Maximum and minimum degree conditions for embedding trees

Guido Besomi, Matías Pavez-Signé[*], and Maya Stein[†]

Departamento de Ingeniería Matemática
Universidad de Chile
Beauchef 851
Santiago, Chile
{gbesomi, mpavez, mstein}@dim.uchile.cl

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Abstract

We propose the following conjecture: For every fixed $\alpha \in [0, \frac{1}{2}]$, each graph of minimum degree at least $(1+\alpha)\frac{k}{2}$ and maximum degree at least $2(1-\alpha)k$ contains each tree with k edges as a subgraph.

Our main result is an approximate version of the conjecture for bounded degree trees and large dense host graphs. We also show that our conjecture is asymptotically best possible.

The proof of the approximate result relies on a second result, which we believe to be interesting on its own. Namely, we can embed any bounded degree tree into host graphs of minimum/maximum degree asymptotically exceeding $\frac{k}{2}$ and $\frac{4}{3}k$, respectively, as long as the host graph avoids a specific structure.

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1   Introduction

A central challenge in extremal graph theory is to determine degree conditions a graph G has to satisfy in order to ensure that it contains a given subgraph H. One of the most interesting open cases are trees. Instead of focusing on the containment of just one specific tree T, one usually asks for containment of all trees of some fixed size $k \in \mathbb{N}$. To this end, bounds on the average degree, the median degree or the minimum degree of the host graph G have been suggested in the literature. Let us give a quick outline of the most relevant directions.

Starting with a simple observation, it is very easy to see that every graph of *minimum* degree at least k contains each tree with k edges. This is not true if we weaken the bound of the minimum degree, even if just by one: It suffices to consider the disjoint union of complete graphs of order k, which does not contain any tree with k edges.
The well known Erdős–Sós conjecture from 1963 (see [6]) states that every graph of *average* degree strictly greater than $k - 1$ contains each tree with k edges. A proof of this conjecture for large graphs was announced by Ajtai, Komlós, Simonovits and Szemerédi in the early 1990's, and for other partial results, see e.g. [4, 20]. The Loebl–Komlós–Sós conjecture from 1992 (see [7]) states that every graph of *median* degree at least k contains each tree with k edges. For progress on the latter conjecture, see [1, 5, 10–16, 22].

A new angle to the problem was introduced in 2016 by Havet, Reed, Stein, and Wood [9], who impose bounds on both the minimum *and* the maximum degree. They suggest that every graph of minimum degree at least $\lfloor \frac{2k}{3} \rfloor$ and maximum degree at least k contains each tree with k edges. We call their conjecture the $\frac{2}{3}$–*conjecture*, for progress see [9, 17, 18].
In [3], the present authors proposed a variation of this approach, conjecturing that every graph of minimum degree at least $\frac{k}{2}$ and maximum degree at least $2k$ contains each tree with k edges. We call this conjecture the $2k$–$\frac{k}{2}$ *conjecture*.

Comparing the two variants of maximum/minimum degree conditions given by the latter two conjectures, it seems natural to ask whether one can allow for a wider spectrum of bounds for the maximum and the minimum degree of the host graph. We believe it might be possible to weaken the bound on the maximum degree given by the $2k$–$\frac{k}{2}$ conjecture, if simultaneously, the bound on the minimum degree is increased.

Quantitatively speaking, we suggest the following.

Conjecture 1.1. *Let $k \in \mathbb{N}$, let $\alpha \in [0, \frac{1}{2}]$ and let G be a graph with $\delta(G) \geq (1 + \alpha)\frac{k}{2}$ and $\Delta(G) \geq 2(1 - \alpha)k$. Then G contains each tree with k edges.*

Note that for $\alpha = 0$, the bounds from Conjecture 1.1 conincide with the bounds from the $2k$–$\frac{k}{2}$ conjecture, and for all $\alpha \in [\frac{1}{3}, \frac{1}{2}]$, Conjecture 1.1 follows from the $\frac{2}{3}$–conjecture. So Conjecture 1.1 can be seen as a unification of the other two conjectures.
We show that Conjecture 1.1 is asymptotically best possible for infinitely many values of α.

Proposition 1.2. *For all odd $\ell \in \mathbb{N}$ with $\ell \geq 3$, and for all $\gamma > 0$ there are $k \in \mathbb{N}$, a k-edge tree T, and a graph G with $\delta(G) \geq (1 + \frac{1}{\ell} - \gamma)\frac{k}{2}$ and $\Delta(G) \geq 2(1 - \frac{1}{\ell} - \gamma)k$ with the property that T does not embed in G.*

We prove this proposition in Section 2. Note that Proposition 1.2 covers all values of $\alpha \in \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$. The tightness of our conjecture for other values of α will be discussed in Section 6.3.
We remark that Proposition 1.2 disproves a conjecture from [19] (see Section 2 for details).

On the positive side, observe that Conjecture 1.1 trivially holds for the star and for the double star. Also, it is not difficult to see that the conjecture holds for paths. In fact, if the tree we wish to embed is the k-edge path P_k, it already suffices to require a minimum degree of at least $\frac{k}{2}$ and a maximum degree of at least k in the host graph G.[1]

We provide further evidence for the correctness of Conjecture 1.1 by proving an approximate version for bounded degree trees, and large dense host graphs.

Theorem 1.3. *For all $\delta \in (0, 1)$ there exist $k_0 \in \mathbb{N}$ such that for all $n, k \geq k_0$ with $n \geq k \geq \delta n$ and for each $\alpha \in [0, \frac{1}{3}]$ the following holds.*
Every n-vertex graph G with $\delta(G) \geq (1+\delta)(1+\alpha)\frac{k}{2}$ and $\Delta(G) \geq 2(1+\delta)(1-\alpha)k$ contains each k-edge tree T with $\Delta(T) \leq k^{\frac{1}{67}}$ as a subgraph.

Theorem 1.3 only holds for $\alpha \in [0, \frac{1}{3}]$, but all other allowed values for α from Conjecture 1.1 are covered by a result from [3] that is an analogy of

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[1]This is enough because the latter condition forces a component of size at least $k + 1$, and thus the statement reduces to a well known result of Erdős and Gallai [8].

Theorem 1.3 for the $\frac{2}{3}$-conjecture (which, as we noticed above, implies Conjecture 1.1 for all $\alpha \in [\frac{1}{3}, \frac{1}{2}]$).

The proof of Theorem 1.3 will be given in Section 5. It relies on another result, namely Theorem 1.4 below, which we already prove in Section 4, making use of a powerful embedding tool from [3] (Lemma 4.2). We believe Theorem 1.4 is interesting in its own right.

Theorem 1.4 is a variant of Theorem 1.3, but with the much weaker conditions $\delta(G) \geq (1+\delta)\frac{k}{2}$ and $\Delta(G) \geq (1+\delta)\frac{4}{3}k$. Because of Proposition 1.2, these bounds are not sufficient to guarantee an embedding of any given tree T, but if we are not able to embed T, then some information about the structure of G can be deduced. We will give an explicit description of the corresponding class of graphs, which we will call (ε, x)-*extremal graphs*, in Definition 4.1, but already state our result here.

Theorem 1.4. *For all $\delta \in (0, 1)$ there is $n_0 \in \mathbb{N}$ such that for all $k, n \geq n_0$ with $n \geq k \geq \delta n$, the following holds for every n-vertex graph G with $\delta(G) \geq (1+\delta)\frac{k}{2}$ and $\Delta(G) \geq (1+\delta)\frac{4k}{3}$.*
If T is a k-edge tree with $\Delta(T) \leq k^{\frac{1}{67}}$, then either

(a) T embeds in G; or

(b) G is $(\frac{\delta^4}{10^{10}}, x)$-extremal for every $x \in V(G)$ of degree at least $(1+\delta)\frac{4k}{3}$.

We discuss some further directions and open problems in Section 6. More precisely, we discuss possible extensions of Theorems 1.3 and 1.4, and the influence of additional assumptions on the host graph, such as higher connectivity, on the degree bounds from Theorem 1.3. Also, we discuss the sharpness of Conjecture 1.1 for those values of α not covered by Proposition 1.2.

2 Sharpness of Conjecture 1.1

This section is devoted to showing the asymptotical tightness of our conjecture, for infinitely many values of α. In order to be able to prove Proposition 1.2, let us consider the following example.

Example 2.1. *Let $\ell, k, c \in \mathbb{N}$ with $1 \leq c \leq \frac{k}{\ell(\ell+1)}$ such that $\ell \geq 3$ is odd and divides k.*
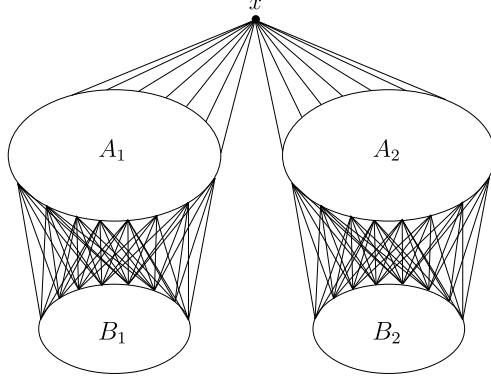
Figure 1: The graph $H_{k,\ell,c}$ from Example 2.1

For $i = 1, 2$, we define $H_i = (A_i, B_i)$ to be the complete bipartite graph with

$$|A_i| = (\ell - 1)\left(\frac{k}{\ell} - 1\right) \quad and \quad |B_i| = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2} - 1.$$

We obtain $H_{k,\ell,c}$ by adding a new vertex x to $H_1 \cup H_2$, and adding all edges between x and $A_1 \cup A_2$. Observe that

$$\delta(H_{k,\ell,c}) = \min\{|A_1|, |B_1| + 1\} = |B_1| + 1 = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2}$$

and

$$\Delta(H_{k,\ell,c}) = |A_1 \cup A_2| = 2(\ell - 1)\left(\frac{k}{\ell} - 1\right).$$

Let $T_{k,\ell}$ be the tree formed by ℓ stars of order $\frac{k}{\ell}$ and an additional vertex v connected to the centres of the stars.

We will use Example 2.1 to prove Proposition 1.2. However, a similar proposition (with slightly weaker bounds) could be obtained by replacing one of the graphs H_i from Example 2.1 with a small complete graph. See Example 2.4 near the end of this section.

Let us now show that the graph $H_{k,\ell,c}$ from Example 2.1 does not contain the tree $T_{k,\ell}$.

Lemma 2.2. *For all $\ell, k, c \in \mathbb{N}$ with $1 \leq c \leq \frac{k}{\ell(\ell+1)}$ such that $\ell \geq 3$ is odd and divides k, the tree $T_{k,\ell}$ from Example 2.1 does not embed in the graph $H_{k,\ell,c}$.*

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Proof. Observe that we cannot embed $T_{k,\ell}$ in $H_{k,\ell,c}$ by mapping v into x, since then, one of the sets B_i would have to accommodate all leaves of at least $\frac{\ell+1}{2}$ of the stars of order $\frac{k}{\ell}$. But these are at least

$$\frac{\ell+1}{2} \cdot \left(\frac{k}{\ell} - 1\right) = \frac{k}{2} + \frac{1}{2\ell}(k - \ell(\ell+1)) \geq \frac{k}{2} + \frac{1}{2}(c-1)(\ell+1) > |B_i|$$

leaves in total, so they will not fit into B_i.

Moreover, we cannot map v into one of the H_i, because then, we would have to embed at least $\ell - 1$ stars into H_i. The leaves of these stars would have to go to the same side as v, but together these are

$$(\ell - 1)\left(\frac{k}{\ell} - 1\right) + 1 > |A_i| \geq |B_i|$$

vertices (note that we count v), so this, too, is impossible. We conclude that the tree $T_{k,\ell}$ does not embed in $H_{k,\ell,c}$. $\qquad\square$

Before we prove Proposition 1.2, let us state a weaker result, Proposition 2.3, which we will prove as a warm-up.

Proposition 2.3. *For all $\alpha \in (0, \frac{1}{2})$ there are $k \in \mathbb{N}$, a k-edge tree T, and a graph G with $\delta(G) = \frac{k}{2}$ and $\Delta(G) \geq 2(1 - \alpha)k$ such that T does not embed in G.*

This result is already sufficient to disprove the conjecture from [19] mentioned earlier.[2]

Proof of Proposition 2.3. Given $\alpha \in (0, 1)$, we set $\ell := 2\lceil \frac{1}{\alpha} \rceil - 1$. Then $\ell \geq 3$ is odd, and we can consider the tree $T_{k,\ell}$ and the graph $H_{k,\ell,c}$ from Example 2.1, where we take $k := \ell(\ell + 1)$ and $c := 1$. By Lemma 2.2, we know that $T_{k,\ell}$ does not embed in $H_{k,\ell,c}$.

Observe that $\delta(H_{k,\ell,c}) = \frac{k}{2}$ and, by our choice of k,

$$\Delta(H_{k,\ell,c}) = 2(\ell - 1)\left(\frac{1}{\ell} - \frac{1}{k}\right)k = 2\left(1 - \frac{2}{\ell+1}\right)k,$$

and therefore, $\Delta(H_{k,\ell,c}) \geq 2(1 - \alpha)k$, which is as desired. $\qquad\square$

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[2]In [19] it was conjectured that a maximum degree of at least $\frac{4}{3}k$ and a minimum degree of at least $\frac{k}{2}$ would be enough to guarantee containment of all trees with k edges. Let us note that Propositions 1.2 and 2.3, Conjecture 1.1 and Theorem 1.4 were found independently of [19] (see [2]).

Let us now prove Proposition 1.2. For this, we will let the constant c go to infinity.

Proof of Proposition 1.2. Let ℓ and γ be given. For any fixed integer $c \geq 1$, set

$$k := c\ell(\ell + 1),$$

and consider the tree $T_{k,\ell}$ and the host graph $H_{k,\ell,c}$ from Example 2.1 for parameters k, ℓ and c.

Observe that

$$\delta(H_{k,\ell,c}) > \left(1 + \frac{(c-1)(\ell+1)}{k}\right)\frac{k}{2} = \left(1 + \frac{c-1}{c\ell}\right)\frac{k}{2} = \left(1 + \frac{1}{\ell} - \frac{1}{c\ell}\right)\frac{k}{2}$$

and

$$\Delta(H_{k,\ell,c}) = 2(1 - \frac{1}{\ell} - \frac{\ell-1}{k})k > 2\left(1 - \frac{1}{\ell} - \frac{1}{c\ell}\right)k$$

So for any given γ we can choose c large enough such that

$$\delta(H_{k,\ell,c}) \geq \left(1 + \frac{1}{\ell} - \gamma\right)\frac{k}{2}$$

and

$$\Delta(H_{k,\ell,c}) \geq 2\left(1 - \frac{1}{\ell} - \gamma\right)k,$$

which is as desired, since by Lemma 2.2, we know that $T_{k,\ell}$ does not embed in $H_{k,\ell,c}$, $\qquad\square$

Let us now quickly discuss an alternative example, which gives worse bounds than the ones given in Proposition 1.2, but might be interesting because of its different structure.

Example 2.4. *Let k, ℓ, c be as in Example 2.1. Let C be a complete graph of order $\frac{k}{2} + \frac{(c-1)(\ell+1)}{2}$. Let $G_{k,\ell,c}$ be obtained by taking C and the bipartite graph $H_1 = (A_1, B_1)$ from Example 2.1, and joining a new vertex x to all vertices from A_1 and to all vertices in C.*
Then $\delta(G_{k,\ell,c}) = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2}$ and $\Delta(G_{k,\ell,c}) = \frac{3\ell-2}{2\ell}k + \frac{(c-3)(\ell+1)}{2} - 2$, and an analogue of Lemma 2.2 holds.
Moreover, in the same way as in the proof of Proposition 1.2, we can show that if k is large enough in terms of (odd) $\ell \geq 3$ and γ, then

$$\delta(G_{k,\ell,c}) \geq (1 + \frac{1}{\ell} - \gamma)\frac{k}{2} \quad and \quad \Delta(G_{k,\ell,c}) \geq \frac{3}{2}(1 - \frac{1}{\ell} - \gamma)k.$$

This example, as well as the examples underlying Propositions 2.3 and 1.2 illustrate that requiring a maximum degree of at least ck, for any $c < 2$ (in particular for $c = \frac{4}{3}$), and a minimum degree of at least $\frac{k}{2}$ is not enough to guarantee that any graph obeying these conditions contains all k-edge tree as subgraphs. Nevertheless, we could not come up with any radically different examples, and it might be that graphs that look very much like the graph $H_{k,\ell,c}$ from Example 2.1 or the graph $G_{k,\ell,c}$ from Example 2.4 are the only obstructions for embedding all k-edge trees. This suspicion is partially confirmed by Theorem 1.4.

3   Regularity

In the proofs of our results, we will make use of the regularity lemma, which we quickly introduce here.

For a bipartite graph $H = (A, B)$, the density of any subpair $(X, Y) \subseteq (A, B)$ is $d(X, Y) := \frac{e(X,Y)}{|X||Y|}$. For fixed $\varepsilon > 0$, the pair (A, B) is said to be ε-regular if $|d(X, Y) - d(A, B)| < \varepsilon$ for all $(X, Y) \subseteq (A, B)$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$. If, moreover, $d(A, B) > \eta$ for some $\eta > 0$, we call the pair (ε, η)-regular. The regularity lemma of Szemerédi [21] states that the vertex set of any large enough graph can be partitioned into a bounded number of sets such that almost all pairs form an ε-regular bipartite graph. We will use the well known degree form of the regularity lemma. Call a vertex partition $V(G) = V_1 \cup \ldots \cup V_\ell$ an (ε, η)-regular partition if

1. $|V_1| = |V_2| = \ldots = |V_\ell|$;

2. V_i is independent for all $i \in [\ell]$; and

3. for all $1 \leq i < j \leq \ell$, the pair (V_i, V_j) is ε-regular with density either $d(V_i, V_j) > \eta$ or $d(V_i, V_j) = 0$.

Lemma 3.1 (Regularity lemma - Degree form). *For all $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ there are N_0, M_0 such that the following holds for all $\eta \in [0, 1]$ and $n \geq N_0$. Any n-vertex graph G has a subgraph G', with $|G| - |G'| \leq \varepsilon n$ and $\deg_{G'}(x) \geq \deg_G(x) - (\eta + \varepsilon)n$ for all $x \in V(G')$, such that G' admits an (ε, η)-regular partition $V(G') = V_1 \cup \ldots \cup V_\ell$, with $m_0 \leq \ell \leq M_0$.*

The (ε, η)-reduced graph R of G, with respect to the (ε, η)-regular partition given by Lemma 3.1, is the graph with vertex set $\{V_i : i \in [\ell]\}$, where $V_i V_j$

is an edge if $d(V_i, V_j) > \eta$. We will often refer to the (ε, η)-reduced graph R without explicitly referring to the associated (ε, η)-regular partition. It turns out that R inherits many properties of G. For instance, it asymptotically preserves the minimum degree of G (scaled to the order of R). Indeed, for every V_i, we have

$$\deg_R(V_i) \geq \sum_{V_j \in N_R(V_i)} d(V_i, V_j) = \sum_{v \in V_i} \frac{\deg_{G'}(v)}{|V_i|} \cdot \frac{|R|}{|G'|}, \tag{1}$$

and so, in particular, one can deduce that

$$\delta(R) \geq \delta(G') \cdot \frac{|R|}{|G'|} \geq \left(\delta(G) - (\varepsilon + \eta)n \right) \cdot \frac{|R|}{n}. \tag{2}$$

4   Maximum degree $\frac{4k}{3}$

In this section we will prove our tree embedding result for host graphs of maximum degree approximately $\frac{4k}{3}$ and minimum degree approximately $\frac{k}{2}$, namely Theorem 1.4. The proof of Theorem 1.4 crucially relies on an embedding result from [3], Lemma 4.2 below. This lemma describes a series of configurations which, if they appear in a graph G, allow us to embed any bounded degree tree of the right size into G.

Before stating Lemma 4.2, and defining the class of (ε, x)-extremal graphs (the graphs that are excluded as host graphs in Theorem 1.4), let us go through some useful notation.

For a fixed $\theta \in (0, 1)$, we say that a vertex x of a graph H θ-sees a set $U \subseteq V(H)$ if it has at least $\theta|U|$ neighbours in U. If C is a component of a reduced graph of $H - x$, then we say that x θ-sees C if it has at least $\theta|\bigcup V(C)|$ neighbours in $\bigcup V(C)$.

A nonbipartite graph H is said to be (k, θ)-small if $|V(H)| < (1 + \theta)k$. A bipartite graph $H = (A, B)$ is said to be (k, θ)-small if

$$\max\{|A|, |B|\} < (1 + \theta)k.$$

If a graph is not (k, θ)-small, we will say that it is (k, θ)-large.

We are now ready to define the excluded hosts from Theorem 1.4.

Definition 4.1 $((\varepsilon, x)$-extremal$)$. *Let $\varepsilon > 0$ and let $k \in \mathbb{N}$. Given a graph G and a vertex $x \in V(G)$, we say that G is (ε, x)-extremal if for every $(\varepsilon, 5\sqrt{\varepsilon})$-reduced graph R of $G - x$ the following conditions hold:*

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(i) *every component of R is $(k \cdot \frac{|R|}{|G|}, \sqrt[4]{\varepsilon})$-small;*

(ii) *x $\sqrt{\varepsilon}$-sees two components C_1 and C_2 of R and x does not see any other component of R;*

and furthermore, assuming that $\deg(x, \bigcup V(C_1)) \geq \deg(x, \bigcup V(C_2))$,

(iii) *C_1 is bipartite and $(\frac{2k}{3} \cdot \frac{|R|}{|G|}, \sqrt[4]{\varepsilon})$-large, with x only seeing the larger side of C_1;*

(iv) *if C_2 is nonbipartite, then C_2 is $(\frac{2k}{3} \cdot \frac{|R|}{|G|}, \sqrt[4]{\varepsilon})$-small, and if C_2 is bipartite, then x sees only one side of the bipartition.*

We will now state Lemma 4.2.

Lemma 4.2. [3, **Lemma 7.3**] *For every $\varepsilon \in (0, 10^{-10})$ and $M_0 \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for all $n, k \geq n_0$ the following holds for every n-vertex graph G of minimum degree at least $(1 + \sqrt[4]{\varepsilon})\frac{k}{2}$.*
Let $x \in V(G)$, and suppose $G - x$ has an $(\varepsilon, 5\sqrt{\varepsilon})$-reduced graph R, with $|R| \leq M_0$, such that at least one of the following conditions holds:

(I) *R has a $(k \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-large nonbipartite component; or*

(II) *R has a $(\frac{2k}{3} \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-large bipartite component such that x $\sqrt{\varepsilon}$-sees both sides of the bipartition; or*

(III) *x $\sqrt{\varepsilon}$-sees two components C_1, C_2 of R and one of the following holds:*

(a) *x sends at least one edge to a third component C_3 of R; or*

(b) *C_i is nonbipartite and $(\frac{2k}{3} \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-large for some $i \in \{1, 2\}$; or*

(c) *$C_i = (A, B)$ is bipartite for some $i \in \{1, 2\}$, and x sees both A and B; or*

(d) *$C_i = (A, B)$ is bipartite for some $i \in \{1, 2\}$, with $\min\{|A|, |B|\} \geq (1 + \sqrt[4]{\varepsilon})\frac{2k}{3} \cdot \frac{|R|}{n}$ and x seeing only one side of the bipartition.*

Then every k-edge tree T of maximum degree at most $k^{\frac{1}{67}}$ embeds in G.

Let us remark that the original result in [3] actually covers even more cases than we chose to reproduce here. It also allows for relaxing the bound of the maximum degree of the tree, if the host graph has minimum degree substantially larger than $(1 + \delta)\frac{k}{2}$.

We are now ready for the proof of Theorem 1.4.

Proof of Theorem 1.4. Given $\delta \in (0,1)$, we set

$$\varepsilon := \frac{\delta^4}{10^{10}}. \tag{3}$$

Let N_0, M_0 be given by Lemma 3.1, with input ε, $\eta := 5\sqrt{\varepsilon}$ and $m_0 := \frac{1}{\varepsilon}$, and let n'_0 be given by Lemma 4.2, with input ε and M_0. We choose

$$n_0 := (1 - \varepsilon)^{-1} \max\{n'_0, N_0\} + 1$$

as the numerical output of Theorem 1.4.

Let G and T be given as in Theorem 1.4. Consider an arbitrary vertex $x \in V(G)$ with $\deg(x) \geq (1+\delta)\frac{4}{3}k$, and apply Lemma 3.1 to $G - x$. We obtain a subgraph $G' \subseteq G - x$ which admits an $(\varepsilon, 5\sqrt{\varepsilon})$-regular partition of $G - x$, with corresponding $(\varepsilon, 5\sqrt{\varepsilon})$-reduced graph R. Note that

$$\delta(G') \geq \left(1 + \frac{\delta}{2}\right)\frac{k}{2} \geq (1 + 100\sqrt[4]{\varepsilon})\frac{k}{2}.$$

If R has a $(k \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-large component, we are in scenario (I) from Lemma 4.2, and we can embed T. So let us assume this is not the case. In particular, we can assume that condition (i) of Definition 4.1 holds.

Since G' misses less than $\varepsilon n + 1$ vertices from G, we have that

$$\deg_G(x, G') \geq \left(1 + \frac{\delta}{2}\right)\frac{4}{3}k \geq (1 + 100\sqrt[4]{\varepsilon})\frac{4}{3}k. \tag{4}$$

It is clear that x has to $\sqrt{\varepsilon}$-see at least one component C_1 of R. Indeed, otherwise, we would have that

$$\frac{4}{3}\delta n \leq \frac{4}{3}k \leq \deg_G(x, G') = \sum_C \deg_G(x, \bigcup V(C)) \leq \sqrt{\varepsilon}n, \tag{5}$$

where the sum is over all components C of R, and this contradicts (3). Now, if x only sees C_1, then, since C_1 is $(k \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-small and $\deg(x, \bigcup V(C)) \geq (1 + \frac{\delta}{2})\frac{4k}{3}$, we are in scenario (II) from Lemma 4.2, and we can embed T. In particular, this implies that

$$\deg_G(x, G' \setminus \bigcup V(C_1)) \geq (1 + 50\sqrt[4]{\varepsilon})\frac{k}{3}. \tag{6}$$

Thus, a computation similar to (5) shows that x $\sqrt{\varepsilon}$-sees at least two components of R. If x sees a third component, then we are in scenario (IIIa) from Lemma 4.2 and T can be embedded.

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Therefore, we know that x actually $\sqrt{\varepsilon}$-sees exactly two components, which we will call C_1 and C_2. (In particular, we know that condition (ii) of Definition 4.1 holds.)

By symmetry, we may assume that $\deg(x, \bigcup V(C_1)) \geq \deg(x, \bigcup V(C_2))$ and thus, by (4),

$$\deg(x, \bigcup V(C_1)) \geq (1 + 100\sqrt[4]{\varepsilon})\tfrac{2k}{3}. \tag{7}$$

Thus, if C_1 is nonbipartite we are in scenario (IIIb) from Lemma 4.2, and therefore, we can assume $C_1 = (A_1, B_1)$ is bipartite. Also, x only sees one side of the bipartition, say A_1, since otherwise we are in scenario (IIIc). Moreover, by (7), and since we may assume we are not in scenario (IIId), we know that

$$|A_1| \geq (1 + 100\sqrt[4]{\varepsilon})\tfrac{2k}{3} \cdot \tfrac{|R|}{|G|} \quad \text{and} \quad |B_1| \leq (1 + \sqrt[4]{\varepsilon})\tfrac{2k}{3} \cdot \tfrac{|R|}{|G|}. \tag{8}$$

So, condition (iii) of Definition 4.1 holds. Furthermore, if C_2 is nonbipartite, then it is $(\tfrac{2k}{3} \cdot \tfrac{|R|}{|G|}, \sqrt[4]{\varepsilon})$-small, as otherwise we are in case (IIIb). If C_2 is bipartite, then x can only see one side of the bipartition, since otherwise we are in scenario (IIIc). Therefore, C_2 satisfies condition (iv) of Definition 4.1, implying that G is $(\tfrac{\delta^4}{10^{10}}, x)$-extremal. $\qquad\square$

5   The proof of Theorem 1.3

This section contains the proof of our main result, Theorem 1.3. We will need some preliminary results.

Our first lemma is folklore. It states that every tree T has a cutvertex which separates the tree into subtrees of size at most $\tfrac{|T|}{2}$. The proof is straightforward, and can be found for instance in [9].

Lemma 5.1. *Every tree T with t edges has a vertex z such that every component of $T - z$ has at most $\lceil \tfrac{t}{2} \rceil$ vertices.*

A vertex z as in Lemma 5.1 is called a $\tfrac{t}{2}$-separator for T and x. We also need the following lemma from [3], which will allow us to conveniently group the components of $T - z$ obtained from Lemma 5.1 when applied to a tree T .

Lemma 5.2.[**3, Lemma 4.4**] *Let $m, t \in \mathbb{N}_+$ and let $(a_i)_{i=1}^m$ be a sequence of integers with $0 < a_i \leq \lceil \tfrac{t}{2} \rceil$, for each $i \in [m]$, such that $\sum_{i=1}^m a_i \leq t$. Then*

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(i) *there is a partition $\{J_1, J_2\}$ of $[m]$ such that $\sum_{i \in J_2} a_i \leq \sum_{i \in J_1} a_i \leq \frac{2}{3}t$; and*

(ii) *there is a partition $\{I_1, I_2, I_3\}$ of $[m]$ such that $\sum_{i \in I_3} a_i \leq \sum_{i \in I_2} a_i \leq \sum_{i \in I_1} a_i \leq \lceil \frac{t}{2} \rceil$.*

Finally, we need another embedding result from [3]. This result will enable us to embed any bounded degree forest into any large enough bipartite graph with an underlying (ε, η)-regular partition of a certain structure.
Let us first define the kind of forest we are interested in.

Definition 5.3. *Let $t_1, t_2 \in \mathbb{N}$ and let $c \in (0,1)$. We say that a forest F with colour classes C_1 and C_2 is a (t_1, t_2, c)-forest if*

1. *$|C_i| \leq t_i$ for $i = 1, 2$; and*

2. *$\Delta(F) \leq (t_1 + t_2)^c$.*

We are now ready for the embedding result.

Lemma 5.4. [**3, Corollary 5.4**] *For all $\varepsilon \in (0, 10^{-8})$ and $d, M_0 \in \mathbb{N}$ there is t_0 such that for all $n, t_1, t_2 \geq t_0$ the following holds. Let G be a n-vertex graph having an $(\varepsilon, 5\sqrt{\varepsilon})$-reduced graph R with $|R| \leq M_0$ such that*

(i) *$R = (X, Y)$ is connected and bipartite;*

(ii) *$\mathrm{diam}(R) \leq d$;*

(iii) *$\deg(x) \geq (1 + 100\sqrt{\varepsilon})t_2 \cdot \frac{|R|}{n}$, for all $x \in X$; and*

(iv) *$|X| \geq (1 + 100\sqrt{\varepsilon})t_1 \cdot \frac{|R|}{n}$.*

Then any $(t_1, t_2, \frac{1}{d})$-forest F, with colour classes C_0 and C_1, can be embedded into G, with C_0 going to $\bigcup X$ and C_1 going to $\bigcup Y$.
Moreover, if F has at most $\frac{\varepsilon n}{|R|}$ roots, then the roots going to $\bigcup X$ can be mapped to any prescribed set of size at least $2\varepsilon |\bigcup X|$ in X, and the roots going to $\bigcup Y$ can be mapped to any prescribed set of size at least $2\varepsilon |\bigcup Y|$ in Y.

We are now ready for the proof of our main theorem, Theorem 1.3.

Proof of Theorem 1.3. Given $\delta \in (0, 1)$, we set

$$\varepsilon := \frac{\delta^4}{10^{10}}$$

and apply Lemma 3.1, with inputs ε, $\eta = 5\sqrt{\varepsilon}$ and $m_0 := \frac{1}{\varepsilon}$, to obtain numbers n_0 and M_0. Next, apply Lemma 5.4, with input ε and further inputs $d := 3$ and M_0 to obtain a number k_0'. Choose k_0 as the larger of n_0, k_0' and the output of Theorem 1.4.

Now, let $k, n \in \mathbb{N}$, let $\alpha \in [0, \frac{1}{3})$, let T be a tree and let G be a graph as in Theorem 1.3. Let x be an arbitrary vertex of maximum degree in G. Note that

$$\deg(x) \geq 2(1 + \delta)(1 - \alpha)k \geq (1 + \delta)\frac{4k}{3}.$$

We apply the regularity lemma (Lemma 3.1) to $G - x$ to obtain a subgraph $G' \subseteq G - x$ which admits an $(\varepsilon, 5\sqrt{\varepsilon})$-regular partition with a corresponding reduced graph R. Moreover, since G' misses only few vertices from G, we know that

$$\deg(x, G') \geq 2(1 + \tfrac{\delta}{2})(1 - \alpha)k \tag{9}$$

and

$$\delta(G') \geq (1 + \tfrac{\delta}{2})(1 + \alpha)\tfrac{k}{2}. \tag{10}$$

Thus,

$$\delta(R) \geq \left(1 + \frac{\delta}{2}\right)(1 + \alpha)\frac{k}{2} \cdot \frac{|R|}{n}. \tag{11}$$

Apply Theorem 1.4 to T and G. This either yields an embedding of T, which would be as desired, or tells us that G is an (ε, x)-extremal graph. We assume the latter from now on.

So, we know that x $\sqrt{\varepsilon}$-sees two components C_1 and C_2 of R, where $C_1 = (A, B)$ is bipartite, say with $|A| \geq |B|$. Moreover, x does not see any other component of R.

Furthermore,

(A) C_i is $(k \cdot \frac{|R|}{n}, \sqrt[4]{\varepsilon})$-small, for $i = 1, 2$;

(B) C_1 is $(\frac{2k}{3} \cdot \frac{|R|}{|G|}, \sqrt[4]{\varepsilon})$-large, and x does not see B.

By (9), and since we assume x sends more edges to $\bigcup V(C_1)$ than to $\bigcup V(C_2)$, we know that

$$\deg(x, \bigcup V(C_1)) \geq \left(1 + \frac{\delta}{2}\right)(1 - \alpha)k, \tag{12}$$

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and thus, by (B),

$$|C_1| \geq |A| \geq \left(1 + \frac{\delta}{2}\right)(1 - \alpha)k \cdot \frac{|R|}{n}, \tag{13}$$

since the vertex x has at least that many neighbours in A, because of inequality (12).

Also, note that because of (A) and because of the bound (11), we know that any pair of vertices from the same bipartition class of C_1 has a common neighbour. Therefore,

$$\text{the diameter of } C_1 \text{ is bounded by 3.} \tag{14}$$

Let us now turn to the tree T. We apply Lemma 5.1 to find a $\frac{t}{2}$-separator z of T, for an arbitrary leaf x. Let $\mathcal{F}$ denote the set of all components of $T - z$. Then

$$\text{each component of } \mathcal{F} \text{ has size at most } \left\lceil \frac{t}{2} \right\rceil. \tag{15}$$

Let V_0 denote the set of all vertices of $T - z$ that lie at even distance to z. We claim that if we cannot embed T, then

$$|V_0| \geq (1 + \alpha)\frac{k}{2}. \tag{16}$$

Indeed, suppose otherwise. Then we can apply Lemma 5.2 (i) to obtain a partition of $\mathcal{F}$ into two sets $\mathcal{J}_1$ and $\mathcal{J}_2$ such that

$$\left| \bigcup \mathcal{J}_2 \right| \leq \frac{k}{2} \quad \text{and} \quad \left| \bigcup \mathcal{J}_1 \right| \leq \frac{2}{3}k.$$

We embed z into x. Our plan is to use Lemma 5.4 with reduced host graph C_1, and with

$$t := \left| \bigcup \mathcal{J}_1 \right| \leq \frac{2}{3}k$$

where we accordingly choose t_1 and t_2 as the sizes of the two partition classes of $\bigcup \mathcal{J}_1$. Note that then

$$k_2 \leq |V_0| \leq (1 + \alpha)\frac{k}{2}$$

(since we assumed (16) not to hold). We can therefore embed $\bigcup \mathcal{J}_1$ into C_1, with the roots of J_2 embedded in the neighbourhood of x. Observe that

conditions (iii) and (iv) of Lemma 5.4 are met because of (11) and (13), respectively, and the neighbourhood of x is large enough to accommodate the roots of the trees from $\mathcal{J}_1$ because of (12). In order to see condition (ii) of Lemma 5.4, it suffices to recall (14).

Also, because of (10), and since x also $\sqrt{\varepsilon}$-sees the component C_2, we can embed the trees from $\mathcal{J}_2$ into C_2. We do this by first mapping the roots of the trees from $\mathcal{J}_2$ into the neighbourhood of x in C_2. We then use the minimum degree of G' to greedily complete the embedding. Thus we have embedded all of T.

So, from now we will assume that (16) holds.

We split the remainder of the proof into two complementary cases, which will be solved in different ways. Our two cases are defined according to whether or not there is a tree $F^* \in \mathcal{F}$ such that $|V(F^*) \cap V_0| > \alpha k$. Let us first treat the case where such an F^* does not exist.

Case 1: $|V(F) \cap V_0| \leq \alpha k$ for each $F \in \mathcal{F}$.

In this case, we proceed as follows. First, we embed z into x. We take an inclusion-maximal subset $\mathcal{F}_1$ of $\mathcal{F}$ such that

$$|\bigcup \mathcal{F}_1 \cap V_0| \leq (1 + \alpha)\frac{k}{2} \tag{17}$$

holds. Then, because of our assumption on $|V(F) \cap V_0|$ for the trees $F \in \mathcal{F}$, we know that

$$|\bigcup \mathcal{F}_1 \cap V_0| \geq (1 - \alpha)\frac{k}{2}. \tag{18}$$

Hence, the trees from $\mathcal{F}_1$ can be embedded into C_1, by using Lemma 5.4, as before, with $t := |\bigcup \mathcal{F}_1|$ and t_1, t_2 chosen appropriately. Indeed, inequalities (17) and (11) ensure that condition (iii) of the lemma holds. Furthermore, because of (13) and (18), we know that

$$|\bigcup \mathcal{F}_1 \setminus V_0| \leq (1 + \alpha)\frac{k}{2} \leq \frac{1}{1 + \frac{\delta}{2}}|\bigcup V(A)|,$$

and hence, it is clear that also condition (iv) of Lemma 5.4 holds.

Condition (ii) of Lemma 5.4 holds because of (14). Finally, inequality (12) ensures we can embed $\mathcal{F}_1$ in C_1 in such a way the roots of $\mathcal{F}_1$ are embedded into the neighbourhood of x in C_1.

Now, the trees from $\mathcal{F}_2 := \mathcal{F} \setminus \mathcal{F}_1$ can be embedded into C_2. First, embed the neighbours of z into the neighbourhood of x in C_2. Then, observe that (18) implies that

$$|\bigcup \mathcal{F}_2| \leq (1 + \alpha)\frac{k}{2} \leq \delta(G').$$

Therefore, we can embed the remainder of the trees from $\mathcal{F}_2$ into C_2 in a greedy fashion.

Case 2: There is a tree $F^* \in \mathcal{F}$ such that $|V(F^*) \cap V_0| > \alpha k$.

In this case, let us set $\mathcal{F}' := \mathcal{F} \setminus \{F^*\}$, and note that

$$|V(\bigcup \mathcal{F}') \cap V_0| \leq (1 - \alpha)k. \tag{19}$$

Our plan is to embed z into a neighbour of x in A, and embed all trees from $\mathcal{F}'$ into C_1. We then complete the embedding by mapping the root of F^* to x, and the rest of F^* to C_2.

For the embedding of $\{z\} \cup \bigcup \mathcal{F}'$, we will use Lemma 5.4 as before, but this time the roles of A and B will be reversed. That is, all of

$$F_0 := (\{z\} \cup \bigcup \mathcal{F}') \cap V_0$$

is destined to go to A, while all of

$$F_1 := (\{z\} \cup \bigcup \mathcal{F}') \setminus V_0$$

is destined to go to B.

We choose $t := |\bigcup \mathcal{F}'| + 1$ and choose t_1, t_2 as the sizes of the bipartition classes of $\{z\} \cup \bigcup \mathcal{F}'$, that is, we set $t_1 := |F_0|$ and $t_2 := |F_1|$. Because of (16), there are at most $(1 - \alpha)\frac{k}{2}$ vertices in $T - z$ at odd distance from z. In particular, $t_2 \leq (1 - \alpha)\frac{k}{2}$. So, by (11), we know that condition (iii) of Lemma 5.4 holds (and condition (i) is obviously true).

Now, condition (iv) of Lemma 5.4 is ensured by inequality (19) together with (13). Condition (ii) of Lemma 5.4 holds because of (14). Therefore, we can embed all of $\{z\} \cup \bigcup \mathcal{F}'$ with the help of Lemma 5.4. Furthermore, we can make sure that z is embedded into a neighbour of x.

It remains to embed the tree F^*. We embed its root $r(F^*)$ into x, and embed all neighbours of $r(F^*)$ into arbitrary neighbours of x in C_2. We then embed

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the rest of F^* greedily inyo C_2. Note that this is possible, since by (15), we know that

$$|F^* - r(F^*)| \leq \left\lceil \frac{k}{2} \right\rceil - 1,$$

and so, our bound (10) guarantees that we can embed the remainder of F^* greedily into C_2. $\qquad\square$

6 Conclusion

6.1 Extensions of Theorem 1.4

In Theorem 1.4, we saw that asymptotically, requiring maximum degree at least $\frac{4}{3}k$ and minimum degree at least $\frac{k}{2}$ is enough to guarantee the appearance of every tree of maximum degree bounded by $k^{\frac{1}{67}}$ as a subgraph, as long as the host graph is large and dense enough to apply the regularity method, and as long as the host graph does not resemble too closely the graph from Example 2.1. It seems natural to ask whether this can be generalised in any of the following directions.
First, we might ask whether the same holds for a larger class of trees (or possibly, all trees).

Problem 6.1. *Does Theorem 1.4 continue to hold if we relax the condition on the maximum degree of T?*

Also, it might be possible to describe the forbidden structure in more explicit terms. Perhaps the graphs $H_{k,\ell,c}$ and $G_{k,\ell,c}$ from Examples 2.1 and 2.4, respectively, do not only appear in the reduced graph, but also in the host graph itself, if we fail to embed some tree T.

Problem 6.2. *Can we describe the forbidden structure from Theorem 1.4 more explicitly, for instance by excluding the graphs $H_{k,\ell,c}$ and $G_{k,\ell,c}$ as subgraphs of G, for $\ell \geq 3$ (or for odd $\ell \geq 3$)?*

Or, instead of forbidding these graphs as subgraphs, it might be enough to forbid them as components of the host graph G.
It is also not clear what an analogue of Theorem 1.4 for sparse graphs might look like.

Problem 6.3. *Find a version of Theorem 1.4 for sparse graphs.*

6.2 Extensions of Theorem 1.3

6.2.1 Lower bounds on the minimum degree

Let us now discuss why variants of Conjecture 1.1 (or of Theorem 1.3) with bounds on the minimum degree that are below the threshold $\frac{k}{2}$ are not possible. In fact, if we do not add further restrictions, and the minimum degree of the host graph G is only bounded by some function $f(k) < \lfloor \frac{k}{2} \rfloor$, then no maximum degree bound can make G contain all k-edge trees. In order to see this, it suffices consider K_{n_1,n_2}, the complete bipartite graph with classes of size $n_1 := \lfloor \frac{k-1}{2} \rfloor$ and $n_2 := n - \lfloor \frac{k-1}{2} \rfloor$, respectively. No perfectly (or almost perfectly) balanced k-edge tree embeds into K_{n_1,n_2}, since one would need to use at least $\lfloor \frac{k+1}{2} \rfloor$ vertices from each class.

One might think that perhaps, the situation changes if we require the minimum degree bound $f(k)$ to be at least as large as the smaller bipartition class of the tree. But that is not true: Let $\ell \in \mathbb{N}$ such that $\ell + 2$ divides $k + 1$, and let T be obtained from a $2\frac{k+1}{\ell+2}$-edge path by adding $\ell - 2$ new leaf neighbours to every other vertex on the path. This tree has bipartition classes of sizes $\frac{1}{\ell}(k+1)$ and $\frac{\ell-1}{\ell}(k+1)$. However, it cannot be embedded into the graph obtained by joining a universal vertex to a disjoint union of (any number of) complete graphs of order $c := \lfloor \frac{k-1}{2} \rfloor$, since for every $v \in V(T)$, at least one component of $T - v$ has at least $\lceil \frac{k}{2} \rceil > c$ vertices.

6.2.2 Higher connectivity

The examples from the previous section, as well as Examples 2.1 and 2.4 from Section 2, all have a cutvertex. So one might think that in a c-connected host graph, for some $c \geq 2$ which might even depend on k, we could cope with lower bounds on the minimum (or maximum) degree.

However, we would not gain much by requiring higher connectivity, as the following variation of Example 2.1 illustrates.

Example 6.4. *Let $H_{k,\ell,c}$ be as in Example 2.1, with slightly adjusted size of the sets A_i, namely, we choose*

$$|A_i| = (\ell - 1)\left(\frac{k}{\ell} - 2\right) \text{ and } |B_i| = \frac{k}{2} + \frac{(c-1)(\ell+1)}{2} - 1.$$

Now, add a matching of size $|B_1|$ between the sets B_1 and B_2, and call the new graph $H'_{k,\ell,c}$.

The graph $H'_{k,\ell,c}$ is $\frac{k}{2}$-connected, and for any given γ there is a number c such that $\delta(H'_{k,\ell,c}) \geq (1 + \frac{1}{\ell} - \gamma)\frac{k}{2}$ and $\Delta(H'_{k,\ell,c}) \geq 2(1 - \frac{1}{\ell} - \gamma)k$. But, similarly as in Lemma 2.2, we can show that the tree $T_{k,\ell}$ from Example 2.1 does not embed in $H'_{k,\ell,c}$.

It is not clear what happens if we require a connectivity of $(1 + \varepsilon)\frac{k}{2}$, for some $0 < \varepsilon \leq \alpha$. It is possible that then, the bound on the maximum degree can be weakened, perhaps to $\Delta(G) \geq 2(1 - 2\alpha)k$.

6.3 Is Conjecture 1.1 tight for all values of α?

Finally, we believe it would be very interesting to generalise Proposition 1.2 to even ℓ, if this is possible. Or even better, find examples so that the term $\frac{1}{\ell}$ from the proposition can be replaced with any $\alpha \in [0, \frac{1}{2}]$.

Question 6.5. *Is Conjecture 1.1 asymptotically tight for all $\alpha \notin \{\frac{1}{\ell}\}_{\ell \geq 3, \ell \text{ odd}}$?*

We believe that Conjecture 1.1 might be tight (or close to tight) in the range $\alpha \in [0, \frac{1}{3}]$. Indeed, for any $\alpha \in [0, \frac{1}{3}]$ and given $\gamma > 0$ small, we can construct examples of graphs with minimum degree at least $(1 + \alpha - \gamma)\frac{k}{2}$ and maximum degree at least $2(1 - g(\alpha) - \gamma)k$, where $g(\alpha)$ is a function which is bigger than α but reasonably close to it. In particular, $g(\alpha)$ satisfies $|\alpha - g(\alpha)| = O(\alpha^2)$, and, more explicitly, for any even $\ell \geq 3$ we obtain $g(\frac{1}{\ell}) = \frac{1}{\ell} + \frac{1}{\ell(\ell-2)}$. These examples are very similar to Example 2.1. The difference is that the small stars that make up the tree may have different sizes (more precisely, one star is smaller than the other ones). The host graph is the same, with slightly adjusted size of the sets A_i.

However, for $\alpha \in [0, \frac{1}{2}]$ the situation might be very different. We are inclined to believe that Conjecture 1.1 is not sharp in the range $[\frac{1}{3}, \frac{1}{2}]$, and that in this range, the $\frac{2}{3}$–conjecture gives the 'correct' bounds.

References

[1] AJTAI, M., KOMLÓS, J., AND SZEMERÉDI, E. On a conjecture of Loebl. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ. Wiley, New York, 1995, pp. 1135–1146.

[2] BESOMI, G. Tree embeddings in dense graphs. Master's thesis, University of Chile, 2018.

[3] Besomi, G., Pavez-Signé, M., and Stein, M. Degree conditions for embedding trees. Preprint 2018, arXiv:1805.07338.

[4] Brandt, S., and Dobson, E. The Erdős-Sós conjecture for graphs of girth 5. *Discrete Mathematics 150*, 1 (1996), 411 – 414.

[5] Cooley, O. Proof of the Loebl–Komlós–Sós Conjecture for Large, Dense Graphs. *Discrete Math. 309*, 21 (2009), 6190–6228.

[6] Erdős, P. Extremal problems in graph theory. In *Theory of graphs and its applications, Proc. Sympos. Smolenice* (1964), pp. 29–36.

[7] Erdős, P., Füredi, Z., Loebl, M., and Sós, V. Discrepancy of trees. *Studia Sci. Math. Hungar. 30* (1995), 47 – 57.

[8] Erdős, P., and Gallai, T. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar 10* (1959), 337–356.

[9] Havet, F., Reed, B., Stein, M., and Wood, D. R. A Variant of the Erdős-Sós Conjecture. Preprint 2016, arXiv:1606.09343.

[10] Hladký, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M., and Szemerédi, E. The Approximate Loebl–Komlós–Sós Conjecture I: The sparse decomposition. *SIAM Journal on Discrete Mathematics 31*, 2 (2017), 945–982.

[11] Hladký, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M., and Szemerédi, E. The Approximate Loebl–Komlós–Sós Conjecture II: The Rough Structure of LKS Graphs. *SIAM Journal on Discrete Mathematics 31*, 2 (2017), 983–1016.

[12] Hladký, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M., and Szemerédi, E. The Approximate Loebl–Komlós–Sós Conjecture III: The Finer Structure of LKS Graphs. *SIAM Journal on Discrete Mathematics 31*, 2 (2017), 1017–1071.

[13] Hladký, J., Komlós, J., Piguet, D., Simonovits, M., Stein, M., and Szemerédi, E. The Approximate Loebl–Komlós–Sós Conjecture IV: Embedding Techniques and the Proof of the Main Result. *SIAM Journal on Discrete Mathematics 31*, 2 (2017), 1072–1148.

[14] HLADKÝ, J., AND PIGUET, D. Loebl–Komlós–Sós Conjecture: dense case. *J. Comb. Theory Ser. B 116*, C (2016), 123–190.

[15] HLADKÝ, J., PIGUET, D., SIMONOVITS, M., STEIN, M., AND SZE-MERÉDI, E. The approximate Loebl-Komlós-Sós conjecture and embedding trees in sparse graphs. *Electronic Research Announcements in Mathematical Sciences 22* (2015), 1–11.

[16] PIGUET, D., AND STEIN, M. J. An approximate version of the Loebl–Komlós–Sós conjecture. *J. Combin. Theory Ser. B 102*, 1 (2012), 102–125.

[17] REED, B., AND STEIN, M. Spanning trees in graphs of high minimum degree with a universal vertex I: An approximate asymptotic result. In Preparation.

[18] REED, B., AND STEIN, M. Spanning trees in graphs of high minimum degree with a universal vertex II: A tight result. In Preparation.

[19] ROHZOŇ, V. A local approach to the Erdős-Sós conjecture. Preprint 2018, arXiv:1804.06791.

[20] SACLÉ, J.-F., AND WOŹNIAK, M. The Erdős-Sós Conjecture for Graphs without C_4. *Journal of Combinatorial Theory, Series B 70*, 2 (1997), 367 – 372.

[21] SZEMERÉDI, E. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, vol. 260 of *Colloq. Internat. CNRS*. CNRS, Paris, 1978, pp. 399–401.

[22] ZHAO, Y. Proof of the $(n/2 - n/2 - n/2)$ Conjecture for Large n. *Electr. J. Comb. 18* (2011).