Computing Shintani Fundamental Domains

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Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por el candidato

Alex Gustavo Capuñay Gonzales

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Director de Tesis:

Dr. Eduardo Friedman Eduardo Friedman......

Comisión de Evaluación de la Tesis

Dr. Luis Arenas ....

Dr. Francisco Díaz y Díaz ....

Dra. María Inés Icaza ....

Para mi familia.

“ *All human cognition begins with intuitions, proceeds from there to concepts, and ends with ideas.*”
I. Kant, Kritik der reinen Vernunft (Critique of Pure Reason), 1781.

BIOGRAFÍA

Nací el 26 de Abril de 1989, en un distrito llamado Monsefú perteneciente a la región de Lambayeque ubicado al noroeste de Perú. Mi acercamiento hacia las matemáticas fue dos años antes de terminar la escuela secundaria, influenciado por varios profesores que impartían matemáticas y también por un ambiente donde la competencia académica entre estudiantes era lo bastante interesante como para terminar envuelto en ello. Faltando unos meses para dejar la escuela, nuestros profesores nos preguntaban si queriamos estudiar alguna carrera o qué queriamos hacer con nuestras vidas. En cuanto a mí empezaba a tener interés más por los números que por las letras, así lo común era estudiar alguna carrera en ingeniería, pero eso implicaría obtener más puntos en el examen de Admisión dado por la única Universidad pública de mi región. Dado que no me sentía tan preparado para estudiar alguna carrera en ingeniería, empecé a buscar otras opciones, entre las que me parecieron idóneas fueron la carrera en matemáticas o la carrera en educación matemática y computación, debido a mi poca influencia computacional en aquel momento, opté por la carrera en matemáticas. Así me inscribí en tal carrera en la Universidad Nacional Pedro Ruíz Gallo, sin saber realmente de qué se trataría tal carrera (pues no sabía de su existencia, solo pensé que algo de matemáticas iba a ver, mas para mi sorpresa fue vasto), habiendo dado el examen de admisión pude conseguir un cupo para estudiar matemáticas en tal Universidad. Fue así como en 2006 empecé más enserio con las matemáticas, muchos de los cursos que iba tomando en la carrera me encauzaba aún más hacia un repertorio inagotable de teorías matemáticas. Después de culminar mi pregrado en la Universidad, me interesó el álgebra y la teoría de números. El profesor Rubén Burga, mi tutor de pregrado, me sugirió hacer estudios de postgrado. Por aquel tiempo, algunos de los profesores que tuve habían estudiado algún postgrado en Chile. Ellos conocían algunos programas de postgrado donde yo podría postular. Fue así como en 2013 llegué a la Universidad del Bío-Bío en Concepción-Chile, donde hice una maestría en Matemática con mención en matemática aplicada, teniendo como tutor al profesor Nicolas Thériault, quien me recomendo cursar un doctorado. Culminado mi maestría, tuve la oportunidad de ser aceptado en el programa de Doctorado en Ciencias, mención matemáticas en la Universidad de Chile, empezando mis estudios en 2015. Teniendo claro de mi interés en álgebra y teoría de números, he tenido la grata oportunidad de conocer y trabajar bajo la dirección del profesor Eduardo Friedman. Después de muchos años he comprendido que las matemáticas es buscar ideas, para resolver problemas, y que estas llegan después de un razonamiento constante. Deseo poder continuar en la senda de las matemáticas y conocer más sobre su historia con el objetivo de ampliar mi visión sobre esta.

Agradecimientos

Primeramente quisiera agradecer al profesor Eduardo Friedman por todo el tiempo que ha invertido en mí para aconsejarme no solo académicamente sino también humanamente. Sabiendo pues que se podía aprender de él, tuve el atrevimiento de preguntarle si podría dirigirme en mi tesis de doctorado. Agradezco pues infinitamente todos sus consejos vertidos en mí, la cual me ha fortalecido tanto en lo académico como en lo personal.

Agradezco también a Becas CONICYT-CHILE (Comisión Nacional de Investigación Científica y Tecnológica) por haberme otorgado una beca (N° 21150751) para hacer estudios de doctorado en la Universidad de Chile, estoy completamente agradecido por el apoyo económico brindado desde Marzo de 2015 hasta Septiembre de 2019, la que me ha permitido una mejor estadía en el país. También quiero agradecer el apoyo financiero brindado por el Proyecto FONDECYT Regular 1170176 de octubre 2019 a marzo 2020.

También quisiera agradecer a mi familia, por todo el apoyo brindado desde que inicié mi travesía por las matemáticas. En especial a mi madre y hermanos que siempre me han dado toda su confianza y motivación.

Finalmente quisiera agradecer a los profesores y funcionarios que laboran en el departmento de Matemáticas de la Facultad de Ciencias de la Universidad de Chile, por toda la disponibilidad que tienen para ayudarte y darte algún consejo. Igualmente agradecer a los funcionarios de la Escuela de Postgrado de la Facultad de Ciencias, por toda la ayuda burocrática brindada, para que cada año pudiéra presentar la documentación necesaria para renovar mi beca Conicyt.

Resumen

Sea k un cuerpo de números de grado $n = r_1 + 2r_2$ teniendo r_1 incrustaciones reales y r_2 pares de incrustaciones complejas. Sea $G \subseteq E_+$ cualquier subgrupo de índice finito del grupo E_+ de las unidades totalmente positivas de k, así G actúa en $\mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}$, donde $\mathbb{R}_+$ denote los números reales positivos y $\mathbb{C}^*$ los números complejos no nulos. Diaz y Diaz, Espinoza y Friedman introdujeron la noción de dominio fundamental con signo y han dado un algoritmo para determinarlo explícitamente desde un conjunto de generadores de G si k no es totalmente complejo (es decir, $r_1 > 0$). Su dominio fundamental con signo consiste de a lo más $(n-1)!3^{r_2}$ conos poliédricos k-racionales. Aquí damos un algoritmo para extraer un dominio fundamental $\mathfrak{F}$ desde un dominio fundamental con signo. Tal $\mathfrak{F}$ es de nuevo una unión finita de conos poliédricos k-racionales. Excepto para cuerpos cuadráticos y cúbicos ambos totalmente reales, un tal algoritmo no era conocido previamente. Aunque nuestro algoritmo es teóricamente bastante lento debido a la gran cantidad de conos involucrados, en la práctica funciona bien si el grado del cuerpo es menor que seis. También, para cuerpos séxticos totalmente reales nuestro algoritmo es a veces exitoso.

Abstract

Let k be a number field of degree $n = r_1 + 2r_2$ having r_1 real embeddings and r_2 pairs of complex conjugate embeddings. Let $G \subseteq E_+$ be any subgroup of finite index of the group E_+ of totally positive units of k, so that G acts on $\mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}$, where $\mathbb{R}_+$ denotes the positive real numbers and $\mathbb{C}^*$ the non-zero complex numbers. Diaz y Diaz, Espinoza and Friedman introduced the notion of signed fundamental domain and gave an algorithm to determine these explicitly from a given set of generators of G if k is not totally complex (*i.e.*, $r_1 > 0$). Their signed fundamental domain consists of at most $(n-1)!3^{r_2}$ k-rational polyhedral cones. Here we give an algorithm to extract a true fundamental domain $\mathfrak{F}$ from such a signed fundamental domain. Here again, $\mathfrak{F}$ is a finite union of k-rational polyhedral cones. Except for totally real quadratic and cubic fields, no such algorithm was previously known. Although our algorithm is theoretically rather slow due to the great number of cones involved, in practice it works well if the degree of the number field is at most five. Also, for totally real sextic fields our algorithm is sometimes successful.

Contents

Introduction

0.1 Shintani Domains

Let k be a number field of degree $n = r_1 + 2r_2$, with r_1 real embeddings and r_2 pairs of complex conjugate embeddings. Arbitrarily choosing one from each pair of complex conjugate embeddings, we obtain the standard mapping of k into the real vector space $V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. This induces an action by component-wise multiplication of the group of totally positive units E_+ of k on the open subset $V_+ := \mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}$. Shintani [S76] [S79] [N99] proved that there is a fundamental domain $\mathfrak{F} = \bigcup_{j \in J} C_j$ for V_+/E_+ which is a disjoint finite union of (open) k-rational simplicial cones C_j, where by definition $C_j := C_j\left(v_{j1}, \ldots, v_{jr(j)}\right) \subseteq V_+$ consists of all positive linear combinations of the linearly independent vectors $v_{j1}, \ldots, v_{jr(j)} \in k_+ := V_+ \cap k$.[1] Such an $\mathfrak{F}$ is now known as a Shintani domain. Thus,

$$V_+ := \mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2} = \bigcup_{j \in J} \bigcup_{\mu \in E_+} \mu C_j \quad \text{(disjoint union)}.$$

This allowed Shintani to study Hecke L-functions attached to abelian extensions of k. In particular he showed that the values of certain zeta functions at non-positive integers are rational [S76, Theorem 1]. This result was shown by Siegel [S69] whose proof (different from Shintani's) was based on the theory of elliptic modular forms. Shintani also used his domain to give an affirmative answer to the Hecke conjecture that the relative class number of a totally complex quadratic extension of a totally real number field admits an elementary arithmetic expression [S76, Theorem 2].

Shintani's proof does not give a practical procedure to construct the cones that determine his fundamental domain. Soon after Shintani's proof this issue was addressed by Nakamula [N77] for Galois cubic fields, and then by Thomas and Vasquez [TV80] for any totally real cubic field. An efficient algorithm for totally real cubics was found by Diaz y Diaz and Friedman [DF12]. The general case was addressed by Okazaki [O93] and Halbritter and Pohst [HP00], but they only arrived at conjectural algorithms. Aside from its interest in number theory, Shintani domains are important in the explicit resolution of singularities of certain moduli spaces [E75] [TV80] [S86] [G99].

A first advance toward explicit cones in the case of a general totally real number field was provided by Colmez [C88] [C89]. He proved the existence of units having special geometric properties, which generate a finite-index subgroup

[1] For notational simplicity, throughout this thesis we regard k as a subset of V.

$G \subseteq E_+$ for which he gave an explicit construction of a Shintani domain for V_+/G consisting of $(n-1)!$ k-rational cones $\{C_\sigma\}_{\sigma \in S_{n-1}}$. Colmez's cones require the prior construction of his special units. Unfortunately, there is no known practical procedure to find them, except in the quadratic and cubic cases [DF12].

In 2014 Diaz y Diaz and Friedman [DF14] got rid of the need for Colmez's special units at the cost of allowing signed fundamental domains. Their idea was to regard a fundamental domain as a list of cones C_j satisfying

$$\sum_{j\in J} \#(C_j \cap G\cdot x) = 1 \qquad (\forall x \in V_+),$$

and to generalize this by permitting the subtraction of cones.[2]

Definition 1. *A signed fundamental domain* $(\mathcal{N};\mathcal{P})$ *for the action of a countable group* G *acting freely*[3] *on a set* X *is by definition a pair of lists* $\mathcal{N} = (N_1, \dots, N_m)$ *and* $\mathcal{P} = (\Pi_1, \dots, \Pi_\ell)$ *of subsets of* X *such that*

$$\sum_{i=1}^{\ell} \#\big(\Pi_i \cap (G\cdot x)\big) - \sum_{j=1}^{m} \#\big(N_j \cap (G\cdot x)\big) = 1 \qquad (\forall x \in X), \tag{1}$$

where we also require that $\#(\Pi_i \cap G\cdot x)$ *and* $\#(N_j \cap G\cdot x)$ *are finite and bounded independently of* $x \in X$*. We shall call the* N_j *the "negative subsets" and the* Π_i *the "positive subsets."*

Diaz y Diaz and Friedman showed that for the purpose of calculating Hecke L-functions, signed fundamental domains are just as convenient as true fundamental domains. The former have the advantage that they are easily constructed given any set of independent generators of a subgroup $G \subseteq E_+$ of finite index [DF14] [EF20], and this for any number field that is not totally complex. Thus, there is no longer any need for Colmez's special units as long as we calculate with L-functions (except in the totally complex case). Nonetheless, aside from their intrinsic interest, Shintani domains are still useful to resolve singularities.

Leaving aside the trivial case where E_+ is finite, algorithms producing Shintani domains are known only for the case of totally real cubic and quadratic fields. Our goal in this thesis is to give and implement an algorithm for producing a true fundamental domain from a signed one.

0.2 From signed to true fundamental domains: An algorithm

We assume given a signed fundamental domain $(\mathcal{N};\mathcal{P})$ for X/G, as defined above, and wish to modify it to produce a true fundamental domain. The idea for such an algorithm becomes clear after the following observations.

[2] Here and below # denotes cardinality.

[3] *i.e.* the stabilizer of any $x \in X$ is assumed trivial. This essential condition was mistakenly omitted in [DF14, Definition 4], but it holds for their application to units of number fields.

Observation 1. *If all the "negative subsets" N_j in the list $\mathcal{N}$ are empty, then $\mathcal{F} := \bigcup_{i=1}^{\ell} \Pi_i$ is a true fundamental domain for the action of G on X, and the Π_i are disjoint.*

Observation 1 follows immediately from (1). Thus, our goal will be to remove the N_j so as to eventually arrive at an empty list of negative subsets. We will do this bit by bit.

Observation 2. *If for some i and j we have $N_j \cap \Pi_i \neq \varnothing$, then on replacing N_j by $N_j \setminus (N_j \cap \Pi_i)$ and Π_i by $\Pi_i \setminus (N_j \cap \Pi_i)$, the new $(\mathcal{N}; \mathcal{P})$ is a signed fundamental domain for X/G.*

This is because the contributions of $(G \cdot x) \cap (N_j \cap \Pi_i)$ just cancel out in (1).

However, the N_j's will not in general be contained in $\bigcup_{i=1}^{\ell} \Pi_i$, as the following shows.

Observation 3. *If for any $g \in G$ and any $N_j \in \mathcal{N}$ we replace N_j by $g \cdot N_j$, then the new $(\mathcal{N}; \mathcal{P})$ is a signed fundamental domain for X/G.*

The reason is that $\#\big(N_j \cap (G \cdot x)\big) = \#\big((g \cdot N_j) \cap (G \cdot x)\big)$ in (1).

Observation 4. *If $L := (g \cdot N_j) \cap \Pi_i \neq \varnothing$, then on replacing N_j by $N_j \setminus (g^{-1} \cdot L)$ and Π_i by $\Pi_i \setminus L$, the new $(\mathcal{N}; \mathcal{P})$ is a signed fundamental domain for X/G.*

This follows on combining Observations 2 and 3.

Thus, to remove pieces of the N_j's it suffices to find a $g \in G$ and a $\Pi_i \in \mathcal{P}$ such that $(g \cdot N_j) \cap \Pi_i \neq \varnothing$. It is an essential feature of signed fundamental domain that this is always possible.

Observation 5. *Given any (non-empty!) N_j, there is a $g \in G$ and some i such that $(g \cdot N_j) \cap \Pi_i \neq \varnothing$.*

The reason is that an even stronger property holds. Since for any $x \in N_j$ we have $\sum_{h=1}^{m} \#\big(N_h \cap (G \cdot x)\big) \geq 1$, it follows from (1) that $\sum_{i=1}^{\ell} \#\big(\Pi_i \cap (G \cdot x)\big) \geq 2$. Thus there is some g and Π_i such that $(g \cdot N_j) \cap \Pi_i \neq \varnothing$.

Combining the above observations we see that we need to successively crop off pieces of the signed fundamental domain $(\mathcal{N}; \mathcal{P})$ for X/G so as to eventually arrive at a new signed fundamental domain $(\mathcal{N}', \mathcal{P}')$, where $\mathcal{N}'$ is a list of empty sets. Then a true fundamental domain is simply the union of the subsets in the list $\mathcal{P}'$.

This leads to the following idea for an algorithm producing a true fundamental domain $\mathfrak{F}$ from a signed one $(\mathcal{N}; \mathcal{P})$, where $\mathcal{N} = (N_1, \ldots, N_m)$, $\mathcal{P} = (\Pi_1, \ldots, \Pi_\ell)$.

Step 1 (done?). Read $(\mathcal{N}; \mathcal{P})$. If $\mathcal{N}$ is a list of empty sets, then RETURN $\mathcal{F} := \bigcup_{i=1}^{\ell} \Pi_i$. Otherwise, go to Step 2.

Step 2 (crop). Find some $N_j \in \mathcal{N}$, $N_j \neq \varnothing$, $g \in G$ and $\Pi_i \in \mathcal{P}$ such that $L := (g \cdot N_j) \cap \Pi_i \neq \varnothing$. Replace N_j in $\mathcal{N}$ by $N_j \setminus g^{-1} \cdot L$ and Π_i in $\mathcal{P}$ by $\Pi_i \setminus L$. Go to Step 1 with the new $(\mathcal{N}; \mathcal{P})$.

It is not hard to show (see Chapter 3) that the algorithm terminates provided we are able to discard g from future use. This avoids the trap of having to try the same g indefinitely at later steps of the algorithm. When g is used for the first time, we will discard g after using it to crop all remaining $N_{j'}$ in $\mathcal{N}$.[4]

Although Steps 1 and 2 will be our basic strategy, in our case there are several obstacles. Firstly, we are not interested in just any fundamental domain. We want a fundamental domain which is a finite disjoint union of k-rational polyhedral cones. Unfortunately, the difference of two such cones is no longer polyhedral, so we will have to write the set-theoretical differences $N_j \setminus (g^{-1} \cdot L)$ and $\Pi_i \setminus L$ in Step 2 as a (possibly empty) finite union of k-rational polyhedral cones.

The boundary of the cones in the signed fundamental domains deserves mention. When we crop off pieces of cones and rewrite the result as a union of cones, we would normally have to keep track of which common boundary piece is assigned to which cone. Fortunately, we are able to avoid all consideration of boundary contributions until there are no more negative cones left. For this we use Colmez's (unpublished) trick of choosing a special vantage point to select which boundary pieces to include in the fundamental domain (see [EF20, § 4.3] and Chapters 3 and 4 of this thesis). As happened already to Espinoza and Friedman, only here do we need the hypothesis that k is not totally complex, as in general no vantage point exists in this case.

As is usual in algorithmic number theory, some steps that seem trivial in theory can take some effort to carry out. An example of this is determining if $(g \cdot N_j) \cap \Pi_i \neq \varnothing$. A more important problem is that picking an N_j and then finding g and Π_i as in Step 2 is slow. In practice, we list the g in the (countable) group $G = \{g_1, g_2, \ldots\} \subseteq E_+$ (see Chapter 4), and for each g_h (beginning with $h = 1$), we apply the cropping process (Step 2) to all N_j and Π_i such that $(g_h \cdot N_j) \cap \Pi_i \neq \varnothing$. We then move to g_{h+1} if any negative cones remain.

0.3 On the running time

For convenience, we work in somewhat more generality than we need for our application. We take as given a signed fundamental domain, consisting of pointed rational polyhedral cones,[5] for the free linear action of a countable group G on a subset $\mathfrak{O} \subseteq V$ of an n-dimensional real vector space V endowed with a rational structure (see Definition 34 for details).

In Chapter 3 we describe an algorithm that finds a true fundamental domain for $\mathfrak{O}/G$ consisting of a finite number of pointed rational polyhedral cones. We implemented our algorithm to find a Shintani domain (see §0.1) for number fields that are not totally complex. Here we relied on the Espinoza-Friedman algorithm [EF20] for producing a signed fundamental domain.

Our algorithm starts by computing an Espinoza-Friedman signed fundamental domain, which has $(n-1)!3^{r_2}$ cones.[6] If $n > 15$, the $(n-1)! > 1.3 \cdot 10^{12}$ cones

[4] The algorithm terminates essentially because $\#(\Pi_i \cap G \cdot x)$ and $\#(N_j \cap G \cdot x)$ are finite and bounded independently of $x \in X$ (see Definition 1).

[5] A cone is pointed if it contains no 1-dimensional vector subspace. A rational cone is one generated by elements in a fixed $\mathbb{Q}$-vector space $V_{\mathbb{Q}} \subseteq V$, the rational structure.

[6] In practice it can have fewer cones since cones of dimension less than n are discarded.

are just too many to do anything with. Thus, since most computations in our algorithm involve operating with $n \times n$ rational matrices and n is quite small, say $n \leq 15$, the running time for a given field depends mainly on the number of cones and units involved.[7]

The main step of our algorithm is to crop cones using a unit g. Taking a negative cone N_j and a positive cone Π_i, cropping with g replaces them respectively by $N_j' := N_j \setminus (N_j \cap g^{-1} \cdot \Pi_i)$ and $\Pi_i' := \Pi_i \setminus (g \cdot N_j \cap \Pi_i)$. Unfortunately, in general N_j' and Π_i' are not polyhedral cones, so we have to write them as a union of cones, disjoint except along common boundary faces. This could replace each N_j and Π_i by as many as $h(N_j) + h(\Pi_i)$ cones, where $h(N) \geq n$ is the minimal number of inequalities determining the polyhedral cone N. If we had ℓ negative cones before cropping with g, the number of positive cones could increase by a factor of n^ℓ, or more, after g is processed.

Hence we would expect running time to worsen as the number of cones rises, and as the number of units needed in the cropping process increases. Of course, as the finite set of units g involved is used up, the number of negative cones must decrease and eventually reach 0, but we may run out of time or memory before this happens.

The above analysis could be turned into an upper bound on the time and space required by our algorithm. The bound would naturally be in terms of the discriminant or regulator of the number field k. However, our empirical runs showed in practice no connection between running time and discriminant or regulator (see the Appendix for graphs of our results against the discriminant), so we have dispensed with producing these (very bad) bound. Rather, it seems to depend in some complicated way on how the orbits of the positive and negative cones intersect.

0.4 Experimental Results

Here we present some data from our implementation in Pari/GP of Algorithm 45, which produces a true fundamental domain from a signed one. We tested lists of number fields obtained from the database in https://www.lmfdb.org/ using a LINUX computer with an Intel Core i7-8700 CPU 3.20GHz x 6 processor. The degrees that we have considered are $n = r_1 + 2r_2 = 3, 4$ and 5, for any signature (r_1, r_2) with $r_1 > 0$. We also tested our algorithm for some totally real sextic number fields.

In practice, no implementation of the Espinoza-Friedman algorithm [EF20, §2] was available, so our first task was to write the corresponding program. Starting from an irreducible polynomial $f \in \mathbb{Q}[x]$, we use the PARI function *bnfinit* to obtain the standard invariants of the abstract number field $k := \mathbb{Q}[x]/\big(f(x)\big)$. This includes a integral basis and a set of fundamental units.[8] We then computed

[7] Of course, if the fundamental units are huge and therefore not computable, our algorithm cannot even get started since we cannot compute the signed fundamental domain. In assuming that we are given a signed fundamental domain, we are tacitly supposing that fundamental units can be computed.

[8] Or an error message if it fails to find the ring of integers or fundamental units, but this

free generators for the totally positive units E_+ of k using *tpu*, a short program given in the PARI user's manual. Twisters were computed as suggested in [EF20, p. 487, footnote 2].

We implemented the Espinoza-Friedman algorithm, producing a signed fundamental $(\mathcal{N};\mathcal{P})$ for the action of E_+ on $\mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}$, up to the E_+-orbit of the boundaries of the cones.[9] This consists of at most $3^{r_2}(n-1)!$ rational n-dimensional polyhedral cones divided into two lists, the positive cones $\mathcal{P}$ and negative cones $\mathcal{N}$.

When the Espinoza-Friedman algorithm outputs no negative cones, we shall say that we are in the Colmez case. Then the signed fundamental domain is already a true one, so Algorithm 45 is not needed except for applying the Colmez trick to determine the boundary pieces.

Table 1 gives an overview of the time it took to obtain fundamental domains for roughly 750 000 fields of degree up to 5, about 90% of which were Colmez cases owing to the many fields of small degree considered. The first column of

Table 1: Time spent on true and signed fundamental domains for degree ≤ 5.

Fields	|Discrim.| $\leq$	(r_1, r_2)	Colmez cases	Non Colmez	True domain	Signed domain
112444	2×10^6	(3,0)	111376 (99%)	1068	3m	9m
370444	2×10^6	(1,1)	370443 (99.99%)	1	47m	5h 26m
169371	10^7	(4,0)	147548 (87%)	21823	1h 36m	33m
90671	10^6	(2,1)	49929 (55%)	40742	1h 17m	4h 3m
4863	6×10^6	(5,0)	1735 (36%)	3128	31h 50m	4m 50s
2197	3×10^5	(3,1)	1474 (67%)	723	29h 18m	48m
4142	2×10^5	(1,2)	4001 (96.6%)	141	145h 47m	3h 44m

Table 1 shows the number of fields tested for the signature (r_1, r_2) in column 3. Fundamental domains for all number fields of that signature with discriminant up to the value in the second column were obtained using Algorithm 45. The fourth column gives the number and percentage of Colmez cases.[10] The fifth column gives the number of non Colmez cases (where our algorithm is fully applied). The sixth column gives the total time taken to find a true fundamental domain (cones and boundary) from a signed one for all the fields considered for this signature. Even in the Colmez cases, some time was taken finding the boundary pieces. The seventh column gives the total time taken by the Espinoza-Friedman algorithm to find a signed fundamental domain, without determining the boundary pieces.

On examining the results of Algorithm 45 for the (non Colmez) fields in Table 1, we found large variations from field to field. Most fields were quickly dispatched, but a few took extraordinarily much longer than the average. For example, although totally real (non Colmez) quintics took an average of about half a minute to process, at least one quintic took 57 minutes. This was not an isolated exception, for the standard deviation of the time taken in this signature

never happened for the fields we considered.

[9] After producing the generators for the positive and negative cones in the signed fundamental domain, we did not waste time determining the correct boundary pieces. The reason is that this information, *i.e.* an explicit description of $\mathfrak{H}$ in (3.6), is not needed for Algorithm 45.

[10] Surprisingly enough, for complex cubic fields only one non Colmez case appeared, and it was for discriminant -23, the first field in that signature.

was more than six times the average value. Moreover, as the initial number of cones increases, this effect became more pronounced. For the 141 non Colmez quintics with two complex places that we ran, over 98% of the time was spent on 14 fields.

As the graphs in the Appendix show, the running time of Algorithm 45 for a field does not seem related to its discriminant.[11] This is not altogether unexpected as the algorithm spends most of its time removing cones of the type $\Pi \cap g \cdot N$, where Π is a positive cone, g is a unit and N is a negative cone. Each time such a cone is removed, the remaining pieces of N and Π, *i.e.* $N \setminus (N \cap (g^{-1} \cdot \Pi))$ and $\Pi \setminus (g \cdot (N \cap \Pi))$, are not in general cones, and so must be replaced (if not empty) by a union of cones. This causes the number of cones to multiply, slowing down the algorithm. Moreover, this is done until there are no more negative cones left.

One is therefore let to expect that the number of cones in the fundamental domain output, as well as the number of units g used in the process, should be strongly correlated with running time. This is borne out by the data on sextic fields in Table 2. Our algorithm often fails in practice (*i.e.* does not terminate within a week) for sextic fields. We therefore studied 1860 totally real sextic fields

Table 2: Totally real sextics with at most five negative cones.

Discriminant $\leq 3 \times 10^7$, Fields: 1860, Colmez cases: 50, Non Colmez: 1810					
Negative cones:	1	2	3	4	5
Fields	47	54	78	95	87
Done in less than 15 minutes:	45	48	62	68	41
Average number of cones in $\mathcal{F}$:	274.42	495.47	780.37	1018.97	1152.78
Maximum number of cones:	1642	1698	2457	2472	2352
Average number of units:	2.04	2.66	3.43	3.80	3.97
Maximum number of units:	6	6	7	8	9
Average time for success:	25s	1m 5s	2m 40s	3m 25s	4m 21s

as the number of negative cones in the Espinoza-Friedman signed fundamental domain increased. Of course, for the 50 Colmez cases (*i.e.* no negative cones, and so in general about $120 = 5!$ positive cones) the algorithm terminated immediately. Of the 47 fields having only one negative cone, 45 terminated in less that 15 minutes (each of the remaining 2 fields was skipped after 15 minutes), and the average running time for these 45 fields is about 25 seconds, as shown in the first column in Table 2. The number of cones grew on the average to nearly 275, although the worst case was 1642 cones. On the average 2.04 units were used (so 1 or 2 units frequently sufficed), but up to 6 units were processed. As Table 2 shows, there is a clear pattern as the number of negative cones increases: The proportion of fields terminating in less than 15 minutes drops while the average and maximal number of cones (or units processed) increase.

Although it is hard to make (let alone prove) a precise statement, the running time should depend on how the E_+-orbits of negative and positive cones intersect.

[11] Nor to its regulator. We omitted regulator data from the Appendix after deciding that one appendix with negative results was enough.

The number of negative cones in the signed fundamental domain should be an important factor, as well as the number of units taking negative cones to positive ones. The following tables tabulate these matters for all signatures with $r_1 > 0$ up to degree 5.

In Table 3 we show the maximal and average time for each signature (for all non Colmez cases), as well as the standard deviation of the time taken to find a fundamental domain. *Since our algorithm does nothing new in the Colmez cases, in all of the following tables and graphs we only consider non Colmez cases.*

Table 3: Time taken to obtain fundamental domain.

Signature (r_1, r_2):	(3,0)	(1,1)	(4,0)	(2,1)	(5,0)	(3,1)	(1,2)
Fields:	1068	1	21823	40742	3128	723	141
Total time:	3s	0.039s	18m	51m	31h 48m	29h 8m	144h 54m
Average time:	0.002s	0.039s	0.048s	0.075s	37s	2m 49s	1h 2m
Maximum time:	0.004s	0.039s	3s	1.79s	57m	2h 1m	46h 39m
Standard deviation (σ) :	0.0005s	0s	0.065s	0.062s	2m 54s	9m 45s	5h 35m

In Table 4 we give statistics on the number of cones in the fundamental domain produced by Algorithm 45.

Table 4: Number of cones in fundamental domain.

Signature (r_1, r_2):	(3,0)	(1,1)	(4,0)	(2,1)	(5,0)	(3,1)	(1,2)
Fields:	1068	1	21823	40742	3128	723	141
Average number of cones:	2	5	11.414	23.878	242.46	676.240	4216.44
Maximum number of cones:	2	5	80	85	4652	9175	79845
Standard deviation (σ):	0	0	3.992	5.296	413.526	1027.973	12570.12

Recall that the initial number of cones in the signed fundamental domain is at most (and often equals) $3^{r_2}(n-1)!$. From the table we deduce that for quartic fields the final number of cones is only about twice the initial number on the average, whereas for quintics it is from 10 to 20 times the initial number (depending on the signature). Presumably this proliferation of cones, rather than the increase in the original numbers, accounts for the increasing difficulty of applying the algorithm as the degree of the number field rises. In §4.4 and in the Appendix we give some more tables and graphs for quartic and quintic fields.

Table 5 shows some statistics on the number of negative cones in the signed fundamental domain initially provided by the Espinoza-Friedman algorithm. Table 6 does the same for the number of units needed.

Table 5: Number of negative cones in the initial signed fundamental domain.

Signature (r_1, r_2):	(3,0)	(1,1)	(4,0)	(2,1)	(5,0)	(3,1)	(1,2)
Fields:	1068	1	21823	40742	3128	723	141
Average number of negative cones:	1	1	1.261	1.691	2.651	8.359	16.06
Maximum number of negative cones:	1	1	4	9	13	36	90
Standard deviation (σ):	0	0	0.541	1.190	2.028	7.443	17.82

In conclusion, empirically the algorithm runs well and quickly for degree up to 4, as it usually also does for quintics. Some quintics, however, take very long

Table 6: Number of units processed to obtain a fundamental domain.

Signature (r_1, r_2):	(3,0)	(1,1)	(4,0)	(2,1)	(5,0)	(3,1)	(1,2)
Fields:	1068	1	21823	40742	3128	723	141
Average number of units:	1	2	1.706	1.075	3.368	1.669	1.858
Maximum number of units:	1	2	10	2	18	7	8
Standard deviation (σ):	0	0	0.952	0.264	2.491	0.952	1.686

and result in a huge number of cones in the fundamental domain. For sextics the algorithm runs only sometimes, with the probability of success being high if the number of negative cones is very small.

Many open questions remain on how to improve our algorithm. The algorithm makes several choices rather blindly. Experiments show that modifying them can lead to large changes in running time and in the number of cones in the fundamental domain produced. These choices involve mainly the order in which units are used to crop the cones, and the order of the cones themselves. The challenge is to find heuristics leading to fewer cones and so a faster algorithm. Using a measure, such as spherical volume, of the section cropped would probably be a good way to order the operations. Unfortunately, calculating volumes can itself be a difficult task, so we have not yet implemented this strategy.

Lastly, we describe the structure of this thesis. In Chapters 1 and 2 of this thesis we review in detail various algorithms for working with polyhedral cones, mostly following Fukuda and Prodon [FP96]. We give full proofs both for the reader's convenience and to make sure that rationality is preserved. In Chapter 3 we give a general algorithm for passing from a signed fundamental domain consisting of rational cones in a real vector space (with some rational structure) to a true fundamental domain. In Chapter 4 we first show that the main algorithm in Chapter 3 applies to number fields having at least one real place. Then we give a few tables describing our experimental results for quartic and quintic fields. The Appendix features a number of graphs designed to visualize some features of our runs, and to show that the running time of the algorithm does not correlate well with the discriminant, the traditional parameter associated to a number field.

Chapter 1

Review of rational cones

As explained in Step 2 in §0.2, we need an algorithm for subtracting one k-rational cone from another. Moreover, we will need to express this as a finite union of k-rational cones. For this we need to review how to pass from the representation of rational polyhedral cones by generators to its representation by inequalities and vice-versa. This is treated in §1.1 and §1.2. In §1.4 we shall give a method to remove the intersection of two cones, obtaining the desired union of k-rational cones. These procedures are well-known in the theory of polyhedra [L13] [Z95] [Sc86], but they are rarely mentioned in number-theoretical contexts, so we give detailed proofs following [FP96], paying careful attention to rationality. For convenience, we work in a slightly more general context of $\mathbb{Q}$-subspaces of real vector spaces rather than dealing directly with the number fields.

In our case, we are lucky to be able to use the Colmez trick mentioned in the Introduction (see §3.1 below) to ignore lower-dimensional cones. In this chapter we will therefore only need to work with n-dimensional polyhedral cones $P \subseteq V$ contained in an n-dimensional real vector space, and we will assume that the cone is pointed (*i.e.* P contains no line). For the same reason, when we deal with intersections, we will only consider subtracting pointed cones whose intersection is n-dimensional.

1.1 Rational cones and duality

Throughout this chapter, we assume that V is an n-dimensional $\mathbb{R}$-vector space endowed with a fixed $\mathbb{Q}$-structure $V_{\mathbb{Q}}$ and a fixed bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$, which we assume is non-degenerate, symmetric and rational. That is, we fix an n-dimensional $\mathbb{Q}$-vector subspace $V_{\mathbb{Q}} \subseteq V$ containing a $\mathbb{Q}$-basis which is also an $\mathbb{R}$-basis for V, and assume that $\langle\ ,\ \rangle$ restricted to $V_{\mathbb{Q}} \times V_{\mathbb{Q}}$ takes values in $\mathbb{Q}$ and is non-degenerate.[1] To avoid trivialities, we always assume (tacitly) that $n \geq 1$.

In our application, we will start with a number field k of degree n having r_1 real and r_2 complex embeddings ($n = r_1 + 2r_2$), and let $V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. We let $V_{\mathbb{Q}} \subseteq V$ be the image of k in V, which we identify with k for convenience.

[1] Note that it follows that any $\mathbb{Q}$-basis of $V_{\mathbb{Q}}$ is an $\mathbb{R}$-basis of V. Hence we do not need to distinguish $\mathbb{Q}$-linear independence from $\mathbb{R}$-linear independence.

For $\alpha, \beta \in k$, we let $\langle \alpha, \beta \rangle := \mathrm{Trace}_{k/\mathbb{Q}}(\alpha\beta) \in \mathbb{Q}$, which extends of course to an $\mathbb{R}$-valued form on V.

A (closed) half-space in V is a subset $H \subseteq V$ of the form

$$H := \{x \in V : \ L(x) \geq 0\},$$

where L is a non-zero linear form on V. A (closed) polyhedral cone P is the intersection of a finite number of such half-spaces. That is,

$$P := \{x \in V : L_i(x) \geq 0, \ 1 \leq i \leq \ell\},$$

where the L_i are non-zero linear forms on V. We shall often abbreviate this as $P = \mathrm{H}(\mathcal{T})$, where $\mathcal{T} = \{L_1, ..., L_\ell\}$ or simply as $P = \mathrm{H}(L_1, \dots, L_\ell)$, and call it an H-representation of P. We shall say that $\mathrm{H}(\mathcal{T})$ is a rational H-representation of P if every L_i assumes rational values on $V_{\mathbb{Q}}$.

We shall often need to pass to a description of cones by generators. A finite subset $B := \{r_1, ..., r_t\} \subseteq Q$ of a cone Q in V is said to be a ray representation of Q if every element $x \in Q$ can be written as $x = \sum_{j=1}^{t} \lambda_j r_j$ with $\lambda_j \geq 0, \ 1 \leq j \leq t$. We shall abbreviate this as $Q := C[B]$ or $Q := C[r_1, ..., r_t]$ and say that $C[B]$ is an R-representation of Q. If $B \subseteq V_{\mathbb{Q}}$ we shall say that $C[B]$ is a rational R-representation of Q.

Our first task will be to give an algorithm to produce a rational R-representation from a rational H-representation of a cone. More precisely, given a rational H-representation $P = \mathrm{H}(\mathcal{T})$, where $\mathcal{T} = \{L_1, \dots, L_m\}$ and $L_j(x) := \langle v_j, x \rangle$ for some $v_j \in V_{\mathbb{Q}}$, we want to find $r_1, \dots, r_t$ in $V_{\mathbb{Q}}$, such that $P = C[r_1, \dots, r_t]$. We shall later see that duality allows us to use essentially the same algorithm to produce an H-representation from an R-representation.

We use the well-known double description method [MRTT53] [Z95], which is a dual version of the classical Fourier-Motzkin elimination algorithm [F26] [M36], but we give all the details for the reader's convenience. In fact, most of the time we just take Fukuda and Prodon's improved version of the double description method [FP96], replacing the real vector space V by $V_{\mathbb{Q}}$ and the dot product by a bilinear form.

First we consider the simplest case, *i.e.* when P has exactly n independent generators.

Lemma 2. *If $P = \mathrm{H}(L_1, \dots, L_n) \subseteq V$, with $L_j(x) := \langle v_j, x \rangle$ and $\{v_1, \dots, v_n\}$ a basis of $V_{\mathbb{Q}}$, then $P = C[r_1, \dots, r_n]$, where $\{r_1, \dots, r_n\}$ is the basis dual to $\{v_1, \dots, v_n\}$ with respect to the bilinear form $\langle \ , \ \rangle$.*

Conversely, if $P = C[r_1, \dots, r_n]$ and $\{r_i\}$ is a basis of $V_{\mathbb{Q}}$, let $\{v_j\}$ be the basis dual to $\{r_i\}$ and let $L_j(x) := \langle v_j, x \rangle$. Then $P = \mathrm{H}(L_1, \dots, L_n)$.

Finding the dual basis is easily implemented in exact (*i.e.* rational) arithmetic. Indeed, we shall prove that

$$r_i = \sum_{h=1}^{n} \Lambda_{ih} v_h, \qquad \Lambda = (\Lambda_{ih})_{1 \leq i,h \leq n} =: \Gamma^{-1}, \qquad \Gamma := (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n}. \tag{1.1}$$

Proof. Let $r_1, ..., r_n \in V$ be the basis dual to $v_1, ..., v_n$, *i.e.* $\langle v_j, r_j \rangle = 1$ and $\langle v_j, r_i \rangle = 0$ if $i \neq j$. We claim that $P = C[r_1, \dots, r_n]$. Indeed, $C[r_1, \dots, r_n] \subseteq P$ as $L_j(r_i) \geq 0$. On the other hand, since V is generated over $\mathbb{R}$ by $\{r_1, \dots, r_n\} \subseteq V_{\mathbb{Q}}$, for $y \in V$ there exist $\lambda_1, \dots, \lambda_n$ in $\mathbb{R}$ such that $y = \sum_{i=1}^n \lambda_i r_i$. If $y \in P$, we have $0 \leq \langle v_j, y \rangle = \sum_{i=1}^n \lambda_i \langle v_j, r_i \rangle = \lambda_j$. Hence $P \subseteq C[r_1, \dots, r_n]$, as claimed.

We now show that $r_i \in V_{\mathbb{Q}}$. The (rational) matrix Γ in (1.1) is invertible since the inner product is non-degenerate and $\{v_j\}$ is assumed to be a basis of $V_{\mathbb{Q}}$. Let $\Lambda := \Gamma^{-1}$, a rational matrix. Then

$$\left\langle \sum_{h=1}^n \Lambda_{ih} v_h, v_j \right\rangle = \sum_{h=1}^n \Lambda_{ih} \langle v_h, v_j \rangle = [\Lambda\Gamma]_{ij} = \mathrm{Id}_{ij}.$$

Thus the r_i in (1.1) do indeed give the dual basis. □

If $P = C[B] \subseteq V$ is an R-representation, we say that $b \in B$ is redundant if $C[B \setminus \{b\}] = C[B]$. We say that B is redundant if B contains a redundant element. By successively removing redundant elements, we can always write $P = C[B]$ with B irredundant. We shall see below in Lemma 15 d) that such B is unique (up to multiplying by positive scalars and permutations of its elements) if P is pointed, *i.e.* if P does not contain a non-trivial vector subspace of V.

Likewise, if $P = \mathrm{H}(\mathcal{T})$ is an H-representation, we say that $L_j \in \mathcal{T}$ is redundant if $\mathrm{H}(\mathcal{T} \setminus L_j) = \mathrm{H}(\mathcal{T})$. We say that $\mathcal{T}$ is redundant if it contains a redundant linear form. We can always write $P = \mathrm{H}(\mathcal{T})$ with $\mathcal{T}$ irredundant. If P is pointed and n-dimensional the irredundant B and $\mathcal{T}$ are unique (up to multiplication by positive scalars). We aim to prove this and to find an algorithm to compute B from $\mathcal{T}$ and vice-versa.

The following lemma gives a well-known characterization of a pointed polyhedral cone by the rank of a linear transformation [L13, Prop. 3.9].

Lemma 3. *Let V be an n-dimensional real vector space and let $\mathcal{T} = \{L_1, \dots, L_m\}$ be linear forms on V. The polyhedral cone $P := \mathrm{H}(\mathcal{T})$ is pointed if and only if $\mathrm{rank}(\mathcal{L}) = n$, where $\mathcal{L} : V \to \mathbb{R}^m$ is the linear transformation $\mathcal{L}(x) := \big(L_1(x), \dots, L_m(x)\big)$.*

More concretely, writing $L_j(x) = \langle v_j, x \rangle$ for some $v_j \in V$, P is pointed if and only if there are n linearly independent elements $v_{i_1}, \dots, v_{i_n}$.

Proof. Suppose $\mathrm{rank}(\mathcal{L}) < n$. Then $\dim\big(\ker(\mathcal{L})\big) > 0$, so there exists $x_0 \in \ker(\mathcal{L})$, $x_0 \neq 0$. But $L_j(\lambda x_0) = 0$ for all $\lambda \in \mathbb{R}$, so P is not pointed. Conversely, if there exists a non-trivial subspace $S \subseteq P$, there exists $x_0 \in S$, $x_0 \neq 0$. But then $-x_0 \in S$, so x_0 and $-x_0$ are in P. Then $L_j(x_0) \geq 0$ and $L_j(-x_0) \geq 0$ for all j. Therefore $L_j(x_0) = 0$ for all j, so $\mathcal{L}(x_0) = 0$. That is, $\dim\big(\ker(\mathcal{L})\big) > 0$, which implies that $\mathrm{rank}(\mathcal{L}) < n$.

We now prove the last claim in the Lemma. Let W be the subspace generated by $\{v_1, \dots, v_m\}$. If W is a proper subspace of V, then there exists a non-zero element w_0 in V such that $\langle v_j, w_0 \rangle = 0$ for $j = 1, \dots, m$. So $w_0 \in \ker(\mathcal{L})$. As $w_0 \neq 0$ we have that $P = \mathrm{H}(\mathcal{T})$ is not pointed. Conversely, if $v_{i_1}, \dots, v_{i_n}$ are linearly independent, let y be any element in V and write $y = \sum_{j=1}^n \lambda_j v_{i_j}$, $\lambda_j \in \mathbb{R}$. If $x \in \ker(\mathcal{L})$, we have $\langle x, y \rangle = \sum_j \lambda_j \langle x, v_{i_j} \rangle = \sum_j \lambda_j \langle v_{i_j}, x \rangle = 0$. Thus $\langle x, y \rangle = 0$

for any $y \in V$. This implies that $x = 0$ as $\langle \, , \rangle$ is assumed non-degenerate. Hence $\ker(\mathcal{L}) = \{0\}$, and so $P = \mathrm{H}(\mathcal{T})$ is pointed. □

For a cone $S \subseteq V$, its dual cone $S^* \subseteq V$ is defined as

$$S^* := \{x \in V : \langle x, y \rangle \geq 0 \quad \forall y \in S\}.$$

Note that if $S_1 \subseteq S_2$, then $S_2^* \subseteq S_1^*$. We now show that taking the dual switches the H- and R- representations of a cone.

Lemma 4. *Let $P = C[r_1, \dots, r_\ell]$ be an* R*-representation of the cone P and let $L_j(x) := \langle r_j, x \rangle$ $(1 \leq j \leq \ell)$. Then its dual P^* has an* H*-representation $P^* = \mathrm{H}(L_1, \dots, L_\ell)$.*

Proof. If $x \in P^*$, then $\langle x, y \rangle \geq 0$ for all y in P. In particular, $L_j(x) = \langle r_j, x \rangle = \langle x, r_j \rangle \geq 0$. Thus $P^* \subseteq \mathrm{H}(L_1, \dots, L_\ell)$. Now suppose $x \in \mathrm{H}(L_1, \dots, L_\ell)$, so $\langle x, r_j \rangle \geq 0$. If $y = \sum_j \lambda_j r_j \in P$ (*i.e.* $\lambda_j \geq 0$), then $\langle x, y \rangle = \sum_j \lambda_j \langle x, r_j \rangle \geq 0$. Thus $x \in P^*$. □

Lemma 5. *Let $P = \mathrm{H}(L_1, \dots, L_m)$ be an* H*-representation of the cone P, where $L_j(x) := \langle v_j, x \rangle$ $(1 \leq j \leq m)$. Then its dual P^* has an* R*-representation $P^* = C[v_1, \dots, v_m]$.*

Proof. $C[v_1, \dots, v_m] \subseteq P^*$ is clear since $v_j \in P^*$. Suppose there is an $x_0 \in P^*$, $x_0 \notin C[v_1, \dots, v_m]$. By Minkowski's separation theorem [L13, p. 134], there exists $c \in V$ such that $\langle c, x_0 \rangle < 0$ but $\langle c, x \rangle > 0$ for all $x \in C[v_1, \dots, v_m]$. Since $v_j \in C[v_1, \dots, v_m]$, we have $\langle c, v_j \rangle > 0$, and so $c \in P = \mathrm{H}(L_1, \dots, L_m)$. Thus, $\langle x_0, c \rangle \geq 0$ as $x_0 \in P^*$, contradicting $\langle c, x_0 \rangle < 0$. □

The next lemma assumes that a (a priori not necessarily polyhedral) cone P has an R-representation. Later we shall see that such a cone is necessarily polyhedral and that any polyhedral cone $P = \mathrm{H}(\mathcal{T})$ has an R-representation.

Lemma 6. *Let $P := C[r_1, \dots, r_\ell]$, where $r_i \in V$, and let $n := \dim(V)$. Then the following hold.*

(i) $(P^*)^* = P$.

(ii) P is an n-dimensional cone if and only if P^* is a pointed cone.

(iii) P is a pointed if and only if P^* is an n-dimensional cone.

Proof. Claim (i) follows directly from Lemmas 4 and 5. To prove (ii), note that by Lemma 4, $P^* = \mathrm{H}(L_1, \dots, L_\ell)$ where $L_j(x) = \langle r_j, x \rangle$. By Lemma 3, we need to show that $x \to \big(L_1(x), \dots, L_\ell(x)\big)$ has rank n. Since P is assumed n-dimensional, there is an $\mathbb{R}$-basis of V of the form $\{r_{i_1}, \dots, r_{i_n}\}$. By duality, $x \to \big(L_{i_1}(x), \dots, L_{i_n}(x)\big)$ has rank n. The converse is obtained by again applying Lemma 3 and duality.

To prove (iii), we may assume $P \neq \{0\}$ (for otherwise $P^* = V$, an n-dimensional cone). As above, $P^* = \{x \in V : \langle r_j, x \rangle \geq 0\}$. Hence if P^* has no interior, $P^* \subseteq \mathcal{H} := \{x \in V : \langle r_j, x\rangle = 0\}$ for some j with $r_j \neq 0$. Taking duals,

$$\mathcal{H}^* \subseteq (P^*)^* = P.$$

As the subspace $\mathcal{H} \neq V$, $\mathcal{H}^*$ contains a line (in fact, $\mathcal{H}^* = \{\lambda r_j : \lambda \in \mathbb{R}\} \neq \{0\}$ by Lemma 5), contradicting the assumption that P does not. Conversely, we assume that P^* is an n-dimensional cone. Hence if P is not pointed, then there exists a line $L = \{\lambda v : \lambda \in \mathbb{R}\}$ contained in P with $v \in V$, $v \neq 0$, so taking duality we have $P^* \subseteq L^*$. As $L \neq \{0\}$, we have that $L^* = \{x \in V : \langle v, x\rangle = 0\}$ has dimension $n-1$, and so P^* is not n-dimensional. $\square$

Note that Lemmas 3, 5 and 6 imply that a cone $P \subseteq V$ given by either an H- or R-representation, is pointed if and only if its dual P^* is n-dimensional (where $n = \dim V$). Similarly, Lemmas 4, 5 and 6 show that $P = C[r_1, \dots, r_\ell]$ is an irredundant R-representation if and only if $P^* = \mathrm{H}(L_1, \dots, L_\ell)$ is an irredundant H-representation, where $L_i(x) := \langle r_i, x\rangle$.

1.2 From H- to R-representations and vice-versa

In this Section we give a rational version of Fukuda and Prodon's account of the Fourier-Motzkin double description method [FP96] for passing from an H- to an R-representation of a pointed cone. Using Lemmas 4 and 5 we immediately deduce a method for passing from an R- to an H-representation. Indeed, if $P = C[r_1, \dots, r_m]$ spans V (*i.e.* P is n-dimensional), then the dual $P^* = \mathrm{H}(L_{r_1}, \dots, L_{r_m})$ is a pointed cone, where $L_r(x) = \langle r, x\rangle$. Applying the double description method we can find an R-representation $P^* = C[v_1, \dots, v_\ell]$ for its dual, a pointed cone. Duality then gives the H-representation $P = (P^*)^* = \mathrm{H}(L_{v_1}, \dots, L_{v_\ell})$.

In Lemma 2 we showed how to pass from rational H- to rational R- representations (and vice-versa) when the cone $P \subseteq V$ is given by $n = \dim(V)$ independent linear inequalities, or equivalently, is generated by n linearly independent elements. In this Section we consider the general case, assuming only that P is a pointed rational cone. As above, we assume that V is an n-dimensional real vector space equipped with a fixed $\mathbb{Q}$-structure $V_\mathbb{Q} \subseteq V$ and a non-degenerate symmetric rational bilinear form $\langle\ ,\ \rangle$ on V.

We assume $P = \mathrm{H}(L_1, \dots, L_{n+m})$ is a rational H-representation of a pointed cone P. As P is pointed, we may assume that $L_1, \dots, L_n$ are independent linear forms. Our final aim is to find a rational R-representation $P = C[r_1, \dots, r_\ell]$, but we find the r_i inductively. By the dual basis construction of Lemma 2, there is a rational R-representation of the cone $P_0 := \mathrm{H}(L_1, \dots, L_n) = C[B_0]$ where $B_0 := \{b_1, \dots, b_n\} \subseteq V_Q$. Inductively we assume that $P_i := \mathrm{H}(L_1, \dots, L_{n+i}) = C[B_i]$, where $B_i := \{s_1, \dots, s_{j_i}\} \subseteq V_Q$.

Lemma 7 (Double Description Method)**.** *Given a rational* R*-representation of a pointed rational cone $P = C[B]$ and a non-trivial rational linear form $L : V \to \mathbb{R}$ given by $L(x) := \langle v, x\rangle$, where $v \in V_\mathbb{Q}$, define the (possibly empty) subsets of B*

$$B_+ := \{b : L(b) > 0\}, \quad B_- := \{b : L(b) < 0\}, \quad B_0 := \{b : L(b) = 0\}. \tag{1.2}$$

For each pair $(b,b') \in B_+ \times B_-$*, define* $v_{bb'} := L(b)b' - L(b')b \in V_{\mathbb{Q}}$*. Then we have the rational* R*-representation*

$$P' := P \cap \{x \in V : \ L(x) \geq 0\} = C[B'],$$

where

$$B' := B_+ \cup B_0 \cup \{v_{bb'} : \ (b,b') \in B_+ \times B_-\} \subseteq V_{\mathbb{Q}}. \tag{1.3}$$

Proof. We follow the proof in [FP96, Lemma 3], taking care that all constructions are rational. Since $v_{bb'}$ is a positive combination of $b \in B_+ \subseteq P$ and $b' \in B_- \subseteq P$, we have $v_{bb'} \in P$. Also, $L(v_{bb'}) = 0$, so $v_{bb'} \in P'$. Thus $C[B'] \subseteq P'$.

To show the reverse inclusion, let $x \in P'$. Thus $x \in P = C[B]$ and $L(x) \geq 0$. Normalizing the elements of B we can suppose that

$$L(b) = \begin{cases} +1 & \text{if} \quad b \in B_+, \\ 0 & \text{if} \quad b \in B_0, \\ -1 & \text{if} \quad b \in B_-. \end{cases}$$

As $x \in C[B]$ we can write

$$x = \sum_{b \in J_+} \lambda_b b + \sum_{b \in J_0} \lambda_b b + \sum_{b \in J_-} \lambda_b b, \tag{1.4}$$

where $\lambda_b > 0$, $J_+ \subseteq B_+$, $J_- \subseteq B_-$, $J_0 \subseteq B_0$ $\big($see (1.2)$\big)$. Since $L(x) \geq 0$, we have

$$0 \leq L(x) = \sum_{b \in J_+} \lambda_b L(b) + \sum_{b \in J_-} \lambda_b L(b) = \sum_{b \in J_+} \lambda_b - \sum_{b \in J_-} \lambda_b. \tag{1.5}$$

If J_- is empty, then $x \in C[B_+ \cup B_0] \subseteq C[B']$ by (1.3). If J_- is not empty, choose any $b' \in J_-$. For each $b \in J_+ \subseteq B_+$,

$$v_{bb'} := (L(b))b' - (L(b'))b = b + b'.$$

By (1.5),

$$\sum_{b \in J_+} \lambda_b \geq \sum_{b \in J_-} \lambda_b \geq \lambda_{b'} > 0.$$

Thus, there are positive numbers $\{t_{bb'}\}_{b \in J_+}$ such that $\lambda_b \geq t_{bb'}$ for all $b \in J_+$ and $\sum_{b \in J_+} t_{bb'} = \lambda_{b'}$. Now we can write x in (1.4) as

$$\begin{aligned} x &= \sum_{b \in J_+} (\lambda_b - t_{bb'})b + \sum_{b \in J_0} \lambda_b b + \sum_{b \in J_-} \lambda_b b + \sum_{b \in J_+} t_{bb'}(b + b') - \Big(\sum_{b \in J_+} t_{bb'}\Big) b' \\ &= \sum_{b \in \widetilde{J}_+} \lambda'_{bb'} b + \sum_{b \in J_0} \lambda_b b + \sum_{b \in J_- \setminus \{b'\}} \lambda_b b + \sum_{b \in J_+} t_{bb'} v_{bb'}, \end{aligned}$$

where $\lambda'_{bb'} := \lambda_b - t_{bb'} \geq 0$ and $\widetilde{J}_+ := \big\{b \in J_+ : \ \lambda'_{bb'} > 0\big\}$. Comparing with (1.4), we see that we have succeeded in removing one element of J_-, namely b', by adding generators of the form $v_{bb'}$ and as $L(v_{bb'}) = 0$, we can repeat the process until all elements of J_- have been removed. □

1.3 Deleting redundancies in an R-representation

An R-representation $P = C[B]$ produced by Lemma 7 from an H-representation could have many redundant generators, even if the H-representation has no redundancies. In this Subsection we give an account of Fukuda and Prodon's method for finding an irredundant R-representation [FP96, Lemma 8] if P is pointed.

Let $P = \mathrm{H}(L_1, \dots, L_m)$ be an H-representation of a polyhedral cone, with $L_j(x) := \langle v_j, x\rangle$. For any subset $S \subseteq P$, we define

$$Z_P(S) := \big\{j \in \{1, \dots, m\} : \; L_j(r) = 0 \; \forall r \in S\big\}. \tag{1.6}$$

Note that $Z_P(S) = \bigcap_{r\in S} Z_P(r)$, where $Z_P(r) := Z_P(\{r\})$, and that $Z_P(r) \neq \{1, \dots, m\}$ if P is pointed and $r \neq 0$.

Recall that a subset $\mathcal{F} \subseteq P$ is a *face* of P if either $\mathcal{F} = \varnothing$ or $\mathcal{F} = P$, or if there exists a linear form $L : V \to \mathbb{R}$ such that $\mathcal{F} = \{x \in P : L(x) = 0\}$ and $P \subseteq \{x \in V : L(x) \geq 0\}$. A *facet* of P is a *maximal proper face* of P. The intersection of a finite number of faces of P is still a face of P [G03, §2.4, Theorem 10]. Therefore for any subset $J \subseteq \{1, \dots, m\}$ of cardinality ℓ,

$$\mathcal{F}_J := \ker(\mathcal{L}_J) \cap P, \qquad \mathcal{L}_J : V \to \mathbb{R}^\ell, \qquad \mathcal{L}_J(x) := \big(L_j(x)\big)_{j\in J}, \tag{1.7}$$

is a face of P. Note that a face of a pointed polyhedral cone is again a pointed polyhedral cone. If J is empty, we regard $\mathcal{L}_J$ as the zero map and so $\mathcal{F}_\varnothing = P$.

For $Q \subseteq V$ a convex set, by *the relative interior* $\mathrm{relint}(Q)$ of Q we mean its interior within the smallest affine subspace containing Q, and we shall also denote by Q° the interior of Q with respect to the vector space V.

Proposition 8. *Let $Q_1, \dots, Q_\ell$ be a finite list of convex subsets of V. Suppose that the relative interiors $\mathrm{relint}(Q_i)$ have at least one point in common. Then $\mathrm{relint}\big(\bigcap_{i=1}^\ell Q_i\big) = \bigcap_{i=1}^\ell \mathrm{relint}(Q_i)$.*

Proof. See [R70, Theorem 6.5]. □

The next proposition shows that any face of P is of the form (1.7) for a certain subset $J \subseteq \{1, \dots, m\}$.

Proposition 9. *Let $P = \mathrm{H}(L_1, \dots, L_m) \subseteq V$ be an H-representation of a polyhedral cone. Then the following hold.*

(i) Either P contains an interior point or P lies in a proper subspace of V.

(ii) If $\mathcal{F} \subseteq P$ is a non-empty face of P then $\mathcal{F} = \mathcal{F}_{Z_P(v)}$ for any v in the relative interior of $\mathcal{F}$, with notation as in (1.6) and (1.7).

Proof. This is a particular case of Theorem 4.15 in [B08]. □

We shall call $\mathcal{F}_J$ a t-dimensional face, or simply a t-face of P, if the $\mathbb{R}$-span of $\mathcal{F}_J$ has dimension t. Note that if I and J are subsets of $\{1, \dots, m\}$, then $\mathcal{F}_I \cap \mathcal{F}_J = \mathcal{F}_{I\cup J}$. Thus, for any subset $S \subseteq P$ there is a minimal face of P containing it.

Lemma 10. *Let $P = \mathrm{H}(L_1, \dots, L_m) \subseteq V$ be an* H*-representation of a polyhedral cone, and let $S \subseteq P$ be any subset of P. Then the minimal face of P containing S is $\mathcal{F}_{Z_P(S)}$, with notation as in* (1.6) *and* (1.7).

Proof. If $r \in S$, it is clear that $r \in \mathcal{F}_{Z_P(S)}$, so $S \subseteq \mathcal{F}_{Z_P(S)}$. Let $\mathcal{F}_J$ be any face of P such that $S \subseteq \mathcal{F}_J = \ker(\mathcal{L}_J) \cap P$. Thus $L_j(r) = 0$ for all $j \in J$ and $r \in S$. So $J \subseteq Z_P(S)$, and thus $\mathcal{F}_{Z_P(S)} \subseteq \mathcal{F}_J$. $\square$

Note that by Proposition 9 (i), any polyhedral cone $P = \mathrm{H}(\mathcal{T})$ where $\mathcal{T} := \{L_1, \dots, L_m\}$ can be written as $P = \mathrm{H}(T_1, \dots, T_\ell, \pm M_1, \dots, \pm M_t)$ and there exists $x_0 \in V$ with $T_j(x_0) > 0$ $(1 \le j \le \ell)$ and $M_s(x_0) = 0$ $(1 \le s \le t)$. If $P = \{0\}$ is trivial, we regard $\ell = 0$ and conditions involving T_j become vacuously valid, of course. We record this as follows (see [Sc86, Section 8.2] for a complete proof).

Lemma 11. *Let $P = \mathrm{H}(T_1, \dots, T_\ell, \pm M_1, \dots, \pm M_t)$ be a polyhedral cone and suppose there is some $x_0 \in V$ with $T_j(x_0) > 0$ $(1 \le j \le \ell)$, $M_s(x_0) = 0$ $(1 \le s \le t)$. Then the linear span of P is*

$$\mathrm{span}(P) = \{x \in V : \ M_s(x) = 0, \ 1 \le s \le t\} =: S$$

In particular $\dim(P) = \dim(S) = \dim\big(\bigcap_{s=1}^{t} \ker(M_s)\big)$.

Let T and M be non-zero linear forms on V. Then $\mathrm{relint}(\mathrm{H}(T)) = \{x \in V : T(x) > 0\}$ and $\mathrm{relint}(\mathrm{H}(\pm M)) = \mathrm{H}(\pm M)$ [R70, Theorem 6.7].

Corollary 12. *Let P and x_0 be as in Lemma 11. Then the following hold.*

(i) $\mathrm{relint}(P) = \{x \in V : T_j(x) > 0\} \cap \mathrm{span}(P)$.

(ii) For $1 \le i \le \ell$, define $P_i := \mathrm{H}(T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_\ell, \pm M_1, \dots, \pm M_t)$. Then $\mathrm{relint}(P_i) \cap \big\{x \in V : T_i(x) = 0\big\} \neq \varnothing$ if and only if $P \neq P_i$.

Proof. As in the remark just before this Corollary, (i) is consequence of Proposition 8 and Lemma 11. To prove the *if* part in (ii), assume that $P \neq P_i$, *i.e.* $P \subsetneq P_i$. Then there exists $x_i \in V$ such that $T_j(x_i) \ge 0$ $(1 \le j \neq i \le \ell)$, $M_s(x_i) = 0$ $(1 \le s \le t)$ and $T_i(x_i) < 0$. Let $z := T_i(x_0)x_i - T_i(x_i)x_0$, so $T_i(z) = 0$. Moreover, $T_j(z) = T_j(x_i)T_i(x_0) - T_j(x_0)T_i(x_i) > 0$ for $1 \le j \neq i \le \ell$ and $M_s(z) = 0$ for $1 \le s \le t$. Hence $z \in \mathrm{relint}(P_i) \cap \{x \in V : T_i(x) = 0\}$.

Conversely, if $\mathrm{relint}(P_i) \cap \{T_i(x) = 0\} \neq \varnothing$, then by (i) there is $z \in V$ with $T_j(z) > 0$ $(1 \le j \neq i \le \ell)$ and $M_s(z) = 0 = T_i(z)$ $(1 \le s \le t)$. Consider $p_\lambda(x_0) := x_0 + \lambda(z - x_0)$. We claim that there is $\lambda > 1$ such that $p_\lambda(x_0) \in P_i \setminus P$. Indeed, $T_j(p_\lambda(x_0)) = T_j(x_0) + \lambda\big(T_j(z) - T_j(x_0)\big)$ for $1 \le j \le \ell$. Thus, $T_i(p_\lambda(x_0)) = (1-\lambda)T_i(x_0) < 0$ for $\lambda > 1$. If $T_j(z) \ge T_j(x_0)$ for all $j \in \{1, \dots, \ell\}$ with $j \neq i$ then $T_j(p_\lambda(x_0)) > 0$. Otherwise, if $T_{j_0}(z) < T_{j_0}(x_0)$ for some $j_0 \in \{1, \dots, \ell\}$ with $j_0 \neq i$, we consider

$$\lambda := \min\left\{ \frac{T_j(x_0)}{T_j(x_0) - T_j(z)} \ : \ 0 < T_j(z) < T_j(x_0) \right\} > 1.$$

Then $T_j(p_\lambda(x_0)) > 0$ for all $j \in \{1, \dots, \ell\}$ with $j \neq i$. Finally, it is clear that $M_s(p_\lambda(x_0)) = 0$ for $1 \le s \le t$. Hence $P \neq P_i$. $\square$

The following Proposition gives a characterization of the facets of a polyhedral cone in terms of its irredundant H-representation [Sc86, Theorem 8.1].

Proposition 13. *Let $P = \mathrm{H}(T_1, \ldots, T_\ell, \pm M_1, \ldots, \pm M_t) \neq \{0\}$ be an irredundant* H*-representation of a non-trivial polyhedral cone and suppose there is some $x_0 \in V$ with $T_j(x_0) > 0$ $(1 \leq j \leq \ell)$, $M_s(x_0) = 0$ $(1 \leq s \leq t)$. Then $\mathcal{F}$ is a facet of P if and only if $\mathcal{F} = \big\{x \in P : T_i(x) = 0\big\}$ for some $1 \leq i \leq \ell$.*

Proof. Let $\mathcal{F}$ be a facet of P, then by Proposition 9 (ii) $\mathcal{F} = \mathcal{F}_{Z_P(v)}$ for any v in the relative interior of $\mathcal{F}$, using notation as in (1.6) and (1.7). As the representation is irredundant, if the cardinality $\#Z_p(v) > 1$, then $\mathcal{F}$ is a proper face of the face $\{x \in P : T_j(x) = 0\} \neq P$ for any $j \in Z_p(v)$. But a facet is a maximal proper face, so $\#Z_p(v) = 1$ and thus $\mathcal{F} = \{x \in P : T_i(x) = 0\}$ for some $1 \leq i \leq \ell$. Conversely, suppose that $\mathcal{F} := \{x \in P : T_i(x) = 0\}$ for $1 \leq i \leq \ell$. It is sufficient to prove that $\dim(\mathcal{F}) = \dim(P) - 1$, so $\mathcal{F}$ is a facet of P. Indeed, let P_i be as in Corollary 12. Since the H-representation of P is irredundant, $P \neq P_i$, and hence by Corollary 12 (ii), $\mathrm{relint}(P_i) \cap \{x \in V : T_i(x) = 0\} \neq \varnothing$. Thus there is a point $z \in V$ such that $T_i(z) = 0 = M_s(z)$ for $1 \leq s \leq t$ and $T_j(z) > 0$ for $1 \leq j \neq i \leq \ell$. Therefore by Lemma 11, $\dim(\mathcal{F}) = \dim(P) - 1$ as $\mathcal{F} = \big\{x \in P : T_i(x) = 0\big\} = P_i \cap \big\{x \in V : T_i(x) = 0\big\}$. □

We will be specially interested in one- and two-dimensional faces of pointed cones. Write $r \sim r'$, and call r and r' equivalent $(r,\ r' \in V)$, if $r = \lambda r'$ for some $\lambda > 0$. An element r of a cone P is called extreme (for P) if $r \neq 0$ and

$$\big[r = \lambda_1 r_1 + \lambda_2 r_2,\ \lambda_i \geq 0,\ r_i \in P,\ r_1 \not\sim r_2\big] \quad \Longrightarrow \quad \big[\lambda_1 = 0 \text{ or } \lambda_2 = 0\big].$$

We shall call two inequivalent extreme elements $r, r' \in P$ adjacent (for P) if the minimal face of P containing r and r' has no extreme elements (for P) other than those equivalent to r or r'.

Proposition 14 (Fukuda-Prodon [FP96, Prop. 4])**.** *Let $P = \mathrm{H}(L_1, \ldots, L_m) \subseteq V$ be an* H*-representation of a pointed cone P. For $r \in P$, $r \neq 0$, let $A := Z_P(r)$ and let $\mathcal{L}_A$ be as in (1.6) and (1.7). Then the following hold.*

(i) $\mathrm{rank}(\mathcal{L}_{A \cup \{j\}}) = \mathrm{rank}(\mathcal{L}_A) + 1$ *for all $j \notin A$.*

(ii) The face $\mathcal{F}_A := \ker(\mathcal{L}_A) \cap P$ of P contains a basis of $\ker(\mathcal{L}_A)$.

(iii) If $\dim\big(\ker(\mathcal{L}_A)\big) \geq 2$, *then r is a positive combination of two inequivalent, non-zero w_1, $w_2 \in P$, with* $\mathrm{rank}(\mathcal{L}_{Z_P(w_i)}) > \mathrm{rank}(\mathcal{L}_A)$ $(i = 1, 2)$.

Proof. To prove (i), note that L_j is not a linear combination of $\{L_i\}_{i \in A}$, for otherwise $L_j = \sum_{i \in A} \lambda_i L_i$ for some $\lambda_i \in \mathbb{R}$. Hence, $L_j(r) = 0$, contradicting $j \notin A := Z_P(r)$. Hence $\mathrm{rank}(\mathcal{L}_{A \cup \{j\}}) = \mathrm{rank}(\mathcal{L}_A) + 1$.

To prove (ii), note that since $\mathcal{L}_A(r) = 0$, the subspace $\ker(\mathcal{L}_A)$ contains a basis $r, v_2, v_3, \ldots, v_t$ of $\ker(\mathcal{L}_A)$. For $\alpha_i \neq 0$, $\alpha_i \in \mathbb{R}$ $(2 \leq i \leq t)$, let $w_i := r + \alpha_i v_i$. Then $r, w_2, \ldots, w_t$ is also a basis of $\ker(\mathcal{L}_A)$. If $v_i \in P$, take $\alpha_i := 1$. If $v_i \notin P$, there exists some j with $L_j(v_i) < 0$. As $v_j \in \ker(\mathcal{L}_A)$, we have $j \notin A$. Hence we can take α_i such that

$$0 < \alpha_i \leq \min\{-L_j(r)/L_j(v_i) : \text{ for } j \text{ with } L_j(v_i) < 0\}.$$

Thus, there is a basis $r, w_2, \ldots, w_t$ of $\ker(\mathcal{L}_A)$ consisting of elements of $\mathcal{F}_A$.

To prove (iii), let $F := \ker(\mathcal{L}_A)$. We first prove that there exists $v \in F$ with $v \notin P$ and $-v \notin P$. Indeed, as $\dim(F) \geq 2$ we have $\operatorname{rank}(\mathcal{L}_A) \leq n-2$, where $n := \dim(V)$. Since P is pointed, Lemma 3 yields $n = \operatorname{rank}(\mathcal{L}_{\{1,\ldots,m\}})$. Thus $2+\operatorname{rank}(\mathcal{L}_A) \leq \operatorname{rank}(\mathcal{L}_{\{1,\ldots,m\}})$. Therefore there exist $\ell \notin A$ and $j \notin (A \cup \{\ell\})$ such that $\operatorname{rank}(\mathcal{L}_{A\cup\{\ell,j\}}) = \operatorname{rank}(\mathcal{L}_A)+2$, which implies that $\dim(F) = \dim\big(\ker(\mathcal{L}_A)\big) = \dim\big(\ker(\mathcal{L}_{A\cup\{\ell,j\}})\big) + 2$. Hence there exists $b_1, b_2 \in F$ with $L_\ell(b_1) = 0$, $L_j(b_1) > 0$, $L_j(b_2) = 0$, $L_\ell(b_2) < 0$. Thus, $v := b_1 + b_2 \in F$ satisfies $v \notin P$ and $-v \notin P$.

Returning to the proof of (iii), note that r and v are linearly independent, since $v = \lambda r$ for some $\lambda \in \mathbb{R}$ implies that $v \in P$ (if $\lambda \geq 0$) or $-v \in P$. We can therefore choose α_1 and α_2 small enough positive real numbers so that, as in the proof of (ii), $w_1 := r + \alpha_1 v \in \mathcal{F}_A$ and $w_2 := r - \alpha_2 v \in \mathcal{F}_A$. As

$$r = \frac{\alpha_2}{\alpha_1+\alpha_2} w_1 + \frac{\alpha_1}{\alpha_1+\alpha_2} w_2, \qquad v = \frac{1}{\alpha_1+\alpha_2} w_1 - \frac{1}{\alpha_1+\alpha_2} w_2,$$

the w_i are linearly independent. Since $w_i \in \mathcal{F}_A$, we have $A \subseteq Z_P(w_i)$. Moreover, from the proof above of the claim, $\ell \in Z_P(w_1) \setminus A$ and $j \in Z_P(w_2) \setminus A$. Hence, (i) implies that $\operatorname{rank}(\mathcal{L}_{Z_P(w_i)}) > \operatorname{rank}(\mathcal{L}_A)$ $(i = 1, 2)$. □

The following Lemma appears in [FP96], although our proof is a bit different.

Lemma 15 (Fukuda-Prodon [FP96, p. 99])**.** *Let $P = \mathrm{H}(L_1, \ldots, L_m) \subseteq V$ be an* H*-representation of a pointed cone P. Then the following hold.*

a) If $b \in P$, $b \neq 0$, then $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) \leq n-1$, where $n := \dim(V)$. Moreover, b is extreme for P if and only if $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) = n-1$, and this happens if and only if the minimal face of P containing b is one-dimensional.

b) Any non-zero $b \in P$ is a positive combination of extreme elements for P.

c) If b and b' are two inequivalent extreme elements for P, then $\operatorname{rank}\big(\mathcal{L}_{Z_P(\{b,b'\})}\big) \leq n-2$. Moreover, b and b' are adjacent for P if and only if $\operatorname{rank}\big(\mathcal{L}_{Z_P(\{b,b'\})}\big) = n-2$, and this happens if and only if the minimal face of P containing b and b' is two-dimensional.

d) $P = C[B]$ and B is irredundant if and only if every element of B is extreme for P and every extreme element for P is equivalent to exactly one element of B.

Proof. By Lemma 10, $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) = n-1$ (resp., $\operatorname{rank}\big(\mathcal{L}_{Z_P(\{b,b'\})}\big) = n-2$) if and only if the minimal face containing b (resp., b and b') is one-dimensional (resp., two-dimensional). Hence the final claims in a) and c) follow from the second claims. To prove the first claim in a), note that as P is pointed and $b \in P$, $b \neq 0$, we have $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) \leq n-1$ by Lemma 3 and Proposition 14(i).

To prove the second claim in a), suppose b is extreme. If $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) < n-1$, then $\dim\big(\ker(\mathcal{L}_{Z_P(b)})\big) \geq 2$, so by (iii) in Proposition 14, b is not extreme. Conversely, if $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) = n-1$, then $\dim\big(\ker(\mathcal{L}_{Z_P(b)})\big) = 1$, and $b \neq 0$ as P

is pointed. Therefore $\ker(\mathcal{L}_{Z_P(b)}) = \{\lambda b : \lambda \in \mathbb{R}\}$. Thus, $\mathcal{F}_{Z_P(b)} := \ker(\mathcal{L}_{Z_P(b)}) \cap P = \{\lambda b : \lambda \geq 0\}$, since $b \in \mathcal{F}_{Z_P(b)}$ and P is pointed. Suppose $b = \lambda_1 w_1 + \lambda_2 w_2$ with $w_i \in P$ and $\lambda_i > 0$. For $j \in Z_P(b)$ we have $0 = L_j(b) = \lambda_1 L_j(w_1) + \lambda_2 L_j(w_2)$ and $L_j(w_i) \geq 0$ as $w_i \in P$. Hence $L_j(w_i) = 0$ and so $w_i \in \mathcal{F}_{Z_P(b)}$. Hence $w_1 \sim b \sim w_2$, and so b is extreme.

Next we prove b). Let D be the set of non-zero elements of P which cannot be written as a positive combination of extreme elements for P. If $D \neq \varnothing$, let $b \in D$ be such that $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) \geq \operatorname{rank}(\mathcal{L}_{Z_P(d)})$ for all $d \in D$. If $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) = n-1$, then b is extreme by a), contradicting $b \in D$. If $\operatorname{rank}(\mathcal{L}_{Z_P(b)}) < n-1$, then by (iii) in Proposition 14, $b = \alpha_1 w_1 + \alpha_2 w_2$ with $\alpha_i > 0$ and $w_i \in P$ inequivalent and non-zero. Moreover $\operatorname{rank}(\mathcal{L}_{Z_P(w_i)}) > \operatorname{rank}(\mathcal{L}_{Z_P(b)})$. By the maximality of $\operatorname{rank}(\mathcal{L}_{Z_P(b)})$, each w_i can be written as a positive combination of extreme elements for P, contradicting $b \in D$. Hence $D = \varnothing$, as claimed.

We now prove c). Note that if b is extreme, then a) implies that b generates $\ker(\mathcal{L}_{Z_P(b)})$. As P is pointed, this shows that $Z_P(b) \neq Z_P(b')$ if b and b' are inequivalent extreme elements for P. Hence the first claim in c) from a repeated application of Proposition 14(i). To prove the second claim in c), let b and b' be adjacent. By Lemma 10, $\mathcal{F} := \ker(\mathcal{L}_{Z_P(\{b,b'\})}) \cap P$ is the minimal face of P containing b and b'. If we have $\operatorname{rank}(\mathcal{L}_{Z_P(\{b,b'\})}) < n-2$, then $k := \dim(\mathcal{F}) \geq 3$. By b), applied to the polyhedral cone $\mathcal{F}$, at least $k \geq 3$ extreme inequivalent elements are necessary to generate $\mathcal{F}$. Thus b and b' are not adjacent for P.

Conversely, if $\operatorname{rank}(\mathcal{L}_{Z_P(\{b,b'\})}) = n-2$, then $\dim\bigl(\ker(\mathcal{L}_{Z_P(\{b,b'\})})\bigr) = 2$. Since P is pointed and b, b' are inequivalent extreme elements for P which generate $\ker(\mathcal{L}_{Z_P(\{b,b'\})})$, any $\gamma \in \mathcal{F}$ can be written $\gamma = \alpha_1 b + \alpha_2 b'$ with $\alpha_1, \alpha_2 \in \mathbb{R}$. Since $Z_P(b) \neq Z_P(b')$, let $i \in Z_P(b) \setminus Z_P(b')$ and $j \in Z_P(b') \setminus Z_p(b)$. Thus, $L_i(b) = 0 = L_j(b')$, $L_i(b') > 0$, $L_j(b) > 0$. Moreover, $0 \leq L_i(\gamma) = \alpha_1 L_i(b) + \alpha_2 L_i(b') = \alpha_2 L_i(b')$, so $\alpha_2 \geq 0$, similarly $0 \leq L_j(\gamma) = \alpha_1 L_j(b) + \alpha_2 L_j(b') = \alpha_1 L_j(b)$, so $\alpha_1 \geq 0$. Hence $\gamma \in \mathcal{F}$ is a non-negative combination of b and b' and there are no other extreme elements in $\mathcal{F}$. Thus b and b' are adjacent for P.

To prove d), assume first that $P = C[B]$ and B is irredundant. To begin with, we show that every extreme element $r \in P$ is equivalent to an element of B. Indeed, write $r = \sum_{b \in B} \lambda_b b$ with $\lambda_b \geq 0$. As $r \neq 0$, we can write

$$r = \lambda_{b_0} b_0 + \sum_{\substack{b \in B \\ b \neq b_0}} \lambda_b b =: w_1 + w_2,$$

where $\lambda_{b_0} > 0$. We cannot have $w_1 = 0$ since $b_0 = 0$ cannot belong to any irredundant set. If $w_1 \sim w_2$, then $C[B] = C[B \setminus \{b_0\}]$, contradicting the non-redundancy of B. Since $r = w_1 + w_2$ is assumed extreme, we must have $w_2 = 0$, and so $r = w_1 = \lambda_{b_0} b_0$ is equivalent to b_0, an element of B as claimed. Let $B' \subseteq B$ consist of the extreme elements for P that belong B, so any extreme element for P is equivalent to some element of B'. Then by b), $P = C[B']$. As B is irredundant, $B' = B$, *i.e.* all elements of B are extreme. No two of them can be equivalent, as B is irredundant.

To prove the converse, note that b) implies $P = C[B]$ since B is assumed to contain a positive multiple of every extreme element. Let us show that B is irredundant. Let $c \in B$ be a redundant element of B. Thus we can write

$c = \sum_{i=1}^{\ell} \lambda_i b_i$ with $b_i \in B \setminus \{c\}$ and $\lambda_i > 0$. Pick c such that ℓ is minimal. Since B consists by assumption only of extreme elements, $c \neq 0$, so $\ell > 0$. If $\ell = 1$, then c would be an extreme element equivalent to two distinct elements of B (namely c and b_1), contradicting our uniqueness hypothesis. As $\ell \geq 2$ we can write

$$c = \lambda_\ell b_\ell + \sum_{i=1}^{\ell-1} \lambda_i b_i =: w_1 + w_2.$$

We cannot have $w_2 = 0$, for then $0 = \sum_{i=1}^{\ell-1} \lambda_i L_j(b_i)$, which implies $L_j(b_i) = 0$ for all linear forms L_j and all $i \leq \ell - 1$. As $\ell > 1$, this contradicts the assumption that P is pointed. Furthermore, $w_1 \not\sim w_2 := \sum_{i=1}^{\ell-1} \lambda_i b_i$ by the minimality of ℓ. Hence $c = w_1 + w_2$ is not extreme, contradicting our hypothesis on B. Thus, B is irredundant. $\square$

Remark 16. *Let $\mathcal{L} : V \to \mathbb{R}^m$ be the linear transformation $x \mapsto \mathcal{L}(x) := (L_1(x), \dots, L_m(x))$, $L_j(x) := \langle v_j, x \rangle$, $v_j \in V_{\mathbb{Q}}$. Recall that we assumed that V has a fixed $\mathbb{Q}$-structure $V_{\mathbb{Q}}$, that is, V has a $\mathbb{Q}$-vector subspace $V_{\mathbb{Q}} \subseteq V$ such that any $\mathbb{Q}$-basis for $V_{\mathbb{Q}}$ is an $\mathbb{R}$-basis for V. Thus if we consider the restriction of $\mathcal{L}$ to $V_{\mathbb{Q}}$, $\mathcal{L}_{\mathbb{Q}} := \mathcal{L}|_{V_{\mathbb{Q}}} : V_{\mathbb{Q}} \to \mathbb{Q}^m$, we have that $\dim_{\mathbb{R}}(\ker(\mathcal{L})) = \dim_{\mathbb{Q}}(\ker(\mathcal{L}_{\mathbb{Q}}))$, and so $\mathrm{rank}_{\mathbb{R}}(\mathcal{L}) = \mathrm{rank}_{\mathbb{Q}}(\mathcal{L}_{\mathbb{Q}})$. So given a rational* H*-representation of a cone P, to determine when a generator $b \in V_{\mathbb{Q}}$ is extreme for P, it is sufficient to apply Lemma 15 a). Similarly, given two extreme elements b, $b' \in P \cap V_{\mathbb{Q}}$ both inequivalent, to determine when b and b' are adjacent for P, it is sufficient apply Lemma 15 c).*

Let $P = C[B]$ be an irredundant R-representation of a pointed rational cone $P \subseteq V$. By Lemma 15, any element of B is extreme for P. Also we assume P has H-representation $P = \mathrm{H}(L_1, \dots, L_m)$ not necessarily irredundant. Let $\mathrm{H}(L) := \{x \in V : L(x) := \langle v, x \rangle \geq 0\}$ be a rational half-space in V for some $v \in V_{\mathbb{Q}}$. The following Lemma is a strengthening of Lemma 7, which determine an irredundant rational R-representation of $P' = P \cap \mathrm{H}(L)$ from the irredundant rational R-representation of $P = C[B]$.

Lemma 17 (Fukuda-Prodon [FP96, Lemma 8])**.** *Given an irredundant rational* R*-representation of a pointed rational cone $P = C[B]$ and a non-trivial rational linear form $L : V \to \mathbb{R}$ given by $L(x) := \langle v, x \rangle$, where $v \in V_{\mathbb{Q}}$, define the subsets of B,*

$$B_+ := \{b : L(b) > 0\}, \qquad B_- := \{b : L(b) < 0\}, \qquad B_0 := \{b : L(b) = 0\}.$$

Define $v_{bb'} := L(b)b' - L(b')b \in V_{\mathbb{Q}}$, for each pair $(b, b') \in \Lambda$, where

$$\Lambda := \{(b, b') \in B_+ \times B_- : b \text{ and } b' \text{ are adjacent for } P\}. \tag{1.8}$$

Then $P' := P \cap \mathrm{H}(L) = C[\widetilde{B}]$, where

$$\widetilde{B} := B_+ \cup B_0 \cup \left\{v_{bb'} : (b, b') \in \Lambda\right\} \subseteq V_{\mathbb{Q}}, \tag{1.9}$$

is an irredundant rational R*-representation of the cone P'.*

Proof. By Lemma 15 d), it is sufficient to prove that every element in $\widetilde{B}$ is extreme for P', and that every extreme element for P' is equivalent to exactly one element of $\widetilde{B}$. Firstly, we will prove that every element of $\{v_{bb'} : (b, b') \in \Lambda\}$ is extreme for P'. Indeed, $P' = \mathrm{H}(L_1, \ldots, L_m, L_{m+1})$ is a rational H-representation of $P' = P \cap \mathrm{H}(L)$, where $P = \mathrm{H}(L_1, \ldots, L_m)$ and $L_{m+1} := L$. As $L_{m+1}(v_{bb'}) := L(v_{bb'}) = L(b)L(b') - L(b')L(b) = 0$, by definition of $Z_{P'}(v_{bb'})$ $\big($see (1.6)$\big)$, we have

$$Z_{P'}(v_{bb'}) = \big\{j \in \{1, \ldots, m\} : L_j(v_{bb'}) = 0\big\} \cup \big\{m + 1\big\}.$$

Moreover for each $j \in \{1, \ldots, m\}$, $L_j(v_{bb'}) = 0$ if and only if $L_j(b) = 0 = L_j(b')$. Therefore $Z_{P'}(v_{bb'}) = \big(Z_P(b) \cap Z_P(b')\big) \cup \{m + 1\}$. Also note that L_{m+1} is not a linear combination of $\{L_j\}_{j\in S}$, for any subset $S \subseteq Z_P(b) \cap Z_P(b')$. So $\mathrm{rank}\big(\mathcal{L}_{Z_{P'}(v_{bb'})}\big) = \mathrm{rank}\big(\mathcal{L}_{Z_P(b)\cap Z_P(b')}\big) + 1$. By Lemma 15, $\mathrm{rank}\big(\mathcal{L}_{Z_P(b)\cap Z_P(b')}\big) = n - 2$ for each $(b, b') \in \Lambda$, so

$$\mathrm{rank}\big(\mathcal{L}_{Z_{P'}(v_{bb'})}\big) = \mathrm{rank}\big(\mathcal{L}_{Z_P(b)\cap Z_P(b')}\big) + 1 = (n - 2) + 1 = n - 1.$$

Therefore for any $(b, b') \in \Lambda$, $v_{bb'}$ is extreme for P'.

Now let $b \in B_+ \cup B_0$. If b is not extreme for P', then by definition $b = \lambda_1 b_1 + \lambda_2 b_2$ where $\lambda_i > 0$, $b_i \in P'$, with b_1 and b_2 both inequivalent. Since $b_i \in P' = P \cap \mathrm{H}(L)$, we have $b_i \in P = C[B]$. Therefore b is not extreme for P, contradicting Lemma 15 d). Thus, every element of $B_+ \cup B_0$ is extreme for P'. Hence every element of $B' = B_+ \cup B_0 \cup \big\{v_{bb'} : (b, b') \in \Lambda\big\}$ is extreme for P'.

Finally we want to prove that every extreme element for P' is equivalent to exactly one element of $\widetilde{B}$. By Lemma 7, we know that $P' = C[B']$ where $B' := B_+ \cup B_0 \cup \big\{v_{bb'} : (b, b') \in B_+ \times B_-\big\}$. So by Lemma 15 b), any extreme element x for $P' = C[B']$ is equivalent to exactly one element of B'. Thus if x is not equivalent to one element in $\widetilde{B}$, then $x = \lambda v_{bb'}$ for some $\lambda > 0$, such that $(b, b') \in B_+ \times B_-$ with b and b' not adjacent for P. Lemma 15 implies $\mathrm{rank}(\mathcal{L}_{Z_P(b)\cap Z_P(b')}) < n - 2$. As $Z_{P'}(x) = Z_{P'}(v_{bb'}) = (Z_P(b) \cap Z_P(b')) \cup \{m+1\}$, and $L_{m+1} := L$ is not a linear combination of $\{L_i\}_{i\in S}$ for any S subset in $Z_P(b) \cap Z_P(b')$, we have

$$\mathrm{rank}(\mathcal{L}_{Z_{P'}}(x)) = \mathrm{rank}(\mathcal{L}_{Z_{P'}}(v_{bb'})) = \mathrm{rank}(\mathcal{L}_{Z_P(b)\cap Z_P(b')}) + 1 < n - 2 + 1 = n - 1.$$

So x is not extreme for P', contradicting our assumption. □

Using Lemma 17 repeatedly, we can obtain an irredundant rational R-representation of any pointed rational cone P in V from its rational H-representation.

Proposition 18. *Let $P = \mathrm{H}(L_1, \ldots, L_{n+m}) \subseteq V$ be an* H*-representation of a pointed rational cone P, $n = \dim(V)$. Then there exists an irredundant rational* R*-representation $P = C[B]$, $B \subseteq V_{\mathbb{Q}}$.*

Proof. Since P is pointed, by Lemma 3 we can assume that $L_1, \ldots, L_n$ are linearly independent rational linear forms. Lemma 2, applied to $P_0 := \mathrm{H}(L_1, \ldots, L_n)$, shows that there exists an irredundant rational R-representation $P_0 = C[B_0]$, $B_0 \subseteq V_{\mathbb{Q}}$. For $j \geq 1$, let $P_j := P_{j-1} \cap \mathrm{H}(L_{n+j})$, where $\mathrm{H}(L_{n+j}) := \{x \in V : L_{n+j}(x) \geq 0\}$. Induction on m and Lemma 17 conclude the proof. □

In §2 we re-state the above proof as an algorithm, and show that duality allows us to apply it to obtain an H-representation from an R-representation.

1.4 Difference of two rational cones

Suppose we are given irredundant rational R- and H- representations of two pointed (closed) n-dimensional rational cones $P, P' \subseteq V$, with $n = \dim(V)$, such that $P \cap P'$ is n-dimensional. In our algorithm to determine a true fundamental domain from a signed one, we will need to express the closure $\overline{P \setminus (P \cap P')}$ (which is not necessarily a polyhedral cone) as a finite union of (possibly empty) pointed n-dimensional rational cones. The following Lemma gives this closure as a finite union of explicit polyhedral cones. In §2 we will re-express the lemma as an algorithm.

Lemma 19. *Let $P = \mathrm{H}(L_1, \ldots, L_m)$ and $P' = \mathrm{H}(T_1, \ldots, T_\ell)$ be rational polyhedral cones in V such that P is pointed. Then the following hold.*

(i) $P \cap P'$ is n-dimensional if and only if for each $j = 1, \ldots, \ell$, the cone

$$P^{(j)} := P \cap \mathrm{H}(T_1, \ldots, T_j) = H(L_1, \ldots, L_m, T_1, \ldots, T_j) \tag{1.10}$$

is n-dimensional.

(ii) For each $j = 1, \ldots, \ell$, define the cones

$$Q_j := P^{(j-1)} \cap \big\{x \in V : T_j(x) < 0\big\}, \tag{1.11}$$

where $P^{(0)} := P$. Then $P \setminus (P \cap P') = \bigcup_{j=1}^{\ell} Q_j$ (disjoint union).

(iii) Assume $P^{(j-1)}$ is n-dimensional and $Q_j \neq \varnothing$. Then the closure $\overline{Q_j} = P^{(j-1)} \cap \big\{x \in V : T_j(x) \leq 0\big\}$ is pointed and n-dimensional. Moreover, $\overline{Q_i} \cap \overline{Q_j}$ has no interior for $1 \leq i \leq \ell,\ i \neq j$.

(iv) Suppose $P \cap P'$ is n-dimensional and $P \not\subseteq P \cap P'$. Then

$$\overline{P \setminus (P \cap P')} = \bigcup_{j,\, Q_j \neq \varnothing} \overline{Q_j}$$

is a finite union of pointed n-dimensional rational polyhedral cones $\overline{Q_j} \subseteq P$ with disjoint interiors.

Proof. We first prove (i). By definition of $P^{(j)}$ in (1.10), it is clear that

$$P \cap P' = P^{(\ell)} \subseteq P^{(\ell-1)} \subseteq \ldots \subseteq P^{(2)} \subseteq P^{(1)} \subseteq P^{(0)} := P.$$

Thus $P \cap P' = P^{(\ell)}$ is n-dimensional if and only if $P^{(j)}$ is so for $j = 1, \ldots, \ell$.

To prove (ii), note that $x \in P \setminus (P \cap P')$ iff $x \in P$ and $T_j(x) < 0$ for some $j \in \{1, \ldots, \ell\}$. Define $j_0(x) := \min\{j : T_j(x) < 0\}$ if some $T_j(x) < 0$ and $j_0(x) := \infty$ otherwise. Then by definition of Q_j in (1.11), we have that $Q_j = \{x \in P : j_0(x) = j\}$. Then

$$P \setminus (P \cap P') = \bigcup_{j=1}^{\ell} \{x \in P : j_0(x) = j\} = \bigcup_{j=1}^{\ell} Q_j. \tag{1.12}$$

Moreover, it is clear that $Q_i \cap Q_j = \varnothing$ for $i \neq j$.

We now prove (iii). By definition of the closure of a set,

$$\overline{Q_j} \subseteq A := P^{(j-1)} \cap \{x \in V : T_j(x) \leq 0\}.$$

To prove the reverse inclusion, let $x \in A$. Since $Q_j \neq \varnothing$, by (1.11) there exists $y \in P^{(j-1)}$ such that $T_j(y) < 0$. We consider $P_x(t) := ty + (1-t)x$, for $t \in (0,1)$. It is clear that $P_x(t) \in Q_j$. Moreover, if $t \to 0$ then $P_x(t) \to x$. Thus $x \in \overline{Q_j}$, so $\overline{Q_j} = A$.

Next we shall prove that $\overline{Q_j}$ is n-dimensional. Suppose the interior $(\overline{Q_j})^\circ = \varnothing$. Recall that in any topological space we have the identity $(A \cap B)^\circ = A^\circ \cap B^\circ$. Since $(\{x \in V : T_j(x) \leq 0\})^\circ = \{x \in V : T_j(x) < 0\}$ [R70, Thm. 6.7], we have

$$\varnothing = (\overline{Q_j})^\circ = (P^{(j-1)})^\circ \cap \{x \in V : T_j(x) < 0\}. \tag{1.13}$$

Since $P^{(j-1)} = (P^{(j-1)})^\circ \cup \partial(P^{(j-1)})$, the disjoint union of its interior with its boundary, by (1.11) and (1.13) we have that

$$\begin{aligned} Q_j &= P^{(j-1)} \cap \{x \in V : T_j(x) < 0\}, \\ &= (\overline{Q_j})^\circ \cup \Big(\partial(P^{(j-1)}) \cap \{x \in V : T_j(x) < 0\}\Big) \\ &= \partial(P^{(j-1)}) \cap \{x \in V : T_j(x) < 0\}. \end{aligned} \tag{1.14}$$

Since $Q_j \neq \varnothing$, by (1.14) there is $y_0 \in \partial(P^{(j-1)})$ such that $T_j(y_0) < 0$. Moreover by assumption $P^{(j-1)}$ is n-dimensional, so there exists $x_0 \in (P^{(j-1)})^\circ$. But as assumed in (1.13) that $(\overline{Q_j})^\circ = \varnothing$, we have $T_j(x_0) \geq 0$. Let $P_{y_0}(t) := tx_0 + (1-t)y_0 \in (P^{(j-1)})^\circ$, $t \in (0,1)$. For some sufficiently small positive number t, $T_j(P_{y_0}(t)) < 0$. So $P_{y_0}(t) \in (\overline{Q_j})^\circ$ contradicting (1.13). Hence $\overline{Q_j}$ is n-dimensional. Also, $\overline{Q_j}$ is pointed as P is pointed and $\overline{Q_j} \subseteq P$.

If $i < j$, we have (with no assumption on Q_j or Q_i)

$$\begin{aligned} \overline{Q_i} &= \mathrm{H}(T_1, \ldots, T_{i-1}) \cap \{x \in P : T_i(x) \leq 0\}, \\ \overline{Q_j} &= \mathrm{H}(T_1, \ldots, T_i, \ldots, T_{j-1}) \cap \{x \in P : T_j(x) \leq 0\}. \end{aligned}$$

Hence $\overline{Q}_i \cap \overline{Q}_j \subseteq \{x \in V : T_i(x) = 0\}$, and so $\overline{Q_i} \cap \overline{Q_j}$ has no interior, and this also holds if $i > j$. Finally, (iv) follows directly from (1.12) and (i)-(iii). $\square$

Remark 20. *By Lemma 19 (iv), the closure $\overline{P \setminus P \cap P'}$ has a decomposition as a union of at most ℓ pointed n-dimensional rational cones, where ℓ is the number of facets of $P' = \mathrm{H}(T_1, \ldots, T_\ell)$ (irredundant* H*-representation).*

Chapter 2

Algorithms for rational cones

In this chapter describe the algorithms needed in the next chapter to determine a true fundamental domain from a signed one, working always in the setting $V_{\mathbb{Q}} \subseteq V$ described at the beginning of §1.1. All of the algorithms in this chapter work with exact arithmetic, as calculations are reduced to elementary operations with rational matrices. Thus we assume that any element of $V_{\mathbb{Q}}$ is coded as an n-tuple of rational numbers (the coordinates with respect to a fixed basis), and that the bilinear form $\langle\ ,\ \rangle$ is given by a fixed (invertible, symmetric) $n \times n$ rational matrix.

Algorithm 21 (H-to-R)**.** *Obtains an irredundant rational* R*-representation of a pointed cone from a rational* H*-representation.*

Input: A finite set $\{v_1, \ldots, v_{n+m}\} \subseteq V_{\mathbb{Q}}$ defining a pointed rational polyhedral cone $Q := \mathrm{H}(L_1, \ldots, L_{n+m}) \subseteq V$, where $L_i(x) = \langle v_i, x\rangle$.
Output: An irredundant set $B \subseteq V_{\mathbb{Q}}$ such that $Q = C[B]$.

Step 1. Since Q is pointed, by Lemma 3 we can reorder the L_i if necessary so that $L_1, \ldots, L_n$ are linearly independent. Concretely, we compute linearly independent $v_1, \ldots, v_n$.

Step 2. Using Lemma 2, we obtain an irredundant rational R-representation of the cone $P_0 := \mathrm{H}(L_1, \ldots, L_n) = C[B_0]$, $B_0 \subseteq V_{\mathbb{Q}}$. Concretely, we compute $B_0 = \{r_1, \ldots, r_n\}$ by inverting a matrix as in (1.1).

Step 3. For each $j = 1, \ldots, m$, define $P_j := C[B_{j-1}] \cap \mathrm{H}(L_{n+j})$ and use (1.9) in Lemma 17 to compute $B_j \subseteq V_{\mathbb{Q}}$ so that $P_j = C[B_j]$ is an irredundant rational R-representation.

Step 4. RETURN $B := B_m$.

Proof. This is just an algorithmic form of Proposition 18. $\square$

Algorithm 22 (R-to-H)**.** *Obtains an irredundant rational* H*-representation of an n-dimensional cone from a rational* R*-representation.*

Input: A finite set $B = \{v_1, \ldots, v_\ell\} \subseteq V_{\mathbb{Q}}$ determining $P = C[B]$ an n-dimensional rational cone.

Output: A rational H-representation $P = \mathrm{H}(L_1, \dots, L_m)$, $L_j(x) = \langle r_j, x\rangle$ where $r_j \in V_{\mathbb{Q}}$.

Step 1. Apply the H-to-R algorithm 21 to the pointed cone $Q := \mathrm{H}(L_{v_1}, \dots, L_{v_\ell})$, where $L_v(x) := \langle v, x\rangle$ to obtain the irredundant R-representation $Q = C[r_1, ..., r_m]$ with $r_i \in V_{\mathbb{Q}}$.

Step 2. RETURN $P = \mathrm{H}(L_{r_1}, \dots, L_{r_m})$.

Proof. By Lemma 4, the dual cone $P^* = \mathrm{H}(L_{v_1}, \dots, L_{v_\ell}) =: Q$. By Lemma 6 (ii), Q is pointed. By Lemmas 6 (i) and 5,

$$P = (P^*)^* = Q^* = (C[r_1, ..., r_m])^* = \mathrm{H}(L_{r_1}, \dots, L_{r_m})$$

is a rational H-representation for P. It is irredundant by the remark immediately following the proof of Lemma 6. □

Next we give algorithm for making an H- or R-representation irredundant H- or R-representation.

Algorithm 23 (R-irredundant)**.** *Obtains an irredundant rational* R*-representation of a pointed n-cone from a rational* R*-representation.*

Input: A rational R-representation $P = C[B]$ of a pointed n-dimensional rational cone.
Output: An irredundant rational R-representation $P = C[B']$.

Step 1. Apply the R-to-H algorithm 22 to $P = C[B]$, obtaining an (irredundant) H-representation $P = \mathrm{H}(L_1, \dots, L_m)$.

Step 2. Apply the H-to-R algorithm 21 to $P = \mathrm{H}(L_1, \dots, L_m)$, yielding the output $P = C[B']$.

Proof. Algorithms 21 and 22 reverse H- and R-representations, yielding irredundant ones. Hence, appplying them in succession returns the original type of representation, but made irredundant. □

The next algorithm has the same proof.

Algorithm 24 (H-irredundant)**.** *Obtains an irredundant rational* H*-representation of a pointed n-cone from a a rational* H*-representation.*

Input: A rational H-representation $P = \mathrm{H}(L_1, ..., L_m)$ of a pointed n-dimensional rational cone.
Output: An irredundant rational H-representation $P = \mathrm{H}(L'_1, ..., L'_{m'})$.

Step 1. Apply the H-to-R algorithm 21 to $P = \mathrm{H}(L_1, \dots, L_m)$, obtaining an (irredundant) R-representation $P = C[B]$.

Step 2. Apply the R-to-H algorithm 22 to $P = C[B]$, yielding the output $P = \mathrm{H}(L'_1, ..., L'_{m'})$.

Algorithm 25 (Cone Dimension)**.** *Checks if a pointed rational cone given by an* H*-representation has maximal dimension.*

Input: A finite subset $\{v_1, \ldots, v_m\}$ of $V_{\mathbb{Q}}$ determining a pointed rational cone $P = \mathrm{H}(L_1, \ldots, L_m) \subseteq V$, where $L_i(x) = \langle v_i, x\rangle$.
Output: 1 if P has dimension n, 0 otherwise.

Step 1. Apply the H-to-R algorithm 21 to $P = \mathrm{H}(L_1, \ldots, L_m)$ to obtain an irredundant R-representation $P = C[r_1, \ldots, r_\ell]$ where $r_i \in V_{\mathbb{Q}}$.

Step 2. To express each $r_j \in V_Q$ as an n-tuple of rational numbers (with a fixed basis for $V_{\mathbb{Q}}$). So this return a $n \times \ell$ rational matrix. RETURN 1 if the rank of such matrix is n, 0 otherwise.

Algorithm 26 (Cone Containment)**.** *Checks if a pointed rational cone given by an* H*-representation is contained in another.*

Input: Two pointed rational polyhedral cones $P = \mathrm{H}(L_1, \ldots, L_m)$ and $P' = \mathrm{H}(T_1, \ldots, T_\ell)$, where $L_i(x) = \langle v_i, x\rangle$ and $T_j(x) = \langle w_j, x\rangle$.
Output: 1 if $P \subseteq P'$, 0 otherwise.

Step 1. In each cone $P^{(j-1)} := \mathrm{H}(L_1, \ldots, L_m, T_1, \ldots, T_{j-1})$, $P^{(0)} := P$ apply the H-to-R algorithm 21 to obtain an irredundant R-representation $P^{(j-1)} = C[B_{j-1}]$ with $B_{j-1} \subseteq V_{\mathbb{Q}}$ finite.

Step 2. If $T_j(b) \geq 0$ for all $b \in B_{j-1}$, return $Q_j := P^{(j-1)} \cap \{x \in T_j(x) < 0\} = \varnothing$, otherwise $Q_j \neq \varnothing$.

Step 3. If $Q_j = \varnothing$ for all $j = 1, \ldots, \ell$, then $P \subseteq P'$ and RETURN 1, 0 otherwise.

Proof. By Lemma 19 (ii) we known that $\bigcup_{j=1}^{\ell} Q_j = P \setminus P \cap P'$, so the Step 3 is justified. □

An alternative to Algorithm 26 is to apply the H-to-R algorithm to obtain $P = C[b_1, \ldots, b_s]$ and then verify if $T_j(b_i) \geq 0$ for all j and i.

Algorithm 27 (Fat Intersection)**.** *Checks if the intersection of two pointed n-dimensional rational cones, both given in their* H*-representations, has dimension n.*

Input: Two finite subsets $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_\ell\}$ of $V_{\mathbb{Q}}$ determining pointed n-dimensional rational cones $P = \mathrm{H}(L_1, \ldots, L_m)$ and $P' = \mathrm{H}(T_1, \ldots, T_\ell)$, where $L_i(x) = \langle v_i, x\rangle$ and $T_j(x) = \langle w_j, x\rangle$.
Output: 1 if $P \cap P'$ has dimension n, 0 otherwise.

Step 1. Apply Cone Dimension algorithm 25 to $P^{(j)} := \mathrm{H}(L_1, \ldots, L_m, T_1, \ldots, T_j)$ to determine if $P^{(j)}$ is n-dimensional or not.

Step 2. If $P^{(j)}$ is n-dimensional for all j, then $P \cap P'$ is n-dimensional RETURN 1, 0 otherwise.

Proof. The Step 2 is justified by Lemma 19 (i). □

As explained in the Introduction, in producing a true fundamental domain $\mathfrak{F}$ from a given signed one $(\mathcal{N};\mathcal{P})$, the main step is to remove all the "negative" polyhedral cones, *i.e.* those in the list $\mathcal{N} = (N_1, \ldots, N_m)$. The main difficulty that arises in using our main tool, is that the difference of two cones is in general no longer a polyhedral cone. Rather, it is a finite union of cones with disjoint interiors. Thus, when replacing a given cone N_j in the list $\mathcal{N}$ by $N_j \setminus (N_j \cap \Pi_i)$, or a cone Π_i in the list $\mathcal{P}$ by $\Pi_i \setminus (N_j \cap \Pi_i)$, we obtain a (possible empty) finite union of cones with disjoint interiors instead of single cones. To express this it will prove useful to make the following two *ad hoc* definitions.

Definition 28. *A list $(P_1, \ldots, P_m)$ of cones $P_i \subseteq V$ is clean if the interiors $(P_i \cap P_j)^\circ = \varnothing$ for all $i \neq j$.*

In particular, an empty list is clean, as is a list consisting of cones with empty interiors.

Definition 29. *A (possibly empty) subset $\mathcal{D} \subseteq V$ of an n-dimensional real vector space V is called a rational polyhedral semi-complex if $\mathcal{D} = \bigcup_{j=1}^{t} C_j$, where the C_j are pointed n-dimensional rational polyhedral cones and the (possibly empty) list $(C_1, \ldots, C_t)$ is clean.*

A semi-complex is for computational purposes the list $(C_1, \ldots, C_t)$, but we use the same term for the subset $\mathcal{D} = \bigcup_j C_j \subseteq V$.

Algorithm 30 (Fat Difference)**.** *Expresses the closure $\mathcal{D} := \overline{P \setminus (P \cap P')}$ of the difference of two pointed n-dimensional rational cones as a rational polyhedral semi-complex $\mathcal{D} = \bigcup_j C_j$ if $\mathcal{D}$ has a non-empty interior. Otherwise outputs the empty set, meaning that $\mathcal{D} = \varnothing$ is the empty semi-complex.*

Input: Same Input as in the Fat Intersection algorithm 27.
Output: If $\mathcal{D}$ has a non-empty interior, a finite list $(C_1, \ldots, C_r)$ of H-representations of pointed n-dimensional rational cones $C_i \subseteq V$ such that $(C_i \cap C_j)^\circ = \varnothing$ for $i \neq j$ and $\mathcal{D} = \bigcup_{i=1}^{r} C_i$. Otherwise, the empty set.

Step 1. Apply the Cone Containment algorithm 26 to determine if $P \not\subseteq P'$. If so, go to step 2. Otherwise RETURN the empty set.

Step 2. Apply the Fat Intersection algorithm 27 to see if $P \cap P'$ is n-dimensional. If so, go to Step 3. Otherwise RETURN the cone $P = \mathrm{H}(L_1, \ldots, L_m)$.

Step 3. As $P \cap P'$ is n-dimensional and $P \not\subseteq P \cap P'$, we have $\mathcal{D} = \bigcup_{j,\ Q_j \neq \varnothing} \overline{Q_j}$ (union of n-dimensional cones). RETURN the non-empty list $(\overline{Q_1}, \ldots, \overline{Q_r})$ $(r \leq \ell)$ in Lemma 19 (iv) of pointed n-dimensional rational cones.

Proof. Lemma 19. □

Since algorithms 21 and 22 allow us to pass from an H- to an R- representation and vice-versa, for the remainder of this thesis when we assume that a pointed rational cone is given, we will always assume that either of these is given, and that the other one is computed whenever necessary. Using algorithms 23 and 24, we can assume the representations to be irredundant,

Algorithm 31 (Removing One Cone)**.** *Expresses the closure* $\overline{\mathcal{D}\setminus C}$ *of the difference of a rational polyhedral semi-complex* $\mathcal{D}$ *and a a pointed* n*-dimensional rational polyhedral cone* C *as a rational polyhedral semi-complex* $\mathcal{D}'$*.*

Input: A rational polyhedral semi-complex $\mathcal{D} \subseteq V$ and $C \subseteq V$ a pointed n-dimensional rational polyhedral cone.
Output: A (possibly empty) rational polyhedral semi-complex $\mathcal{D}' = \overline{\mathcal{D}\setminus C}$.

Step 1. Given $\mathcal{D} = \bigcup_{j=1}^{t} C_j$, where $(C_1,\dots,C_t)$ is a clean list of pointed n-dimensional rational polyhedral cones, apply the Fat Difference algorithm 30 to each pair (C_j, C) to obtain t rational polyhedral semi-complexes $\mathfrak{S}(C_j) = \overline{C_j\setminus(C_j\cap C)}$.
Step 2. RETURN the rational polyhedral semi-complex $\mathcal{D}' := \big(\mathfrak{S}(C_1),\dots,\mathfrak{S}(C_t)\big)$.

Proof. Note that

$$\overline{\mathcal{D}\setminus C} = \overline{\left(\bigcup_{j=1}^{t} C_j\right)\setminus C} = \overline{\bigcup_{j=1}^{t}(C_j\setminus C)} = \bigcup_{j=1}^{t}\overline{C_j\setminus(C_j\cap C)} = \bigcup_{j=1}^{t}\mathfrak{S}(C_j) =: \mathcal{D}'.$$

Also as the list $(C_1,\dots,C_t)$ is clean, *i.e.* the interior $(C_i\cap C_j)^\circ = \varnothing$ for $i\neq j$, the interiors of the cones in $\mathfrak{S}(C_j)$ are disjoint from those of $\mathfrak{S}(C_i)$. Thus $\mathcal{D}'$ is a rational polyhedral semi-complex. $\square$

The next algorithm generalizes the previous one.

Algorithm 32 (Removing Intersections)**.** *Expresses the closure* $\overline{\mathcal{D}\setminus\mathcal{C}}$ *of the difference of two rational polyhedral semi-complexes as a rational polyhedral semi-complex* $\mathcal{D}'$*.*

Input: Two rational polyhedral semi-complexes $\mathcal{D} = (D_1, ..., D_r)$ and $\mathcal{C} = (C_1, ..., C_s)$.
Output: A rational polyhedral semi-complex $\mathcal{D}' = \overline{\mathcal{D}\setminus\mathcal{C}}$.

Step 1. $\mathcal{D} = \bigcup_{j=1}^{r} D_j$ and $\mathcal{C} = \bigcup_{i=1}^{s} C_i$, where $(D_1,\dots,D_r)$ and $(C_1,\dots,C_s)$ are clean lists of pointed n-dimensional rational polyhedral cones. Use the Fat Intersection algorithm 27 to compute for $1\le j\le r$ the subsets of $\{1,2,\dots,s\}$

$$I_j := \big\{\ell\in\{1,\dots,s\} :\ (D_j\cap C_\ell)^\circ \neq\varnothing\big\}. \tag{2.1}$$

Step 2. If $I_j=\varnothing$, set $\mathcal{S}(D_j) := (D_j)$, a polyhedral semi-complex consisting of one cone. Otherwise, pick an $h_1\in I_j = \{h_1,\dots,h_{e_j}\}$ and apply the Fat Difference algorithm 30 to the pair (D_j, C_{h_1}) to obtain a rational polyhedral semi-complex $\mathcal{S}_1 = \overline{D_j\setminus(D_j\cap C_{h_1})}$. Next, apply the Removing One Cone algorithm 31 to the rational polyhedral semi-complex $\mathcal{S}_1$, removing the cone C_{h_2} to obtain a rational polyhedral semi-complex $\mathcal{S}_2 = \overline{\mathcal{S}_1\setminus C_{h_2}}$. Continue using algorithm 31 to construct $\mathcal{S}_\ell$ by removing C_{h_ℓ} from $\mathcal{S}_{\ell-1}$ until $\ell = e_j$. Set $\mathcal{S}(D_j) := \mathcal{S}_{e_j}$.
Step 3. RETURN the rational polyhedral semi-complex $\mathcal{D}' := \mathcal{S}(D_1)\cup\dots\cup\mathcal{S}(D_r)$.

Proof. The semi-complexes $\mathcal{S}(D_j)$ computed in Step 2 satisfy for $1 \le j \le r$,

$$\mathcal{S}(D_j) = \overline{D_j \setminus \bigcup_{h \in I_j} (D_j \cap C_h)} = \overline{D_j \setminus \mathcal{C}}.$$

Hence,

$$\overline{\mathcal{D} \setminus \mathcal{C}} = \overline{\bigcup_{j=1}^{r} (D_j \setminus \mathcal{C})} = \bigcup_{j=1}^{r} \overline{D_j \setminus \mathcal{C}} = \bigcup_{j=1}^{r} \mathcal{S}(D_j) := \mathcal{D}',$$

which is easily checked to be a semi-complex. □

Remark 33. *Remark 20 shows that the length of the list in the output of Algorithm 32, i.e. the number of polyhedral cones obtained by concatenating the lists $\mathcal{S}(D_j)$, is at most*

$$r \prod_{i=1}^{s} \#\{\mathfrak{F}(C_i)\}, \tag{2.2}$$

where $\#\{\mathfrak{F}(C_i)\}$ denotes the number of facets of the cone C_i.

Chapter 3

From signed to true fundamental domains

In this chapter we give an algorithm whose input is a signed fundamental domain (see Definition 1 in the Introduction) consisting of finitely many rational polyhedral cones, and whose output is a true fundamental domain of the same kind. Thus, we assume that we are given a countable group G acting freely on a subset $\mathfrak{D} \subseteq V$, and lists $\mathcal{N} = (N_1, \ldots, N_m)$ and $\mathcal{P} = (\Pi_1, \ldots, \Pi_\ell)$ of rational polyhedral cones in V giving a signed fundamental domain for $\mathfrak{D}/G$.[1] We will make some additional assumptions on the action of G, making it a Colmez action." In chapter 4 we show how to associate Colmez actions to any number field that is not totally complex.

Definition 34 (Colmez action)**.** *Let G be a group and let $\rho : G \to GL(V)$ be a linear representation of G on an n-dimensional $\mathbb{R}$-vector space V endowed with a fixed $\mathbb{Q}$-structure $V_\mathbb{Q}$ and a fixed non-degenerate, symmetric, rational bilinear form $\langle\, ,\rangle : V \times V \to \mathbb{R}$ (see §1.1). Let $\mathfrak{D} \subseteq V$ be a non-empty open cone in V (i.e. $t \in \mathbb{R}_+,\ w \in \mathfrak{D}$ imply $tw \in \mathfrak{D}$) and let $v_0 \in V,\ v_0 \notin \mathfrak{D}$. For simplicity, we shall write $\rho(g)(v)$ as $g \cdot v$ and call v_0 the vantage point.*

The 4-tuple $(G, V, \mathfrak{D}, v_0)$ is a Colmez action if the following hold.

(i) *G is countable, infinite and acts freely on $\mathfrak{D}$, i.e. if $g \cdot x = x$ for some $x \in \mathfrak{D}$ and some $g \in G$, then $g = e_G$, the identity element of G.*

(ii) *$g \cdot V_\mathbb{Q} \subseteq V_\mathbb{Q},\ g \cdot \mathfrak{D} \subseteq \mathfrak{D}$ and $\langle g \cdot v, w\rangle = \langle v, g \cdot w\rangle\ \forall v, w \in V,\ \forall g \in G$.*

(iii) *For any compact subset $K \subseteq \mathfrak{D}$ and any pointed rational polyhedral cone $P \subseteq \mathfrak{D} \cup \{0_V\}$, there are at most finitely many $g \in G$ such that $(g \cdot K) \cap P \neq \varnothing$.*

(iv) *For each $g \in G$, there exists $\lambda = \lambda(g) > 0$ such that $g \cdot v_0 = \lambda v_0$.*

(v) *v_0 is not contained in the real span of any proper $\mathbb{Q}$-subspace of $V_\mathbb{Q}$.*

For the remainder of this chapter, we will assume (not always explicitly) that we have fixed a Colmez action $(G, V, \mathfrak{D}, v_0)$.

[1] It will prove convenient to simplify boundary issues by removing from consideration a lower dimensional G-stable subset $\mathfrak{H} \subseteq \mathfrak{D}$ which includes the G-orbits of all facets of the cones N_j or Π_i involved. The Colmez Trick, Lemma 39 below, will allow us to restore boundary pieces in the end.

In §3.1 we explain Colmez's vantage point trick for dealing with boundary issues. In §3.2 we show how the algorithms of Chapter 2 allow us to crop a piece of any negative cone N_i (always cropping a corresponding piece of some positive cone P_j) and still have a signed fundamental domain. In §3.3 we show how a finite number of croppings result in a true fundamental domain.

3.1 Piercing and the Colmez Trick

In this section we start from a G-action on a set $\mathfrak{D} \subseteq V$ and a finite set of cones $\{Q_\lambda\}$ in V whose union is nearly a fundamental domain for $\mathfrak{D}/G$, *i.e.* there is a lower-dimensional set of G-orbits where this may fail. Following Colmez's idea (unpublished, but see [DF14] [EF20], or this section), we show that the vantage point v_0 allows us to select certain boundary points of the cones to obtain a true fundamental domain for the whole of $\mathfrak{D}$. Briefly put, Colmezs idea was to add to the interior of each cone all boundary points which are reached via the inside of the cone when coming straight from v_0.

For later reference we note the following result, omitting its obvious proof.

Lemma 35. *Let $g \in G$ and let P be a pointed n-dimensional rational cone in V, with* R- *and* H-*representations $P = C[B] = \mathrm{H}(T_1, \dots, T_\ell)$, where $T_j(x) = \langle v_j, x\rangle$, $v_j \in V_\mathbb{Q}$, and $B \subseteq V_\mathbb{Q}$. Then $g \cdot P$ is a pointed n-dimensional rational cone with* R- *and* H-*representations $g \cdot P = C[g \cdot B] = \mathrm{H}(T_1', \dots, T_\ell')$, where $T_j'(x) := \langle g^{-1} \cdot v_j, x\rangle$.*

As in [EF20], for any subset $P \subseteq V$, $x, y \in V$ we shall say that $\overrightarrow{x,y}$ pierces P if $y \in P$ and if the closed line segment $\overrightarrow{x,y}$ connecting x and y intersects the interior of P. The following lemma gives a characterization of piercing of an n-dimensional polyhedral cone $P = \mathrm{H}(L_1, \dots, L_m)$ in terms of the linear forms L_i. It is similar to Lemma 14 in [DF14] and to Lemma 4.13 in [EF20], except that they only dealt with simplicial[2] n-cones and so could use barycentric coordinates. These coordinates are not available when the cone is not simplicial.

Lemma 36. *Let $P = \mathrm{H}(L_1, \dots, L_m)$ be an* H-*representation of an n-dimensional polyhedral cone P in V, where $n := \dim(V)$, and let $x, y \in V$. Then, $\overrightarrow{x,y}$ pierces P if and only if $L_j(y) \geq 0$ for $1 \leq j \leq m$ and for $1 \leq i \leq m$ we have $\big[L_i(y) = 0 \Rightarrow L_i(x) > 0\big]$.*

Moreover, if $\overrightarrow{x,y}$ pierces P and $s \in \overrightarrow{x,y}$ is an interior point of P, then every point of $\overrightarrow{s,y}$ is an interior point of P, except possibly for y.

Proof. Suppose that $\overrightarrow{x,y}$ pierces P. Then $y \in P$, so $L_j(y) \geq 0$ for $1 \leq j \leq m$. Moreover, $\overrightarrow{x,y}$ intersects the interior of P, so there exists $t \in (0,1)$ such that $L_j((1-t)x + ty) > 0$ for all j. So if $L_i(y) = 0$, we have $(1-t)L_i(x) > 0$. Thus, $L_i(x) > 0$. Conversely, suppose $L_j(y) \geq 0$ for all j (so $y \in P$) and that $L_i(y) = 0$ implies $L_i(x) > 0$. We need to prove that the line segment $\overrightarrow{x,y}$ intersects the interior of P. For this, we note that for some δ sufficiently near 1, and for any $t \in (\delta, 1)$ the point $p_t := (1-t)x + ty$ lies in the interior of P. Indeed, if

[2] A cone $P \subseteq V$ contained in an n-dimensional real vector space is simplicial if it is generated by ℓ elements which are linearly independent, with $\ell \leq n$.

$L_j(y) = 0$ then $L_j(p_t) > 0$ for any $t \in (0,1)$, while if $L_j(y) > 0$, it suffices to take t sufficiently close to 1. To prove the last claim let $t \in (0,1)$. Since s is an interior point of P, we have $L_j(s) > 0$ for all j. But $L_j(y) \geq 0$, so for $0 \leq t < 1$ we have $L_j((1-t)s + ty) = (1-t)L_j(s) + tL_j(y) \geq (1-t)L_j(s) > 0$. ☐

The following lemma establishes a piercing invariance of the vantage point v_0.

Lemma 37. *Suppose $(G, V, \mathfrak{D}, v_0)$ is a Colmez action, let $P = \mathrm{H}(L_1, \ldots, L_m)$ be an n-dimensional polyhedral cone in V, suppose $y \in P$ and $g \in G$. Then $\overrightarrow{v_0, y}$ pierces P if and only if $\overrightarrow{g \cdot v_0, y}$ pierces P.*

Proof. By (iv) in Definition 34, we have $g \cdot v_0 = \lambda v_0$ for some $\lambda > 0$. Lemma 36 concludes the proof. ☐

Lemma 38. *Suppose $(G, V, \mathfrak{D}, v_0)$ is a Colmez action, $y \in \mathfrak{D}$, and $P \subseteq \mathfrak{D} \cup \{0_V\}$ is an n-dimensional pointed rational polyhedral cone. Define*

$$A_P(y) := \left\{g \in G : \overrightarrow{v_0, g \cdot y} \text{ pierces } P\right\}, \qquad \ell(t) := (1-t)v_0 + ty \ \ (0 \leq t \leq 1).$$

Then $A_P\big(\ell(t)\big)$ stabilizes for t near 1. More precisely, there exists $T_0 \in (0,1)$ such that for all $t \in [T_0, 1]$ we have $A_P(y) = A_P\big(\ell(t)\big)$ and $\ell(t) \in \mathfrak{D}$.

Moreover, if we are given a finite number of pointed d_j-dimensional rational polyhedral cones $F_j \subseteq \mathfrak{D} \cup \{0_V\}$ with $d_j < n$, and let $\mathfrak{H} := \bigcup_j (G \cdot F_j)$ be the union of their G-orbits, we can choose $T_0 < 1$ so that $\ell(t) \notin \mathfrak{H}$ for all $t \in [T_0, 1)$.

Proof. The proof is essentially that of [EF20, Lemma 4.14], but we give all the details for the reader's convenience. As $\mathfrak{D} \subseteq V$ is open and $\ell(1) = y \in \mathfrak{D}$, it is clear that $\ell(t) \in \mathfrak{D}$ for all t sufficiently near 1. Note that by property (iii) of Definition 34 applied to $K := \{y\}$, $A_P(y) \subseteq G$ is finite.

Since $v_0 \notin \mathfrak{D}$ by definition of a Colmez action, $\ell(t)$ is not constant. Suppose $g \in A_P(y)$, so $\overrightarrow{v_0, g \cdot y}$ pierces P. By Lemma 37, $\overrightarrow{g \cdot v_0, g \cdot y}$ pierces P. By the last part of Lemma 36, there exists $s = s(g) \in (0,1)$ such that for all $t \in [s, 1)$ the point $g \cdot \ell(t) = (1-t)(g \cdot v_0) + t(g \cdot y)$ is an interior point of P. Thus, $g \in A_P\big(\ell(t)\big)$, for all $t \in [s, 1]$. Thus, taking $T_0 := \sup\big\{s(g) : g \in A_P(y)\big\}$, we have $A_P(y) \subseteq A_P(\ell(t))$ for all $t \in [T_0, 1]$. After possibly increasing $T_0 < 1$, we have $\ell(t) \in \mathfrak{D}$ for all $t \in [T_0, 1]$.

Now we prove the reverse inclusion, *i.e.* $A_P(\ell(t)) \subseteq A_P(y)$ for all $t \in [1-\epsilon, 1]$ for some $\epsilon > 0$. If this is false there is a sequence $t_j \in (0,1)$ converging to 1, and $g_j \in A_P(\ell(t_j))$ but $g_j \notin A_P(y)$. By (iii) in Definition 34, if $\Theta \subseteq \mathfrak{D}$ is a neighbourhood of y whose closure is compact and contained in $\mathfrak{D}$, there are finitely many $g \in G$ such that $(g \cdot \Theta) \cap (P \setminus \{0_V\}) \neq \varnothing$. Hence the set of such g_j's is finite. Passing to a subsequence, we may assume $g_j = g$ is fixed, $g \in A_P\big(\ell(t_j)\big)$, $g \notin A_P(y)$. Thus $\overrightarrow{v_0, g \cdot \ell(t_j)}$ pierces P. In particular $g \cdot \ell(t_j) \in P$ for all t_j. As $\ell(t_j)$ converges to y (as $t_j \to 1$) and as P is closed in V, so $g \cdot y \in P$. Also, as $\overrightarrow{v_0, g \cdot \ell(t_j)}$ pierces P, Lemma 37 implies $\overrightarrow{g \cdot v_0, g \cdot \ell(t_j)}$ pierces P. But

$$\overrightarrow{g \cdot v_0, g \cdot \ell(t_j)} = g \cdot (\overrightarrow{v_0, \ell(t_j)}) \subseteq g \cdot (\overrightarrow{v_0, \ell(1)}) = g \cdot (\overrightarrow{v_0, y}) = \overrightarrow{g \cdot v_0, g \cdot y},$$

implies that $\overrightarrow{g\cdot v_0, g\cdot y}$ contains an interior point of P. As $g\cdot y\in P$, we have proved that $\overrightarrow{g\cdot v_0, g\cdot y}$ pierces P. Again by Lemma 37, $\overrightarrow{v_0, g\cdot y}$ pierces P, *i.e.* $g\in A_P(y)$, contradicting our choice of g.

To prove the last claim in the Lemma, suppose there is a sequence $t_i\in(0,1)$ converging to 1 such that $\ell(t_i)\in\bigcup_j(G\cdot F_j)$. So for each t_i, there exist $g_i\in G$ and F_{j_i} such that $\ell(t_i)\in g_i\cdot F_{j_i}$. Thus, $g_i^{-1}\ell(t_i)\in F_{j_i}$. Again by (iii) in Definition 34, the g_i belong to a finite set. Passing to a subsequence, we may assume that $g_i=g$ and $F_{j_i}=F$ are fixed. Therefore, for some $g\in G$ and some pointed rational polyhedral cone F with $\dim(F)<n$, we have that $\ell(t_i)\in g\cdot F$. Since F is rational and $\dim(F)<n$, F is contained in the real span W of some proper $\mathbb{Q}$-subspace of $V_{\mathbb{Q}}$. Thus, $\ell(t_i)\in W$. Any two distinct points $\ell(t_i),\ell(t_{i'})\in W$, so the entire line passing through $\ell(t_i)$ and $\ell(t_{i'})$ lies on W. As $\ell(t_i)\in\overrightarrow{v_0, y}$, this line includes v_0. But then $v_0\in W$, contradicting (v) in Definition 34. $\square$

We can now describe the Colmez Trick. Let $Q=\mathrm{H}(L_1,\dots,L_m)$ be an irredundant H-representation of a pointed n-dimensional rational polyhedral cone $Q\subseteq\mathfrak{D}\cup\{0_V\}$. Since $\ker(L_j)$ is the real span of a rational proper subspace of $V_{\mathbb{Q}}$, we have $L_j(v_0)\neq 0$ for $1\le j\le m$. Define subsets $I_+,I_-\subseteq\{1,\dots,m\}$ by

$$I_\pm := \big\{j\in\{1,\dots,m\}:\ \pm L_j(v_0)>0\big\}. \tag{3.1}$$

They are non-empty since $v_0\notin Q$ and $-v_0\notin Q$. Indeed, since G takes the 1-cone generated by v_0 (and the 1-cone generated by $-v_0$) to itself, assumption (iii) of Definition 34 rules out that either cone could be in $Q\subseteq\mathfrak{D}$. Define the semi-closed cone

$$\widetilde{Q} := \big\{y\in V:\ L_j(y)\ge 0 \text{ for } j\in I_+ \text{ and } L_i(y)>0 \text{ for } i\in I_-\big\}\subseteq Q. \tag{3.2}$$

Thus, $\widetilde{Q}$ is the interior of Q together with part of its boundary. By Lemma 36,

$$\widetilde{Q}=\big\{y\in Q:\ \overrightarrow{v_0, y} \text{ pierces } Q\big\}. \tag{3.3}$$

Lemma 39 (Colmez Trick)**.** *Suppose $(G,V,\mathfrak{D},v_0)$ is a Colmez action, $n:=\dim(V)$, and we are given a finite list of n-dimensional pointed rational polyhedral cones $Q_\lambda\subseteq\mathfrak{D}\cup\{0_V\}$ such that*

$$\sum_\lambda \#\big(Q_\lambda\cap(G\cdot y)\big)=1 \qquad\qquad (\forall y\in\mathfrak{D}\setminus\mathfrak{H}), \tag{3.4}$$

where $\mathfrak{H}:=\bigcup_j G\cdot F_j\subseteq V$ is as in Lemma 38 and contains all facets of all cones Q_λ. Then $\bigcup_\lambda\widetilde{Q}_\lambda$ is a fundamental domain for the action of G on $\mathfrak{D}$, with $\widetilde{Q}_\lambda$ as in (3.2).

Proof. For all $y\in\mathfrak{D}$ we must show

$$\sum_\lambda \#\big(\widetilde{Q}_\lambda\cap(G\cdot y)\big)=1, \tag{3.5}$$

given that it holds for all $y \in \mathfrak{D} \setminus \mathfrak{H}$ (since $Q_\lambda \setminus \mathfrak{H} = \widetilde{Q}_\lambda \setminus \mathfrak{H}$ as we assumed that all facets of Q_λ are contained in $\mathfrak{H}$). From the definition of $A_P(y)$ in Lemma 38 and (3.3), we have for all $y \in \mathfrak{D}$

$$A_{Q_\lambda}(y) := \{g \in G : \overrightarrow{v_0, g \cdot y} \text{ pierces } Q_\lambda\} = \{g \in G : g \cdot y \in \widetilde{Q}_\lambda\}.$$

As the action of G is free (see (i) in Definition 34), $\#\big(A_{Q_\lambda}(y)\big) = \#\big(\widetilde{Q}_\lambda \cap (G \cdot y)\big)$. Since the set of λ's is finite, Lemma 38 shows the existence of some $x = \ell(t) \in \mathfrak{D} \setminus \mathfrak{H}$ such that $A_{Q_\lambda}(y) = A_{Q_\lambda}(x)$ for all λ. Hence,

$$1 = \sum_\lambda \#\big(\widetilde{Q}_\lambda \cap (G \cdot x)\big) = \sum_\lambda \#\big(A_{Q_\lambda}(x)\big) = \sum_\lambda \#\big(A_{Q_\lambda}(y)\big) = \sum_\lambda \#\big(\widetilde{Q}_\lambda \cap (G \cdot y)\big),$$

proving (3.5). $\square$

3.2 Cropping a signed fundamental domain

Given a signed fundamental domain $(\mathcal{N}; \mathcal{P})$ for $\mathfrak{D}/G$, where $\mathcal{N} := (N_1, \ldots, N_m)$ and $\mathcal{P} := (\Pi_1, \ldots, \Pi_\ell)$ are lists of n-dimensional rational polyhedral cones, Observation 4 in the Introduction suggests finding $g \in G$ such that $(g \cdot N_j) \cap \Pi_i \neq \varnothing$ for some i and j, and replacing N_j by $N_j \setminus g^{-1}\big((g \cdot N_j) \cap \Pi_i\big)$ and Π_i by $\Pi_i \setminus \big((g \cdot N_j) \cap \Pi_i\big)$. The motivation is that this results in a smaller signed fundamental domain for $\mathfrak{D}/G$, and so is closer to a true fundamental domain. In the next subsection we shall prove that repeating this cropping process eventually removes all negative cones. In this subsection we concentrate on a single cropping.

The difference of two polyhedral cones is not in general a polyhedral cone, but rather a finite union of such cones intersecting along pieces of facets. This makes it cumbersome to keep track of boundary points in the decomposition $N_j \setminus g^{-1}\big((g \cdot N_j) \cap \Pi_i\big) = \bigcup_u N_{ju}$. It is easier to remove from $\mathfrak{D}$ the G-orbits of all facets and use the Colmez Trick (Lemma 39) at the end to attach certain faces to obtain a true fundamental domain for $\mathfrak{D}/G$.

We will also need to keep track of the fact that the cones $N_{ju} \subseteq N_j$ share the property that $(g \cdot N_{ju}) \cap \Pi_i = \varnothing$. If we lost this information and listed the cones N_{ju} with the $N_{j'}$, the algorithm might wastefully try to re-use the same g to remove a piece mapped by g to $(g \cdot N_{ju}) \cap \Pi_i$. Hence we group the N_{ju} into a semi-complex $\mathfrak{N}_j = \{N_{ju}\}$. Recall that a semi-complex is either empty or is a finite list of pointed n-dimensional rational cones meeting only along their boundaries. Thus, instead of working with lists $(\mathcal{N}; \mathcal{P})$ of negative and positive cones, we will work with lists of negative and positive semi-complexes $(\mathfrak{N}; \mathfrak{P})$.

The following Lemma shows that for any negative n-cone N there is some $g \in G$ so that $g \cdot N$ intersects the interior of some positive cone Π. As explained above, it is useful to phrase this in terms of semi-complexes instead of cones.

Lemma 40. *Suppose $(G, V, \mathfrak{D}, v_0)$ is a Colmez action and we are given two finite lists $\mathfrak{N} := (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ of rational polyhedral semi-complexes contained in $\mathfrak{D} \cup \{0_V\}$ satisfying*

$$\sum_{i=1}^{\ell} \#\big(\mathcal{P}_i \cap (G \cdot x)\big) - \sum_{j=1}^{m} \#\big(\mathcal{N}_j \cap (G \cdot x)\big) = 1 \qquad (\forall x \in \mathfrak{D} \setminus \mathfrak{H}), \tag{3.6}$$

where $\mathfrak{H} := \bigcup_h (G \cdot F_h)$*, the* F_h *are a finite number of pointed* d_h*-dimensional rational polyhedral cones* $F_h \subseteq \mathfrak{D} \cup \{0_V\}$ *with* $d_h < n$*, and* $\mathfrak{H}$ *contains all facets of the* N_{ju} *and of the* Π_{it} *in* $\mathcal{N}_j := \bigcup_{u=1}^{\alpha_j} N_{ju}$ *and* $\mathcal{P}_i := \bigcup_{t=1}^{\beta_i} \Pi_{it}$ *for all* i, j*.*

Then, if $\mathcal{N}_j$ *is non-empty, for each* N_{ju} *there exists* $g \in G$ *and some* Π_{it} *such that* $(g \cdot N_{ju}) \cap \Pi_{it}$ *is an* n*-dimensional pointed rational polyhedral cone.*

Proof. The only non-obvious claim is that $(g \cdot N_{ju}) \cap \Pi_{it}$ has dimension n. Since $\mathfrak{H}$ includes all facets of the N_{ju} and of the Π_{it}, we have $N_{ju} \cap N_{ju'} \subseteq \mathfrak{H}$ for $u \neq u'$ and $\Pi_{it} \cap \Pi_{it'} \subseteq \mathfrak{H}$ for $t \neq t'$. Hence, for $x \in \mathfrak{D} \setminus \mathfrak{H}$,

$$\#\big(\mathcal{N}_j \cap (G \cdot x)\big) = \sum_{u=1}^{\alpha_j} \#\big(N_{ju} \cap (G \cdot x)\big), \ \#\big(\mathcal{P}_i \cap (G \cdot x)\big) = \sum_{t=1}^{\beta_i} \#\big(\Pi_{it} \cap (G \cdot x)\big).$$

Since $N_{ju} \neq \varnothing$ and n-dimensional, choose some x in the interior of N_{ju}, $x \notin \mathfrak{H}$. By (3.6) there exist at least two cones Π_{it} such that $(G \cdot x) \cap \Pi_{it} \neq \varnothing$. Pick one of them. Then there is some $g \in G$ such that $g \cdot x \in \Pi_{it}$. But as $x \notin \mathfrak{H}$, we have $g \cdot x \notin \mathfrak{H}$, and so $g \cdot x$ in the interior of Π_{it}. Thus, by Lemma 35, $(g \cdot N_{ju}) \cap \Pi_{it}$ is a pointed n-dimensional rational polyhedral cone. □

Fixing $g \in G$, the following algorithm allows us to remove all N_{ju} and Π_{it} for which $(g \cdot N_{ju}) \cap \Pi_{it}$ has a non-empty interior, with the new lists of semi-complexes still satisfying (3.6).

Algorithm 41 (Removing pieces related by g)**.** *Given* $g \in G$ *and two lists* $(\mathfrak{N}; \mathfrak{P})$ *of semi-complexes satisfying* (3.6) *for* $x \notin \mathfrak{H}$*, produces new lists* $(\mathfrak{N}'; \mathfrak{P}')$ *with* $\mathfrak{N}' \subseteq \mathfrak{N}$*,* $\mathfrak{P}' \subseteq \mathfrak{P}$*, satisfying* (3.6) *for* $x \notin \mathfrak{H}'$*, and such that* $(g \cdot \mathfrak{N}') \cap \mathfrak{P}'$ *has empty interior. Moreover,* $\mathfrak{H}'$ *contains the* G*-orbits of all facets of all cones appearing in the lists of semi-complexes* $\mathfrak{N}'$ *or* $\mathfrak{P}'$*.*

Input: Some $g \in G$ and lists of rational polyhedral semi-complexes $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ satisfying (3.6) for some $\mathfrak{H}$ as in Lemma 40.
Output: Lists of rational polyhedral semi-complexes $\mathfrak{N}' = (\mathcal{N}'_1, \ldots, \mathcal{N}'_m)$ and $\mathfrak{P}' := (\mathcal{P}'_1, \ldots, \mathcal{P}'_\ell)$ for which $\mathcal{N}'_j \subseteq \mathcal{N}_j$, $\mathcal{P}'_i \subseteq \mathcal{P}_i$, $(g \cdot \mathcal{N}'_j) \cap \mathcal{P}'_i$ has empty interior for all i and j, and

$$\sum_{i=1}^{\ell} \#\big(\mathcal{P}'_i \cap (G \cdot x)\big) - \sum_{j=1}^{m} \#\big(\mathcal{N}'_j \cap (G \cdot x)\big) = 1 \qquad (\forall x \in \mathfrak{D} \setminus \mathfrak{H}'), \tag{3.7}$$

where $\mathfrak{H}'$ is the union of $\mathfrak{H}$ and the G-orbits of all facets of all cones in the lists $\mathcal{N}'_j$ and $\mathcal{P}'_i$.

Step 1. Let $\widetilde{\mathfrak{N}} := g \cdot \mathfrak{N} = (\widetilde{\mathcal{N}}_1, \ldots, \widetilde{\mathcal{N}}_m)$ where $\widetilde{\mathcal{N}}_j := g \cdot \mathcal{N}_j$. Apply the Removing Intersections algorithm 32 to $\widetilde{\mathcal{N}}_1$ and $\mathcal{P}_1$ to obtain rational polyhedral semi-complexes $\widehat{\mathcal{N}}_1 := \widetilde{\mathcal{N}}_1 \setminus \mathcal{P}_1$ and $\widehat{\mathcal{P}}_1 := \mathcal{P}_1 \setminus \widetilde{\mathcal{N}}_1$. In $\widetilde{\mathfrak{N}}$ replace $\widetilde{\mathcal{N}}_1$ by $\widehat{\mathcal{N}}_1$ and in $\mathfrak{P}$ replace $\mathcal{P}_1$ by $\widehat{\mathcal{P}}_1$ (and still denote by $\mathfrak{P}$ and $\widetilde{\mathfrak{N}}$ the new lists of semi-complexes). Thus, with the new values, $\widetilde{\mathcal{N}}_1 \cap \mathcal{P}_1$ consists of cones of dimension strictly lower than n. Repeat this with $\mathcal{P}_2, \mathcal{P}_3, ..., \mathcal{P}_\ell$ so that in the new lists $\widetilde{\mathcal{N}}_1 \cap \mathcal{P}_i$ consists of cones of dimension strictly lower than n for all i. Now repeat this for $\widetilde{\mathcal{N}}_2$ and $\mathcal{P}_1, ..., \mathcal{P}_\ell$. Continue doing this for all the $\widetilde{\mathcal{N}}_j$.

Step 2. RETURN $(\mathfrak{N}'; \mathfrak{P}') := (g^{-1} \cdot \widetilde{\mathfrak{N}}; \mathfrak{P})$.

Proof. The Removing Intersections algorithm 32 ensures that $(g \cdot \mathfrak{N}') \cap \mathfrak{P}'$ has empty interior, $\mathcal{N}'_j \subseteq \mathcal{N}_j$, $\mathcal{P}'_i \subseteq \mathcal{P}_i$, and that the facets of the cones in $\mathfrak{N}'$ and $\mathfrak{P}'$ are as claimed. Observation 4 in the Introduction ensures that $(\mathfrak{N}'; \mathfrak{P}')$ satisfies (3.7). $\square$

Remark 42 (Upper Bound on the Number of Polyhedral Cones)**.** *Given the lists $\mathfrak{N} := (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ of rational polyhedral semi-complexes contained in $\mathfrak{D} \cup \{0_V\}$ satisfying (3.6) for all $x \in \mathfrak{D} \setminus \mathfrak{H}$, denote by α_j and β_i the number of polyhedral cones that define each rational polyhedral semi-complex $\mathcal{N}_j$ and $\mathcal{P}_i$ respectively. Thus the number of polyhedral cones in $(\mathfrak{N}; \mathfrak{P})$ is $\sum_{j=1}^{m} \alpha_j + \sum_{i=1}^{\ell} \beta_i$. By (2.2), after applying the Removing Rational Polyhedral semi-complexes algorithm 41 to $(\mathfrak{N}; \mathfrak{P})$, the number of polyhedral cones in $(\mathfrak{N}'; \mathfrak{P}')$ is at most*

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \Big(\alpha_j \prod_{t=1}^{\beta_i} \#\mathfrak{F}(\Pi_{it}) + \beta_i \prod_{u=1}^{\alpha_i} \#\mathfrak{F}(N_{ju}) \Big),$$

where $\#\mathfrak{F}(C)$ denotes the number of facets in a polyhedral cone C and $\mathcal{P}_i := \bigcup_{t=1}^{\beta_i} \Pi_{it}$, $\mathcal{N}_j = \bigcup_{u=1}^{\alpha_j} N_{ju}$. So we expect that the number of polyhedral cones to grow in passing from $(\mathfrak{N}; \mathfrak{P})$ to $(\mathfrak{N}'; \mathfrak{P}')$.

3.3 A true fundamental domain from a signed one

Recall that in the previous section we fixed a Colmez action $(G, V, \mathfrak{D}, v_0)$ (see Definition 34) and showed how, given a $g \in G$ and a pair $(\mathfrak{N}; \mathfrak{P})$, we could crop the pair to obtain $(\mathfrak{N}'; \mathfrak{P}')$ so that $(g \cdot \mathfrak{N}') \cap \mathfrak{P}'$ has no interior. Our first aim is to show that cropping is needed only for finitely many g.

Lemma 43. *Let N and Π be two pointed n-dimensional rational polyhedral cones in $\mathfrak{D} \cup \{0_V\}$. Then the set $\{g \in G : g \cdot N \cap (\Pi \setminus \{0_V\}) \neq \varnothing\}$ is finite.*

Proof. Since N is pointed and $N \neq \{0\}$,[3] there exists a polytope[4] $K \subseteq N \setminus \{0_V\}$ so that $N = \{tx : t \geq 0,\ x \in K\}$ is the conic hull of K [B08, proof of Theorem 4.11]. Since $K \subseteq \mathfrak{D}$ is compact, the set $\{g \in G : (g \cdot K) \cap \Pi \neq \varnothing\}$ is finite by (iii) in Definition 34. If $g_0 \cdot x \in (\Pi \setminus \{0_V\})$ for some $g_0 \in G$ and $x \in N$, then $x = ty$ for some $y \in K$ and $t > 0$. Thus $t^{-1}x = y \in K$, and as Π is a cone, $t^{-1}(g_0 \cdot x) \in t^{-1}(\Pi) = \Pi$. So $g_0 \cdot y = g_0 \cdot (t^{-1}x) = t^{-1}(g_0 \cdot x) \in \Pi$. Thus $g_0 \in \{g \in G : (g \cdot K) \cap \Pi \neq \varnothing\}$, a finite set. $\square$

[3] N is n-dimensional and $n \geq 1$ since G acts freely on $\mathfrak{D} \neq \varnothing$.

[4] By definition, a polytope K is compact and $K = \{x \in V : L_j(x) \geq b_j,\ 1 \leq j \leq m\}$ for some linear forms $L_1, , ..., L_m$ and real numbers $b_1, ..., b_m$.

Corollary 44. *Let $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ be finite lists of rational polyhedral semi-complexes contained in $\mathfrak{D} \cup \{0_V\}$. Then the set*

$$S(\mathfrak{N}; \mathfrak{P}) := \big\{ g \in G : \ \exists\, j, \exists\, i, \ ((g \cdot \mathcal{N}_j) \cap \mathcal{P}_i)^\circ \neq \varnothing \big\} \tag{3.8}$$

is finite.

We can now give the abstract form of the algorithm which is the aim of this thesis. In the next chapter we will apply it to number fields and give experimental results.

Algorithm 45 (From signed to true fundamental domain)**.** *Given a Colmez action $(G, V, \mathfrak{D}, v_0)$, two lists $(\mathfrak{N}; \mathfrak{P})$ of rational semi-complexes satisfying (3.6), and an algorithm for listing the elements of $S(\mathfrak{N}; \mathfrak{P})$ in (3.8), produces a finite list of rational n-dimensional semi-closed cones whose union is a fundamental domain for the action of G on $\mathfrak{D}$.*

Input:
- A Colmez action $(G, V, \mathfrak{D}, v_0)$.
- Lists of rational polyhedral semi-complexes $\mathfrak{N} := (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ contained in $\mathfrak{D} \cup \{0_V\}$, where the semi-complexes $\mathcal{N}_j := \bigcup_{u=1}^{\alpha_j} N_{ju}$ and $\mathcal{P}_i := \bigcup_{t=1}^{\beta_i} \Pi_{it}$ satisfy

$$\sum_{i=1}^{\ell} \#\big(\mathcal{P}_i \cap (G \cdot x)\big) - \sum_{j=1}^{m} \#\big(\mathcal{N}_j \cap (G \cdot x)\big) = 1 \qquad (\forall x \in \mathfrak{D} \setminus \mathfrak{H}). \tag{3.9}$$

Here $\mathfrak{H} := \bigcup_h (G \cdot F_h)$ (as in Lemma 40) contains the G-orbits of all facets of the polyhedral cones that define $\mathcal{N}_j$ and $\mathcal{P}_i$ for all i and j.
- A list of the elements of $S(\mathfrak{N}; \mathfrak{P})$.[5]

Output: A finite list $\{\widetilde{Q}_\lambda\}_\lambda$ of semi-closed pointed n-dimensional rational polyhedral cones, such that their union $\mathfrak{F} = \bigcup_\lambda \widetilde{Q}_\lambda$ is a fundamental domain for the action of G on $\mathfrak{D}$.[6]

Step 0. Set $s := 0$ and go to Step 1.

Step 1. If $\mathcal{N}_j = \varnothing$ for $1 \leq j \leq m$, RETURN the list $\{\widetilde{\Pi}_{it}\}_{\substack{1 \leq i \leq \ell \\ 1 \leq t \leq \beta_i}}$, where the semi-closed cone $\widetilde{Q}$ attached to the closed cone Q was defined in (3.2). Otherwise go to Step 2.

Step 2. Increment s by 1 and set $g := g_s$. Apply the Removing Rational Polyhedral Semi-complexes algorithm 41 to $(\mathfrak{N}; \mathfrak{P})$ and g to obtain $(\mathfrak{N}'; \mathfrak{P}')$. Redefine $\mathfrak{N} := \mathfrak{N}'$, $\mathfrak{P} := \mathfrak{P}'$ and go to Step 1.

---

[5] More precisely, an algorithm with input $s \in \mathbb{N}$ and output $g_s \in G$ so that $S(\mathfrak{N}; \mathfrak{P}) \subseteq \bigcup_{s=1}^{M} g_s$ for some $M \in \mathbb{N}$. This M need not be computed by the algorithm, but its existence ensures that the present algorithm stops.

[6] An output of a semi-closed cone $\widetilde{Q}$ consists of its closure $Q = H(L_1, \ldots, L_m)$, and the vector $\big(\mathrm{sign}(L_1(v_0)), \mathrm{sign}(L_2(v_0), \ldots, \mathrm{sign}(L_m(v_0))\big)$ needed in (3.1) and (3.2).

Proof. If $\mathcal{N}_j = \varnothing$ for $1 \leq j \leq m$, then the Colmez Trick (Lemma 39) applies, showing that $\bigcup_{\substack{1\leq i\leq \ell \\ 1\leq t\leq \beta_i}} \widetilde{\Pi}_{it}$ is a fundamental domain for $\mathfrak{O}/G$. Thus the algorithm stops after Step 1, with the correct output, if $\mathcal{N}_j = \varnothing$ for $1 \leq j \leq m$.

We now show that after a finite number of steps, the algorithm reaches Step 1 with $\mathcal{N}_j = \varnothing$ for $1 \leq j \leq m$. By assumption, there exists $M \in \mathbb{N}$ large enough that $S(\mathfrak{N}; \mathfrak{P}) \subseteq \{g_1, \ldots, g_M\}$. Suppose that after M applications of Step 2 we still had some $\mathcal{N}_j \neq \varnothing$. Then N_{ju} is n-dimensional and, by Lemma 40, there exists some $g \in G$ and some i and t so that $(g \cdot N_{ju}) \cap \Pi_{it}$ has a non-empty interior. Thus $(g \cdot \mathfrak{N}) \cap \mathfrak{P}$ has non-empty interior, and so $g \in S(\mathfrak{N}; \mathfrak{P})$. But this means that Step 2 has been applied to g and some pair $(\mathfrak{N}''; \mathfrak{P}'')$, with the current $\mathfrak{N} \subseteq \mathfrak{N}''$, $\mathfrak{P} \subseteq \mathfrak{P}''$. This contradicts the fact that algorithm 41 results in $(g \cdot \mathfrak{N}'') \cap \mathfrak{P}''$ having empty interior. $\square$

Chapter 4

Application to number fields

In this chapter we show how the algorithm of the previous chapter applies to number fields having at least one real place. After a general introduction, we specialize to totally real fields as signed fundamental domains for these fields are easily described. After recalling Espinoza and Friedman's work [EF20] for general number fields, we describe and tabulate the results of our algorithm for a large sample of number fields of degree up to 5. Then we briefly discuss a sample of sextic fields.

4.1 The Colmez action for number fields

Let k be an algebraic number field of degree $n = r_1 + 2r_2$, with r_1 real and $2r_2$ complex embeddings. We assume $r_1 > 0$ and $r_1 + r_2 > 1$, so the group E_k of units of k is infinite. Arbitrarily selecting one embedding from each complex conjugate pair, we map k to the n-dimensional $\mathbb{R}$-vector space $V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

For convenience, we identify k with its image in V. We fix a $\mathbb{Q}$-structure on V by setting $V_{\mathbb{Q}} := k \subseteq V$. Let $\mathfrak{O} := \mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2} \subseteq V$, $k_+ := k \cap \mathfrak{O}$, and $E_+ := E_k \cap k_+$, the totally positive units of k. Let $G \subseteq E_+$ be a torsion-free subgroup of finite index in E_+. Then G acts freely by component-wise multiplication on V, on $V_{\mathbb{Q}} = k$ and on the cone $\mathfrak{O} := \mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}$.

We let $\langle\ ,\ \rangle$ be the symmetric $\mathbb{R}$-bilinear form on V

$$\langle v, w\rangle := \sum_{j=1}^{r_1} v^{(j)} w^{(j)} + 2 \sum_{j=r_1+1}^{r_1+r_2} \mathrm{Re}(v^{(j)} w^{(j)}), \tag{4.1}$$

where $v := \left(v^{(1)}, \ldots, v^{(r_1)}, v^{(r_1+1)}, \ldots, v^{(r_1+r_2)}\right) \in V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ (similarly for w). Note that $\langle v, w\rangle = \mathrm{Trace}_{k/\mathbb{Q}}(vw)$ for $v, w \in k$, so the form is $\mathbb{Q}$-valued on $V_{\mathbb{Q}}$ and non-degenerate. Moreover, the action is self-adjoint, *i.e.* $\langle g \cdot v, w\rangle = \langle v, g \cdot w\rangle$. Thus, properties (i) and (ii) of Definition 34 of a Colmez action hold. Property (iii) was proved in [EF20, Lemma 4.10].

We define our vantage point $v_0 := (1, 0, 0, \ldots, 0) \in V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Note that $v_0 \notin \mathfrak{O}$ as $k \neq \mathbb{Q}$. Also, for $g \in G$ we have $g \cdot v_0 = g^{(1)} v_0$ and $g^{(1)} > 0$, as g is a totally positive unit and $r_1 > 0$. Hence property (iv) in Definition 34 holds. Property (v) was proved in [EF20, Lemma 4.12].

Henceforth, we fix a non totally complex number field k and the above Colmez action.

4.2 Totally real number fields

In the case of totally real number fields of degree $n = r_1$, Colmez [C88][C89] proved that there exist special subgroups of E_+ for which he could give an explicit Shintani fundamental domain $\{\widetilde{P}_\sigma\}_{\sigma\in S_{n-1}}$.[1] However, as mentioned in the Introduction, no practical algorithm for finding such special Colmez subgroups is known if $n > 3$. In 2014 [DF14] Diaz y Diaz and Friedman proved, for any finite-index subgroup of E_+, that the Colmez cones $\{\widetilde{P}_\sigma\}_\sigma$ determine a signed fundamental domain for $\mathfrak{O}/G = \mathbb{R}_+^n/G$. Special Colmez subgroups are exactly those for which the signed domain is a true one.

More precisely, let $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be free generators of a subgroup $G \subseteq E_+$, let S_{n-1} denote the group of permutations of $\{1, 2, \ldots, n-1\}$, and for $\sigma \in S_{n-1}$ let

$$f_{i,\sigma} := \prod_{j=1}^{i-1} \varepsilon_{\sigma(j)} \qquad (2 \le i \le n), \qquad\qquad f_{1,\sigma} := 1 = (1, 1, \ldots, 1) \in \mathbb{R}_+^n,$$
$$\omega_\sigma := \frac{(-1)^{n-1} \cdot \mathrm{sign}(\sigma) \cdot \mathrm{sign}(\det(f_{1,\sigma}, \ldots, f_{n,\sigma}))}{\mathrm{sign}(\det(\mathrm{Log}\,\varepsilon_1, \mathrm{Log}\,\varepsilon_2, \ldots, \mathrm{Log}\,\varepsilon_{n-1}))} \in \{-1, 0, +1\},$$

with $\mathrm{Log}\,\varepsilon_i \in \mathbb{R}^{n-1}$, $(\mathrm{Log}\,\varepsilon_i)^{(j)} := \log \varepsilon_i^{(j)}$ $(1 \le j \le n-1)$. Recall that we have embedded k in $\mathbb{R}^n$, so we regard $(f_{1,\sigma}, \ldots, f_{n,\sigma})$ as an $n \times n$ real matrix. Thus, $w_\sigma = 0$ if and only if the $f_{i,\sigma}$ fail to be linearly independent. The determinant in the denominator of ω_σ is a non-zero integral multiple of the regulator of k, so it does not vanish. The Colmez cones are defined as

$$\widetilde{P}_\sigma := \Big\{ \sum_{i=1}^{n} \lambda_i f_{i,\sigma} : \ \lambda_i \in \mathfrak{R}_{i,\sigma} \Big\} \subseteq \mathfrak{O} := \mathbb{R}_+^n \qquad\qquad (\sigma \in S_{n-1},\ w_\sigma \ne 0),$$

where

$$\mathfrak{R}_{i,\sigma} := \begin{cases} [0, +\infty) & \text{if } v_0 \in H_{i,\sigma}^+, \\ (0, +\infty) & \text{if } v_0 \in H_{i,\sigma}^-, \end{cases} \qquad (1 \le i \le n,\ v_0 := (1, 0, \ldots, 0) \in \mathbb{R}^n).$$

Here $\mathbb{R}^n = H_{i,\sigma}^+ \cup H_{i,\sigma} \cup H_{i,\sigma}^-$ is a disjoint union of the rational $(n-1)$-dimensional hyperplane $H_{i,\sigma} := \sum_{\substack{1\le j\le n\\ j\ne i}} \mathbb{R} \cdot f_{j,\sigma}$ and the two open half-spaces $H_{i,\sigma}^\pm$, labeled so that $f_{i,\sigma} \in H_{i,\sigma}^+$. Diaz y Diaz and Friedman proved [DF14]

$$\sum_{\substack{\sigma\in S_{n-1}\\ w_\sigma=+1}} \#\big(\widetilde{P}_\sigma \cap (G \cdot x)\big) - \sum_{\substack{\sigma\in S_{n-1}\\ w_\sigma=-1}} \#\big(\widetilde{P}_\sigma \cap (G \cdot x)\big) = 1 \qquad (\forall x \in \mathbb{R}_+^n). \tag{4.2}$$

[1] His original articles were not explicit as to the boundary of his cones, but he later fixed this in unpublished lectures.

If $w_\sigma \neq 0$, let $P_\sigma = C[f_{1,\sigma}, \ldots, f_{n,\sigma}] \subseteq \mathbb{R}^n_+ \cup \{0_V\}$ be the rational, pointed polyhedral n-cone with generators $f_{1,\sigma}, \ldots, f_{n,\sigma}$. Note that $\widetilde{P}_\sigma \subseteq P_\sigma$ and that $P_\sigma \setminus \widetilde{P}_\sigma$ is contained in the boundary of P_σ.

To each cone P_σ with $w_\sigma = -1$ we attach a semi-complex $\mathcal{N}_j$ which consists of the single polyhedral cone P_σ $(1 \leq j \leq m := \#\{\sigma \in S_{n-1} : w_\sigma = -1\})$. Similarly, to each P_σ with $w_\sigma = +1$ we attach a semi-complex $\mathcal{P}_i$ $(1 \leq i \leq \ell := \#\{\sigma \in S_{n-1} : w_\sigma = +1\})$. Let $(\mathfrak{N}_0; \mathfrak{P}_0)$ be the corresponding pair of lists of semi-complexes $\mathfrak{N}_0 := (\mathcal{N}_1, \ldots, \mathcal{N}_m)$, $\mathfrak{P}_0 := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$. Thus $(\mathfrak{N}_0; \mathfrak{P}_0)$ satisfies (4.2), except for x in the G-orbits of the facets of the P_σ. Hence we can use $(\mathfrak{N}_0; \mathfrak{P}_0)$ as input in Algorithm 45 to obtain a true fundamental domain for the case of totally real number fields, provided we specify a listing algorithm for the elements of $S(\mathfrak{N}_0; \mathfrak{P}_0)$ in (3.8). We will do this in the next subsection.

4.3 General non-totally complex number fields

Espinoza [E14] was able to generalize the signed fundamental domain in §4.2 from the totally real case to number fields with exactly one complex place (*i.e.* $r_2 = 1$). More recently, Espinoza and Friedman [EF20] extended this work to all number fields that have at least one real embedding (*i.e.* $r_1 > 0$). Except in the totally real case, there is no easy description of the Espinoza-Friedman signed fundamental domain. Instead, there is an inductive construction which depends rather subtly on the arguments of certain units at each of the complex places.[2]

As in [DF14] and [E14], from an input of free generators $\varepsilon_1, \ldots, \varepsilon_r$ (with $r := r_1 + r_2 - 1$) of a subgroup $G \subseteq E_+$, Espinoza and Friedman gave an algorithm that constructs a signed fundamental domain $(\mathcal{N}; \mathcal{P})$ for $\mathfrak{O}/G$, where $\mathcal{N} := (\widetilde{N}_1, \ldots, \widetilde{N}_m)$ is a list of negative" semi-closed n-dimensional pointed rational polyhedral cones and $\mathcal{P} := (\widetilde{\Pi}_1, \ldots, \widetilde{\Pi}_\ell)$ is a list of positive" ones. The total number of cones (*i.e.* $m + \ell$) is at most $3^{r_2}(n-1)!$.[3]

Regarding the closure $N_j \subseteq V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ of the negative semi-closed cone $\widetilde{N}_j$ as a semi-complex $\mathcal{N}_j$ (a list consisting of the single cone N_j), and similarly for the positive semi-closed cones Π_i, the output of the Espinoza-Friedman algorithm [EF20, §2] is a pair $(\mathfrak{N}_0; \mathfrak{P}_0)$, where $\mathfrak{N}_0 := (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and $\mathfrak{P}_0 := (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$ are lists of semi-complexes satisfying

$$\sum_{i=1}^{\ell} \#\big((G \cdot x) \cap \mathcal{P}_i\big) - \sum_{j=1}^{m} \#\big((G \cdot x) \cap \mathcal{N}_j\big) = 1, \qquad (\forall x \in \mathfrak{O} \setminus \mathfrak{H}), \tag{4.3}$$

[2] For totally real fields the Espinoza-Friedman algorithm reduces to applying the formulas in §4.2, so is much faster than the general case (for fields of the same degree).

[3] Since the output of the Espinoza-Friedman algorithm is the input for the main algorithm of this thesis, we implemented it in PARI/GP. The construction of the fundamental domain in [EF20] is more complex than in the totally real case, mainly due to the non-convexity of $\mathfrak{O} := \mathbb{R}^{r_1}_+ \times (\mathbb{C}^*)^{r_2}$ and the lack of sufficient units to generate an n-cone (if $r_2 > 0$). Espinoza and Friedman introduced "twisters" as substitutes for the missing units needed to generate the cones. Twisters are sets of 3^{r_2} totally positive elements of k whose arguments at complex places are sufficiently well distributed to force the generators of the cones into convex subregions of $\mathfrak{O}$. To our knowledge, ours was the first implementation of the Espinoza-Friedman algorithm when $r_2 > 1$.

where $\mathfrak{H} \subseteq \mathfrak{O}$ consists of the G-orbits of the facets of all cones involved. Hence again we can use $(\mathfrak{N}_0; \mathfrak{P}_0)$ as input in Algorithm 45 to obtain a true fundamental domain for any non-totally complex number field, provided we implement some way to list the elements of $S(\mathfrak{N}_0; \mathfrak{P}_0)$ in (3.8).

Our listing algorithm depends mainly on the weight

$$W(g) := \sum_{s=1}^{r} |b_s| \qquad \left(g = \prod_{s=1}^{r} \varepsilon_s^{b_s},\ \ b = b(g) := (b_1, \dots, b_r) \in \mathbb{Z}^r\right). \tag{4.4}$$

We begin with weight 0 (the identity element), then weight 1 (generators and their inverses), and so on until the algorithm terminates because there are no more negative cones. After discarding elements $g \notin S(\mathfrak{N}_0; \mathfrak{P}_0)$, we arranged elements g of equal weight simply in increasing lexicographic order of $b(g) = (b_1, \dots, b_r)$. Thus, the elements were listed

$$1, \varepsilon_1^{-1}, \varepsilon_2^{-1}, \dots, \varepsilon_r^{-1}, \varepsilon_r^{1}, \varepsilon_{r-1}^{1}, \dots, \varepsilon_1^{1}, \varepsilon_1^{-2}, \varepsilon_1^{-1}\varepsilon_2^{-1}, \dots .$$

We tried replacing lexicographic order by a seemingly more clever one, but the results were worse on the average. Namely, we tried to order the units g by the number of pairs i and j such that $(g \cdot \mathcal{N}_j) \cap \mathcal{P}_i$ has non-empty interior. Improving on the lexicographic order might greatly improve the effectiveness of our algorithm.

4.4 Tables for quartics and quintics

The next few tables provide more detailed information on the distribution of the number of units processed to obtain a fundamental domain $\mathcal{F}$, and on the distribution of the number of cones in $\mathcal{F}$ for the different signatures in degree 4 and 5. For example, Table 4.5 shows that for signature $(r_1, r_2) = (1, 2)$, for 127 fields $\mathcal{F}$ consisted of at most 6554 cones, but for 14 fields there were between 8840 and 79845 cones in $\mathcal{F}$.

Table 4.1: Cones in $\mathcal{F}$ and units processed for totally real quartics.

Units used:	-	1	2	3	4	5	6	7	8	9	10
Fields:	-	11885	6263	2087	1438	96	44	3	4	1	2
Range of cones in $\mathcal{F}$:	Min.	6	13	21	28	36	43	50	58	73	
	Max.	12	20	27	35	42	49	57	64	80	
Fields:	-	17221	3826	637	88	32	5	4	8	2	

Table 4.2: Cones in $\mathcal{F}$ and units processed for quartics with one complex place.

Units used:	-	1	2								
Fields:	-	37661	3081								
Range of cones in $\mathcal{F}$:	Min.	20	27	33	40	46	53	59	66	72	79
	Max.	26	32	39	45	52	58	65	71	78	85
Fields:	-	33609	4217	1726	658	425	54	30	13	8	2

Table 4.3: Cones in $\mathcal{F}$ and units processed for totally real quintics.

Units used:	-	1	2	3	4	5	6	7	8	9
Fields:	-	882	606	299	617	196	181	129	85	44
Units used:	-	10	11	12	13	14	15	16	17	18
Fields:	-	34	19	15	9	2	4	4	1	1
Range of cones in $\mathcal{F}$:	Min.	26	489	951	1414	1876	2339	2802	3264	4189
	Max.	488	950	1413	1875	2338	2801	3263	3726	4652
Fields:	-	2721	233	87	44	19	11	7	4	2

Table 4.4: Cones in $\mathcal{F}$ and units processed for quintics with one complex place.

Units used:	-	1	2	3	4	5	6	7	
Fields:	-	414	193	67	42	5	1	1	
Range of cones in $\mathcal{F}$:	Min.	77	987	1897	2806	3716	4626	5536	6446
	Max.	986	1896	2805	3715	4625	5535	6445	9175
Fields:	-	613	59	17	15	6	4	5	4

Table 4.5: Cones in $\mathcal{F}$ and units processed for quintics with two complex places.

Units used:	-	1	2	4	5	6	8			
Fields:	-	104	9	15	4	6	3			
Range of cones in $\mathcal{F}$:	Min.	236	8840	29992	46364	46960	53751	60421	72497	79845
	Max.	6554	16157	29992	46364	46960	53751	60421	72497	79845
Fields:	-	127	7	1	1	1	1	1	1	1

Chapter 5

Appendix

The figures in this appendix are meant to give a graphical overview of the output of our algorithm for quartic and quintic fields, and to show that, empirically at least, the discriminant of a field tells us nothing about how the algorithm will perform for that field.

We begin with the running time of the algorithm for the 21823 non Colmez totally real quartics summarized on line 4 of Table 1. The discriminant D_k ranges from about 10^3 to 10^7, so it is wise to plot running time against $\log D_k$. On doing this, the results were so wild that most of the points on the graph were blurred together at the bottom with a few stray ones far up. Indeed, as Table 3 shows, although the average running time was about $\frac{1}{20}$ of a second, the worst case took about 60 times longer. This could be solved by a log-log graph. The problems was that the graph still looked very wild because fields with nearly the same discriminant had very different running times.

We decided to tame the data by dividing up the fields into packets of 10 consecutive discriminants. Thus, in Figure 5.1 there is one point for every 10 fields, plotting average log discriminant against average running time for each packet of ten fields. This averaging process reduces the range of the running time so that we did not have to use a log-log plot. It still showed very clearly that knowledge of the discriminant gives no information on running time, even on the average. Figures 5.2-5.4 show the same phenomenon for other signatures.[1]

Next we replaced running time by the number of cones in the fundamental domain produced by our algorithm. Since the results are almost as wild as the running time, we used the above plotting procedure. Lastly, we plotted log discriminant against the number of units processed. Since the number of units is relatively small, these graphs are less wild, but still show no relation between discriminant and the number of units.

[1] For quintics with 2 complex places we ran only 141 non Colmez cases, so for this one signature we did not group fields in packages of 10. Each dot represents one field, rather than 10 as is the case for other signatures.

Figure 5.1: Log discriminant against running time for totally real quartics.

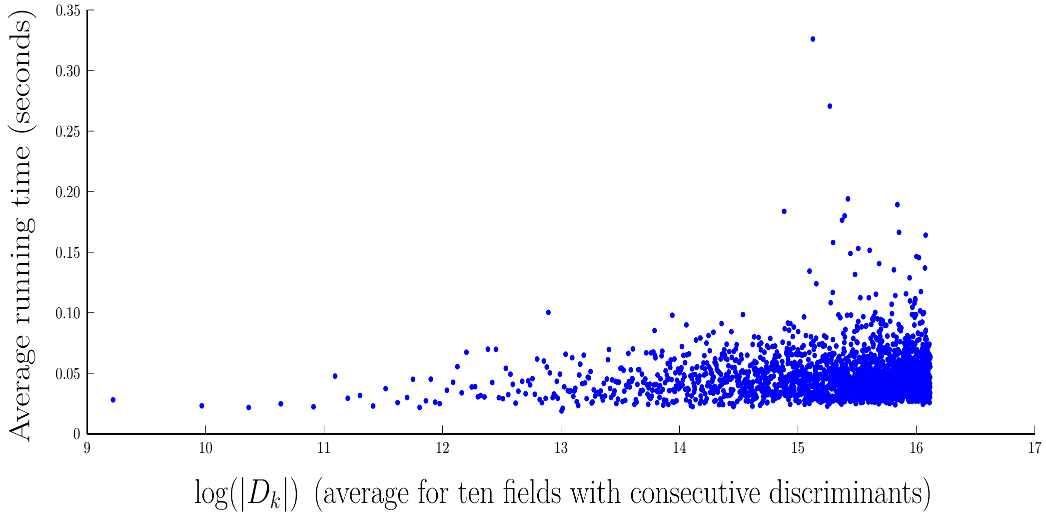


Figure 5.2: Log discr. against running time for quartics with one complex place.

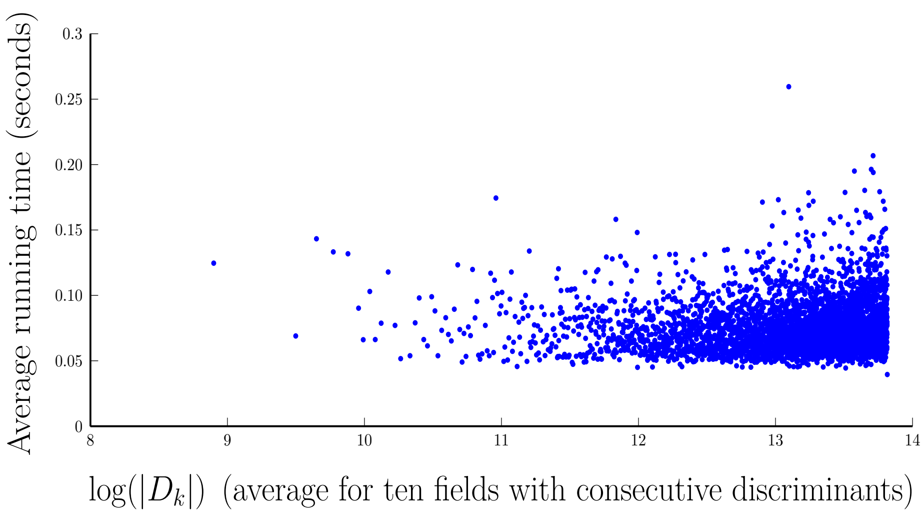


Figure 5.3: Log discr. against running time for totally real quintics.

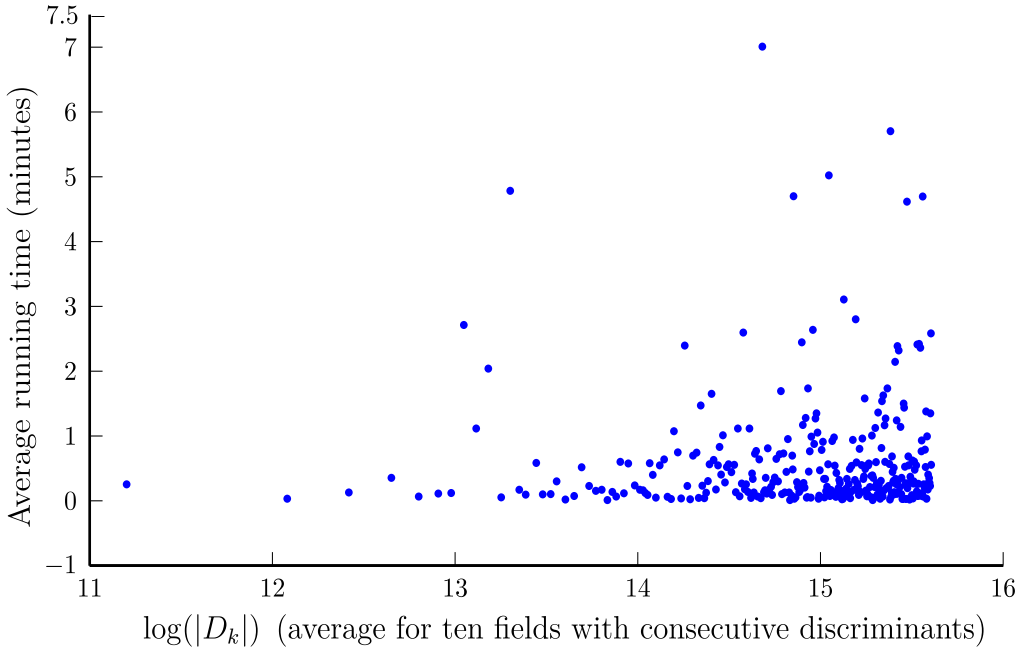


Figure 5.4: Log discr. against running time for quintics with $r_2 = 1$.

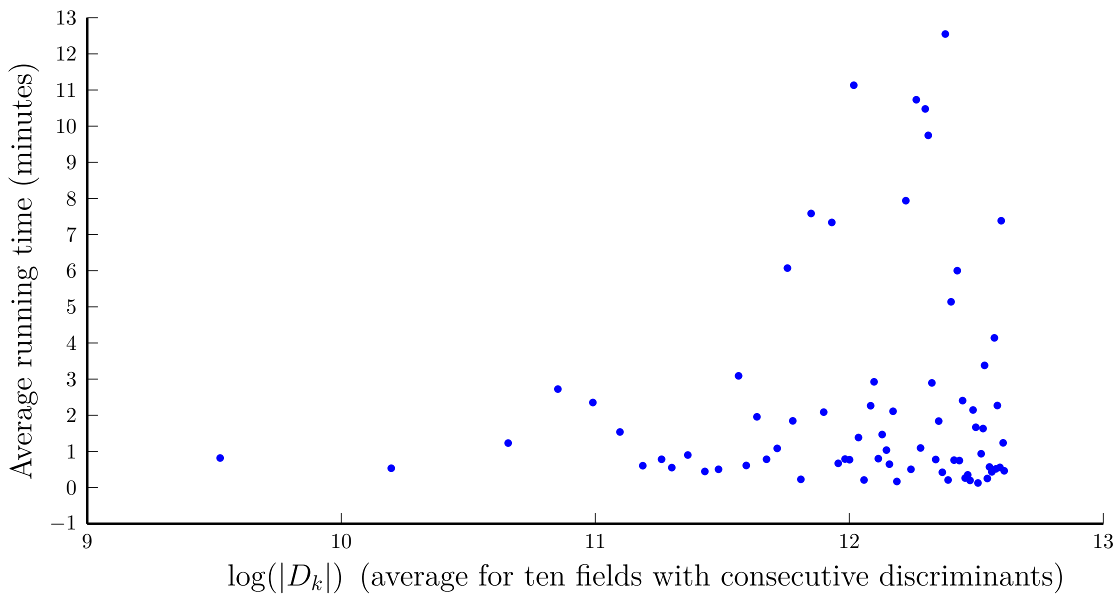


Figure 5.5: Log discr. against running time for quintics with $r_2 = 2$.

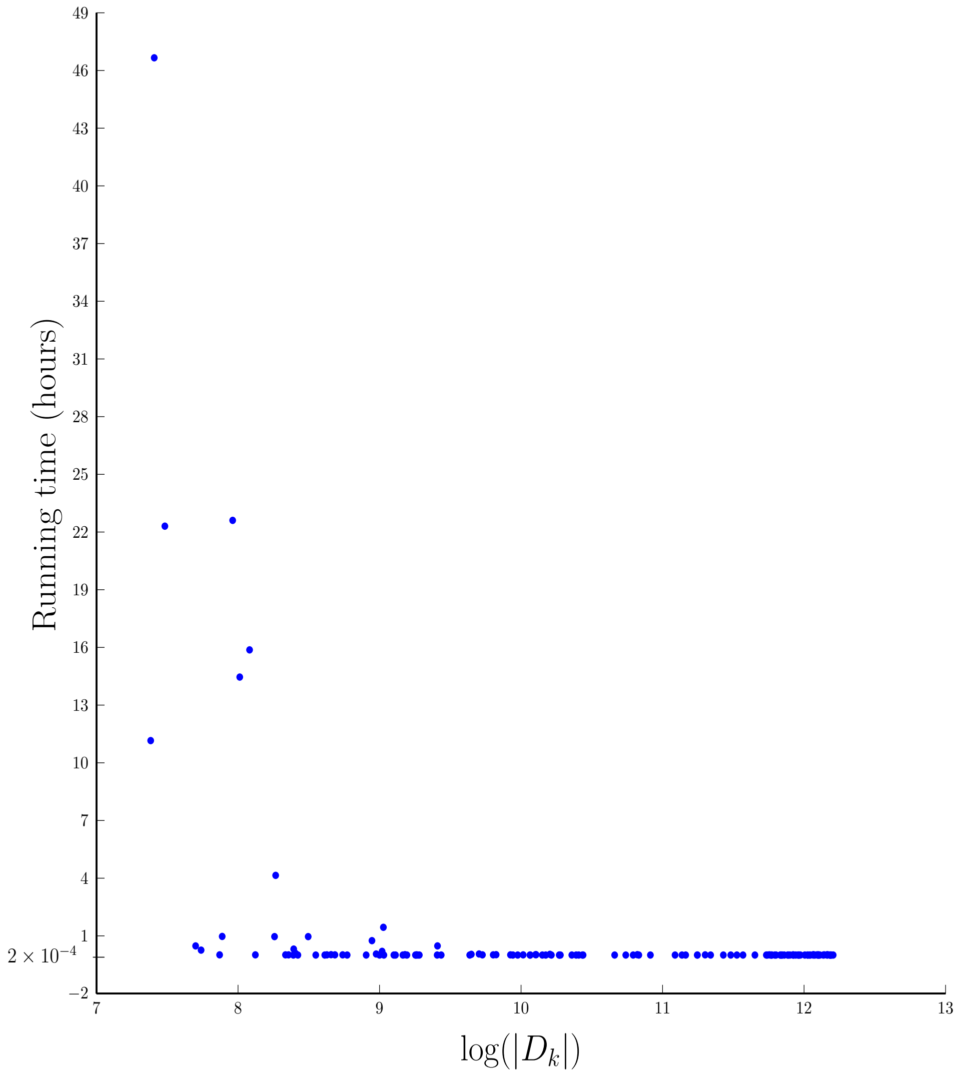


Figure 5.6: Log discr. against number of cones for totally real quartics.

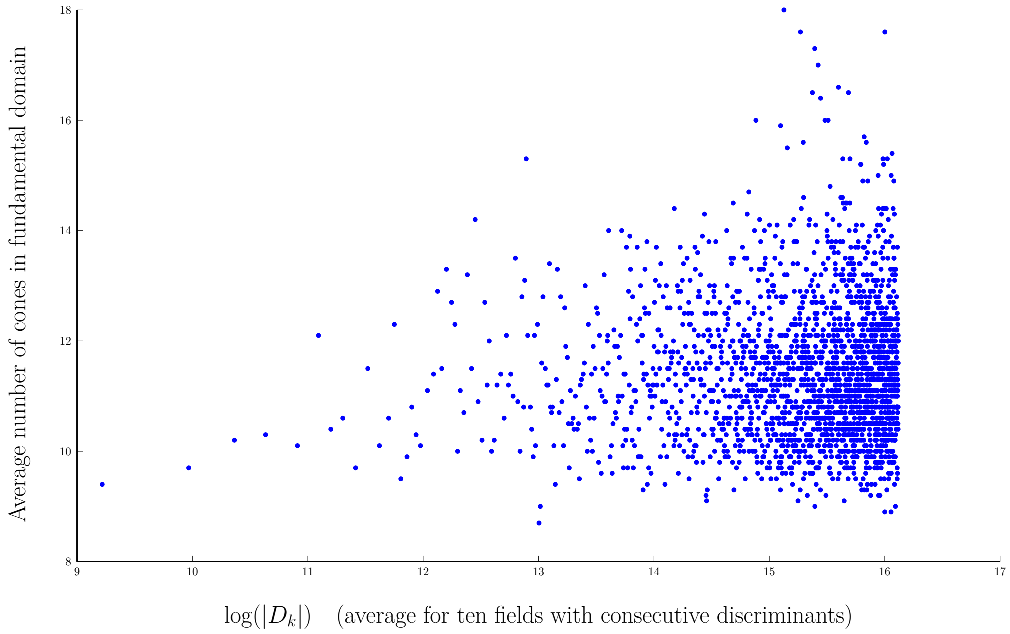


Figure 5.7: Log discr. against number of units processed for totally real quartics.

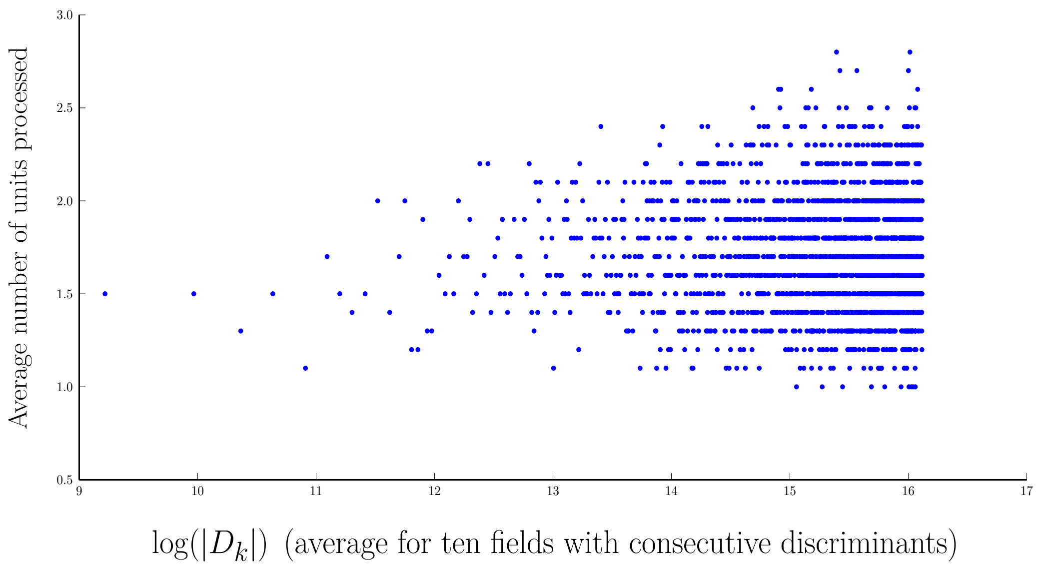


Figure 5.8: Log discr. against number of cones for quartics with $r_2 = 1$.

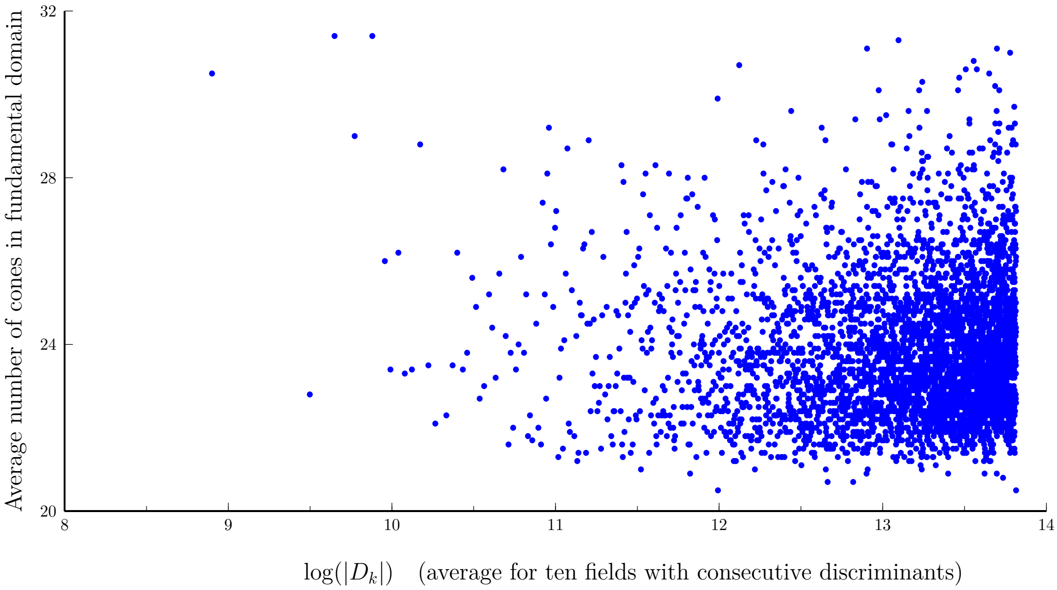


Figure 5.9: Log discr. against number of units processed for quartics with $r_2 = 1$.

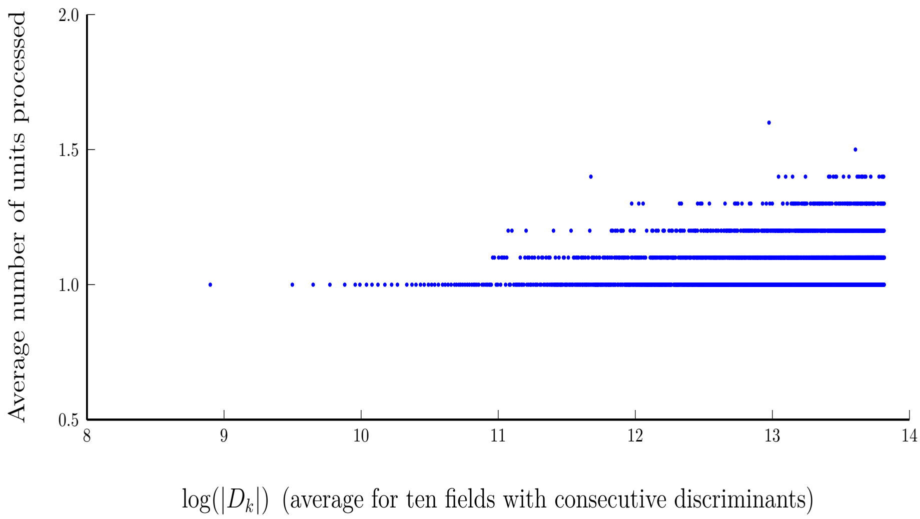


Figure 5.10: Log discr. against number of cones for totally real quintics.

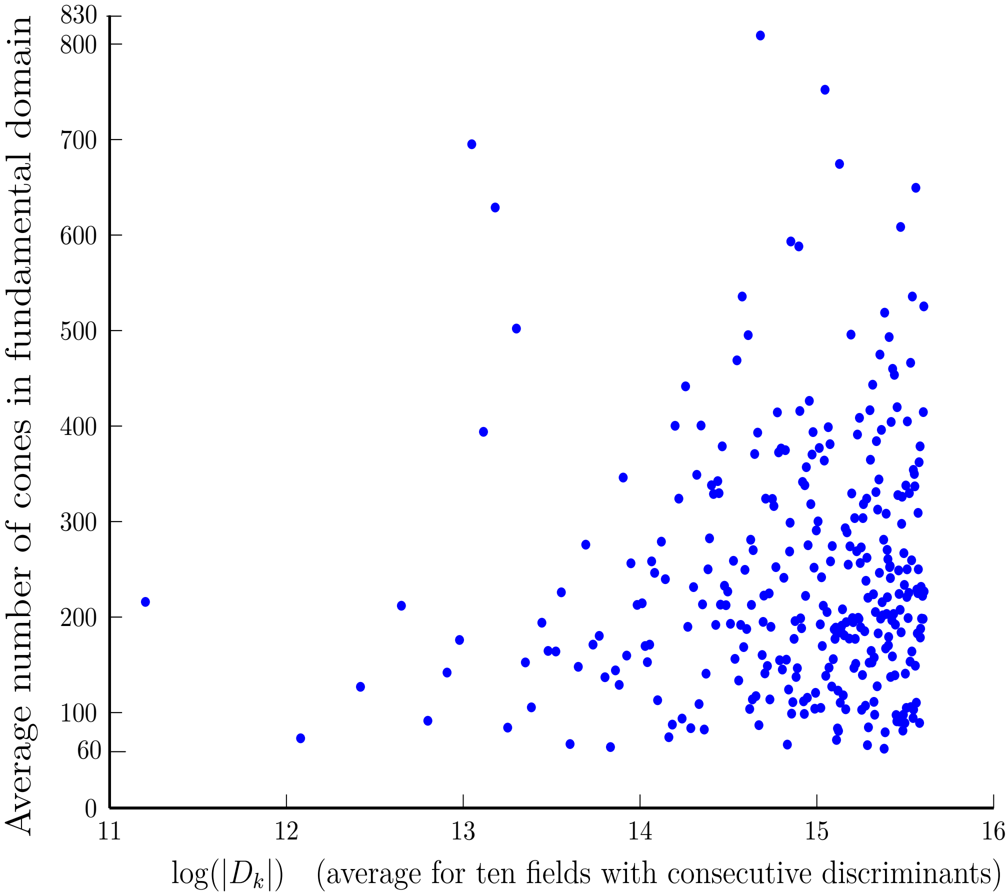


Figure 5.11: Log discr. against number of units processed for totally real quintics.

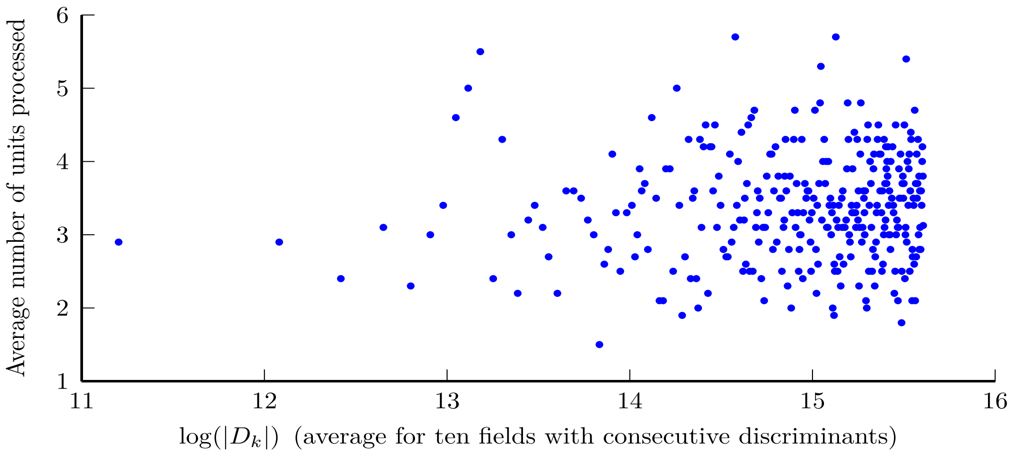


Figure 5.12: Log discr. against number of cones for quintics with $r_2 = 1$.

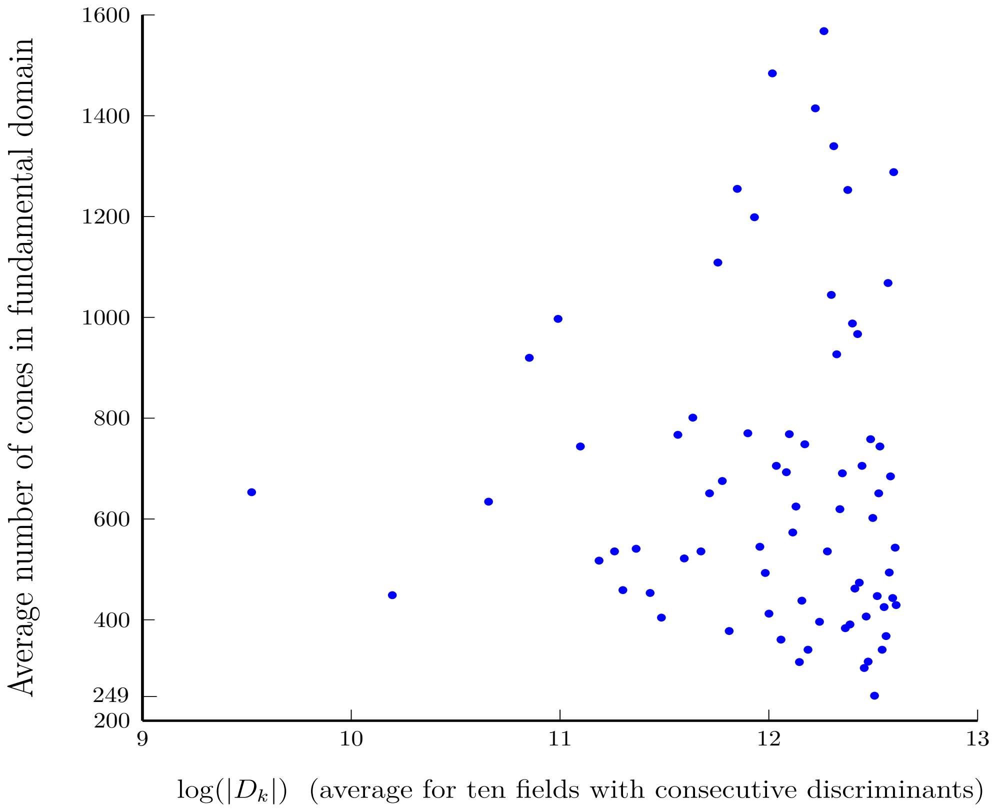


Figure 5.13: Log discr. against number of units processed for quintics with $r_2 = 1$.

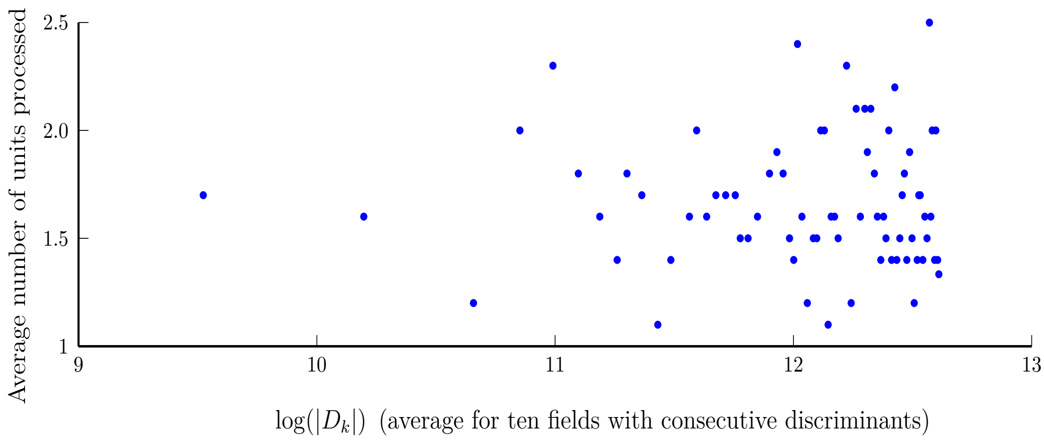


Figure 5.14: Log discr. against **log**(number of cones) for quintics with $r_2 = 2$.

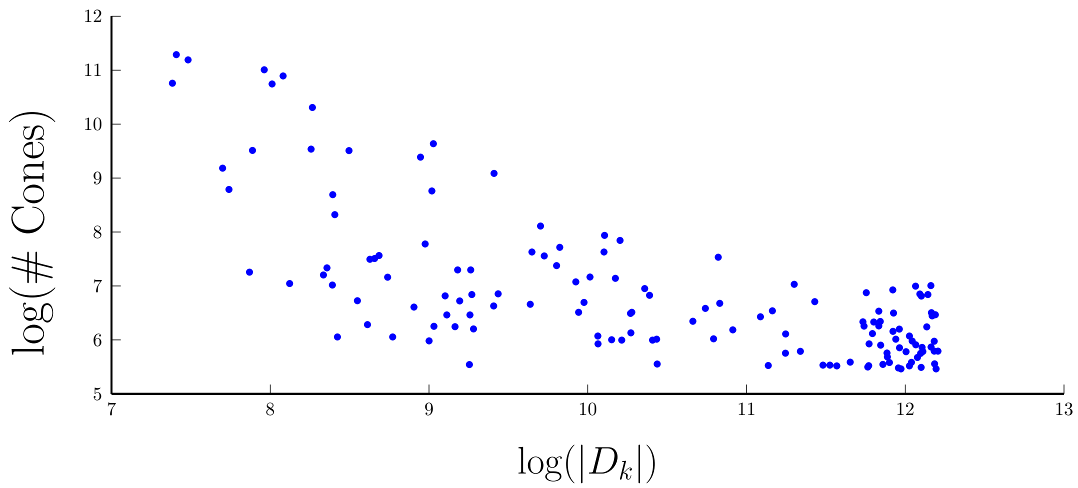
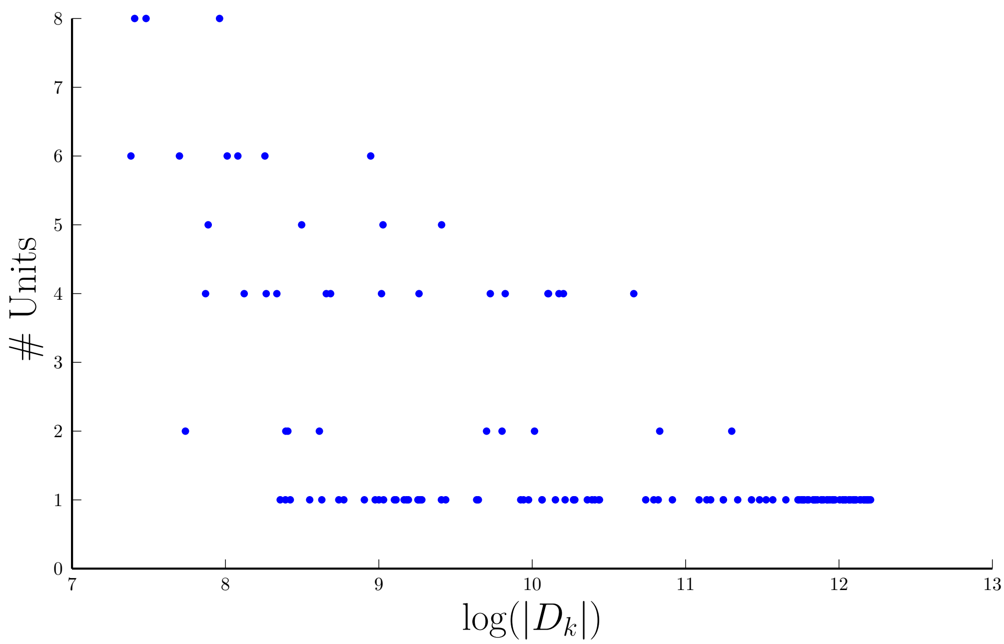


Figure 5.15: Log discr. against number of units processed for quintics with $r_2 = 2$.



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