# Spectral Theory of the Thermal Hamiltonian.

Tesis
entregada a la
Universidad de Chile
en cumplimiento parcial de los requisitos
para optar al grado de
Magíster en Ciencias con Mención en Matemáticas
Facultad de Ciencias

por

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Julio 2020

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FACULTAD DE CIENCIAS

UNIVERSIDAD DE CHILE

INFORME DE APROBACIÓN

TESIS MAGÍSTER

Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Magíster presentada por el candidato

**Vicente Lenz Burnier**

Ha sido aprobada por la Comisión de Evaluación de la Tesis como requisito para optar al grado de Magíster en Ciencias con mención en Matemáticas, en el examen de Defensa de Tesis rendido el día.

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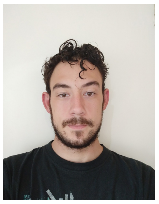
## BIOGRAFÍA



Nací un 5 de enero de 1995, en Santiago, Chile. Me crié con mi madre, en una infancia bastante tranquila, con un pie en el campo, en la actual región de los Ríos, y con otro pie en Santiago; tan así que hay mas fotos mías comiendo maqui en un potrero que en la ciudad. Fue una infancia tranquila, sin ver mucho a mi mamá de día por su trabajo nocturno. Pasando todos los veranos, y varios inviernos, y más de algún fin de semana largo en el sur tiñeron mis recuerdos de verde y azul. Corderos, terneros, caballos, un poni (o dos, dependiendo de como uno cuente) se roban varias de las historias de estos años. Cómo olvidar jugar con mi primo en los esteros hasta que ambos volvíamos con ambas botas llenas de agua; cuando eso pasaba recuerdo que me llegaba menor reto que a él. En Santiago tampoco faltó fauna, con mi mamá rodeada de caballos, y hasta en algún punto, llevándome a su trabajo en el zoológico metropolitano; y a esto después sumémosle las idas a Polpaico donde la Charo.
Dicho esto, era un niño feliz con la vida urbana, y más aún, reacio al cambio; así que cuando nos instalamos definitivamente en el sur por temas laborales el 2006 no me lo tomé muy bien. La soledad, frío, e incertezas del devenir marcaron los primeros años allá, e incluso al pasar, cambiaron como nos enfrentamos a futuros desafíos. Pero no puedo no mencionar que distinto hubiera sido sin la Charo, los Sommer, Recordón y Heinsohn en esos años. Con el tiempo dificultades iniciales pasaron, pero no terminé de aclimatarme al medio; con los grandes amigos que mantengo de ese tiempo, todos eramos postizos en Osorno de una forma u otra, y en la casa de cada uno de ellos encontré un espacio para mi. Fue también en estos años que empecé a conocer las matemáticas, en el contexto de distintas olimpiadas: de la región, de los distintos colegios y eventualmente de las nacionales. En estas últimas tuve una bella parábola con concavidad negativa de rendimiento que me enseñó bastante de mi mismo, y de mi temple. Durante cuarto medio en una visita a la Austral, tener un primer acercamiento real con lo que es dedicarse a las matemáticas; y esto sumado a venir a conocer nuestro departamento en Las Palmeras 3425, terminó de convencerme de estudiar matemáticas y no ceder en este punto. Aún así algo de revuelo se hizo en el colegio de perder un nacional en algo que no sea ingeniería.
Mi vida tras volver a Santiago se ha caracterizado por relativamente constantes cambios. A través de estos cambios, en general a mejor siempre, encontré libertades previamente mucho mas difíciles a experimentar, viviendo entre el campo y con mi abuela. A partir de esto, y junto a los lazos que formé, el yo más adulto empezó a gestarse con sus códigos, principios y prioridades. Los años pasaron, y a través de la carrera, en general fui encantándome más con las matemáticas, eso si, aprendiendo que la vocación no es un parámetro constante, si no una curva, dependiente de muchas variables, y pareciera que un componente estocástico también. Durante mi pregrado tomé una variedad de electivos, una buena parte de estos tendiendo hacia el área del Álgebra. Llegando a último año, a las alturas de octubre caí en cuenta que me quedaban sólo meses en pregrado, y era el momento de tomar determinaciones para el futuro. Gratamente decidí proceder con el Magíster en la misma casa de estudios. Para esto postulé a la beca Conicyt de Magister Nacional, la cual obtuve, y con eso pude financiar los dos primeros años de postgrado. Durante el primer semestre de este programa, asistí al curso "Métodos Matemáticos de la Mecánica Cuántica", dictado por quien sería mi tutor de tesis, Giuseppe de Nittis. Posterior a esto, seguí vertiendome en esta área asistiendo a cursos dictados por Marius Mantoiu en la Universidad de Chile, y Olivier Bourget en la UC, confirmando mi elección de área hacia la física matemática. Estos años en magíster, un poco más de tiempo previsto, y con dificultades imprevistas, ha sido de profundo aprendizaje: personal, del mundo, y de lo que significa ser un científico. Al momento de escribir esto planeo seguir con determinación por este camino.

# AGRADECIMIENTOS

En todos estos años los desafíos no han faltado. Pero tampoco el apoyo y amor de mi medio. El primer agradecimiento es hacia mi madre, María Eliana, que siempre ha estado a mi lado, no hubiese llegado a este punto de no ser por los múltiples desafíos y difíciles decisiones que ha superado, y no dudo que seguiré contando con ella. Le agradezco profundamente también a mi madrina Charo, gran cultivadora de mi sentido del humor, siempre cómplice, y como si fuera poco, estuvo hasta en las más difíciles, con humor, cariño y comprensión. Agradezco también a mi familia, en especial a la Carmen por su ayuda, y esos meses de comer rico, contarnos el día comiendo algo rico y ver pasapalabra.

En mi educación, no puedo dejar de agradecerles a todos los profesores que han participado en mi formación. En mis años de colegio, un especial agradecimiento a mis profesores de Matemáticas, ambos estupendos docentes, Paola Muñoz y José Goldsmith, quién además nunca dejó de jugársela fuera de la sala de clases, de desayunos con Seremis hasta subidas a Antillanca.

En mi paso por la educación superior, le agradezco al departamento de Matemáticas de la Universidad de Chile, mi casa de estudios dónde cursé mis estudios de pregrado y postgrado. Mis especiales agradecimientos para Anita Rojas, Marius Mantoiu, Gonzalo Robledo y Alicia Labra, que siempre me dieron su incansable apoyo, cariño, y comprensión estos años, y conversaciones para el recuerdo. De mi paso por el departamento de Matemáticas de la Universidad Católica, le agradezco enormemente a mi tutor Giuseppe de Nittis que ha sabido guiarme a través de estos años y este proceso, no podría haber pedido un mejor acercamiento a la Física Matemática que junto a él. También le agradezo a Claudio Fernández, Gregorio Moreno y Olivier Bourget por varias estupendas y esclarecedoras conversaciones.

A mis amigos en estos 20 tantos años de existencia, el infaltable hogar lejos del hogar, me llena de orgullo conocerlos y quererlos a cada uno de ustedes. A Daniel L. y Xime de mis tiempos en Santiago, con quien incluso con distancia y tiempo entremedio, siempre cercanos, son mis más antiguos amigos que forman parte del día a día. De mis años en Osorno, a Claudio, Aníbal, JM y Verónica, cuánto hemos crecido cada uno, y cómo nos hemos sabido reencontrar en las etapas que hemos vivido.

En mis primeros años en Santiago como universitario conocí maravillosas personas que me han sido parte de mi crecimiento. Khris, Jacob, Camila M. y Mirko, no me imagino como sería al día de hoy sin ustedes al lado mío en este tiempo y las aventuras vividas. A Matilde, se que estoy siempre en su equipo y ella en el mío, y a Loreto, segundo hogar en el sur más al sur; ambas eternas compañeras y confidentes. A Trino, por la infinita confianza; a Pablo Q., mejor compañero de viajes; a Lucas, constante compañía, que me ha instado a desarrollarme en nuevas formas; a Felipe, que nunca olvidaremos, y siempre me sacará una sonrisa su recuerdo; a TP, que hace demasiado no veo; a Alvin, por las infinitas conversaciones y reflexiones; a Cami G, que la risa y la música no nos deje nunca; a Matias, por su apoyo y humor; a Esteban, con quien entre risas el 2019 y 2020, la tesis y las crisis existenciales pasaron harto más fácil. A Camila T, con que hemos sobrevivido dos aparentes Apocalipsis, siempre con apañe en el espacio compartido. A Daniel G, Felipe G y Alejandro C, ya será hora de volverse a ver, cerveza en mano, y jugar algo. Al grupo de estudiantes de postgrado en matemáticas, que bien poco nos vemos, y que (ojalá) más adelante encontremos las instancias.

Siempre queda algo en el tintero, anécdotas, personas, recuerdos, pero ya llené mi página. Muchas gracias.

# RESUMEN

Se estudiará el operador $H_T$, llamado el Hamiltoniano Termal, originalmente propuesto por Luttinger para estudiar el efecto de un gradiente térmico en la materia. Primero le daremos definiciones rigurosas al inicialmente formalmente autoadjunto $H_T$, y a operadores una serie de operadores unitariamente equivalentes a este. Posterior a esto se estudiarán las propiedades espectrales de estos, y se calculará la dinámica generada por este operador libre de perturbaciones, junto a su función de Green y familia resolvente. En esta sección concluimos encontrando una familia de potenciales de convolución para los cuales las condiciones de scattering se satisfacen. Finalmente estudiaremos la dinámica del caso clásico.

# ABSTRACT

We will study the operator $H_T$, called the Thermal Hamiltonian, originally proposed by Luttinger to study the effects of a thermal gradient in the matter. We will start by rigurously defining the initially formally self-adjoint operator $H_T$, as well as some unitarily equivalent operators. Then we will study their spectral properties, and compute their unperturbed time evolution, as their Green functions and resolvent family. We will conclude that section by presenting a convolution potentials family for which the scattering conditions are satisfied. Finally we will study the dynamics defined by the classical case.

Contents

## 1. Introduction

1.1. **Main Results.** Our principal subject of study is the operator $H_T(\lambda, \gamma)$, from now on called the *Thermal Hamiltonian*, initially in the space of Schwarz functions as

$$H_T \equiv H_T(\lambda, \gamma) := p^2 + \frac{\lambda}{2}\{p^2, \gamma \cdot x\} ,$$

As given in equation (11). $H_T$ can be simplified through translations and rotations to the operator $T$, given formally by

$$T \equiv \frac{1}{2}\{p^2, x_1\} = \frac{1}{2}\big(p^2 x_1 + x_1 p^2\big)$$

as seen in equation (17). Both of these operators are thorougly studied in section 3. Further conjugating withe the Fourier transform we obtain $\widehat{T}$

$$\widehat{T} \equiv -\frac{1}{2}\{x^2, p_1\} = -\frac{1}{2}\big(x^2 p_1 + p_1 x^2\big)$$

described in equation (19), studied in section 2. We will restrict ourselves to the one dimensional case. We fully describe the self-adjoint extensions of these operators initially defined on $\mathcal{S}(\mathbb{R})$, further computing their spectral measures, resolvents, unitary propagators and Green functions. We will also compute the Wave and Scattering operators for $H_T$ and $T$, for a class of convolution potentials. In section 4 we will study the classical dinamics through both Hamiltonian and Lagrangian formalism.

1.2. **Motivation.** The motion of an electron in a static magnetic field $B$ inside a medium is described by the (one-particle) Hamiltonian

$$H(A, V) := K(A) + V . \tag{1}$$

with

$$K(A) := \frac{1}{2m}\Big(p - \frac{e}{c}A\Big)^2 . \tag{2}$$

The parameters $m$, $e$ and $c$ describe the mass and the charge of the electron and the speed of the light, respectively. The fixed (effective) potential $V$ takes care of the interaction of the electron with the atomic structure of the medium and causes only *elastic scattering*. The magnetic field enters in the *kinetic* term $K(A)$ through its vector potential ($B = \nabla \times A$). In Quantum Mechanics the Hamiltonian $H(A, V)$ is interpreted as a differential operator acting on the Hilbert space $L^2(\mathbb{R}^d)$ where the differential part is provided by the momentum operator $p := -\hbar\,\mathrm{i}\,\nabla$ ($\hbar$ being the Planck constant). The potentials $V = V(x)$ and $A = A(x)$ are functions of the position operator and act as multiplication operators.

The *transport phenomena* in matter are described by considering the response of the system to an external *perturbation* $F = F(x)$ [Lut1, Lut2]. In the stationary regime, that is when all the transient effects due to the switching-on of the perturbation are suppressed, the system reacts by generating a (stationary) drift current. The latter can be computed (at least in the linear response regime, see *e.g.* [DL]) starting from the full dynamics generated by the *perturbed* Hamiltonian

$$H(A, F, V) := H_0(A, F) + V . \tag{3}$$

In (3) the "free" Hamiltonian

$$H_0(A, F) := K(A) + F \tag{4}$$

describes the motion of an electron that moves in the empty space under the influence of the (external) fields generated by $A$ and $F$. The potential $V$ in (3) describes the interaction with the matter which generates elastic scattering of the particle. Once the "free" dynamics generated by $H_0(A,F)$ is known one can study the influence of the matter by means of the *scattering theory* [RS3, Yaf] for the pair of operators $H_0(A,F)$ and $H(A,F,V)$.

The best studied case concerns the response of the system to the perturbation induced by a uniform electric field $E$. In this case the perturbation potential is given by $F_E(x) := -eE \cdot x$ (the "dot" denotes the product in the Euclidean space $\mathbb{R}^d$) and the associate perturbed Hamiltonian takes the form

$$H(A,F_E,V) \;:=\; H_{\mathrm{S}}(A) \;+\; V \tag{5}$$

where the "free" part is given by

$$H_{\mathrm{S}}(A) \;:=\; K(A) \;-\; eE \cdot x \tag{6}$$

according to (4). The operator $H_{\mathrm{S}}(A)$ is known as *(magnetic) Stark Hamiltonian*. The non-magnetic case $H_{\mathrm{S}}(A=0) = \frac{p^2}{2m} - eE\cdot x$ has been extensively studied since the dawn of the Quantum Mechanics. Among the vast literature we will refer to [AH] for a concise and rigorous presentation of the spectral theory of $H_{\mathrm{S}}(0)$ and the related scattering theory when the matter potential $V$ is taken in consideration. The spectral theory of $H_{\mathrm{S}}(A)$ in presence of a uniform magnetic field is discussed in [DP, ADF], among others.

In order to study the *thermal transport* in matter Luttinger proposed a model which allows a "mechanical" derivation of the thermal coefficients [Lut2]. Such a model has been then applied and generalized successfully by other authors like in [SS, VMT]. The essential point of the Luttinger's model is that the effect of the thermal gradient in the matter is replaced by a "fictitious" gravitational field, which can be easily described by a perturbation of the Hamiltonian in the spirit of (3) and (4). More precisely one assumes that the particles are subject to a force which has the direction of the thermal gradient $\nabla T$ (wher $T$ is the distribution of temperature) and which is proportional to the local content of energy divided by $c^2$ (in view of the mass-energy equivalence). The latter is given by the Hamiltonian (2) itself. Such a thermal-gravitational field is given by the potential

$$\begin{aligned} F_T \;:=&\; \frac{1}{2}\left[\left(\frac{\nabla T}{c^2}\cdot x\right)\; H(A,V) \;+\; H(A,V)\;\left(\frac{\nabla T}{c^2}\cdot x\right)\right] \\ =&\; \frac{\nabla T}{c^2}\cdot\frac{1}{2}\big\{H(A,V),x\big\} \end{aligned} \tag{7}$$

where the anti-commutator $\{\;,\;\}$ between $H(A,V)$ and $x$ is needed to make $F_T$ formally self-adjoint (*i.e.* symmetric). The total perturbed Hamiltonian $H(A,F_T,V)$ computed according to (3) can be written as

$$H(A,F_T,V) \;=\; H_T(A) \;+\; W(V) \tag{8}$$

where the "free" part, called *(magnetic) thermal Hamiltonian*[1], is given by

$$H_T(A) \;:=\; K(A) \;+\; \frac{\nabla T}{c^2}\cdot\frac{1}{2}\big\{K(A),x\big\} \tag{9}$$

[1]An equivalently appropriate name for $H_T(A)$ could be (magnetic) *Luttinger Hamiltonian*.

and the effective *gravitational-matter potential* reads

$$W(V) \ := \ \left(1 + \frac{\nabla T}{c^2} \cdot x\right) \ V \tag{10}$$

The thermal Hamiltonian $H_T(A)$ is the analog of the Stark Hamiltonian when the system is perturbed by a gravitational-thermal field instead of an electric field. For this reason it seems natural to look for the extension of the results valid for the Stark Hamiltonian (*e. g.* [AH, DP, ADF]) to the case of the thermal Hamiltonian. This consists of two consecutive problems: (i) the analysis of the spectral theory of the "free" operator $H_T(A)$; (ii) the study of the scattering theory for the pair $H_T(A)$ and $H(A, F_T, V)$. Both of these problems seem not to have been studied yet in the literature, at least to the best of our knowledge. For this reason we devote this work at the analysis of the questions (i) and (ii) above in the one-dimensional case.

1.3. **Mathematical Formulation.** In order to formulate the problems sketched above in a rigorous mathematical setting we will make some simplifications. The most relevant concerns the absence of the magnetic field: From here on, unless otherwise indicated, we will fix $A = 0$. It is worth mentioning that this is not a major restriction as long as one wants to consider only the one-dimensional regime. Indeed in one spatial dimension the magnetic field is a pure gauge and can be removed with a unitary transformation[2] . The magnetic (multi-dimensional) case will be the subject of a future work.

As usual in mathematics, we will normalize all the physical units: $2m = \hbar = c = e = 1$. Moreover, we will denote with $\lambda := |\nabla T|$ the the strength of the thermal gradient and with $\gamma := \lambda^{-1}\nabla T \in \mathbb{S}^{d-1}$ its direction. With these simplifications the thermal Hamiltonian reads

$$H_T \ \equiv \ H_T(\lambda, \gamma) \ := \ p^2 \ + \ \frac{\lambda}{2}\big\{p^2, \gamma \cdot x\big\} \ . \tag{11}$$

However, the quantum formalism need the Hamiltonian governing the time evolution to be self-adjoint. Then, our first question is to prove the selfadjointness of $H_T$ on suitable domain $\mathcal{D}(H_T)$.

## 2. The operator $\widehat{T}$

The expression (11) is essentially formal until the domain of definition of $H_T$ is specified. However, $H_T$ is evidently well defined on the space of the compactly supported smooth function $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ or on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. On these dense domains the operator (11) acts as

$$\big(H_T\psi\big)(x) \ := \ -\left(1 + \lambda\, \gamma \cdot x\right) (\Delta\psi)(x) \ - \ \lambda\, (\gamma \cdot \nabla\psi)(x) \ , \qquad \psi \in \mathcal{S}(\mathbb{R}^d) \tag{12}$$

where $\Delta := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator and and $\gamma \cdot \nabla := \sum_{j=1}^d \gamma_j \frac{\partial}{\partial x_j}$. We can simplify the last expression with the help of two unitary transformations of the Hilbert space $L^2(\mathbb{R}^d)$. The first one is the rotation

$$(R_\gamma\psi)(x) \ := \ \psi\big(O_\gamma^{-1}x\big) \ , \qquad \psi \in L^2(\mathbb{R}^d) \tag{13}$$

[2]This fact can be interpreted as a consequence of the *Stone-von Neumann theorem* (see *e. g.* [Ros]). Indeed, in one spatial dimension the pair $x$, $\pi_f := p + f(x)$ necessarily meets the canonical commutation rule and so it is unitarily equivalent to the canonical pair $x, p$.

where the orthogonal matrix $O_\gamma$ meets the condition $O_\gamma \gamma = (1, 0, \ldots, 0)$. A short computation shows that

$$\big(R_\gamma H_T R_\gamma^* \psi\big)(x) \;=\; -\,(1 + \lambda\, x_1)\,(\Delta\psi)(x) \;-\; \lambda\, \frac{\partial \psi}{\partial x_1}(x)\;, \qquad\qquad \psi \in \mathcal{S}(\mathbb{R}^d)$$

where $x_1$ denotes the first component of the position vector $x = (x_1, x_\perp) \in \mathbb{R}^d$ and $x_\perp := (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$ is its orthogonal component. Evidently, the rotation $R_\gamma$ has the role of aligning the thermal-gravitational field along the $x_1$-axis[3]. The second transformation is the translation

$$(S_\lambda \psi)(x_1, x_\perp) \;:=\; \psi\left(x_1 - \frac{1}{\lambda}, x_\perp\right)\;, \qquad\qquad \psi \in L^2(\mathbb{R}^d) \tag{14}$$

and a direct calculation provides

$$\big(S_\lambda R_\gamma H_T R_\gamma^* S_\lambda^* \psi\big)(x) \;=\; \lambda \left[-x_1 (\Delta\psi)(x) \;-\; \frac{\partial \psi}{\partial x_1}(x)\right]\;, \qquad\qquad \psi \in \mathcal{S}(\mathbb{R}^d)\;. \tag{15}$$

The operator on the square brackets

$$(T\psi)(x) \;:=\; -x_1 (\Delta\psi)(x) \;-\; \frac{\partial \psi}{\partial x_1}(x)\;, \qquad\qquad \psi \in \mathcal{S}(\mathbb{R}^d) \tag{16}$$

agrees with the formal anti-commutator

$$T \;\equiv\; \frac{1}{2}\big\{p^2, x_1\big\} \;=\; \frac{1}{2}\big(p^2 x_1 \;+\; x_1 p^2\big) \tag{17}$$

when evaluated on sufficiently regular functions. With a slight abuse of notation we will often use the representation (17) for the operator $T$ instead of the more precise definition (16).

The unitary equivalence between $H_T$ and $\lambda T$ given by $S_\lambda R_\gamma$ implies that the spectral theory of the thermal Hamiltonian coincides with the spectral theory of the operator $T$ and in fact leaves $\mathcal{S}(\mathbb{R}^d)$ invariant. For this reason one is led to investigate if the operator $T$ defined by (16) (or by (17)) admits some self-adjoint extension and to compute the related spectrum. For technical reasons it results easier to face the equivalent problems reformulated in the Fourier space. Let $\mathscr{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the Fourier transform defined by

$$(\mathscr{F}\psi)(k) \;:=\; \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{d}x\; \mathrm{e}^{-\,\mathrm{i}\, k\cdot x}\; \psi(x)$$

on the dense subspace $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let $\widehat{T} := \mathscr{F} T \mathscr{F}^*$ be the Fourier transformed version of the operator (16). A direct computation provides

$$\big(\widehat{T}\psi\big)(x) \;:=\; \mathrm{i}\left[x_1 \psi(x) \;+\; x^2 \frac{\partial \psi}{\partial x_1}(x)\right]\;, \qquad\qquad \psi \in \mathcal{S}(\mathbb{R}^d) \tag{18}$$

where $x^2 := \sum_{j=1}^d x_j^2$. The operator defined by (18) agrees with the formal expression

$$\widehat{T} \;\equiv\; -\frac{1}{2}\big\{x^2, p_1\big\} \;=\; -\frac{1}{2}\big(x^2 p_1 \;+\; p_1 x^2\big) \tag{19}$$

on sufficiently regular functions[4].

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[3]Clearly, in dimension $d = 1$ the thermal-gravitational field is trivially aligned with the only reference axis and therefore $R_\gamma$ reduces to the identity.

[4]Formula (19) can be formally derived from (17) by using the well known transformations of the canonical operators $\mathscr{F} p_j \mathscr{F}^* = x_j$ and $\mathscr{F} x_j \mathscr{F}^* = -p_j$ for all $j = 1, \ldots, d$.

The representation (19) is quite intriguing if one compares the operator $\widehat{T}$ with the typical generator of $C_0$-groups associated to $\mathcal{C}^\infty$-flows [ABG, Chapter 4]. At first glance it would seem that the general theory of $C_0$-groups applies to $\widehat{T}$. However, a closer inspection to the $\mathbb{R}$-flow associated to $\widehat{T}$ shows that this is not the case in general (see Section 2.2 for more details). Therefore, the question of the self-adjointness of $\widehat{T}$ needs to be investigated with other tools.

The first fundamental question is whether the operator $\widehat{T}$ admits self-adjoint extensions or not. This is fortunately true.

**Proposition 2.1.** *The operator $\widehat{T}$ as defined by* (18)*, is* symmetric *(hence closable) on* $\mathcal{S}(\mathbb{R}^d)$

*Proof.* Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^d)$. We then have

$$\begin{aligned}
\langle \widehat{T}\psi, \varphi\rangle &= \int_{\mathbb{R}^d} \mathrm{d}x\ \overline{\widehat{T}\psi(x)}\varphi(x) \\
&= \int_{\mathbb{R}^d} \mathrm{d}x\ \overline{\left(ix^2\frac{\partial\psi}{\partial x_1}(x) + ix_1\psi(x)\right)}\varphi(x) \\
&= \int_{\mathbb{R}^d} \mathrm{d}x\ \left(i\frac{\partial\psi}{\partial x_1}\left(\psi(x)x\right)\right)\cdot\left(\varphi(x)x\right)
\end{aligned}$$

with $\cdot$ being the usual inner product in $\mathbb{C}^d$, anti-linear on the left side. Now using integration by parts on the $x_1$ coordinate we have, as both $\varphi(x)x$ and $\psi(x)x$ are regular enough, we have

$$\begin{aligned}
&= -\int_{\mathbb{R}^d} \mathrm{d}x\ \left(i\psi(x)x\right)\cdot\left(\frac{\partial\psi}{\partial x_1}\left(\varphi(x)x\right)\right) \\
&= \int_{\mathbb{R}^d} \mathrm{d}x\ \overline{\psi(x)}\left(ix^2\frac{\partial\varphi}{\partial x_1}(x) + ix_1\varphi(x)\right) \\
&= \langle\psi, \widehat{T}\varphi\rangle
\end{aligned}$$

As this holds true for arbitrary vectors in the operator's domain $\widehat{T}$ is symmetric. This observation allows us to identify $\widehat{T}$ with its closure (still denoted with the same symbol) defined on the domain

$$\mathcal{D}_0 := \overline{\mathcal{S}(\mathbb{R}^d)}^{\|\ \|_{\widehat{T}}} \tag{20}$$

obtained by the closure of $\mathcal{S}(\mathbb{R}^d)$ with respect to the graph-norm

$$\|\psi\|^2_{\widehat{T}} := \|\psi\|^2 + \|\widehat{T}\psi\|^2 .$$

The existence of self-adjoint extensions of $\widehat{T}$ is justified by the von Neumann's criterion as shown in A.5. Let $C$ be the complex-conjugation (anti-linear) operator on $L^2(\mathbb{R}^d)$ defined by $C\psi(x) = \overline{\psi(x)}$. Clearly the domains $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$ are left unchanged by $C$ and a direct check shows that $C\widehat{T} = \widehat{T}C$ on these domains. This is sufficient to prove the following preliminary result:

**Proposition 2.2.** *The closed symmetric operator $\widehat{T}$ with domain $\mathcal{D}_0$ has self-adjoint extensions.*

The latter result is preparatory for a precise definition of the family of thermal Hamiltonians.

**Definition 2.3** (Thermal Hamiltonian)**.** *Let $\widehat{T}_\theta$ be a given self-adjoint extension of the operator $\widehat{T}$ with domain $\mathcal{D}(\widehat{T}_\theta) \supset \mathcal{D}_0$. Let $\mathscr{U}(\lambda,\gamma) := \mathscr{F} S_\lambda R_\gamma$ be the unitary operator given by the product of the Fourier transform $\mathscr{F}$, the translation $S_\lambda$ defined by* (14) *and the rotation $R_\gamma$ defined by* (13)*. Then, the associated thermal Hamiltonian is the self-adjoint operator*

$$H_{T,\theta}(\lambda,\gamma) \;:=\; \lambda\, \mathscr{U}(\lambda,\gamma)^* \widehat{T}_\theta \mathscr{U}(\lambda,\gamma)\;, \qquad \lambda > 0,\;\; \gamma \in \mathbb{S}^{d-1}$$

*defined on the domain $\mathcal{D}(H_{T,\theta}) := \mathscr{U}(\lambda,\gamma)^*[\mathcal{D}(\widehat{T}_\theta)]$.*

Definition (2.3) reduces the question of the spectral theory of the thermal Hamiltonian to the analysis of the self-adjoint realizations of the operator $\widehat{T}$. This is usually done by studying the deficiency subspaces

$$\mathscr{K}_\pm \;:=\; \mathrm{Ker}(\mathrm{i} \mp \widehat{T}^*)\;.$$

The existence of the conjugation $C$ for $\widehat{T}$ implies the equality of the *deficiency indices* $n_\pm := \dim(\mathscr{K}_\pm)$ [RS2, Theorem X.3] which in turn ensures the existence of self-adjoint extensions. In order to build the spaces $\mathscr{K}_\pm$ and to compute $n_\pm$ one needs to solve the equations $\widehat{T}^*\psi = \pm\,\mathrm{i}\,\psi$ which, in view of (18), is equivalent of finding the *weak solutions* [RS1, Section V.4] to the differential equations

$$(21)\;\; (x_1^2 + x_\perp^2)\frac{\partial \psi}{\partial x_1}(x_1,x_\perp) \;+\; (x_1 \mp 1)\psi(x_1,x_\perp) \;=\; 0 \qquad \psi \in L^2(\mathbb{R}^d)\cap \mathcal{S}'(\mathbb{R}^d)\;.$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of *tempered distributions*[5]. This problem will be solved for the one-dimensional case in detail.

2.1. **Self Adjoint Realizations.** In the 1-dimensional case the operator $\widehat{T}$ is initially defined by

$$(22)\qquad \begin{aligned} (\widehat{T}\psi)(x) \;&=\; \mathrm{i}\left[x\psi(x) \;+\; x^2\frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\right] \\ &=\; \mathrm{i}\,x\frac{\mathrm{d}}{\mathrm{d}x}\left[x\psi(x)\right] \end{aligned} \qquad \psi \in \mathcal{S}(\mathbb{R})\;.$$

**Remark 2.4.** The last equality allow us to formally identify

$$\widehat{T} \;\equiv\; -xpx$$

on sufficiently regular functions. ◂

The operator (22) is symmetric, hence closable and its closure (still denoted with $\widehat{T}$) has domain $\mathcal{D}_0$ given by (20). In order to give a more precise characterization of $\mathcal{D}_0$ we will benefit from the transformation

$$(B\psi)(x) \;:=\; \frac{1}{x}\psi\left(\frac{1}{x}\right)\;, \qquad \psi \in L^2(\mathbb{R})\;.$$

**Lemma 2.5.** *$B$ is a unitary involution.*

[5] Similarly, one can consider weak solutions in $L^2(\mathbb{R}^d)\cap\mathcal{D}'(\mathbb{R}^d)$ where $\mathcal{D}'(\mathbb{R}^d)\supset\mathcal{S}'(\mathbb{R}^d)$ is the space of distributions.

*Proof.* A direct computation shows that

$$||B\psi||^2 = \int_{\mathbb{R}} \frac{1}{x^2}\left|\psi\left(\frac{1}{x}\right)\right|^2 \mathrm{d}x = \int_{-\infty}^{+\infty}\left|\psi\left(\frac{1}{x}\right)\right|^2 \mathrm{d}\left(\frac{1}{x}\right) = \int_{+\infty}^{-\infty} |\psi(s)|^2 \,\mathrm{d}s = ||\psi||^2.$$

Then $B$, initially defined on any good dense domain, extends to an isometry on the whole $L^2(\mathbb{R})$. From its definition it follows that $B^2\psi = \psi$. This shows that $B$ is an involution, and in particular it is invertible. □

Instead of $\widehat{T}$ let us consider the conjugated operator $\wp := B\widehat{T}B$ defined on the domain $\mathcal{D}(\wp) := B[\mathcal{D}_0]$. Using standard notation let $H^k(\Omega) := W^{k,2}(\Omega) \subset L^2(\Omega)$ be the $k$-th Sobolev space[6] with respect to the open set $\Omega \subseteq \mathbb{R}$. Let us introduce the space

$$H_0^1(\mathbb{R}) \;:=\; \left\{\phi \in H^1(\mathbb{R}) \;\middle|\; \phi(0) = 0\right\}$$

of the Sobolev functions on $\mathbb{R}$ vanishing in $x = 0$. This condition makes sense, as Sobolev functions can be uniquely identified with continuous functions [Bre, Theorem 8.2]. In view of this we will continue to identify Sobolev functions with their continuous representative so that the following inclusions $H_0^1(\mathbb{R}) \subset H^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$ hold.

**Proposition 2.6.** *The closed symmetric operator $\wp$ coincides with the momentum operator on $H_0^1(\mathbb{R})$, namely*

$$(\wp\phi)(x) \;=\; -\,\mathrm{i}\,\phi'(x)\,, \qquad\qquad \phi \in \mathcal{D}(\wp) \;=\; H_0^1(\mathbb{R})$$

*where $\phi'$ is the weak derivative of $\phi$.*

*Proof.* The unitarity of $B$ implies that the graph norms of $\wp$ and $\widehat{T}$ are related by $||\phi||_{\wp} = ||B\phi||_{\widehat{T}}$ for all $\phi \in \mathcal{D}(\wp)$. In fact

$$||\phi||_{\wp} = ||\phi|| + ||\wp\phi|| = ||\phi|| + ||B\widehat{T}B\phi|| = ||B\phi|| + ||\widehat{T}B\phi|| = ||B\phi||_{\widehat{T}}$$

This provides that

$$\mathcal{D}(\wp) \;=\; B(\mathcal{D}_0) \;=\; B\left[\overline{\mathcal{S}(\mathbb{R})}^{\,||\;||_{\widehat{T}}}\right] \;=\; \overline{B[\mathcal{S}(\mathbb{R})]}^{\,||\;||_{\wp}}\;.$$

Let $\phi \in B[\mathcal{S}(\mathbb{R})]$. Since $B\phi \in \mathcal{S}(\mathbb{R})$, one has from (22)

$$(\widehat{T}B\phi)(x) \;=\; \mathrm{i}\,x\frac{\mathrm{d}}{\mathrm{d}x}\left[x(B\phi)(x)\right] \;=\; \mathrm{i}\,x\frac{\mathrm{d}}{\mathrm{d}x}\left[\phi\left(\frac{1}{x}\right)\right] \;=\; -\frac{\mathrm{i}}{x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\left(\frac{1}{x}\right)\;.$$

Therefore

$$(\wp\phi)(x) \;=\; (B(\widehat{T}B\phi))(x) \;:=\; -\,\mathrm{i}\,\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)$$

acts as the momentum operator on $B[\mathcal{S}(\mathbb{R})]$. This implies that the domain of the closed operator $\wp$ is given by the closure of $B[\mathcal{S}(\mathbb{R})]$ with respect the Sobolev norm $||\phi||^2_{H^1} := ||\phi||^2 \;+\; ||\phi'||^2$. Let $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\backslash\{0\})$ be the set of smooth functions having compact support separated from the origin. Let us prove that

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\backslash\{0\}) \;\subset\; B[\mathcal{S}(\mathbb{R})] \;\subset\; H_0^1(\mathbb{R})\;. \tag{23}$$

For that, let $\psi \in C^\infty_c(\mathbb{R}\backslash\{0\})$ supported in $[-b,-a] \cup [a,b]$ and $\phi := B\psi$. A direct inspection shows that $\phi$ is a smooth function supported in $[-a^{-1},-b^{-1}] \cup [b^{-1},a^{-1}]$. This allows to conclude that $B[\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\backslash\{0\})] \subseteq \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\backslash\{0\})$. By exploiting

---

[6] For more details on the theory of Sobolev spaces we refer the reader to [Bre, Chapter 8 & Chapter 9].

the involutive character of $B$ one gets $B[\mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\})] = \mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\}) \subset \mathcal{S}(\mathbb{R})$ and in turn $\mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\}) \subset B[\mathcal{S}(\mathbb{R})]$. For the second contention let us take $\phi \in B(\mathcal{S}(\mathbb{R}))$ so that $\phi(x) = x^{-1}\psi(x^{-1})$ for some $\psi \in \mathcal{S}(\mathbb{R})$. Clearly, $\phi$ is smooth in $\mathbb{R} - \{0\}$, and again one can check that $\phi$ extends to a smooth function on $\mathbb{R}$ such that $\phi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. In particular $\phi \in H_0^1(\mathbb{R})$ implying the second inclusion $B[\mathcal{S}(\mathbb{R})] \subset H_0^1(\mathbb{R})$. To conclude the proof it is enough to show that the closure of the space $\mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\})$ with respect to the Sboolev norm $\|\ \|_{H^1}$ is (identifiable with) $H_0^1(\mathbb{R})$. Let $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$ and observe that

$$
\begin{aligned}
\overline{\mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\})}^{\|\ \|_{H^1}} &= \overline{\mathcal{C}_c^\infty(\mathbb{R}_-) \oplus \mathcal{C}_c^\infty(\mathbb{R}_+)}^{\|\ \|_{H^1}} \\
&= \overline{\mathcal{C}_c^\infty(\mathbb{R}_-)}^{\|\ \|_{H^1}} \oplus \overline{\mathcal{C}_c^\infty(\mathbb{R}_+)}^{\|\ \|_{H^1}} \\
&= W_0^{1,2}(\mathbb{R}_-) \oplus W_0^{1,2}(\mathbb{R}_+) = H_0^1(\mathbb{R})
\end{aligned}
\tag{24}
$$

where the notation for $W_0^{1,2}(\Omega)$ was borrowed from [Bre, Section 8.3]. The last equality in (24) is a consequence of the fact that every element of $W_0^{1,2}(\mathbb{R}_\pm)$ can be uniquely identified with a continuous function that vanishes on the boundary $x = 0$ [Bre, Theorem 8.12]. The identification (24), along with the double inclusion (23), implies $\mathcal{D}(\wp) = \overline{B[\mathcal{S}(\mathbb{R})]}^{\|\ \|_{H^1}} = H_0^1(\mathbb{R})$. □

As a first consequence of Proposition 2.1 we have a precise description of the domain of the closed operator $\widehat{T}$, *i.e.*

$$
\mathcal{D}_0 \ = \ B[\mathcal{D}(\wp)] \ = \ \left\{ \psi \in L^2(\mathbb{R}) \ \middle|\ \psi(x) = \frac{1}{x}\phi\left(\frac{1}{x}\right), \quad \phi \in H_0^1(\mathbb{R}) \right\} .
\tag{25}
$$

Unlike the functions in $H_0^1(\mathbb{R})$, the elements of the domain $\mathcal{D}_0$ are generally not continuous and can show singularities in $x = 0$. As an example consider $\phi(x) := (1+x^2)^{-\frac{1}{3}}\mathrm{e}^{-\frac{1}{x^2}}$ which evidently an element of $H_0^1(\mathbb{R})$. Its image $\psi(x) := (B\phi)(x) = (x^3 + x)^{-\frac{1}{3}}\mathrm{e}^{-x^2}$ is divergent in $x = 0$. A better characterization of the elements of $\mathcal{D}_0$ is provided in the following result.

**Proposition 2.7.** *Let $\psi \in \mathcal{D}_0$. Then it holds true that* $\lim_{x\to\pm\infty}(|x|\psi(x)) = 0$*;*

*Proof.* Property (i) follows from

$$
\lim_{x\to\pm\infty}(x\psi(x)) \ = \ \lim_{x\to\pm\infty}\phi\left(\frac{1}{x}\right) \ = \ \lim_{t\to 0^\pm}\phi(t) \ = \ 0
$$

where in the last equality we used the fact that $\phi \in H_0^1(\mathbb{R})$ is continuous and $\phi(0) = 0$. The equality (25) follows from $\mathcal{D}_0 = B[\mathcal{D}(\wp)]$ along with the description of $\mathcal{D}(\wp)$ provided in Proposition 2.1. □

We are now in position to study the self-adjoint realizations of $\widehat{T}$. Again, we can take advantage of the unitary transform $B$ to study the self-adjoint realization of *singular* momentum operator $\wp$. The latter is a classical problem strongly related with the study of singular delta interactions for one-dimensional Dirac operators [GS, BD, CMP] (see also [AGHG, Appendix J]).

It is of our interest to use Neumann's method of deficiency subspaces shown in A.9. to proceed, we will need an explicit realization of $\wp^*$.

**Proposition 2.8.** *The operator $\wp^*$ acts as $p_- \oplus p_+$ in $H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$ with $p_\pm\psi = -i\psi' \quad \forall \psi \in H^1(\mathbb{R}_\pm)$, acting as the weak derivative in the respective Sobolev space.*

*Proof.* As $\wp$ was defined as the closure of the derivative acting in $\mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\})$, we can profit the general fact $\overline{A}^* = A^*$ to simplify computations. For $\psi = \psi_- + \psi_+ \in H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$, $\phi = \phi_- + \phi_+ \in \mathcal{C}_c^\infty(\mathbb{R}\backslash\{0\}) = \mathcal{C}_c^\infty(\mathbb{R}_-) \oplus \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\langle \wp\,\phi \cdot \psi\rangle = \langle \wp\,\phi_+ \cdot \psi_+\rangle + \langle \wp\,\phi_- \cdot \psi_-\rangle = i(\langle \phi'_+ \cdot \psi_+\rangle + \langle \phi'_- \cdot \psi_-\rangle)$$

using integration by parts, as $\psi_\pm \in H^1(\mathbb{R}_\pm)$

$$= -i(\langle \phi_+ \cdot \psi'_+\rangle + \langle \phi_- \cdot \psi'_-\rangle) = \langle \phi \cdot -i(\psi'_- + \psi'_+)\rangle$$

Thus $\psi \in \mathcal{D}(\wp^*)$ and $\wp^*\psi = -i(\psi'_- + \psi'_+) = p_-\psi_- + p_+\psi_+$. Now let $\psi \in \mathcal{D}(\wp^*)$ and let $\varphi_\pm \in \mathcal{C}_c^\infty(\mathbb{R}_\pm) \subset \mathcal{D}(\wp)$. Let $\Pi_\pm$ be the projection operator acting as the restriction over $\mathbb{R}_\pm$. We then have

$$\langle -i(\varphi_\pm)' \cdot \psi\rangle = \langle \wp\,\varphi_\pm \cdot \psi\rangle = \langle \varphi_\pm \cdot \wp^*\psi\rangle$$

As $\varphi_\pm$ its first derivative have support containted in $\mathbb{R}\pm$ this is implies

$$\langle -i(\varphi_\pm)' \cdot \Pi_\pm\psi\rangle = \langle \varphi_\pm \cdot \Pi_\pm\wp^*\psi\rangle$$

We conclude of this that $\Pi_\pm\psi$ has a weak derivative and in fact $\Pi_\pm\psi \in H^1(\mathbb{R}_\pm)$. Computing that derivative we have $-i(\Pi_\pm\psi)' = \Pi_\pm\wp^*\psi$, and knowing both $\Pi_\pm\wp^*\psi$ completely determines $\wp^*\psi$, we conclude

$$\wp^*\psi = -i(\Pi_+\psi)' + -i(\Pi_-\psi)' \tag{26}$$

And

$$\mathcal{D}(\wp^*) = H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+) \tag{27}$$

□

Now having an explicit descritption of $\wp^*$ we will completely characterize the self adjoint extensions of $\wp$, all which are restrictions of $\wp^*$

**Proposition 2.9.** *The closed symmetric operator $\wp$ has deficiency indices equal to 1. Therefore, the self-adjoint extensions of $\wp$ are in one-to-one correspondence with the angles $\theta \in \mathbb{S} \simeq [0, 2\pi)$. The self-adjoint extension $\wp_\theta$ has domain*

$$\mathcal{D}(\wp_\theta) \;:=\; \left\{\varphi \in L^2(\mathbb{R}) \,\middle|\, \varphi = \phi + c\eta_\theta \;, \phi \in H_0^1(\mathbb{R}) \;, c \in \mathbb{C}\right\}$$

*where*[7]

$$\eta_\theta(x) \;:=\; \begin{Bmatrix} \mathrm{e}^{-\left(x - \mathrm{i}\,\frac{\theta}{2}\right)} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ \mathrm{e}^{+\left(x - \mathrm{i}\,\frac{\theta}{2}\right)} & \text{if } x < 0 \end{Bmatrix} \;=\; \mathrm{e}^{-|x|}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}}$$

*and acts has*

$$\wp_\theta\,(\phi + c\eta_\theta) \;:=\; -\,\mathrm{i}\,\phi' \;+\; c\eta_{\theta+\pi}\;. \tag{28}$$

*In addition, $\mathcal{D}(\wp) \oplus \mathbb{C}\eta_\theta$ is a core for $\wp_\theta$. Finally, $\wp_0$ agrees with the standard momentum operator $p$ with domain $H^1(\mathbb{R})$.*

---

[7]The *sign function* is defined by $\mathrm{sgn}(x) \;:=\; \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$

*Proof.* As shown in in proposition 2.8 the adjoint of $\wp$ acts as the weak derivative on its domain $\mathcal{D}(\wp^*) := H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$. We will now determine the deficiency subspaces, as given by the eigenvalues equations $\wp^* \phi_\pm = \pm \mathrm{i}\, \phi_\pm$. These correspond to the differential equations $\phi'_\pm = \mp \phi_\pm$ which admit in $\mathcal{D}(\wp^*)$ the unique (normalized) weak solutions

$$\phi_+(x) \ := \ \begin{cases} \sqrt{2}\mathrm{e}^{-x} & \text{if } \ x > 0 \\ 0 & \text{if } \ x < 0 \end{cases} , \qquad \phi_-(x) \ := \ \begin{cases} 0 & \text{if } \ x > 0 \\ \sqrt{2}\mathrm{e}^{+x} & \text{if } \ x < 0 \end{cases} .$$

According to Theorem A.9 one has that the self-adjoint extensions of $\wp$ are parametrized by the unitary maps from $\mathscr{K}_+ = \mathbb{C}[\phi_+] \simeq \mathbb{C}$ to $\mathscr{K}_- = \mathbb{C}[\phi_-] \simeq \mathbb{C}$. The later are identified by the angle $\theta \in \mathbb{S}^1 \simeq [0, 2\pi)$ according to $U_\theta \phi_+ := \mathrm{e}^{-\mathrm{i}\,\theta} \psi_-$. One also has that the domain of the self-adjoint extension $\wp_\theta$ is made by functions of the type $\phi + c'(\phi_+ + \mathrm{e}^{-\mathrm{i}\,\theta}\phi_-) = \phi + c\eta_\theta$ with $\phi \in H^1_0(\mathbb{R})$ and $c, c' \in \mathbb{C}$ suitable complex coefficients. The action of $\wp_\theta$ on the elements of its domain is given by

$$\wp_\theta\big(\phi \ + \ c'(\phi_+ \ + \ \mathrm{e}^{-\mathrm{i}\,\theta}\phi_-)\big) \ = \ -\mathrm{i}\,\phi' \ + \ \mathrm{i}\,c'(\phi_+ \ - \ \mathrm{e}^{-\mathrm{i}\,\theta}\phi_-)$$

which translates in the equation (28) in terms of the function $\eta_\theta$. Evidently, the standard momentum operator $p$ is a self-adjoint extension of $\wp$ since $H^1_0(\mathbb{R}) \subset H^1(\mathbb{R})$. This extension corresponds to $\wp_0$ in view of the fact that $\eta_0 \in H^1(\mathbb{R})$. Finally, from A.10 one obtains the core for $\wp_\theta$ □

Although the symmetric operator $\wp$ admits several self-adjoint realizations, all these realization are in a sense equivalent. To express this fact in a precise way we need to introduce the family of unitary operators $L_\theta$ defined by

$$(L_\theta \psi)(x) \ := \ \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}} \psi(x) \, , \qquad \psi \in L^2(\mathbb{R}) \, .$$

**Proposition 2.10.** *The unitary operators $L_\theta$ intertwine all the self-adjoint realizations of the operator $\wp$. More precisely one has that*

$$\wp_\theta \ = \ L_\theta p L_\theta^* \, , \qquad \theta \in \mathbb{S}^1$$

*where $p = \wp_0$ is the standard momentum operator. As a consequence one has that*

$$\sigma(\wp_\theta) \ = \ \sigma_{\text{a.c.}}(\wp_\theta) \ = \ \mathbb{R} \, , \qquad \theta \in \mathbb{S}^1 \, .$$

*Proof.* A direct computation shows

$$L_\theta^* = L_{-\theta} \text{ and } L_\beta \eta_\gamma = \eta_{\beta+\gamma}. \tag{29}$$

We will proceed by proving $L_\theta p L_\theta^* \subset \wp_\theta$. We start by checking the domain contention, $\mathcal{D}(L_\theta p L_\theta^*) = L_\theta \mathcal{D}(p) \subset \mathcal{D}(\wp_\theta)$.
Let $L_\theta \psi \in L_\theta \mathcal{D}(p)$. As $\psi \in H^1(\mathbb{R}) = \mathcal{D}(p)$, $\psi$ has a continious representative and we can write $\psi = (\psi - \psi(0)\eta_0) + \psi(0)\eta_0$. Let us note that $L_\theta(\psi - \psi(0)\eta_0) \in H^1_0(\mathbb{R})$; as it satisfies the border condition, and for $g \in \mathcal{S}(\mathbb{R})$ we have

$$\langle g' \cdot L_\theta(\psi - \psi(0)\eta_0) \rangle = \int_{-\infty}^{+\infty} \overline{g'(x)} \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}} (\psi(x) - \psi(0)\eta_0(x))\,\mathrm{d}x$$

$$= \mathrm{e}^{-\mathrm{i}\,\frac{\theta}{2}} \int_{-\infty}^{0} \overline{g'(x)} (\psi'(x) - \psi(0)\mathrm{e}^{x})\,\mathrm{d}x + \mathrm{e}^{\mathrm{i}\,\frac{\theta}{2}} \int_{0}^{+\infty} \overline{g'(x)} (\psi(x) - \psi(0)\mathrm{e}^{-x})\,\mathrm{d}x$$

integrating by parts

$$= b - \mathrm{e}^{-\mathrm{i}\,\frac{\theta}{2}} \int_{-\infty}^{0} \overline{g(x)} (\psi'(x) - \psi(0)\mathrm{e}^{x})\,\mathrm{d}x - \mathrm{e}^{\mathrm{i}\,\frac{\theta}{2}} \int_{0}^{+\infty} \overline{g(x)} (\psi'(x) + \psi(0)\mathrm{e}^{-x})\,\mathrm{d}x$$

with $b = \mathrm{e}^{-\mathrm{i}\,\frac{\theta}{2}}\overline{g(x)}(\psi(x)-\psi(0)\mathrm{e}^{x})\Big|_{-\infty}^{0} + \mathrm{e}^{\mathrm{i}\,\frac{\theta}{2}}\overline{g(x)}(\psi(x)-\psi(0)\mathrm{e}^{-x})\Big|_{0}^{\infty} = 0$, thus

$$\langle g' \cdot L_\theta(\psi - \psi(0)\eta_0)\rangle = -\int_{-\infty}^{+\infty} \overline{g(x)}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}} \left(\psi'(x) + \psi(0)\mathrm{sgn}(x)\mathrm{e}^{-|x|}\right) \mathrm{d}x$$

This in turn gives us the derivative in question:
(30)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(L_\theta(\psi - \psi(0)\eta_0)\right)(x) = \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}}\left(\psi'(x) + \psi(0)\mathrm{sgn}(x)\mathrm{e}^{-|x|}\right) = \left(L_\theta \psi'\right)(x) + -i\psi(0)\eta_{\theta+\pi}$$

which is square integrable as both $\psi'$ and $\mathrm{e}^{-|x|}$ are. Also $L_\theta(\psi(0)\eta_0) = \psi(0)\eta_\theta$ and as such $L_\theta\,\psi \in \mathcal{D}(\wp_\theta)$.
We will now prove that both operators coincide in $L_\theta\mathcal{D}(p)$. We have

$$(L_\theta p L_\theta^*)L_\theta\psi = -iL_\theta\psi',$$

and

$$\begin{aligned}\wp_\theta L_\theta\psi =& \wp_\theta L_\theta\left((\psi - \psi(0)\eta_0) + \psi(0)\eta_0\right)\\ =& \wp_\theta\left(L_\theta(\psi - \psi(0)\eta_0) + \psi(0)\eta_\theta\right)\end{aligned}$$

As we already saw $L_\theta(\psi - \psi(0)\eta_0) \in H_0^1(\mathbb{R})$, so by equation (28) and (30)

$$\begin{aligned}\wp_\theta L_\theta\psi =& -i\frac{\mathrm{d}}{\mathrm{d}x}L_\theta(\psi - \psi(0)\eta_0) + \psi(0)\eta_{\theta+\pi}\\ =& -i\left(L_\theta\psi' + -i\psi(0)\eta_{\theta+\pi}\right) + \psi(0)\eta_{\theta+\pi}\\ =& -iL_\theta\psi'\end{aligned}$$

We conclude $L_\theta p L_\theta^* \subset \wp_\theta$, and as both are self adjoint, they must be equal. Finally as two unitarily equivalent operators must have the same spectral decomposition, we have $\sigma(\wp_\theta) = \sigma_{\mathrm{a.c.}}(\wp_\theta) = \mathbb{R}$ as can be seen in appendix A.1. $\square$

We are now in position to provide a complete description of the self-adjoint extensions of the one-dimensional version of the operator $\widehat{T}$.

**Theorem 2.11** (Self-adjoint extensions: one-dimensional case)**.** *The self-adjoint extensions of the closed symmetric operator $\widehat{T}$ initially defined by (22) are in one-to-one correspondence with the angles $\theta \in \mathbb{S}$. The self-adjoint extension $\widehat{T}_\theta$ has domain*

$$\mathcal{D}(\widehat{T}_\theta) \;:=\; \left\{\varphi \in L^2(\mathbb{R}) \,\middle|\, \varphi = \psi + c\zeta_\theta \,, \psi \in \mathcal{D}_0 \,, c \in \mathbb{C}\right\}$$

*where*

$$\zeta_\theta(x) \;:=\; \left\{\begin{array}{ll} \dfrac{1}{x}\,\mathrm{e}^{-\left(\frac{1}{x}+\mathrm{i}\,\frac{\theta}{2}\right)} & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \dfrac{1}{x}\,\mathrm{e}^{+\left(\frac{1}{x}+\mathrm{i}\,\frac{\theta}{2}\right)} & \text{if } x<0 \end{array}\right\} \;=\; \frac{1}{x}\mathrm{e}^{-\frac{1}{|x|}}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}}$$

*and acts has*

$$\widehat{T}_\theta\left(\psi + c\zeta_\theta\right) \;:=\; \widehat{T}\psi \;+\; c\zeta_{\theta+\pi}\;.$$

*All the self-adjoint realization are unitarily equivalent, i.e. $\widehat{T}_\theta = L_\theta\widehat{T}_0L_\theta^*$ for all $\theta \in \mathbb{S}$. In addition, $\mathcal{D}(\widehat{T}) \oplus \mathbb{C}\zeta_\theta$ is a core for $\widehat{T}_\theta$ Finally one has that*

$$\sigma(\widehat{T}_\theta) \;=\; \sigma_{\mathrm{a.c.}}(\widehat{T}_\theta) \;=\; \mathbb{R}\,, \qquad \theta \in \mathbb{S}^1\;.$$

*Proof.* The theorem is a direct consequence of the unitary equivalence established in Proposition which allows to define the self-adjoint realizations of $\widehat{T}$ by $\widehat{T}_\theta := B\wp_\theta B$. At this point the claim of the theorem is nothing more than a re-writing of Proposition 2.9 and Proposition 2.10. The formula $\widehat{T}_\theta = L_\theta \widehat{T}_0 L_\theta^*$ is justified by the commutation relation $L_\theta B = BL_\theta$ and as such

$$\widehat{T}_\theta = B\wp_\theta B = BL_\theta\wp_0 L_\theta^* B = L_\theta B\wp_0 BL_\theta^* = L_\theta \widehat{T}_0 L_\theta^*$$

□

**Remark 2.12** (Standard realization)**.** In view of the unitary equivalence among all the self-adjoint realizations proved in Theorem 2.11 we can focus our attention only in a preferred member of the family of the self-adjoint extension. We will refer to $\widehat{T}_0$ as the *standard* realization of the operator initially defined by (22). ◂

**Remark 2.13** (Boundary triplets)**.** The business of the determination of the self-adjoint realizations of $\wp$ or $\widehat{T}$ can be also investigated inside the theory of the *boundary triplets*, in subsection A.2. Let us start with the operator $\wp$ and its adjoint $\wp^*$. According to definiton A.11, a boundary triplet for $\wp^*$ is a triplet $(\mathbb{H}, \Gamma_0, \Gamma_1)$ made by an Hilbert space $\mathbb{H}$ and linear maps $\Gamma_0, \Gamma_1$ from $\mathcal{D}(\wp^*)$ to $\mathbb{H}$ that satisfy the *abstract Green's identity*

$$\langle \wp^*\varphi, \psi\rangle - \langle \varphi, \wp^*\psi\rangle = \langle \Gamma_0\varphi, \Gamma_1\psi\rangle_{\mathbb{H}} - \langle \Gamma_1\varphi, \Gamma_0\psi\rangle_{\mathbb{H}} , \qquad \forall\, \varphi, \psi \in \mathcal{D}(\wp^*)$$

and the mapping $\mathcal{D}(\wp^*) \ni \varphi \mapsto (\Gamma_0\varphi, \Gamma_1\varphi) \in \mathbb{H}\times\mathbb{H}$ is surjective. Since the operator $\wp^*$ acts as the weak derivative on its domain $\mathcal{D}(\wp^*) := H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$, an integration by parts provides

$$\langle \wp^*\varphi, \psi\rangle - \langle \varphi, \wp^*\psi\rangle = \mathrm{i}\left(\overline{\varphi(0^-)}\psi(0^-) - \overline{\varphi(0^+)}\psi(0^+)\right) , \qquad \forall\, \varphi, \psi \in \mathcal{D}(\wp^*)$$

where $\varphi(0^\pm) := \lim_{x\to 0^\pm}\varphi(x)$ and similarly for $\psi(0^\pm)$. A comparison with the abstract Green's identity shows that the triplet $(\mathbb{H}, \Gamma_0, \Gamma_1)$ can be fixed in the following way: $\mathbb{H} := \mathbb{C}$;

$$\Gamma_0\varphi := \frac{\varphi(0^+) - \varphi(0^-)}{\mathrm{i}\sqrt{2}} , \qquad \Gamma_1\varphi := \frac{\varphi(0^+) + \varphi(0^-)}{\sqrt{2}} ,$$

The surjectivity condition is obviously satisfied. Observe that $\mathrm{Ker}(\Gamma_0)\cap\mathrm{Ker}(\Gamma_1) = H^1_0(\mathbb{R}) = \mathcal{D}(\wp)$. By theorem A.13, the self-adjoint extensions of $\wp$ are parametrized by the self-adjoint operators acting on the closed subspaces of $\mathcal{K} = \mathbb{C}$. For the particular trivial operator acting on the $\{0\}$ space we have:

$$\mathcal{D} := \left\{\varphi \in \mathcal{D}(\wp^*) \;\middle|\; \Gamma_0\varphi = 0\right\} = H^1(\mathbb{R}) \tag{31}$$

Recovering the usual momentum operator. The other cases are all given by operators acting as multiplication by a real number $\gamma \in \mathbb{R}$ defining a restriction $\wp_\gamma := \wp^*|_{\mathcal{D}_\gamma}$ where the domain $\mathcal{D}_\gamma \subset \mathcal{D}(\wp^*)$ is defined by

(32)
$$\begin{aligned}
\mathcal{D}_\gamma &:= \left\{\varphi \in \mathcal{D}(\wp^*) \;\middle|\; \gamma\Gamma_0\varphi = \Gamma_1\varphi\right\} \\
&= \left\{\varphi \in H^1(\mathbb{R}_-)\oplus H^1(\mathbb{R}_+) \;\middle|\; (\gamma - \mathrm{i})\varphi(0^+) = (\gamma + \mathrm{i})\varphi(0^-)\right\} \\
&= \left\{\varphi \in H^1(\mathbb{R}_-)\oplus H^1(\mathbb{R}_+) \;\middle|\; \mathrm{e}^{-\mathrm{i}\arctan\left(\frac{1}{\gamma}\right)}\varphi(0^+) = \mathrm{e}^{+\mathrm{i}\arctan\left(\frac{1}{\gamma}\right)}\varphi(0^-)\right\} .
\end{aligned}$$

A comparison with Proposition 2.9 shows that the self-adjoint extensions $\wp_\theta$ and $\wp_\gamma$ are related by the equation $\theta(\gamma) = \arctan\left(\frac{1}{\gamma}\right)$. In particular, the standard momentum is reidentified by $\gamma = \infty$ which corresponds to $\theta = 0$. The definition (32) provides the description of the domain of $\wp_\theta$ in therms of *boundary conditions*. The same can be done for the the self-adjoint extensions $\widehat{T}_\theta$ with the help of the unitary operator $B$. A direct computation shows that

$$\mathcal{D}(\widehat{T}_\theta) \ := \ \left\{\varphi \in B[H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)] \mid \mathrm{e}^{-\,\mathrm{i}\,\frac{\theta}{2}}(x\varphi)(+\infty) = \mathrm{e}^{+\,\mathrm{i}\,\frac{\theta}{2}}(x\varphi)(-\infty)\right\}$$

where $(x\varphi)(\pm\infty) := \lim_{x\to\pm\infty} x\varphi(x)$. It is interesting to note that for $\widehat{T}$ to be self adjoint, boundary conditions in infinity must be satisfied, similarly to the case of momentum operator over a finite interval.

2.2. **Unitary propagator.** Let

$$U_\theta(t) \ := \ \mathrm{e}^{-\,\mathrm{i}\,t\widehat{T}_\theta}\ , \qquad t \in \mathbb{R} \tag{33}$$

be the *unitary propagator* defined by the self-adjoint operator $\widehat{T}_\theta$ on $L^2(\mathbb{R})$. The description of $U_\theta(t)$ is provided in the following theorem.

**Theorem 2.14.** *Let $U_\theta(t)$ be the unitary group defined by* (33)*. It holds true that*

$$\big(U_\theta(t)\psi\big)(t) \ = \ \frac{\mathrm{e}^{\frac{\mathrm{i}}{2}\left(1-\mathrm{sgn}(1-tx)\right)\mathrm{sgn}(x)\theta}}{1-tx}\psi\left(\frac{x}{1-tx}\right)\ , \qquad \psi \in L^2(\mathbb{R})\ . \tag{34}$$

*Proof.* We can use the unitary equivalence $\widehat{T}_\theta = BL_\theta pL_\theta^* B$ proved in Theorem 2.11 which implies that $U_\theta(t) = BL_\theta \mathrm{e}^{-\,\mathrm{i}\,tp}L_\theta^* B$ along with the well-known fact $(\mathrm{e}^{-\,\mathrm{i}\,tp}\psi)(x) = \psi(x-t)$. The proof of the claim follows by a direct computation. ☐

For each $t \in \mathbb{R}$ let us consider the map $f_t : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ defined by

$$f_t(x) \ := \ \begin{cases} \dfrac{x}{1-tx} & \text{if } \ x \in \mathbb{R}\backslash\{t^{-1}\} \\ \infty & \text{if } \ x = t^{-1} \\ -\,t^{-1} & \text{if } \ x = \infty\ . \end{cases} \tag{35}$$

with the convention that $\pm 0^{-1} \equiv \infty$. The family of these maps defines an $\mathbb{R}$-*flow* in the sense, a direct check shows that the following relations hold:

$$\begin{cases} f_0 \ = \ \mathrm{Id} \\ f_{t_1}\circ f_{t_2} \ = \ f_{t_1+t_2} \\ f_t^{-1} \ = \ f_{-t} \end{cases} \qquad \forall\ t, t_1, t_2 \in \ \mathbb{R}\ . \tag{36}$$

The flow $f_t$ allows to rewrite the action unitary propagator $U_\theta(t)$ in the form

$$\big(U_\theta(t)\psi\big)(t) \ = \ \mathrm{e}^{\frac{\mathrm{i}}{2}\left(1-\mathrm{sgn}(1-tx)\right)\left(\mathrm{sgn}(x)\theta+\pi\right)}\ \sqrt{(\partial_x f_t)(x)}\ \psi\left(f_t(x)\right) \tag{37}$$

When $\theta = \pi$ equation (37) agrees with the definition of the $C_0$-*group* associated to the flow $f_t$ as defined in [ABG, Section 4.2]. It is interesting to notice that the flow $f_t$ is not of class $\mathcal{C}^\infty$ and the generator of the flow

$$F(x) \ := \ \left.\frac{\mathrm{d}f_t}{\mathrm{d}t}\right|_{t=0}(x) \ = \ x^2$$

has an unbounded first derivative. Therefore the flow $f_t$ doesn't meet the conditions of [ABG, Lemma 4.2.2 & Proposition 4.2.3]. The latter fact explains why [ABG, Proposition 4.2.3] doesn't apply to the operator $\widehat{T} \equiv -\frac{1}{2}(pF(x) + F(x)p)$ which indeed is not essentially self-adjoint on $\mathcal{C}^\infty_{\rm c}(\mathbb{R})$.

2.3. **Resolvent and Green Function.** The resolvent of the of the operator $\widehat{T}_\theta$ can be derived from the resolvent of the standard momentum operator $p$ by exploiting the various unitary equivalences described in Section 2.1. For every $\zeta \in \mathbb{C}\backslash\mathbb{R}$ the resolvent of $\widehat{T}_\theta$ at $\zeta$ is defined as

$$R_\zeta(\widehat{T}_\theta) \;:=\; \left(\widehat{T}_\theta - \zeta\mathbf{1}\right)^{-1} \;=\; L_\theta B\,(p-\zeta\mathbf{1})^{-1}\,BL_\theta^* \;. \tag{38}$$

The explicit form of the resolvent is then computable and explicit:

**Proposition 2.15.** *Let $\zeta := \epsilon \pm {\rm i}\,\delta \in \mathbb{C}\backslash\mathbb{R}$ with $\delta > 0$. The resolvent $R_\zeta(\widehat{T}_\theta)$ acts as*

$$\big(R_\zeta(\widehat{T}_\theta)\psi\big)(x) \;=\; \int_{\mathbb{R}} {\rm d}y\, \mathscr{R}^\theta_\zeta(x,y)\,\psi(y)\;, \qquad\qquad \psi \in L^2(\mathbb{R})$$

*with kernel given by*

$$\mathscr{R}^\theta_{\epsilon\pm{\rm i}\,\delta}(x,y) \;:=\; \frac{{\rm e}^{{\rm i}\left(\mathrm{sgn}(x)-\mathrm{sgn}(y)\right)\frac{\theta}{2}}}{\mp{\rm i}\,xy}\,\Theta\left(\pm\left(\frac{1}{x}-\frac{1}{y}\right)\right)\;{\rm e}^{{\rm i}\,\epsilon\left(\frac{1}{x}-\frac{1}{y}\right)}{\rm e}^{-\delta\left|\frac{1}{x}-\frac{1}{y}\right|}$$

*where $\Theta$ is the* Heaviside function. [8]

*Proof.* The integral kernel $\mathscr{R}^0_\zeta$ of the resolvent of $\widehat{T}_0$ can be obtained from the Green's function $\mathscr{G}^0_\zeta$ of the standard momentum operator (see Appendix A.1). A direct computation provides

$$\big(R_\zeta(\widehat{T}_0)\psi\big)(x) \;=\; \big(B\,(p-\zeta\mathbf{1})^{-1}\,B\psi\big)(x) \;=\; \frac{1}{x}\int_{\mathbb{R}}{\rm d}u\,\mathscr{G}^0_\zeta\left(\frac{1}{x},u\right)\,\frac{1}{u}\psi\left(\frac{1}{u}\right)\;.$$

After the change of variables $y = u^{-1}$ The explicit expression of $\mathscr{G}^0_\zeta$ given in (104) and a change of variable in the integral provide

$$\mathscr{R}^0_\zeta(x,y) \;:=\; \frac{1}{xy}\mathscr{G}^0_\zeta\left(\frac{1}{x},\frac{1}{y}\right)\;.$$

Since $L_\theta$ is a multiplication operator, the relation between the kernels for $\theta = 0$ and $\theta \neq 0$ is simply given by

$$\mathscr{R}^\theta_\zeta(x,y) \;:=\; {\rm e}^{{\rm i}\left(\mathrm{sgn}(x)-\mathrm{sgn}(y)\right)\frac{\theta}{2}}\mathscr{R}^0_\zeta(x,y) \;=\; \frac{{\rm e}^{{\rm i}\left(\mathrm{sgn}(x)-\mathrm{sgn}(y)\right)\frac{\theta}{2}}}{xy}\mathscr{G}^0_\zeta\left(\frac{1}{x},\frac{1}{y}\right)\;.$$

This concludes the proof. $\square$

[8] The *Heaviside function* is defined by $\Theta(x) \;:=\; \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0\,. \end{cases}$

2.4. **Spectral measure.** Let $\mu^\theta_\psi$ be the spectral measure of the operator $\widehat{T}_\theta$ associated with the normalized state $\psi \in L^2(\mathbb{R})$. We know from Theorem 2.11 that $\widehat{T}_\theta$ as a purely absolutely continuous spectrum which coincides with $\mathbb{R}$, and as such, the spectral measure $\mu^\theta_\psi$ is purely absolutely continuous and can be written as

$$\mu^\theta_\psi(\,\mathrm{d}\epsilon) \;:=\; f^\theta_\psi(\epsilon)\;\mathrm{d}\epsilon$$

with $f^\theta_\psi \in L^1(\mathbb{R})$ a non-negative function whose integral equals to 1. The next result provides a description of $f^\theta_\psi$.

**Proposition 2.16.** *Let $\mu^\theta_\psi$ be the spectral measure of the operator $\widehat{T}_\theta$ associated with the (normalized) state $\psi \in L^2(\mathbb{R})$. Then $\mu^\theta_\psi$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\epsilon$ in $\mathbb{R}$ and*

$$\mu^\theta_\psi(\,\mathrm{d}\epsilon) \;:=\; |\widehat{\phi}_\theta(\epsilon)|^2\;\mathrm{d}\epsilon \tag{39}$$

*where $\widehat{\phi}_\theta := \mathscr{F}(\phi_\theta)$ is the Fourier transform of the function*

$$\phi_\theta(x) \;:=\; (L^*_\theta B\psi)(x) \;=\; \frac{\mathrm{e}^{-\,\mathrm{i}\,\mathrm{sgn}(x)\frac{\theta}{2}}}{x}\psi\left(\frac{1}{x}\right)\,.$$

*Proof.* Starting from the unitary equivalence $\widehat{T}_\theta = BL_\theta p L^*_\theta B$ one gets

$$F^\theta_\psi(\zeta) \;:=\; \langle\psi,(\widehat{T}_\theta-\zeta\mathbf{1})^{-1}\psi\rangle \;=\; \langle\psi, BL_\theta(p-\zeta\mathbf{1})^{-1}L^*_\theta B\psi\rangle \;=\; F^p_{\phi_\theta}(\zeta)\,.$$

Following the arguments in Appendix **??** on gets

$$\lim_{\delta\to 0^+}\frac{1}{\pi}\,\mathrm{Im}\left(F^\theta_\psi(\epsilon+\mathrm{i}\,\delta)\right) \;=\; f^p_{\phi_\theta}(\epsilon) \;=\; |\widehat{\phi}_\theta(\epsilon)|^2\,.$$

where the last equality is justified by (106). This concludes the proof. □

## 3. The operators $T$ and $H_T$

In this section we will study the operators $T$ and $H_T$. As we are treating the 1-dimensional case, the rotation $R_\gamma$ becomes either the identity or the inversion on the spatial domain. Both cases can be covered considering $\lambda > 0$ and $\lambda < 0$. Then equation (15) becomes

$$H_T \;=\; \lambda S^*_\lambda T S_\lambda\,. \tag{40}$$

Also, from the relation $\widehat{T} := \mathscr{F}T\mathscr{F}^*$ comes that every self-adjoint extension of $T$ is unitarily equivalent to a self-adjoint extension of $\widehat{T}$. The following proposition illustrates this point

**Proposition 3.1** (Self-adjoints extensions of $T$)**.** *Every self adjoint extension of $T$ is of the form $T_\theta = \mathscr{F}^*\widehat{T}_\theta\mathscr{F}$ with domain $\mathcal{D}(T_\theta) = \mathscr{F}^*\mathcal{D}(\widehat{T}_\theta)$ with $\theta \in \mathbb{S}$.*
*Moreover, all the self-adjoint realization are unitarily equivalent, i.e. $T_\theta \;=\; S_\theta T_0 S^*_\theta$ for all $\theta \in \mathbb{S}$ With the $S_\theta$ being given by*

$$\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathcal{H} \tag{41}$$

*With $\mathcal{H}$ the Hilbert transform, acting as*

$$\mathcal{H}\psi(x) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\psi(y)}{x-y}\,\mathrm{d}y \tag{42}$$

*Over sufficiently regular functions, with the integral taken as a Cauchy principal value. Finally one has that*

$$\sigma(\widehat{T}_\theta) \;=\; \sigma_{\text{a.c.}}(\widehat{T}_\theta) \;=\; \mathbb{R}\,, \qquad\qquad \theta \in \mathbb{S}^1\,.$$

*Proof.* From theorem 2.11 we have

$$T_\theta = \mathscr{F}^*\widehat{T}_\theta\mathscr{F} = \mathscr{F}^*L_\theta\widehat{T}_0L_\theta^*\mathscr{F} = \mathscr{F}^*L_\theta\mathscr{F}T_0\mathscr{F}^*L_\theta^*\mathscr{F}$$

We now set

$$S_\theta = \mathscr{F}^*L_\theta\mathscr{F} \tag{43}$$

As $L_\theta$ acts as multiplication by $e^{\operatorname{sign} x\,\mathrm{i}\,\theta/2} = \cos\theta/2 + \operatorname{sign} x\,\mathrm{i}\,\sin\theta/2$. As in seen in equation (100) $\mathscr{F}^*f(q)\mathscr{F}\psi = f(p)\psi = \frac{1}{\sqrt{2\pi}}\check{f} * \psi$. Finally It is known that $\widetilde{(\operatorname{sign} x)}(k) = i\sqrt{\frac{2}{\pi}}k^{-1}$. we have.

$$S_\theta\psi(x) = \mathscr{F}^*L_\theta\mathscr{F}\psi = \cos\frac{\theta}{2}\psi(x) + i\sin\frac{\theta}{2}\operatorname{sign}(p)\psi(x) = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\frac{1}{\pi}\int_{\mathbb{R}}\frac{\psi(y)}{x-y}\,\mathrm{d}y.$$

□

**Remark 3.2** (Self-adjoint extensions of $H_T$). *Analogously every self adjoint extension of $H_{T,\theta}$ is of the form $H_{T,\theta} = \lambda S_\lambda^* T_\theta S_\lambda$. Aditionally, as translations commute with the Hilbert transform, the different self adjoint extensions of $H_T$ are related by $H_{T,\theta} \;=\; S_\theta H_{T,0} S_\theta^*$*

From this point most explicit calculations will be done for $T_0$ and $H_{T,0}$ as to simplify them.

3.1. **Description of the domain.** By construction the domain of $T_0$ is given by

$$\mathcal{D}(T_0) \;:=\; \mathscr{F}^*[\mathcal{D}(\widehat{T}_0)] \;=\; (\mathscr{F}^*B)[H^1(\mathbb{R})] \;=\; \mathscr{F}^*B\mathscr{F}\mathcal{D}(x) = \widehat{B}\mathcal{D}(x),$$

and analogously, as $\mathcal{D}(H_{T_0}) = S_\lambda^*\mathcal{D}(T_0)$

$$\mathcal{D}(H_{T,0}) = S_\lambda^*\widehat{B}\mathcal{D}(x)$$

with $\widehat{B} := \mathscr{F}^*B\mathscr{F}$. These equalities are justified by $\mathcal{D}(\widehat{T}_0) = B[\mathcal{D}(p)]$ and $\mathcal{D}(p) = H^1(\mathbb{R})$. Finally the domain of the momentum operator is the Fourier transform of the domain of the position operator [RS2, Chapter IX]. Therefore, the domain of $T_0$ is made by functions in $\mathcal{D}(x)$ transformed by the operator $\widehat{B}$. To compute the action of $\widehat{B}$ we will first have to compute a particular type of integral.

**Lemma 3.3.** *On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the operator $\widehat{B} = \mathscr{F}^*B\mathscr{F}$ acts as an integral operator with kernel given by*

$$\mathscr{B}(x,y) \;:=\; \mathrm{i}\,\frac{\operatorname{sgn}(x) - \operatorname{sgn}(y)}{2}\; J_0\left(2\sqrt{|xy|}\right) \tag{44}$$

*Proof.* Let us start with the computation of the kernel of $\widehat{B}$ acting on $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then $\mathscr{F}(\psi) \in L^2(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$, namely $\mathscr{F}(\psi)$ is a square-integrable continuous function that vanishes at infinity. For every $n \in \mathbb{N}$, let $\chi_{I_n}$ be the characteristic function of the interval $I_n := [-n, -n^{-1}] \cup [n^{-1}, n]$ Since $\mathscr{F}(\psi) - \mathscr{F}(\psi)\chi_{I_n} = \mathscr{F}(\psi)\chi_{I_n^c}$, where $I_n^c$ is the complement of $I_n$, one can prove that $\mathscr{F}(\psi)\chi_{I_n} \to \mathscr{F}(\psi)$ in the $L^2$-topology when $n \to +\infty$. Thus, the unitarity of the Fourier transform implies that $\psi_n \to \psi$ in the $L^2$-topology where $\psi_n := \mathscr{F}^*(\mathscr{F}(\psi)\chi_{I_n}) = \psi * \mathscr{F}^*(\chi_{I_n})$

and $*$ denotes the convolution. Since $\widehat{B}$ is a unitary operator one gets $\widehat{B}\psi_n \to \widehat{B}\psi$ with respect to the $L^2$-topology. An explicit computation provides

$$
\begin{aligned}
(\widehat{B}\psi_n)(x) &= (\mathscr{F}^* B \mathscr{F}\psi_n)(x) \\
&= (\mathscr{F}^* B \mathscr{F}\,(\psi \ * \mathscr{F}^*\chi_{I_n}))\,(x) \\
&= (\mathscr{F}^* B((\mathscr{F}\psi)\chi_{I_n}))(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}u\; \mathrm{e}^{\,\mathrm{i}\,ux} \frac{1}{u} (\mathscr{F}\psi)\left(\frac{1}{u}\right) \chi_{I_n}\left(\frac{1}{u}\right) \\
&= \frac{1}{2\pi} \int_{I_n} \mathrm{d}u\; \mathrm{e}^{\,\mathrm{i}\,ux} \frac{1}{u} \left( \int_{\mathbb{R}} \mathrm{d}y\; \mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}} \psi(y) \right)
\end{aligned}
$$

where in the last two equalities we used the fact that $I(\mathscr{F}\psi)\chi_{I_n}$ and $\psi$ are $L^1$-functions (this justifies the use of the integral representation of $\mathscr{F}$ and $\mathscr{F}^*$) and the equality $\chi_{I_n}(u^{-1}) = \chi_{I_n}(u)$. Since the function $g_x(y,u) := \frac{1}{u}\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{\frac{-\,\mathrm{i}\,y}{u}}\psi(y)$ is absolutely integrable in $\mathbb{R} \times I_n$ one can invoke the Fubini-Tonelli theorem to change the order of integration. This provides

$$
(\widehat{B}\psi_n)(x) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}y\; \psi(y) \left( \int_{I_n} \mathrm{d}u\; \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \right) . \tag{45}
$$

Corollary B.2 says that

$$
\lim_{n\to\infty} \int_{I_n} \mathrm{d}u\; \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; 2\pi\; \mathscr{B}(x,y) .
$$

And also gives the bound

$$
\left| \int_{I_n} \mathrm{d}u\; \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \right| \;\leqslant\; 4\pi
$$

for all $n > n_0$. In view of the bound above one can use the Lebesgue's dominated convergence theorem in (45) providing the formula

$$
\lim_{n\to+\infty} (\widehat{B}\psi_n)(x) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}y\; \mathscr{B}(x,y)\; \psi(y) . \tag{46}
$$

Equation (46) says that $B\psi_n$ converges pointwise to the integral in the right-hand side. Since $B\psi_n$ converges to $B\psi$ in the $L^2$-topology it follows there exists a subsequence $B\psi_{n_k}$ which converges pointwise (almost everywhere) to $\widehat{B}\psi$ [Bre, Theorem 4.9 (a)]. Then the unicity of the limit assures that $\widehat{B}\psi$ coincides with the right-hand side of (46), thus completing the proof. □

**Remark 3.4.** Much in the same sense of the Fourier transform, Lemma 3.3 states that $\widehat{B}$ can be expressed as an integral operator only on the dense domain $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. For function in $\psi \in L^2(\mathbb{R})\backslash L^1(\mathbb{R})$ in principle, we do not have the right to write $\widehat{B}\psi$ using the integral kernel. However, in the following, we will tacitly use the following convention

$$
(\widehat{B}\psi)(x) \;\equiv\; \lim_{R\to\infty} \int_{-R}^{+R} \mathrm{d}y\; \mathscr{B}\,(x,y)\,\psi(y) , \quad \text{if } \psi \in L^2(\mathbb{R})\backslash L^1(\mathbb{R}) .
$$

This identification must be understood as follows: (i) As the product $\psi_R := \psi\chi_{[-R,+R]}$ is in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ $\widehat{B}\psi_R$ can be computed (pointwise) through the integral formula; (ii) $\psi_R \to \psi$, and in turn $\widehat{B}\psi_R \to \widehat{B}\psi$, in the $L^2$-topology; (iii)

Then, the identification above makes sense almost everywhere on subsequences [Bre, Theorem 4.9 (a)]. ◂

Lemma 3.3 allows to describe the domain of $T$ as follows:

$$\mathcal{D}(T) \;=\; \left\{ \psi \in L^2(\mathbb{R}) \;\middle|\; \psi(x) = \int_{\mathbb{R}} \mathrm{d}y\; \mathscr{B}(x,y)\phi(y)\;, \quad \phi \in \mathcal{D}(x) \right\} \;.$$

An explicit computation (made of several changes of integration variable) shows that the generic element $\psi$ in $\mathcal{D}(T_0)$ has the form

$$\psi(x) \;=\; \frac{1}{x}\int_0^{+\infty} \mathrm{d}s\; J_0(\sqrt{s})\;\phi\left(\frac{s}{x}\right)\;, \quad \phi \in \mathcal{D}(x)\;,$$

and for an element $\psi$ in $\mathcal{D}(H_{T,0})$

$$\psi(x) \;=\; \frac{1}{x+\frac{1}{\lambda}}\int_0^{+\infty} \mathrm{d}s\; J_0(\sqrt{s})\;\phi\left(\frac{s}{x+\frac{1}{\lambda}}\right)\;, \quad \phi \in \mathcal{D}(x)\;.$$

Theorem 2.11 provides an explicit of the action of $\widehat{T}_0$ along a core. The next proposition will give us such a statement for our prefered realization for $T_0$

**Proposition 3.5.** *Action of $T_0$ and $H_{T,0}$ along a core. The domain*

$$\mathcal{D}_0(T) \;:=\; \mathcal{S}(\mathbb{R}) \;+\; \mathbb{C}[\kappa_0] \tag{47}$$

*with $\kappa_0 := \mathscr{F}^*\zeta_0 = (\widehat{B}\mathscr{F}^*)\eta_0$ is a core for $T_0$. The action of $T_0$ is described by*

$$T_0(\psi + c\kappa_0) = T\psi + c\kappa_1 \tag{48}$$

*with $\kappa_1 := \mathscr{F}^*\zeta_\pi = (\widehat{B}\mathscr{F}^*)\eta_\pi$. Finally $\kappa_0$ and $\kappa_1$ explicitly given by:*

$$\kappa_0 \;=\; -\,\mathrm{i}\sqrt{\frac{8}{\pi}}\;\mathrm{sgn}(x)\;\mathrm{kei}\left(2\sqrt{|x|}\right) \tag{49}$$

*and*

$$\kappa_1 \;=\; \mathrm{i}\sqrt{\frac{8}{\pi}}\;\mathrm{ker}\left(2\sqrt{|x|}\right)\;. \tag{50}$$

*For $H_{T,0}$, as $\mathcal{S}(\mathbb{R})$ is invariant under translations, the domain*

$$\mathcal{D}_0(H_T) \;:=\; \mathcal{S}(\mathbb{R}) \;+\; \mathbb{C}[S_\lambda^*\kappa_0] \tag{51}$$

*Is a core for $H_{T,0}$. The action of $H_{T,0}$ along this ore is given by*

$$H_{T,0}(\psi + cS_\lambda^*\kappa_0) = H_{T,0}\psi + c\lambda S_\lambda^*\kappa_1 \tag{52}$$

*Proof.* From (20) and Theorem 2.11 one infers that $\mathcal{S}(\mathbb{R}) \subset \mathcal{D}_0 \subset \mathcal{D}(\widehat{T}_0)$ and $\mathcal{S}(\mathbb{R})+\mathbb{C}[\zeta_0]$ is a core for $\widehat{T}_0$. Applying the Fourier transform, in view of the invariance of the Schwartz space under it, we conclude $\mathcal{D}_0(T)$ is a core. By construction we have $T_0\kappa_0 = \kappa_1$. To compute an explicit formula for $\kappa_0$ and $\kappa_1$ we start by noting that $\eta_0$ and $\eta_\pi$ have known (inverse) Fourier transforms:

$$(\mathscr{F}^*\eta_0)(x) \;=\; \sqrt{\frac{2}{\pi}}\frac{1}{1+x^2}\;, \quad (\mathscr{F}^*\eta_\pi)(x) \;=\; -\sqrt{\frac{2}{\pi}}\frac{x}{1+x^2}\;.$$

Then Lemmas B.3 and 3.3 provide

$$(\widehat{B}\mathscr{F}^*\eta_0)(x) \;=\; -\,\mathrm{i}\sqrt{\frac{8}{\pi}}\;\mathrm{sgn}(x)\;\mathrm{kei}\left(2\sqrt{|x|}\right).$$

Since $\mathscr{F}^*\eta_\pi \in L^2(\mathbb{R})\backslash L^1(\mathbb{R})$, the transformed function $\widehat{B}\mathscr{F}^*\eta_\pi$ as to be computed according to the prescription of Remark 3.4. In this case one has

$$(\widehat{B}\mathscr{F}^*\eta_\pi)(x) \;=\; -\sqrt{\frac{2}{\pi}} \lim_{R\to+\infty} \int_{-R}^{+R} \mathrm{d}y \; \frac{\mathscr{B}(x,y)\, y}{1+y^2} \;.$$

However, as shown in the proof of Lemma B.3, the integrant is absolutely integrable for every values of $x$. This allows to forget the limit and one gets

$$(B\mathscr{F}^*\eta_\pi)(x) \;=\; \mathrm{i}\sqrt{\frac{8}{\pi}}\, \ker\left(2\sqrt{|x|}\right) \;.$$

□

3.2. **Unitary propagator.** Let

$$U_{T_0}(\tau) \;:=\; \mathrm{e}^{-\mathrm{i}\,\tau T_0}$$

the *unitary propagator* associated with the self-adjoint operator $T_0$. From the unitary equivalence with $\widehat{T}_0$ we have

$$U_T(\tau) \;=\; \mathscr{F}^*\mathrm{e}^{-\mathrm{i}\,\tau\widehat{T}_0}\mathscr{F}$$

With methods similar to Lemma 3.3 we can compute the integral kernel of $U_T(t)$.

**Proposition 3.6.** *On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the unitary propagator $U_{T,0}(\tau)$ with $(\tau \neq 0)$ acts as an integral operator given by*

$$(U_{T_0}(\tau)\psi)(x) := \int_{\mathbb{R}} \mathrm{d}y\; \mathscr{U}_\tau(x,y)\psi(y) \tag{53}$$

*with*

$$\mathscr{U}_\tau = -\,\mathrm{i}\,\frac{\mathrm{sign}(x)+\mathrm{sign}(y)}{2\tau}\mathrm{e}^{\mathrm{i}\,\frac{(x+y)}{\tau}} J_0\left(\frac{2}{\tau}\sqrt{|xy|}\right). \tag{54}$$

*The unitary propagator of the Thermal Hamiltonian $U_{H_{T,0}}$ acts on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ as an integral operator given by*

$$(U_{H_{T,0}}(\tau)\psi)(x) = \int_{\mathbb{R}} \mathrm{d}y\; \mathscr{U}_{\lambda\tau}\left(x+\frac{1}{\lambda}, y+\frac{1}{\lambda}\right)\psi(y) \tag{55}$$

*Proof.* Let $I_n = \left[-n, -n^{-1}\right] \cup \left[n^{-1}, n\right]$ and $J_n = |t|^{-1}I_n - t^{-1}$. For $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ we define $\psi_n = \mathscr{F}^*(\chi_{J_n}) \star \psi$ converging to $\psi$ in the $L^2$ topology. computing $U_T(\tau)$ over $\psi_n$ we have, using the explicit formula for $\mathrm{e}^{-\mathrm{i}\,t\widehat{T}_0}$ found in (2.14)

$$\begin{aligned}
(U_{T_0}(\tau)\psi_n)(x) \;&=\; (\mathscr{F}^*\mathrm{e}^{-\mathrm{i}\,\tau\widehat{T}_0}\mathscr{F}\psi_n)(x) \\
&=\; \left(\mathscr{F}^*\mathrm{e}^{-\mathrm{i}\,\tau\widehat{T}_0}\mathscr{F}\left(\psi\; * \mathscr{F}^*\chi_{J_n}\right)\right)(x) \\
&=\; (\mathscr{F}^*\mathrm{e}^{-\mathrm{i}\,\tau\widehat{T}_0}((\mathscr{F}\psi)\chi_{J_n}))(x) \\
&=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}u\; \mathrm{e}^{\mathrm{i}\,ux}\frac{1}{1-\tau u}(\mathscr{F}\psi)\left(\frac{u}{1-\tau u}\right)\chi_{J_n}\left(\frac{u}{1-\tau u}\right)
\end{aligned}$$

with the change of variables $w = u - \tau^{-1}$ and checking $w \in I_n \Leftrightarrow u(1-\tau u)^{-1} \in J_n$

$$=\; \frac{1}{2\pi}\int_{I_n} \mathrm{d}w\; \mathrm{e}^{\mathrm{i}\,(w+\frac{1}{\tau})x}\frac{1}{-\tau w}\left(\int_{\mathbb{R}} \mathrm{d}y\; \mathrm{e}^{\mathrm{i}\,y\left(\frac{1}{\tau}+\frac{1}{\tau^2 w}\right)}\psi(y)\right)$$

As the integrand is absolutely integrable, by Fubini-Tonelli we can change the integration order, giving us

$$(U_{T_0}(\tau)\psi_n)(x) \ = \int_{\mathbb{R}} \mathrm{d}y \frac{\mathrm{e}^{\,\mathrm{i}\,\frac{x+y}{\tau}}}{-2\pi\tau}\psi(y)\left(\int_{I_n} \mathrm{d}w \frac{\mathrm{e}^{\,\mathrm{i}\left(xw+\frac{y}{\tau^2 w}\right)}}{w}\right)$$

Which gives us an integral of the type seen in Corollary B.2. Repeating the arguments used in Lemma 3.3 the proof for $U_{T,0} is finished.$ For the unitary propagator of $H_{T,0}$, from the relation $H_{T,0} = \lambda S_\lambda^* T_0 S_\lambda$ one gets $U_{H_{T,0}} = S_\lambda^* U_T(\lambda\tau) S_\lambda$, giving us equation (55). ☐

**Remark 3.7** (). *It is interesting to note that $U_{T_0}$ leaves both $L^2(\mathbb{R}_0^+)$ and $L^2(\mathbb{R}_0^+)$ invariant. In fact, if $\psi_+ \in L^2(\mathbb{R}_0^+)$ and $x < 0$:*

$$\begin{aligned}(U_{T_0}(\tau)\psi_+)(x) &= -\,\mathrm{i}\int_{\mathbb{R}} \mathrm{d}y\ \frac{\mathrm{sign}(x)+\mathrm{sign}(y)}{2\tau}\mathrm{e}^{\,\mathrm{i}\,\frac{(x+y)}{\tau}} J_0\left(\frac{2}{\tau}\sqrt{|xy|}\right)\psi_+(y) \\ &= -\,\mathrm{i}\int_0^{+\infty} \mathrm{d}y \frac{-1+1}{2\tau}\mathrm{e}^{\,\mathrm{i}\,\frac{(x+y)}{\tau}} J_0\left(\frac{2}{\tau}\sqrt{|xy|}\right)\psi_+(y) = 0\end{aligned}$$

*In the same manner, $U_{H_{T,0}}(\tau)$ leaves invariant $L^2((-1/\lambda,+\infty))$ and $L^2((-\infty,-1/\lambda))$, thus the dynamics of both sides of $-1/\lambda$ are independent from one another.*

3.3. **Resolvent and Green function.** The resolvent of $T_0$ can be computed as the Laplace transform of the unitary propagator $U_{T_0}(\tau)$ according to the well known formula [Kat, eq. (1.28), p. 484]. For every $\zeta \in \mathbb{C}\backslash\mathbb{R}$ let

$$R_\zeta(T_0) \ := \ (T_0 - \zeta\mathbf{1})^{-1}$$

be the resolvent of $T_0$. Using the unitary equivalence between $T_0$ and -x given by $\widehat{B}$, we can write

$$R_\zeta(T_0) \ := \ \widehat{B}\frac{1}{-x-\zeta\mathbf{1}}\widehat{B}$$

The following proposition consists in an explicit expressiof for $R_\zeta(T_0)$ as an integral operator

**Proposition 3.8.** *On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the resolvent $R_\zeta(T_0)$ with $0 < \phi < \pi$, $\zeta = |\zeta|\mathrm{e}^{\pm i\phi}$ acts as an integral operator given by*

$$R_\zeta(T_0) \ = \ \int_{\mathbb{R}} \mathrm{d}y\big(\mathrm{sgn}(x)+\mathrm{sgn}(y)\big)\ F_\zeta(x,y)\psi(y) \tag{56}$$

*Where*

$$\begin{aligned}F_\zeta(x,y) \ &:= I_0\left(2\sqrt{|\zeta|\min\{|x|,|y|\}}\mathrm{e}^{\pm\,\mathrm{i}\left[\frac{\phi}{2}-\frac{\pi}{4}\left(\mathrm{sgn}(x)+1\right)\right]}\right) \\ &\qquad \times\ K_0\left(2\sqrt{|\zeta|\max\{|x|,|y|\}}\mathrm{e}^{\pm\,\mathrm{i}\left[\frac{\phi}{2}-\frac{\pi}{4}\left(\mathrm{sgn}(x)+1\right)\right]}\right)\ .\end{aligned} \tag{57}$$

*While the resolvent $R_\zeta/H_{T,0}$ is given by*

$$R_\zeta(H_{T,0}) \ = \ \frac{1}{\lambda}\int_{\mathbb{R}} \mathrm{d}y\left(\mathrm{sgn}\left(x+\frac{1}{\lambda}\right)+\mathrm{sgn}\left(y+\frac{1}{\lambda}\right)\right)\ F_{\frac{\zeta}{\lambda}}\left(x+\frac{1}{\lambda},y+\frac{1}{\lambda}\right)\psi(y) \tag{58}$$

*Proof.* Let us start with $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Let us note that as $J(|u|)$ asymptotically behaves as $|u^{-1/2}|$ when u $\to \infty$ we have

$$\int_R \mathrm{d}x \left| \frac{1}{-x-\zeta} \widehat{B}\psi(x) \right| \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}y \frac{1}{|x+\alpha|} \left| J_0(2\sqrt{|xy|})\psi(y) \right| < \infty$$

As such, $\frac{1}{-x-\zeta\mathbf{1}} \widehat{B}\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $R_\zeta(T_0)\psi$ can be expressed as

$$\begin{aligned} R_\zeta(T_0)\psi(x) = \int_{\mathbb{R}} \mathrm{d}z\, \mathrm{i}\, \frac{\operatorname{sgn}(x) - \operatorname{sgn}(z)}{2}\, J_0\left(2\sqrt{|xz|}\right) \frac{1}{-z-\zeta} \times \\ \left( \int_{\mathbb{R}} \mathrm{d}y\, \mathrm{i}\, \frac{\operatorname{sgn}(z) - \operatorname{sgn}(y)}{2}\, J_0\left(2\sqrt{|yz|}\right) \psi(y) \right) \end{aligned}$$

Due to the decay of $J_0$ the function is absolutely integrable, then invoking Fubini-Tonelli we can swap the integration order. Furthermore, using the identity

$$(\operatorname{sign}(x) - \operatorname{sign}(z))\,(\operatorname{sign}(z) - \operatorname{sign}(y)) = (\operatorname{sign}(x) + \operatorname{sign}(y))\,(\operatorname{sign}(z) - \operatorname{sign}(x))$$

We have

$$\begin{aligned} R_\zeta(T_0)\psi(x) = \int_{\mathbb{R}} \mathrm{d}y\ (\operatorname{sign}(x) + \operatorname{sign}(y))\, \psi(y) \times \\ \left( \int_{\mathbb{R}} \mathrm{d}z\ \frac{\operatorname{sign}(z) - \operatorname{sign}(x)}{4} \frac{1}{z+\zeta} J_0\left(2\sqrt{|xz|}\right) J_0\left(2\sqrt{|yz|}\right) \right) \end{aligned}$$

with the change of variables $u = -\operatorname{sign}(x)z$ the inner integral, which we will from know on call $F_\alpha(x,y)$, becomes

$$\begin{aligned} F_\alpha(x,y) &= \int_{\mathbb{R}} \mathrm{d}u\ \frac{-\operatorname{sign}(x)(u+1)}{4} \frac{1}{-\operatorname{sign}(x)u+\zeta} J_0\left(2\sqrt{|xu|}\right) J_0\left(2\sqrt{|yu|}\right) \\ &= \frac{1}{2} \int_0^\infty \mathrm{d}u \frac{1}{u - \operatorname{sign}(x))\zeta} J_0\left(2\sqrt{|xu|}\right) J_0\left(2\sqrt{|yu|}\right) \end{aligned}$$

now setting $u = w^2$, noting $\left(\sqrt{|\zeta|}\mathrm{e}^{\pm\,\mathrm{i}\left[\frac{\phi}{2} - \frac{\pi}{4}\left(\operatorname{sgn}(x)+1\right)\right]}\right)^2 = -\operatorname{sign}(x))\zeta$

$$F_\alpha(x,y) = \int_0^\infty \mathrm{d}w \frac{w J_0\left(2\sqrt{|x|}w\right) J_0\left(2\sqrt{|y|}w\right)}{w^2 + \left(\sqrt{|\zeta|}\mathrm{e}^{\pm\,\mathrm{i}\left[\frac{\phi}{2} - \frac{\pi}{4}\left(\operatorname{sgn}(x)+1\right)\right]}\right)^2}.$$

We have $-\pi/2 < \frac{\phi}{2} - \frac{\pi}{4}\big(\operatorname{sgn}(x)+1\big) < \pi/2$, then using formula [GR, 6.541 (1)] we obtain the stated expression. The expression for $R_\zeta(H_{T,0})$ comes from the relation

$$R_\zeta(H_{T,0}) = S_\lambda^* R_\zeta(\lambda T_0) S_\lambda = \frac{1}{\lambda} S_\lambda^* R_{\frac{\zeta}{\lambda}}(T_0) S_\lambda.$$

□

3.4. **Scattering by a convolution potential.** In this section we will find a class of convolution perturbations of $T_0$ for which the scattering and wave operators exist. A brief introduction to some relevant definitions may me found in appendix A.3. Let $g \in L^1(\mathbb{R})$ such that $\overline{g(-x)} = g(x)$ and $||g||_1 \leqslant 1$, and and consider the associated convolution potential $W_g$ defined by

$$N_g\varphi(x) \coloneqq (g * \varphi)(x) = \int_{\mathbb{R}} \mathrm{d}y\ g(x-y)\varphi(y) \tag{59}$$

By equation (100) in appendix A.2 we have

$$N_g = \sqrt{2\pi}\ \widehat{g}(p) \tag{60}$$

With $p$ the momentum operator. The following Lemma gives us a rigurous definition for $T_0 + N_g$

**Lemma 3.9.** *The operator $T_0 + N_g$ with with domain $\mathcal{D}(T_0)$ is self adjoint.*

*Proof.* We start to check that $N_g$ is in fact a bounded symmetric operator. As $\mathscr{F} N_g \mathscr{F}^* = \sqrt{2\pi}\widehat{g}(x)$ we have $||N_g|| = ||\sqrt{2\pi}\widehat{g}(x)|| = \sqrt{2\pi}||\widehat{g}||_\infty \leqslant ||g||_1$.To check for it being symmetric, for $\psi, \varphi \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ we have

$$\langle N_g\varphi, \psi\rangle = \int_{\mathbb{R}} \mathrm{d}x \ \overline{\int_{\mathbb{R}} \mathrm{d}y\ g(x-y)\varphi(y)}\psi(x)$$

swapping the integration order as the integrand is absolutely integrable and using $\overline{g(-u)} = g(u)$

$$\begin{aligned}\langle N_g\varphi, \psi\rangle &= \int_{\mathbb{R}} \mathrm{d}y\ \overline{\varphi(y)} \int_{R} \mathrm{d}x\ g(y-x)\psi(x) \\ &=\langle \varphi, N_g\psi\rangle\end{aligned}$$

Invoking the Kato-Rellich theorem stated in $A.15$, as $N_g$ is bounded then $T_0 + N_g$ is well defined as a self-adjoint operator on the domain $\mathcal{D}(T_0)$. □

Due to this it makes sense to consider the scattering theory of the pair $(T_0, T_0 + N_g)$. However, rather than working directly with $T_0$, we will make use of relation $T_0 = (B\mathscr{F})^* p (B\mathscr{F})$ and study the scattering theory of the pair $(p, p_g)$). where $p_g := p + M_g$ is the perturbation of the momentum given by the potential.

$$M_g \ \coloneqq\ B\mathscr{F} N_g \mathscr{F}^* B\ .$$

**Lemma 3.10.** *The potential $M_g$ is the multiplication operator defined by*

$$(M_g\psi)(x)\ :=\ \sqrt{2\pi}\hat{g}\left(\frac{1}{x}\right)\psi(x)\ , \qquad \psi \in L^2(\mathbb{R})\ .$$

*where $\hat{g}$ denotes the Fourier transform of $g$.*

*Proof.* As $W_g = \sqrt{2\pi}\ \widehat{g}(p)$, we have $\mathscr{F} W_g \mathscr{F}^* = \sqrt{2\pi}\widehat{g}(x)$. Then for $\psi \in L^2(\mathbb{R})$ and recalling the definition of $B$

$$M_g\psi(x) = \sqrt{2\pi}\left(B\widehat{g}(x)B\right)(\psi)(x) = \sqrt{2\pi}\frac{1}{x}\widehat{g}\left(\frac{1}{x}\right)(B\psi)\left(\frac{1}{x}\right) = \sqrt{2\pi}\widehat{g}\left(\frac{1}{x}\right)\psi(x)$$

finishing the proof □

We are now in condition to prove the following theorem fully describing the scattering of the pair $(T_0, T_0 + N_g)$.

**Theorem 3.11** (Scattering theory for convolution perturbations in $d = 1$)**.** *Let $g \in L^1(\mathbb{R})$ and $N_g$ the associated convolution perturbation defined by* (59)*. Then:*

- *(i) The perturbed operator $T_0 + N_g$ is self-adjoint with domain $\mathcal{D}(T_0)$, unitary equivalent to $T_0$ and*

$$\sigma\big(T_0 + N_g\big)\ =\ \sigma_{\mathrm{a.c.}}\big(T_0 + N_g\big)\ =\ \mathbb{R}\ .$$

*Let $\widehat{g}$ be the Fourier transform of $g$ such that $\int_{\mathbb{R}} \mathrm{d}s\ \frac{|\widehat{g}(s)|}{s^2}\ < \infty$. Then:*

*(ii) The wave operators $W_\pm$ defined by*

$$W_\pm \ := \ \operatorname*{s-lim}_{t\to\pm} \mathrm{e}^{\mathrm{i}\, t(T_0+N_g)}\ \mathrm{e}^{-\mathrm{i}\, tT_0} \tag{61}$$

*exist and are complete;*

*(iii) The S-matrix $S_g := (W_+)^* W_-$ is a constant phase given by*

$$S_g \ = \ \mathrm{e}^{-\mathrm{i}\sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}s\ \frac{\hat{g}(s)}{s^2}}\ .$$

*Proof.* We start by the scattering theory of the pair $(p, p+q(x))$ with $q$ a bounded integrable real function, originally found in [Kat, Example 3.1, p. 530]. As is known $\mathrm{e}^{-\mathrm{i}\, tp}$, acts as a translation by $t$. To find the group generated by $p+q(x)$ we first define the unitary operator

$$I_q\psi(x) \coloneqq \mathrm{e}^{\mathrm{i}\, Q(x)}\psi(x); \quad Q(x) \coloneqq \int_0^x \mathrm{d}y\ q(y). \tag{62}$$

Let us note that $I_q$, $I_q^*$ and $Q(x)$ leave $H^1(\mathbb{R}) = \mathcal{D}(p)$ invariant as $I_q\psi(x)$ has a square integrable derivative given by $iq(x)I_q\psi(x) + I_q\psi'(x)$. Then the commutator $[Q(x), p] = q(x)$ has a well defined sense, and as $[Q(x), [Q(x), p]] = 0$ we have by the Baker-Campbell-Hausdorff formula

$$I_g^* p I_g = p + [-iQ(x), p] = p + q(x) \tag{63}$$

and in consequence

$$\mathrm{e}^{\mathrm{i}\, t(p+q(x))}\psi(x) = I_g^*\mathrm{e}^{\mathrm{i}\, tp} I_g\psi(x) = \mathrm{e}^{-\mathrm{i}\, Q(x)}\mathrm{e}^{\mathrm{i}\, Q(x+t)}\psi(x+t). \tag{64}$$

Then to compute the wave operators we first write

$$W(t)\psi(x) = \mathrm{e}^{\mathrm{i}\, t(p+q(x))}\mathrm{e}^{-\mathrm{i}\, tp}\psi(x) = \mathrm{e}^{\mathrm{i}\,(Q(x+t)-Q(x))}\psi(x) \tag{65}$$

Computing let us note that $Q(x+t) - Q(x) = \int_x^{x+t} \mathrm{d}y\ q(y)$. Then for $\psi, \phi \in L^2(\mathbb{R})$, as $q(x)$ is absolutely integrable,

$$\langle\phi, W(t)\psi\rangle = \int_{\mathbb{R}} \mathrm{d}x\ \overline{\phi(x)}\mathrm{e}^{\mathrm{i}\int_x^{x+t} \mathrm{d}y\ q(y)}\psi(x) \xrightarrow[t\to+\infty]{} \int_{\mathbb{R}} \mathrm{d}x\overline{\phi(x)}\mathrm{e}^{\mathrm{i}\int_x^{+\infty} \mathrm{d}y\ q(y)}\psi(x)$$

Using the dominated convergence theorem using $|\psi(x)\phi(x)|$ as a bound. Taking the analogous limit as $t \to -\infty$ we finally get the wave operators

$$W_+(p+q(x), p) = \exp i\int_x^{+\infty} \mathrm{d}y\ q(y), \quad W_-(p+q(x), p) = \exp -i\int_{-\infty}^x \mathrm{d}y\ q(y) \tag{66}$$

Let us note that with an analogous computation the limits $W_\pm(p, p+q(x)) = \text{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\, tp}\mathrm{e}^{-\mathrm{i}\, t(p+q(x))}$ exist. as such the wave operators are complete. We only need to compute the scattering matrix

$$S(p+q(x), p) = W_+^*(p+q(x), p)W_-(p+q(x), p)$$
$$= \exp\left(-i\int_x^{+\infty} \mathrm{d}y\ q(y)\right)\exp\left(-i\int_{-\infty}^x \mathrm{d}y\ q(y)\right) = \exp\left(-i\int_{-\infty}^{+\infty} \mathrm{d}y\ q(y)\right)$$

Which acts as multiplication by a phase. Now for the pair $(T_0, T_0+N_g)$, we have

$$(T_0+N_g) = B\mathscr{F}(p+M_g)\mathscr{F}^*B,\ (T_0) = B\mathscr{F}(p)\mathscr{F}^*B,$$

and if the wave operators exist,

$$W_\pm(T_0+N_g, T_0) \ = B\mathscr{F}W_\pm(p+M_g, p)\mathscr{F}^*B.$$

Thus we need to check if $M_g$ satisfies the same conditions as $q(x)$. As $M_g$ acts as multiplication by $\sqrt{2\pi}\widehat{g}\left(\frac{1}{x}\right) < \infty$. Moreover, both integrals

$$\int_x^{\infty} \mathrm{d}y \sqrt{2\pi}\widehat{g}\left(\frac{1}{y}\right), \int_{-\infty}^{x} \mathrm{d}y \sqrt{2\pi}\widehat{g}\left(\frac{1}{y}\right)$$

are finite as

$$\int_{\mathbb{R}} \mathrm{d}x \left|\sqrt{2\pi}\widehat{g}\left(\frac{1}{x}\right)\right| = \int_{\mathbb{R}} \mathrm{d}x \frac{1}{x^2}\sqrt{2\pi}|\widehat{g}(x)| < \infty.$$

Thus $M_g$ satisfies the same conditions as $q(x)$ and the wave operators exist and are complete. The Scattering Matrix is given by

$$S(T_0 + N_g, T_0) = B\mathscr{F}S(p + M_g, p)(B\mathscr{F})^* = \exp\left(-\mathrm{i}\sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}s\ \frac{\hat{g}\,(s)}{s^2}\ .\right)$$

From the equivalence between $p$ and $p+q(x)$ we also obtain the equivalence between $T_0$ and $T_0 + N_g$. □

**Remark 3.12** (Scattering with a convolution potential for$H_{T,0}$)**.** *From the relation* $H_T = \lambda S_\lambda^* T S_\lambda$ . *and as convolution operators commute with translations, one has under the same conditions for g*

$$W_\pm(H_{T,0} + N_g, H_{T,0}) = S_\lambda^* W_{\pm\,\mathrm{sign}(\lambda)}(T_0 + N_{\lambda^{-1}g}, T_0)S_\lambda,$$

*and*

$$\begin{aligned} S(HT,_0 + N_g, H_{T,0}) &= S_\lambda^* W_{\mathrm{sign}(\lambda)}(T_0 + N_{\lambda^{-1}g}, T_0)^* W_{-\,\mathrm{sign}(\lambda)}(T_0 + N_{\lambda^{-1}g}, T_0)S_\lambda \\ &= \exp\left(-\mathrm{i}\,\frac{\sqrt{2\pi}}{|\lambda|}\int_{\mathbb{R}} \mathrm{d}s\ \frac{\hat{g}\,(s)}{s^2}\ .\right). \end{aligned}$$

*In this last calculation we use the fact that* $S^* = W_-^* W_+$

With this we close the study of the quantum Thermal Hamiltonian, and we give a similar study to the Classical version.

## 4. Classical Case

In this last section we will study the classical dynamics induced by a thermal gradient. The classic analogue of the Luttinger's model is provided by the the Hamiltonian function

$$H_T(x,p) \ :=\ (1 + \lambda\,\gamma\cdot x)\,\frac{p^2}{2m}\ =\ K(p)\ +\ \lambda\,T_\gamma(x,p)\ , \tag{67}$$

with parameters $\lambda > 0$ and $\gamma \in \mathbb{S}^{d-1}$. The Hamiltonian $H_T$ can be seen as the sum of the Hamiltonian of a *free* $d$-dimensional particle of mass $m$

$$K(p)\ :=\ \frac{p^2}{2m}\ =\ \frac{1}{2m}\sum_{j=1}^{N} p_j^2$$

coupled through the *coupling constant* $\lambda > 0$ with the *thermal potential*

$$T_\gamma(x,p)\ :=\ (\gamma\cdot x)\,K(p)\ =\ \left(\frac{p^2}{2m}\right)\ \sum_{j=1}^{d}\gamma_j\,x_j$$

along the direction $\gamma \in \mathbb{S}^{d-1}$. The coupling constant has the dimension of the inverse of a distance, namely $\lambda = \ell^{-1}$ with $\ell > 0$ the *typical length* of the thermal field. Therefore, the limit $\lambda \to 0$ describes the situation in which the typical length of the field is much larger than the typical length of the system (*e.g.* the size of the particle). The potential $T_\gamma$ is an example of what is known as a *generalized potential*, namely a potential which depends not only on the position but also on the the velocity.

4.1. **Hamiltonian Formalism and Newton equation.** The Hamilton equations associated to (67) read

$$\begin{cases} \dot{x} \;=\; +\nabla_p H_T \;=\; \dfrac{(1+\lambda\,\gamma\cdot x)}{m}\, p \\ \dot{p} \;=\; -\nabla_x H_T \;=\; -\lambda\,\dfrac{p^2}{2m}\,\gamma\,. \end{cases} \tag{68}$$

The first equation can be inverted out of the *critical plane*

$$\Xi_c \;:=\; \left\{x \in \mathbb{R}^d \mid \gamma\cdot x + \ell = 0\right\} \tag{69}$$

and provides

$$p(x,\dot{x}) \;=\; \frac{m}{(1+\lambda\,\gamma\cdot x)}\,\dot{x} \tag{70}$$

One can restore the usual relation $p = m_T\dot{x}$ between momentum and velocity by introducing the *position-dependent mass* (PDM)

$$m_T(x) \;:=\; \frac{m}{(1+\lambda\,\gamma\cdot x)}\,.$$

It is interesting to notice that the Hamiltonian (67) can be rewritten as

$$H_T(x,p) \;=\; \frac{p^2}{2m_T(x)}\,, \tag{71}$$

namely as the Hamiltonian of a free particle with a PDM. The second equation of (68) can be rewritten as

$$\dot{p} \;=\; -\lambda\,\nabla_x T_\gamma\,. \tag{72}$$

A straightforward computation allows to derive the Newton's laws from (68):

$$m\,\ddot{x} \;=\; \lambda(\gamma\cdot\dot{x})p \;-\; \lambda\,(1+\lambda\,\gamma\cdot x)\,\frac{p^2}{2m}\,\gamma\,.$$

After introducing (70) in the las expression one obtains the Newton's equation

$$m\,\ddot{x} \;=\; \lambda\,F_T(x,\dot{x})$$

where the *thermal force* (which has the dimensions of a force times a distance) is given by

$$F_T(x,\dot{x}) \;=\; m_T(x)\,\left[(\gamma\cdot\dot{x})\dot{x} \;-\; \frac{\dot{x}^2}{2}\,\gamma\right]. \tag{73}$$

A way of interpreting this Newton's Equation is to say that the motion of the PDM-particle is influenced by the effect of its own internally self-produced force field generated by the spatial dependence of the mass. The relation between the force $F_T$ and the potential $T_\gamma$ can be deduced by observing that

$$-\,\lambda\nabla_x T_\gamma \;=\; -\lambda\,\frac{m_T(x)^2}{m}\,\frac{\dot{x}^2}{2}\,\gamma \tag{74}$$

in view of the (72), (68) and (70), respectively. After some manipulation and the use of equation (70) one gets

$$F_T(x,p) \ = \ -\nabla_x T_\gamma(x,p) \ + \ R_T(x,p) \tag{75}$$

which shows that the thermal force is not simply given by $-\nabla_x T_\gamma$, as for ordinary conservative forces, but it includes an extra *reacting* term

$$R_T(x,p) \ := \ \frac{\mathrm{d}}{\mathrm{d}t}\,(\gamma\cdot x)\,p \ = \ m\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\nabla_p T_\gamma(x,p)) \tag{76}$$

which is generally not aligned with the direction $\gamma$ of the field.

4.2. **Qualitative analysis.** Let us start with the analysis of the qualitative behavior of the solution of the Hamiltonian system (67). To simplify the study let us fix convenient notations. The unit vector $\gamma$ can be completed to an orthonormal basis by adding other $d-1$ orthonormal vectors $e_1,\ldots,e_{d-1}$. This allows to fix the generalized coordinates $x_0 := \gamma\cdot x$, $x_j := e_j\cdot x$, and the generalized momenta $p_0 := \gamma\cdot p$, $p_j := e_j\cdot p$ with $j=1,\ldots,d-1$. In this coordinates the Hamiltonian (67) reads

$$H_T(x_0,p_1,\ldots,p_d) \ = \ (1+\lambda x_0)\ \frac{p^2}{2m} \tag{77}$$

and the Hamilton equations (68) become

$$\begin{cases} \dot{x}_j \ = \ (1+\lambda\, x_0)\ \dfrac{p_j}{m} \\ \dot{p}_j \ = \ -\delta_{0,j}\ \lambda\ \dfrac{p^2}{2m} \end{cases} \qquad\qquad j=0,\ldots,d-1\ . \tag{78}$$

The integration of the equations for the "orthogonal" components of the momentum immediately leads to

$$p_j(t) \ = \ \wp_j \ = \ \text{const.}\,, \qquad\qquad j=1,\ldots,d-1\ .$$

This can be seen as a consequence of the Noether's theorem applied to the invariance under translations of the Hamiltonian $H_T$ along all the directions orthogonal to $\gamma$. Let us introduce the constant of motion

$$\wp_\perp := \left(\sum_{j=1}^{d-1}\wp_j^2\right)^{\frac{1}{2}}$$

which quantifies the momentum in the orthogonal plane to the direction of the thermal field. The square of the momentum at any time takes the form

$$p^2(t) \ = \ p_0^2(t) \ + \ \wp_\perp^2\ . \tag{79}$$

The value of the parameter $\wp_\perp$ strongly determines the behavior of the solutions of the system (78). To see this, one can observe that the Hamiltonian $H_T$ is time-independent and therefore the Noether's theorem provides a further constant of motion, *i.e.* the (total) energy

$$E_0 \ := \ (1+\lambda\,\varrho_0)\ \frac{\wp_0^2+\wp_\perp^2}{2m}$$

which is completely specified by the initial conditions

$$\varrho_0 \ := \ x_0(t=0)\,, \qquad\qquad \wp_0 \ := \ p_0(t=0)\ .$$

The constraint

$$H_T(x(t),p(t)) \;=\; E_0 \;, \qquad\qquad \forall\; t \in \mathbb{R} \tag{80}$$

can be used to obtain the equation

$$x_0(t) \;=\; \frac{1}{\lambda}\left(\frac{2mE_0}{p^2(t)} \;-\; 1\right) \;=\; \frac{\wp_0^2+\wp_\perp^2}{p_0^2(t)+\wp_\perp^2}\left(\frac{1}{\lambda} \;+\; \varrho_0\right) \;-\; \frac{1}{\lambda}\;, \tag{81}$$

which provides the time evolution of $x_0$ once it is known the form of $p_0^2(t)$ and the initial conditions $\varrho_0$ and $\wp_0, \wp_1, \ldots, \wp_{N-1}$. In addition to this, the constraint (80) also provides useful information for a qualitative study of the trajectory $x(t)$ of the particle. A comparison between (77) and (80) shows that the sign of $E_0$ only depends on the quantity $1 + \lambda\varrho_0$. More precisely, one has that

$$\pm E_0 \;\geqslant\; 0 \qquad \Leftrightarrow \qquad \pm\varrho_0 \;\geqslant\; \mp\ell \;\;.$$

Thus, the critical plane $\Xi_c \subset \mathbb{R}^N$ separates the space into two regions labelled by the sign of the energy $E_0$. The full trajectory $x(t)$ of the particle is fully contained in only one of these two half-spaces according to the initial position $\varrho_0$ along the direction $\gamma$ at the initial time $t = 0$. Moreover, the trajectory can touch the critical plane only at the cost of a divergence in the value of the total momentum, $p^2 \to \infty$.

The existence of this critical impenetrable plane can be justified on the basis of the Newton's law $m\ddot{x}_j = \lambda\, F_{T,j}$ where the force (73) is given for components by

$$F_{T,j} \;=\; \begin{cases} \dfrac{E_0}{2} \;-\; (1+\lambda x_0)\,\dfrac{\wp_\perp^2}{m} & \text{if} \quad j=0 \\ (1+\lambda x_0)\;\dfrac{p_0\wp_j}{m} & \text{if} \quad j=1,\ldots,d-1\;. \end{cases} \tag{82}$$

In the derivation of (82) from (73) we made use of (71) along with $m_T\dot{x} = p$ and the conservation laws (79) and (80). The component $F_{T,0}$ is proportional to $E_0$ very close to the critical plane ($1 + \lambda x_0 \sim 0$) and force the particle to stay inside the half-space where the particle was at the initial time. When $\wp_\perp^2 \neq 0$ the component $F_{T,0}$ changes sign sufficiently far from the critical plane and begins to attract the particle towards $\Xi_c$. This suggests that the motion of the particle must be bounded in the direction $\gamma$ provided that the momentum has a non-vanishing component orthogonal to $\gamma$ at the initial time. The components $F_{T,1}, \ldots, F_{T,d-1}$ are due to the reaction term $R_T$ (76). The conservation of the energy implies that $|p_0| \propto |1+\lambda x_0|^{-\frac{1}{2}}$ for $x_0 \to -\ell$. Therefore the orthogonal components of $F_T$ vanish when the particle approaches the critical plane.

4.3. **Exceptional solutions.** The Hamilton equations (78) (or equivalently (68)) admit the *exceptional* family of solutions $p(t) = 0$ and $x(t) = \varrho$ for all $t \in \mathbb{R}$ parametrized by all the possible initial positions $\varrho \in \mathbb{R}^d \backslash \Xi_c$ not belong to the critical plane. In this case the particle is at every moment at rest in a configuration of total zero energy $E_0 = 0$. This is not surprising even though the particle is immersed in the thermal field. In fact the force $F_T$ produced by the field vanishes when $p = 0$. If at the initial time one has $\wp_j = 0$ for all $j = 0, \ldots, d-1$ and $\varrho_0 \neq -\ell$, then $p^2 = 0$ for all $t \in \mathbb{R}$ (as a consequence of energy conservation) and therefore the particle is not subject to any force. This allows the particle to stay in equilibrium forever at the position $\varrho$.

Another family of exceptional solutions is again described by $x(t) = \varrho$ for all $t \in \mathbb{R}$ with the initial positions $\varrho \in \Xi_{\rm c}$. Also in this case the particle remains at rest in a configuration of total zero energy $E_0 = 0$. However, since the particle lies in the critical plane the total momentum is not forced to be zero. While the component of the momentum orthogonal to $\gamma$ is constant and quantified by $\wp_\perp$ the component $p_0(t)$ evolves in time according to the Hamilton equation (78) (with solutions (90) if $\wp_\perp = 0$ or (83) when $\wp_\perp \neq 0$).

4.4. **The general solution.** Let us derive the general solution of the Hamiltonian system (78) under the generic assumption $\wp_\perp \neq 0$. In this case the differential equation for $p_0$ reads

$$\dot{p}_0 \;=\; -\lambda\, \frac{p_0^2 \;+\; \wp_\perp^2}{2m}$$

and is solved by

$$p_0(t) \;=\; \wp_\perp\, \tan\left(\phi \;-\; \lambda \frac{\wp_\perp}{2m} t\right) \tag{83}$$

where $\phi := \arctan\left(\frac{\wp_0}{\wp_\perp}\right)$ is determined by the initial conditions. Equation (83) shows that $p_0(t)$ diverges periodically at the critical times $t_{\rm c}^{(n)} := t_{\rm c} + nT$, $n \in \mathbb{Z}$, where

$$t_{\rm c} \;:=\; (2\phi - \pi)\, \frac{\ell m}{\wp_\perp}\;, \qquad\qquad T \;:=\; 2\pi \frac{\ell m}{\wp_\perp}$$

and $\ell = \lambda^{-1}$.

From (83) and (79) one immediately gets

$$p^2(t) \;=\; \frac{\wp_\perp^2}{\cos\left(\phi - \lambda \frac{\wp_\perp}{2m} t\right)^2}$$

and after some manipulations, equation (81) provides

$$x_0(t) \;=\; \varrho_0 \;+\; A_\lambda \left[\cos\left(\phi - \lambda \frac{\wp_\perp}{2m} t\right)^2 \;-\; \cos(\phi)^2\right] \tag{84}$$

where we the amplitude $A_\lambda$ is given by

$$A_\lambda \;:=\; \ell \frac{2mE_0}{\wp_\perp^2} \;=\; \frac{\ell + \varrho_0}{\cos(\phi)^2}\;.$$

Equation (84) shows that the motion along the direction $\gamma$ is bounded and more precisely is confined between the critical plane $\Xi_{\rm c}$ which is reached periodically at the critical times $t_{\rm c}^{(n)}$ and the *extremal* plane

$$\Xi_{\rm e} \;:=\; \left\{ x \in \mathbb{R}^d \mid \gamma \cdot x = \varrho_0 + \left(\frac{\wp_0}{\wp_\perp}\right)^2 (\ell + \varrho_0)\right\} \tag{85}$$

which is reached periodically at the *extremal* times $t_{\rm e}^{(n)} := t_{\rm e} + nT$ where $t_{\rm e} := 2\phi \frac{\ell m}{\wp_\perp}$.

By inserting the solution (84) in the differential equations for the other components of the position one gets

$$\dot{x}_j(t) \;=\; \lambda \frac{\wp_j}{m} A_\lambda \cos\left(\phi - \lambda \frac{\wp_\perp}{2m} t\right)^2\;, \qquad\qquad j = 1, \ldots, d-1\;.$$

For each $j$, the corresponding differential equation is integrated by

$$x_j(t) \ = \ \varrho_j \ + \ \lambda \frac{\wp_j}{2m} A_\lambda t \ - \ \frac{A_\lambda}{2} \ \frac{\wp_j}{\wp_\perp} \ \left[ \sin\left( 2\phi - \lambda \frac{\wp_\perp}{m} t \right) \ - \ \sin(2\phi) \right] \ . \tag{86}$$

Evidently the motion in the directions $e_j$ is unbounded when $\wp_j \neq 0$ due to the linear term in $t$ which describes a uniform motion with constant velocity $v_{j,\lambda} := \lambda A_\lambda \frac{\wp_j}{2m}$.

Let us introduce the unit vector $\nu := \wp_\perp^{-1} \sum_{j=1}^{d-1} \wp_j e_j$. By construction $\nu$ is orthogonal to $\gamma$ and $\wp := \wp_0 \gamma + \wp_\perp \nu$ describes the initial momentum of the particle at $t = 0$. From (84) and (86) one gets that

$$x(t) \ = \ \varrho \ + \ A_\lambda \big( f_0(t) \gamma \ + \ f_\perp(t) \nu \big) \tag{87}$$

with $\varrho := \varrho \gamma + \sum_{j=1}^{d-1} \rho_j e_j$ the initial position and

$$\begin{aligned} f_0(t) \ &:= \ \cos\left( \phi - \lambda \frac{\wp_\perp}{2m} t \right)^2 \ - \ \cos(\phi)^2 \\ f_\perp(t) \ &:= \ \lambda \frac{\wp_\perp}{2m} t \ - \ \frac{1}{2} \left[ \sin\left( 2\phi - \lambda \frac{\wp_\perp}{m} t \right) \ - \ \sin(2\phi) \right] \ . \end{aligned}$$

Equation (87) shows that the motion of the particle is essentially two-dimensional. In fact the orbit $x(t)$ lies entirely in the affine plane spanned by $\mu$ and $\nu$ and passing through the initial position $\rho$.

**Remark 4.1** (2D-case)**.** In view of (87) the general motion of a particle in the thermal field is a two-dimensional motion provided that the initial momentum is not aligned with the direction of the field. Therefore, one can always identify the direction $\gamma$ of the field and the direction $\nu$ of the orthogonal component of the initial momentum with the $x$-axis and the $y$-axis of $\mathbb{R}^2$, respectively. This allows us to use the "cozy" notation $x(t)$ and $y(t)$ for the two projections of the trajectory along the direction $\gamma$ y $\nu$, respectively. Let $\wp = (\wp_x, \wp_y)$ be the components of the initial momentum projected along the two coordinate direction $\gamma$ and $\nu$. Let us consider here the special situation in which the total momentum is completely orthogonal to $\gamma$. This means that $\wp_0 = \wp_x = 0$ and $\wp_\perp = |\wp_y| = |\wp|$. This also implies that $\phi = \arctan(0) = 0$ and $A_\lambda = \ell + \varrho_x$ with $\varrho_x = \varrho_0$ is the $x$-component of the initial position $\varrho = (\varrho_x, \varrho_y)$. In this case the equations of motion for the position simplify to

$$\begin{aligned} x(t) \ &= \ \varrho_x \ + \ (\ell + \varrho_x) \left[ \cos\left( \lambda \frac{|\wp|}{2m} t \right)^2 \ - \ 1 \right] \ , \\ y(t) \ &= \ \varrho_y \ + \ (\ell + \varrho_x) \left[ \lambda \frac{|\wp|}{2m} t \ + \ \frac{1}{2} \ \sin\left( \lambda \frac{|\wp|}{m} t \right) \right] \ . \end{aligned} \tag{88}$$

The time evolution of the momentum is described by the equations

$$p_x(t) = -|\wp| \ \tan\left( \lambda \frac{|\wp|}{2m} t \right)$$

and

$$p_y(t) = \wp_y .$$

◂

4.5. **The one-dimensional case.** As discussed at the end of Section 4.4 (see Remark 4.1) the general motion of a particle in the thermal field is two-dimensional whenever $\wp_\perp \neq 0$. Therefore the condition $\wp_\perp = 0$, $\wp_0 \neq 0$ corresponds to considering the one-dimensional case. In fact, under these conditions, one immediately gets from (78) that $p_j(t) = \wp_j = 0$ for all $j = 1, \ldots, d-1$. This in turn implies $\dot{x}_j = 0$ for $j = 1, \ldots, d-1$ and so

$$x_j(t) \ := \ \varrho_j \ = \ \text{const.}\,, \qquad\qquad j = 1, \ldots, d-1\ .$$

This means that the only possible motion could take place exclusively in the direction $\gamma$, namely it is one-dimensional.

Without loss of generality let us assume that $\varrho_1 = \ldots = \varrho_{d-1} = 0$ which means that $x_j(t) = 0 = p_j(t)$ for all $j = 1, \ldots, d-1$. Given that, the only interesting degrees of freedom are $x_0$ and $p_0$ and we can simplify the notation identifying $x_0$ with $x$ and $p_0$ with $p$. With this notation the (non-trivial) one-dimensional system of Hamilton equations reads

$$\left\{\begin{aligned} \dot{x} \ &= \ (1+\lambda\, x)\,\frac{p}{m} \\ \dot{p} \ &= \ -\lambda\frac{p^2}{2m}\ . \end{aligned}\right. \tag{89}$$

The equation for the momentum immediately integrated by

$$p(t) \ = \ \ell\,\frac{\wp}{\frac{\wp}{2m}\,t+\ell} \tag{90}$$

with $\wp = p(0)$ the initial momentum. Notice that the value of the momentum diverges at the critical time $t_{\rm c} := -\ell\frac{2m}{\wp}$.

The time evolution of the position can be derived directly from equation (81) which, after some algebraic manipulation, provides

$$x(t) \ = \ \frac{(\ell+\varrho)}{\ell^2}\left(\frac{\wp}{2m}\,t+\ell\right)^2 \ - \ \ell \tag{91}$$

with $\varrho = x(0)$ the initial position. The long time behavior of the trajectory is determined by the sign of the coefficient of $t^2$ in (91), namely by the sign of $\ell + \varrho$. It follows that

$$\lim_{|t|\to\infty} x(t) \ = \ \pm\infty \qquad\quad \text{if} \qquad \pm\,\varrho \ > \ \mp\ell\ .$$

The turning time in which the velocity changes sign is determined by $\dot{x}(t) = 0$ and a simple computation shows that this time coincides with the critical time $t_{\rm c}$. Moreover, one has that $x(t_{\rm c}) = -\ell$ independently of the initial value $\varrho_0 \neq -\ell$. In conclusion the critical plane $\Xi_{\rm c}$ separates the space into two regions and the trajectory $x(t)$ is fully contained in only one of these two half-spaces according to the initial position $\varrho$. Moreover, the trajectory can touch the critical plane only once at the critical time $t_{\rm c}$. These results are in accordance with the qualitative analysis of Section 4.2.

4.6. **The Lagrangian Formalism.** By using the Legendre transformation $\mathscr{L}_T(x,\dot{x}) = \dot{x}\cdot p \ - \ H_T(x,p)$ one can compute the Lagrangian of the system:

$$\mathscr{L}_T(x,\dot{x}) \ := \ \frac{1}{2}\frac{m}{(1+\lambda\,\gamma\cdot x)}\,\dot{x}^2 \ = \ m_T(x)\,\frac{\dot{x}^2}{2}\ . \tag{92}$$

Expressions of the type (92) are well studied in the literature under the name of *quasi-free PDM Lagrangian* (see [MM, BDGP, Mu] and references therein). The canonical momentum

$$p(x,\dot{x}) \ := \ \nabla_{\dot{x}}\mathscr{L}_T(x,\dot{x}) \ = \ m_T(x)\,\dot{x}$$

is exactly that given by equation (70). To compute the Euler-Lagrange equations of motion we need also

$$\nabla_x\mathscr{L}_T(x,\dot{x}) \ = \ \nabla_x m_T(x)\,\frac{\dot{x}^2}{2} \ = \ -\lambda\,\frac{m_T(x)^2}{m}\,\frac{\dot{x}^2}{2}\,\gamma\ .$$

A comparison with (74) shows that

$$\nabla_x\mathscr{L}_T \ = \ \dot{p} \ = \ -\nabla_x H_T$$

and this assures that the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_{\dot{x}}\mathscr{L}_T\right) \ = \ \nabla_x\mathscr{L}_T$$

is equivalent to the Hamilton system (68). An explicit computation provides

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_{\dot{x}}\mathscr{L}_T\right) \ &= \ m_T(x)\,\ddot{x} \ + \ \frac{\mathrm{d}}{\mathrm{d}t}\left(m_T(x)\right)\,\dot{x}\\ &= \ m_T(x)\,\ddot{x} \ - \ \lambda\,\frac{m_T(x)^2}{m}\,\left(\gamma\cdot\dot{x}\right)\dot{x}\end{aligned}$$

and putting all the pieces together one gets

$$m_T(x)\,\ddot{x} \ = \ \lambda\,\frac{m_T(x)^2}{m}\,\left(\gamma\cdot\dot{x}\right)\dot{x} \ - \ \lambda\,\frac{m_T(x)^2}{m}\,\frac{\dot{x}^2}{2}\,\gamma \tag{93}$$

which is equivalent to the Newton's equation $m\ddot{x} = \lambda F_T$ with the force (73).

In the one-dimensional case it is useful to use the change of Lagrangian coordinates $(x,\dot{x}) \mapsto (q,\dot{q})$ implemented by

$$x(q) \ := \ \mathrm{e}^{\lambda q} \ - \ \frac{1}{\lambda}\ , \qquad\qquad \dot{x}(q,\dot{q}) \ := \ \lambda\,\mathrm{e}^{\lambda q}\,\dot{q}\ .$$

The inverse is given by

$$q(x) \ := \ \frac{1}{\kappa}\log\left(x+\frac{1}{\lambda}\right)$$

and shows that the change of coordinates between $x$ and $q$ is one-to-one only when $x \geqslant -\ell$. However, as seen in Section 4.2, this is exactly the range of values of interest for the problem. With this change of coordinates the Lagrangian becomes

$$\mathscr{L}'_T(q,\dot{q}) \ := \ m\,\lambda\,\mathrm{e}^{\lambda q}\,\frac{\dot{q}^2}{2}\ . \tag{94}$$

and the associated Euler-Lagrange equation reads

$$\ddot{q} \ := \ -\lambda\,\frac{\dot{q}^2}{2}\ .$$

This equation immediately provides the time-behavior of the generalized velocity

$$\dot{q}(t) \ := \ \frac{\dot{q}_0}{1+\frac{\dot{q}_0\lambda}{2}t}$$

and a further integration gives

$$q(t) \ := \ q_0 \ + \ \frac{2}{\lambda}\log\left(1+\frac{\dot{q}_0\lambda}{2}t\right)$$

where $q_0, \dot{q}_0$ are the initial conditions. By coming back to the original variable one can recover the expression (91) for $x(t)$.

4.7. **parallels between the Classical and Quantum Dynamics.** Let us note that the critical plane $\Xi_c$ as described in (69) also plays a role in the quantum case, as seen in remark 3.7. In both cases, a particle (or an element of the Hilbert space) localized in one side of $\Xi_c$ (in the quantum case, the point $x = -1/\lambda$) doesn't leave that space. Moreover, we can further link the existance of this critical plane, with the rather anomalous property of $H_T$ of not having the Schwarz functions as a core. As we saw in section 3, $H_T$ has as a core $\mathcal{S}(\mathbb{R}) + \mathbb{C}\,[S^*_\lambda(\kappa_0)]$, composed of sufficiently regular functions, except for a jump in $x = -1/\lambda$ given by $S^*_\lambda(\kappa_0)$; Just describing how $H_T$ acts over $\mathcal{S}(\mathbb{R})$ is not enough for it's closure to be self adjoint. To be able to give a satisfactory answer to the question of the transition quantum-to-ckassic, one possible way lies in the WKB analysis, studying the limit of the quantum case as $\hbar$ tends to 0. This is one of the open questions to be tackled in future works.

## Appendix A. Appendix: Spectral Theory

### A.1. The Momentum operator.

**Definition A.1** (The Momentum operator $p$)**.** *The 1-dimensional momentum operator $p$ is defined by*

$$p\varphi(x) = -i\varphi'(x), \qquad \varphi \in H^1(\mathbb{R}) \tag{95}$$

*with $\varphi'$ being the weak derivative of $\varphi$, and $H^1(\mathbb{R})$ the first Sobolev space.*

This definition of $p$ ensures one of the many possible quantizations of the classical momentum operator, and the most usual one in the continous case, for it holds the canonical commutation relation $i[p, x] = I$ with the position operator $x$ acting as multiplication, and notably $p$ is the generator of the translation group acting on $L^2(\mathbb{R}) : U(t)\psi(x) = \psi(x - t)$. The spectral theory of the Momentum operator is greatly simplified due to the following result.

**Theorem A.2** (Diagonalization of the Momentum operator)**.** *The operator $p$ is unitarily equivalent to the position operator $x$, thoroughly defined by*

$$(x\varphi)(x) = x\varphi(x); \qquad \mathcal{D}(x) = \left\{ \varphi \in L^2(\mathbb{R}) \middle| \int_{\mathbb{R}} |x\varphi(x)|^2 \,\mathrm{d}x < \infty \right\} \tag{96}$$

*The unitary equivalence is given by the Fourier transform:*

$$(\mathscr{F}\psi)(k) \;:=\; \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{d}x \; \mathrm{e}^{-\,\mathrm{i}\,k\cdot x} \; \psi(x), \tag{97}$$

*the inverse Fourier transform:*

$$(\mathscr{F}^{-1}\phi)(x) = (\mathscr{F}^*\phi)(x) \;:=\; \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{d}k \; \mathrm{e}^{\,\mathrm{i}\,x\cdot k} \; \phi(k), \tag{98}$$

*and the relation*

$$\mathscr{F} p \mathscr{F}^* = x \tag{99}$$

From this unitary equivalence we can also compute the spectrum of $p$, obtaining $\sigma(p) = \sigma_{a.c.}(p) = \mathbb{R}$ and also the functional calculus for p

$$f(p)\varphi = \mathscr{F}^* f(x) \mathscr{F} \varphi$$

fixing $\mathscr{F}\varphi = \widehat{\varphi}$, and as $f(x)$ acts by multiplication

$$= \mathscr{F}^* f(x)\widehat{\varphi} = \mathscr{F}^*(f\widehat{\varphi})$$

Using the convolution theorem we finally have

$$f(p)\varphi = \frac{1}{\sqrt{2\pi}}\check{f} * \varphi. \tag{100}$$

An important family of functions to compute are the resolvent functions for $p$. Let $\epsilon \in \mathbb{R}$, $\lambda > 0$ and

$$R_{\epsilon \pm \mathrm{i}\,\lambda}(p) = \frac{1}{p - (\epsilon \pm \mathrm{i}\,\delta)}. \tag{101}$$

From the previous formula, and as $f(x) = (\epsilon \pm \mathrm{i}\,\lambda - k)^{-1}$ has an explicit inverse Fourier transform

$$\mathscr{F}^*\left(\frac{1}{k - (\epsilon \pm \mathrm{i}\,\delta)}\right)(x) = \pm\,\mathrm{i}\, e^{\mathrm{i}\,\epsilon x} e^{-\delta|x|}\Theta(\pm x) \tag{102}$$

As such, we have the following proposition.

**Proposition A.3.** *The resolvent family of operators $\mathcal{R}_\zeta(p)$ are integrals operators given by*

$$R_{\epsilon \pm \mathrm{i}\,\delta}(p)\psi(x) = \int_{\mathbb{R}} \mathrm{d}y \mathscr{G}^0_\zeta(x,y)\psi(y), \tag{103}$$

*with integral kernel*

$$\mathscr{G}^0_{\epsilon \pm \mathrm{i}\,\delta}(x,y) \ = \ \pm\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\epsilon(x-y)}\ \mathrm{e}^{-\delta\ |x-y|}\Theta\big(\pm(x-y)\big). \tag{104}$$

We will now proceed to find an explicit expression for the expectral measure. Let $\mu^A_\psi$ be the spectral measure of the self-adjoint operator $A$ associated with the state $\psi \in L^2(\mathbb{R})$. The function $F^A_\psi : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ defined by the scalar product

$$F^A_\psi(\zeta) \ := \ \langle \psi, (A - \zeta\mathbf{1})^{-1}\psi\rangle \ = \ \int_{\mathbb{R}} \mathrm{d}\mu^A_\psi(\epsilon)\ \frac{1}{\epsilon - \zeta}$$

is called the *Borel-Stieltjes transformation* of the finite Borel measures $\mu^A_\psi$. Since

$$\mathrm{Im}\left(F^A_\psi(\zeta)\right) \ = \ \mathrm{Im}(\zeta) \int_{\mathbb{R}} \mathrm{d}\mu^A_\psi(\epsilon)\ \frac{1}{|\epsilon - \zeta|^2},$$

it follows that $F^A_\psi : \mathbb{C}^+ \to \mathbb{C}^+$ is is a holomorphic map from the upper half plane $\mathbb{C}^+$ into itself. Such functions are called *Herglotz* or *Nevanlinna functions* (see [DK, Section 1.4] or [AW, Appendix]). A classical result by de la Vallée-Poussin assures that the limit $F^A_\psi(\epsilon) := \lim_{\delta \to 0^+} F^A_\psi(\epsilon + \mathrm{i}\,\delta)$ exists and is finite for Lebesgue-almost every $\epsilon \in \mathbb{R}$. Moreover, the absolutely continuous part of the spectral measure $\mu^A_\psi$ can be recovered from the imaginary part of $F^A_\psi(\rho)$ according to the classical formula [DK, Theorem 1.4.16.]

$$\mu^A_\psi|_{\mathrm{a.c.}}(\,\mathrm{d}\epsilon) \ = \ f^A_\psi(\epsilon)\ \mathrm{d}\epsilon,$$

with

$$f^A_\psi(\epsilon) \ := \ \lim_{\delta \to 0^+} \frac{1}{\pi}\ \mathrm{Im}\left(F^A_\psi(\epsilon + \mathrm{i}\,\delta)\right)\ . \tag{105}$$

It is known that the momentum operator has purely absolutely continuous spectrum, *i. e.* $\mu_\psi^p = \mu_\psi^p|_{\text{a.c.}}$. using the Fourier transform $\mathscr{F}$ one obtains that

$$F_\psi^p(\epsilon + \mathrm{i}\,\delta) \ := \ \langle \psi, (p - (\epsilon + \mathrm{i}\,\delta)\mathbf{1})^{-1}\psi\rangle \ = \ \int_{\mathbb{R}} \mathrm{d}k \ \frac{|\widehat{\psi}(k)|^2}{(k-\epsilon) - \mathrm{i}\,\delta},$$

where $\widehat{\psi} := \mathscr{F}(\psi)$ is the Fourier transform of $\psi$. The application of the formula (105) provides

$$f_\psi^p(\epsilon) \ = \ \lim_{\delta\to 0^+} \int_{\mathbb{R}} \mathrm{d}k \ \frac{1}{\pi}\frac{\delta}{(k-\epsilon)^2 + \delta^2}|\widehat{\psi}(k)|^2 \ = \ |\widehat{\psi}(\epsilon)|^2$$

where in the last equality one used that $\frac{1}{\pi}\frac{\delta}{x^2+\delta^2}$ converges in the distributional sense to $\delta(x)$ when $\delta \to 0^+$. In this way one recovers the well-known result

$$\mu_\psi^p(\,\mathrm{d}\epsilon) \ = \ |\widehat{\psi}(\epsilon)|^2 \ \mathrm{d}\epsilon\,. \tag{106}$$

A.2. **Self-Adjoint and Symmetric Operators.** As the axioms of QM state, every physical observable is given by a (possibly unbounded) self adjoint operator acting on a Hilbert space $\mathcal{H}$, thus the question of determining which operators are self adjoint is one of the first to be asked.
We first start by stating the definition of the adjoint of an operator, in the particular case of Hilbert spaces.

**Definition A.4** (Adjoint of an operator)**.** *Let $\mathcal{H}$ be a Hilbert space and $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ a densely defined linear operator. We start defining $A^\star$ by its domain*

$$D(A^\star) = \{v \in \mathcal{H}; \quad \exists w \in \mathcal{H}\colon \ (v \cdot Au) = (w \cdot u) \quad \forall u \in D(A)\}$$

*. $A^\star$ is then the operator acting on the domain $\mathcal{D}(A^\star)$ given by $A^\star v = w$.*

In a sense, the relation between $A$ and $A^\star$ is essentialy

$$(v \cdot Au) = (A^\star v \cdot u) \quad \forall u \in D(A), v \in D(A^\star)$$

where $D(A^\star)$ is the maximal domain where such property can take place. Naturally an operator is self adjoint exactly when $D(A) = D(A^\star)$ and

$$(v \cdot Au) = (Av \cdot u) \quad \forall u, v \in D(A).$$

If an operator B acting on $\mathcal{H}$ only satisfies this last property, it is called symmetric and a simple computation shows that $\mathcal{D}(A) \subset \mathcal{D}(A^\star)$. In general checking if an operator is symmetric over a certain domain is a much more straightforward task, thus we are interested in the question of determining the self adjoint extensions (if there are any) of a given symmetric operator A. The following theorem due to Von Neumann is a clasical result and usually one of the first criterions used.

**Theorem A.5** (Von Neumann's Theorem)**.** *Let $A$ be a symmetric operator acting on a domain $\mathcal{D}(A)$ and suppose that there exist a conjugation $C : \mathcal{D}(A) \mapsto D(A)$ (that it, an antilinear operator that preserves the norm and satisfies $C^2 = I$) with $CA = AC$. Then $A$ has equal deficiency indices, and thus self adjoint extensions.*

It's proof can be found in [RS2, Theorem X.3]. The main appeal of this theorem is the relative simplicity of the conjugation $C$. An usual example $C$ is $Cf(x) = \overline{f(x)}$, thus earning the name of conjugation. Let us now proceed with some tools for determining the self adjoint extensions. We start with an usual definition.

**Definition A.6** (Closed operator)**.** *Let $\mathcal{H}$ be a Hilbert space and $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ a densely defined linear operator. We say $A$ is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ we have $x \in \mathcal{D}(A)$ and $Ax = y$.*

This is equivalent to the graph of $A$ $\Gamma(A)$ being closed with the product topology. Given an operator, we may be interested in the smallest closed extension it has, if any. This motivates the following definition.

**Definition A.7** (Closure of an Operator)**.** *Given $A$ acting in $\mathcal{H}$, we say $A$ is closable if and only if $\overline{\Gamma(A)}$ has the single-valued property, that is, $(0, y) \in \overline{\Gamma(A)} \implies y = 0$. In that case, we can define the closure of $A$, denoted by $\overline{A}$ and given by*

$$\begin{aligned}
&\overline{A}\colon \mathcal{D}(\overline{A}) = \overline{\mathcal{D}(A)}^{\|\,\|_A} = \left\{x \in \mathcal{H}\colon \exists\, y \in \mathcal{H}\,,\, (x, y) \in \overline{\Gamma(A)}\right\} \to \mathcal{H} \\
&\overline{A}x := y
\end{aligned} \tag{107}$$

Not every operator is closable, but every symmetric operator is closable. The following definition and result characterises the degrees of freedom available when defining a self adjoint extension of a symmetric operator.

**Definition A.8** (Deficiency subspaces)**.** *Suppose that $A$ is symmetric. Let*

$$\mathcal{K}_\pm = \ker(\mathrm{i}\,\mathrm{Id} \mp A^*) = \mathrm{Ran}(iI \pm A)^\perp, \tag{108}$$

*and*

$$n_\pm(A) = \dim \mathcal{K}_\pm. \tag{109}$$

*$\mathcal{K}_+$ and $\mathcal{K}_-$ are called the deficiency subspaces of $A$. The respective dimensions $n_+$ and $n_-$ are called the deficiency indices of $A$.*

**Theorem A.9** (Self-adjoint extensions of a closed symmetric operator)**.** *Let $A$ be a closed symetric operator with deficiency indices $n_+$ and $n_-$, then:*

*(a) $A$ is self-adjoint if and only if $n_+ = n_- = 0$.*
*(b) $A$ has self adjoint extensions if and only if $n_+ = n_-$.*

*Moreover, every self-adjoint extension of $A$ is determined by an isometry $U : \mathcal{K}_+ \to \mathcal{K}_-$, and given by*

$$\begin{aligned}
&A_U\colon \mathcal{D}(A) + (U + I)\mathcal{K}_+ \to \mathcal{H} \\
&A_U(\phi + (U + I)\psi_+) = A\phi + i(I - U)\psi_+, \quad \phi \in \mathcal{D}(A)\ , \psi_+ \in \mathcal{K}_+
\end{aligned} \tag{110}$$

**Remark A.10.** *If a symmetric non-closed operator $A$ is such that $\overline{A}$ satisfies the conditions of the last theorem, one has that $A_U\colon \mathcal{D}(A) + (U + I)\mathcal{K}_+$ is instead a core for $A_U$, that is, the closure of $A_U$ is self-adjoint.*

These results on deficiency spaces and their proofs can be found in [RS2, Section X]. This way, the self-adjoint extensions of a closed symmetric operator with finite and equal deficiency indices $n_- = n_+ = N$ can be parametrized by an element of $GL(N, \mathbb{C})$ after choosing orthonormal bases for $\mathcal{K}_\pm$.
Another formulation for self adjoint extensions is given in [Sch, Chapter 14] in the form of boundary triples, by the following definition

**Definition A.11** (Boundary Triple)**.** *Let $T$ be a densely defined symmetric operator on a Hilbert space $\mathcal{H}$. A boundary triplet for $T^*$ is a triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$ of a Hilbert* *space* $(\mathcal{K}, \langle\cdot\rangle_{\mathcal{K}})$ *and linear mappings* $\Gamma_0 : \mathcal{D}(T^*) \to \mathcal{K}$ *and* $\Gamma_1 : \mathcal{D}(T^*) \to \mathcal{K}$ *such that*

$$\langle T^*x \cdot y\rangle - \langle x \cdot T^*y\rangle = \langle \Gamma_1 x \cdot \Gamma_0 y\rangle_{\mathcal{K}} - \langle \Gamma_0 x \cdot \Gamma_1 y\rangle_{\mathcal{K}} \quad \text{for } x, y \in \mathcal{D}(T^*) \tag{111}$$

*holds and the mapping* $\mathcal{D}(T^*) \ni x \mapsto (\Gamma_0 x, \Gamma_1 x)$ *is surjective.*

The name boundary triple is due to that, when $T$ is a differential operator, $\Gamma_0$ and $\Gamma_1$ usually take the form of Boundary evaluations of the argument or it's derivatives. In this vein equation (111) is called the abstract Green's Identity. the following proposition coming from [Sch, 14.5] to links the existance of boundary triples and the existance of self adjoint extensions, taking a step through deficiency indices.

**Proposition A.12.** *There exists a boundary triplet* $(\mathcal{K}, \Gamma_0, \Gamma_1)$ *for* $T^*$ *if and only if the symmetric operator* $T$ *has equal deficiency indices. We then have* $n_-(T) = n_+(T) = \dim \mathcal{K}$

The characterization of all self-adjoint extensions comes in the form of the following theorem:

**Theorem A.13.** *Suppose* $(\mathcal{K}, \Gamma_0, \Gamma_1)$ *is a boundary triplet for* $T^*$*. Then* $S$ *is a self adjoint extension of* $T$ *on* $\mathcal{H}$ *if and only if there is a self-adjoint operator* $B$ *acting on a closed subspace* $\mathcal{K}_B \subset \mathcal{K}$ *and* $S$ *is a restriction of* $T^*$ *acting on the domain*

$$\mathcal{D}(S) = \{x \in \mathcal{D}(T^*) \colon \Gamma_0 x \in \mathcal{D}(B) \text{ and } B\Gamma_0 x = P_B \Gamma_1 x\} \tag{112}$$

*Where* $P_B$ *is the orthogonal projector over the domain of* $B$*.*

Whose proof and a more extensive formulation can be found in [Sch, 14.10]. We will finish this section with a classical theorem, that relates to small perturbations of self-adjoint perturbations. For this goal we need the following definition.

**Definition A.14.** *Let* $A$ *and* $B$ *be densely defined linear operators on a Hilbert space* $\mathcal{H}$*. Suppose that* $\mathcal{D}(A) \subset \mathcal{D}(B)$ *and for some* $a, b \in \mathbb{R}$ *and for al* $\psi \in \mathcal{D}(A)$*,*

$$\|B\phi\| \leqslant a\|A\phi\| + b\|\phi\| \tag{113}$$

*Then* $B$ *is said to be* $A$*-bounded. The infimum of such* $A$ *is called the relative bound of* $B$ *with respect to* $A$*. In particular, if* $B$ *is bounded, as*

$$\|B\phi\| \leqslant \|B\|\|\phi\|$$

*then it is also* $A$*-bounded with relative bound equal equal to 0.*

Now we are ready to state the Kato-Rellich Theorem.

**Theorem A.15** (Kato-Rellich)**.** *Suppose* $A$ *and* $B$ *are self-adjoint and symmetric operators respectively, and* $B$ *is* $A$*-bounded with relative bound* $a < 1$*. Then* $A + B$ *is self-adjoint with* $\mathcal{D}(A + B) = \mathcal{D}(A)$ *and essentialy self adjoint on any core of* $A$

The theorem can be found in [RS2, Theorem X.12] with it's complete proof.

A.3. **Scattering Theory.** In a broad sense, the basis of Scattering Theory is the comparison of a system governed by a Hamiltonian $H$, and a different, and hopefully more well understood, Hamiltonian $H_0$. Supposing for example, that the difference between $H$ and $H_0$ is localized in a region of space, we can reasonably expect that it's effects will be negligible in the dynamics for starting from a given time. This argument can be made more formal asking if there is a certain $\rho_-$ such that for a given $\rho$:

$$\lim_{t\to+\infty} \|\mathrm{e}^{-itH_0}\rho_+ - \mathrm{e}^{-itH}\rho\| = 0$$

A way to interpret this condition is: For a given initial condition $\rho$ evolving with the dynamics determined by $H$, is there a state $\rho_+$ such that the evolution of $\rho_+$ by a similar, and usually simpler Hamiltonian $H_0$ is arbitrarily close to it? Using the unitarity of evolution groups, the last condition may be rewritten as

$$\operatorname*{s-lim}_{t\to+\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0}\rho_+ = \rho.$$

To make this notions formal, we start with the following definition, from [RS3, Section XI.3].

**Definition A.16** (Generalized wave operators)**.** *Let $H$ and $H_0$ be self-adjoint operators on a Hilbert space $\mathscr{H}$ and let $P_{\text{a.c.}}(H_0)$ be the projection onto the absolutely continous subspace of $H_0$. We say the generalized wave operators $W_\pm(H, H_0)$ exist if the strong limits*

$$W_\pm(H, H_0) = \operatorname*{s-lim}_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}\,Ht}\mathrm{e}^{-\mathrm{i}\,H_0 t}P_{\text{a.c.}}(H_0) \tag{114}$$

*exist. When they exist we also define*

$$\mathscr{H}_+ = \operatorname{Ran} W_+ \quad \textit{and} \quad \mathscr{H}_- = \operatorname{Ran} W_-$$

By construction, the wave operators are partial isometries from $\operatorname{Ran} P_{ac}(H_0)$ to $\mathscr{H}_+$. Also, one can show

$$\mathrm{e}^{\mathrm{i}\,sH}W_\pm(H, H_0) = \operatorname*{s-lim}_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}\,H(t+s)}\mathrm{e}^{-\mathrm{i}\,H_0 t}P_{\text{a.c.}}(H_0) = W_\pm(H, H_0)\mathrm{e}^{\mathrm{i}\,-sH_0},$$

which can then be generalized up to

$$HW_\pm(H, H_0) = W_\pm(H, H_0)H_0P_{\text{a.c.}}(H_0)$$

As we can already see, the wave operators are almost unitary equivalences between the absolutely continuous part of $H_0$ and $H$. To reach the desired equivalence just the existance of generalized wave operators is not strong enough. The following definition gives us conditions strong enough for such statements.

**Definition A.17** (Complete wave operator)**.** *Suppose that $W_\pm(H, H_0)$ exist. We say they are complete if and only if*

$$\mathscr{H}_+ = \mathscr{H}_- = P_{\text{a.c.}}(H)$$

*Or equivalently, $W_\pm(H, H_0)$ are complete if and only if $W_\pm(H_0, H)$ exist.*

Directly from this we get the following theorem

**Theorem A.18.** *Let both $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ exist, then*

$$W_\pm(H, H_0)^* = W_\pm(H_0, H). \tag{115}$$

Finally, if both exist we can also define the Scattering Operator or Scattering Matrix:

$$S := W_+(H, H_0)^* W_-(H, H_0) \tag{116}$$

Completing our initial analogy, when studying the dynamics of $H$, a perturbation of $H_0$, the scattering operator represents the effect of the perturbation in the original dynamics.
Most of times, finding conditions for completeness of wave operators is easier said than done; in general one does not have an explicit expression for $\mathrm{e}^{-\,\mathrm{i}\,tH}$. The tools to circumvent this issue are many, which we will not be reviewing in this appendix.

## Appendix B. Appendix: Technical Tools

B.1. **Some principal value integrals.** All the integrals in this section will be used in calculations for integral kernels. To streamline the other proofs, they were placed in their own subsection.

**Proposition B.1.** *Let*

$$G_s^{\pm}(u) \;:=\; \frac{\mathrm{e}^{\,\mathrm{i}\,s\left(u \pm \frac{1}{u}\right)}}{u}\;, \qquad s \in \mathbb{R},$$

*and* $\mathbb{I}_{R,r} = [-R, -r] \cup [r, R]$. *Then the following principal value integral exists and is given by*

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_s^{\pm}(u) := \lim_{\substack{R \to +\infty \\ r \to 0^+}} \int_{\mathbb{I}_{R,r}} \mathrm{d}u\; G_s^{\pm}(u) \;=\; \mathrm{i}\,(1 \pm 1)\pi\; \mathrm{sign}(s)\; J_0\left(2|s|\right) \tag{117}$$

*where $J_0$ is the 0-th Bessel function of the first kind. Moreover, the following bound holds for $r < 1 < R$*

$$\left| \int_{\mathbb{I}_{R,r}} \mathrm{d}u\; G_s^{\pm}(u) \right| \leqslant 4\pi \tag{118}$$

*Proof.* For the trivial case $s = 0$ one has that $G_0^{\pm}(u) = u^{-1}$ and

$$\int_{\mathbb{I}_{R,r}} \frac{\mathrm{d}u}{u} \;=\; 0\;, \qquad \forall\, R > r > 0$$

since the function $u^{-1}$ is odd and the integration domain $\mathbb{I}_{R,r}$ is symmetric with respect to the origin. For $s \neq 0$ we have the symmetry

$$G_{-|s|}^{\pm}(u) \;=\; -\,G_{|s|}^{\pm}(-u),$$

which provides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_{-|s|}^{\pm}(u) \;=\; \mathcal{P}\int_{\mathbb{R}} \mathrm{d}(-u)\; G_{|s|}^{\pm}(-u) \;=\; -\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_{|s|}^{\pm}(u)\;. \tag{119}$$

The relation (119) guarantees that we can focus only on the case $s > 0$. In this case the computation of the principal value of $G_s^{\pm}$ requires the Cauchy's residue theorem. The function $G_s^{\pm}$ has a holomorphic extension to every bounded open subset of $\mathbb{C}\backslash\{0\}$ and has a singularity in 0. For $u \neq 0$ expanding in series each exponential we have

$$G_s^{\pm}(u) = \frac{\mathrm{e}^{\,\mathrm{i}\,su}\mathrm{e}^{\pm\,\mathrm{i}\,su^{-1}}}{u} = \frac{1}{u}\sum_{n=0}^{\infty}\frac{(\,\mathrm{i}\,su)^n}{n!}\sum_{m=0}^{\infty}\frac{(\pm\,\mathrm{i}\,su^{-1})^m}{m!}$$

As the exponential series is absolutely convergent we can manipulate the summation order

$$G_s^{\pm}(u) = \sum_{n,m\geqslant 0} \frac{(\pm 1)^m(\mathrm{i}\, s)^{n+m}u^{n-m-1}}{n!\, m!}$$

changing $n$ for $l = n - m$

$$G_s^{\pm}(u) = \sum_{l=-\infty}^{+\infty}\sum_{m=0}^{+\infty} \frac{(\pm 1)^m(\mathrm{i}\, s)^{l+2m}}{(m+l)!\, m!}u^{l-1}$$

The series following two series are absolutely summable and equal to

$$\sum_{m=0}^{+\infty}\frac{(\mathrm{i}\, s)^{l+2m}}{(m+l)!\, m!} = J_l(2s); \ \sum_{m=0}^{+\infty}\frac{(-1)^m(\mathrm{i}\, s)^{l+2m}}{(m+l)!\, m!} = I_l(2s)$$

With $J_l$ the Bessel Functions of the first kind, and $I_l$ the modified Bessel Function of the first kind. We then have the writing as Laurent series for $G_s^{pm}$

$$G_s^+(u) \ = \ \sum_{n\in\mathbb{Z}} J_n(2s)\ u^{n-1}\ ; G_s^-(u) = \sum_{n\in\mathbb{Z}} I_n(2s)\ u^{n-1}$$

where the $J_n$ are the Bessel function of the first kind and the $I_n(z) := (-\mathrm{i}\,)^n J_n(\mathrm{i}\, z)$ are the modified Bessel functions of the first kind.
By definition, the residue of $G_s^{\pm}$ is the coefficient of its Laurent series for $n = -1$. This provides

$$\mathrm{Res}_{u=0}(G_s^-) \ = \ I_0(2s)\ , \qquad \mathrm{Res}_{u=0}(G_s^+) \ = \ J_0(2s)\ .$$

From the Cauchy's residue theorem one gets

$$\mathrm{i}\, 2\pi \mathrm{Res}_{u=0}(G_s^+) \ = \ \oint_{\Gamma_{R,r}} \mathrm{d}z\ G_s^+(z) \ = \ \left(\int_{\mathbb{I}_{R,r}} + \int_{C_R^+} + \int_{C_r^-}\right) \mathrm{d}z\ G_s^+(z)$$

And

$$0 \ = \ \oint_{\Gamma'_{R,r}} \mathrm{d}z\ G_s^-(z) \ = \ \left(\int_{\mathbb{I}_{R,r}} + \int_{C_R^+} + \int_{C_r^+}\right) \mathrm{d}z\ G_s^(z)$$

where $\Gamma_{R,r}$ is a positively (counterclockwise) oriented simple closed curve composed by the union of the domain $\mathbb{I}_{R,r}$ on the real line, the semicircle $C_r^- := \{r\mathrm{e}^{\mathrm{i}\,\theta} \mid \theta \in [-\pi, 0]\}$ in the lower half-plane and the semicircle $C_R^+ := \{R\mathrm{e}^{\mathrm{i}\,\theta} \mid \theta \in [0,\pi]\}$ in the upper half-plane and $\Gamma_{R,r}$ also a positively oriented simple closed curve composed by $\mathbb{I}_{R,r}, C_R^+$ and $C_r^+$. An explicit computation provides

$$\int_{\mathcal{C}_R^+} \mathrm{d}z\ G_s^{\pm}(z) \ = \ \mathrm{i}\int_0^{+\pi} \mathrm{d}\theta\ \mathrm{e}^{\mathrm{i}\, s\left(R\pm R^{-1}\right)\cos\theta}\mathrm{e}^{-s\left(R\mp R^{-1}\right)\sin\theta}\ ,$$

and consequently one has the following estimate for $R > 1$.

$$\left|\int_{\mathcal{C}_R^+} \mathrm{d}z\ G_s^{\pm}(z)\right| \ \leqslant \ \int_0^{+\pi} \mathrm{d}\theta\ \mathrm{e}^{-s\left(R\mp R^{-1}\right)\sin\theta} \leqslant \pi.$$

Since $\mathrm{e}^{-s\left(R\mp R^{-1}\right)\sin\theta} \to 0$ when $R \to +\infty$ for all $\theta \in (0,\pi)$, it follows from the Lebesgue's dominated convergence theorem that

$$\lim_{R\to+\infty}\int_{\mathcal{C}_R^+} \mathrm{d}z\ G_s^{\pm}(z) \ = \ 0\ . \tag{120}$$

A similar computation for the integral along $\mathcal{C}_r^-$ provides

$$\int_{\mathcal{C}_r^-} \mathrm{d}z\; G_s^+(z) \;=\; \mathrm{i}\int_{-\pi}^{0} \mathrm{d}\theta\; \mathrm{e}^{\mathrm{i}\, s\left(r+r^{-1}\right)\cos\theta}\mathrm{e}^{-s\left(r-r^{-1}\right)\sin\theta}\;.$$

One gets the following bound for $r<1$

$$\left|\int_{\mathcal{C}_r^-} \mathrm{d}z\; G_s^+(z)\right| \;\leqslant\; \int_{-\pi}^{0} \mathrm{d}\theta\; \mathrm{e}^{s\left(r^{-1}-r\right)\sin\theta} \leqslant \pi\;.$$

The latest inequality along with the Lebesgue's dominated convergence theorem provides

$$\lim_{r\to 0^+}\int_{\mathcal{C}_r^-} \mathrm{d}z\; G_s^+(z) \;=\; 0. \tag{121}$$

Finally along $C_r^+$ we have

$$\int_{\mathcal{C}_r^+} \mathrm{d}z\; G_s^-(z) \;=\; \mathrm{i}\int_{0}^{\pi} \mathrm{d}\theta\; \mathrm{e}^{\mathrm{i}\, s\left(r-r^{-1}\right)\cos\theta}\mathrm{e}^{-s\left(r+r^{-1}\right)\sin\theta}\;,$$

providing for $r<1$

$$\left|\int_{\mathcal{C}_r^+} \mathrm{d}z\; G_s^-(z)\right| \;\leqslant\; \int_{0}^{\pi} \mathrm{d}\theta\; \mathrm{e}^{-s\left(r+r^{-1}\right)\sin\theta} \leqslant \pi\;,$$

and also giving us

$$\lim_{r\to 0^+}\int_{\mathcal{C}_r^+} \mathrm{d}z\; G_s^-(z) \;=\; 0. \tag{122}$$

Putting together (120), (121) and the formula of the residue theorem one gets

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_s^+(u) \;=\; \mathrm{i}\,2\pi J_0(2s)\;, \qquad s>0\;. \tag{123}$$

For the case $s<0$ the relation (119) immediately provides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_s^+(u) \;=\; -\,\mathrm{i}\,2\pi J_0(2|s|)\;, \qquad s<0\;. \tag{124}$$

And analogously for $G_s^-$ putting together (120) and (122) and the residue theorem we have

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; G_s^-(u) \;=\; 0 \tag{125}$$

Covering the cases. As $J_0$ is bounded, reaching its maximum of 1 in 0, and from the previous bounds we obtain (118). $\square$

**Corollary B.2.** *The formula*

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\; \frac{\mathrm{e}^{\mathrm{i}\,xu}\mathrm{e}^{-\mathrm{i}\,\frac{y}{u}}}{u} \;=\; \mathrm{i}\,2\pi\left(\frac{\mathrm{sgn}(x)-\mathrm{sign}(y)}{2}\right)\; J_0\left(2\sqrt{|xy|}\right)$$

*and the bound*

$$\int_{\mathbb{I}_{R,r}} \mathrm{d}u\; \frac{\mathrm{e}^{\mathrm{i}\,xu}\mathrm{e}^{-\mathrm{i}\,\frac{y}{u}}}{u} \leqslant 4\pi$$

*holds true for all $(x,y)\in\mathbb{R}^2$.*

*Proof.* Let us start by considering the singular situations $xy = 0$. The case $x = 0 = y$ corresponds to

$$\mathcal{P}\int_{\mathbb{R}} \frac{\mathrm{d}u}{u} \;=\; 0$$

as proved at the beginning of Lemma B.1. The case $y = 0$ is proportional to the (well known) Fourier transform of the function $u^{-1}$ and provides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\, \frac{\mathrm{e}^{\,\mathrm{i}\,xu}}{u} \;=\; -\sqrt{2\pi}\,\mathscr{F}\left(\frac{1}{u}\right) \;=\; \mathrm{i}\,\pi\,\mathrm{sign}(x)\;.$$

The case $x = 0$ can be treated with the change of variables $u \mapsto -v^{-1}$ which provides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\, \frac{\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; -\mathcal{P}\int_{\mathbb{R}} \mathrm{d}v\, \frac{\mathrm{e}^{\,\mathrm{i}\,yv}}{v} \;=\; -\,\mathrm{i}\,\pi\,\mathrm{sign}(y)\;.$$

The non singular situation $xy \neq 0$ can be separated in two different cases: (a) $xy > 0$, and (b) $xy < 0$.

**Case (a).** Let $a := \sqrt{xy}$. Then, after the change of variables $v := \frac{a}{|y|}u$, one has

$$\int_{\mathbb{I}_{R,r}} \mathrm{d}u\, \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; \int_{\mathbb{I}_{R',r'}} \mathrm{d}v\, \frac{\mathrm{e}^{\,\mathrm{i}\,\frac{x|y|}{a}v}\mathrm{e}^{-\,\mathrm{i}\,\mathrm{sign}(y)\,\frac{a}{v}}}{v} \;=\; \int_{\mathbb{I}_{R',r'}} \mathrm{d}v\; G_s^-(v)$$

where $R' := a|y|^{-1}R$, $r' := a|y|^{-1}r$ and $s = a\,\mathrm{sign}(y)$. Then, Lemma **??**rovides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\, \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; \mathcal{P}\int_{\mathbb{R}} \mathrm{d}v\; G_s^-(v) \;=\; 0\;.$$

**Case (b).** Let $b := \sqrt{|xy|}$. Then, after the change of variables $v := \frac{b}{|y|}u$, one has

$$\int_{\mathbb{I}_{R,r}} \mathrm{d}u\, \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; \int_{\mathbb{I}_{R',r'}} \mathrm{d}v\, \frac{\mathrm{e}^{\,\mathrm{i}\,\frac{x|y|}{b}v}\mathrm{e}^{-\,\mathrm{i}\,\mathrm{sign}(y)\,\frac{b}{v}}}{v} \;=\; \int_{\mathbb{I}_{R',r'}} \mathrm{d}v\; G_s^+(v)$$

where $R' := b|y|^{-1}R$, $r' := b|y|^{-1}r$ and $s = -b\,\mathrm{sign}(y)$. Again Lemma B.1 provides

$$\mathcal{P}\int_{\mathbb{R}} \mathrm{d}u\, \frac{\mathrm{e}^{\,\mathrm{i}\,xu}\mathrm{e}^{-\,\mathrm{i}\,\frac{y}{u}}}{u} \;=\; \mathcal{P}\int_{\mathbb{R}} \mathrm{d}v\; G_s^+(v) \;=\; -\,\mathrm{i}\,2\pi\mathrm{sign}(y)J_0(2\sqrt{|xy|})\;.$$

The observation that $-2\mathrm{sign}(y) = \mathrm{sign}(x) - \mathrm{sign}(y)$ when $xy < 0$ completes the point. Finally, the bound comes straightforwardly from lemma B.1. $\square$

B.2. **Irregular Kelvin functions.** The following integrals result in the irregular Kelvin functions kei and ker, defined over $\mathbb{R}_0^+$ as

$$\mathrm{ker}(x) = \mathrm{Re}\,K_0\left(e^{\frac{\mathrm{i}\,\pi}{4}}x\right) \quad \text{and} \quad \mathrm{kei}(x) = \mathrm{Im}\,K_0\left(e^{\frac{\mathrm{i}\,\pi}{4}}x\right)$$

Both $\mathrm{ker}(x)$ and $\mathrm{kei}(x)$ have an exponential decay of the type $\sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-\frac{x}{\sqrt{2}}}$ when $x \to +\infty$. The function $\mathrm{kei}(x)$ is regular in the origin where it takes the value $\mathrm{kei}(0) = -\frac{\pi}{4}$. The function $\mathrm{ker}(x)$ diverges at the origin as $\sim -\log(x)$. In particular one has that both Kelvin functions are in $L^2(\mathbb{R}_+)$.

**Lemma B.3.** *Let $\mathscr{B}(x,y)$ the kernel* (44)*. Then, the following formulas hold true:*

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}y\, \frac{\mathscr{B}(x,y)}{1+y^2} &= -\mathrm{i}\,2\,\mathrm{sign}(x)\,\mathrm{kei}\left(2\sqrt{|x|}\right)\\ \int_{\mathbb{R}} \mathrm{d}y\, \frac{\mathscr{B}(x,y)\,y}{1+y^2} &= -\mathrm{i}\,2\,\mathrm{ker}\left(2\sqrt{|x|}\right)\end{aligned}$$

*Proof.* After the change of variable $s := xy$ one gets

$$\mathcal{I}_1(x) := \int_{\mathbb{R}} \mathrm{d}y\, \frac{\mathscr{B}(x,y)}{1+y^2} = \mathrm{i}\,x \int_{-\infty}^{0} \mathrm{d}s\, \frac{J_0\left(2\sqrt{|s|}\right)}{x^2+s^2}\,.$$

A second change of variable $s := -t^2$ provides

$$\mathcal{I}_1(x) = \mathrm{i}\,2x \int_0^{+\infty} \mathrm{d}t\, t \frac{J_0\,(2t)}{x^2+t^4}$$

setting $\sqrt{|x|}u = t$

$$\begin{aligned}\mathcal{I}_1(x) &= \mathrm{i}\,2\,\mathrm{sign}(x) \int_0^{+\infty} \mathrm{d}u\, u\, \frac{J_0\left(2\sqrt{|x|}u\right)}{u^4+1}\\ &= -\mathrm{i}\,2\,\mathrm{sign}(x)\,\mathrm{kei}\left(2\sqrt{|x|}\right)\end{aligned}$$

where the last equality is justified by [OMS, eq. 55:3:6].
The second formula can be proved with similar changes of variable and one gets

$$\begin{aligned}\mathcal{I}_2(x) &:= \int_{\mathbb{R}} \mathrm{d}y\, \frac{\mathscr{B}(x,y)\,y}{1+y^2} = \mathrm{i} \int_{-\infty}^{0} \mathrm{d}s\, \frac{J_0\left(2\sqrt{|s|}\right)s}{x^2+s^2}\\ &= -\mathrm{i}\,2 \int_0^{+\infty} \mathrm{d}t\, t^3 \frac{J_0\,(2t)}{x^2+t^4}\\ &= -\mathrm{i}\,2 \int_0^{+\infty} \mathrm{d}u\, u^3\, \frac{J_0\left(2\sqrt{|x|}u\right)}{u^4+1}\\ &= -\mathrm{i}\,2\,\mathrm{ker}\left(2\sqrt{|x|}\right)\end{aligned}$$

where the last equality comes from [OMS, eq. 55:3:5]. □

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