \in \operatorname{int} D(A)$ (see [14, Theorem 1.14] or [8, Theorem 3.11.15]). Moreover, if $(x_i, x_i^*)_{i\in I}$ is a net in $G(A)$ such that $x_i \to x$ and $x_i^* \overset{*}{\rightharpoonup} x^*$ then $(x_i, x_i^*)_{i\in I}$ is *eventually bounded* (see [15, Theorem 4.1]), i.e., there exists $i_0 \in I$ and $M > 0$ such that

$$\|x_i\| + \|x_i^*\| \leq M, \quad \forall i \succeq_I i_0.$$

When X is reflexive, $D(A)$ is *nearly convex* (see [16, Corollary 3.4]), i.e., $\overline{D(A)}$ is convex. Moreover, if X^* is strictly convex then for every $x \in D(A)$, since Ax is nonempty closed and convex, there exists a unique point $x^*_{\min} \in Ax$ such that

$$\|x^*_{\min}\| = \min\{\|x^*\| : x^* \in Ax\}$$

(see [17, Exercise 3.32]). Therefore, the single-valued operator

$$A^\circ : D(A) \subset X \to X^*, \quad A^\circ x := x^*_{\min}$$

is well-defined; it is called the *minimal-norm selection* of A. Let us end this section by recalling some results related to the Yosida approximation of a maximal monotone operator (see [10, Proposition 2.2] and [18, Problems 2.161–2.164]).

Proposition 2.3 *Let X be a real reflexive Banach space such that both X and X^* are strictly convex and let $A : X \rightrightarrows X^*$ be a maximal monotone operator. For every $x \in X$ and $\lambda > 0$, there exists a unique $x_\lambda \in X$ such that*

$$0 \in J(x_\lambda - x) + \lambda A x_\lambda.$$

If $x \in D(A)$ then $x_\lambda \to x$ and $\lambda^{-1} J(x - x_\lambda) \rightharpoonup A^\circ x$ as $\lambda \to 0$. Moreover, if X^* is locally uniformly convex, then $\lambda^{-1} J(x - x_\lambda) \to A^\circ x$ as $\lambda \to 0$ for every $x \in D(A)$.

3 Representation Formulas in Reflexive Locally Uniformly Convex Spaces

In this section, we will establish representation formulas for faces and supports functions in reflexive locally uniformly convex spaces with locally uniformly convex duals. First, we recall the notion of the face associated with direction of the values of a maximal monotone operator.

Definition 3.1 Let X be a real Banach space and $A : X \rightrightarrows X^*$ be a maximal monotone operator. For $x \in X$ and $v \in X$, we define the set

$$A(x; v) := \begin{cases} \{x^* \in Ax : \langle x^*, v\rangle = \sigma_{Ax}(v)\}, & \text{if } \ x \in D(A), \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $x \in D(A)$ and $v \neq 0$, the set $A(x; v)$ is called the face associated with direction v of the value Ax.

Remark 3.1 By the definition of the support function, $x^* \in A(x; v)$ if and only if $v \in N(x^*; Ax)$, i.e., $x^* \in Ax$ and

$$\langle y^* - x^*, v\rangle \le 0, \quad \forall y^* \in Ax.$$

If Ax is singleton then $A(x; v) = Ax$ for all $v \in X$, identifying a singleton with its element. For any $v \in X \setminus \{0\}$, any element of $A(x; v)$ is a support point of Ax (and so an element of $\mathrm{bd}(Ax)$), being supported by the functional $\varphi_v \in X^{**}$ with $\varphi_v(x^*) := \langle x^*, v\rangle$. Consequently, $\bigcup_{v \in X\setminus\{0\}} A(x; v)$ is the set of support points of Ax when X is reflexive. Moreover, $A(x; v)$ is the subdifferential of the convex function σ_{Ax} at v (see [6, Proposition 2.121]) and so it is convex and weakly star closed.

We give two examples of faces associated with directions of values of maximal monotone operators.

Example 3.1

– Let $X = \mathbb{R}$ and $A = \partial|\cdot|$. Then, A is a maximal monotone operator and

$$Ax = \begin{cases} \{-1\} & \text{if } \ x < 0, \\ [-1, 1] & \text{if } \ x = 0, \\ \{1\} & \text{if } \ x > 0, \end{cases} \qquad A(0; v) = \begin{cases} \{1\} & \text{if } \ v > 0, \\ [-1, 1] & \text{if } \ v = 0, \\ \{-1\} & \text{if } \ v < 0. \end{cases}$$

Clearly, for $x \neq 0$, Ax is singleton, and so $A(x; v) = Ax$ for all $v \in \mathbb{R}$.

– Let X be a real Banach space and $\overline{\mathbb{B}}$ a closed unit ball in X. We consider a maximal monotone operator $A = \partial I_{\overline{\mathbb{B}}}(\cdot) = N(\cdot; \overline{\mathbb{B}})$ from X to X^*. Clearly, if $\|x\| < 1$ then $A(x; v) = Ax = \{0\}$ and if $\|x\| > 1$ then $A(x; v) = \emptyset$. Moreover, for $\|x\| = 1$, we have $Ax = \mathbb{R}_+ J(x)$ and

$$\sigma_{Ax}(v) < \infty \Leftrightarrow \sigma_{Ax}(v) = 0 \Leftrightarrow \sigma_{J(x)}(v) \le 0,$$

where J is the duality mapping on X. Therefore,

$$A(x; v) = \begin{cases} \mathbb{R}_+ S & \text{if } \ \sigma_{J(x)}(v) = 0, \\ \{0\} & \text{if } \ \sigma_{J(x)}(v) < 0, \\ \emptyset & \text{if } \ \sigma_{J(x)}(v) > 0, \end{cases}$$

where $S := \{\xi \in X^* : \xi \in J(x), \langle \xi, v\rangle = 0\}$.

Definition 3.2 Let X be a real Banach space and $A : X \rightrightarrows X^*$ be a maximal monotone operator. For every $x \in D(A)$ and $v \in X$ we define the following sets

$$\operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw) := \Big\{x^* \in X^* \,|\exists \text{ sequences } w_n \to v, t_n \downarrow 0 \text{ and } x_n^* \to x^* \text{ with } x_n^* \in A(x+t_n w_n) \text{ for all } n \in \mathbb{N}\Big\},$$

$$\mathrm{w}-\operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw) := \Big\{x^* \in X^* \,|\exists \text{ sequences } w_n \to v, t_n \downarrow 0 \text{ and } x_n^* \rightharpoonup x^* \text{ with } x_n^* \in A(x+t_n w_n) \text{ for all } n \in \mathbb{N}\Big\}.$$

Remark 3.2 Observe that we have the following inclusions

$$\operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw) \subset \mathrm{w}-\operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw) \subset A(x; v). \tag{5}$$

The first inclusion follows from Definition 3.2 while the second one is proved similarly as in the proof of [2, Theorem 3.2]. On reflexive and locally uniformly convex spaces with locally uniformly convex duals we have the equalities in (5).

Theorem 3.1 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let $A : X \rightrightarrows X^*$ be a maximal monotone operator. For every $x \in D(A)$ and $v \in X \setminus \{0\}$ we have*

$$A(x; v) = \operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw) = \mathrm{w}-\operatorname*{Lim\,sup}_{w\to v, t\downarrow 0} A(x+tw). \tag{6}$$

Proof From (5), to get (6) it suffices to check

$$A(x; v) \subset \operatorname*{Limsup}_{w\to v, t\downarrow 0} A(x+tw). \tag{7}$$

Suppose that $x^* \in A(x; v)$. Since X and X^* are reflexive and locally uniformly convex, by Proposition 2.1, both J and J^{-1} are single-valued and continuous with respect to strong topologies on X and X^*, respectively. Consider the operator $B : X \rightrightarrows X^*$ given by

$$By := Ay - J(v) - x^*, \quad \forall y \in X.$$

Clearly, B is maximal monotone and $D(B) = D(A)$. We first show that $B^\circ x = -J(v)$. Indeed, since $x^* \in A(x; v)$ we have $x^* \in Ax$ and so

$$-J(v) = x^* - J(v) - x^* \in Ax - J(v) - x^* = Bx.$$

Moreover, for every $y^* \in Ax$ we have $\langle x^* - y^*, v\rangle \geq 0$ and

$$\begin{aligned} \|-J(v)\| &= \|v\|^{-1}\langle J(v), v\rangle \\ &\leq \|v\|^{-1}\langle J(v) + x^* - y^*, v\rangle \\ &\leq \|v\|^{-1}\|y^* - J(v) - x^*\|\|v\| \\ &= \|y^* - J(v) - x^*\|. \end{aligned}$$

Applying Proposition 2.3 for the maximal monotone operator B and $x \in D(B)$, we can construct a sequence $\{x_n\} \subset X$ such that

$$0 \in J(x_n - x) + \frac{1}{n}Bx_n, \tag{8}$$

$$\lim_{n\to\infty} x_n = x, \quad \text{and} \quad \lim_{n\to\infty}[nJ(x - x_n)] = -J(v). \tag{9}$$

Consider the sequence $\{w_n\} \subset X$ given by $w_n := n(x_n - x)$ for every $n \in \mathbb{N}$. Then, by (8) and (9), we have

$$\begin{aligned} &-J(w_n) + J(v) + x^* \in Bx_n + J(v) + x^* = Ax_n = A(x + (1/n)w_n), \\ &\lim_{n\to\infty} w_n = \lim_{n\to\infty}[J^{-1}J(n(x_n - x))] = J^{-1}[J(v)] = v, \\ &\lim_{n\to\infty}[-J(w_n) + J(v) + x^*] = x^*. \end{aligned}$$

It follows that $x^* \in \operatorname*{Limsup}_{w\to v, t\downarrow 0} A(x + tw)$ and so (7) holds. □

Remark 3.3 – Theorem 3.1 generalizes [2, Theorem 3.2] from Hilbert spaces to reflexive locally uniformly convex Banach spaces having locally uniformly convex duals. Our proof is based on the properties of Yosida approximation of maximal monotone operators and it is simpler than the proof of [2, Theorem 3.2] where the authors used the properties of solutions of differential inclusions governed by maximal monotone operators.

– We formulate in the context of reflexive locally uniformly convex spaces since our proof strongly depends on the single valuedness, the strong continuity of the duality mapping and its inverse, and the pointwise strong convergence of the Yosida approximation of a maximal monotone operator stated in Proposition 2.3.

The formulas in (6) allow us to characterize the boundaries of the values of maximal monotone operators, by means only of the values at nearby points, which are close enough to the reference point but distinct of it (see [2, Theorem 3.1] in Hilbert setting).

Corollary 3.1 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let $A : X \rightrightarrows X^*$ be a maximal monotone operator. Then, for every $x \in D(A)$, we have*

$$\operatorname{bd}(Ax) = \operatorname*{Lim\,sup}_{y\to x, y\neq x} Ay := \Big\{x^* \in X^* \,|\, \exists \textit{ sequences } y_n \to x \textit{ and } y_n^* \to x^* \textit{with } y_n \neq x \textit{ and } y_n^* \in Ay_n \textit{ for all } n \in \mathbb{N}\Big\}. \tag{10}$$

Proof Let $x \in D(A)$. Observe that $\mathrm{bd}(Ax)$ can be represented by as

$$\mathrm{bd}(Ax) = \overline{\left(\bigcup_{v \neq 0} A(x; v) \right)}. \tag{11}$$

Indeed, as observed in Remark 3.1, since X is reflexive, $\bigcup_{v \neq 0} A(x; v)$ is the set of support points of Ax. Therefore, (11) is an immediate consequence of Bishop−Phelps theorem [19, Theorem 1].

Now we use (6) and (11) to get (10). We have,

$$\mathrm{bd}(Ax) = \overline{\left(\bigcup_{v \neq 0} \operatorname*{Limsup}_{w \to v, t \downarrow 0} A(x + tw) \right)} \subset \overline{\operatorname*{Lim\,sup}_{y \to x, y \neq x} Ay} = \operatorname*{Lim\,sup}_{y \to x, y \neq x} Ay.$$

Suppose that $x^* \in \operatorname*{Lim\,sup}_{y \to x, y \neq x} Ay$. Then, there exist sequences $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ such that $x_n \to x$, $x_n^* \to x^*$ and $x_n \neq x$, $x_n^* \in Ax_n$ for every $n \in \mathbb{N}$. By the maximal monotonicity of A, we have $x^* \in Ax$. We will show that $x^* \in \mathrm{bd}(Ax)$. Suppose on the contrary that $x^* \in \mathrm{int}(Ax)$. Then, for sufficiently large n, we have $x_n^* \in \mathrm{int}(Ax) \subset Ax$ and by the monotonicity of A, we have

$$x_n^* \in A(x; x_n - x) \quad \text{with} \quad x_n - x \neq 0 \tag{12}$$

By (11), for sufficiently large n, (12) implies that $x_n^* \in \mathrm{bd}(Ax)$ which is a contradiction. □

Now, we use Theorem 3.1 to obtain a representation for the support function of the values of A via its minimal-norm selection A°.

Lemma 3.1 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let $A : X \rightrightarrows X^*$ be a maximal monotone operator, $x \in D(A)$ and $v \in X \setminus \{0\}$ be such that* $\mathrm{w} - \operatorname*{Lim\,sup}_{w \to v, t \downarrow 0} A(x + tw) \neq \emptyset$. *Then,*

$$\mathrm{w} - \operatorname*{Limsup}_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D(A)}} A^\circ(x + tw) := \Big\{ x^* \in X^* \,|\, \exists \text{ sequences } w_n \to v,\ t_n \downarrow 0 \text{ such that } x + t_n w_n \in D(A) \text{ for all } n \in \mathbb{N} \text{ and } A^\circ(x + t_n w_n) \rightharpoonup x^* \Big\} \tag{13}$$

is a nonempty subset of $\mathrm{w} - \operatorname*{Lim\,sup}_{w \to v, t \downarrow 0} A(x + tw)$.

Proof Since A° is a selection of A, for every $x \in D(A)$ and $v \in X \setminus \{0\}$, we have

$$\mathrm{w} - \operatorname*{Limsup}_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D(A)}} A^\circ(x + tw) \subset \mathrm{w} - \operatorname*{Lim\,sup}_{w \to v, t \downarrow 0} A(x + tw).$$

Suppose that $\mathrm{w}-\operatorname*{Lim\,sup}_{w\to v,t\downarrow 0} A(x+tw)\neq\emptyset$ for some $x\in X$ and $v\in X\setminus\{0\}$. Then, there exist $x^*\in X^*$ and sequences $t_n\downarrow 0$, $w_n\to v$, $x_n^*\rightharpoonup x^*$ with $x_n^*\in A(x+t_nw_n)$ for every $n\in\mathbb{N}$. It follows that $x+t_nw_n\in D(A)$ for every $n\in\mathbb{N}$ and $\{x_n^*\}$ is bounded in X^*. Since $\|A^\circ(x+t_nw_n)\|\leq\|x_n^*\|$, $\{A^\circ(x+t_nw_n)\}$ is also bounded in X^*. By the reflexivity of X^*, the sequence $\{A^\circ(x+t_nw_n)\}$ has a subsequence converging weakly to some $\bar{x}^*\in \mathrm{w}-\operatorname*{Limsup}_{\substack{w\to v,t\downarrow 0\\ x+tw\in D(A)}} A^\circ(x+tw)$. □

The next example shows that the inclusion $\mathrm{w}-\operatorname*{Limsup}_{\substack{w\to v,t\downarrow 0\\ x+tw\in D(A)}} A^\circ(x+tw)\subset \mathrm{w}-\operatorname*{Lim\,sup}_{w\to v,t\downarrow 0} A(x+tw)$ may be strict.

Example 3.2 Let X be a real Hilbert space and $A=N(\cdot;\overline{\mathbb{B}})$. Let $x_0, v_0\in X$ be such that

$$\|x_0\|=\|v_0\|=1 \quad\text{and}\quad \langle x_0,v_0\rangle=0.$$

Clearly, $\mathrm{w}-\operatorname*{Limsup}_{\substack{w\to v_0,t\downarrow 0\\ x_0+tw\in D(A)}} A^\circ(x_0+tw)=\{0\}$ (since $A^\circ x=0$ for all $x\in\overline{\mathbb{B}}$) while

$$\mathrm{w}-\operatorname*{Lim\,sup}_{w\to v_0,t\downarrow 0} A(x_0+tw)=A(x_0;v_0)=Ax_0=\mathbb{R}_+x_0$$

since $\langle v_0,x^*\rangle=0$ for all $x^*\in Ax_0$.

Theorem 3.2 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let $A:X\rightrightarrows X^*$ be a maximal monotone operator. For every $x\in D(A)$ and $v\in X\setminus\{0\}$, we have*

$$\sigma_{Ax}(v)=\begin{cases}\displaystyle\liminf_{\substack{w\to v,t\downarrow 0\\ x+tw\in D(A)}}\langle A^\circ(x+tw),w\rangle, & \textit{if}\quad v\in T(x;\overline{D(A)}),\\ +\infty, & \textit{otherwise}.\end{cases}\tag{14}$$

Proof By the reflexivity of X, $\overline{D(A)}$ is convex. Let $x\in D(A)$ and $v\in X\setminus\{0\}$. If $v\notin T(x;\overline{D(A)})$ then there exists $x^*\in N(x;\overline{D(A)})$ such that $\langle x^*,v\rangle>0$. Then, for every $y^*\in Ax$ and $t>0$, we have $y^*+tx^*\in Ax+N(x;\overline{D(A)})=Ax$ by the maximal monotonicity of A. It follows that $\sigma_{Ax}(v)\geq\langle y^*,v\rangle+t\langle x^*,v\rangle$. Taking $t\to+\infty$ in the latter inequality, we get $\sigma_{Ax}(v)=+\infty$.

Suppose now that $v\in T(x;\overline{D(A)})$. It follows from (2) that there exist sequences $w_n\to v$, $t_n\downarrow 0$ with $x+t_nw_n\in D(A)$ for all $n\in\mathbb{N}$. Let $\{t_n\}$ and $\{w_n\}$ be any such sequences. Then, we have

$$\sigma_{Ax}(w_n)\leq\langle A^\circ(x+t_nw_n),w_n\rangle.\tag{15}$$

Indeed, by the monotonicity of A, for every $x^* \in Ax$, we have

$$\langle A^\circ(x + t_n w_n) - x^*, w_n\rangle = t_n^{-1}\langle A^\circ(x + t_n w_n) - x^*, x + t_n w_n - x\rangle \geq 0.$$

Hence, $\langle A^\circ(x + t_n w_n), w_n\rangle \geq \langle x^*, w_n\rangle$ for every $x^* \in Ax$ and so (15) holds. Taking $n \to \infty$ in (15), by the lower semicontinuity of σ_{Ax}, we get

$$\sigma_{Ax}(v) \leq \liminf_{n\to\infty}\langle A^\circ(x + t_n w_n), w_n\rangle.$$

Hence,

$$\sigma_{Ax}(v) \leq \liminf_{\substack{w\to v, t\downarrow 0\\ x+tw\in D(A)}} \langle A^\circ(x + tw), w\rangle. \tag{16}$$

Now we establish the reverse inequality when $\sigma_{Ax}(v) < +\infty$. To do this, we only need to point out the existence of the sequences $t_n \downarrow 0$, $w_n \to v$ with $x + t_n w_n \in D(A)$ for every $n \in \mathbb{N}$ such that

$$\langle A^\circ(x + t_n w_n), w_n\rangle \to \sigma_{Ax}(v). \tag{17}$$

Applying Proposition 2.2 for the proper lower semicontinuous convex function σ_{Ax} and $v \in \operatorname{dom}\sigma_{Ax}$, we can find a sequence $\{v_n\} \subset X$ such that $v_n\to v$, $\sigma_{Ax}(v_n)\to\sigma_{Ax}(v)$ and $A(x; v_n) = \partial\sigma_{Ax}(v_n) \neq \emptyset$. By Theorem 3.1 and Lemma 3.1, we have $\mathrm{w}-\operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D(A)}} A^\circ(x + tw) \neq \emptyset$. Hence, for every $n \in \mathbb{N}$, there exists sequences $t_m^n \downarrow 0$, $w_m^n \to v_n$ as $m \to \infty$ with $x + t_m^n w_m^n \in D(A)$ for every $m \in \mathbb{N}$ such that $\langle A^\circ(x+t_m^n w_m^n), w_m^n\rangle \to \sigma_{Ax}(v_n)$ as $m \to \infty$. For every $n \in \mathbb{N}$, choosing m such that

$$t_m^n \leq \frac{1}{n}, \ \|w_m^n - v_n\| \leq \frac{1}{n}, \ \left|\langle A^\circ(x + t_m^n w_m^n), w_m^n\rangle - \sigma_{Ax}(v_n)\right| \leq \frac{1}{n}$$

and setting $t_n := t_m^n$, $w_n := w_m^n$. Then, $t_n \downarrow 0$, $w_n \to v$ with $x + t_n w_n \in D(A)$ for every $n \in \mathbb{N}$ and $\langle A^\circ(x + t_n w_n), w_n\rangle \to \sigma_{Ax}(v)$. Hence, we have the equality in (16). □

Remark 3.4 It follows from (14) that $(x, x^*) \in G(A)$ if and only if $x \in D(A)$ and the following inequality

$$\langle x^* - A^\circ y, x - y\rangle \geq 0 \tag{18}$$

holds for all $y \in D(A) \cap U$, where U is some neighborhood of x. Indeed, by Theorem 3.2, if $v \in T(x; \overline{D(A)})$ then

$$\begin{aligned}\sigma_{Ax}(v) &= \liminf_{\substack{w\to v, t\downarrow 0\\ x+tw\in D(A)}} \langle A^\circ(x + tw), w\rangle \\ &\geq \liminf_{\substack{w\to v, t\downarrow 0\\ x+tw\in D(A)}} \langle x^*, w\rangle \\ &= \langle x^*, v\rangle.\end{aligned}$$

Therefore, $\sigma_{Ax}(v) \geq \langle x^*, v\rangle$ for all $v \in X$ and so $(x, x^*) \in G(A)$ by the classical separation theorem.

The formula (14) helps us to establish a local reconstruction of a maximal monotone operator from its minimal-norm selection.

Corollary 3.2 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let A_1 and A_2 be maximal monotone operators from X to X^*. If there exist $x_0 \in D(A_1) \cap D(A_2)$ and $r > 0$ such that $D(A_1)\cap B(x_0; r)=D(A_2)\cap B(x_0; r)$ and $A_1^\circ = A_2^\circ$ on $D(A_1)\cap B(x_0; r)$ then $A_1 = A_2$ on $D(A_1) \cap B(x_0; r)$. In particular, if $D(A_1) = D(A_2)$ and $A_1^\circ = A_2^\circ$ then $A_1 = A_2$.*

Proof Let $x_0\in D(A_1)\cap D(A_2)$ and $r>0$ be such that $D(A_1)\cap B(x_0; r)=D(A_2)\cap B(x_0; r)$ and $A_1^\circ=A_2^\circ$ on $D(A_1)\cap B(x_0; r)$. Let $x \in D(A_1) \cap B(x_0; r)$. By Theorem 3.2 and our assumptions, we obtain $\sigma_{A_1x} = \sigma_{A_2x}$. Hence, we have

$$\begin{aligned} A_1x = A_1(x; 0) &= \partial\sigma_{A_1x}(0) \\ &= \partial\sigma_{A_2x}(0) = A_2(x; 0) = A_2x. \end{aligned}$$

□

The next corollary presents a local decomposition of maximal monotone operator provided that its minimal-norm selection is locally bounded. As a consequence, if the minimal-norm selection of a maximal monotone operator is bounded with some modulus around some interior point of the domain then the whole values of the maximal monotone operator are also bounded with the same modulus around that point.

Corollary 3.3 *Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let A be a maximal monotone operator from X to X^* and $x \in D(A)$. Suppose that there exist $r > 0$ and $\rho > 0$ such that*

$$\|A^\circ y\| \leq \rho, \quad \forall y \in B(x; r) \cap D(A). \tag{19}$$

Then, for every $y \in B(x; r) \cap D(A)$, we have

$$Ay \subset N(y; \overline{D(A)}) + \rho\overline{\mathbb{B}^*}. \tag{20}$$

In particular, if $B(x; r) \subset D(A)$ then $Ay \subset \rho\overline{\mathbb{B}^*}$ for every $y \in B(x; r)$.

Proof Let $y \in B(x; r) \cap D(A)$ and $y^* \in Ay$. We first show that

$$\langle y^*, z - y\rangle \leq \rho\|z - y\|, \quad \forall z \in \overline{D(A)}. \tag{21}$$

Indeed, for every $z \in \overline{D(A)} \setminus \{y\}$, $z - y \in T(y; \overline{D(A)}) \setminus \{0\}$, and by Theorem 3.2 and (19)

$$\begin{aligned}\langle y^*, z-y\rangle &\le \sigma_{Ay}(z-y)\\ &= \liminf_{\substack{w\to z-y, t\downarrow 0\\ y+tw\in D(A)}} \langle A^\circ(y+tw), w\rangle\\ &\le \liminf_{\substack{w\to z-y, t\downarrow 0\\ y+tw\in D(A)}} \|A^\circ(y+tw)\|\|w\|\\ &\le \rho\|z-y\|.\end{aligned}$$

Hence, (21) holds and so $y^* \in N(y; \overline{D(A)}) + \rho\overline{\mathbb{B}^*}$. Therefore, (20) is satisfied.

If $B(x;r) \subset D(A)$ then $N(y; \overline{D(A)}) = \{0\}$ for every $y \in B(x;r)$. By (20) we have $Ay \subset \rho\overline{\mathbb{B}^*}$ for all y in this set. □

4 Representation Formulas in Reflexive Spaces

In this section, we will work with maximal monotone operators having their domains with nonempty interiors in real Banach spaces. Under these assumptions, we could refine the formulas obtained in the previous section. First, we make a relationship of the faces and the limit values of any selection of maximal monotone operators.

Lemma 4.1 *Let X be a real Banach space and $A : X \rightrightarrows X^*$ be a maximal monotone operator such that $\operatorname{int}(D(A)) \neq \emptyset$. Let D be a dense subset of $D(A)$ and $\tilde{A}$ be any selection of A. For every $x \in D(A)$ and $v \in \operatorname{int}(D(A) - x)$, the following set*

$$\mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x+tw) := \Big\{x^* \in X^* \,|\, \exists \text{ a net } (w_i, t_i)_{i\in I} \to (v, 0^+) \text{ such that } x + t_i w_i \in D \text{ for all } i \in I \text{ and } \tilde{A}(x+t_i w_i) \overset{*}{\rightharpoonup} x^*\Big\} \tag{22}$$

is a nonempty subset of $A(x; v)$.

Proof Let $x \in D(A)$ and $v \in \operatorname{int}(D(A) - x)$. We first show that the set in (22) is nonempty. Since $x + v \in \operatorname{int}(D(A))$, A is locally bounded around $x + v$, i.e., there exist $r, M > 0$ such that $Ay \neq \emptyset$ and $Ay \subset M\mathbb{B}^*$ for all $y \in x + v + 4r\mathbb{B}$. According to [15, Lemma 4.1], there exists $K > 0$ such that

$$\emptyset \neq Az \subset K\mathbb{B}^*, \ \forall z \in (x, x + v + 2r\mathbb{B}], \tag{23}$$

where

$$(x, x + v + 2r\mathbb{B}] := \{\lambda x + (1-\lambda)z : \lambda \in [0, 1[, z \in x + v + 2r\mathbb{B}\}.$$

Pick the sequences $\{t_n\} \subset]0, 1[$ and $\{w_n\} \subset X$ converging to 0 and v, respectively. Since $w_n \to v$ and

$$x + t_n w_n = (1 - t_n)x + t_n(x + w_n) \in (x, x + w_n),$$

we can assume that $x+t_n w_n \in (x, x+v+r\mathbb{B}]$ for every $n\in\mathbb{N}$. By (23), $x+t_n x_n \in D(A)$. Since D is dense in $D(A)$, one has $(x + t_n w_n + (t_n/n)\mathbb{B}) \cap D \neq \emptyset$. Hence, we can find $\nu_n \in (1/n)\mathbb{B}$ such that $x + t_n(w_n + \nu_n) \in D$. Without loss of generality, we can assume that $w_n + \nu_n \in v + 2r\mathbb{B}$. Again, by (23), we have $w_n + \nu_n \to v$ as $n \to +\infty$ and

$$\tilde{A}(x + t_n(w_n + \nu_n)) \in A(x + t_n(w_n + \nu_n)) \subset K\mathbb{B}^*, \quad \forall n \in \mathbb{N}.$$

By the Banach–Alaoglu theorem, we can find a subnet $\{\xi_i\}_{i\in I}$ of $\{\tilde{A}(x+t_n(w_n+\nu_n))\}$ such that $\xi_i \overset{*}{\rightharpoonup} \xi \in X^*$. Moreover, the corresponding nets $\{w_i\}_{i\in I}, \{t_i\}_{i\in I}, \{\nu_i\}_{i\in I}$ are subnets of $\{w_n\}, \{t_n\}, \{\nu_n\}$, respectively, and satisfy $\xi_i = \tilde{A}(x + t_i(w_i + \nu_i))$ for all $i \in I$. Since $x + t_n(w_n + \nu_n) \in D$ for every $n \in \mathbb{N}$ and $w_n + \nu_n \to v, t_n \downarrow 0$, we have $x + t_i(w_i + \nu_i) \in D$ for every $i \in I$ and $w_i + \nu_i \to v, t_i \downarrow 0$. This yields that $\xi \in \mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x + tw)$.

Next, we show that $\mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x + tw) \subset A(x; v)$. Let $x^* \in \mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x + tw)$. Then, we can find nets $w_i \to v, t_i \downarrow 0$ such that $x + t_i w_i \in D$ for all $i \in I$ and $\tilde{A}(x + t_i w_i) \overset{*}{\rightharpoonup} x^*$. Since A is maximal monotone and $\mathrm{int}(D(A)) \neq \emptyset$, by [15, Theorem 4.1], the net $\{\tilde{A}(x+t_i w_i)\}_{i\in I}$ is eventually bounded, and by [15, Corollary 4.1], $x^* \in Ax$. It follows from the monotonicity of A, for every $\xi \in Ax$, we have

$$\langle \tilde{A}(x + t_i w_i) - \xi, w_i\rangle = \frac{1}{t_i}\langle \tilde{A}(x + t_i w_i) - \xi, x + t_i w_i - x\rangle \geq 0.$$

Therefore, by taking the limit along the net, we get

$$\langle x^* - \xi, v\rangle \geq 0, \quad \forall \xi \in Ax,$$

which shows that $x^* \in A(x; v)$. □

Second, we use Lemma 4.1 to improve the representation formula (14) in Theorem 3.2.

Theorem 4.1 *Let X be Banach space and $A : X \rightrightarrows X^*$ be a maximal monotone operator such that* $\mathrm{int}(D(A)) \neq \emptyset$. *Let D be a dense subset of $D(A)$ and $\tilde{A}$ be any* *selection of* A. *For every* $x \in D(A)$ *and* $v \in X \setminus \{0\}$,

$$\sigma_{Ax}(v) = \begin{cases} \langle \xi, v \rangle, & \textit{if} \quad v \in \operatorname{int}\left(T(x; \overline{D(A)})\right), \\ \liminf\limits_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D}} \langle \tilde{A}(x + tw), w \rangle, & \textit{if} \quad v \in \operatorname{bd}\left(T(x; \overline{D(A)})\right), \\ +\infty, & \textit{otherwise}, \end{cases} \tag{24}$$

where ξ *is any vector in the set* $\mathrm{w}^* - \operatorname*{Limsup}\limits_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D}} \tilde{A}(x + tw)$.

Proof Since $D(A)$ has nonempty interior, $\overline{D(A)}$ is convex and $\operatorname{int}(D(A)) = \operatorname{int}(\overline{D(A)})$ (see [11, Theorem 27.1 and Theorem 27.3]). Let $x \in D(A)$ and $v \in X \setminus \{0\}$. We consider three cases of v.

Case 1. $v \notin T(x; \overline{D(A)})$

Repeating the first part of the Proof of Theorem 3.2, we get $\sigma_{Ax}(v) = +\infty$.

Case 2. $v \in \operatorname{int}\left(T(x; \overline{D(A)})\right)$

By [7, Proposition 4.2.3], the interior of the tangent cone can be expressed as

$$\operatorname{int}\left(T(x; \overline{D(A)})\right) = \bigcup_{h > 0} \left(\frac{\operatorname{int}(\overline{D(A)}) - x}{h} \right) = \bigcup_{h > 0} \left(\frac{\operatorname{int}(D(A)) - x}{h} \right)$$
$$= \bigcup_{h > 0} \frac{\operatorname{int}(D(A) - x)}{h}.$$

Hence, there exists $h > 0$ such that $hv \in \operatorname{int}(D(A) - x)$. Applying Lemma 4.1, the following sets are nonempty and

$$\mathrm{w}^* - \operatorname*{Limsup}_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D}} \tilde{A}(x + tw) = \mathrm{w}^* - \operatorname*{Limsup}_{\substack{w \to hv, t \downarrow 0 \\ x + tw \in D}} \tilde{A}(x + tw) \subset A(x; hv) = A(x; v).$$

Then, for every $\xi \in \mathrm{w}^* - \operatorname*{Limsup}\limits_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D}} \tilde{A}(x + tw)$, we have $\sigma_{Ax}(v) = \langle \xi, v \rangle$.

Case 3. $v \in \operatorname{bd}\left(T(x; \overline{D(A)})\right)$

Since A is monotone, for every $w \in X$ and $t > 0$ such that $x + tw \in D \subset D(A)$, we have

$$\langle x^*, w \rangle \leq \langle \tilde{A}(x + tw), w \rangle, \quad \forall x^* \in Ax.$$

This yields that

$$\sigma_{Ax}(v) \leq \liminf_{\substack{w \to v, t \downarrow 0 \\ x + tw \in D}} \langle \tilde{A}(x + tw), w \rangle. \tag{25}$$

Pick $v_0 \in \operatorname{int}\left(T(x; \overline{D(A)})\right)$. From Case 2., we have $\sigma_{Ax}(v_0) < +\infty$. Consider the sequence $\{v_n\}$ given by

$$v_n := \frac{1}{n}v_0 + \frac{n-1}{n}v, \quad \forall n \in \mathbb{N}.$$

On one hand, since σ_{Ax} is convex, we have

$$\sigma_{Ax}(v_n) \leq \frac{1}{n}\sigma_{Ax}(v_0) + \frac{n-1}{n}\sigma_{Ax}(v).$$

Taking the superior limit both sides of the above inequality, we get

$$\limsup_{n\to\infty} \sigma_{Ax}(v_n) \leq \sigma_{Ax}(v).$$

Then, the lower semicontinuity of σ_{Ax} implies that

$$\lim_{n\to\infty} \sigma_{Ax}(v_n) = \sigma_{Ax}(v). \tag{26}$$

On the other hand, the convexity of $T(x; \overline{D(A)})$ implies that $v_n \in \operatorname{int}\left(T(x; \overline{D(A)})\right)$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Since $v_n \in \operatorname{int}\left(T(x; \overline{D(A)})\right)$, as in the proof of Case 2., we have

$$\emptyset \neq \mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v_n, t\downarrow 0 \\ x+tw\in D}} \tilde{A}(x+tw) \subset A(x; v_n).$$

Then, there exist nets $w_i \to v_n, t_i \downarrow 0$ such that $x + t_i w_i \in D$ for all $i \in I$ and $\tilde{A}(x + t_i w_i) \overset{*}{\rightharpoonup} x^* \in A(x; v_n)$. Since A is maximal monotone and $\operatorname{int}(D(A)) \neq \emptyset$, by [15, Theorem 4.1], the net $\{\tilde{A}(x + t_i w_i)\}_{i\in I}$ is eventually bounded. It follows that there exist $t_n \in \{t_i\}_{i\in I}$, $w_n \in \{w_i\}_{i\in I}$ such that

$$t_n \leq \frac{1}{n}, \quad \|w_n - v_n\| \leq \frac{1}{n}, \quad x + t_n w_n \in D$$

and

$$|\sigma_{Ax}(v_n) - \langle \tilde{A}(x + t_n w_n), w_n\rangle| = |\langle x^*, v_n\rangle - \langle \tilde{A}(x + t_n w_n), w_n\rangle| \leq \frac{1}{n}.$$

Therefore, we can choose sequences $t_n \downarrow 0$, $w_n \to v$ and, by (26), such that

$$\lim_{n\to\infty} \langle \tilde{A}(x + t_n w_n), w_n\rangle = \sigma_{Ax}(v).$$

Combining this and (25), we get

$$\sigma_{Ax}(v) = \liminf_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \langle \tilde{A}(x+tw), w\rangle.$$

□

Remark 4.1 Clearly, by Theorem 4.1, for $x \in D(A)$ and $v \in \operatorname{int}\left(T(x; \overline{D(A)})\right)$, we have $\sigma_{Ax}(v) = \langle \xi, v\rangle$ whenever $\xi \in \mathrm{w}^* - \operatorname*{Limsup}_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x+tw)$.

Finally, we employ Theorem 4.1 and [15, Proposition 5.1] to get the global decompositions for maximal monotone operators. Our proof follows the technique of [15, Theorem 5.2]. From now, we denote $\overline{S}^{\mathrm{w}^*}$ is the closure with respect to weak star topology on X^* of a subset $S \subset X^*$.

Corollary 4.1 *Let X be a real Banach space and $A : X \rightrightarrows X^*$ be a maximal monotone operator such that $\operatorname{int}(D(A)) \neq \emptyset$. Let D be a dense subset of $D(A)$ and $\tilde{A}$ be any selection of A. Then, for every $x \in X$,*

$$Ax = \overline{\operatorname{conv}\left\{\bigcup_{v\in \operatorname{int}(D(A))-x} \mathrm{w}^* - \limsup_{\substack{w\to v, t\downarrow 0\\ x+tw\in D}} \tilde{A}(x+tw)\right\}}^{\mathrm{w}^*} + N(x; \overline{D(A)}) \qquad (27)$$

$$= \overline{\operatorname{conv}\left\{\mathrm{w}^* - \operatorname*{Limsup}_{y\xrightarrow{D}x} \tilde{A}y\right\}}^{\mathrm{w}^*} + N(x; \overline{D(A)}), \qquad (28)$$

where

$$\mathrm{w}^* - \operatorname*{Limsup}_{y\xrightarrow{D}x} \tilde{A}y := \left\{x^* \in X^* | \ \exists \text{ a net } y_i \to x \text{ such that } y_i \in D \text{ for all } i \in I \text{ and } \tilde{A}y_i \xrightarrow{*} x^*\right\}.$$

Proof Let $x \in X$. Since A is maximal and $\operatorname{int}(D(A)) \neq \emptyset$, by [15, Corollary 4.1], $\mathrm{w}^* - \operatorname*{Limsup}_{y\xrightarrow{D}x} \tilde{A}y \subset Ax$. Therefore,

$$\overline{\operatorname{conv}\left\{\bigcup_{v\in\operatorname{int}(D(A))-x} \mathrm{w}^*-\limsup_{\substack{w\to v,t\downarrow 0\\ x+tw\in D}} \tilde{A}(x+tw)\right\}}^{\mathrm{w}^*} + N(x;\overline{D(A)})$$
$$\subset \overline{\operatorname{conv}\left\{\mathrm{w}^*-\operatorname*{Limsup}_{y\xrightarrow{D}x} \tilde{A}y\right\}}^{\mathrm{w}^*} + N(x;\overline{D(A)})$$
$$\subset Ax + N(x;\overline{D(A)}) = Ax.$$

Since both sides of the latter inclusions are empty sets if $x \notin D(A)$, to show the reverse, it suffices to consider $x \in D(A)$. Let

$$K := \overline{\operatorname{conv}\left\{\bigcup_{v\in\operatorname{int}(D(A))-x} \mathrm{w}^*-\limsup_{\substack{w\to v,t\downarrow 0\\ x+tw\in D}} \tilde{A}(x+tw)\right\}}^{\mathrm{w}^*}.$$

Observe that, for every $v \in \operatorname{int}(T(x;\overline{D(A)})$, we have

$$\sigma_{Ax}(v) \le \sigma_K(v). \tag{29}$$

Indeed, by Theorem 4.1, (29) holds for all $v \in \operatorname{int}(D(A)-x)$. Moreover, by the formula for the interior of tangent cone, $\operatorname{int}(T(x;\overline{D(A)}) = \mathbb{R}^*_+ \operatorname{int}(D(A)-x)$. Hence, (29) is satisfied for all $v \in \operatorname{int}(T(x;\overline{D(A)})$. According to [15, Proposition 5.1], we get

$$Ax \subset \overline{K + N(x;\overline{D(A)})}^{\mathrm{w}^*}.$$

To finish our proof for the theorem, one has to prove that

$$\overline{K + N(x;\overline{D(A)})}^{\mathrm{w}^*} \subset K + N(x;\overline{D(A)}).$$

The latter inclusion is proved by applying the same argument as in the proof of [15, Theorem 5.2]. □

Remark 4.2 – The formula (28) has a similar form to the representation formula in [15, Theorem 5.2]. The first term on the right hand-side of the representation in [15, Theorem 5.2] is the closed convex hull of the limit values of the given maximal monotone operator on a dense subset of its domain while the first term on the right hand-side of (28) is only represented by the closed convex hull of the limit values of any selection of that maximal monotone operator. One of the usefulness of this representation is to allow us to prove the unique determination of maximal monotone operators on dense subsets of their domains and characterize the Lipschitz continuity of a convex function.

– If X is reflexive, the nets defined in $\mathrm{w}^* - \operatorname*{Limsup}_{\substack{w \to v, t \downarrow 0 \\ x+tw \in D}} \tilde{A}(x+tw)$ in Lemma 4.1, Theorem 4.1 and Corollary 4.1 can be replaced by sequences, and the weak star limits can be also replaced by the weak ones. Moreover, we can change the closures with respect to weak star topology in (27) and (28) into the closures with respect to strong (or weak) topology on X^*.

Corollary 4.2 *Let X be a real Banach space and $A, B : X \rightrightarrows X^*$ be two maximal monotone operators such that $\operatorname{int}(D(A)) = \operatorname{int}(D(B)) \neq \emptyset$. If there exists a dense subset D of $D(A)$ such that*

$$Ax \cap Bx \neq \emptyset \quad \forall x \in D, \tag{30}$$

then $A = B$.

Proof Since $\operatorname{int}(D(A)) = \operatorname{int}(D(B))$, we have

$$\overline{D(A)} = \overline{\operatorname{int}(D(A))} = \overline{\operatorname{int}(D(B))} = \overline{D(B)}.$$

From (30), we can find selections $\tilde{A}$ of A and $\tilde{B}$ of B such that $\tilde{A} = \tilde{B}$ on D. Applying Corollary 4.1, for every $x \in X$, we have

$$\begin{aligned} Ax &= \overline{\operatorname{conv}\left\{ \mathrm{w}^* - \operatorname*{Limsup}_{y \xrightarrow{D} x} \tilde{A}y \right\}}^{\mathrm{w}^*} + N(x; \overline{D(A)}) \\ &= \overline{\operatorname{conv}\left\{ \mathrm{w}^* - \operatorname*{Limsup}_{y \xrightarrow{D} x} \tilde{B}y \right\}}^{\mathrm{w}^*} + N(x; \overline{D(B)}) \\ &= Bx. \end{aligned}$$

□

We end this section by the following example.

Example 4.1 Let X be a real Banach space and $f : X \to \mathbb{R}$ a lower semicontinuous convex function. Suppose that there exist a dense subset D of X and $\ell \geq 0$ such that

$$\partial f(x) \cap \ell \overline{\mathbb{B}^*} \neq \emptyset, \quad \forall x \in D.$$

Then, f is $\ell-$Lipschitz continuous on X, i.e.,

$$|f(x) - f(y)| \leq \ell \|x - y\|, \quad \forall x, y \in X.$$

Indeed, under our assumptions, it follows from (28) that

$$\partial f(x) \subset \ell \overline{\mathbb{B}^*}, \quad \forall x \in X,$$

which implies that f is $\ell-$Lipschitz continuous on X.

5 Conclusions

We have provided representation formulas for faces and support functions for the values of maximal monotone operators in real Banach spaces. The obtained representation formulas help us to prove the local unique determination of a maximal monotone operator from its minimal-norm selection or on a dense subset of its domain. Some local and global decompositions for maximal monotone operators are also established.

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