

Refining the partition for multifold conic optimization problems

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Abstract

In this paper we give a unified treatment of two different definitions of complementarity partition of multifold conic programs introduced independently in [J. F. Bonnans and H. Ramírez C., *Math. Program.* 104 (2005), no. 2-3, Ser. B, 205–227] for conic optimization problems, and in [J. Peña and V. Roshchina, *Math. Program.* 142 (2013), no 1-2, Ser. A, 579–589] for homogeneous feasibility problems. We show that both can be treated within the same unified geometric framework, and extend the latter notion to optimization problems. We also show that the two partitions do not coincide, and their intersection gives a seven-set index partition. Finally, we demonstrate that the partitions are preserved under the application of nonsingular linear transformations, and in particular that a standard conversion of a second-order cone program into a semidefinite programming problem preserves the partitions.

1 Introduction

The complementarity partitions for linear programming and linear complementarity problems are well understood and are utilized heavily in the analysis of optimization problems as well as in numerical methods. In addition to strong duality (i.e., when both primal and dual problems have optimal solutions, and the values of their objective functions coincide), linear programming problems exhibit strict complementarity: in an appropriately formulated pair of primal and dual problems each primal variable has a dual counterpart, and there exists an optimal solution that is also a fully complementary pair, such that exactly one variable in each pair is positive and the other one is zero. The situation is much more complex for general linear conic problems. In a multifold system, where in place of the componentwise inequalities we have cone inclusions, it may happen that the relevant components of the primal and dual variables lie on the boundary of a corresponding cone. Moreover, there are several different definitions of complementarity that do not coincide and lead to different characterizations.

In some special cases, specifically when a homogeneous feasibility problem is considered, complementarity partition can be identified via an algorithm. The polyhedral (linear programming) case was considered in [20, 23], and it was shown in [21] that it is possible to identify the partition of [2] via a variant of an interior point method for second-order cone programming.

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The complementarity partition is related to generalized condition measures for multifold conic systems, see [4]. Different approaches to complementarity partition for optimization problems based on faces that contain primal and dual solutions can be found in [22, 25]. These studies commonly treat the cone as a whole, treating partition as an intrinsic geometric notion. Our take on the problem is entirely different, because we impose a multifold structure of the cone explicitly and define the partition in terms of the block indices.

Recall that in the linear case complementarity partition has a simple and well-studied structure. Consider the primal-dual pair of linear programming problems

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \geq b, \end{array} \quad (\text{LP}) \qquad \begin{array}{ll} \max_{y \in \mathbb{R}^m} & b^\top y \\ \text{s.t.} & A^\top y = c, \\ & y \geq 0. \end{array} \quad (\text{LD})$$

Here $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The classical Goldman-Tucker theorem [8] (also see [1, Proposition 17.16] for modern treatment) states that when either one of the problems has a finite optimal value, there exists a unique partition (B, N) of the index set $J = \{1, \dots, m\}$ that is of maximal complementarity. In other words, there is a pair (\bar{x}, \bar{y}) of optimal solutions to (LP) and (LD) such that for $\bar{s} = A\bar{x} - b$ and \bar{y} we have $\bar{s}_B > 0$, $\bar{y}_N > 0$, and moreover for any primal-dual optimal solution (x, y) , with $s := Ax - b$, it always holds that $y_B = 0$, $s_N = 0$ (here we have used the notation standard in linear programming literature: by v_I we denote the vector of entries of $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ with indices from a subset $I \subseteq J$ of $J = \{1, 2, \dots, m\}$).

The structure of optimal partition of monotone linear complementarity problems is also well-known. Indeed, let Q and R be $n \times n$ matrices, and $h \in \mathbb{R}^n$. Recall that the linear complementarity problem

$$\begin{cases} x^\top s = 0, \\ Qx + Rs = h, \\ x \geq 0, \quad s \geq 0, \end{cases}$$

is called monotone if $Qu + Rv = 0$ yields $u^\top v \geq 0$. For this type of problems the optimal partition is of the form (B, N, T) , where T corresponds to the subset of indices for which both s and x components are zero for every solution; see, e.g., [11, 12, 14].

This linear complementarity partition is reminiscent of the situation encountered in multifold conic systems. When the system of linear inequalities is replaced by a product of general closed convex cones (as in the case of second-order cone programming for instance), the partition of indices becomes more complex since the component variables may end up on the boundary of the cones.

In [2] a four-set partition was introduced for general multifold conic systems, based on regularity conditions involving normal cones. In [16] a refined six-set partition was suggested for homogeneous feasibility problems and a geometric characterization of such partition was obtained. Our goal is to generalize the latter partition to multifold conic optimization problems and compare the two partitions.

We focus on the case when strong duality holds for the primal-dual pair of conic optimization problems. We note that this is a minimal assumption required for the notion of complementarity to make sense: the optimal solutions must exist, and this in particular can be guaranteed by the presence of Slater points in both primal and dual problems. In principle, the assumption of strong duality can always be guaranteed by performing a facial reduction preprocessing step if necessary; see [13, 15, 24]. Another possible approach is to use Ramana Duals [17, 18]. Both approaches need to assume that the optimal value of the primal is attained. As pointed out

by a reviewer, another way to remedy the failure of strong duality, and thus to ensure the existence of the optimal partition, is to cast the primal problem into a self-dual embedding format, for which the Slater condition always holds, see e.g., [6, 7]. Then, the optimal partition of the original problem, if exists, can be recovered from the optimal partition of the embedding problem.

We introduce the two types of partitions in Section 2, show the relations between them in Theorem 2, treat the special case of second-order cones in Proposition 3 and provide an illustrative example. In Section 3, we link the geometric relations of [16] to the four-set partition of [2] in Theorem 6. We conclude the paper with a study of the relations between partitions of reformulated (lifted) problems in Section 4, strengthening some results of [2].

Throughout the paper, we work in the finite-dimensional Euclidean setting of a real vector space \mathbb{R}^n endowed with the standard Euclidean norm $\|x\| = \sqrt{x^\top x}$.

2 Partitions for multifold conic optimization problems

Consider a general linear conic optimization problem with constraints in product form, i.e.,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & A^j x - b^j \in K_j \quad \forall j \in J, \end{aligned} \tag{P}$$

where K_j is a closed convex regular cone in \mathbb{R}^{q_j} , $q_j \in \mathbb{N}$ for every $j \in J$, and $J = \{1, \dots, r\}$ is a finite index set (a regular cone has a nonempty interior and is pointed, i.e. it does not contain lines). We set $K := K_1 \times \dots \times K_r$, and define $A = (A^1; \dots; A^r)$ as the matrix whose rows are those of A^1 to A^r , and $b := \text{vec}(b^1, \dots, b^r)$ so that (P) is equivalent to

$$\min_{x \in \mathbb{R}^n} \{c^\top x; Ax - b \in K\}.$$

The dual problem is

$$\begin{aligned} \max_{y^1, \dots, y^r} \quad & \sum_{j=1}^r (b^j)^\top y^j \\ \text{s.t.} \quad & \sum_{j=1}^r (A^j)^\top y^j = c, \\ & y^j \in K_j^+ \quad \forall j \in J, \end{aligned} \tag{D}$$

where the (positive) polar of a set $C \subset \mathbb{R}^m$ is defined as $C^+ := \{y \in \mathbb{R}^m; y^\top z \geq 0, \forall z \in C\}$. Generally speaking, such primal-dual pairs of linear conic problems may have a nonzero duality gap (see [9, Section 11.6] for a detailed discussion and [19, Section 3.2] for examples). The Slater condition (i.e., the existence of \hat{x} such that $A^j \hat{x} - b^j \in \text{int} K_j$, $\forall j \in J$, for its primal version, or an analogous condition for the dual) guarantees the absence of the duality gap. Together with the bounded objective for either (P) or (D) this also yields the existence of the optimal solution for the relevant primal or dual counterpart. Also note that zero duality gap alone does not guarantee the existence of optimal solutions for both primal and dual problems (see, e.g., [5]). So, we make the following standing assumption throughout the paper:

Standing assumption: *strong duality holds for the pair (P)-(D)*, i.e, the optimal solution sets of (P) and (D) are nonempty, and the duality gap is zero.

A pair (x, y) of optimal solutions to primal and dual problems is characterized by the complementarity system

$$\begin{aligned} A^j x - b^j &\in K_j, \\ y^j &\in K_j^+, \\ (y^j)^\top (A^j x - b^j) &= 0 \quad \forall j \in J, \\ A^\top y &= c. \end{aligned} \tag{C}$$

We denote by $S(C)$ the set of solutions for relations (C). Observe that $S(C)$ is nonempty if and only if strong duality holds (which is our standing assumption). We also denote by $S(P)$ and $S(D)$ the set of solutions to the problems (P) and (D), respectively, and by $F(P)$ and $F(D)$ their respective feasible sets. Notice that strong duality implies the equality $S(C) = S(P) \times S(D)$.

We say that *strict primal (resp. dual) feasibility* holds for $j \in J$ if there exists $x \in F(P)$ such that $A^j x - b^j \in \text{int } K_j$ (resp. $y \in F(D)$ such that $y^j \in \text{int } K_j^+$).

One says (e.g., [3, Def. 4.74]) that the *strict complementarity condition* holds for problem (P) if there exists a pair (x, y) solution of the optimality system (C), such that $-y \in \text{ri } N_K(Ax - b)$, where N_K is the standard normal cone of convex analysis (see [10, Section A.5.2]). Since K is a closed convex cone, we have for $s \in K$ that

$$N_K(s) = \begin{cases} (-K^+) \cap s^\perp, & s \in \partial K, \\ \emptyset, & s \notin K, \\ \{0\}, & s \in \text{int } K, \end{cases} \tag{1}$$

where s^\perp denotes the orthogonal complement to the linear span of s .

For problems with constraints in the product form as in (P), we introduce the notion of *componentwise strict complementarity*, which means that for each component j there exists a pair $(x, y) \in S(C)$, such that $-y^j \in \text{ri } N_{K_j}(A^j x - b^j)$. As we will see later, the two notions are equivalent.

In [2] the notion of optimal partition, well-known for linear programming and monotone linear complementarity problems (see, e.g., [1, Section 18.2.4]), was extended to our abstract framework. Let

$$\begin{aligned} B &= \{j \mid \exists (x, y) \in S(C) \text{ s.t. } A^j x - b^j \in \text{int } K_j\} \\ N &= \{j \mid \exists (x, y) \in S(C) \text{ s.t. } y^j \in \text{int } K_j^+\} \\ R^0 &= \{j \mid \exists (x, y) \in S(C) \text{ s.t. } -y^j \in \text{ri } N_{K_j}(A^j x - b^j)\}. \end{aligned}$$

Note that $B \cup N \subseteq R^0$. Define $R := R^0 \setminus (B \cup N)$, $T := J \setminus R^0$. It was shown in [2, Lemma 3] that if $S(C)$ is not empty, the partition (B, N, R, T) is a disjoint partition of the index set J .

Definition 1. Any pair $(x, y) \in S(C)$ satisfying the relations below is said to be of *maximal complementarity*:

$$\begin{cases} \text{(i)} & A^j x - b^j \in \text{int } K_j, \quad \forall j \in B, \\ \text{(ii)} & y^j \in \text{int } K_j^+, \quad \forall j \in N, \\ \text{(iii)} & -y^j \in \text{ri } N_{K_j}(A^j x - b^j), \quad \forall j \in R. \end{cases} \tag{2}$$

For each $j \in B \cup N \cup R$ let $(x(j), y(j)) \in S(C)$ be such that

$$\begin{cases} A^j x(j) - b^j \in \text{int } K_j & \text{if } j \in B, \\ y^j(j) \in \text{int } K_j & \text{if } j \in N, \\ -y^j(j) \in \text{ri } N_{K_j}(A^j x(j) - b^j) & \text{if } j \in R, \end{cases}$$

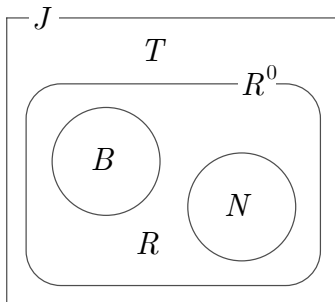


Figure 1: The four-partition (B, N, R, T) .

where $x^j(j)$ and $y^j(j)$ stand for the block j of vectors $x(j)$ and $y(j)$, respectively.

We define

$$\hat{x} := (|B| + |R|)^{-1} \sum_{j \in B \cup R} x(j); \quad \hat{y} := (|N| + |R|)^{-1} \sum_{j \in N \cup R} y(j). \quad (3)$$

It was shown in [2, Lemma 7] that the pair (\hat{x}, \hat{y}) defined in (3) is of maximal complementarity, and that any pair $(x, y) \in \text{ri } S(\text{P}) \times \text{ri } S(\text{D})$ is of maximal complementarity. An immediate consequence of this result is that componentwise strict complementarity yields strict complementarity (observing that $\text{ri } N_K(Ax - b) = \text{ri } N_{K_1}(A^1x - b^1) \times \text{ri } N_{K_2}(A^2x - b^2) \times \dots \times \text{ri } N_{K_r}(A^rx - b^r)$), therefore the two notions are equivalent.

In a separate development [16], a different idea was used to define a partition for a homogeneous feasibility problem. In the case when $b = 0$ and $c = 0$ the pair (P)-(D) is reduced to a pair of homogeneous feasibility problems (FP)-(FD) below. In this case the primal problem consists of finding an x such that

$$A^j x \in K_j \quad \forall j \in J, \quad (\text{FP})$$

and the dual problem consists of finding y satisfying

$$\sum_{j=1}^r (A^j)^\top y^j = 0, \quad y^j \in K_j^+ \quad \forall j \in J. \quad (\text{FD})$$

Either one of the problems has a strictly interior solution if and only if the other one only has a zero solution; the most interesting situation in terms of the partition indices occurs when both problems have boundary solutions. We can assign an index $j \in J$ to the set B^0 if all solutions to the dual problem (FD) have zero j -th component, and to N^0 if for every primal solution to the problem (FP) the j -th component is zero. The remaining two partition sets are the intersection $O = B^0 \cap N^0$ and the complement $C = J \setminus B^0 \cup N^0$. In what follows we extend this definition to optimization problems. Define the index subsets B^0 and N^0 :

$$\begin{aligned} B^0 &= \{j \mid \forall (x, y) \in S(C) \quad y^j = 0\}, \\ N^0 &= \{j \mid \forall (x, y) \in S(C) \quad A^j x - b^j = 0\}, \end{aligned}$$

and define the following four sets:

$$B' = B^0 \setminus (N^0 \cup B), \quad N' = N^0 \setminus (B^0 \cup N), \quad O = B^0 \cap N^0, \quad C = J \setminus (B^0 \cup N^0).$$

It follows immediately from the complementarity conditions of (C) that $B \subset B^0$ and $N \subset N^0$, and that $B' \cap N' = \emptyset$. Therefore, it is not difficult to observe that the sets B, N, B', N', O, C form a disjoint partition of the index set J . To avoid confusion, we will refer to this partition as the six-partition, whereas the four-set partition introduced before will be referred to as the four-partition.

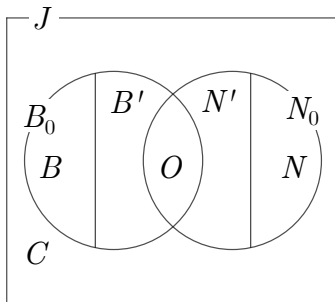


Figure 2: The six-partition.

Theorem 2. *The following relations between the four- and six-partitions hold.*

$$T \supset B' \cup N' \cup O = (N^0 \cup B^0) \setminus (N \cup B); \quad R \subset C = J \setminus (B^0 \cup N^0); \quad (4)$$

Proof. The first relation follows directly from the fact that the six-partition is indeed a partition of set J and $R \subset C$. Thus, it only remains to show that $R \subset C$. We argue by contradiction and suppose the existence of an index $j^* \in R \setminus C$. Since $C = J \setminus (B^0 \cup N^0)$, $j^* \in B^0 \cup N^0$.

Since $j^* \in R$, there exists a solution $(x(j^*), y(j^*)) \in S(C)$ with

$$-y^{j^*}(j^*) \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}). \quad (5)$$

First assume that $j^* \in B^0 \cap R$. This yields $y_{j^*}(j^*) = 0$, and hence

$$0 \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}).$$

This is only possible when either $N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}) = \{0\}$ or $N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*})$ contains lines. In the first case $A^{j^*}x(j^*) - b^{j^*} \in \text{int } K_{j^*}$, and $j^* \in B$, and that contradicts the definition of R . The second case is impossible by our assumption that all cones K_j , $j \in J$ have a nonempty interior.

Now assume that $j^* \in N^0 \cap R$. We have $A^{j^*}x(j^*) - b^{j^*} = 0$, hence, (5) yields

$$-y^{j^*}(j^*) \in \text{ri } N_{K_{j^*}}(A^{j^*}x(j^*) - b^{j^*}) = \text{ri } N_{K_{j^*}}(0) = \text{ri}(-K_{j^*}^+) = -\text{int } K_{j^*}^+,$$

which yields $j^* \in N$, and contradicts the fact that j^* is in R . We therefore conclude that

$$R \subset C = J \setminus (B^0 \cup N^0). \quad (6)$$

□

We demonstrate the relations of Theorem 2 using a Venn diagram in Fig. 3.

We next show that in the case of second-order cone programming the complementarity partition has special properties. Recall that a second-order cone is a product of Lorentz cones (closed convex cones L_n in \mathbb{R}^{n+1} for some $n \geq 1$ defined as follows,

$$L_n = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n \mid x_0 \geq \|\bar{x}\|\},$$

i.e., such cones are the epigraphs of the Euclidean norm). For consistency, we also let $L_0 := \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Second-order cone programs are of type (P) and (D), where each cone K_j , for $j \in J$, is a Lorentz cone. Notice that linear programming case can be cast as a special case of second-order cone programming when $K_j = L_0$ for all $j \in J$.

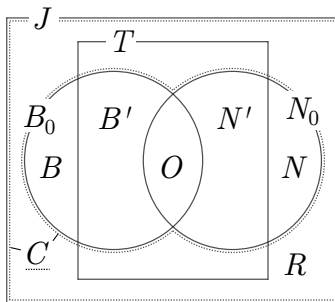


Figure 3: The relations between index sets (see Theorem 2).

Proposition 3. *If for every $j \in J$ the cone K_j is a Lorentz cone, then*

$$R = C = J \setminus (B^0 \cup N^0). \quad (7)$$

Proof. Let $j \in C$. Then, there exists a solution $(x(j), y(j)) \in S(C)$ with both $y_j(j)$ and $s = A^j x(j) - b^j$ nonzero. Since $y_j(j) \perp s$, we have $y_j(j) \in (K_j^+) \cap s^\perp = -N_{K_j}(s)$. Observe that for a second-order cone the normal cone $N_{K_j}(s)$ is one-dimensional provided $s \neq 0$, hence, $\text{ri } N_{K_j}(s) = N_{K_j}(s) \setminus \{0\} \ni y_j(j)$, and therefore $j \in R$. We have then shown that $C \subset R$. The reverse inclusion follows from Theorem 2. \square

We next show a simple example of mixed polyhedral and Lorentz cones in which it can happen that $R \neq C$.

Example 4. *We consider a feasibility problem given by $b = 0$, $c = 0$ and*

$$A^\top = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

First, we consider

$$K = L_3 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1.$$

Solving the problem analytically, we obtain all possible solutions as the parametric family

$$x = (0, 0, \alpha)^\top, \quad Ax = (\alpha, 0, 0, \alpha, 0, 0, 0, \alpha)^\top, \quad y = (\beta, 0, 0, -\beta, \gamma, 0, \gamma, 0)^\top,$$

where $\alpha, \beta, \gamma \geq 0$. We thus obtain that $1 \in C$, $2, 4 \in N$, $3 \in O$, $5 \in B$, and $R = \{1\}$, since the first cone is a Lorentz cone. Hence, as expected from Proposition 3, in this case $R = C$. However, if we let $K = L_3 \times \mathbb{R}_+^1 \times \mathbb{R}_+^3$, we directly deduce that $1, 3 \in C$, $2 \in N$. Now observe that

$$N_{K_3}(A_3 x) = N_{\mathbb{R}_+^3}(0, 0, \alpha) = -\mathbb{R}_+^2 \times \{0\},$$

provided $\alpha > 0$, and hence

$$\text{ri } N_{K_3}(A_3 x) = -\text{int } \mathbb{R}_+^2 \times \{0\}.$$

Hence, since the only possible solution is $y_3 = (0, \gamma, 0)$, with $\gamma \geq 0$, it clearly follows that $-y_3 \notin \text{ri } N_{K_3}(A_3 x)$. Hence, $3 \notin R$.

3 Homogeneous feasibility problems

In this section we focus on homogeneous feasibility problems (P)–(D) introduced earlier in Section 2. Observe that in this case the strong duality property coincides with solvability of problems (FP) and (FD). The relevant complementarity conditions can be obtained from (C),

$$\begin{aligned} A^j x &\in K_j, \\ y^j &\in K_j^+, \\ (y^j)^\top A^j x &= 0 \quad \forall j \in J, \\ A^\top y &= 0. \end{aligned} \tag{FC}$$

In [16] a dual characterization of the homogeneous six-partition was obtained.

Theorem 5 ([16, Theorem 1]). *The sets B , N , B^0 and N^0 can be described as follows.*

$$\begin{aligned} B &= \{j \in J \mid A_j^\top (K_j^+ \setminus \{0\}) \cap \text{Lin}(\overline{A^\top K^+}) = \emptyset\}, \\ N &= \{j \in J \mid \text{ri } A_j^\top K_j^+ \cap \text{Lin}(A^\top K^+) \neq \emptyset\}, \\ B^0 &= \{j \in J \mid A_j^\top (K_j^+ \setminus \{0\}) \cap \text{Lin}(A^\top K^+) = \emptyset\}, \\ N^0 &= \{j \in J \mid \text{ri } A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+}) \neq \emptyset\}. \end{aligned}$$

Here by $\text{Lin } C$ we denote the lineality space of a convex set C .

It appears that in the case of feasibility problem the set R has a similar dual property. The next result extends the relations of Theorem 5.

Theorem 6. *In the case of a homogeneous feasibility problem*

$$R^0 \subseteq \tilde{R} := \{j \in J \mid A_j^\top K_j^+ \cap \text{Lin}(A^\top K^+) = A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+})\}.$$

Proof. We argue by contradiction and assume that, for some pair of problems (FP)–(FD), there exists $j \in R^0 \setminus \tilde{R}$.

Consider the linear subspaces

$$L = \{y \mid A_j^\top y \in \text{Lin}(A^\top K^+)\}, \quad \bar{L} = \{y \mid A_j^\top y \in \text{Lin}(\overline{A^\top K^+})\},$$

and let

$$F = L \cap K_j^+, \quad \bar{F} = \bar{L} \cap K_j^+.$$

Here observe that $F \subsetneq \bar{F}$: indeed, since $j \notin \tilde{R}$, the sets

$$A_j^\top K_j^+ \cap \text{Lin}(A^\top K^+) \quad \text{and} \quad A_j^\top K_j^+ \cap \text{Lin}(\overline{A^\top K^+})$$

are different, and hence there exists some $p \in A_j^\top K_j^+$ such that $p \in \text{Lin}(\overline{A^\top K^+}) \setminus \text{Lin}(A^\top K^+)$. Subsequently there exists $u \in K_j^+$ such that $A^\top u = p$. For every such u we have $u \in \bar{L} \setminus L$, yielding $F \neq \bar{F}$.

Pick an arbitrary solution x to (FP) and any $v \in \bar{F}$. Since $v \in \bar{F}$, we have $A_j^\top v \in \text{Lin}(\overline{A^\top K^+})$, hence, $-A_j^\top v \in \text{Lin}(\overline{A^\top K^+}) \subseteq \overline{A^\top K^+}$. Then there exist sequences $\{v_k^+\}$, $\{v_k^-\}$ such that $v_k^-, v_k^+ \in K^+$, $A^\top v_k^-, A^\top v_k^+ \in A^\top K^+$, $A^\top v_k^\pm \rightarrow \pm A_j^\top v$. Since $A^\top v_k^-, A^\top v_k^+ \in A^\top K^+$, and since x is a solution (i.e. $Ax \in K$), we have for some $y \in K^+$, $A^\top v_k^- = A^\top y$

$$(Ax)^\top v_k^- = x^\top A^\top v_k^- = x A^\top y = (Ax)^\top y \geq 0,$$

and analogously $0 \leq (Ax)^\top v_k^+$ for all k . Hence, passing to the limit, we have $(A_j x)^\top v = 0$. Therefore, $\bar{F} \subseteq (A_j x)^\perp$. By definition, $\bar{F} \subseteq K_j^+$, hence,

$$\bar{F} \subseteq K_j^+ \cap (A_j x)^\perp.$$

Since $j \in R^0$, there exists a pair of feasible solutions (x, y) to (FP)-(FD) such that

$$y_j \in -\text{ri } N_{K_j}(A_j x), \quad \text{where } N_{K_j}(A_j x) = -K_j^+ \cap (A_j x)^\perp. \quad (8)$$

Let $\bar{y} \in \mathbb{R}^n$ be such that $\bar{y}_i = 0$ for all $i \neq j$, and $\bar{y}_j = y_j$. Observe that $\bar{y} \in K^+$, hence, $A_j^\top \bar{y}_j = A^\top \bar{y} \in A^\top K^+$, and since $A^\top y = 0$, we have $-A_j^\top y_j \in A^\top K^+$, hence, $A_j^\top y_j \in \text{Lin}(AK^+)$, and therefore $y_j \in F$.

Since $y_j \in F \subsetneq \bar{F} \subseteq K_j^+ \cap (A_j x)^\perp$, $F = \bar{F} \cap L$, and all three sets F , \bar{F} and $K_j^+ \cap (A_j x)^\perp$ are cones, their affine hulls are linear subspaces, moreover, we have the following representations

$$\text{Aff } F = L_1, \quad \text{Aff } \bar{F} = L_2, \quad \text{Aff}(K_j^+ \cap (A_j x)^\perp) = L_3,$$

where L_1, L_2, L_3 are linear subspaces, $L_1 \subsetneq L_2 \subseteq L_3$. Observe that $L_1 \subsetneq L_2$ follows from the fact that $F = \bar{F} \cap L$, where L is a linear subspace and $F \neq \bar{F}$. Also note that since $F = L \cap \bar{F}$, and $\bar{F} = \bar{L} \cap (K^+ \cap (A_j x)^\perp)$, we have $F = \bar{F} \cap L_1$, $\bar{F} = L_2 \cap (K^+ \cap (A_j x)^\perp)$.

From (8) we deduce that there exists $\varepsilon > 0$ such that $(y_j + B_\varepsilon) \cap L_3 \subseteq K_j^+ \cap (A_j x)^\perp$. Observe that since $\text{Aff } \bar{F} = L_1 \oplus (L_2 \cap L_1^\perp)$, there exists a unit vector u in $L_1^\perp \cap L_2$ such that $y_j + \varepsilon u \in \bar{F} \subseteq L_2 \subseteq L_3$, therefore $y_j - \varepsilon u \in L_3$, and since $\pm \varepsilon u \in B_\varepsilon$, we have $y_j \pm \varepsilon u \in L_2 \cap (K_j^+ \cap (A_j x)^\perp)$, hence, $y_j \pm \varepsilon u \in \bar{F}$, as $\bar{F} = \bar{L}_2 \cap (K^+ \cap (A_j x)^\perp)$.

Therefore, $A_j^\top (y_j \pm \varepsilon u) \in A^\top K^+$. Since $A_j^\top y_j \in \text{Lin}(A^\top K^+)$, we have $-A_j^\top y_j \in A^\top K^+$, and therefore we deduce that $-A_j^\top (y_j \pm \varepsilon u) \in A^\top K^+$, and hence $A_j^\top (y_j \pm \varepsilon u) \in \text{Lin}(A^\top K)$, which means that $y_j \pm \varepsilon u \in F$, which contradicts the construction of u . Therefore, our assumption that there exists $j \in R^0 \setminus \tilde{R}$ is wrong, and hence $R^0 \subseteq \tilde{R}$. \square

In the following example we show that the reverse inclusion in $R^0 \subseteq \tilde{R}$ does not always hold.

Example 7. Consider a feasibility problem given by $b = 0$, $c = 0$,

$$A^\top = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$K = L_3 \times \mathbb{R}_+^1 \times \mathbb{R}_+^3.$$

Solving the problem analytically, we obtain all possible solutions as the parametric family

$$x = (0, 0, \alpha)^\top, \quad Ax = (\alpha, 0, 0, \alpha, 0, 0, 0, \alpha)^\top, \quad y = (\beta, 0, 0, -\beta, \gamma, 0, \gamma, 0)^\top,$$

where $\alpha, \beta, \gamma \geq 0$. We thus obtain that $1 \in R$, $2 \in N$, and $3 \in T$. Consequently, $R^0 = \{1, 2\}$. However, for this concrete example it is possible to prove that $A^\top K^+$ is closed. Then, from the definition of \tilde{R} , it follows that $\tilde{R} = \{1, 2, 3\}$, which thus turns to be strictly larger than R^0 . Indeed, let us denote by $u \in L_3$, $w \in \mathbb{R}_+^1$ and $v \in \mathbb{R}_+^3$ the components of $y \in K$. Hence,

$$\begin{aligned} A^\top K^+ &= \{(u_1 + v_1, u_2 + v_2 - w, u_0 + u_3 + v_3)^\top : u \in L_3, w \geq 0, v_i \geq 0 \forall i = 1, 2, 3\} \\ &= \{(x_1, x_2, x_3)^\top : x_2 \in \mathbb{R}, x_3 \geq 0, x_1 + x_3 \geq 0\}. \end{aligned}$$

In the computation above we have used that: i) $v_2 - w$ can be any real number when v_2 and w are nonnegative; ii) the third component $x_3 = u_0 + u_3 + v_3$ is nonnegative because $u \in L_3$ and $v_3 \geq 0$, and iii) the first component $x_1 = u_1 + v_1$ can be negative because of u_1 . However, since $u \in L_3$, u_0 is always greater or equal than $-u_1$. This necessarily implies that $x_1 + x_3 \geq 0$. We thus conclude that $\tilde{R} = \{1, 2, 3\}$, and consequently $R^0 \neq \tilde{R}$.

4 Equivalent optimization problems

We now introduce another problem related to (P). Let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_r$ be another finite family of regular closed convex cones in \mathbb{R}^{r_j} , $j \in J$, and M^j be $r_j \times q_j$ matrices such that

$$s^j \in \mathcal{K}_j \quad \text{iff} \quad M^j s^j \in \mathcal{K}_j, \quad j = 1, \dots, J. \quad (9)$$

The latter is inspired by the relation between second-order cones and the cone of the positive semidefinite matrices (denoted by \mathbb{S}_+^{n+1}). Indeed, consider, for some $n \geq 1$, the cones L_n and \mathbb{S}_+^{n+1} . For $s = (s_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^n$ it is easy to check that

$$s \in L_n \quad \text{iff} \quad \text{Arw}(s) := \begin{pmatrix} s_0 & \bar{s}^\top \\ \bar{s} & s_0 I_n \end{pmatrix} \in \mathbb{S}_+^{n+1}. \quad (10)$$

Linear function $\text{Arw}(\cdot)$ is the *arrow function* and it has several useful properties that will be exploited in what follows.

Let $M = (M^1; \cdots; M^r)$ be the matrix whose rows are those of M^j . Then, (P) is equivalent to the linear conic problem

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad M^j (A^j x - b^j) \in \mathcal{K}_j, \quad j \in J, \quad (\text{MP})$$

whose dual is

$$\max_{z \in \mathcal{K}^+} \sum_{j=1}^r (b^j)^\top (M^j)^\top z^j \quad \text{s.t.} \quad \sum_{j=1}^r (A^j)^\top (M^j)^\top z^j = c; \quad z^j \in \mathcal{K}_j^+, \quad j \in J. \quad (\text{MD})$$

If the strong duality holds for this problem, an optimal pair (x, y) of the primal and dual problems is characterized by the optimality system

$$\begin{cases} M^j (A^j x - b^j) \in \mathcal{K}_j, \quad z^j \in \mathcal{K}_j^+, \quad (z^j)^\top M^j (A^j x - b^j) = 0, \quad j \in J; \\ \sum_{j=1}^r (A^j)^\top (M^j)^\top z^j = c. \end{cases} \quad (\text{MC})$$

In what follows, let (B_P, N_P, R_P, T_P) denotes the optimal partition of (P), and adopt a similar convention for (MP).

Our goal in this section is to extend the results of [2] to the six-partition. This permits to end this section by showing that relevant pair of second-order and semidefinite programming problems has the same partitions. Before proceeding in this direction we need to recall some useful claims proved in Lemmas 8 and 9 of [2].

Lemma 8. *The following relations hold:*

- (i) $S(\text{P}) = S(\text{MP})$, $M^\top \mathcal{K}^+ \subset K^+$, and $M^\top S(\text{MD}) \subset S(\text{D})$.
- (ii) If $M^\top \mathcal{K}^+$ is closed, then $M^\top \mathcal{K}^+ = K^+$ and $M^\top S(\text{MD}) = S(\text{D})$.
- (iii) Closedness of $M^\top \mathcal{K}^+$ holds if M^\top is coercive on \mathcal{K}^+ , i.e., if there exists $\gamma > 0$ such that $\|M^\top z\| \geq \gamma \|z\|$ for all $z \in \mathcal{K}^+$. In that case, $S(\text{MD})$ is bounded iff $S(\text{D})$ is bounded.

(iv) Assume in addition that the linear mapping defined by the matrix M is one to one. Then, $M^\top \text{int } \mathcal{K}^+ = \text{int } K^+$ and $M^\top \text{int } S(\text{MD}) = \text{int } S(\text{D})$. Moreover, for all $s \in K$, $M^\top \text{ri}(\mathcal{K}^+ \cap (Ms)^\perp) \subset \text{ri}(K^+ \cap s^\perp)$.

The next proposition strengthens [2, Lemma 10] by replacing the inclusions $N_P \supseteq N_{MP}$ and $T_P \subseteq T_{MP}$ with equalities, under the same assumptions. It states conditions for which the four-partition of problems (P) and (MP) coincide.

Proposition 9. *Assume that $M^\top \mathcal{K}^+$ is closed, that M is one to one, and that*

$$\text{For all } s^j \in K_j, M^j s^j \in \partial \mathcal{K}_j \text{ iff } s^j \in \partial K_j. \quad (11)$$

Then, the four-partition of problems (P) and (MP) coincide, that is

$$\begin{cases} B_P = B_{MP}, & N_P = N_{MP}, \\ R_P = R_{MP}, & T_P = T_{MP}. \end{cases} \quad (12)$$

In particular, the strict complementarity condition holds for (P) iff it holds for (MP).

Proof. As mentioned above, due to Lemma 10 of [2] it only remains to prove that $R_P \subseteq R_{MP}$. Let $j \in R_P$. This implies the existence of $(x, y) \in S(C)$ such that $s^j := A^j x^j - b^j \in \partial K_j \setminus \{0\}$ and $y^j(j) \in \partial K_j^+ \setminus \{0\}$. It follows from Lemma 8, Parts (i) and (ii), that $x \in S(\text{MP})$ and that there exists $z \in S(\text{MD})$ such that $y = M^\top z$, respectively. In particular, $y^j = (M^j)^\top z^j$ and

$$0 = (s^j)^\top y^j = (s^j)^\top (M^j)^\top z^j = M^j (s^j)^\top z^j. \quad (13)$$

Moreover, from (11) and the fact that M is one to one, we deduce that $M^j s^j$ is a nonzero element of $\partial \mathcal{K}_j$. This in particular implies that $z^j \notin \text{int}(\mathcal{K}_j^+)$ (because otherwise (13) yields $M^j s^j = 0$). Finally, since $(M^j)^\top z^j = y^j \neq 0$ it directly follows that $z^j \neq 0$. Summarizing, we have proved that $(x, z) \in S(\text{MC})$, and $M^j s^j$ and z^j are nonzero elements of $\partial \mathcal{K}_j$ and $\partial \mathcal{K}_j^+$, respectively. This means that $j \in R_{MP}$, which concludes our proof. \square

We now state the equivalence between six-partitions of problems (P) and (MP).

Theorem 10. *Under the assumptions of Proposition 9,*

$$\begin{cases} B_P = B_{MP}, & N_P = N_{MP}, \\ N_P^0 = N_{MP}^0, & B_P^0 \supseteq B_{MP}^0. \end{cases} \quad (14)$$

Additionally, suppose that $z = 0$ is the only element in \mathcal{K}^+ such that $M^\top z = 0$. Then, $B_P^0 = B_{MP}^0$. The last relation in particular implies that the six-partition of problems (P) and (MP) coincide.

Proof. Relations $B_P = B_{MP}$ and $N_P = N_{MP}$ were established in Proposition 9. From Lemma 8 (i) we have $MS(P) = S(\text{MP})$. Hence, the inclusion $N_P^0 \subseteq N_{MP}^0$ is obvious. Moreover, since M is one to one, $M_j(A_j x - b_j) = 0$ yields $A_j x - b_j = 0$, and consequently, the equality $N_P^0 = N_{MP}^0$.

On the other hand, relation $B_{MP}^0 \subseteq B_P^0$ follows directly from Lemma 8(ii). Indeed, if for every $(x, z) \in S(\text{MC})$ we have $z_j = 0$, then $y_j = M_j^\top(z_j) = 0$, obtaining that $y_j = 0$ for all $y \in S(D)$. Now, for the opposite inclusion, assume that $j \in B_P^0$. Then every solution $(x, z) \in S(\text{MP})$ satisfies $M_j^\top z_j = 0$. By assumption this yields $z_j = 0$. We thus conclude that $B_P^0 \subseteq B_{MP}^0$. The result follows. \square

We show in the next example that the assumptions of Theorem 10 are crucial for the equality between partitions.

Example 11. Consider a pair of feasibility problems

$$Ax \in K, \quad (P) \quad A^\top y = 0, y \in K^+, \quad (D)$$

where

$$K = K^+ = L_2 \times \mathbb{R}_+^1,$$

and L_2 is the three-dimensional Lorentz cone. Let

$$A^\top = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We can solve this feasibility problem directly and obtain the parametric family of solutions

$$x^* = (0, \alpha)^\top, \alpha \geq 0, \text{ with } Ax^* = (\alpha, 0, \alpha, 0)^\top; \quad y^* = (\beta, 0, -\beta, 0)^\top, \beta \geq 0.$$

It is not difficult to see that for the first index the solutions lie on the boundary of the cones, hence, $1 \in C = R$. For the second index, both primal and dual components are zero, hence, $2 \in O$.

Now we transform our problem into a higher-dimensional one. We let

$$M_1 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the transformed problem, we let $\mathcal{K}_1 = K_1 = L_2$, $\mathcal{K}_2 = L_1$. Observe that for every $y \in \mathbb{R}$ we have

$$M_2 y = (y, y)^\top,$$

and hence $M_2 y \in \mathcal{K}_2 = L_1$ iff $y \in K_2 = \mathbb{R}_+^1$. Further,

$$\mathcal{A}_1 = M_1 A_1 = A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}_2 = M_2 A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $\mathcal{A} = [\mathcal{A}_1; \mathcal{A}_2]$, then we can write the transformed feasibility problem as

$$\mathcal{A}u \in \mathcal{K}, \quad (P') \quad \mathcal{A}^\top v = 0, v \in \mathcal{K}^+. \quad (D')$$

Solving the problem directly, we obtain the family of optimal solutions

$$u = (\lambda, 0), \lambda \geq 0 \text{ with } \mathcal{A}u = (\lambda, 0, \lambda, 0, 0); \quad v = (\mu, 0, -\mu, \gamma, -\gamma), \mu, \gamma \geq 0.$$

Therefore, we conclude that $1_M \in C$, $2_M \in N^0 \setminus (B^0 \cup N) = N'$, and hence the partition has changed.

The situation above occurs because M_2 does not satisfy that $M_2^\top z = 0$ yields $z = 0$ for all $z \in \mathcal{K}^+ = L_1$. Indeed, it is enough to take $z = (1, -1)^\top$ to check this hypothesis fails.

The next corollary follows directly from Theorem 10 and definition of the index set C . This result provides alternative hypotheses to those in Proposition 9 in order to ensure that the four-partition of problems (P) and (MP) coincide.

Corollary 12. Suppose that the assumptions of Lemma 9 are fulfilled. Suppose also that for all $z \in \mathcal{K}^+$ such that $M^\top z = 0$ yields $z = 0$, $R_P = C_P$ and $R_{MP} = C_{MP}$. Then

$$R_P = R_{MP}, \quad T_P = T_{MP}.$$

We end this section by applying previous results to the case of a linear second-order cone program and its semidefinite programming equivalent representation constructed via (10). Indeed, those problems fits in the framework given by problems (P) and (MP) provided, for all $j \in J$, we consider the following sets $K_j := L_{n_j}$, $\mathcal{K}_j := \mathbb{S}_+^{n_j+1}$, and $M^j s^j = \text{Arw}^j s^j$. Here, the arrow function $\text{Arw}^j : \mathbb{R}^{n_j+1} \rightarrow \mathbb{S}^{n_j+1}$ was defined in (10).

In what follows consider a generic nonnegative integer value n and omit index j from the arrow function. Note that we can write

$$\text{Arw}(s) = (s_0 - \|\bar{s}\|)I_{n+1} + \begin{pmatrix} \|\bar{s}\| & \bar{s}^\top \\ \bar{s} & \|\bar{s}\|I_n \end{pmatrix}. \quad (15)$$

This shows that for $s \in L_n \setminus \{0\}$, $\text{Arw}(s)$ is of rank n iff $s \in \partial L_n$, and of rank $n+1$ otherwise. In particular, $\text{Arw} \partial L_n \subset \partial \mathbb{S}_+^{n+1}$, and $\text{Arw} \text{int } L_n \subset \text{int } \mathbb{S}_+^{n+1}$. Therefore (11) holds. Also Arw is clearly one to one.

Let us decompose any matrix $Y \in \mathbb{S}^{n+1}$ as follows

$$Y = \begin{pmatrix} Y_{00} & \bar{Y}_0^\top \\ \bar{Y}_0 & \bar{Y} \end{pmatrix}, \quad (16)$$

where $Y_{00} \in \mathbb{R}$, $\bar{Y}_0 \in \mathbb{R}^n$ and $\bar{Y} \in \mathbb{S}^n$. We note that for any $s \in \mathbb{R}^{n+1}$ we get

$$\text{Arw}(s) \cdot Y = s_0 \text{Tr}(Y) + 2\bar{s} \cdot \bar{Y}_0. \quad (17)$$

It follows that $\text{Arw}^\top : \mathbb{S}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is nothing but

$$\text{Arw}^\top Y := \begin{pmatrix} \text{Tr}(Y) \\ 2\bar{Y}_0 \end{pmatrix}. \quad (18)$$

Consequently

$$M^\top(Y^1, \dots, Y^r) = \text{vec} \left(\begin{pmatrix} \text{Tr}(Y^1) \\ 2\bar{Y}_0^1 \end{pmatrix}, \dots, \begin{pmatrix} \text{Tr}(Y^r) \\ 2\bar{Y}_0^r \end{pmatrix} \right). \quad (19)$$

Then, $M^\top \mathcal{K}$ is clearly closed. Moreover, since $\text{Arw}^\top Y = 0$ implies $\text{Tr}(Y) = 0$, when $Y \in \mathbb{S}_+^{n+1}$ the latter yields $Y = 0$. Hence, all the assumptions of Proposition 9, and of Theorem 10, are fulfilled. We thus conclude that a second-order program and its respective semidefinite programming representation (given by (10)) have the same four- and six-partitions.

5 Conclusions

Our results demonstrate what one cannot expect to obtain simple geometric characterizations of the kind that exist for the four-partition of Bonnans and Ramírez [2] for the practically useful partition of Peña and Roshchina [16] in a general case of a multifold conic system. However, in the case of second-order cone programs the complement $T \setminus C$ is empty (cf. Fig. 3), and $R = C$ (see Proposition 3). This is in contrast to Example 4 in which we lumped three copies of the nonnegative orthant together, deliberately disrespecting the natural multifold structure of the problem.

In the case of a homogeneous feasibility problem, we have obtained a geometric condition satisfied by the set R^0 that originates from the six-partition of Bonnans and Ramírez. It would be valuable to generalize this result (stated in Theorem 6) as well as the original geometric result (in Theorem 5) to optimization problems.

Also, motivated by a question from one of the referees that we were not able to answer, we would like to ask the reader if there is an example of a primal-dual pair of problems for which $B' \cup N' \neq \emptyset$, but $C = \emptyset$.

Finally, we have shown that the canonical transformation of second-order cone programs into a positive semidefinite program preserves the complementarity partitions. It is an interesting question to determine other specific classes of problems for which this is also the case.

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