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ENDOGENOUS TIMING IN DUOPOLIES

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In this thesis we aim to understand how does leadership emerge in duopolistic competitions. In particular, we want to know which are the key features, of the firms and the market, that explain that some interactions are simultaneous while others are sequential. In order to do so, we develop a general model of duopolistic competition in which the timing of movements is not exogenously given, but is part of the equilibrium, this is, depends on the actions of the players. More precisely, we consider two firms that are absolutely identical except for one single characteristic that makes them different. To fix ideas, it is possible to think this feature as the marginal cost or capacity of production. We interpret this difference as the consequence of different levels of investment made prior to the competition. This investment variable might be *tough* or *soft*, which means that the total effect of the investment on the payoff of the other player is negative or positive, respectively. After the investment, firms engage in supermodular or submodular competition. This competition can be (a priori) simultaneous or sequential, allowing us to endogenously obtain the timing of movements in equilibrium. In order to do so, we use the extension models from Hamilton and Slutsky (1990), namely, the Game with Observable Delay (GOD) and the Game with Action Commitment (GAC). When there is multiple equilibria, we base our refinement on the risk dominance concept from Harsanyi and Selten (1998).

For the supermodular case, we found that simultaneous competition is never the outcome of the interaction, neither with GOD nor GAC. In the GOD extension model this result comes from the fact that the existence theorem, in our setting, predicts that only sequential play is an equilibrium. In the GAC model the result comes from the refinement process based on risk considerations and the nature of the investment. Also, our results predict that, when the investment variable is tough, the firm with the largest investment is more likely to become the risk dominant leader, for both extension models. When the investment variable is soft, we provide sufficient (but not necessary) conditions for the leadership of the firm with the largest investment. Regarding this last point, we still need to work further on finding necessary hypotheses to characterize the leadership.

For the submodular case, we fully characterize which equilibrium will emerge when the extension model is GOD: simultaneous competition. This result holds regardless of the type nor level of investment. On the other hand, for the GAC extension model, we find that the simultaneous equilibrium is never the risk dominant (and therefore it should never emerge). Also, when the investment variable is tough, the firm with the largest investment is more likely to become the leader. In the case of soft investment, as with supermodular competition, we give sufficient conditions for the leadership of the player with the largest investment. Considering the results obtained in this setting, we also provide an interpretation of the differences between both extension models, GOD and GAC, based on risk considerations.

RESUMEN DE LA TESIS PARA OPTAR
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En esta tesis buscamos comprender cómo emerge el liderazgo en competencia duopólica, en particular cuales son los elementos esenciales de las firmas y del mercado que explican que algunas interacciones sean simultáneas y otras secuenciales. Para realizar aquello, desarrollamos un modelo general de competencia duopólica en el cual el timing de movimientos no viene dado exógenamente, sino que es parte del equilibrio, es decir, depende de las acciones de los jugadores. Más precisamente, consideramos dos firmas idénticas, excepto por una sola característica que las diferencia. Para fijar ideas, se puede pensar esta característica como el costo marginal o la capacidad de producción. En el modelo interpretamos esta diferencia como consecuencia de distintos niveles de una inversión hecha antes de la competencia. Esta variable de inversión puede ser de tipo *tough* o *soft*, lo que significa que el efecto total de la inversión en el pago del rival es negativo o positivo, respectivamente. Luego de la etapa de inversión, las firmas enfrentan competencia que puede ser supermodular o submodular. Esta competencia a su vez puede ser simultánea o secuencial, lo que nos permite obtener de forma endógena el *timing*. Para llevar a cabo esto último, usamos los modelos de extensión de Hamilton and Slutsky (1990): *Game with Observable Delay (GOD)* y *Game with Action Commitment (GAC)*. En caso de múltiples equilibrios, usamos como refinamiento la dominancia en riesgo de Harsanyi and Selten (1998).

Para el caso supermodular, encontramos que la competencia simultánea nunca es el resultado de la interacción, ni con GOD ni GAC. En el caso del modelo GOD, este resultado viene del teorema de existencia, que en nuestro modelo, predice que únicamente la competencia secuencial es la que emerge. En el modelo GAC, el resultado es consecuencia del proceso de refinamiento basado en consideraciones de riesgo y el tipo de inversión. Además, nuestros resultados predicen que, cuando la variable de inversión es *tough*, es más probable que la firma con la mayor inversión emerja como líder (basado en riesgo), en ambos modelos de extensión. Cuando la variable de inversión es *soft*, entregamos condiciones suficientes (pero no necesarias) para el liderazgo de la firma con mayor inversión. Con respecto a este último punto, aún necesitamos trabajar en encontrar condiciones que sean necesarias para caracterizar el liderazgo.

En el caso submodular, caracterizamos completamente cual equilibrio emerge cuando el modelo de extensión es GOD: competencia simultánea. Este resultado es válido para todo tipo y nivel de inversión. Por otro lado, para el modelo de extensión GAC, encontramos que el que el equilibrio simultáneo nunca es el que domina en riesgo (y en consecuencia, nunca debería emerger). Además, cuando la variable de inversión es de tipo *tough*, la firma con la mayor inversión inversión es más probable que se convierta en líder. En el caso de inversión *soft*, al igual que con competencia supermodular, entregamos condiciones suficientes para el liderazgo de la firma con la mayor inversión. Considerando en estos resultados, damos una interpretación de las diferencias entre ambos modelos de extensión, basados en consideraciones de riesgo.

"Life's a piece of shit
When you look at it
Life's a laugh and death's a joke, it's true
You'll see it's all a show
Keep 'em laughin' as you go
Just remember that the last laugh is on you
And always look on the bright side of life"

- Eric Idle, Graham Chapman, Terry Jones,
Michael Palin, John Cleese & Terry Gilliam.

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As I approach to the end of this acknowledgements section, I would like to make a random comment. In case of you are young person at some point in the future that somehow came across with this work and is reading this pages while deciding what to do in life; here is an advice for you, mate: do what you love. Do not be afraid of quitting what you are doing, in case it is not making you happy. Going deeper in the hole is not how you find the way out of it. The pain does not vanish, trust me. Time flies, do not waste it doing things that don't light the fire inside you.

Finally, a big thank to myself, because I'm just amazing.

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Introduction

One of the key features that distinguishes different models of oligopolistic competition is whether firms take actions simultaneously or sequentially. In the former case, each firm decides what to do without knowing what the other does, and in the latter, a firm acts as a leader and the other as a follower, meaning that the follower knows what the leader did before deciding her own action. When firms compete in quantities, these models are commonly known as Cournot and Stackelberg competition respectively[1], and when competition is in prices, Bertrand and Price Leadership. The key difference between simultaneous and sequential games, driving the difference in equilibrium outcomes, is how to calculate the equilibria. In the simultaneous case, the equilibria are obtained by intersecting reaction curves. On the other hand, in the sequential framework, the follower computes her best response and therefore reacts to what the leader does; but the leader plays anticipating that the follower will do so. This anticipation means that the leader incorporates the best response of the follower in his own decision problem. Of course this previous explanation is considering that both players are rational. The traditional oligopoly literature treated this feature as exogenously given, meaning that the game was already defined as simultaneous or sequential, and if it was sequential, the roles of leader follower were also defined exogenously. For instance, in textbook security Stackelberg games, it is assumed that the defendant commits to a strategy first, and then the attacker plays in response to that strategy.

In late 80's and early 90's, economists started to think that whether duopolist playing a simultaneous or sequential game should not be exogenous, but the result of their own prior decisions. At the beginning, this mainly led to the question if firms preferred to move first and be a leader, or second and be a follower, depending on their characteristics. For instance, it can be proved that if firms are absolutely symmetrical, the role of leader is preferred in the case of quantity competition, and the role of follower is better if the competition is in prices. Two of the most important frameworks to work on making the timing endogenous were provided by Hamilton and Slutsky (1990). Their models will be described in detail further in this introduction but, broadly speaking, the general idea is the following: for a basic interaction (namely price or quantity competition), they define an extended game by adding a pre play stage, in which the players have to take an action related to the timing; this action is indeed simultaneous. Now, when the equilibria of the extended game are computed, this equilibria induce a timing of movements on the basic interaction. A different approach to make the timing of movements endogenous, is the continuous scheme of Robson (1990).

[1] Although the original Stackelberg model only specifies that competition is sequential, not the competition variable.

In his model, the players can decide when to take their actions in a continuum of time, and in that sense, is not discrete as the models of Hamilton and Slutsky (1990). Since Hamilton and Slutsky (1990), there have been several papers addressing the issue of which firm will come up as a leader if certain conditions are fulfilled. The most common insight of these models is that, in some cases having lower marginal cost or more capacity of production or greater investment in R&D than your rival, could be beneficial to you in the sense of that it could give you the best role in the market.

Before properly reviewing the literature on endogenous timing, it is necessary to present the tools needed to prove the existence, compute and refine the equilibria. In order to do so, we will first introduce the models from Hamilton and Slutsky (1990), and then the most used refinement criteria from Harsanyi and Selten (1998).

1 Endogenous Timing Models

As we have said before, two of the classical schemes to model endogenous timing of decisions in oligopolistic contexts were provided by Hamilton and Slutsky (1990). For a *basic* game (for instance Cournot or Bertrand), they define two models in order to *extend* the game and make the timing of movements endogenous: the Game with Action Commitment (GAC from now on) and the Game with Observable Delay (GOD from now on). Both models share the characteristic that a pre play stage is added, but the difference relies on the actions that players can take in that pre play stage. As it is crucial to our work to understand these models, the next two subsections describe each one of them.

1.1 Extended Game with Observable Delay

Given a basic game, in the pre play stage of the GOD model players have two possible actions to take: to move *Early* or *Late* in such basic game. After they make their choices, this becomes public knowledge and the basic game is played as follows: if both players chose to move early, they play a simultaneous game; the same if both of them chose to move late. If the choices were not the same, they play a sequential game in the order defined by their actions.

Formally, consider a basic game with two players $N = \{A, B\}$, actions spaces α and β (convex and compact subsets of $\mathbb{R}$), and payoff functions $a : \alpha \times \beta \to \mathbb{R}$ and $b : \alpha \times \beta \to \mathbb{R}$. As we said before, the *Extended Game with Observable Delay* is defined by adding a pre play stage, in which the players can choose the time to posteriorly select an action on the basic interaction. The set of possible times is $T = \{F, S\}$ (where F and S stands for *first* and *second* respectively). Let $i \in T$ be the period in which A has chosen to select an element of α, and $j \in T$ the period in which B has chosen to select an element of β. The set of strategies for players A and B in the extended game are:

$$S_A = \{F, S\} \times \Phi_A.$$
$$S_B = \{F, S\} \times \Phi_B.$$

where Φ_A is the set of functions that map $\{(F,F),\ (F,S),\ (S,F)\times\beta,\ (S,S)\}$ into α, and Φ_B is the set of functions that map $\{(F,F),\ (F,S)\times\alpha,\ (S,F),\ (S,S)\}$ into β. Given a pair of strategies $s_A=(i,\phi_A)\in S_A$ and $s_B=(j,\phi_B)\in S_B$, the payoff for player A in this extended game is defined as follows (analogous for player B):

$$P_A(s)=\begin{cases} a[\phi_A(i,j),\ \phi_B(i,j)] & \text{if } (i,j)\in\{(F,F),\ (S,S)\},\\ a[\phi_A(F,S),\ \phi_B(F,S,\ \phi_A(F,S))] & \text{if } (i,j)=(F,S),\\ a[\phi_A(S,F,\ \phi_B(S,F)),\ \phi_B(S,F)] & \text{if } (i,j)=(S,F).\end{cases}$$

The normal form of the Extended Game with Observable Delay is:

A/B	F	S
F	$a_s,\ b_s$	$a_l,\ b_f$
S	$a_f,\ b_l$	$a_s,\ b_s$

Where a_s, a_l and a_f denote simultaneous, leader and follower equilibrium payoffs in the basic game, respectively. Analogous for b.

For this model of extended game, the authors proved the following results:

- If $a_s > a_f$ and $b_s > b_f$, then the unique subgame perfect equilibrium is with both players choosing F, which results in simultaneous play in the basic game.

- If $a_f > a_s$ and $b_f > b_s$, then the two sequential configurations ((L,F) and (F,L)) are equilibria of the extended game. There is also a mixed strategy equilibrium, in which players randomize between their two options.

- Sequential play is the unique subgame perfect equilibrium if and only if, one player choosing F and the other S, has an outcome which Pareto dominates the simultaneous play game, and the reverse order of movements yields an outcome which does not.

Observation It is important to mention that, in a posterior note, Amir (1995) highlighted that the previous results also require monotonicity on the best response functions in order to be true.

1.2 Extended Game with Action Commitment

In the case of the GAC model, if players want to move early they have to commit to an action. For instance, if the basic game is in quantities, they have to set a quantity $\bar{q}$ if they want to achieve the leader position (or more precisely to avoid the follower position). The other option they have is to choose *wait*. Therefore, in this case the action space on the pre play stage is not finite as in the GOD model, here is in general $S_i \cup \{W\}$, where S_i is the strategy space of the basic game and W represents waiting. If both players commit, then a simultaneous game is played, the same if both choose to wait; and if one firm commits and the other one does not, the latter can observe the action took by the first one before choosing its own action.

Formally, consider again a basic game with two players $N = \{A, B\}$, actions spaces α and β (convex and compact subsets of $\mathbb{R}$), and payoff functions $a : \alpha \times \beta \to \mathbb{R}$ and $b : \alpha \times \beta \to \mathbb{R}$. Let W be the action of waiting until the second period to choose an action. The set of strategies for each player is now:

$$S_A = \{\alpha_i,\ W \times \Psi_A(\beta_i)\}.$$
$$S_B = \{\beta_i,\ W \times \Psi_B(\alpha_i)\}.$$

Where $\alpha_i \in \alpha$, $\beta_i \in \beta$, Ψ_A is the set of functions that map $\beta \cup \{W\}$ into α , and Ψ_B is the set of functions that map $\alpha \cup \{W\}$ into β.

Given a pair of strategies $s_A \in S_A$ and $s_B \in S_B$, we define:

$$\hat{\alpha}_i = \begin{cases} \alpha_i & \text{if } s_A = \alpha_i. \\ \psi_A(\beta_i) & \text{if } s_A = W \times \psi_A(\beta_i). \end{cases}$$

$$\hat{\beta}_i = \begin{cases} \beta_i & \text{if } s_B = \beta_i. \\ \psi_B(\alpha_i) & \text{if } s_B = W \times \psi_B(\alpha_i). \end{cases}$$

Then, payoffs are $P_A(s) = a[\hat{\alpha}_i, \hat{\beta}_i]$ and $P_B(s) = b[\hat{\alpha}_i, \hat{\beta}_i]$.

Observation Note that the commitment feature also implies that if a player chose an action on the pre play stage, she cannot "correct" her decision after observing what the other player did. For instance, if the basic interaction is Curnot duopoly, and a player commits to a quantity q_1 on the pre play stage, she cannot produce an additional quantity q_2 afterwards.

For this model of extended game the authors proved that, in general, there are three equilibria in pure strategies:

- Both playing their simultaneous equilibrium actions in the first period. It is important to mention that this strategy is weakly dominated by waiting.

- One player waiting and the other committing to her leader action in the first period.

In order to point out the difference between both models, consider a model of quantity competition with two firms as in Hua-Yang et al. (2009). If we want to make the timing endogenous and the scheme is the GOD model, we have the following situation in the pre play stage:

	E	L
E	$\Pi_1^N,\ \Pi_2^N$	$\Pi_1^L,\ \Pi_2^F$
L	$\Pi_1^F,\ \Pi_2^L$	$\Pi_1^N,\ \Pi_2^N$

Where E and L represents the possible strategies for each firm (move *early* or *late*), Π_i^L and Π_i^F are the equilibrium profits of the leader and follower in a sequential game, that is a textbook Stackelberg game, and Π_i^N is the equilibrium profit in the simultaneous case. After a brief inspection, it is possible to notice that the unique equilibrium of the game is (E, E), which results in a simultaneous basic game in which both firms move early.

Observation Actually as we mention before, in the GOD model this result (both firms moving simultaneously in the basic game) is valid for every basic game, no matter which is the competition variable or the nature of firms, in which the following inequality holds:

$$\Pi_i^F < \Pi_i^N < \Pi_i^L.$$

Now if we want to use the GAC model, the results are quite different. First of all, we can not represent the pre play stage as a matrix game because the strategy spaces are now not finite, and also in this framework, there are three pure strategy equilibria (not just one as in the GOD model): (W_1, q_2^L), (q_1^L, W_2) and (q_1^N, q_2^N); where q_i^L, q_i^F and q_i^N are the equilibrium quantities of leader, follower and simultaneous. In Chapter 1 we present a model that will allow us to go further in the analysis of the differences between both extension models.

As we have seen up to this moment, and we will notice throughout this document, it is natural to find multiple equilibria for these kind of extended games. In such cases we will need a refinement concept in order to obtain the desired results, and due to the nature of the games, it will be impossible to apply the most classical payoff dominance criteria. Given this, we introduce the following refinement concept which will be useful not only to select an equilibrium, but also will allow us to interpret the differences between GOD and GAC in terms of incentives.

2 Risk Dominance and The Tracing Procedure

A natural criteria to refine multiple equilibria in a game is payoff dominance. Frequently in duopoly theory, when there are multiple equilibria, is not possible to apply this criteria because of the different preferences of the players. In such cases, an interesting concept from Harsanyi and Selten (1998) applies. They develop a refinement concept, based on the idea that some equilibrium might entail more risk to be played than others. This idea of "risk dominance" has quite interesting properties, and gives insightful interpretations to some results in the literature.

Suppose that you have a game with two equilibria. Intuitively, the concept of risk dominance captures the idea that, since players do not know which of the equilibria would emerge, they will measure the risk associated to each strategy they may play in the different equilibria. Risk is measured as how much they loose by playing the strategy of one equilibrium when the other player played the strategy of a different equilibrium . They will measure the risk involved in playing each of these equilibria and they will coordinate expectations on the less risky one, which is called the risk dominant equilibrium. In order to define this risk dominance concept, Harsanyi and Selten (1998) develop an axiomatic theory that concludes, for the case of 2×2 games, in only one order relation that satisfies the axioms: deviation losses. In the case of general games is much more complicated, and the definition of risk dominance requires two previous concepts: the *bicentric prior* and the *tracing procedure*. The *bicentric prior* describes the initial assessment of the players about the situation. If this initial assessment is not an equilibrium of the game, it can not be the final view of the players and then, they have to adjust their plans until they are in equilibrium. The *tracing procedure* is a formal model of this adjustment process.

To fix ideas, let us consider the following 2×2 example, which also exhibits the difference between payoff dominance and risk dominance.

	A	B
A	80,80	80,0
B	0,80	100,100

This game has two equilibria in pure strategies: (A, A) and (B, B). Note first that (B, B) is the payoff dominant equilibrium. On the other hand, if we denote L_i the deviation losses associated to equilibrium i, we have that:

$$L_{(A,A)} = (80 - 0) \times (80 - 0).$$
$$L_{(B,B)} = (100 - 80) \times (100 - 80).$$

Therefore, the risk dominant equilibrium is (B, B).

Formally, let be $g = (S_1, S_2, u_1, u_2)$ a game of two players with strategy spaces S_1 and S_2, and payoff functions u_1 and u_2. Suppose that there two equilibria in g, $s = (s_i, s_j)$ and $s' = (s'_i, s'_j)$. Harsanyi and Selten (1998) argue that when players are uncertain about which of these two equilibria should be played, their initial beliefs should be constructed as follows:

- Player j believes that i will play s_i with probability z_j and s'_i with probability $(1 - z_j)$. Therefore, j will play a best response against $z_j s_i + (1 - z_j) s'_i$. Denote this best response by $b_j(z_j)$.

- Player i does not know z_j, and then she assumes that $z_j \sim U([0, 1])$. Consequently, if we consider a random variable $Z_j \sim U([0, 1])$, we can state that i will believe she is playing against $m_j = b(Z_j)$. This m_j is the *prior belief* of player i about j's behaviour.

- Analogously, we can construct the prior belief m_i of player j about i's behaviour.

- The pair $m = (m_i, m_j)$ is called the *bicentric prior* associated to s and s'.

Now we describe the *tracing procedure*, which, intuitively, is a map converting initial beliefs into equilibria of the game. Let m_i be a mixed strategy for player i. This mixed strategy represents the initial uncertainty of player j about i's behaviour at the beginning of the game. For a pair of mixed strategies $m = (m_i, m_j)$, and every $t \in [0,1]$, we define the perturbed game $g^{t,m} = (S_1, S_2, u_1^{t,m}, u_2^{t,m})$ which has the same strategies and players of g, but different payoff functions. Specifically, this new payoffs are given by:

$$u_i^{t,m}(s_i, s_j) = (1-t)u_i(s_i, m_j) + tu_i(s_i, s_j).$$

Observation For $t = 0$, we have a game in which the payoff of each player depends only on his own behaviour and prior beliefs. For $t = 1$, the game $g^{t,m}$ coincides with g.

For this family of perturbed games, it is possible to define the equilibrium correspondence graph Γ^m, which is the representation of the equilibria of the perturbed game for each value t:

$$\Gamma^m \doteq \{(t, s):\ t \in [0,1],\ s \text{ is an equilibrium of } g^{t,m}\}.$$

In Schanuel et al. (1991) it is shown that if g is finite, then for almost any[2] pair of prior beliefs m, Γ^m contains a unique distinguished curve that connects the equilibrium $s^{0,m}$ with an equilibrium $s^{1,m}$ of $g^{1,m}$. The interpretation is as follows: suppose that the original game has two equilibria $\bar{s}_1$ and $\bar{s}_2$. If players have initial beliefs about their opponent actions given by m, and they adjust this beliefs according to the tracing procedure in terms of $u^{t,m}$, eventually converging to $\bar{s}_1$, we say that $\bar{s}_1$ *risk dominates* $\bar{s}_2$. If the procedure converges to $\bar{s}_2$, we have the opposite situation. If the process does not reach neither $\bar{s}_1$ nor $\bar{s}_2$, we can not establish a comparison in terms of risk. This idea leads to the following formal definition.

Definition 1 *If $s^{1.m} = s$, the equilibrium s risk dominates s'. If $s^{1,m} \neq s, s'$, neither of the equilibriums risk dominates the other one.*

Observation A final observation to make is that this relation between equilibria is not necessarily transitive. This is, if s^1 risk dominates s^2, and s^2 risk dominates s^3, it could happen that s^1 would not risk dominate s^3.

3 Literature Review

Having covered the endogenous timing models and the main refinement concept useful to deal with multiple equilibria, now is time to review the literature that seeks to make the timing of movements endogenous for specific oligopoly models. An approach that will be suitable for our purposes, and a contribution of this thesis, is to organize the literature according to

[2]The set of prior beliefs for which Γ^m contains more than one curve has measure zero.

the strategic space, and the model used to make the timing endogenous, that is, quantity (or submodular competition) and price (or supermodular competition) competition models, and GAC vs GOD. We will see that a common denominator of many articles is that firms are absolutely symmetrical, except for one single characteristic that makes them different. This difference is what allows to perform a risk analysis, in order to refine the possible multiple equilibria.

Let us start by reviewing the papers related to price competition. Li (2014) presents a model of a vertically differentiated duopoly (quality exogenously different), in which firms with identical costs choose prices, and there is a continuum of consumers with different taste for quality indexed by $\theta \in [0, 1]$. He uses the GOD model to make the timing of movements endogenous (price choice; quality is exogenous), and finds that there are two equilibria of the extended game: sequential price setting with each firm being a leader, and the other one a follower. Simultaneous play is not an equilibrium. Using the risk dominance criteria, shows that the risk dominant equilibrium is with the high quality firm being a leader. In a more recent work, Lambertini and Tampieri (2017) study a vertically differentiated duopoly, in which the timing of movement at the quality stage is determined endogenously using GOD (unlike Li (2014)). They find that firms always select sequential play at the quality stage, and when there is full market coverage, the high-quality firm emerges as the leader. When the relevant variable is location, Lambertini (1997) describes a two stage game of a spatial duopoly with consumers distributed along a linear city of finite length. Firms produce a homogenous good, and they have zero production costs (firms are symmetrical). In the first stage they choose their locations, and in the second one they choose their prices. The author uses the GOD model to add a pre play stage for both decisions, location and prices, and finds that the unique subgame perfect equilibrium is with firms moving simultaneously in location stage, and sequentially in the price stage (up to a permutation since firms are symmetrical). The leader in the price stage locates at one border of the city, and the follower beyond the opposite end of the city. In a similar context, Meza and Tombak (2009) present a Hotelling model in which firms with different marginal costs can determine their locations and prices. Their insight is quite different from the others in the literature, since the time is modelled by a continuum $[0, \infty)$, and they do not use neither GOD nor GAC to make the timing endogenous. What they do is to define a three stage game: in the first stage firms decide when to enter (the roles of leader and follower are determined at the end of this stage), in the second stage they enter into the market in the moment defined previously and choose their location (if there is a follower, she knows the leader position before deciding); finally in the third stage, they choose prices simultaneously. It is shown that, when cost differences are large enough, the game yields Stackelberg behaviour where the high cost firm will delay choosing a location until the low cost firm commits to its position. In the case of firms with different capacities of production, there are two main papers that work on endogenous timing of movements. Deneckere and Kovenock (1992) present a price duopoly with different capacity constraints, firms have equal marginal costs and face a common demand function $\mathrm{d}(p)$, with $p = (p_1, p_2)$ (i.e. they provide a homogenous good). Consumers buy from the cheapest firm until that firm complete its capacity. The authors show that the firm with greater capacity is indifferent between being a leader, follower or simultaneous competition; while the small firm is indifferent between being leader or simultaneous competition, but strictly prefers being a follower. Given these results, they argue that it should be natural for the large firm to emerge as a leader, and they provide a theoretical model in which this happens. In Furth and Kovenock (1993), the authors present a very similar model but with the difference that the goods are imperfect substitutes. In that context, they analyse the preference of each firm for each role, and conclude that the large firm should emerge as a leader. They also provide a theoretical model in which this occurs. An approach that is particularly interesting for the work that we develop in this thesis, is the one from Amir et al. (1999). They study a Bertrand duopoly with differentiated products and different marginal costs. The authors compare the payoffs of three different games: simultaneous play, and two sequential with each firm being a leader. They make the timing endogenous using the GOD model, and find different results depending on the demand function characteristics. If prices are strategic complements, they show that both firms prefer sequential play (in any order of movements), before simultaneous. On the other hand, if prices are strategic substitutes, both firms prefer simultaneous play over being a follower. On a posterior paper, Amir and Stepanova (2006) using the risk dominance criteria, proved the same result but in a more general context, by relaxing some assumptions on the functions involved. In the same context, meaning firms with different marginal costs and differentiated products, van Damme and Hurkens (2004) use the GAC model to show that the more efficient one will emerge as a leader in the price stage. First, they show that there are three possible equilibria of the extended game: simultaneous and both sequential. After that, they select the equilibrium with the efficient firm being a leader by performing a risk dominance analysis based on Harsanyi and Selten (1998). It is important to mention that, to our knowledge, this is the only paper that applies the risk dominance concept to a GAC model in price competition.

Now let us focus the attention to quantity competition. Amir et al. (2000) presents a two stage R& D/Cournot model, in which asymmetrical firms (different marginal costs) invest in R&D, resulting in lower marginal costs. After doing so, firms compete in quantities. The model considers linear demand, differentiated products, general costs of R&D and also spillovers, that is, a fraction α of the investment contributes to the improvement of the other firm. The authors use the GOD model in the R&D stage, and show that, depending on certain conditions on the spillovers and the demand functions, sequential and simultaneous play can emerge in the R& D stage. The results depend on the relation between the variables (strategic substitutes or complements). Also in the R&D context, Tesoriere (2008) studies the same model of Amir et al. (2000) but with different assumptions. In this case, the firms are symmetrical and spillovers only exists if there is sequential play in the R&D stage, flowing only from the leader to the follower. The author compares the payoffs of three different games to make the timing of movements endogenous in the R&D stage: G_{sim}, $G_{seq,i}$ and $G_{seq,j}$ which represent simultaneous play, sequential play with firm i being a leader, and sequential play with firm j being a leader, respectively. The author uses the GOD model, and finds that the only structure sustainable as a subgame perfect equilibrium is with simultaneous play in the R&D stage. In a working paper, Lambertini and Tampieri (2011) study a vertically differentiated duopoly in which firms have different qualities, which they can improve assuming a cost (a convex function of the investment). More specifically, their model has two stages: on the first one, firms choose their quality, while on the second one, they compete (simultaneously) in quantities. The authors make the timing of movements endogenous in the quality stage using the GOD model, and find that the unique subgame perfect equilibrium is with the low quality firm acting as a leader. In a very similar context,

Investment/Variable	Price	Quantity
Quality	Li (2014) Lambertini & Tampieri (2017)	Lambertini and Tampieri (2011) Jinji (2004)
Location	Lambertini (1997) Meza and Tombak (2009)	
Capacity	Deneckere and Kovenock (1992) Furth and Kovenock (1993)	Lu and Poddar (2009)
Marginal Cost	Amir et al. (1999) van Damme and Hurkens (2004) Amir and Stepanova (2006)	Amir & Grilo (1999) van Damme and Hurkens (1999)
R&D		Amir et al. (2000) Tesoriere (2008)

Table 1: Endogenous Timing Summary (Price/Quantity)

Jinji (2004) presents a more general model, considering two extended games using GOD. The first one consists of three stages: on the first one, firms simultaneously chooses the order of movements in quality stage (this is the GOD part), while on the second stage, firms choose quality in the order defined previously; finally in the third stage, firms compete simultaneously in quantities. The second game that the author presents is similar to the the first one, but with a "stage zero" in which firms can simultaneously choose a relative position in the quality space. For the first game it is shown that there are two equilibria, and both result in simultaneous play in the quality stage. For the second game there are also two equilibria, both resulting in leadership of the low quality firm. Lu and Poddar (2009) present a duopoly in which firms can choose their capacities of production and after that their quantities. They use the GOD model for both decisions, and find that the unique subgame perfect equilibrium is with both firms moving simultaneously in both stages. If firms differ in marginal costs and the goods are homogeneous, van Damme and Hurkens (1999) show that the efficient firm (the low cost one) will move first. This paper is quite different from the others that analyse quantity competition, because the authors use the GAC model to extend the basic interaction. They find that there are three equilibria of the extended game in pure strategies: simultaneous play and the two sequential settings. After that, they perform a risk analysis using the tracing procedure of Harsanyi and Selten (1998), in order to show that risk dominant equilibrium is the one in which the low cost firm act as a leader. In the same context of different marginal cost, Amir and Grilo (1999) study minimum conditions on the demand and cost functions that lead to simultaneous and sequential equilibria. They show that the simultaneous equilibrium predominates, but the sequential can also appear under very restrictive conditions, depending on the shape of the reaction curves. It is possible to summarize the previous review in Table 1, where "investment" refers to the source of difference between the firms, and "variable" to the nature of the competition.

What we do in this doctoral project, is to define a general model that captures the incentives and underlying dynamics that yield (and generalize) most results in the literature, that is, a model such that every model in the previous Table 1 may be understood as a particular case of ours. Our claim is that there are essential features or assumptions of the models that drive the results, and therefore, endogenous leadership can, in general, be assessed in different economic situations, just by observing those features. The first step of our research is, then, to re classify the existing articles from the point of view of these relevant dimensions or features. The essential features that we identify are three: Type of competition (submodular or supermodular), model of endogenous timing (GAC and GOD), and finally the nature of the variable that makes firms different. We refer to this last characteristic as the *nature of investment*. To classify this investment, we are inspired by Fudenberg and Tirole (1984). In Fudenberg and Tirole (1984) the authors present two models to show that in some cases it is beneficial for an incumbent to underinvest, in order to deter entry of new firms. They exhibit this effect using investment in *advertising* and *R& D* but, in principle, investment could be in any relevant feature of the firms. Naturally, to stablish that there is underinvestment, they need a benchmark to compare. If entry is deterred, they use a monopolist investment to do so, and in the case of accommodation, they compare the investment to that in a "open loop" equilibrium, in which the incumbent takes the entrant actions as given, and does not try to influence her through its pre entry investment. This is, in the open loop equilibrium, investment is simultaneously decided along with the other strategic variable (such as price or quantity). In that framework, Fudenberg and Tirole (1984) provide a taxonomy to characterize the behaviour of a firm if it wants to accommodate or deter entry in terms of the nature of its previous investment, which could make the firms *tough* or *soft*. Intuitively, an investment (in some variable) makes a firm *tough*, if marginally adding more of that variable decreases the profit of the other; and makes it *soft* if the opposite holds. When we consider these three key features, then the literature can be re classified as it is presented in Table 2.

What immediately strikes the reader when looking at Table 2 is that, if we classify papers according to these three key features, each cell contains papers that have results that are similar in terms of leadership. For instance, if we look at supermodular competition with tough investment and using the GOD model, in all five papers the firm with the largest investment emerges as leader. In the case of submodular competition with GOD, all seven papers find simultaneous play regardless of the type of investment. Having looked at this, our objective is to "complete" Table 2. More formally, to answer the following question: *Is it possible to develop a general duopoly model, that captures the underlying dynamics of the previous models and generalize their results?* In a nutshell, is it possible to create a general duopoly model that allows to complete Table 2 directly with an endogenous leadership result, so that any specific duopoly game, corresponding to specific economic situations, can be quickly analysed? To tackle that leading question, we need to understand which are the essential characteristics, and economic incentives that lead to each possible outcome and then try to generalize. Finally, it is important to highlight that, to our knowledge, there are no articles addressing models that involve GAC and soft investment, as can be seen in the two empty cells on Table 2.

The following sections are organized as follows: in Chapter one we present a particular model of competition, in which firms with different marginal costs compete in quantities with a degree of differentiation. Our main goal with that model is to explore the differences between both models of extension, GOD and GAC, for the exact same basic interaction. We analize the differences in incentives of both extension models, and give an interpretation in terms of risk. Also, in that environment, we try to understand how does the degree of differentiation affects the timing, and also how the firms behave in a prior investment phase, knowing that their investment not only diminishes their marginal costs, but also could give

	Supermodular		Submodular	
	Tough	**Soft**	**Tough**	**Soft**
GOD Model	Li (2014) Lambertini & Tampieri (2017) Amir & Grilo (1999) Amir et al. (1999) Amir & Stepanova (2006)	Lambertini (1997) Amir et al. (2000)	Jinji (2014) Lu & Poddar (2009) Tesoriere (2008) Amir & Grilo (1999) Amir et al. (2000) Amir et al. (1999)	Jinji (2014)
	Tough	**Soft**	**Tough**	**Soft**
GAC Model	v. Damme & Hurkens (2004)		v.Damme & Hurkens (1999)	

Table 2: Re classification of the literature.

them the best role in the market in terms of leadership. In Chapters two and three, we study a general model of competition between two firms. First, they decide their level of investment in some variable that may be *tough* or *soft*, as in Fudenberg and Tirole (1984). After the investment is made, they engage in supermodular and submodular competition respectively. This latter competition can be simultaneous or sequential to allow the study of endogenous timing, and we will use GOD and GAC in order to tackle this last point. With this models, we would like to understand which player will emerge as leader under GOD and GAC, based on the nature of the variable that makes firms different, and possible more hypothesis. As we have said before, we want to understand and complete Table 2 based on the results that we obtain in Chapter 2 and Chapter 3.

Chapter 1

Endogenous Timing in a differentiated Cournot Duopoly

In this Chapter, we study a model in which two firms with different marginal costs compete in quantities, selling differentiated products. In this setting, we want to fully characterize which equilibrium will arise for the GOD and GAC extension models, considering risk dominance to refine possible multiple equilibria. The main motivation to tackle this model comes from the fact that it will allow us to compare the (potential) differences between GOD and GAC extension models, when they are applied to the exact same basic interaction. As it can be seen from Table 2, this is something that has never been done before.

The model presented in this Chapter is also interesting because it has not been covered in the literature. While van Damme and Hurkens (1999) study a similar model with homogeneous products, the difference is that our considers a degree of differentiation. In this sense, one of our secondary goals is to understand if this coefficient plays a role on the results, meaning that if there is a difference with the results obtained by the authors in van Damme and Hurkens (1999). Finally, in the last section of the Chapter, we study how the firms behave in a prior investment phase, knowing that this investment not only diminishes their marginal costs, but also could give them the best role in the market; this analysis has been seldom done in the literature Basso and Jara-Moroni (2013).

Formally, let us consider two firms which compete in quantities selling a differentiated product. They decide to produce q_i and q_j respectively, and face linear demands of the form:

$$p_i(q_i, q_j) = A - q_i - \alpha q_j.$$
$$p_j(q_i, q_j) = A - q_j - \alpha q_i.$$

Where A is a constant that represents the size of the market, and $\alpha \in (0, 1)$ is the degree of differentiation. At the beginning, firms have equal constant marginal cost $c > 0$. We consider that the interaction occurs in two stages:

- On the first stage, firms can make an investment in process R&D, i.e. in variables $b_i, b_j \in [0, c]$ that decreases c. If firm i invest b_i, she has to pay a cost $F(b_i)$, where $F(\cdot)$ is a convex function. Analogously for player j.

- On the second stage, firms compete in quantities and we consider that this interaction occurs in two periods, to allow the possibility of simultaneous or sequential play.

Given the previous set up, once all the decisions are made, the payoffs of firms i and j are:

$$\Pi_i(q_i, q_j, b_i, b_j) = (A - q_i - \alpha q_j - c_i(b_i))q_i - F(b_i).$$
$$\Pi_j(q_i, q_j, b_i, b_j) = (A - q_j - \alpha q_i - c_j(b_j))q_j - F(b_j).$$

Observations

- When looking at Table 2, this model would be included in the cells of tough investment and submodular competition (with GOD and also GAC, since we will cover both of them).
- Note that $\Pi_i(q_i, q_j, b_i, b_j)$ does not depend directly on b_j (analogous for firm j), which means that we are not considering spillovers due to the investment.
- $c_i < c$ is the marginal cost obtained after the investment b_i is performed (analogous for firm j).

As we have said before, we have two main goals in this chapter: in the first place, discover which timing will emerge after the investment (which is assumed to be simultaneous), and second, analyse the investment $b = (b_i, b_j)$ that firms would perform if they knew that this investment could give them the best (most preferred by them) role in the market.

1.1 Endogenous Timing and Equilibrium Selection

In this section, we make the timing of movements endogenous using the extended games defined in Hamilton and Slutsky (1990). We will be assuming that firms have already decided their level of variable b, and therefore, the function $F(\cdot)$ plays no role in determining the timing of the basic game. We assume $F(\cdot) \equiv 0$ without lost of generality. We first stablish which are the equilibrium strategies and payoffs, in case that competition is exogenously determined as simultaneous or sequential.

The best responses of the firms are given by:

$$q_i(q_j) = \frac{a_i - \alpha q_j}{2}.$$
$$q_j(q_i) = \frac{a_j - \alpha q_i}{2}.$$

Where $a_i = A - c_i$ and $a_j = A - c_j$. Considering this, we can compute the equilibrium in each case.

- If competition in q is simultaneous, equilibrium actions are:

$$q_i^N = \frac{A(2-\alpha) - 2c_i + \alpha c_j}{4-\alpha^2} = \frac{2a_i - \alpha a_j}{4-\alpha^2}.$$
$$q_j^N = \frac{A(2-\alpha) - 2c_j + \alpha c_i}{4-\alpha^2} = \frac{2a_j - \alpha a_i}{4-\alpha^2}.$$

  And subsequently, equilibrium payoffs are:

$$\Pi_i^N \doteq \Pi_i(q_i^N, q_j^N) = \left[\frac{2a_i - \alpha a_j}{4-\alpha^2}\right]^2.$$
$$\Pi_j^N \doteq \Pi_j(q_i^N, q_j^N) = \left[\frac{2a_j - \alpha a_i}{4-\alpha^2}\right]^2.$$

  The upper index N stands for *Nash* (simultaneous) equilibrium.

- On the other hand, if competition in q is sequential, equilibrium actions are:

$$q_i^L = \frac{2a_i - \alpha a_j}{2(2-\alpha^2)} \qquad q_j^F = \frac{4a_j - \alpha^2 a_j - 2\alpha a_i}{4(2-\alpha^2)}.$$
$$q_j^L = \frac{2a_j - \alpha a_i}{2(2-\alpha^2)} \qquad q_i^F = \frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2-\alpha^2)}.$$

  Where the upper indexes L and F, stand for *Leader* and *Follower* respectively. The respective equilibrium payoffs are:

$$\Pi_i^L \doteq \Pi_i(q_i^L, q_j^F) = \frac{[2a_i - \alpha a_j]^2}{8(2-\alpha^2)} \qquad \Pi_j^F \doteq \Pi_j(q_i^L, q_j^F) = \left[\frac{4a_j - \alpha^2 a_j - 2\alpha a_i}{4(2-\alpha^2)}\right]^2.$$
$$\Pi_j^L \doteq \Pi_j(q_i^F, q_j^L) = \frac{[2a_j - \alpha a_i]^2}{8(2-\alpha^2)} \qquad \Pi_i^F \doteq \Pi_i(q_i^F, q_j^L) = \left[\frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2-\alpha^2)}\right]^2.$$

Observation Notice that if we set $c_i = c_j = c$, then we recover the classical results from differentiated Cournot competition.

Using the previous payoffs, now it is possible to start the endogenous timing analysis. First we consider the GOD model, and then the GAC model.

	F	S
F	Π_i^N, Π_j^N	Π_i^L, Π_j^F
S	Π_i^F, Π_j^L	Π_i^N, Π_j^N

Table 1.1: Extended Game with Observable Delay

1.1.1 Extended Game with Observable Delay (GOD)

For the extended GOD model, we have to consider two possible actions in the pre play stage: to move first (F) or to move second (S). This allow us to model the extended game as a 2×2 game represented on Table 1.1.

In the next two sub sections, we first obtain the equilibria of this extended game, and after that, we perform a risk analysis in order to refine the (possible) multiple equilibria that could arise.

Equal Marginal Costs

In this sub section, we assume that $c_i = c_j = c$ (meaning that $b_i = b_j = \bar{b}$, with some $\bar{b} \leq c$). Therefore, we recover the profits from the classic Cournot competition. In particular, if $\alpha \in (0, 1)$ we obtain that:

$$\Pi_i^L > \Pi_i^N > \Pi_i^F.$$

Consequently, the unique equilibrium in this case is with both firms moving simultaneously in the first stage. We formalize that in the following theorem.

Theorem 1.1 *If $c_i = c_j$, the unique SPE of the extended game is with both firms choosing F, which results in simultaneous play.*

Observation Note that if $\alpha = 0$, then all payoffs would be equal, and every configuration would be an equilibrium.

Different Marginal Costs

Assume now, without loss of generality, that $c_i < c_j$ and $\alpha > 0$.

Theorem 1.2 *If $c_i < c_j$ and $\alpha \in (0, 1)$, then:*

(i) *(F, S) is never a SPE.*

(ii) *(S, F) is a SPE if $c_j < A \leq \dfrac{2c_j - \alpha c_i}{2 - \alpha}$. This condition implies that $q_j^L \leq 0$, therefore in this case the inefficient firm does not participate of the market.*

(iii) *(F, F) is the unique SPE of the extended game, if the market is large enough.*

Observation The notation (F, S) stand for firm i choosing F and firm j choosing S. Analogous for the case of (S, F) and (F, F).

PROOF. **(i)** Suppose that (F, S) is an equilibrium of the extended game. Then, in order to secure the non existence of profitable deviations, the following inequalities must hold:

- $\Pi_i^L \geq \Pi_i^N$.
- $\Pi_j^F \geq \Pi_j^N$.

Let us analyse the implications of the second inequality:

$$\Pi_j^F \geq \Pi_j^N \quad \Rightarrow \quad \left[\frac{4a_j - \alpha^2 a_j - 2\alpha a_i}{4(2-\alpha^2)}\right]^2 \geq \left[\frac{2a_j - \alpha a_i}{4-\alpha^2}\right]^2 \quad \Rightarrow \quad \alpha a_j \geq 2a_i.$$

Which can not be true, since $a_i > a_j$ and $\alpha < 1$. Consequently, (F, S) (the sequential configuration with the efficient firm leading) is never an equilibrium.

(ii) To sustain (S, F) as an equilibrium of the extended game, we need in particular that $\Pi_j^L \geq \Pi_j^N$ (which is always true) and $\Pi_i^F \geq \Pi_i^N$. Let us find under which conditions the last inequality holds.

$$\Pi_i^F \geq \Pi_i^N \quad \Longleftrightarrow \quad \left[\frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2-\alpha^2)}\right]^2 \geq \left[\frac{2a_i - \alpha a_j}{4-\alpha^2}\right]^2.$$

Which is true if $\alpha = 0$, or if $\alpha \in (0, 1]$ and $c_j < A \leq \dfrac{2c_j - \alpha c_i}{2-\alpha}$.

Observation Note that the condition over A implies that the inefficient firm will produce zero. In fact:

$$\begin{aligned} q_j^L &= \frac{2a_j - \alpha a_i}{2(2-\alpha^2)} \\ &= \frac{2A - 2c_j - \alpha A + \alpha c_i}{2(2-\alpha^2)} \\ &\leq \frac{\dfrac{2c_j - \alpha c_i}{2-\alpha} \cdot (2-\alpha) - 2c_j + \alpha c_i}{2(2-\alpha^2)} = 0. \end{aligned}$$

(iii) We have to analyse under which conditions, simultaneous equilibrium payoffs are bigger than follower payoffs. We start with the efficient firm:

$$\Pi_i^F \leq \Pi_i^N \quad \Longleftrightarrow \quad \left[\frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2-\alpha^2)}\right]^2 \leq \left[\frac{2a_i - \alpha a_j}{4-\alpha^2}\right]^2 \quad \Longleftrightarrow \quad A \geq \frac{2c_j - \alpha c_i}{2-\alpha}.$$

While for the inefficient player:

$$\Pi_j^F \leq \Pi_j^N \iff \left[\frac{4a_j - \alpha^2 a_j - 2\alpha a_i}{4(2-\alpha^2)}\right]^2 \leq \left[\frac{2a_j - \alpha a_i}{4-\alpha^2}\right]^2$$
$$\iff A \geq \frac{c_j(32 - 16\alpha^2 + \alpha^4) - c_i(16\alpha - 6\alpha^2)}{(2-\alpha)(16 - \alpha^2(8+\alpha))}.$$

This is a more restrictive condition over A than the one obtained for the efficient firm, and therefore the result follows.

□

Summarizing, in the case of the GOD model, regardless of the cost difference (and consequently the level of investment), the unique SPE is both firms choosing (F, F), which induce simultaneous play in the basic interaction. Note that in this case the degree of differentiation plays no role in the result, which supports the idea that the underlying dynamic ruling the result is related to the sign of the best response, and not its magnitude.

1.1.2 Extended Game with Action Commitment (GAC)

In the case of this extension model, the strategy spaces on the pre play stage are not finite, as in GOD. Here, if firms want to achieve the leader the position, or more precisely, avoid the follower position, they need to commit to a specific action. In this setting, there are three possible equilibria of the extended game: simultaneous competition, and both of the sequential configurations.

Proposition 1.3 *The extended game, using the GAC model, has three equilibria in pure strategies:* $S_i \doteq (q_i^L, W_j)$, $S_j \doteq (W_i, q_j^L)$ *and* $N \doteq (q_i^N, q_j^N)$. *Here* q_i^L *is the leader quantity in the sequential game equilibrium,* W_i *is to wait, and* q_i^N *is the Nash equilibrium quantity of the simultaneous game.*

PROOF. When modelled using the GAC extension model, this game fits perfectly in the framework Hamilton and Slutsky (1990). Therefore the proof comes directly from that article.

□

In this case there is no payoff dominance between the equilibria, and therefore, in order to select which equilibrium will arise, we will use a risk dominance criteria as described in Harsanyi and Selten (1998). As we have seen before, the proper way to perform a risk analysis for non finite games is to construct a bicentric prior, and then update those beliefs by the tracing procedure. That will be the general scheme in each proof.

Equal Marginal Costs

In this subsection, we study which equilibrium will emerge when firms have identical marginal cost after the investment stage, this is, we assume $c_i = c_j = c$. We find that players prefer sequential move (in any order) before simultaneous, but is not possible to discriminate between the two possible sequential configurations. These results are presented in the following theorems.

Theorem 1.4 *Any of the Stackelberg equilibria risk dominates the simultaneous one, i.e.* S_i *and* S_j *risk dominates* N.

PROOF. Without loss of generality, we prove that S_i risk dominates N. In order to complete the proof we need to build the bicentric prior, and then apply the tracing procedure.

- Let us start with the bicentric prior.
  - Firm j thinks that is playing against $z_j q_i^L + (1-z_j)q_i^N$. Therefore, the best she can do is to wait for all $z_j \in (0,1)$. This is the prior belief of Firm i about the behaviour of Firm j.

  - Firm i believes that is playing against $z_i W_j + (1-z_i)q_j^N$, that is, thinks that j will wait with probability z_i and play q_j^N with the complementary probability. By waiting, she will certainly obtain $\Pi_i^N = \Pi_i(q_i^N, q_j^N)$. Otherwise, she can commit to a quantity slightly greater than q_i^N by solving the following problem:

    $$\max_{q_i} \Pi_i(q_i, z_i W_j + (1-z_i)q_j^N).$$

    We obtain the optimal $q_i(z_i)$ by imposing the first order condition of the previous problem:

    $$q_i(z_i)(2 - z_i\alpha^2) = a_i - \frac{z_i\alpha a_i}{2} - \frac{(1-z_i)\alpha a_i}{2+\alpha}.$$

    Which leads to:

    $$q_i(z_i) = \frac{a_i}{2 - z_i\alpha^2} - \frac{z_i\alpha a_i}{2(2 - z_i\alpha^2)} - \frac{(1-z_i)\alpha a_i}{(2+\alpha)(2 - z_i\alpha^2)}.$$

    Since firm j does not know the weight z_i that firm i puts in waiting, the best approximation she can do is to assume that $z_i \sim U(0,1)$. Considering that, the prior belief of Firm j is that i will play $\mu_i = \mathbb{E}(q_i(z_i)) > q_i^N$.

    In summary, Firm i believes that Firm j will certainly wait, and Firm j believes that i will commit to a quantity greater than q_i^N.

- Now we proceed with the tracing procedure. The starting point ($t = 0$) is defined by the best response to the prior belief:
  - Firm i commits to q_i^L.
  - Firm j waits.

This means that in $t = 0$ the unique equilibrium is S_i. As S_i is an equilibrium of the original game, it is also an equilibrium $\forall t \in [0, 1]$, and it is the risk dominant equilibrium.

$\square$

We refer to the comparison between S_i and S_j in the next observation.

Observation Due of the symmetry of the problem, it is not possible to prove that neither S_i risk dominates S_j, nor S_i dominates S_j. Let us suppose that we came up with a proof of that S_i risk dominates S_j, then we could just simply change the sub indexes and we would obtain the proof in the opposite direction. This is true only because in this sub section we are considering equal marginal costs, and therefore, firms are absolutely identical. We will see in the next section, that this situation changes when we consider different marginal costs.

Different Marginal Costs

In this subsection, we analyse which timing combination will emerge in the basic game, when firms end up the investment stage with different marginal costs. The key difference with the previous subsection is that in this case, due to the different marginal costs, it will be feasible to perform a risk analysis to compare the two sequential equilibria. Let us start by comparing the simultaneous and sequential equilibria. Without lost of generality, during this section we assume that $c_i < c_j$.

Theorem 1.5 *Any of the Stackelberg equilibrium risk dominates the simultaneous one, i.e. S_i and S_j risk dominates N.*

PROOF. The proof is analogous to that in Theorem 1.4, but we go further in the details because it will be useful for future developments. Without loss of generality, we prove that S_i risk dominates N. As we have said before, the proof goes in two steps: first we build the bicentric prior, and after that we apply the tracing procedure to conclude.

It is important to mention that, because of the linear quadratic nature of the payoffs involved, the only relevant characteristics of a mixed strategy (or a bicentric prior) are the probability to wait, the mean of the commitment quantity and its variance. These characteristic are denoted respectively w_i, μ_i and ν_i, for firm i, and analogously for firm j.

- We start with the bicentric prior.

- Firm j believes that firm i plays $z_j \cdot q_i^L + (1 - z_j) \cdot q_i^N$. Therefore, the best firm j can do is to wait. This is, for all $z_j \in (0,1)$, $w_j = 1$. Given the previous, the prior belief of firm i is that j will certainly wait.

- Firm i thinks that j will play $z_i \cdot W_j + (1 - z_i) \cdot q_j^N$. Therefore, if firm i waits, the payoff is Π_i^N, no matter the value of z_i. If $z_i > 0$, and firm i commits to a quantity slightly above q_i^N, gets higher payoff. Then, the best for firm i is to commit to a quantity $q_i(z_i)$.

After some algebra, it is possible to find out that the optimal commitment q_i is the one that solves:

$$a_i - \alpha(1 - z_i)\left[\frac{2a_j - \alpha a_i}{4 - \alpha^2}\right] - \frac{\alpha z_i a_j}{2} = q_i(2 - z_i\alpha^2).$$

Where $a_i = A - c_i$ and $a_j = A - c_j$.

Observation Note that, if $z_i = 1$, it is possible to recover q_i^L, i.e. if firm i knows that its rival will wait, the best option is to commit to the leader equilibrium quantity.

Summarizing, firm i believes that firm j will wait certainly, and firm j believes that firm i will play as:

$$w_i = 0, \ \mu_i > q_i^N, \ \nu_i > 0.$$

Meaning that firm i certainly will not wait, and it will commit to a quantity greater than the simultaneous equilibrium quantity.

- Now we apply the tracing procedure. The starting point ($t = 0$) is defined by the best response to the prior belief:
  - Firm i commits to q_i^L.
  - Firm j waits.

  This means that in $t = 0$ the unique equilibrium is S_i. As S_i is an equilibrium of the original game, it is also an equilibrium $\forall t \in [0,1]$, and consequently is the risk dominant.

□

The next step is to compare sequential equilibria between them, and we present the results in the same order as they were obtained. We begin analysing the case when the cost difference is large and α is close to one (Theorem 1.6), which gives the idea that the differentiation parameter is actually playing a role, but then we present Theorem 1.9 which shows that the result hold regardless of the magnitude of the parameter neither the cost difference.

Theorem 1.6 *If $a_i = 2a_j$ and $\alpha \in (0.73, 1)$, then S_i risk dominates S_j.*

Observation Note that the hypothesis $a_i = 2a_j$ means that the cost difference is quite large in favour of firm i. This fact comes from:

$$a_i = 2a_j \Rightarrow A - c_i = 2(A - c_j).$$

Therefore, c_i is so "small" that its difference versus A is twice as big the difference between c_j and A.

PROOF. As we have seen in the proof of Theorem 1.5, we start by constructing the bicentric prior of the players. Firm j puts weight on the two possible actions of firm i, and therefore will play against $zq_i^L + (1-z)W_i$, and the expected payoff due to this is:

$$\begin{aligned} u_j(zq_i^L + (1-z)W_i, q_j) &= zu_j(q_i^L, q_j) + (1-z)u_j(W_i, q_j) \\ &= q_j \left[a_j - q_j - z\alpha q_i^L - (1-z)\alpha q_i(q_j)\right]. \end{aligned}$$

Where $q_i(q_j) = \dfrac{a_i - \alpha q_j}{2}$ is the best response of player i to the actions of player j. Optimizing the previous expression with respect to q_j, we find that the optimal commitment is:

$$q_j^*(z) = a_j \left[\frac{4 - \alpha^2(2-z)}{2(2-\alpha^2)(2-\alpha^2(1-z))}\right] - a_i \left[\frac{2\alpha - \alpha^3 + z\alpha^3}{2(2-\alpha^2)(2-\alpha^2(1-z))}\right].$$

The payoff due to the optimal commitment is:

$$u_j(zq_i^L + (1-z)W_i, q_j^*(z)) = \frac{\left[a_j(4-\alpha^2(2-z)) - \alpha a_i(2-\alpha^2(1-z))\right]^2}{8(2-\alpha^2)^2(2-\alpha^2(1-z))}.$$

Observation Is possible to verify that $q_j(z) < q_i(z)\ \forall z \in [0,1]$, which means that the inefficient firm is less willing to commit to a higher quantity.

It is easy to see that, if $z = 1$, the best firm j can do is to wait (because firm i will certainly commit to her leader quantity). On the other hand, if $z = 0$, firm j knows that the other firm will certainly wait, and in that case, the best she can do is to commit to her optimal leader quantity. Given this, it is logical that the bicentric prior will have a threshold structure. Therefore, to complete the prior belief, we have to analyse in which case committing is better than waiting, i.e.

$$u_j(zq_i^L + (1-z)W_i, q_j^*(z)) \geq u_j(zq_i^L + (1-z)W_i, W_j).$$

Which in turn is true if:

$$z \leq z_j(\alpha) \doteq \frac{2(2a_j - \alpha a_i)^2(2-\alpha^2)}{8\alpha^3 a_i a_j - 2\alpha^4 a_i^2 + a_j^2(16 - 16\alpha^2 + \alpha^4)} \quad \wedge \quad \left[a_i \leq 2a_j \vee 2a_j < a_i \leq \frac{7a_j}{2}\right].$$

Therefore, the best response of player j to $zq_i^L + (1-z)W_i$ is given by:

$$b_j(z) = \begin{cases} W_j & \text{if } \; z > z_j(\alpha). \\ q_j^*(z) & \text{if } \; z \leq z_j(\alpha). \end{cases}$$

We will simply denote $z_j(\alpha)$ by z_j. It is possible to show that $0 < z_j < z_i$, therefore is more likely for the efficient firm to commit.

Summarizing, the prior belief of firm i about firm j behaviour is:

$$m_j = b_j(Z), \;\; Z \sim U([0,1]).$$

As we have said before, due to the nature of the payoffs, the relevant characteristics of a mixed strategy are the probability to wait, its mean and its variance. In this case we have:

$$w_j \doteq \mathbb{P}(\text{waiting}) = 1 - z_j.$$

$$\mu_j \doteq \mathbb{E}(q_j^*(z)|z < z_j) = \frac{a_j - \alpha a_i}{2(2-\alpha^2)} + \frac{a_j}{2\alpha^2 z_j} \cdot \ln\left(\frac{2-\alpha^2+\alpha^2 z_j}{2-\alpha^2}\right).$$

$$\nu_j \doteq \mathbb{V}(q_j^*(z)) =$$
$$\frac{a_j^2}{4\alpha^4 z_j^3}\left[\frac{\alpha^2 z_j^2(1-\alpha^2+\alpha^2 z_j)}{2-\alpha^2+\alpha^2 z_j} + 2z_j \ln\left(\frac{2-\alpha^2+\alpha^2 z_j}{2-\alpha^2}\right)\ln\left(\frac{2-\alpha^2}{2-\alpha^2+\alpha^2 z_j}\right) + z_j \ln^2\left(\frac{2-\alpha^2+\alpha^2 z_j}{2-\alpha^2}\right)\right].$$

Since we have completed the construction of the bicentric prior, we can move on to apply the tracing procedure. We start by analysing the case in which the cost difference is big between the firms. First we prove that is always better for the inefficient player to wait at the beginning of the procedure, and then, that the opposite holds for the efficient firm.

Lemma 1.7 *If the cost difference is large enough ($a_i = 2a_j$), waiting is a dominant strategy for the inefficient firm.*

PROOF. In order to prove the Lemma, we have to study under which conditions it is true that

$$u_j(m_i, q_j) < u_j(m_i, W_j) \;\forall q_j.$$

Where m_i is the prior belief of firm j about the behaviour of firm i. Note that:

$$
\begin{aligned}
u_j(m_i, q_j) &= z_i(a_j - q_j - \alpha\mu_i)q_j + (1 - z_i)\left(a_j - q_j - \alpha q_i(q_j)\right) q_j \\
&= z_i(a_j - q_j - \alpha\mu_i)q_j + (1 - z_i)\left(a_j - q_j - \frac{\alpha a_i - \alpha^2 q_j}{2}\right) q_j \\
&= z_i(a_j - q_j - \alpha\mu_i)q_j + (1 - z_i)\left(\frac{2a_j - q_j(2 - \alpha^2) - \alpha a_i}{2}\right) q_j.
\end{aligned}
$$

Hence, the optimal commitment is given by

$$
q_j^* = \frac{a_j - \frac{\alpha a_i}{2} + z_i\left(\frac{\alpha a_i}{2} - \alpha\mu_i\right)}{2 - \alpha^2 + \alpha^2 z_i}. \tag{1.1}
$$

We know, from Hamilton and Slutsky (1990), that any quantity below the simultaneous equilibrium quantity is dominated by waiting for each player (given that the other player uses a non degenerated mixed strategy). Therefore, the results follows if $q_j^* \leq q_j^S$, which is equivalent to ask that:

$$
\frac{a_j - \frac{\alpha a_i}{2} + z_i\left(\frac{\alpha a_i}{2} - \alpha\mu_i\right)}{2 - \alpha^2 + \alpha^2 z_i} \leq \frac{2a_j - \alpha a_i}{4 - \alpha^2}.
$$

And this condition holds if:

$$
\ln\left(\frac{2 - \alpha^2 + \alpha^2 z_i}{2 - \alpha^2}\right) \quad \geq \quad \frac{a_i^3\alpha^6 - 4a_i^3\alpha^4 + 8a_i^3\alpha^2 - 2a_i^2 a_j\alpha^5 - 8a_i^2 a_j\alpha^3 + 10a_i a_j^2\alpha^4 - 2a_j^3\alpha^5}{a_i^3\alpha^4 - 16a_i^3\alpha^2 + 16a_i^3 + 8a_i^2 a_j\alpha^3 - 2a_i a_j^2\alpha^4}.
$$

As we are assuming that $a_i = 2a_j$, the inequality holds if:

$$
\ln\left(1 + \frac{\alpha^2}{2 - \alpha^2} \cdot \left[\frac{32 - \alpha^4 + 8\alpha^3 - 14\alpha^2 - 16\alpha}{32 + \alpha^4 + 8\alpha^3 - 32\alpha^2}\right]\right) \quad \geq \quad \frac{4\alpha^6 - 3\alpha^5 - 22\alpha^4 - 32\alpha^3 + 32\alpha^2}{2\alpha^4 + 16\alpha^3 - 64\alpha^2 + 64}.
$$

Which is always true since $\alpha \in [0, 1]$. The result follows.

$\square$

Up to this point, we have proved that, when the cost difference is large enough, it is always dominant for the inefficient player to wait. In the next Lemma we prove that under the same hypothesis, the efficient firm will always want to commit to a quantity, and consequently will never wait.

Lemma 1.8 *If $a_i = 2a_j$ and $\alpha \in (0.73, 1)$, then $u_i(W_i, m_j) < \max_{q_i} u_i(q_i, m_j)$.*

We discuss here the idea of the proof, and leave the technical details on the Appendix. As on the proof of the previous Lemma 1.7, it is possible to compute the optimal commitment for firm i:

$$q_i^* = \frac{a_i - \frac{\alpha a_j}{2} + z_j\left(\frac{\alpha a_j}{2} - \alpha\mu_j\right)}{2 - \alpha^2 + \alpha^2 z_j}.$$

Replacing on the payoff function, we find the optimal payoff for committing $u_i(q_i^*, m_j)$. On the other hand, if firm i decides to wait, she will obtain:

$$\begin{aligned} u_i(W_i, m_j) &= z_j\left(\mathbb{E}\left(\frac{[a_i - \alpha q_j]^2}{4}\right)\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2 \\ &= z_j\left(\frac{1}{4}\cdot\mathbb{E}([a_i - \alpha q_j]^2)\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2 \\ &= z_j\left(\frac{(a_i - \alpha\mu_j)^2}{4} + \frac{\alpha^2\nu_j}{4}\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2. \end{aligned}$$

The final step is to calculate the difference between both situations:

$$u_i(q_i^*, m_j) - u_i(W_i, m_j).$$

And is possible to show that (details on the Appendix), when $a_i = 2a_j$, this difference is strictly positive, which proves the Lemma. Combining Lemmas 1.7 and 1.8, we obtain the desired result.

□

At this point, one might think that the differentiation parameter is actually playing a role in the result, but we will see in the next general result that, even when the cost difference is not that large and regardless of the value α, the equilibrium with the firm i playing as leader is the risk dominant one. The main difference with the previous analysis is that, in this case, the best response of the inefficient firm to the prior belief also involves committing.

Theorem 1.9 *If $c_i < c_j$, then S^i risk dominates S^j.*

PROOF. Analogously as in Theorem 1.6, the first step is to build the bicentric prior, but as we are comparing exactly the same equilibria that in such case, we will only focus on the tracing procedure. In order to do so, we start by proving that it is not possible that both firms keep on committing themselves up to the end of the tracing procedure.

Lemma 1.10 *Let be s_t the equilibrium on time t in the tracing procedure. Then exists $k \in \{i, j\}$ and $t < 1$, such that $s_k^t = W_k$.*

PROOF. Without loss of generality, we do the proof for player i. The strategy to prove the result will be to define the gain of player i by committing himself versus waiting, and show that this gain is negative for some $t < 1$. Denote by q_i^t the optimal commitment quantity at the moment t. Since at $t = 1$ the payoff functions coincide with those of the original game, it can be proved that $q_i^1 = q_i^S$. Let be u_i^t the payoff function at the moment t of the tracing procedure, we define:

$$\varphi_i(t) = u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t).$$

Note that if we set $t = 1$, then we have:

$$\begin{aligned}
\varphi_i(1) &= u_i^t(q_i^1, q_j^1) - u_i^t(W_i, q_j^1) \\
&= u_i^t(q_i^1, q_j^1) - u_i^t(W_i, q_j^1) \\
&= u_i^t(q_i^S, q_j^S) - u_i^t(W_i, q_j^S) \\
&= 0.
\end{aligned}$$

Therefore, $\varphi_i(1) = 0$, i.e. there is no gain associated to commit versus waiting at time $t = 1$. We will prove that $\varphi_i'(1) > 0$, which implies that $\varphi_i(t) < 0$ for some $t < 1$. The derivative of $\varphi_i(t)$ is given by:

$$\begin{aligned}
\varphi_i'(t) &= \frac{\partial[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)]}{\partial t} + \frac{\partial[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)]}{\partial q_j} \cdot \frac{\partial q_j^t}{\partial t} + \underbrace{\frac{\partial[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)]}{\partial q_i}}_{=0} \cdot \frac{\partial q_i^t}{\partial t} \\
&= \frac{\partial[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)]}{\partial t} + \frac{\partial[u_i^t(q_i^t, q_j^t) - u_i^t(W_i, q_j^t)]}{\partial q_j} \cdot \frac{\partial q_j^t}{\partial t}.
\end{aligned}$$

Now we have to analyze both of the terms in the previous expression when $t = 1$. We start with the first one:

$$\underbrace{u_i(q_i^S, q_j^S) - u_i(W_i, q_j^S)}_{=0} - u_i(q_i^S, m_j) + u_i(W_i, m_j) = u_i(W_i, m_j) - u_i(q_i^S, m_j) > 0.$$

The second term is:

$$\frac{\partial[u_i^t(q_i^t,q_j^t)-u_i^t(W_i,q_j^t)]}{\partial q_j}\cdot\frac{\partial q_j^t}{\partial t}=\left[t\cdot\frac{\partial u_i(q_i^t,q_j^t)}{\partial q_j}-t\cdot\frac{\partial u_i(W_i,q_j^t)}{\partial q_j}\right]\cdot\frac{\partial q_j^t}{\partial t}$$
$$=\left[-\alpha t q_i^t-t\cdot\frac{1}{4}\cdot[2\alpha^2 q_j^t-2\alpha a_i]\right]\cdot\frac{\partial q_j^t}{\partial t}.$$

Evaluating in $t=1$ we obtain:

$$\left[-\alpha q_i^S+\frac{1}{2}\cdot[\alpha a_i-\alpha^2 q_j^S]\right]\cdot\frac{\partial q_j^t}{\partial t}=\left[-\alpha\cdot\frac{2a_i-\alpha a_j}{4-\alpha^2}+\frac{1}{2}\left[\alpha a_i-\alpha^2\cdot\frac{2a_j-\alpha a_i}{4-\alpha^2}\right]\right]\cdot\frac{\partial q_j^t}{\partial t}=0.$$

Therefore, we have the desired result. □

Observation Note that the Lemma 1.10 says that for both players it is optimal to change her strategy to wait at some point, if the other player keeps committing.

From now on, our strategy is to prove that the inefficient firm will switch first. In order to do so, we will consider a case in which firm i is more pessimistic and prove that even in such case, firm j (the inefficient one) will switch first to waiting. In order to incorporate this "pessimism" characteristic, we must modify the bicentric prior of the game. Specifically, we will analyse the case in which the inefficient firm commits with the same probability that the efficient one. Recall that $m=(m_i,m_j)$ is the original bicentric prior defined previously, and consider $\bar{m}=(\bar{m}_i,\bar{m}_j)$ a new one in which z_j has been replaced by z_i. That is, now the probability to commit for firm j is bigger, in particular it is the same that for firm i.

Observation Hence, player i is more pessimistic under $\bar{m}$ than under m, but for player j the situation does not change considering one bicentric prior or the other.

Let us assume that each player finds it optimal to commit at $t=0$, when the prior is m. Consider $q_i^{t,m}(q_j)$ and $q_j^{t,m}(q_i)$ the best commitment quantities at t, when the other commits to q_j and q_i respectively. We denote the pair of mutual best commitment quantities by (q_i^t,q_j^t). The gains for committing versus waiting are:

$$\varphi_i^t(q_i,q_j)=u_i^{t,m}(q_i,q_j)-u^{t,m}(W_i,q_j).$$
$$\varphi_j^t(q_i,q_j)=u_j^{t,m}(q_i,q_j)-u^{t,m}(q_i,W_j).$$

Then, $\varphi_i^t(q_i,q_j)>0$ and $\varphi_j^t(q_i,q_j)>0$ for sufficiently small t, and (q_i^t,q_j^t) is the equilibrium path on the tracing procedure under m. Analogously, we can define $(\bar{q}_i^t,\bar{q}_j^t)$, $\bar{\varphi}_i^t$ and $\bar{\varphi}_j^t$ using $\bar{m}$ instead of m. Using these definitions we can establish the following lemma:

Lemma 1.11 *Let be $t_k \doteq \sup\{\tau \in [0,1] : \ \varphi_k^t(q_i^t, q_j^t) \geq 0 \ \forall t \in [0,\tau]\}$ for $k \in \{i,j\}$, the last point in time for which is convenient for firm k to commit. Then $t_i > t_j$.*

PROOF. Let us start by noticing the following:

(1) $\bar{q}_i^t < q_i^t \wedge \bar{q}_j^t \geq q_j^t \ \forall t$. The efficient firm is more pessimistic under $\bar{m}$, hence is willing to commit to a lower quantity, and this fact incentives player j to commit to a bigger quantity (since the best responses are decreasing).

(2) $\bar{\varphi}_i^t(q_i, q_j) \leq \varphi_i^t(q_i, q_j) \ \forall t$. If player i is more pessimistic, then committing is less attractive.

(3) $\bar{\varphi}_j^t(q_i, q_j) < \bar{\varphi}_i^t(q_i, q_j) \ \forall t$. Since firm i has a lower marginal cost, committing is more attractive than for firm j.

Using these three observations we can prove the following chain of inequalities:

$$
\begin{aligned}
\varphi_j^t(q_i^t, q_j^t) &= \bar{\varphi}_j(q_i^t, q_j^t) && (1.2)\\
&\leq \bar{\varphi}_j(q_i^t, \bar{q}_j^t) && (1.3)\\
&\leq \bar{\varphi}_j(\bar{q}_i^t, \bar{q}_j^t) && (1.4)\\
&< \bar{\varphi}_i(\bar{q}_i^t, \bar{q}_j^t) && (1.5)\\
&\leq \bar{\varphi}_i(q_i^t, \bar{q}_j^t) && (1.6)\\
&\leq \varphi_i(q_i^t, \bar{q}_j^t) && (1.7)\\
&\leq \varphi_i(q_i^t, q_j^t). && (1.8)
\end{aligned}
$$

Observe that (1.2) is because the prior belief for j is the same in both cases, (1.3) and (1.6) come from observation **(1)**; (1.4) and (1.8) from best response properties; (1.5) from observation **(3)**; and finally (1.7) comes from observation **(2)**. In summary, we have shown that:

$$\varphi_j^t(q_i^t, q_j^t) < \varphi_i^t(q_i^t, q_j^t) \ \forall t.$$

That is, the gain for committing is bigger for the efficient firm, for all time. This inequality proves the lemma.

□

Combining Lemmas 1.11 and 1.10, the Theorem 1.9 is proved. We have shown that the efficient firm will emerge as leader, based on a risk dominance criterium.

□

Theorem 1.9 shows that the differentiation parameter plays no role in the result when using the GAC extension model, since we are obtaining the analogous to that in van Damme and Hurkens (1999). This fact supports our intuition that the endogenous timing results are primary driven by the qualitative behaviour of the reaction curves (increasing or decreasing), instead of their quantitativeness.

At this point we find crucial to highlight the differences between GOD and GAC applied to this basic interaction. First is important to note that, in a setting like this, both players want to avoid the follower position. In the GOD extension model, doing so involves no risk for the players, because they only need to declare "F" on the pre play stage. Under that strategy, the worst that can happen is to end up in simultaneous competition. On the other hand, when considering the GAC extension model, avoiding the follower position would require commitment, specifically, commit to a "large" quantity. Unlike the previous case, this is indeed risky, because if both player do so, they could end up engaged in a Stackelberg warfare (both producing their leader quantity). Given that, they prefer to "wait and see", instead of committing. That is why in the GOD extension model, the unique equilibrium induces simultaneous competition, while in the GAC case, simultaneous competition is never the risk dominant equilibrium. Although, when looking at the articles on Table 2 one could have conjectured that GOD and GAC indeed would have lead to different results, this is the first time that this fact is exhibited using the exact same model as basic interaction.

Having noticed this fact, it is natural to ask the following question: *if GAC and GOD (could) lead to different conclusions even when considering the exact same basic interaction, in which cases would be reasonable to model with GOD and in which others with GAC?* The answer of course is not trivial nor exhaustive, and it would depend on the nature of the game we are modelling. Perhaps a natural first approach to determine which extension model is better to use, is to understand the nature of the market in which the firms are involved. Specifically, how "fast" the firms can modify the features that determine their decisions on the pre play stage. For instance, when the variable that makes firm different is "hard" to modify (namely capacity, marginal cost, etc) it would be more natural to think of GAC as the extension model, because they will need to commit under their current circumstances, with no possibility to change them. On the other hand, when the feature that makes firms different is "easy" to change (for instance location for food trucks), it would be more natural to use GOD as extension model. Other approach might be in terms of the preferred role in the market. In quantity competition, we know that firms want to avoid the follower position and, therefore, GAC would be the more suitable extension model only if we are dealing with firms strong enough to commit to "large" quantities. Otherwise, GOD seems to be the more natural approach.

In the remaining of this chapter, we analyse the investment phase in which players can decide their level of investment in order to diminish their marginal costs, knowing that this affect the equilibrium timing of the game.

1.2 Investment Stage

Let us consider the previous stage of the interaction, in which the players must decide their level of investment, knowing that this level will not only affect the equilibrium strategies, but also the timing of the game. Only for consistency, let us recall the complete timing of the game.

(1) In the first stage, firms decide their level of investment in variables $b_i, b_j \in [0, c]$, to diminish their marginal cost.

(2) Firms compete in quantities q_i and q_j, and this competition might be simultaneous or sequential. We use GOD and GAC to endogenously obtain the timing of movements in equilibrium.

We consider $F(x) = \dfrac{vx}{2}$ as the cost of investment for both firms. After all the decisions are made, firms receive payoffs:

$$\Pi_i(q_i, q_j, b_i, b_j) = (A - q_i - \alpha q_j - (c - b_i))q_i - \frac{1}{2}vb_i^2.$$
$$\Pi_i(q_i, q_j, b_i, b_j) = (A - q_j - \alpha q_i - (c - b_j))q_j - \frac{1}{2}vb_j^2.$$

When we solve the game by backward induction, we find that the payoffs depending on the investment levels are:

$$\Pi_i^N(b_i, b_j) = \left[\frac{2(A - c + b_i) - \alpha(A - c + b_j)}{4 - \alpha^2}\right]^2 - \frac{1}{2}vb_i^2.$$
$$\Pi_i^L(b_i, b_j) = \frac{[2(A - c + b_i) - \alpha(A - c + b_j)]^2}{8(2 - \alpha^2)} - \frac{1}{2}vb_i^2.$$
$$\Pi_i^F(b_i, b_j) = \left[\frac{(4 - \alpha^2)(A - c + b_i) - 2\alpha(A - c + b_j)}{4(2 - \alpha^2)}\right]^2 - \frac{1}{2}vb_i^2.$$

1.2.1 Exogenous timing

In this section, we analyse how the firms would invest if the timing was exogenously determined. We will use this as a benchmark to analyse the investment when the timing is endogenous.

Simultaneous Competition

The first order condition for firms i and j are:

$$2\left[\frac{2(A - c + b_i) - \alpha(A - c + b_j)}{4 - \alpha^2}\right] \cdot \frac{2}{4 - \alpha^2} - vb_i = 0.$$
$$2\left[\frac{2(A - c + b_j) - \alpha(A - c + b_i)}{4 - \alpha^2}\right] \cdot \frac{2}{4 - \alpha^2} - vb_j = 0.$$

Solving for b_i and b_j, we find that the equilibrium investments are:

$$b_i^S = b_j^S = \frac{-4(A-c)}{4+(-2+\alpha)v(2+\alpha^2)}.$$

Which leads to the following equilibrium payoffs:

$$\Pi_i(b_i^S, b_j^S) = \Pi_j(b_i^S, b_j^S) = \frac{v^2\left((\alpha^2-4)^2-8\right)(A-c)^2}{\left((\alpha-2)(\alpha+2)^2v+4\right)^2}.$$

Sequential Competition

Without loss of generality, we assume that firm i is leader, and therefore the payoffs as a function of the investments are:

$$\Pi_i(b_i^L, b_j^F) = \frac{\left[2(A-c+b_i^L)-\alpha(A-c+b_j^F)\right]^2}{8(2-\alpha^2)} - \frac{1}{2}v{b_i^L}^2.$$
$$\Pi_j(b_i^L, b_j^F) = \left[\frac{(4-\alpha^2)(A-c+b_j^F)-2\alpha(A-c+b_i^L)}{4(2-\alpha^2)}\right]^2 - \frac{1}{2}v{b_j^F}^2.$$

The first order condition for player i is:

$$\frac{1}{2(2-\alpha^2)}[2(A-c+b_i)-\alpha(A-c+b_j)] - vb_i = 0.$$

And for player j:

$$\frac{[(4-\alpha^2)(A-c+b_j)-2\alpha(A-c+b_i)]\,(4-\alpha^2)}{8(2-\alpha^2)^2} - vb_j = 0.$$

Solving for b_i and b_j leads to:

$$b_i^L = \frac{2(\alpha-2)(A-c)(2+\alpha+2v(-2+\alpha^2))}{-2\alpha^2(1-4v)^2+8(1-2v)^2+\alpha^4v(-1+8v)}.$$
$$b_j^F = \frac{(\alpha^2-4)(A-c)\left[2+v(-4+\alpha(2+\alpha))\right]}{-2\alpha^2(1-4v)^2+8(1-2v)^2+\alpha^4v(-1+8v)}.$$

And therefore, the equilibrium payoffs are:

$$\begin{aligned}
\Pi_i(b_i^L, b_j^F) &= -\frac{2(\alpha-2)^2 v\left(\left(\alpha^2-2\right)v+1\right)\left(2\left(\alpha^2-2\right)v+\alpha+2\right)^2(A-c)^2}{\left(\alpha^4 v(8v-1)-2\alpha^2(1-4v)^2+8(1-2v)^2\right)^2}. \\
\Pi_j(b_i^L, b_j^F) &= \frac{((A-c)^2 v(-(-4+\alpha^2)^2+8(-2+\alpha^2)^2 v)(2+(-4+\alpha(2+\alpha))v)^2)}{(2(2\alpha^2(1-4v)^2-8(1-2v)^2+\alpha^4(1-8v)v)^2)}.
\end{aligned}$$

1.2.2 Case GOD

In this section we analyse how the firms would decide in the investment stage, if the timing of the quantity stage is determined endogenously and using the GOD model. In this case the analysis is straightforward because, as we know from Theorem 1.2, the firms will compete simultaneously regardless of the level if marginal costs. Therefore, knowing that, the firms will behave exactly as in the benchmark case for simultaneous competition and thus, there will not be neither over nor under investment.

Proposition 1.12 *Under the GOD model, the unique equilibrium on the investment phase is $b_i^* = b_i^S$ and $b_j^* = b_j^S$.*

PROOF. Direct from Theorem 1.2.

□

1.2.3 Case GAC

For the GAC model, the firm with the largest investment will achieve the leader position, which is the preferred one in this setting. We will show that, using this model to extend the basic interaction, the firms will end up over investing.

Proposition 1.13 *Under the GAC extension model, and if v is big enough, the unique equilibrium in the investment phase is $b_i^* = b_j^* = c$, meaning that both firms full invest.*

PROOF. The argument goes in two steps. First, let us assume that (w.l.g) the equilibrium is such that $b_i > b_j$, this is, firm i has a bigger investment and therefore, lower marginal cost. In this context, firm i emerges as leader and obtains:

$$\Pi_i^L(b_i, b_j) = \frac{[2(A-c+b_i)-\alpha(A-c+b_j)]^2}{8(2-\alpha^2)} - \frac{1}{2}vb_i^2.$$

On the other hand, firm j is follower and gets:

$$\Pi_j^F(b_i, b_j) = \left[\frac{(4-\alpha^2)(A-c+b_j)-2\alpha(A-c+b_i)}{4(2-\alpha^2)}\right]^2 - \frac{1}{2}vb_j^2.$$

Since b_i is strictly greater than b_j, firm i can diminish her investment to $b_i - \varepsilon$, and we have:

$$\Pi_i^L(b_i, b_j) - \Pi_i^L(b_i - \varepsilon, b_j) = \frac{(\varepsilon(-2A + \alpha A - 2b_i + \alpha b_j + 2c - \alpha c + \varepsilon + 2(-1 + \alpha^2)(-2b_i + \varepsilon)v))}{(4(-1 + \alpha^2))}.$$

Which is negative if:

$$v \geq \frac{2A - \alpha A + 2b_i - \alpha b_j - 2c + \alpha c - \varepsilon}{2(-1 + \alpha^2)(-2b_i + \varepsilon)}.$$

Taking $\varepsilon \to 0$, we obtain that $v \geq \dfrac{2(A + b_i - c) - \alpha(A + b_j - c)}{4b_i(1 - \alpha^2)}$.

Up to this point, we have that no equilibrium can exist with different levels of investment. Consequently, given that in equilibrium b_i must be exactly equalt to b_j, the unique possibility is full investment, i.e., $b_i = b_j = c$. Let us assume by contradiction that in equilibrium $b_i = b_j < c$. We saw previously that is not possible to discriminate, based on risk considerations, between the two sequential equilibria when the investments are the same. Nevertheless, the risk dominant equilibrium in this case must be a mixed strategy that gives the firms a convex combination between their leader and follower payoffs, this is, $\bar{\Pi}_i = \lambda \Pi_i^L + (1 - \lambda)\Pi_i^F$ (analogous for firm j). Then, for an sufficiently small $\varepsilon > 0$, $b_i + \varepsilon$ is a profitable deviation for firm i, since doing that firm i will obtain $\Pi_i^L > \bar{\Pi}_i$ for $\lambda \neq 1$.

□

1.3 Conclusions

In this Chapter our purpose was to develop a particular model of quantity competition, with differentiated products, in order to exhibit and understand the differences between GOD and GAC, study the influence of the degree of substitution on the timing, and finally analyse if the firms would over or under invest in a previous phase, knowing that their investment affects the equilibrium not only through the actions, but also through the timing of movements.

In the case of the GOD extension model, we found that there was a unique equilibrium of the extended game, which is with both firms choosing F, and therefore, simultaneous play emerge as the induced equilibrium on the basic interaction. This result holds regardless of the costs being equal or different. The intuition is that, as firms want to avoid the follower position, they simply choose F on the pre play stage, and remove that possibility. In the worst case, they will engage in simultaneous competition.

	Supermodular		Submodular	
	Tough	**Soft**	**Tough**	**Soft**
GOD Model	Li (2014) Lambertini & Tampieri (2017) Amir & Grilo (1999) Amir et al. (1999) Amir & Stepanova (2006)	Lambertini (1997) Amir et al. (2000)	Jinji (2014) Lu & Poddar (2009) Tesoriere (2008) Amir & Grilo (1999) Amir et al. (2000) Amir et al. (1999) Chapter 1	Jinji (2014)
	Tough	**Soft**	**Tough**	**Soft**
GAC Model	v. Damme & Hurkens (2004)		v.Damme & Hurkens (1999) Chapter 1	

Table 1.2: Re classification of the literature updated.

The case of GAC model is more complex to analyse. First of all, there are three equilibria of the extended game: the simultaneous and both of the sequential configurations. Given that, we found necessary to perform a risk analysis in order to refine the result. We used risk dominance, because is not possible to stablish payoff dominance in this setting. The first and most interesting result we found is that, in this context, simultaneous competition never emerges as the risk dominant equilibrium, again regardless of the cost difference. As we mentioned before, firms want to avoid being follower. In the case of the GOD model this had no cost or risk whatsoever, they just needed to declare F in the pre play stage. In this GAC model, if they want to avoid the follower position, they need to commit to a specific quantity and, even more, it should be a “large” quantity (above q_i^S). This action, unlike as in the GOD model, is risky because, if both firms commit to a “large” quantity, they could end up in a Stackelberg warfare, which means that both of them produce their leader action. In order to avoid that risk, firms sometimes prefer to wait and see what the rival does.

The next result was the refinement between both sequential configurations, and in that case, we found that the firm with the lowest marginal cost is the one that emerges as the risk dominant leader. The interpretation is that committing is riskier for the inefficient firm, and therefore it prefers to wait. The results in this GAC section allow us to say that the differentiation parameter does not play a role in the timing determination, in the sense that the results are analogous to those in van Damme and Hurkens (1999), in which $\alpha = 1$. We interpret this fact as a signal that the induced timing in equilibrium depends on the reaction curves only through their qualitative behaviour and not their quantitative characteristics. In Chapter 3 we will present model that confirms and generalize this results. Before discussing the results obtained on the last section of the Chapter, let us highlight that the model presented here fits on Table 2 as it is presented on Table 1.2.

The last section of the Chapter analysed the investment phase, in which the firms could invest in a variable b_i, in order to diminish their marginal cost. As a benchmark to assess the investment, we considered the case in which timing was exogenously given. We found that, if the extension model is GOD, firms invest optimally (in the benchmark sense), because they know that, regardless of their investment, the competition would be simultaneous. If the extension model is GAC, firms over invest trying to achieve the leader position in the posterior basic interaction. In this case, the unique investment equilibrium is with both firms "fully investing", that is, until their marginal cost is zero. This analysis is considering marginal deviations only.

Chapter 2

Endogenous Timing with Supermodular Competition

In this Chapter we study a general model of competition between two firms. First, they decide their level of investment in some variable that may be *tough* or *soft*, as in Fudenberg and Tirole (1984). After the investment is made, they engage in supermodular competition (or strategic complements), and this competition can be simultaneous or sequential to allow the study of endogenous timing. We use GOD and GAC in order to tackle this last point.

With this model, we would like to understand which player will emerge as leader under GOD and GAC, based on the nature of the variable that makes firms different, and possible more hypothesis. We will first study the GOD case, but before that, let us properly define the model. Formally, consider two firms, namely i, j. The timing is as follows:

- In the first stage, they decide simultaneously their level of investment a_i and a_j. Without loss of generality, we can think that these variables are chosen from a convex and compact set $A \subseteq \mathbb{R}$.
- In the second stage, they compete in variables x_i and x_j, and this competition can be simultaneous or sequential. Again, we can consider that the variables are chosen from convex and compact set $X \subseteq \mathbb{R}$.

After the game is played, firms obtain payoffs $\Pi_i(x_i, x_j, a_i, a_j)$ and $\Pi_j(x_i, x_j, a_i, a_j)$. We assume that these payoffs are class $\mathcal{C}^2$, and concave on the own variables (competition and investment). Now we formally define the type of competition and the nature of investment, which will be our main assumptions about the model.

Definition 2.1 *We say that competition is supermodular, if the payoffs are supermodular on competition variables, this is, if for all $a_i, a_j \in A$ and $x_i, x_j \in X$:*

$$\text{(A1)} \qquad \frac{\partial^2 \Pi_i}{\partial x_i \partial x_j}(x_i, x_j, a_i, a_j) \geq 0 \qquad \text{and} \qquad \frac{\partial^2 \Pi_j}{\partial x_j \partial x_i}(x_i, x_j, a_i, a_j) \geq 0. \tag{2.1}$$

Observation Assumption (A1) on (2.1) brings as a consequence that the reaction curves of the players are increasing. When this is the case, competition is also known as in strategic complements. This is a generalization of the classical price competition framework (Bertrand duopoly).

Before defining the types of investment, let us make a point about the notation throughout this Chapter. Consider the three possible subgames defined by the competition in variables x_i and x_j: simultaneous competition, sequential with i as leader, and sequential with j as leader. We assume that each one of these subgames has a unique equilibrium, and that those equilibria are different to each other. The notation is as follows:

- In the case that player i is the leader, (x_i^L, x_j^F) are the equilibrium actions.
- When the leader is player j, the equilibrium is denoted (x_i^F, x_j^L).
- Finally, in the case of the simultaneous subgame, the equilibrium is denoted by (x_i^N, x_j^N).[1]

We define the type of investment as follows.

Definition 2.2 *We say that the investment variable a_j is Tough, if for all $a_i, a_j \in A$ and $x_i, x_j \in X$:*

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j) < 0.$$

If the inequality go in the other way, we say that investment a_j is Soft. Analogous for player j and variable a_i.

Observation It is important to mention that:

- *Tough* investment means that the total effect of the investment is negative on the pay-off of the other player. Symmetrically for the *Soft* case. The interpretation is that a tough investment allows the player to be more aggressive in the subsequent competition phase. For instance, an investment intended to diminish marginal costs would be tough in a price competition setting, since it would allow the players to lower their prices. We provide a formal argumentation of this further in this Chapter (section 2.1.3).

- Throughout this chapter, we will be assuming that there are no spillovers. Specifically, we assume that the direct effect term in $\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j)$ is equal to zero. This is

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j) = \frac{\partial\Pi_i}{\partial x_i}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_i}{\partial a_j} + \frac{\partial\Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_j}{\partial a_j}$$
$$+ \frac{\partial\Pi_i}{\partial a_i}(x_i, x_j, a_i, a_j) + \underbrace{\frac{\partial\Pi_i}{\partial a_i}(x_i, x_j, a_i, a_j)}_{=0}.$$

[1] The upper indexes F, L and N represent the follower, leader and Nash equilibrium actions respectively.

In many of the cases that we analyse in this work, we will also have that the first term is equal to zero because of the envelope theorem (since we will be evaluating in equilibrium points).

2.1 Game with Observable Delay

As we have said before, in the pre play stage of the GOD model players have to choose an action from the set $\{F, S\}$. After that, they compete in the order determined by their decisions, and this induces a timing on the basic interaction. We will prove that in this case there could be two equilibria of the extended game.

Lemma 2.3 *Under assumption (A1) on 2.1, and assuming that each payoff is monotone in the action of the other firm, at least one player has a second mover advantage. This means that the payoff for being follower is strictly greater than for being leader. The result is valid for all $a_i, a_j \in A$.*

PROOF. Let us observe that:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \quad > \quad \Pi_i(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_i(x_i^L, x_j^N, a_i, a_j).$$

Where the first inequality comes from the fact that the leader situation is strictly better than the simultaneous equilibrium for the leader player. The second inequality comes from the definition of best response. Therefore, in particular we have that:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) > \Pi_i(x_i^L, x_j^N, a_i, a_j).$$

Analogously we can prove the same for player j and then, in summary, we have:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) > \Pi_i(x_i^L, x_j^N, a_i, a_j). \tag{2.2}$$

$$\Pi_j(x_i^F, x_j^L, a_i, a_j) > \Pi_j(x_i^N, x_j^L, a_i, a_j). \tag{2.3}$$

Now we divide the analysis in cases:

- If $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) > 0$ and $\dfrac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) > 0$, then from (2.2), we have:

$$\Rightarrow x_j^F > x_j^N.$$
$$\Rightarrow x_i^L > x_i^N.$$

This last inequality comes from the fact that the best responses are increasing (assumption (A1) on 2.1).

This leads to:

$$x_i^L > x_i^N \wedge x_j^F > x_j^N.$$

Analogously, if we consider (2.3), we have :

$$x_i^F > x_i^N \wedge x_j^L > x_j^N.$$

Summarizing, we note that the leader and follower optimal actions, for both players, are above the simultaneous equilibrium actions. Considering these inequalities, there are three possible cases:

(i) $x_i^L > x_i^F \wedge x_j^L > x_j^F$. In this case, the following inequalities hold:

$$\Pi_i(x_i^F, x_j^L, a_i, a_j) \quad \geq \quad \Pi_i(x_i^L, x_j^L, a_i, a_j) \quad > \quad \Pi_i(x_i^L, x_j^F, a_i, a_j).$$

The first inequality comes from the definition of best response, and the second one from the fact that $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) > 0$ and $x_j^L > x_j^F$. This shows that there is a second mover advantage for firm i. Analogously, it is trivial to show the same for firm j. Therefore, in this case, both players have a second mover advantage.

(ii) $x_i^L < x_i^F \wedge x_j^L > x_j^F$. In this case, firm i has a second mover advantage.

(iii) $x_i^L > x_i^F \wedge x_j^F < x_j^L$. Firm j has a second mover advantage.

Observation It is important to mention that is not possible that $x_i^L < x_i^F$ and $x_j^F > x_j^L$. This is, the leader equilibrium actions cannot be simultaneously lower than the follower equilibrium actions for both players. This fact comes directly from the increasing best responses.

- If $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) < 0$ and $\dfrac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) < 0$, then from (2.2):

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) > \Pi_i(x_i^L, x_j^N, a_i, a_j) \;\Rightarrow\; x_j^F < x_j^N \;\Rightarrow\; x_i^L < x_i^N \;\Rightarrow\; x_i^L < x_i^N \wedge x_j^F < x_j^N.$$

Analogously, when j is leader we have: $x_i^F < x_i^F \wedge x_j^L < x_j^N$. Again, there are three possible cases:

(i) $x_i^L < x_i^F \wedge x_j^L < x_j^F$. We have that:

$$\Pi_i(x_i^F, x_j^L, a_i, a_j) \quad \geq \quad \Pi_i(x_i^L, x_j^L, a_i, a_j) \quad > \quad \Pi_i(x_i^L, x_j^F, a_i, a_j).$$

Analogously for firm j. Therefore, in this case both firms have a second mover advantage.

(ii) $x_i^L > x_i^F \wedge x_j^L < x_j^F$. In this case firm j has a second mover advantage.

(iii) $x_i^L < x_i^F \wedge x_j^L > x_j^F$. There is a second mover advantage for firm i.

□

Using Lemma 2.3, we can prove the following result about the induced timing on the basic game.

Theorem 2.4 *Under the GOD extension model, assuming condition (A1) on 2.1 and monotonicity of the payoff on the other firm's action, there are two equilibria of the extended game: (F, S) and (S, F).*

PROOF. Direct from Lemma 2.3

□

This result implies that in the case of supermodular competition, using the GOD extension model, simultaneous competition will never emerge as the risk dominant equilibrium (since is not even an equilibrium of the extended game). This result generalizes those obtained by the articles in the respective section of Table 2. In particular, that in the cells corresponding to GOD and supermodular competition, the emerging equilibrium is always simultaneous competition on the basic interaction, no matter the nature of the investment.

The next step is to analyse which one of the sequential equilibria is more likely to appear, and in order to do so, we will refine this multiple equilibria using the risk dominance criteria of Harsanyi and Selten (1998). Perhaps a more natural approach to refine multiple equilibria would be payoff dominance, but in this case it is not possible to do so. To note this, think of the two equilibria of the extended game: (F, S) and (S, F). In the first case, the profits are:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \wedge \Pi_j(x_i^L, x_j^F, a_i, a_j).$$

Meanwhile, in the second case, the payoffs are:

$$\Pi_i(x_i^F, x_j^L, a_i, a_j) \wedge \Pi_j(x_i^F, x_j^L, a_i, a_j).$$

Therefore, in order to (F, S) payoff dominate (S, F), we would need that:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \geq \Pi_i(x_i^F, x_j^L, a_i, a_j) \qquad \text{and} \qquad \Pi_j(x_i^L, x_j^F, a_i, a_j) \geq \Pi_j(x_i^F, x_j^L, a_i, a_j).$$

Which cannot happen (Lemma 2.3). The analysis for the case that (S, F) payoff dominates (F, S) is analogous.

2.1.1 Risk Dominance.

In this section we stablish the sufficient conditions for the leadership of a player, based on the risk dominance criterium. In the two subsequent subsections, we analyse the particular cases of tough and soft investment, and study how our results fit in a particular setting. The first result is the following lemma, which gives sufficient conditions for the leadership of a player (under risk refinement):

Lemma 2.5 *Suppose that a_i, $a_j \in A$ are such that:*

(i) $\Pi_j(x_i^L, x_j^F, a_i, a_j) - \Pi_j(x_i^F, x_j^L, a_i, a_j) \geq \Pi_i(x_i^F, x_j^L, a_i, a_j) - \Pi_i(x_i^L, x_j^F, a_i, a_j)$.

(ii) $\Pi_i(x_i^L, x_j^F, a_i, a_j) - \Pi_i(x_i^N, x_j^N, a_i, a_j) \geq \Pi_j(x_i^F, x_j^L, a_i, a_j) - \Pi_j(x_i^N, x_j^N, a_i, a_j)$.

Then, the equilibrium with the firm i being leader risk dominates the one with the firm j as a leader.

Observation Condition **(i)** says that the total profits are larger when firm i is the leader than when j is. Condition **(ii)** states that the biggest improvement versus the simultaneous scenario is for firm i.

PROOF. Recall that in the case of 2×2 games, the risk dominance concept corresponds to deviation losses, this is, the risk dominant equilibrium will be the one which has associated bigger deviation losses. To make the notation more simple, let us denote:

$$\begin{array}{lll} \Pi_i^L \doteq \Pi_i(x_i^L, x_j^F, a_i, a_j) & \Pi_i^F \doteq \Pi_i(x_i^F, x_j^L, a_i, a_j) & \Pi_i^N \doteq \Pi_i(x_i^N, x_j^N, a_i, a_j). \\ \Pi_j^L \doteq \Pi_j(x_i^F, x_j^L, a_i, a_j) & \Pi_j^F \doteq \Pi_j(x_i^L, x_j^F, a_i, a_j) & \Pi_j^N \doteq \Pi_j(x_i^N, x_j^N, a_i, a_j). \end{array}$$

Now, from hypothesis **(i)** we have that:

$$
\begin{array}{rccc}
 & \underbrace{\Pi_j^F - \Pi_j^L}_{\doteq A} & \geq & \underbrace{\Pi_i^F - \Pi_i^L}_{\doteq B} \\
\Rightarrow & A \cdot [\Pi_j^L - \Pi_j^N] & \geq & B \cdot [\Pi_j^L - \Pi_j^N] \\
\Rightarrow & A \cdot [\Pi_i^L - \Pi_i^N] & \geq & B \cdot [\Pi_j^L - \Pi_j^N].
\end{array}
$$

Where the last inequality comes from the hypothesis **(ii)**. Now, let us note that we can re write the terms A and B in the following manner:

$$
A = [\Pi_j^F - \Pi_j^N - (\Pi_j^L - \Pi_j^N)].
$$
$$
B = [\Pi_i^F - \Pi_i^N - (\Pi_i^L - \Pi_i^N)].
$$

Writing again the last inequality, we have that:

$$
[\Pi_i^L - \Pi_i^N][\Pi_j^F - \Pi_j^N] - [\Pi_i^L - \Pi_i^N][\Pi_j^L - \Pi_j^N] \;\; \geq \;\; [\Pi_i^F - \Pi_i^N][\Pi_j^L - \Pi_j^N] - [\Pi_i^L - \Pi_i^N][\Pi_j^L - \Pi_j^N].
$$

Which in turn implies that:

$$
[\Pi_i^L - \Pi_i^N][\Pi_j^F - \Pi_j^N] \geq [\Pi_i^F - \Pi_i^N][\Pi_j^L - \Pi_j^N].
$$

Which is exactly what we wanted to prove.

□

2.1.2 Risk Dominance: Tough and Soft investment

In this section we investigate under which combination of hypothesis, the conditions on Lemma 2.5 are satisfied. We find sufficient conditions for the leadership of the player with the largest (and smallest) investment, considering both types: tough and soft. At the end of the section, we present a brief discussion about which type of leadership is more likely to appear in classical contexts.

Observation If $a_i = a_j$, then firms are absolutely identical, and the risk dominance criteria cannot determine which one of the sequential equilibria should arise. Therefore, in that case, we are not able to refine using that concept. In the remaining of this section, we focus our attention in the case where $a_i \neq a_j$.

Before presenting the theorems, and to make the notation more tractable, we describe here two conditions that we will assume as hypotheses for those theorems. The conditions are related to the behaviour of payoffs and best responses.

$$\frac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_i(\cdot)}{\partial a_j} \geq 0 \qquad \text{and} \qquad \frac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_j(\cdot)}{\partial a_i} \geq 0. \tag{2.4}$$

$$\frac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_i(\cdot)}{\partial a_j} \leq 0 \qquad \text{and} \qquad \frac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_j(\cdot)}{\partial a_i} \leq 0. \tag{2.5}$$

Condition (2.4) says that the payoffs and best responses are both increasing or decreasing, while condition (2.5) says that they have different behaviour in that respect (one increasing and the other decreasing). Now we proceed with the theorems.

Theorem 2.6 *Let us suppose that assumption (A1) on (2.1) is met, the investment variables are tough and $a_i > a_j$.*

1 *Assuming that (2.4) holds, we have two cases:*

1.1 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right|$, $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right|$, $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm i is the risk dominant leader.*

1.2 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|$, $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|$, $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm j is the risk dominant leader.*

2 *Assuming that (2.5) holds, we also have two cases:*

2.1 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right|$, $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right|$, $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm i is the risk dominant leader.*

2.2 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|$, $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|$, $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm j is the risk dominant leader.*

Despite the fact that we give conditions for the leadership of both players, we will see on Section 2.1.3 that in classical settings, it is more likely for the firm with the largest investment to become the risk dominant leader. In particular, we will analyse the case of price competition with different marginal costs. Now we write the "mirror" version of the Theorem 2.6 for the case of soft investment. We call this a mirror result, because the inequalities on each case go in the opposite direction compared to Theorem 2.6.

Theorem 2.7 *Let us suppose that assumption (A1) on (2.1) is met, the investment variables are soft and $a_i > a_j$.*

1 Assuming that (2.4) holds, we have two cases:

1.1 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|, \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|, \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm i is the risk dominant leader.*

1.2 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right|, \left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right|, \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm j is the risk dominant leader.*

2 Assuming that (2.5) holds, we also have two cases:

2.1 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|, \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|, \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm i is the risk dominant leader.*

2.2 *If* $\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right|, \left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right|, \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|$ *and* $\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|$, *then Firm j is the risk dominant leader.*

We provide the proof of both Theorems in the Appendix. Next, we analyse how the model of price competition from van Damme and Hurkens (2004) fits in our result.

2.1.3 A particular case: price competition and different marginal costs.

In this sub section we study a particular case of a supermodular game to see how it fits on the context of Theorem 2.6. Let us consider the price competition setting defined in van Damme and Hurkens (2004). There are two firms, namely $i = 1, 2$, with different (and constant) marginal costs $c_i \geq 0$. They set prices, and face a linear demand of the form:

$$D(p_i, p_j) = 1 - p_i + \alpha p_j.$$

Where $\alpha \in (0, 1)$. Firms choose price simultaneously and receive profits:

$$u_i(p_i, p_j) = (p_i - c_i)(1 - p_i + \alpha p_j).$$

The best responses are defined by:

$$b_j(p_i) = \frac{1 + \alpha p_i + c_j}{2}.$$

The optimal actions for firm i, in the case of leader, follower and simultaneous respectively, are:

$$
\begin{aligned}
p_i^L &= \frac{2+\alpha+\alpha c_j+(2-\alpha^2)c_i}{2(2-\alpha^2)},\\
p_i^F &= \frac{4+2\alpha-\alpha^2+(4-\alpha^2)c_i+(2\alpha-\alpha^3)c_j}{4(2-\alpha^2)},\\
p_i^N &= \frac{2+\alpha+\alpha c_j+2c_i}{4-\alpha^2}.
\end{aligned}
$$

And the profits due to their actions are:

$$
\begin{aligned}
u_i(p_i^L, p_j^F) &= \frac{(2+\alpha+\alpha c_j+(\alpha^2-2)c_i)^2}{8(2-\alpha^2)},\\
u_i(p_i^F, p_j^L) &= \frac{(4+2\alpha-\alpha^2+(2a-\alpha^3)c_j+(3a^2-4)c_i)^2}{16(2-\alpha^2)^2},\\
u_i(p_i^N, p_j^N) &= \frac{(2+\alpha+\alpha c_j+(\alpha^2-2)c_i)^2}{(4-\alpha^2)^2}.
\end{aligned}
$$

To make it explicit, note that the variable that makes firm different is *tough*, because diminishing marginal costs result in a more aggressive competition. Formally, let us assume that firms can invest in a variable a_i that diminishes marginal cost, then:

$$
\frac{\mathrm{d}u_i}{\mathrm{d}a_j} = \underbrace{\frac{\partial u_i}{\partial p_j}}_{>0} \cdot \underbrace{\frac{\partial p_j}{a_j}}_{<0} < 0.
$$

Observation Note that we are not considering spillovers, and therefore there is no direct effect of the investment i on the profit j. This holds for the case of leader, follower and simultaneous competition.

Now we analyse in which category of Theorem 2.6 this setting fits. Note that

$$
\frac{\partial u_i}{\partial p_j} > 0 \text{ and } \frac{\partial p_i}{\partial a_j} < 0.
$$

And, therefore, we are standing in case 2 of Theorem 2.6. Let us study if the rest of hypotheses are met. In the case of firm i, we have:

$$
\left|\frac{\partial x_i^F}{\partial a_i}\right| = \frac{4-\alpha^2}{4(2-\alpha^2)}, \quad \left|\frac{\partial x_i^L}{\partial a_i}\right| = \frac{1}{2}, \quad \left|\frac{\partial x_i^N}{\partial a_i}\right| = \frac{2}{4-\alpha^2}.
$$

And for firm j:

$$\left|\frac{\partial x_j^F}{\partial a_i}\right| = \frac{2\alpha-\alpha^3}{4(2-\alpha^2)}, \quad \left|\frac{\partial x_j^L}{\partial a_i}\right| = \frac{\alpha}{2(2-\alpha^2)}, \quad \left|\frac{\partial x_j^N}{\partial a_i}\right| = \frac{\alpha}{4-\alpha^2}.$$

Recall that we are considering $c_1 < c_2$, meaning that $a_1 > a_2$. For the leadership of firm 1, we need four conditions to hold:

(i)

$$\left|\frac{\partial x_1^F}{\partial a_1}\right| > \left|\frac{\partial x_1^L}{\partial a_1}\right| \Longleftrightarrow \frac{4-\alpha^2}{4(2-\alpha^2)} > \frac{1}{2}.$$

(ii)

$$\left|\frac{\partial x_1^F}{\partial a_1}\right| > \left|\frac{\partial x_1^N}{\partial a_1}\right| \Longleftrightarrow \frac{4-\alpha^2}{4(2-\alpha^2)} > \frac{\alpha}{4-\alpha^2}.$$

(iii)

$$\left|\frac{\partial x_2^F}{\partial a_1}\right| < \left|\frac{\partial x_2^L}{\partial a_1}\right| \Longleftrightarrow \frac{2\alpha-\alpha^3}{4(2-\alpha^2)} < \frac{\alpha}{2(2-\alpha^2)}.$$

(iv)

$$\left|\frac{\partial x_2^F}{\partial a_1}\right| < \left|\frac{\partial x_2^N}{\partial a_1}\right| \Longleftrightarrow \frac{2\alpha-\alpha^3}{4(2-\alpha^2)} < \frac{\alpha}{4-\alpha^2}.$$

All this four conditions hold, because $\alpha \in (0,1)$.

Based on this analysis, we can say that the risk dominant equilibrium on the GOD extension of the model is with the efficient firm leading. Now, let us confirm that this result is consistent with what we would obtain by performing a risk dominance analysis. Recall that in the pre play stage of the GOD model, firms only choose F or S, and therefore the extended game can be represented by a 2×2 matrix:

	F	S
F	Π_1^N, Π_2^N	Π_1^L, Π_2^F
S	Π_1^F, Π_2^L	Π_1^N, Π_2^N

The equilibrium with firm 1 being leader will be the risk dominant if:

$$(\Pi_1^L - \Pi_1^N)(\Pi_2^F - \Pi_2^N) > (\Pi_1^F - \Pi_1^N)(\Pi_2^L - \Pi_2^N)$$

$$\iff (1+\alpha)(\alpha^2-4)^2(c_1-c_2)\cdot A\cdot B<0. \tag{2.6}$$

Where

$$A \doteq (2+(\alpha-1)(c_1+c_2)),$$

and

$$B \doteq (16\alpha^2(c_1(c_2-1)-c_2)-16(c_1-1)(c_2-1)+2\alpha^4(c_1+c_2-2c_1c_2)+8\alpha(-2+c_1^2+c_2^2)$$
$$+\alpha^5(c_1^2+c_2^2)-\alpha^3(-2+c_1(2+7c_1)+c_2(2+7c_2))).$$

Given that $c_1<c_2$, condition (2.6) holds if and only if, A and B have different sign.

First note that, if $A<0$, then:

$$c_1+c_2>\frac{2}{1-\alpha}.$$

As $\alpha\in(0,1)$, we have that:

$$\frac{2}{1-\alpha}>2.$$

Then, in this case,

$$c_1+c_2>2.$$

Which cannot be true since $c_1<c_2<1$. Summarizing, in this setting, A is strictly positive.

Now, let us analyse the sign of the term B. Note that:

$$\begin{aligned}
B &= c_1c_2(16\alpha^2-16-4\alpha^4)-(c_1+c_2)(16\alpha^2-16-2\alpha^4+2\alpha^3)+(c_1^2+c_2^2)(8\alpha+\alpha^5-7\alpha^3)-16-16\alpha+2\alpha^3\\
&< c_2^2(16\alpha^2-16-4\alpha^4)-(c_1+c_2)(16\alpha^2-16-2\alpha^4+2\alpha^3)+(2c_2^2)(8\alpha+\alpha^5-7\alpha^3)-16-16\alpha+2\alpha^3\\
&= c_2^2(16\alpha^2-16-4\alpha^4+16\alpha+2\alpha^5-14\alpha^3)-(c_1+c_2)(16\alpha^2-16-2\alpha^4+2\alpha^3)-16-16\alpha+2\alpha^3\\
&< 0.
\end{aligned}$$

Therefore, in this case, condition (2.6) holds. Summarizing, in this subsection, we have shown that our result in Theorem 2.6 is consistent with what we would have obtained by performing the risk analysis in terms of deviation losses.

2.2 Game with Action Commitment

In this section, we consider the GAC model in order to extend the basic game, and make the timing of movements endogenous. The first result is related to the existence of multiple equilibria in the extended game.

Theorem 2.8 *Under assumption (A1) on 2.1, there are three equilibria of the extended game:* (x_i^L, W_j, a_i, a_j), (x_i^N, x_j^N, a_i, a_j) *and* (W_i, x_j^L, a_i, a_j). *This holds for all* $a_i, a_j \in A$.

PROOF. See Hamilton and Slutsky (1990)

□

In order to refine this multiple equilibria, we will apply the tracing procedure defined by Harsanyi and Selten (1998). First, we prove that simultaneous competition cannot be the risk dominant equilibrium, and then we make the analysis comparing both sequential configurations.

Theorem 2.9 *Under assumption (A1) on 2.1, any of the sequential equilibria risk dominates the simultaneous one. This result holds for all* $a_i, a_j \in A$, *and for both types of investment (soft and tough).*

PROOF. Without loss of generality, we prove that (x_i^L, W_j, a_i, a_j) risk dominates (x_i^N, x_j^N, a_i, a_j). As we have done previously, we start the proof by building the bicentric prior, which corresponds to the initial beliefs of the players, and then we conclude with the tracing procedure.

Bicentric Prior.

- Player j believes that she is playing against $z_j x_i^L + (1 - z_j)x_i^N$, where $z_j \in (0, 1)$. Given that, is clear that the best player j can do is to wait for all z_j. Consequently, the best response is given by:

$$b_j(z_j) = W_j \ \forall z_j \in (0, 1).$$

  Therefore, the prior belief of player i about the behaviour of player j is that she waits.

- Player i believes that she is playing against $z_i W_j + (1 - z_i)x_j^N$, where $z_i \in (0, 1)$. If player i waits, she obtains:

$$\Pi_i^N = \Pi_i(x_i^N, x_j^N, a_i, a_j),$$

  regardless of z_i. On the other hand, committing to an action $x_i \geq x_i^N$ yields a higher payoff, and therefore, the best response of player i is to commit to a certain action $x_i(z_i)$. Summarizing, the prior belief of firm j about the behaviour of player i, is that she commits to an action higher than x_i^N.

Now we move on to the tracing procedure.

Tracing Procedure.

At the beginning of the tracing procedure, the equilibrium path is defined by the best response to the prior beliefs. In this case, the best response of player i is to commit to x_i^L (since player j will certainly wait), while for player j, the best response is to wait. Therefore, the equilibrium at $t = 0$ is (x_i^L, W_j, a_i, a_j), and since this is an equilibrium of the original game, it is also an equilibrium for any $t \in [0, 1]$.

□

To conclude this section, we must try to establish which of the sequential equilibria is the risk dominant one. In the next two results, we will give sufficient conditions for the leadership of each player, considering both types of investment (soft and tough). As it will be shown, these conditions are rather strong.

Observation We denote by $\mathbb{E}(x_j)$ the expected action of firm j on the bicentric prior, and $x_j(\cdot)$ her best response. Analogous for player i.

Theorem 2.10 *Assume that condition (A2) on (3.1) holds, payoffs are monotone on the rival's action, investment variables are tough, and $a_i > a_j$. If for all $x_i^* \in co(x_i^F, x_i^L)$ and $x_j^* \in co(x_j^F, x_i^L)$*[2]*,*

$$\begin{aligned}
\frac{\mathrm{d}\Pi_i(x_i^*, \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j)}{\mathrm{d}a_i} &\geq 0, \\
\frac{\mathrm{d}\Pi_i(x_i^*, x_j(x_i^*), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} &\geq 0, \\
\frac{\mathrm{d}\Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j^*, a_i, a_j)}{\mathrm{d}a_i} &\leq 0, \\
\frac{\mathrm{d}\Pi_i(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i(x_j^*), x_j^*, a_i, a_j)}{\mathrm{d}a_i} &\leq 0.
\end{aligned}$$

Then, (x_i^L, W_j, a_i, a_j) is the risk dominant equilibrium, and consequently, player i emerges as leader. If the inequalities go in the other way, (W_i, x_j^L, a_i, a_j) is the risk dominant equilibrium.

[2] That is, x_i^* and x_j^* are convex combinations of the respective leader and follower actions. Note that, *a priori*, it could happen that the leader actions are greater than the follower actions or *vice versa*, depending on the monotonicity of the payoffs, as can be seen on the proof of Lemma 2.3.

PROOF. Without loss of generality, we assume that payoffs are increasing on the action of the other player. The analysis for the case of decreasing payoffs is analogous. As we have seen before, first we construct the bicentric prior, and then we perform the tracing procedure.

Bicentric Prior.

- Note that player j believes that she is playing against $zx_i^L + (1-z)W_i$, meaning that i commits to x_i^L with probability z, and waits with the complementary probability. Therefore, if player j waits, obtains:

$$\Pi_j(zx_i^L + (1-z)W_i, W_j, a_i, a_j) = z\Pi_j^F + (1-z)\Pi_j^N[3].$$

On the other hand, by committing to x_j^F, the payoff is:

$$\Pi_j(zx_i^L + (1-z)W_i, x_j^F, a_i, a_j) \quad = \quad z\Pi_j^F + (1-z)\Pi_j(x_i(x_j^F), x_j^F, a_i, a_j).$$

Where $x_i(x_j^F)$ is the best response of firm i to x_j^F. Since the best response is increasing, and $x_j^F \in \left(x_j^N,\ x_j^L\right)$, then

$$\Pi_j(x_i(x_j^F), x_j^F, a_i, a_j) > \Pi_j^N.$$

Summarizing, it is better to commit to some action than waiting, and consequently, the prior belief of firm i about the behaviour of player j is that will commit with probability one.

- The procedure is exactly analogous thinking from the point of view of firm i, and then, the prior belief of firm j about the behaviour of i is that will also commit with probability one.

Summarizing, the bicentric prior (m_i, m_j) is that both players will certainly commit to actions $x_i(Z)$ and $x_j(Z)$ with $Z \sim U(0,1)$, and therefore, the expected actions will be $\mathbb{E}(x_i) \in (x_i^F, x_i^L)$ and $\mathbb{E}(x_j) \in (x_j^F, x_j^L)$.

Tracing Procedure.

At the beginning of the tracing procedure, the equilibrium is determined by the best response to the prior beliefs of the players. Since both of them know that the other will commit with probability one, the best they can do when the process starts is to wait. We will show that there exists a moment of the tracing procedure in which the players change the waiting strategy to leadership, and then find the conditions for each player to be the first one in doing so.

Note that the belief about the behaviour of the other player evolve as:

$$m_j^t = (1-t)m_j + tW_j.$$
$$m_i^t = (1-t)m_i + tW_i.$$

Given that, the gain of committing on each t on the tracing procedure is given by:

$$\varphi_i(m_j^t) = \max_{x_i} \Pi_i(x_i, m_j^t, a_i, a_j) - \Pi_i(W_i, m_j^t, a_i, a_j).$$
$$\varphi_j(m_i^t) = \max_{x_j} \Pi_j(m_i^t, x_j, a_i, a_j) - \Pi_j(m_i^t, W_j, a_i, a_j).$$

Note that, for player i (and analogously for player j), we have that:

- In $t = 0$,

  $$\varphi_i(m_j^0) = \max_{x_i} \Pi_i(x_i, \mathbb{E}(x_j), a_i, a_j) - \Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j) \leq 0.$$

  Where $x_i(\mathbb{E}(x_j))$ is the best response of firm i to $\mathbb{E}(x_j)$.

- In $t = 1$,

  $$\varphi_i(m_j^1) = \max_{x_i} \Pi_i(x_i, x_j(x_i), a_i, a_j) - \Pi_i^N \geq 0.$$

As $\varphi_i(m_j^t)$ and $\varphi_j(m_i^t)$ are continuous, there must be times t_i and t_j, such that from that point in the procedure, it is better to commit and assume the leader position, assuming that the other player is still waiting. In the Appendix, we show that if the assumptions of the theorem are met, then $\forall t \in [0,1], \varphi_i(m_j^t) \geq \varphi_j(m_i^t)$, and consequently $t_i \leq t_j$.

□

Now, we enunciate the respective result for soft investment.

Theorem 2.11 *Assume that condition (A2) on (3.1) holds, payoffs are monotone on the rival's action, investment variables are tough, and $a_i > a_j$. If for all $x_i^* \in co(x_i^F, x_i^L)$ and $x_j^* \in co(x_j^F, x_i^L)$,*

$$\frac{\mathrm{d}\Pi_i(x_i^*, \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j)}{\mathrm{d}a_i} \geq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i^*, x_j(x_i^*), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} \geq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j^*, a_i, a_j)}{\mathrm{d}a_i} \leq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i(x_j^*), x_j^*, a_i, a_j)}{\mathrm{d}a_i} \leq 0.$$

Then, (x_i^L, W_j, a_i, a_j) is the risk dominant equilibrium, and consequently, player i emerges as leader. If the inequalities go in the other way, (W_i, x_j^L, a_i, a_j) is the risk dominant equilibrium.

PROOF. The proof is analogous to the one of Theorem 2.10, but is necessary to make a point about the inequalities. This argumentation can be found in the Appendix.

□

2.3 Conclusions

In this Chapter, we studied a model in which players face supermodular competition. We assumed that firms were absolutely identical, except for one characteristic. To fix ideas, this feature can be thought as the marginal cost or the quality of the product. We made the timing of movements endogenous using two extension models: GOD and GAC and, for each of those models, we considered that the variable that made players different could be *tough* or *soft* as in Fudenberg and Tirole (1984). A *tough* investment variable is one that has a negative total marginal effect on the payoff of the other player. If the effect is positive, the investment variable is said to be *soft*. Classical examples of such types are capacity of production and quality of the product, respectively.

For the GOD extension model, there are two possible equilibria: the two sequential configurations. In order to refine this multiple equilibria, we performed a risk analysis based on the nature investment variable, and found sufficient conditions for the leadership of each firm under different sets of hypothesis. If we think on the classical setting of price competition with different marginal costs, our model predicts that the most efficient firm should be the leader, which is consistent with the literature. This result was based on the fact that the level of investment was different for each firm. If the investment levels are the same, that is $a_i = a_j$, then is not possible to select one of the equilibria using the risk dominance concept.

For the GAC model, there are three possible equilibria: simultaneous and both of the sequential configurations. We first proved that simultaneous competition is never the risk dominant equilibrium. The next step was to compare the two sequential equilibria. As on the GOD case, we found sufficient conditions for the leadership of each player under risk considerations. It is important to mention that, although these conditions ensure the risk dominance of each equilibria, they are rather strong. In this sense, a possible path of future work is to find which are the necessary hypotheses for each result to hold.

A crucial fact about these results, is that both GOD and GAC, lead to similar conclusions. Specifically, if we think about the classic price competition models with different marginal costs and consider our framework, both extension models predict (using risk dominance) that the efficient player will be the leader when the timing is endogenous. When looking at Table 2, this results allow us to confirm that, when the basic interaction is supermodular and the investment variable is tough, the firm with the largest investment will become the risk dominant leader, no matter which extension model we are considering. Note that in this setting, both of the players would prefer being a follower, but still the efficient one takes the leadership, since it is riskier for her to wait and see what the other player does. As we will see further in this document, this behaviour does not hold for the case of submodular competition.

Chapter 3

Endogenous Timing with Submodular Competition

The model studied in here is analogous to that in Chapter 2, with a different assumption on the competition variables. Here, players first decide their level of investment in some variable that may be *tough* or *soft*, as in Fudenberg and Tirole (1984). But now, after the investment is done, they engage in submodular competition (strategic substitutes), and this competition can be simultaneous or sequential to allow the study of endogenous timing. We use GOD and GAC extension models in order to tackle this last point.

With this model, we would like to understand which player will emerge as leader in GOD and GAC, based on the nature of the variable that makes firms different, and possible more hypothesis. Let us properly define the model. Consider two firms, namely i, j. The timing is as follows:

- In the first stage, they decide simultaneously their level of investment a_i and a_j. Without loss of generality, we can think that these variables are chosen from a convex and compact set $A \subseteq \mathbb{R}$.

- In the second stage they compete in variables x_i, x_j, and this competition can be simultaneous or sequential. Again, we can consider that the variables are chosen from convex and compact set $X \subseteq \mathbb{R}$.

After the game is played, firms obtain payoffs $\Pi_i(x_i, x_j, a_i, a_j)$ and $\Pi_j(x_i, x_j, a_i, a_j)$, which we assume to be class $\mathcal{C}^2$ and concave on the own variables (investment and competition). Now let us define the type of competition and the nature of investment, which will be our main assumptions about the model.

Definition 3.1 *We say that competition is submodular, if the payoff functions are submodular on competition variables, that is, if for all $a_i, a_j \in A$ and $x_i, x_j \in X$:*

$$\text{(A2)} \qquad \frac{\partial^2 \Pi_i}{\partial x_i \partial x_j}(x_i, x_j, a_i, a_j) \leq 0 \qquad \textit{and} \qquad \frac{\partial^2 \Pi_j}{\partial x_j \partial x_i}(x_i, x_j, a_i, a_j) \leq 0. \tag{3.1}$$

Observation Assumption (3.1) implies that the reaction curves of the players are decreasing. When this is the case, the competition is known as in strategic substitutes. This is a generalization of the classical quantity competition (Cournot duopoly).

Before defining the types of investment, let us make a point about the notation, which is analogous to that in Chapter 2. Consider the three possible subgames defined by the competition in variables x_i and x_j: simultaneous, sequential with player i as leader, and sequential with j as leader. We assume that each one of these subgames has a unique equilibrium, and that those equilibria are all different from each other. The notation is as follows:

- In the case that player i is the leader, (x_i^L, x_j^F) are the equilibrium actions.
- When the leader is player j, the equilibrium is denoted (x_i^F, x_j^L).
- In the case of the simultaneous subgame, the equilibrium is denoted by (x_i^N, x_j^N).[1]

Now, the types of investment are defined as follows.

Definition 3.2 *We say that the investment variable a_j is Tough, if for all $a_i, a_j \in A$ and for all $x_i, x_j \in X$:*

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j) < 0.$$

If the inequality go in the other way, we say that investment a_j is Soft. Analogously for a_i.

Observation It is important to mention that:

- *Tough* means that the total effect of the investment is negative on the payoff of the other player. Symmetrically for the *Soft* case. The interpretation is that a tough investment allows the player to be more aggressive in the subsequent competition phase. In a context of Cournot competition, a tough investment would be one that allows the player to produce a bigger quantity.

- As in Chapter 2, we will be assuming that there are no spillovers. Specifically, we

[1] The upper indexes F, L and N represent the follower, leader and Nash equilibrium actions respectively.

assume that the direct effect term in $\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j)$ is equal to zero. This is

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j}(x_i, x_j, a_i, a_j) = \frac{\partial \Pi_i}{\partial x_i}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_i}{\partial a_j} + \frac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) \cdot \frac{\partial x_j}{\partial a_j} + \frac{\partial \Pi_i}{\partial a_i}(x_i, x_j, a_i, a_j) + \underbrace{\frac{\partial \Pi_i}{\partial a_i}(x_i, x_j, a_i, a_j)}_{=0}.$$

In many of the cases that we analyse in this work, we will also have that the first term is equal to zero because of the envelope theorem (since we will be evaluating in equilibrium points).

3.1 Game with Observable Delay

When considering the GOD extension model, and unlike Chapter 2, we will show that both firms prefer to move first to being a follower. That result will imply a completely different timing of movements in the extended game, compared to the case in which assumption (A1) on (2.1) holds.

Lemma 3.3 *Under assumption (A2) on (3.1), and assuming that payoff functions are monotone in the action of the rival, both firms have a first mover advantage. This is, their leader equilibrium payoff is greater than their follower payoff. This result holds for all $a_i, a_j \in A$, and for both types of investment: soft and tough.*

PROOF. For any $a_i, a_j \in A$, what we need to prove is that:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \geq \Pi_i(x_i^F, x_j^L, a_i, a_j),$$

and

$$\Pi_j(x_i^F, x_j^L, a_i, a_j) \geq \Pi_j(x_i^L, x_j^F, a_i, a_j).$$

The strategy is to show that the following sequence of inequalities hold (analogous for player j):

$$\begin{aligned} \Pi_i(x_i^L, x_j^F, a_i, a_j) &\geq \Pi_i(x_i^N, x_j^N, a_i, a_j) && (3.2) \\ &\geq \Pi_i(x_i^F, x_j^N, a_i, a_j) && (3.3) \\ &\geq \Pi_i(x_i^F, x_j^L, a_i, a_j). && (3.4) \end{aligned}$$

Inequality (3.2) is true because both points, (x_i^L, x_j^F, a_i, a_j) and (x_i^N, x_j^N, a_i, a_j), lie on the best response of player j, but the former is the one that leaves player i the greatest possible payoff on that curve. Inequality (3.3) is also true, because x_i^N is the best response of player i to x_j^N. Therefore, for the proof to be complete, we only need to argue why inequality (3.4), and its analogous version for player j, are true. This is, we need to show that:

$$\Pi_i(x_i^F, x_j^N, a_i, a_j) \geq \Pi_i(x_i^F, x_j^L, a_i, a_j) \quad \text{and} \quad \Pi_j(x_i^N, x_j^F, a_i, a_j) \geq \Pi_j(x_i^L, x_j^F, a_i, a_j). \tag{3.5}$$

We will see that all we need for this conditions to hold is the payoff functions to be monotone on the action of the other player. For simplicity of exposition, we will separate the case of increasing and decreasing payoffs, but the idea is essentially the same.

Let us start by noticing that the following holds:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \quad \geq \quad \Pi_i(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_i(x_i^L, x_j^N, a_i, a_j).$$

The first inequality was already explained, and the second one comes from the definition of best response. Thus, we have that:

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \geq \Pi_i(x_i^L, x_j^N, a_i, a_j). \tag{3.6}$$

An analogous analysis for player j leads to:

$$\Pi_j(x_i^F, x_j^L, a_i, a_j) \geq \Pi_j(x_i^L, x_j^N, a_i, a_j). \tag{3.7}$$

Now we divide the argument in two cases: decreasing and increasing (on the action of the other player) payoff functions.

- Decreasing payoffs.

  If $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) < 0$ and $\dfrac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) < 0$, then from inequality (3.6) we have:

  $$x_j^N \geq x_j^F \text{ and } x_i^N \leq x_i^L.$$

  From (3.7), we also have:

  $$x_i^N \geq x_i^F \text{ and } x_j^N \leq x_j^L.$$

  Summarizing:

  $$x_i^F \leq x_i^N \leq x_i^L \wedge x_j^F \leq x_j^N \leq x_j^L.$$ [2]

Which is a sufficient condition for (3.5) to be true, provided that the payoff functions are decreasing on the action of the other player.

- Increasing payoffs.

  If $\frac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) > 0$ and $\frac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) > 0$, then from (3.6) and (3.7) we obtain:

$$x_i^L \leq x_i^N \leq x_i^F \wedge x_j^L \leq x_j^N \leq x_j^F.$$

  Which again is a sufficient condition for (3.5) to hold, but in this case provided that the payoff functions are increasing on the action of the rival.

□

Recall that in pre play stage of the GOD model, firms simply choose F or S, indicating if they want to take their action in the "first" or "second" stage (or "early" and "late"), and this naturally induces an order of plays in the extended equilibrium. Given that, the extended game can be presented in a 2×2 matrix as follows:

	F	S
F	Π_1^N, Π_2^N	Π_1^L, Π_2^F
S	Π_1^F, Π_2^L	Π_1^N, Π_2^N

Thus, the equilibria of the extended game can be found by analysing profitable deviations from that 2×2 matrix. Therefore, from Lemma 3.3, we directly obtain the following result which characterizes the endogenous timing in this setting.

Theorem 3.4 *If payoffs are monotone on the action of the other player, then in the GOD model the unique SPE of the extended game is with both firms choosing F, which induces simultaneous play in the basic interaction. This result holds for all $a_i, a_j \in A$ and for both types of investment: tough and soft.*

PROOF. Direct from the Lemma 3.3

□

It should be noted that this existence (and uniqueness) result is diametrically different to the parallel one established in Theorem 2.4. In particular, there is no need for refinement based on the nature of the investment variable. Moreover, Theorem 3.4 implies that if the basic interaction is a Cournot duopoly (quantity competition), the extended GOD model will result in simultaneous competition, no matter what is the nature of the variable that makes firms different. We formalize this idea in the following Corollary which generalizes the results obtained by Jinji (2004), Lu and Poddar (2009), Tesoriere (2008), among others.

Corollary 3.5 *For every duopolistic competition model in which the basic interaction is quantity competition, the extended GOD model has a unique SPE: simultaneous timing.*

3.2 Game with Action Commitment

In this section we consider the GAC model to extend the basic game, and make the timing of movements endogenous. As we have said before, this model is quite different from the GOD model, particularly for two reasons: in this case the action space of the pre play stage is not finite as in GOD, and also, because now if players want to avoid the follower position, they need to commit to an action. The first result is about the existence and characterization of equilibria in the extended game.

Theorem 3.6 *Under the assumption (A2) on (3.1), and for each $a_i, a_j \in A$, there are three equilibria of the extended game: (x_i^L, W_j, a_i, a_j), (x_i^N, x_j^N, a_i, a_j) and (W_i, x_j^L, a_i, a_j).*

PROOF. Since the extension of the game fits the framework of Hamilton and Slutsky (1990), the proof comes directly from their results.

□

Now is necessary to refine this multiple equilibria, and in order to do so, we apply the risk dominance concept from Harsanyi and Selten (1998). We start by proving that the simultaneous equilibrium is never the risk dominant one, and then we move on to the comparison between the sequential equilibria.

Theorem 3.7 *Considering assumption (A2) on (3.1), and assuming payoffs are monotone on the action of other player, any of sequential equilibria risk dominates the simultaneous one. This result holds for all $a_i, a_j \in A$, and for both types of investment (soft or tough).*

PROOF. Without loss of generality, we prove that (x_i^L, W_j, a_i, a_j) risk dominates (x_i^N, x_j^N, a_i, a_j). The proof goes in two steps: first we calculate the bicentric prior, and then we apply the tracing procedure.

Bicentric Prior.

- Firm j believes that firm i plays $zx_i^L + (1-z)x_i^N$. We have that:

  (i) If $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) < 0$ and $\dfrac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) < 0$, then

  $$x_i^F \leq x_i^N \leq x_i^L \qquad \text{and} \qquad x_j^F \leq x_j^N \leq x_j^L.$$

  Therefore

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \quad \geq \quad \Pi_i(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_i(x_i^F, x_j^L, a_i, a_j).$$

Analogously for firm j, we have:

$$\Pi_j(x_i^L, x_j^F, a_i, a_j) \quad \geq \quad \Pi_j(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_j(x_i^F, x_j^L, a_i, a_j).$$

Then, in this case firm j believes that firm i commits to an action $\bar{x}_i \in [x_i^N, x_i^L]$. Note that in particular $\bar{x}_i \geq x_i^N$.

Now, since the best response functions are decreasing, if firm j wants to commit, the best she can do is committing to $\bar{x}_j \leq BR_j(x_i^N) = x_j^N$, but we know this is dominated by waiting (Hamilton and Slutsky, 1990). Therefore, in this case, the best player j can do is to wait.

(ii) If $\dfrac{\partial \Pi_i}{\partial x_j}(x_i, x_j, a_i, a_j) > 0$ and $\dfrac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) > 0$, then

$$x_i^L \leq x_i^N \leq x_i^F \text{ and } x_j^L \leq x_j^N \leq x_j^F.$$

Consequently,

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) \quad \geq \quad \Pi_i(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_i(x_i^F, x_j^L, a_i, a_j).$$

And, for firm j we have:

$$\Pi_j(x_i^L, x_j^F, a_i, a_j) \quad \geq \quad \Pi_j(x_i^N, x_j^N, a_i, a_j) \quad \geq \quad \Pi_j(x_i^F, x_j^L, a_i, a_j).$$

Analogously as in the previous case with decreasing payoffs, it is possible to prove that the best response of firm j is to wait

In summary, the prior belief of firm i is that she plays against a firm that will certainly wait.

- Firm i believes that firm j plays $zW_j + (1-z)x_j^N$. Note that this means that j plays x_j^N with probability $(1-z)$, and waits with the complementary probability. Then, if firm i waits, it will certainly obtain $\Pi_i(x_i^N, x_j^N, a_i, a_j)$.

  On the other hand, if $z > 0$ and firm i commits to an action slightly above x_i^N, it will obtain a greater payoff. Therefore, the best firm i can do is committing to a quantity $x_i(z)$, and the prior belief of firm j is that she is playing against a player that certainly commits to an action greater than x_i^N.

Summarizing the previous analysis, the bicentric prior is that firm j will wait, and firm i commits to a quantity greater than x_i^N.

Tracing Procedure.

At the beginning of the tracing procedure the equilibrium is the best response to the prior belief, and in this case that best response is (x_i^L, W_j, a_i, a_j). As it is also an equilibrium of the basic game, that is the risk dominant equilibrium.

$\square$

Observation Note that the result of Theorem 3.7 does not depend on the investment variable, but only on the behaviour of the payoff function with respect to the competition variables x_i and x_j.

At this point is interesting to set a comparison between the behaviour of the model under supermodularity (as in the previous Chapter) and submodularity as in this Chapter. In the case of supermodular competition (assumption (2.1), Chapter 2), the results of GOD and GAC were not different, in the sense that both models predicted that simultaneous competition would never appear. Now, with strategic substitutes (assumption (3.1)), the GOD model says that competition will be always simultaneous in equilibrium, while GAC establishes that the simultaneous configuration will never appear (regardless of the investment variable). Our interpretation is that this result is capturing a crucial aspect of the submodular competition.

To fix ideas, let us consider a classic Cournot duopoly (quantity competition). In that setting, none of the firms want to be a follower, and so, they will try to avoid this position. In the GOD model, avoiding the follower position has no risk involved, because players just need to declare F in the pre play stage, and by doing so, the worst scenario is ending up engaged in simultaneous play. On the other hand, avoiding the follower position on the GAC model has a high risk involved, because to do that, players need to commit to an action (a specific quantity); and it is possible to prove that committing to "small" quantities is dominated (Hamilton and Slutsky (1990)). Therefore, the only option to avoid the follower position is committing to a "large" quantity, and this last action is risky because, if both players do that, they could end up in a Stackelberg warfare. This explains why in the GAC model, the least aggressive firms would prefer to wait and see what the other player does, before blindly committing to an action.

The final task of this section is determine which one of the sequential equilibria is the risk dominant. We find sufficient conditions for the firm with the largest investment to be the leader, both with soft and tough investment. The result is summarized in Theorems 3.8 and 3.9 .

Observation On the next theorems, we denote $x_j^*(z)$ the optimal commitment action of player j as a function of the weight she puts on the commitment option of player i. Analogous for $x_i^*(z)$. We will also be using the following assumption that we establish here to make the notation simpler.

$$\frac{\partial \Pi_j}{\partial x_i}(x_i, \underline{y}, a_i, a_j) < \frac{\partial \Pi_j}{\partial x_i}(x_i, \bar{y}, a_i, a_j) \quad \forall \underline{y} < x_j^N < \bar{y} \in X,\ \forall x_i \in X,\ \forall a_i > a_j \in A. \tag{3.8}$$

Intuitively, condition (3.8) says that the effect of x_i on Π_j is greater when the action of player j is above the simultaneous equilibrium action.

Theorem 3.8 *Suppose that assumptions (A2) on (3.1) and (3.8) are met, the investment variable is tough, payoffs are monotone on the action of the rival, $a_i > a_j$, and one of the following cases hold.*

(i) *Payoffs are decreasing on the action of the rival,* $\frac{\partial x_i^*(z)}{\partial a_i} > \frac{\partial x_i^F}{\partial a_i}$, $\frac{\partial x_i^*(z)}{\partial a_i} > \frac{\partial x_i^N}{\partial a_i}$.

(ii) *Payoffs are increasing on the action of the rival,* $\frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^F}{\partial a_i}$, $\frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^N}{\partial a_i}$.

Then, (x_i^L, W_j, a_i, a_j) risk dominates (W_i, x_j^L, a_i, a_j), and therefore the firm with the largest investment becomes the risk dominant leader.

PROOF. We start with the bicentric prior, and then we apply the tracing procedure.

Bicentric Prior

Firm j thinks is playing against $zx_i^L + (1-z)W_i$, that is, that i commits with probability z and waits with the complementary probability. If player j commits, obtains payoff:

$$z\Pi_j(x_i^L, x_j, a_i, a_j) + (1-z)\Pi_j(W_i, x_j, a_i, a_j) = z\Pi_j(x_i^L, x_j, a_i, a_j) + (1-z)\Pi_j(x_i(x_j), x_j, a_i, a_j).$$

Where $x_i(x_j)$ is the best response of firm i. The optimal payoff due to this commitment can be obtained by maximizing this expression with respect to x_j. Since payoffs are concave, it is enough to impose the first order condition to find that unique maximum, namely $x^*(z)$. Replacing that maximum, we obtain:

$$z\Pi_j(x_i^L, x_j^*(z), a_i, a_j) + (1-z)\Pi_j(x_i(x_j^*(z)), x_j^*(z), a_i, a_j).$$

If we want to know how the best response to the prior belief behaves, we need to understand in which case committing is better than waiting, i.e.:

$$z\Pi_j(x_i^L, x_j^*(z), a_i, a_j)+(1-z)\Pi_j(x_i(x_j^*(z)), x_j^*(z), a_i, a_j) \geq \underbrace{z\Pi_j(x_i^L, W_j, a_i, a_j) + (1-z)\Pi_j(W_i, W_j, a_i, a_j)}_{z\Pi_j(x_i^L, x_j^F, a_i, a_j)+(1-z)\Pi_j(x_i^N, x_j^N, a_i, a_j)}.$$

Assuming continuous payoffs, this occurs under a threshold condition ($z \leq z_j$, for some z_j), and then, the best response of firm j will be of the form:

$$b_j(z) = \begin{cases} W_j & \text{if } z > z_j. \\ x_j^*(z) & \text{if } z \leq z_j. \end{cases}$$

Therefore, the prior belief of i is that is playing against $b_j(Z)$, where $Z \sim U(0,1)$. For the prior belief of firm j about the behaviour of player i, we proceed analogously and obtain that it is $b_i(Z)$, $Z \sim U(0,1)$, with the corresponding z_i as the threshold value.

Tracing Procedure

For the tracing procedure we will prove two properties. First, that for the firm with the lowest investment is always better to wait, and then, that for the other one is better to commit optimally.

- Let us find the conditions under the following inequality holds for all x_j.

$$\begin{aligned} &z_i\Pi_j(\mathbb{E}(x_i), x_j, a_i, a_j) + (1-z_i)\Pi_j(x_i(x_j), x_j, a_i, a_j) \\ &\leq z_i\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j) + (1-z_i)\Pi_j(x_i^N, x_j^N, a_i, a_j). \end{aligned}$$

  The strategy to prove this condition will be to show that the optimum action for j on the LHS is lower than x_j^N, which cannot happen (see Hamilton and Slutsky (1990)). In order to do this, let us consider that optimization problem:

$$\max_{x_j} z_i\Pi_j(\mathbb{E}(x_i), x_j, a_i, a_j) + (1-z_i)\Pi_j(x_i(x_j), x_j, a_i, a_j). \tag{3.9}$$

  We can see this problem as the optimization of a convex combination of two functions:

$$\begin{aligned} \varphi_1(x_j) &= \Pi_j(\mathbb{E}(x_i), x_j, a_i, a_j). \\ \varphi_2(x_j) &= \Pi_j(x_i(x_j), x_j, a_i, a_j). \end{aligned}$$

  If we consider the problem of optimizing $\varphi_1(x_j)$ and $\varphi_2(x_j)$ separately, we would obtain that the optima are $x_j(\mathbb{E}(x_i))$ and x_j^L respectively. Therefore, the optimum (namely $\bar{x}_j$) of (3.9) must be such that:

$$\bar{x}_j \in \left[x_j(\mathbb{E}(x_i)), x_j^L\right].$$

Since we are on the submodular case, the best responses are decreasing, and therefore $x_j(\mathbb{E}(x_i)) < x_j^N$. As we have said before, to conclude we need to find under which conditions $\bar{x}_j < x_j^N$. Note that the first order condition of (3.9) implies that, on the optimum, the following must hold:

$$\frac{z_i}{1-z_i} = -\frac{\varphi_2'(\bar{x}_j)}{\varphi_1'(\bar{x}_j)} = -\frac{\Pi_j(x_i(\bar{x}_j), \bar{x}_j, a_i, a_j)}{\Pi_j(\mathbb{E}(x_i), \bar{x}_j, a_i, a_j)}. \tag{3.10}$$

Observation Note that the ratio $-\dfrac{\varphi_2'(\bar{x}_j)}{\varphi_1'(\bar{x}_j)}$ is positive, since the numerator is positive and the denominator is negative.

Based on relation (3.10), if we want to ensure $\bar{x}_j$ to be lower that x_j^N, we need to make it closer to $x_j(\mathbb{E}(x_i))$, or equivalently, to make z_i closer to 1. Therefore, we will check under which conditions z_i is increasing.

Recall from the bicentric prior construction, that z_i is the threshold weight at which player i is indifferent between waiting and committing to her leader action, this is, z_i is the greatest z such that:

$$\begin{aligned}
&z\Pi_i(x_i^*(z), x_j^L, a_i, a_j) + (1-z)\Pi_i(x_i^*(z), x_j(x_i^*(z)), a_i, a_j) \\
&\geq z\Pi_i(x_i^F, x_j^L, a_i, a_j) + (1-z)\Pi_i(x_i^N, x_j^N, a_i, a_j).
\end{aligned}$$

If we assume that payoffs are decreasing on the rival's action, then the tough assumption implies that:

$$\frac{\partial x_i}{\partial a_i} > 0.$$
$$\frac{\partial x_j}{\partial a_j} > 0.$$

Consequently, what we need is that $\dfrac{\partial x_i^*(z)}{\partial a_i} > \dfrac{\partial x_i^F}{\partial a_i}$ and $\dfrac{\partial x_i^*(z)}{\partial a_i} > \dfrac{\partial x_i^N}{\partial a_i}$, when a_i is enough greater than a_j.

If the payoffs happen to be increasing on the action of the rival, the tough assumption would imply that:

$$\frac{\partial x_i}{\partial a_i} < 0.$$
$$\frac{\partial x_j}{\partial a_j} < 0.$$

Therefore, in this case we need that $\frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^F}{\partial a_i}$ and $\frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^N}{\partial a_i}$, when a_i is enough greater than a_j.

- Now we focus on the behaviour of the player with the largest investment. Let us find under which conditions it is beneficial for such player to commit instead of waiting. For this, we need that:

$$z_j \Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j) + (1 - z_j)\Pi_i(x_i^N, x_j^N, a_i, a_j) \tag{3.11}$$
$$\leq \max_{x_i} z_j \Pi_i(x_i, \mathbb{E}(x_j), a_i, a_j) + (1 - z_j)\Pi_i(x_i, x_j(x_i), a_i, a_j). \tag{3.12}$$

Let us focus on the RHS of the inequality.

$$\max_{x_i} z_j \Pi_i(x_i, \mathbb{E}(x_j), a_i, a_j) + (1 - z_j)\Pi_i(x_i, x_j(x_i), a_i, a_j). \tag{3.13}$$

Once again, we can see it as the problem of optimizing a convex combination of two concave functions:

$$\psi_1(x_i) = \Pi_i(x_i, \mathbb{E}(x_j), a_i, a_j).$$
$$\psi_2(x_i) = \Pi_i(x_i, x_j(x_i), a_i, a_j).$$

Maximizing $\psi_1(x_i)$ and $\psi_2(x_i)$ separately would bring $x_i(\mathbb{E}(x_j))$ and x_i^L as optima, respectively. Consequently, the optimum of (3.13) (denoted $\bar{x}_i$) must be such that:

$$\bar{x}_i \in \left[x_i(\mathbb{E}(x_j)), x_i^L\right].$$

Observation Since we are in the submodular case, $x_i(\mathbb{E}(x_j)) < x_i^N$.

Note that if $\bar{x}_i = x_i(\mathbb{E}(x_j))$, inequality (3.11) does not hold, while if $\bar{x}_i = x_i^L$ it indeed does. Because of the continuity of the functions, there must be a threshold from which the inequality is true. Therefore, what we need to do is to ensure that the optimum of (3.13) is "close" to x_i^L.

The first order condition of (3.13) implies that, at the optimum, the following equality must hold:

$$\frac{z_j}{1 - z_j} = -\frac{\psi_2'(\bar{x}_i)}{\psi_1'(\bar{x}_i)}. \tag{3.14}$$

Asking for the optimum to be close to x_i^L is equivalent to ask the RHS of (3.14) to be close to zero, which in turn is equivalent to ask for z_j to be close to zero. Recall from the construction of the bicentric prior that z_j is the greatest z such that:

$$z\Pi_j(x_i^L, x_j^*(z), a_i, a_j) + (1 - z)\Pi_j(x_i(x_j^*(z)), x_j^*(z), a_i, a_j)$$
$$\geq z\Pi_j(x_i^L, x_j^F, a_i, a_j) + (1 - z)\Pi_j(x_i^N, x_j^N, a_i, a_j).$$

Since the investment variable is tough, both sides are decreasing if a_i grows (ceteris paribus). Therefore, what we need is the effect on the LHS to be greater than on the RHS, which is true if:

$$\frac{\partial \Pi_j}{\partial x_i}(x_i, \underline{y}, a_i, a_j) < \frac{\partial \Pi_j}{\partial x_i}(x_i, \bar{y}, a_i, a_j) \ \forall \underline{y} < x_j^N < \bar{y}.$$

Note that if the investment variable is soft, both sides would be increasing with a_i, and we would need the same inequality to hold.

□

Observation On Theorem 3.8, we are using the hypotheses essentially through their effect on the probability to commit versus waiting for each player. Specifically, we showed that the probability to commit is increasing as a function of the own investment, while decreasing as a function of the investment of the other player. It is important to mention that our conditions are sufficient but not necessary, since we are imposing strong requirements on the difference in investment for both players.

Now we set the "mirror" result for the case of soft investment. It can be noted that the proof is analogous.

Theorem 3.9 *Suppose that assumptions (A2) on (3.1) and (3.8) are met, the investment variable is soft, payoffs are monotone on the action of the rival, $a_i > a_j$ and one of the following cases hold.*

(i) *Payoffs are decreasing on the action of the rival,* $\frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^F}{\partial a_i}, \frac{\partial x_i^*(z)}{\partial a_i} < \frac{\partial x_i^N}{\partial a_i}$.

(ii) *Payoffs are increasing on the action of the rival,* $\frac{\partial x_i^*(z)}{\partial a_i} > \frac{\partial x_i^F}{\partial a_i}, \frac{\partial x_i^*(z)}{\partial a_i} > \frac{\partial x_i^N}{\partial a_i}$.

Then, (x_i^L, W_j, a_i, a_j) risk dominates (W_i, x_j^L, a_i, a_j), and therefore the firm with the biggest investment becomes the risk dominant leader.

PROOF. Analogous to the proof of Theorem 3.8.

□

3.2.1 A particular case: quantity competition and different marginal costs.

Let us analyse how Theorem 3.8 applies to a particular case. To that purpose, we consider the quantity competition setting defined on van Damme and Hurkens (1999). There are two firms, namely $i = 1, 2$, with different (and constant) marginal costs $c_i \geq 0$. They face a linear market price of the form:

$$p = \max\{0, a - q_1 - q_2\}.$$

Players simultaneously choose quantities, and obtain profits:

$$\Pi_i(q_1, q_2) = (p - c_i) \cdot q_i, \ \forall i = 1, 2.$$

Assume that firm 2 is more efficient, this is, $c_2 < c_1$. The best response of player j against the quantities q_i of player i is unique and given by:

$$b_j(q_i) = \max\{0, \frac{a - c_j - q_i}{2}\}.$$

The optimal equilibrium actions are:

$$\begin{aligned} q_i^L &= \frac{2(a - c_i) - (a - c_j)}{2}. \\ q_i^F &= \frac{3(a - c_i) - 2(a - c_j)}{4}. \\ q_i^N &= \frac{2(a - c_i) - (a - c_j)}{3}. \end{aligned}$$

Given these equilibrium action, the profits of leader, follower and simultaneous competition are respectively:

$$\begin{aligned} \Pi_i^L \doteq \Pi_i(q_i^L, q_j^F) &= \frac{(2(a - c_i) - (a - c_j))^2}{8}. \\ \Pi_i^F \doteq \Pi_i(q_i^F, q_j^L) &= \frac{(3(a - c_i) - 2(a - c_j))^2}{16}. \\ \Pi_i^N \doteq \Pi_i(q_i^N, q_j^N) &= \frac{(2(a - c_i) - (a - c_j))^2}{9}. \end{aligned}$$

Observation Note that $q_i^F < q_i^N < q_i^L$ and $\Pi_i^F < \Pi_i^N < \Pi_i^L$, hence, both players would like to commit and achieve the leader position.

Note that in this case the investment variable that makes firm different is *tough*, since it diminishes marginal cost, and therefore allows a more aggressive competition afterwards. Formally, let us assume that firms can invest in a variable a_i that diminishes marginal cost, then:

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j} = \underbrace{\frac{\partial \Pi_i}{\partial q_j}}_{<0} \cdot \underbrace{\frac{\partial q_j}{\partial a_i}}_{>0} < 0.$$

Observation This holds for the case of leader, follower and simultaneous competition. As we have said before, we are not considering spillovers.

Now let us study in which case of Theorem 3.8 this setting fits. As the investment variable is tough, we need to check that the partial derivatives of the optimal commitment action are greater than the follower and simultaneous action for the leader. Recall that in this model:

$$q_i^*(z) = \frac{(a_i - a_j)}{2} + \frac{a - c + a_i}{2(1+z)}.$$

Therefore,

$$\begin{aligned}
\frac{\partial q_i^F}{\partial a_i} &= \frac{3}{4}. \\
\frac{\partial q_i^N}{\partial a_i} &= \frac{2}{3}. \\
\frac{\partial q_i^*(z)}{\partial a_i} &= \frac{1}{2} + \frac{1}{2(1+z)}.
\end{aligned}$$

And it is trivial to check that $\dfrac{\partial x_i^*(z)}{\partial a_i} > \dfrac{\partial x_i^F}{\partial a_i}$ and $\dfrac{\partial x_i^*(z)}{\partial a_i} > \dfrac{\partial x_i^N}{\partial a_i}$. Now regarding to the hypothesis of the partial derivative of Π_j, we have that:

$$\frac{\partial \Pi_j}{\partial x_i}(x_i, x_j, a_i, a_j) = q_j.$$

Therefore, since $x_j^*(z) > x_j^N > x_j^F$, the hypothesis is met, and we conclude that in this case the player with the largest investment being leader is the risk dominant equilibrium, which is aligned with the result presented by van Damme and Hurkens (1999).

3.3 Conclusions

In this Chapter we presented a model of submodular duopolistic competition. To fix ideas, this is a generalization of the classical Cournot competition. As in Chapter 2, we assumed that firms were absolutely identical except for one single characteristic, and we made the timing of movements endogenous using both GOD and GAC extension models. Recall that the characteristic that made firms different could be *soft* or *tough*as in Fudenberg and Tirole (1984).

For the GOD case, we proved that both players want to move early in the first stage, and consequently, there is a unique equilibrium of the extended game which results in simultaneous competition. This result is driven by the fact that, in submodular competition, both players want to avoid the follower position, and in this extension model, doing so requires only to declare F on the pre play stage of the game. An important feature to highlight of this result is that, it does not depend on the type of variable that make firms different, but only on the submodular characteristic.

In the GAC extension model, there could appear three possible equilibria: both of the sequential configurations, and the simultaneous one. The first refinement result that we proved was that the simultaneous equilibrium is never risk dominant, and this hold regardless of the type or level of investment. While this result is similar to its counterpart in Chapter 2, it is diametrically different to the GOD case in this very same configuration. Despite that the submodular characteristic still places incentives in avoiding the follower position, in the GAC model doing so is not riskless as in GOD. To avoid the follower position in this setting, players must commit to an action knowing that if the other player also commits, there is a chance of ending up in a Stackelberg warfare. Therefore, some firms, particularly the less aggressive ones (in the sense discussed), prefer to wait and see, instead of trying to achieve the leader position. This result, and consequently this intuition, generalize that obtained in Chapter 1. The final result has been divided in two parts for a clearer presentation, but essentially both parts present the conditions needed for the firm with the largest investment to become the leader under risk considerations. This conditions are a "large" difference on the investments levels, monotonicity on the payoffs with respect to the rival's action, and some technical requirements.

Chapter 4

Conclusions

This final Chapter is divided in two sections. First, we present the conclusions obtained from this thesis, with emphasis in how we were able to complete and better understand the results on Table 2. In the second section, we take a look to the future by presenting the most immediate extensions to the work presented in this document.

4.1 Conclusions

When looking at oligopolistic competition, or more particularly duopolies, it is frequently observed that the interaction between players can be simultaneous or sequential. The main motivation to develop this thesis was to understand which are the key features that determine if the interactions are simultaneous or sequential, and in the sequential case, how can be explained the leadership of a specific player. In summary, we wanted to understand leadership.

When we reviewed the literature, we found that the models mostly consisted of particular cases of quantity or price competition, between firms which were different in only one single characteristic: namely marginal cost, capacity of production, location, etc. In those models, the authors usually consider a basic interaction (price or quantity competition), and they try to make the timing endogenous using GOD or GAC from Hamilton and Slutsky (1990). In order to do so, normally it is first proven an "existence" theorem in which it is showed that there could emerge simultaneous or sequential competition. After that, in case of multiple equilibria, it is necessary to refine based on some criteria. When it is not possible to establish payoff dominance, the most used criteria is the risk dominance concept from Harsanyi and Selten (1998). Nevertheless, those models only aimed to understand leadership in their particular settings, but there was no general model that could explain which are the underlying dynamics that explain the timing of movements.

Our intuition after the literature review, was that the relevant characteristics in order to determine the timing of movements were three: the model of endogenous timing, the type of competition (in the literature, price or quantity), and the type of variable that made firms different. Our main objective then, was to develop a model that would help us to formalize this intuitions. In order to do so, we classified the type of competition in submodular and supermodular, which are generalizations of quantity and price competition respectively; and for the variable that made firms different, we took a concept from Fudenberg and Tirole (1984), and classified the type of variable in *tough* or *soft* type. With respect to the endogenous timing model, we considered GOD and GAC from Hamilton and Slutsky (1990).

In Chapter 1, we developed a particular model in which two firms with different marginal costs compete in quantities, selling differentiated products. The main motivation to work this model was to identify and understand the differences between GOD and GAC when applied to the exact same basic interaction, something that, to our knowledge, has not been done before. A second motivation to tackle this model was to compare the results obtained under GAC with those of van Damme and Hurkens (1999). In particular, we wanted to know if the differentiation parameter played a role in the endogenous timing result. Finally, in the last section of the Chapter, we study how the firms behave in a previous investment stage, if they know that their investment decisions affect not only the equilibrium actions, but also the timing of the game. In the GOD model, we showed that the unique equilibrium was simultaneous moving on the basic interaction. The important feature of this result is that it holds regardless of the investment type or level. In the GAC model, we proved that simultaneous competition never arises as the risk dominant, and lately, that the equilibrium in which the efficient firm leads is the risk dominant one between both of the sequentials. It is important to recall that, considering this model, Table 2 now looks like Table 1.2. About these results, the crucial aspects to our purposes are two: first, they do not depend on the differentiation parameter and the underlying intuition is that, both the endogenous timing and the risk dominance results, depend on the partial derivatives of the profit only through their sign (qualitative behaviour) and not their magnitude (quantitativeness). But most important, we can give an interpretation of the difference between GOD and GAC. In this setting (differentiated quantity competition), players want to avoid the follower position, in fact they prefer simultaneous competition before sequential being follower. In the GOD model, avoiding the follower position is riskless, because all they have to do is to declare "F" on the pre play stage, and by doing so, the "worst" can happen is to end up in simultaneous play. On the other hand, avoiding the follower position on the GAC model is risky, because it requires committing to a specific quantity, and even more, it should be a "large" quantity. If both players do so, they could end up in a Stackelberg warfare, and that is why they might prefer to wait and see instead of trying to avoid the follower position. In Chapter 3 we extend and generalize this result, and consequently the intuition. The last analysis performed in this Chapter was to think in a prior investment phase, in which the firms could invest to diminish their marginal cost. In that context, we wanted to know if the firms would under or over invest, compared to an environment in which the timing was exogenously given. Our results showed that in GOD model firms invest optimally, because they know that the further competition will be simultaneous, regardless of their prior decisions. For the GAC model, we showed that the firms over invest to achieve the leader position. This is considering marginal deviations only.

In Chapter 2, we presented a general model of supermodular competition (strategic complements), which is a generalization of the classic price competition. In this setting, firms were absolutely symmetrical, except for one single characteristic that made them different. We classified the nature of this characteristic (that we interpret as an investment) in *tough* or *soft*. We made endogenous the timing of movements in the basic interaction using the GOD and GAC models. For both extension models, we first proved existence results, based on the supermodularity condition, and afterwards, we proceeded with the refinement of possible multiple equilibria using the risk dominant concept, and the nature of the variable that made firms different. In the GOD model, we found that both of the sequential equilibria could emerge, and gave sufficient conditions for the leadership of each player. We found that, when the investment variable is tough, is more likely for the firm with the largest investment to become the leader. In the GAC model, when studying existence, we found that either simultaneous or sequential competition could arise. We proved that regardless of the nature and level of investment, simultaneous competition never risk dominates. Finally, we provided strong sufficient conditions for the leadership of each player (under risk considerations) for both types of investment. As we said, a possible path of future work is to find the necessary conditions for this results to hold.

In Chapter 3, we presented a model analogous of that in Chapter 2, but assuming that the competition was submodular (strategic substitutes), which is a generalization of the well known Cournot competition. We proceeded as in Chapter 2, meaning that the existence results came from the submodular condition, while the refinement was based on the nature of the investment. In the GOD model, we proved that the unique equilibrium of the extended game was simultaneous competition, and this result is regardless of the investment variable. This result implies that any submodular basic interaction which is extended using the GOD model, will result in simultaneous competition no matter what is the variable that makes firm different. For the GAC model, we showed that simultaneous competition is never the risk dominant equilibrium. When comparing the two sequential equilibria, we found sufficient conditions for the firm with the biggest investment to be the leader, both with soft and tough investment. As we mentioned previously, these results are essentially driven by the fact that, in the submodular competition, the follower position is the least preferred by the players, and therefore, they would like to avoid it if it is possible. In the GOD model, doing that involves no risk, since is enough for the player to declare F in the pre play stage. On the other hand, when we consider the GAC model, avoiding the follower position requires committing to a risky action, specifically to an action which is a convex combination of the leader and simultaneous action. Because of that risk, players might prefer to wait and see what the other player does, before trying to take an advantageous position.

On the Introduction of this thesis we stated that our goal was to understand leadership in terms of three key features: type of competition (supermodular or submodular), extension model (GOD or GAC) and what makes firms different (tough or soft variable). More specifically, to generalize and "complete" the results on Table 2. After the work presented in the three Chapters of this thesis, we can summarize our general contributions in Table 4.1.

	Supermodular		Submodular	
	Tough	**Soft**	**Tough**	**Soft**
GOD Model	- Sequential play. - Firm with largest investment more likely to be leader (risk).	- Sequential play.	Simultaneous play. is the unique equilibrium.	Simultaneous play is the unique equilibrium.
	Tough	**Soft**	**Tough**	**Soft**
GAC Model	- Sequential play (risk). -Firm with largest investment more likely to be leader (risk).	- Sequential play (risk).	- Sequential play (risk). -Firm with largest investment more likely to be leader (risk).	- Sequential play (risk).

Table 4.1: Summary of Contribution.

When looking at Table 4.1 it is possible to see that:

- For supermodular competition, we found that simultaneous competition is never the outcome of the interaction. In the GOD extension model this result comes from the fact that the existence theorem, in our setting, predicts that only sequential play emerges as equilibrium. In the GAC case the result comes from the risk refinement process based on the nature of the investment. Also, our results predict that, when the investment variable is tough, the firm with the largest investment is more likely to become the risk dominant leader. When the investment variable is soft, we still need to work further on finding necessary and sufficient conditions to characterize the leadership.

- For submodular competition, we fully characterize which equilibrium will emerge when the extension model is GOD: simultaneous competition. This result holds regardless of the type nor level of investment. On the other hand, for the GAC extension model, we find that the simultaneous equilibrium is never the risk dominant (and therefore it should never emerge). Also, when the investment variable is tough, the firm with the largest investment is more likely to become the leader. In the case of soft investment, as with supermodular competition, we need further work to find necessary and sufficient conditions for leadership. Another important result is the interpretation of the differences between both models that we provide based on risk considerations.

In the next Section of this Chapter, we study the future work related to the investment gamen.

4.2 Future Work: Investment Stage

In this section we describe what is the next step in this research project. In particular, we are interested in what would occur in a previous stage, if players would know that their investment affects not only the equilibrium actions, but also the timing of the interaction. A small sample of this approach was presented previously in Section 1.2, Chapter 1. Our idea is to generalize such model, using the results that we found in Chapters 2 and 3. Formally, we will consider two players that can invest in some variable a, that might be *tough* or *soft*, and then, they compete in strategic substitutes or complements. The timing of the game is as in Chapter 2 or 3, this is:

- In the first stage, they decide simultaneously their level of investment a_i and $a_j \in A$. Where A is a convex, compact subset of $\mathbb{R}$.
- In the second stage, they compete in variables x_i, $x_j \in X$, where X is a convex and compact subset of $\mathbb{R}$. This competition can be simultaneous or sequential, and we will use GOD and GAC to make the timing of movement endogenous.

After the game is played, firms obtain payoffs:

$$\Pi_i(x_i, x_j, a_i, a_j) - F_i(a_i).$$
$$\Pi_j(x_i, x_j, a_i, a_j) - F_j(a_j).$$

Where $F_i(\cdot)$ and $F_j(\cdot)$ are convex functions that represent the investment cost. The main question we want to tackle with this model is *Will the firms sub or over invest, if they have the chance to achieve the most desired position in the market?* Naturally, that question suggests a comparison with a benchmark. To that purpose, we will consider a case in which the timing of movements is exogenously determined, and this fact is known by the players.

4.2.1 Benchmark: exogenously determined timing

In this section we give the ideas that support the benchmark analysis in both, sequential and simultaneous case.

Simultaneous competition

Let us consider that the competition in variables x_i and x_j will be simultaneous. In order to determine the equilibrium investments a_i and a_j, we proceed by backward induction. In the competition phase, the equilibrium is $\left(x_i^S(a_i, a_j), x_j^S(a_i, a_j)\right)$. Therefore, on the investment phase, players must decide their level of investment considering payoffs:

$$\Pi_i\left(x_i^S(a_i, a_j), x_j^S(a_i, a_j), a_i, a_j\right) - F_i(a_i).$$
$$\Pi_j\left(x_i^S(a_i, a_j), x_j^S(a_i, a_j), a_i, a_j\right) - F_j(a_j).$$

The best response of player i comes from solving the first order condition:

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_i}(x_i^S(a_i, a_j), x_j^S(a_i, a_j), a_i, a_j) = F_i'(a_i).$$

$$\Rightarrow \underbrace{\frac{\partial \Pi_i}{\partial x_i}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j)}_{=0} \cdot \frac{\partial x_i^S(a_i,a_j)}{\partial a_i} + \frac{\partial \Pi_i}{\partial x_j}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j) \cdot \frac{\partial x_j^S(a_i,a_j)}{\partial a_i}$$

$$+\frac{\partial \Pi_i(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j)}{\partial a_i} = F_i'(a_i).$$

$$\Rightarrow \boxed{\frac{\partial \Pi_i}{\partial x_j}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j) \cdot \frac{\partial x_j^S(a_i,a_j)}{\partial a_i} + \frac{\partial \Pi_i}{\partial a_i}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j) = F_i'(a_i)}.$$

Analogously, the best response of firm j comes from solving:

$$\boxed{\frac{\partial \Pi_j}{\partial x_i}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j) \cdot \frac{\partial x_i^S(a_i,a_j)}{\partial a_j} + \frac{\partial \Pi_j}{\partial a_j}(x_i^S(a_i,a_j), x_j^S(a_i,a_j), a_i, a_j) = F_j'(a_j).}$$

Sequential competition

Let us assume (w.l.g.) that firm i is the leader in the competition phase. Therefore, the equilibrium in this case is given by $\left(x_i^L(a_i,a_j), x_j^F(a_i,a_j)\right)$, and subsequently, the conditions to obtain (a_i,a_j) in equilibrium are:

$$\boxed{\frac{\partial \Pi_i}{\partial x_j}(x_i^L(a_i,a_j), x_j^F(a_i,a_j), a_i, a_j) \cdot \frac{\partial x_j^F(a_i,a_j)}{\partial a_i} + \frac{\partial \Pi_i}{\partial a_i}(x_i^L(a_i,a_j), x_j^F(a_i,a_j), a_i, a_j) = F_i'(a_i).}$$

And for player j:

$$\boxed{\frac{\partial \Pi_j}{\partial x_i}(x_i^L(a_i,a_j), x_j^F(a_i,a_j), a_i, a_j) \cdot \frac{\partial x_i^L(a_i,a_j)}{\partial a_j} + \frac{\partial \Pi_j}{\partial a_j}(x_i^L(a_i,a_j), x_j^F(a_i,a_j), a_i, a_j) = F_j'(a_j).}$$

4.2.2 Investment with endogenous timing

After the benchmark is set, the next step will be to analyse the case in which the timing of movements in the competition phase also depends on the investment decisions. This dependence is based on the results obtained in Chapter 2 and Chapter 3. Once that is done, the final step is to compare the equilibrium investment obtained in this case, with that from the benchmark.

At this point it is possible to state a conjecture about the results in the submodular competition case, based on the results obtained in Chapter 1, Section 1.2, and Chapter 3.

When we have submodular competition, the follower position is the less preferred by the players, they would like to avoid that position. We proved in Chapter 3, that in the case of GOD extension model, the resulting competition will be simultaneous, regardless of the investment level of the firms. In a complete information environment, as the one we are considering, players know this and therefore, there should be no incentives to over or under invest in order to avoid the follower (or obtain the leader) position. Summarizing, we have the following conjecture.

Conjecture 4.1 *If competition is submodular, and the extension model is GOD, players invest as in Section 4.2.1. This result hold regardless of the type of investment (tough or soft).*

In this same context, when the extension model is GAC, we proved that the firm with the biggest investment should emerge as leader in the risk dominant equilibrium. Actually, we also proved that simultaneous competition never occurs. When players know that, they have incentives to invest more than her rival in order to get the leader position, which is the preferred position in the submodular case. This incentives could lead the players to over invest, because they know that if they have the same investment than the other player, they could achieve the leader position by investing $\varepsilon > 0$ more. We summarize this intuition in the following proposition, which is consistent with the particular result obtained in Chapter 1, section 1.2.

Conjecture 4.2 *If competition is submodular, the extension model is GAC and the investment variable is tough, the firms over invest (compared to the investment that would be obtained as in Section 4.2.1).*

At this point of our research, it is not possible to establish other conjectures about the behaviour in the investment phase. The future work will be to tackle the case with submodular competition and *soft* investment; and also the supermodular case with both types of extension model, and both types of investment variable.

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Appendix

Proof of Lemma 1.8.

PROOF. In the proof of Lemma 1.7, we found that the optimal commitment was:

$$q_i^* = \frac{a_i - \frac{\alpha a_j}{2} + z_j\left(\frac{\alpha a_j}{2} - \alpha\mu_j\right)}{2 - \alpha^2 + \alpha^2 z_j}.$$

Replacing in the utility function, we find the optimal payoff that player i achieves by committing:

$$u_i(q_i^*, m_j) = \frac{(2a_i^3\alpha^5 - 2a_i^2 a_j \alpha^4(5 + Kj) + a_j^3(\alpha^6 + \alpha^2(8 - 16Kj) + \alpha^4(-8 + Kj) + 16Kj) - 2a_i a_j^2 \alpha(16 + \alpha^4 - 4\alpha^2(4 + Kj)))^2}{(8a_j^2\alpha^2(-4 + \alpha^2)^2(2 - \alpha^2)(8a_i a_j \alpha^3 - 2a_i^2\alpha^4 + a_j^2(16 - 16\alpha^2 + \alpha^4)))}.$$

Where $K_j = \ln\left(\frac{2 - \alpha^2 + \alpha^2 z_j}{2 - \alpha^2}\right)$.

On the other hand, if firm i decides to wait, its payoff will be:

$$\begin{aligned} u_i(W_i, m_j) &= z_j\left(\mathbb{E}\left(\frac{[a_i - \alpha q_j]^2}{4}\right)\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2 \\ &= z_j\left(\frac{1}{4} \cdot \mathbb{E}([a_i - \alpha q_j]^2)\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2 \\ &= z_j\left(\frac{(a_i - \alpha\mu_j)^2}{4} + \frac{\alpha^2\nu_j}{4}\right) + (1 - z_j)\left(\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right)^2. \end{aligned}$$

The difference between payoffs, assuming that $a_i = 2a_j$ is:

$$
\begin{aligned}
&u_i(q_i^*, m_j) - u_i(W_i, m_j) \\
&= \frac{[2a_i - \alpha a_j(1 - z_j) - 2\alpha z_j \mu_j]^2}{8(2 - \alpha^2 + \alpha^2 z_j)} - z_j \left(\frac{(a_i - \alpha\mu_j)^2}{4} + \frac{\alpha^2 \nu_j}{4} \right) - (1 - z_j) \left(\frac{2a_i - \alpha a_j}{4 - \alpha^2} \right)^2 \\
&= \frac{1}{(64(-2+\alpha)^2(-1+\alpha)^2\alpha^2(2+\alpha)^2(-2+\alpha^2)(-16+\alpha^2(16+\alpha(-16+7\alpha))))} \times \\
&[a_j^2(-148\alpha^{14} - 4096\alpha K_j^2 + 8\alpha^{13}(85 + 14K_j) \\
&+ 4\alpha^{10}(11895 + 2K_j(1905 + 38K_j - 11T_j)) + 2048\alpha^3(1 + 2K_j(1 + K_j - 2T_j)) - 2048K_j(K_j + 2T_j) \\
&+ 128\alpha^6(73 + K_j + 42K_j^2 - 54K_jT_j) + 1024\alpha^2(-1 + 8K_j^2 + 10K_jT_j) - 256\alpha^4(-4 + 66K_j^2 + 19K_j(4 + T_j)) \\
&+ 8\alpha^{11}(-2311 + 28K_j(-7 + K_j + T_j)) - 256\alpha^7(99 + K_j(189 + 57K_j + 4T_j)) + 512\alpha^5(-10 + K_j(56 + 23K_j + 24T_j)) \\
&- \alpha^{12}(-2088 + K_j(536 + 49K_j + 49T_j)) - 16\alpha^9(4024 + K_j(2728 + 241K_j + 80T_j)) \\
&+ 8\alpha^8(6468 + K_j(8168 + 1355K_j + 378T_j)))].
\end{aligned}
$$

Where $T_j = \ln\left(\dfrac{2 - \alpha^2}{2 - \alpha^2 + \alpha^2 z_j}\right)$. If we simplify the expression, we obtain that the difference between payoffs is:

$$
\begin{aligned}
&u_i(q_i^*, m_j) - u_i(W_i, m_j) = \\
&\left(\frac{a_j^2}{(64(-4+\alpha^2)^2(-2+\alpha^2)}\right) \cdot [-\left(\frac{(4(256+\alpha^2(-512+\alpha(256+\alpha(-1312+\alpha(3456+\alpha(-4712+\alpha(3216+\alpha(-751+\alpha(-96+37\alpha)))))))))}{(-16+\alpha^2(16+\alpha(-16+7\alpha)))}\right) \\
&- \frac{1}{((1-\alpha)^2\alpha^2)} \ln[A_\alpha](-8(-4+\alpha)(-1+\alpha)^2\alpha^3(-1+2\alpha)(-8+\alpha(4+b)) \\
&+ (8+\alpha(16-16\alpha+\alpha^3))(-16+\alpha^2(16+\alpha(-16+7\alpha)))\ln[A_\alpha] + (-4+\alpha^2)^2(-16+\alpha^2(16+\alpha(-16+7\alpha)))\ln[B_\alpha])].
\end{aligned}
$$

Where $A_\alpha = \dfrac{16 - \alpha^2(14 - \alpha(12 - 5\alpha))}{16 - \alpha^2(16 - \alpha(16 - 7\alpha))}$ and $B_\alpha = \dfrac{-16 + \alpha^2(16 + \alpha(-16 + 7\alpha))}{-16 + \alpha^2(14 + \alpha(-12 + 5\alpha))}$.

It is possible to show that this expression is strictly positive when $\alpha > 0.73$.

□

Proof of Theorems 2.6 and 2.7.

PROOF. Note that, since firms are absolutely symmetrical, if $a_i = a_j$, we have that the conditions on Lemma 2.5 are met in equality, this is:

$$\Pi_j(x_i^L, x_j^F, a_i, a_j) + \Pi_i(x_i^L, x_j^F, a_i, a_j) = \Pi_i(x_i^F, x_j^L, a_i, a_j) + \Pi_j(x_i^F, x_j^L, a_i, a_j). \tag{4.1}$$

$$\Pi_i(x_i^L, x_j^F, a_i, a_j) + \Pi_j(x_i^N, x_j^N, a_i, a_j) = \Pi_j(x_i^F, x_j^L, a_i, a_j) + \Pi_i(x_i^N, x_j^N, a_i, a_j). \tag{4.2}$$

Observation To simplify the notation, we will write:

$$\Pi_i^L \doteq \Pi_i(x_i^L, x_j^F, a_i, a_j).$$
$$\Pi_i^F \doteq \Pi_i(x_i^F, x_j^L, a_i, a_j).$$
$$\Pi_i^N \doteq \Pi_i(x_i^N, x_j^N, a_i, a_j).$$

Analogous for firm j.

Consider condition (4.1). Since both sides are equal when $a_i = a_j$, for the proposition to be true we need that, for all $a_i \geq a_j$, the LHS grows faster than the RHS. This is:

$$\frac{\partial \Pi_i^L}{\partial x_j}\frac{\partial x_j^F}{\partial a_i} + \frac{\partial \Pi_i^L}{\partial a_i} + \frac{\partial \Pi_j^F}{\partial x_i}\frac{\partial x_i^L}{\partial a_i} \qquad \geq \qquad \frac{\partial \Pi_i^F}{\partial x_j}\frac{\partial x_j^L}{\partial a_i} + \frac{\partial \Pi_i^F}{\partial a_i} + \frac{\partial \Pi_j^L}{\partial x_i}\frac{\partial x_i^F}{\partial a_i}.$$

Since firms are absolutely symmetrical, we have that $\frac{\partial \Pi_i^L}{\partial x_j} = \frac{\partial \Pi_j^L}{\partial x_i}$, and $\frac{\partial \Pi_i^F}{\partial x_j} = \frac{\partial \Pi_j^F}{\partial x_i}$. Therefore, the previous condition turns into:

$$\frac{\partial \Pi_i^L}{\partial x_j}\left(\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i}\right) + \frac{\partial \Pi_i^F}{\partial x_j}\left(\frac{\partial x_i^L}{\partial a_i} - \frac{\partial x_j^L}{\partial a_i}\right) + \frac{\partial \Pi_i^L}{\partial a_i} - \frac{\partial \Pi_i^F}{\partial a_i} \geq 0. \tag{4.3}$$

An analogous analysis of condition (4.2) allow us to present it as:

$$\frac{\partial \Pi_i^L}{\partial x_j}\left(\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i}\right) + \frac{\partial \Pi_i^N}{\partial x_j}\left(\frac{\partial x_i^N}{\partial a_i} - \frac{\partial x_j^N}{\partial a_i}\right) + \frac{\partial \Pi_i^L}{\partial a_i} - \frac{\partial \Pi_i^N}{\partial a_i} \geq 0. \tag{4.4}$$

If we want firm j to be the leader, we need the inequalities in the other way.

Let us suppose that investment variables are tough and $a_i > a_j$. Recall that this means that:

$$\frac{\mathrm{d}\Pi_j}{\mathrm{d}a_i} = \frac{\partial \Pi_j}{\partial x_i}\frac{\partial x_i}{\partial a_i} < 0.$$
$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j} = \frac{\partial \Pi_i}{\partial x_j}\frac{\partial x_j}{\partial a_j} < 0.$$

We will look for conditions to ensure that (4.3) and (4.4) hold.

1. Let us suppose that $\frac{\partial \Pi_i}{\partial x_j} \geq 0$ and $\frac{\partial \Pi_j}{\partial x_i} \geq 0$. From the tough investment assumption, we have that:

$$\frac{\partial x_j}{a_j} < 0 \wedge \frac{\partial x_i}{\partial a_i} < 0.$$

(1.1) Assume that $\dfrac{\partial x_i}{\partial a_j} < 0 \wedge \dfrac{\partial x_j}{\partial a_i} < 0$. For conditions (4.3) and (4.4) to hold, we need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^L}{\partial a_i} - \frac{x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^N}{\partial a_i} - \frac{x_i^N}{\partial a_i}.$$

Which is true if:

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|}.$$

(1.2) Now, assume that $\dfrac{\partial x_i}{\partial a_j} > 0 \wedge \dfrac{\partial x_j}{\partial a_i} > 0$. For conditions (4.3) and (4.4) to hold, we need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^L}{\partial a_i} - \frac{x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^N}{\partial a_i} - \frac{x_i^N}{\partial a_i}.$$

Which is true if:

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|}.$$

(2) Let us suppose that $\frac{\partial \Pi_i}{\partial x_j} \leq 0$ and $\frac{\partial \Pi_j}{\partial x_i} \leq 0$. From the tough investment assumption, we have that:

$$\frac{\partial x_j}{a_j} > 0 \wedge \frac{\partial x_i}{\partial a_i} > 0.$$

(2.1) Let us assume that $\frac{\partial x_i}{\partial a_j} < 0 \wedge \frac{\partial x_j}{\partial a_i} < 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which is true if:

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right|}.$$

(2.2) Let us assume that $\frac{\partial x_i}{\partial a_j} > 0 \wedge \frac{\partial x_j}{\partial a_i} > 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which is true if:

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_i^F}{\partial a_i}\right| > \left|\frac{\partial x_i^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right|}.$$

If we want firm j to be the leader, we need to impose that the LHS of conditions (4.3) and (4.4) is lower than zero. The results are symmetrical.

Now let us turn the attention to the case in which the investment variable is soft. Recall that in that case, the investment hypotheses said that:

$$\frac{\mathrm{d}\Pi_j}{\mathrm{d}a_i} = \frac{\partial \Pi_j}{\partial x_i}\frac{\partial x_i}{\partial a_i} > 0.$$
$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}a_j} = \frac{\partial \Pi_i}{\partial x_j}\frac{\partial x_j}{\partial a_j} > 0.$$

We investigate the conditions for (4.3) and (4.4) to be true, which is equivalent to firm i being the leader.

(1) Let us suppose that $\frac{\partial \Pi_i}{\partial x_j} \geq 0$ and $\frac{\partial \Pi_j}{\partial x_i} \geq 0$. From the tough investment assumption, we have that:

$$\frac{\partial x_j}{a_j} > 0 \wedge \frac{\partial x_i}{\partial a_i} > 0.$$

(1.1) Assume that $\frac{\partial x_i}{\partial a_j} < 0 \wedge \frac{\partial x_j}{\partial a_i} < 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \geq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which are true if,

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|}.$$

(1.2) Assume that $\dfrac{\partial x_i}{\partial a_j} < 0 \wedge \dfrac{\partial x_j}{\partial a_i} < 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which is true if:

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|}.$$

(2) Let us suppose that $\dfrac{\partial \Pi_i}{\partial x_j} \geq 0$ and $\dfrac{\partial \Pi_j}{\partial x_i} \geq 0$. From the tough investment assumption, we have that:

$$\frac{\partial x_j}{a_j} < 0 \wedge \frac{\partial x_i}{\partial a_i} < 0.$$

(2.1) Assume that $\dfrac{\partial x_i}{\partial a_j} < 0 \wedge \dfrac{\partial x_j}{\partial a_i} < 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which are true if,

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| > \left|\frac{\partial x_j^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|}.$$

(2.2) Assume that $\dfrac{\partial x_i}{\partial a_j} < 0 \wedge \dfrac{\partial x_j}{\partial a_i} < 0$. We need that:

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^L}{\partial a_i} - \frac{\partial x_i^L}{\partial a_i}.$$

And

$$\frac{\partial x_j^F}{\partial a_i} - \frac{\partial x_i^F}{\partial a_i} \leq \frac{\partial x_j^N}{\partial a_i} - \frac{\partial x_i^N}{\partial a_i}.$$

Which are true if,

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^L}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^L}{\partial a_i}\right|}.$$

$$\boxed{\left|\frac{\partial x_j^F}{\partial a_i}\right| < \left|\frac{\partial x_j^N}{\partial a_i}\right| \wedge \left|\frac{\partial x_i^F}{\partial a_i}\right| < \left|\frac{\partial x_i^N}{\partial a_i}\right|}.$$

If we want firm j to be the leader, the analysis is symmetrical.

$\square$

Proof of Theorem 2.10.

PROOF. Here we look for the conditions to ensure that, assuming that $a_i > a_j$, then

$$\varphi_i(m_j^t) \geq \varphi_j(m_i^t) \ \forall t \in [0,1].$$

Recall that

$$\varphi_i(m_j^t) = \underbrace{\max_{x_i} \Pi_i(x_i, m_j^t, a_i, a_j)}_{\doteq I} - \underbrace{\Pi_i(W_i, m_j^t, a_i, a_j)}_{\doteq II}.$$
$$\varphi_j(m_i^t) = \underbrace{\max_{x_j} \Pi_j(m_i^t, x_j, a_i, a_j)}_{\doteq III} - \underbrace{\Pi_j(m_i^t, W_j, a_i, a_j)}_{\doteq IV}.$$

Where

$$m_i^t = (1-t)\mathbb{E}(x_j) + W_j.$$
$$m_j^t = (1-t)\mathbb{E}(x_i) + W_i.$$

Let us start noticing that, if $a_i = a_j$:

$$\varphi_i(m_j^t) = \varphi_j(m_i^t).$$

Note that, if the investment variables are tough, the total effect of a_i is positive on Π_i and negative on Π_j. Therefore, in order to get $\varphi_i(m_j^t) \geq \varphi_j(m_i^t)$, it is sufficient that the following inequalities hold as a_i increases:

$$|I| \geq |IV| \text{ and } |II| \leq |III|. \tag{4.5}$$

Note that:

$$I = (1-t)\Pi_i(x_i^*, \mathbb{E}(x_j), a_i, a_j) + t\Pi_i(x_i^*, x_j(x_i^*), a_i, a_j).$$
$$II = (1-t)\Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j) + t\Pi_i(x_i^N, x_j^N, a_i, a_j).$$
$$III = (1-t)\Pi_j(\mathbb{E}(x_i), x_j^*, a_i, a_j) + t\Pi_j(x_i(x_j^*), x_j^*, a_i, a_j).$$
$$IV = (1-t)\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j) + t\Pi_j(x_i^N, x_j^N, a_i, a_j).$$

Where $x_i^* \in \left(x_i(\mathbb{E}(x_j)), x_i^L\right)$ and $x_j^* \in \left(x_j(\mathbb{E}(x_i)), x_j^L\right)$ are the optima of the respective maximization problems. Therefore, for condition (4.5) to hold it is sufficient to have the following inequalities:

$$\frac{\mathrm{d}\Pi_i(x_i^*, \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j)}{\mathrm{d}a_i} \geq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i^*, x_j(x_i^*), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} \geq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j^*, a_i, a_j)}{\mathrm{d}a_i} \leq 0,$$
$$\frac{\mathrm{d}\Pi_i(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} + \frac{\mathrm{d}\Pi_j(x_i(x_j^*), x_j^*, a_i, a_j)}{\mathrm{d}a_i} \leq 0.$$

Which are met under the hypotheses of Theorem 2.10 for the tough case. Of the inequalities go in the other way, player j as leader becomes the risk dominant equilibrium.

□

Proof of Theorem 2.11.

PROOF. Note that when the investment variable is soft, the total effect of a_i is positive both on Π_i and Π_j. Therefore, using the same notation that the proof of Theorem 2.11, for the leadership of player i it is sufficient that the following holds as a_i increases:

$$|I| \geq |III| \text{ and } |II| \leq |IV|. \tag{4.6}$$

Then, it is sufficient to ask that:

$$\begin{aligned}
\frac{\mathrm{d}\Pi_i(x_i^*, \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} - \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j^*, a_i, a_j)}{\mathrm{d}a_i} &\geq 0, \\
\frac{\mathrm{d}\Pi_i(x_i^*, x_j(x_i^*), a_i, a_j)}{\mathrm{d}a_i} - \frac{\mathrm{d}\Pi_j(x_i(x_j^*), x_j^*, a_i, a_j)}{\mathrm{d}a_i} &\geq 0, \\
\frac{\mathrm{d}\Pi_i(x_i(\mathbb{E}(x_j)), \mathbb{E}(x_j), a_i, a_j)}{\mathrm{d}a_i} - \frac{\mathrm{d}\Pi_j(\mathbb{E}(x_i), x_j(\mathbb{E}(x_i)), a_i, a_j)}{\mathrm{d}a_i} &\leq 0, \\
\frac{\mathrm{d}\Pi_i(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} - \frac{\mathrm{d}\Pi_j(x_i^N, x_j^N, a_i, a_j)}{\mathrm{d}a_i} &\leq 0.
\end{aligned}$$

Which hold under the hypotheses of Theorem 2.11. Again, if the inequalities go in the other way, player j being leader is the risk dominant equilibrium.

□