UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

DESCRIPTION OF DYNAMICS FOR THREE BOUSSINESQ MODELS AND TWO HIGH-ENERGY PHYSICS MODELS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

CHRISTOPHER HUMBERTO MAULÉN MARCHANT

PROFESOR GUÍA:
CLAUDIO MUÑOZ CERÓN

MIEMBROS DE LA COMISIÓN:
EDUARDO CERPA JERIA (PUC)
ANDREW LAWRIE (MIT)
FELIPE LINARES RAMIREZ (IMPA)
JAIME ORTEGA PALMA (PRESIDENTE)
JEAN-CLAUDE SAUT (U. PARIS SACLAY)

Esta tesis ha sido parcialmente financiada por FONDECYT 1191412 y 1150202, CONICYT PFCHA/DOCTORADO NACIONAL/2016-21160593 y CMM ANID PIA AFB170001

SANTIAGO DE CHILE
2021

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: CHRISTOPHER HUMBERTO MAULÉN MARCHANT
FECHA: 2021
PROF. GUÍA: CLAUDIO MUÑOZ CERÓN

DESCRIPTION OF DYNAMICS FOR THREE BOUSSINESQ MODELS AND TWO HIGH-ENERGY PHYSICS MODELS

This thesis is devoted to the study of long-time asymptotic properties of five models appearing in Physics. These are the **Improved, Good, and** *abc*d **Boussinesq** models, and the **Skyrme and Adkins-Nappi** models. The first part of this thesis deals with the Boussinesq models, and the second one with the remaining equations.

After a brief introduction, in Chapter 2 we consider the decay problem for the generalized improved (or regularized) Boussinesq model with power type nonlinearity, a modification of the originally ill-posed shallow water waves model derived by Boussinesq. The associated decay problem has been studied by Liu, and more recently by Cho-Ozawa, showing scattering in weighted spaces provided the power of the nonlinearity p is sufficiently large. We remove that condition on the power p and prove decay to zero in terms of the energy space norm $L^2 \times H^1$, for any $p > 1$, in two almost complementary regimes: (i) outside the light cone for all small, bounded in time $H^1 \times H^2$ solutions, and (ii) decay on compact sets of arbitrarily large bounded in time $H^1 \times H^2$ solutions.

In Chapter 3 we consider the Cauchy problem for (*abc*d)-Boussinesq system posed on one- and two-dimensional Euclidean spaces. This model, initially introduced by Bona, Chen, and Saut, describes a small-amplitude waves on the surface of an inviscid fluid, and is derived as a first order approximation of incompressible, irrotational Euler equations. We mainly establish the ill-posedness of the system under various parameter regimes, which generalize the result of one-dimensional BBM-BBM case by Chen-Liu. The proof follows from an observation of the *high to low frequency cascade* present in nonlinearity, motivated by Bejenaru and Tao.

In Chapter 4 we consider the generalized Good-Boussinesq model in one dimension, with power nonlinearity and data in the energy space $H^1 \times L^2$. This model has solitary waves with speeds $-1 < c < 1$. When $|c|$ approaches 1, Bona and Sachs showed orbital stability of such waves. It is well-known from a work of Liu that for small speeds solitary waves are unstable. We consider in more detail the long time behavior of zero speed solitary waves, or standing waves. By using virial identities, in the spirit of Kowalczyk, Martel and Muñoz, we construct and characterize a manifold of even-odd initial data around the standing wave for which there is asymptotic stability in the energy space.

In Chapter 5 we consider the decay problem for the Skyrme and Adkins-Nappi equations. We prove that the energy associated to any bounded energy solution of the Skyrme (or Adkins-Nappi) equation decays to zero outside the light cone (in the radial coordinate). Furthermore, we prove that suitable polynomial weighted energies of any small solution decays to zero when these energies are bounded. The proof consists of finding three new virial type estimates, one for the exterior of the light cone, based on the energy of the solution, and a more subtle virial identity for the weighted energies, based on a modification of momentum type quantities.

Finally, in Chapter 6 we conclude with some open problems to be considered in the future.

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA
POR: CHRISTOPHER HUMBERTO MAULÉN MARCHANT
FECHA: 2021
PROF. GUÍA: CLAUDIO MUÑOZ CERÓN

DESCRIPTION OF DYNAMICS FOR THREE BOUSSINESQ MODELS AND TWO HIGH-ENERGY PHYSICS MODELS

Esta tesis está dedicada al estudio de las propiedades asintóticas de cinco modelos que aparecen en Física. Estos son los modelos **Improved, Good y** *abc*d **Boussinesq**, y los modelos de **Skyrme y Adkins-Nappi**. La primera parte de esta tesis trata los modelos de Boussinesq y la segunda el resto de las ecuaciones.

Después de una breve introducción, en el Capítulo 2 consideramos el problema de decaimiento para el modelo generalizado Improved B con no linealidad del tipo de potencia, una modificación del modelo de ondas de aguas poco profundas originalmente mal planteado derivado por Boussinesq. El problema de decaimiento asociado ha sido estudiado por Liu, y más recientemente por Cho-Ozawa, mostrando scattering en espacios con peso siempre que la potencia p de la no linealidad sea suficientemente grande. Eliminamos esa condición en la potencia p y probamos decaimiento a cero en el espacio de energía $L^2 \times H^1$, para cualquier $p > 1$, en dos regímenes casi complementarios: (i) fuera del cono de luz para todas las soluciones pequeñas, acotadas en tiempo en $H^1 \times H^2$, y (ii) en conjuntos compactos para soluciones arbitrariamente grandes acotadas en tiempo en tiempo $H^1 \times H^2$.

En el Capítulo 3 consideramos el problema de Cauchy para el sistema (*abc*d)-Boussinesq planteado en $\mathbb{R}^1$ y $\mathbb{R}^2$. Este modelo, introducido inicialmente por Bona, Chen y Saut, describe ondas de pequeña amplitud en la superficie de un fluido no viscoso y se deriva como una aproximación de primer orden de ecuaciones de Euler irrotacionales e incompresibles. Establecemos el mal posicionamiento del sistema en varios regímenes, generalizando el resultado del caso unidimensional BBM-BBM de Chen-Liu.

En el Capítulo 4 consideramos el modelo generalizado de Good-Boussinesq en una dimensión, con no linealidad del tipo de potencia y datos en el espacio de energía $H^1 \times L^2$. Este modelo tiene ondas solitarias con velocidades $|c| < 1$. Cuando $|c|$ se acerca a 1, Bona y Sachs probaron la estabilidad orbital de tales ondas. Liu demuestra que para velocidades pequeñas, las ondas solitarias son inestables. Consideramos con más detalle el comportamiento a largo plazo de las ondas solitarias de velocidad cero. Mediante el uso de identidades viriales, en el espíritu de Kowalczyk, Martel y Muñoz, construimos una variedad de datos iniciales alrededor de la onda estacionaria para los cuales hay estabilidad asintótica en el espacio de energía.

En el Capítulo 5 consideramos el problema de decaimiento para las ecuaciones de Skyrme y Adkins-Nappi. Demostramos que la energía asociada a cualquier solución de energía acotada de la ecuación de Skyrme (o Adkins-Nappi) decae a cero fuera del cono de luz (en la coordenadas radiales). Además, demostramos que las energías con pesos polinomiales de cualquier solución pequeña decaen a cero cuando estas energías son acostadas. La prueba consiste en encontrar tres nuevas estimaciones del tipo virial, una para el exterior del cono de luz, basada en la energía de la solución, y una identidad virial más sutil para las energías ponderadas, basada en una modificación del momentum.

Concluimos en el Capítulo 6 con algunos problemas abiertos para ser considerados en el futuro.

A Claudia, Génesis y Virginia

Agradecimientos

Primero que todo me gustaría agredecerle a mi madre Virginia y mi hermana Génesis por su constante apoyo y preocupación. A mi compañera de aventuras, Claudia, por ser un soporte en este viaje, por saber como darme ánimo y motivarme a emprender nuevos desafios. Muchas gracias por estar ahí cuando las he necesitado.

A mis amigos de toda la vida, al trípode Rodrigo y Diego, por saber escuchar y darme una mano en los momentos difíciles.

A los amigos que nacieron durante este viaje, Evelyn, Rodrigo y Maria Eugenia, por los gratos momentos que pasamos en la oficina 416, por saber escuchar y aguantar mi mal humor.

Quiero agradecerle a mi Profesor Guía, Claudio Muñoz, por su constante apoyo, así como, su sincera preocupación. Agradezco su dedicación, esfuerzo y simpatía en este camino. Por insipirarme, darme la confianza y empujar mis límites como estudiante.

Además, quisiera a gradecerle a los y las funcionarias por sus esfuerzo y trabajo para crear un agradable ambiente en el DIM. En particular, le agradezco a Karen Hernandez, Natacha Astromujoff, Silvia Mariano y Cookie por sus amigables saludos todas las mañanas.

I would like to thank Prof. Didier Pilod, Prof. Juan Soler, and Prof. Francisco Gancedo for their kindness and hospitality during my visit to their universities.

I should like to express my gratitude to Prof. Eduardo Cerpa, Prof. Andrew Lawrie, Prof. Felipe Linares, Prof. Jaime Ortega, Prof. Jean-Claude Saut for accepted belong to the commission of this thesis. Thank you for your time and dedication to reading this work.

Finally, I deeply thankful Miguel Angel Alejo and Chulkwang Kwak for all the things learned in our collaboratives works. Furthermore, for their sympathy and kindness during these difficult times.

Contents

I Introduction **1**

1 Introduction **2**

1.1 Preliminaries . . . 2

1.1.1 Dispersion and Decay . . . 3

1.2 Three Boussinesq Models . . . 4

1.2.1 Basic properties of the Improved Boussinesq model . . . 6

1.2.2 Basic properties of the (*abc*d)-Boussinesq system . . . 7

1.2.3 Basic properties of the Good-Boussinesq equation . . . 9

1.3 The Adkins-Nappi and Skyrme models . . . 10

1.4 The Virial Technique . . . 11

1.5 Results in this thesis . . . 12

1.5.1 Asymptotic dynamics of small solutions in Improved- Boussinesq . . . 12

1.5.2 Ill-posedness in *abc*d–Boussinesq model . . . 14

1.5.3 Asymptotic manifolds around the good-Boussinesq standing wave . . . 16

1.5.4 Decay in Skyrme and Adkin-Nappi field theories . . . 18

Bibliography . . . 21

II Boussinesq Models **26**

2 Decay in the one dimensional generalized Improved Boussinesq equation **27**

2.1 Introduction . . . 28

2.1.1 Setting . . . 28

2.1.2 Main results . . . 30

2.1.3 More about solitons . . . 32

2.2 Virial identities . . . 34

2.3 Decay in exterior light cones. Proof of Theorem 2.1 . . . 37

2.3.1 End of proof of Theorem 2.1 . . . 41

2.4 Decay in compact sets: Proof of Theorem 2.2 . . . 42

2.4.1 End of proof of Theorem 2.2 . . . 44

Bibliography . . . 46

3 Ill-posedness issues on (*abc*d)-Boussinesq system **49**

3.1 Introduction . . . 50

3.1.1 Setting . . . 50

3.1.2 Main results . . . 52

3.2 Preliminaries . . . 55
3.2.1 Bejenaru-Tao's abstract theory . . . 55
3.2.2 Equivalent representation of *abc*d systems . . . 56
3.3 Proof of Theorem 3.2 . . . 62
3.3.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$ . . . 62
3.3.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$ . . . 66
3.4 Proof of Theorem 3.3 . . . 67
3.4.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$ . . . 67
3.4.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$ . . . 70
3.4.3 BBM-BBM regime: $a = c = 0$ and $b = \mathrm{d} = 1/6$ . . . 71
Appendices . . . 73
3.A Local Well-Posedness . . . 73
3.A.1 Notations . . . 73
3.A.2 Littlewood-Paley Decomposition . . . 73
3.A.3 Bilinear estimates . . . 74
3.B Decomposition of (Q_1, Q_2) on the generic regime: one-dimensional case . . . 76
3.C Decomposition of (Q_1, Q_2, Q_3): generic case . . . 77
3.D Decomposition of (Q_1, Q_2, Q_3): $a = c$ and $b = \mathrm{d} \geqslant 0$ . . . 78
Bibliography . . . 80

4 Asymptotic stability manifolds for solitons in the generalized Good Boussinesq equation **83**
4.1 Introduction . . . 84
4.1.1 Setting . . . 84
4.1.2 Standing waves . . . 86
4.1.3 Main results . . . 87
4.1.4 Idea of the proof . . . 89
4.2 A virial identity for the (gGB) system . . . 91
4.2.1 Decomposition of the solution in a vicinity of the soliton . . . 91
4.2.2 Notation for virial argument . . . 92
4.2.3 Virial estimate . . . 93
4.2.4 End of Proposition 4.3 . . . 99
4.3 Transformed problem and second virial estimates . . . 99
4.3.1 The transformed problem . . . 99
4.3.2 Virial functional for the transformed problem . . . 99
4.3.3 Proof of Proposition 4.8: first computations . . . 100
4.3.4 First technical estimates . . . 102
4.3.5 Controlling error and nonlinear terms . . . 106
4.3.6 End of proof of Proposition 4.8 . . . 108
4.4 Gain of derivatives via transfer estimates . . . 109
4.4.1 A virial estimate related to $\mathcal{M}(t)$ . . . 110
4.4.2 Second set of technical estimates . . . 111
4.4.3 Start of proof of Proposition 4.19 . . . 114
4.4.4 Controlling error terms . . . 116
4.4.5 Controlling nonlinear terms . . . 118
4.4.6 End of proof Proposition 4.19 . . . 119
4.5 A second transfer estimate . . . 120

4.5.1 A virial identity for $\mathcal{N}(t)$ . . . 120
4.5.2 Start of proof of Proposition 4.25 . . . 121
4.5.3 Control of nonlinear terms . . . 124
4.5.4 End of proof of Proposition 4.25 . . . 124
4.6 Proof of Theorem 4.1 . . . 125
4.6.1 Coercivity . . . 125
4.6.2 Proof of Theorem 4.1 . . . 129
4.7 Proof of Theorem 4.2 . . . 133
4.7.1 Conservation of Energy . . . 133
4.7.2 Construction of the graph . . . 134
4.7.3 Uniqueness and Lipschitz regularity . . . 136
Appendices . . . 140
4.A Linear spectral theory for $-\partial_x^2\mathcal{L}$ . . . 140
4.A.1 Asymptotic behavior of fundamental solutions of $-\partial_x^2\mathcal{L}(u) = 0$ . . . 144
4.B Proof of Claims 4.21 and 4.22 . . . 145
4.B.1 Relation between $\partial_x z_{\mathrm{i}}$ and $\partial_x v_{\mathrm{i}}$ . . . 145
4.B.2 Relation between $\partial_x^2 z_{\mathrm{i}}$ and $\partial_x^2 v_{\mathrm{i}}$ . . . 147
4.C Proof of Lemma 4.27 . . . 149
Bibliography . . . 150

III High Energy Models 153

5 On the decay problem for the Skyrme wave maps 154
5.1 Introduction . . . 155
5.1.1 Main results . . . 155
5.1.2 Idea of the proof . . . 158
5.2 Virial Identities . . . 158
5.2.1 Virial identities for the Skyrme Model . . . 159
5.2.2 Virial identities for the Adkins-Nappi Model . . . 164
5.3 Decay in exterior light cones for the Skyrme and Adkins-Nappi models . . . 167
5.3.1 Proof of Theorem 5.1: Skyrme and Adkins-Nappi equations . . . 169
5.4 Decay of weighted energies . . . 170
5.4.1 Growth rate for the modified energy in the Skyrme and Adkins-Nappi equations . . . 170
5.4.2 Decay to zero for modified Energies: Proof of the Theorem 5.2 . . . 171
5.4.3 End of proof of Theorem 5.2 . . . 181
Bibliography . . . 182

IV Conclusion 184

6 Conclusions and Perspectives 185
6.1 Conclusions . . . 185
6.2 Future Work . . . 186
6.2.1 Stability and Asymptotic Stability of Kinks in Wave-Cahn-Hilliard equation . . . 186

6.2.2 Asymptotic stability of even data perturbations of solitary waves in the (abcd)-Boussinesq system 187
6.2.3 Stability of the solitary wave in the Improved-Boussinesq equation . . 187
Bibliography 189

Part I

Introduction

Chapter 1

Introduction

1.1 Preliminaries

Physics is an essential ingredient in sciences and has a key role in explaining and describing natural phenomena. When the phenomenon to explain is highly complex, a strong background in advanced mathematics is needed. For example, partial differential equations have often been used to describe the dissipation of heat or the behavior of waves in several media, which originated the heat equation, the Schrödinger equation, the Eulerian formulation for fluids, among other models. On the other side, there is a wide range of phenomena without satisfactory explanations, requiring an equilibrium between assumptions and the precision of the measurements, for a suitable description.

One of these challenging phenomena is to describe **the behavior of a fluid under certain conditions, called the water wave problem**, first introduced by Lagrange. Some basic assumptions are the following: the fluid is delimited below by a flat bottom and above by a free surface; it is homogeneous, inviscid, incompressible and irrotational. These assumptions imply that the incompressible Euler equations govern the fluid.

The understanding of the dynamics in the water waves model is a hard mathematical problem. It consists of a quasilinear system of equations (in the Zakharov-Craig-Sulem formulation) that contains several canonical simpler models as representatives in certain asymptotic regimes.

Among these models, those of **dispersive type** (i.e., those in which waves of different wavelength propagate at different speeds) are of great relevance, for instance: Korteweg-de Vries, Benjamin–Bona–Mahony, Benjamin-Ono, and the family of Boussinesq equations. These models have different properties: some are integrable, others are Hamiltonian, and others are not. This makes the water waves problem very interesting from the point of view of modelling.

The first aim of this thesis is to study differences between a family of Boussinesq equations, which are obtained under similar hypotheses, despite the substantial contrast between the properties and the dynamics. These are the **Good, Improved and abcd Boussinesq**

systems. Secondly, we want to understand the long-time behavior in **Skyrme and Adkins-Nappi models**, which are high-energy equations intended to describe interactions between nucleons and π mesons. More details on these models will be given in next sections. The following table summarizes the models that will be studied in this thesis:

Model	Equation
Good Boussinesq	$\partial_t^2 u + \partial_x^4 u - \partial_x^2 u - \partial_x^2(\lvert u\rvert^{p-1}u) = 0.$
Improved Boussinesq	$\partial_t^2 u - \partial_x^2\partial_t^2 u - \partial_x^2 u - \partial_x^2(\lvert u\rvert^{p-1}u) = 0.$
*abc*d Boussinesq	$\begin{cases}(1-b\,\Delta)\partial_t\eta + \nabla\cdot(a\,\Delta\vec{u} + \vec{u} + \vec{u}\eta) = 0,\\ (1-\mathrm{d}\,\Delta)\partial_t\vec{u} + \nabla\big(c\,\Delta\eta + \eta + \frac{1}{2}\lvert\vec{u}\rvert^2\big) = 0.\end{cases}$
Skyrme	$\Big(1+\frac{2\alpha^2\sin^2 u}{r^2}\Big)(u_{tt}-u_{rr}) - \frac{2}{r}u_r + \frac{\sin 2u}{r^2}\Big[1+\alpha^2\Big(u_t^2 - u_r^2 + \frac{\sin^2 u}{r^2}\Big)\Big] = 0.$
Adkins-Nappi	$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin 2u}{r^2} + \frac{(u-\sin u\cos u)(1-\cos 2u)}{r^4} = 0.$

Table 1.1: Models & equations

What is the relationship between these five models? Precisely, their wave-like character, with more influence of the Korteweg-de Vries dynamics in the Boussinesq case, and from Klein-Gordon in the case of Skyrme and Adkins-Nappi. Each model will have a different nature, but, as we shall see in this thesis, the techniques used here will be transversal to all these models and produce interesting results.

This thesis was made with important collaborations and research visits. A great part was developed in three long visits that I did during past years. I acknowledge professors **Didier Pilod, Juan Soler, and Francisco Gancedo** for their help making these travels possible.

Visit	Professor
Universidad de Sevilla (4 months, 2020)	Francisco Gancedo (Sevilla) Miguel Angel Alejo (Córdoba)
Universidad de Granada (1 month, 2019)	Juan Soler (Granada)
University of Bergen (1 month, 2019)	Didier Pilod (Bergen)

Table 1.2: Research visits during this PhD thesis.

Before describing the models considered in this thesis, we shortly recall some important notions. We concentrate ourselves in the notion of dispersion and decay.

1.1.1 Dispersion and Decay

As said before, models in Table 1.1 are essentially of dispersive type. We cannot understand the notion of **dispersion** without understanding the notion of **decay**.

The most classical example to describe the interconnection between dispersion and decay of solutions is the linear Schrödinger equation (see [44, 52]). Recall that this equation is given by

$$\mathrm{i}\partial_t u + \Delta u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^{\mathrm{d}}, \quad u \in \mathbb{C}.$$

It is well-known that, by applying the Fourier transform, if the initial data u_0 lies in some Sobolev space, the corresponding solution has the form $u(t) = S(t)u_0$, where $S(t)$ is the Schrödinger group given by $S(t) = \mathrm{e}^{\mathrm{i}\Delta t}$. Furthermore, the operator $S(t)$ is unitary (in L^2), and if the initial data u_0 belongs to the Sobolev space $H^s(\mathbb{R}^{\mathrm{d}})$ then $S(t)u_0$ lies in $\mathcal{C}(\mathbb{R}; H^s(\mathbb{R}^{\mathrm{d}}))$.

One can observe, using the Young inequality and the properties of semigroups, that

$$\|u\|_{L^\infty} \lesssim \|u_0\|_{L^1}\|S_t\|_{L^\infty} \lesssim |t|^{-\mathrm{d}/2}\|u_0\|_{L^1},$$

which means that our solutions decay in time at a rate $|t|^{-\mathrm{d}/2}$ as t tends to infinity, and in low-dimension the weaker the rate of decay. This type of estimate is called **decay or dispersive estimate**, key in the analysis of dispersive PDEs. On the other hand, by applying the Plancherel Theorem, one has

$$\|u\|_{L^2} = \|u_0\|_{L^2}.$$

However, the mass is locally dispersed: for any $R > 0$,

$$\int_{|x|\leqslant R} |S(t)u_0(x)|^2 \mathrm{d}x \lesssim R^{\mathrm{d}}\|S(t)u_0\|_{L^\infty}^2 \lesssim \frac{R^{\mathrm{d}}}{|t|^{\mathrm{d}}}.$$

Then, the localized mass tends to zero as t tends to infinity. One says then that the mass is dispersed to infinity.

In nonlinear physical models, this apparently simple property is far from trivial since there are many possible behaviors for general nonlinear solutions. Even worse, this estimate could be false because of the existence of nondecaying solutions such as **solitons, multi-solitons**, or the presence of even stranger objects called breathers, kinks, or lumps.

In this thesis, by using the Virial technique, we will overcome these difficulties and prove decay properties in four of five models mentioned before.

Now we start by describing the models considered in this thesis. First we consider the Boussinesq family.

1.2 Three Boussinesq Models

In the 1870's, J. Boussinesq [8] deduced a system of equations to describe two-dimensional irrotational and inviscid fluids in a uniform rectangular channel with flat bottom. He was the first to give a favorable explanation to the traveling-waves, solitons, or solitary waves solutions discovered by Scott Rusell thirty years earlier [49], which remained in their form and travelled with constant velocity. He made an approximation of the Eulerian problem to describe the two-way propagation of small amplitude gravity waves on the surface of the water in a canal, and obtained the following simplified equation:

$$\partial_t^2\phi - \partial_x^4\phi - \partial_x^2\phi - \partial_x^2(\phi^2) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

However, this equation is strongly linearly ill-posed; it is called the **Bad–Boussinesq** equation. This bad behavior is not present when the plus sign is considered in the approximation, obtaining

$$\partial_t^2\phi + \partial_x^4\phi - \partial_x^2\phi - \partial_x^2(\phi^2) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.2}$$

which is called **Good–Boussinesq**.

Another way to overcome the unpleasant character of Bad–Boussinesq was proposed by V. G. Makhankov [54], who followed the Boussinesq procedure, and used the wave-like zeroth order fluid relation $\partial_x \sim \partial_t$ to deduce the model

$$\partial_t^2\phi - \partial_x^2\partial_t^2\phi - \partial_x^2\phi - \partial_x^2(|\phi|^{p-1}\phi) = 0, \quad \text{for } p>1,\ (t,x)\in\mathbb{R}\times\mathbb{R},$$

which is no longer strongly ill-posed as (1.1). This is the so-called **Improved–Boussinesq** equation.

A third option to regularize Bad–Boussinesq is to change the scalar character of the equation. This method was proposed by Bona, Chen and Saut [11, 12] (see also Bona, Colins and Lannes [13] for the two-dimsensional case). They introduced the (abcd)**-Boussinesq** equation:

$$(abc\mathrm{d})\begin{cases}(1-b\,\Delta)\partial_t\eta + \nabla\cdot(a\,\Delta\vec{u} + \vec{u} + \vec{u}\eta) = 0,\\ (1-\mathrm{d}\,\Delta)\partial_t\vec{u} + \nabla\left(c\,\Delta\eta + \eta + \frac{1}{2}|\vec{u}|^2\right) = 0.\end{cases} \quad (t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^{\mathrm{d}},\ \mathrm{d}=1,2, \tag{1.3}$$

where η is the **elevation from the equilibrium position** of the fluid, and $\vec{u} = \vec{u}_\theta$ is the **horizontal velocity** of the flow at height θh, where h is the undisturbed depth of the fluid.

The parameters (a,b,c,d) in (1.3) are not arbitrary and follow the condition $a+b+c+\mathrm{d} = \frac{1}{3} - \tau$, where $\tau \geqslant 0$ is the surface tension. This three-degree freedom in the parameters makes the (abcd)–Boussinesq to contain a wide variety of regimes, for example: the Classical Boussinesq system, the Kaup system, the Bona–Smith, BBM–BBM, KdV–KdV, coupled KdV–BBM, coupled BBM-KdV system. Therefore, one can expect that the dispersive properties of these models will vary depending on the choice of parameters.

The deductions above mentioned are summarized in Table 1.3.

In this thesis, my main focus is the well-understanding of small solutions, solitary waves (or traveling waves) and the inherent properties of these equations (well- and ill-posedness). In the history of the water waves problem, solitary waves have a long history, started with Scott Rusell's horseback observation [49]. Solitary waves are solutions of type $Q_c(x-ct)\in H^1(\mathbb{R})$, $c\in\mathbb{R}$, that maintain their form and travel at a constant velocity. They are essential for the well-understanding the coherent wave structures. Also, their properties change depending on the model to be studied. In particular, their stability properties have been extensively studied sometimes, but some questions have remained open for a long time.

The first model that we will describe is the Improved Boussinesq model.

Model	Equation	Origin
Bad Boussinesq	$\partial_t^2 u - \partial_x^4 u - \partial_x^2 u - \partial_x^2(\|u\|^{p-1}u) = 0$	Original model deduced by Boussinesq
Good Boussinesq	$\partial_t^2 u + \partial_x^4 u - \partial_x^2 u - \partial_x^2(\|u\|^{p-1}u) = 0$	Considering the plus sign
Improved Boussinesq	$\partial_t^2 \phi - \partial_x^2 \partial_t^2 \phi - \partial_x^2 \phi - \partial_x^2(\|\phi\|^{p-1}\phi) = 0$	Using $\partial_x \sim \partial_t$
(*abc*d) Boussinesq	$\begin{cases} (1 - b\,\Delta)\partial_t \eta + \nabla \cdot (a\,\Delta \vec{u} + \vec{u} + \vec{u}\eta) = 0, \\ (1 - \mathrm{d}\,\Delta)\partial_t \vec{u} + \nabla\big(c\,\Delta\eta + \eta + \frac{1}{2}\|\vec{u}\|^2\big) = 0. \end{cases}$	First order approximation *a la* Boussinesq

Table 1.3: List of deductions of the equations in this work.

1.2.1 Basic properties of the Improved Boussinesq model

Chapter 2 is concerned with the so-called generalized *Improved Boussinesq* equation (gIB) [63, 22]

$$\partial_t^2 u - \partial_x^2 \partial_t^2 u - \partial_x^2 u - \partial_x^2(|u|^{p-1}u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.4}$$

where $u = u(t, x)$ is a real-valued function, and $p > 1$. Sometimes referred as the Pochhammer-Chree equation [47], this model was first introduced by Pochhammer [63] in its linear version in 1876, and in its complete nonlinear form by Chree [22], in 1886. It was derived as a model of the longitudinal vibration of an elastic rod, as well as a model of nonlinear waves in weakly dispersive media, and shallow water waves.

Although (1.4) is no longer strongly ill-posed as bad Boussinesq (1.1), it is still shares some of its unpleasant behavior, but also some nice surprising properties. In order to explain this in detail, we write (1.4) as the system

$$\text{(gIB)} \quad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = (1 - \partial_x^2)^{-1} \partial_x (u + |u|^{p-1}u). \end{cases} \tag{1.5}$$

This 2×2 system is Hamiltonian, but as far as we understand, not integrable. Its Hamiltonian character leads to the conservation of energy and momentum, given by

$$H(u, v) = \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x + \frac{1}{p+1} \int_{\mathbb{R}} |u|^{p+1} \mathrm{d}x, \tag{1.6}$$

$$P(u, v) = \int_{\mathbb{R}} (uv + \partial_x u \partial_x v) \mathrm{d}x. \tag{1.7}$$

Note in particular the complex character of energy and momentum for gIB: the energy is always nonnegative, and makes sense e.g. for $u \in L^2 \cap L^{p+1}$, and $v \in H^1$. On the other hand, the momentum needs even more regularity than expected, and it is only well-defined for $(u, v) \in H^1 \times H^1$ (or $L^2 \times H^2$). Given this lack of concordance, completely contrary to classical linear waves, understanding the well-posedness problem in gIB is far from trivial.

The pioneering work by Liu [48] showed local and global well-posedness for (1.4) for data $(u_0, v_0) \in H^s \times H^{s+1}$ and $s \geqslant 1$. In addition, the energy and momentum (1.6)-(1.7) are conserved by the flow, or in other words, the $L^2 \times H^1$ norm of the solution remains bounded in time. Note however that the $H^1 \times H^2$ norm of the solution need not be globally bounded in time. The method employed by Liu was essentially based in the Sobolev inclusion H^1 into L^∞ in one dimension, since no useful dispersive decay estimates are available for gIB. The fact that the solvability space differs from the energy space is a property standard in quasilinear models, and gIB has the flavor of a standard one. Consequently, we believe that this weakly ill-posed behavior in gIB is deeply motivated and inherited by the original strongly ill-posed bad Boussinesq equation (1.1). Additionally, Liu also showed blow up of negative energy solutions of (1.5) but with focusing nonlinearities (minus sign in $|u|^{p-1}u$ instead of plus sign). Finally, the controllability problem for gIB in a finite interval (1.5) has been recently studied by Cerpa and Crépeau [18].

However, the gIB system (1.5) also enjoys some nice properties. Indeed, this model is also characterized by the existence of *super-luminal* solitary waves, or just naively solitons, of the form

$$(u, v) = (Q_c, -cQ_c)(x - ct - x_0), \quad x_0 \in \mathbb{R}, \quad |c| > 1. \tag{1.8}$$

The super-luminal character is represented by the condition $|c| > 1$ on the speed. Here, the scaled soliton is slightly different from generalized Korteweg-de Vries (gKdV): $Q_c(s) = (c^2-1)^{1/(p-1)} Q\left(\sqrt{\frac{c^2-1}{c^2}}\, s\right)$, and

$$Q(s) = \left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)s}{2}\right)}\right)^{\frac{1}{p-1}} > 0 \tag{1.9}$$

is the soliton that solves $Q'' - Q + Q^p = 0$, $Q \in H^1(\mathbb{R})$. Note that Q_c must solve the modified elliptic equation

$$c^2 Q_c'' - (c^2 - 1)Q_c + Q_c^p = 0. \tag{1.10}$$

Since the speed of solitons can be arbitrarily large, it clearly implies that (1.5) *possesses infinite speed* of propagation, a fact not present in standard wave-like equations. Note also that solitons with speeds $|c| \downarrow 1$ are small in $L^\infty \cap H^1$, but they do not decay to zero as time evolves, in any standard norm.

1.2.2 Basic properties of the $(abc\mathrm{d})$-Boussinesq system

As a rigorous derivation from the free Eulerian formulation of water waves, Bona, Chen, and Saut [12] proposed the model called one-dimensional $(abc\mathrm{d})$-Boussinesq, as

$$1D\ (abc\mathrm{d}) \quad \begin{cases} (1 - b\partial_x^2)\partial_t \eta + \partial_x(a\partial_x^2 u + u + u\eta) = 0, \\ (1 - \mathrm{d}\partial_x^2)\partial_t u + \partial_x(c\partial_x^2 \eta + \eta + \frac{1}{2}u^2) = 0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1.11}$$

As two-dimensional model, Bona, Colin and Lannes [13], formulated 2D $(abc\mathrm{d})$ as

$$2D\ (abc\mathrm{d}) \quad \begin{cases} (1 - b\,\Delta)\partial_t \eta + \nabla \cdot (a\,\Delta\vec{u} + \vec{u} + \vec{u}\eta) = 0, \\ (1 - \mathrm{d}\,\Delta)\partial_t \vec{u} + \nabla\left(c\,\Delta\eta + \eta + \frac{1}{2}|\vec{u}|^2\right) = 0, \end{cases} \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.12}$$

Here, the unknowns η and u (also $\vec{u}$) describe the free surface and the horizontal velocity of fluid, respectively. Both systems (1.11) and (1.12) are all first-order approximations of the incompressible and irrotational Euler equations assuming the small parameters defined by

$$\alpha = \frac{A}{h} \ll 1, \quad \beta = \frac{h^2}{\ell^2} \ll 1, \quad \alpha \sim \beta,$$

where A and ℓ are typical wave amplitude and wavelength, and h is the constant depth. Such assumptions sometimes referred to as small-amplitude long waves or Boussinesq or simply shallow water waves regimes (see [8]). In the two-dimensional case, the irrotational hypothesis can be (mathematically) characterized as

$$\nabla \wedge \vec{u} = 0, \tag{1.13}$$

which is preserved by the evolution. Note that the condition (1.13) is not necessary in the one-dimensional case since there is a single horizontal direction. See also [4] for relevant result.

As said before, the parameters (a, b, c, d) in both (1.11) and (1.12) are not arbitrary. Specifically, they holds the relations (see [12])

$$\begin{aligned} a &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\nu, \quad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\nu), \\ c &= \frac{1}{2}\left(1-\theta^2\right)\nu - \tau, \quad \mathrm{d} = \frac{1}{2}(1-\theta^2)(1-\mu), \end{aligned}$$

where $\theta \in [0, 1]$ appears in the change of scaled horizontal velocity corresponding to the depth $(1-\theta)h$ below the undisturbed surface, τ is the surface tension ($\tau \geqslant 0$), and ν, μ are arbitrary real numbers ensuring

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + \mathrm{d} = \frac{1}{2}(1-\theta^2) - \tau, \quad a + b + c + \mathrm{d} = \frac{1}{3} - \tau.$$

The dispersive properties of the systems depend on the choice of the parameters. Precisely, the pair (a, c) enhances the dispersion, while the pair (b, d) weakens it (see [17]). This versatility makes the $(abc\mathrm{d})$-Boussinesq model interesting and challenging.

Two systems 1D $(abc\mathrm{d})$ and $2D(abc\mathrm{d})$ allow the following energies

$$E_{1D}[u, \eta](t) = \frac{1}{2}\int_{\mathbb{R}} (-au_x^2 - c\eta_x^2 + u^2(1+\eta) + \eta^2)(t, x)\mathrm{d}x,$$

and

$$E_{2D}[\vec{u}, \eta](t) = \frac{1}{2}\int_{\mathbb{R}^2} (-a|\nabla\vec{u}|^2 - c|\nabla\eta|^2 + |\vec{u}|^2(1+\eta) + \eta^2)(t, x)\mathrm{d}x,$$

respectively, that both are conserved in time when $b = \mathrm{d}$ and $a, c < 0$. Thus local well-posedness in H^1-level space is immediately extended to the global one at least for small data. Note that Sobolev embedding in two-dimensional case is not enough to control L^∞ norm of η, but Gagliardo-Nirenberg interpolation inequality can control $\eta|\vec{u}|^2$.

1.2.3 Basic properties of the Good-Boussinesq equation

Recall that the Good Boussinesq model, in its simplified form, is given by:

$$\partial_t^2\phi + \partial_x^4\phi - \partial_x^2\phi + \partial_x^2(f(\phi)) = 0, \tag{1.14}$$

and if formally $u = \phi$ and $v = \partial_x^{-1}\partial_t\phi$, has the following representation as 2×2 system:

$$(g\text{GB}) \qquad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x(-\partial_x^2 u + u - f(u)). \end{cases} \tag{1.15}$$

This will be the exact model worked in this chapter, which is Hamiltonian, and has the following associated conserved quantities:

$$\begin{aligned} E[u,v] &= \frac{1}{2}\int \left[v^2 + u^2 + (\partial_x u)^2 - 2F(u)\right] \qquad &\text{(Energy)},\\ P[u,v] &= \int uv &\text{(Momentum)}. \end{aligned} \tag{1.16}$$

(Here $\int$ means $\int_{\mathbb{R}} dx$.) These laws define a standard energy space $(u,v) \in H^1 \times L^2$. As well as the Korteweg-de Vries (KdV) equation, (gGB) is considered as a canonical model of shallow water waves, see [71]. In addition, (gGB) arises in the so-called "nonlinear string equation" describing small nonlinear oscillations in an elastic beam (see [25]).

The study of the Boussinesq-type equations has increased recently, mainly due to the versatility of these models when describing nonlinear phenomena. There are several authors that focus on the good Boussinesq equation. The fundamental works Bona and Sachs [15], using abstract techniques of Kato, proved that the Cauchy problem is locally and globally well-posed for small data, and showed the existence of solitary waves for velocities $c^2 < 1$. Linares [43, 45], using Stricharz estimates, proved that the Cauchy problem is globally well-posed in the energy space in the case of small data. Kishimoto [32], in the case of a quadratic nonlinearity, proved that the Cauchy problem is globally well-posed in $H^s(\mathbb{R})$, for $s \geqslant -1/2$, and ill-posed for $s < -1/2$. In [62], it was proved that small solutions in the energy space must decay to zero as time tends to infinity in proper subsets of space. Recently, Charlier and Lenells [19] developed the inverse scattering transform and a Riemann-Hilbert approach for the quadratic (gGB), which is integrable.

A solitary wave is a solution to (1.14) of the form

$$(u,v) = (Q_c, -cQ_c)(x - ct - x_0), \quad |c| < 1, \quad x_0 \in \mathbb{R},$$

with Q_c solving $(c^2-1)Q_c + Q_c'' + f(Q_c) = 0$ in $H^1(\mathbb{R})$

In the case that f is a pure power nonlinearity of the form $f(s) = |s|^{p-1}s$ for $p > 1$, it is well-known that (up to shifts) standing solitary waves have the form

$$u(t,x) = Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)}, \quad v(t,x) = 0. \tag{1.17}$$

Here, Q satisfies the equation

$$Q''(x) - Q(x) + f(Q(x)) = 0. \tag{1.18}$$

In general, solitons (solitary waves in integrable equations) are stable objects. However, this is not the case of good Boussinesq (similar to Klein-Gordon). Indeed, small perturbations of solitons may decay or form singularities in finite time, see [25, 47, 9, 72].

1.3 The Adkins-Nappi and Skyrme models

Now we consider two *nonlinear* quantum field models, known in the literature as *Skyrme and Adkins-Nappi equations*. Physically these models intend to describe interactions between nucleons and π mesons. Classical nonlinear field theories played an important role in the description of particles as solitonic objects. A well known example of these nonlinear theories is the $SU(2)$ sigma model [27], obtained as a formal critical point from the action

$$S(\psi) = \int_{\mathbb{R}^{1,\mathrm{d}}} \eta^{\mu\nu}(\psi^* g)_{\mu\nu} = \int_{\mathbb{R}^{1,\mathrm{d}}} \eta^{\mu\nu}\partial_\mu\psi^A\partial_\nu\psi^B g_{AB}\circ\psi. \tag{1.19}$$

Here ψ is a map from a $(1+\mathrm{d})$-dimensional Minkowski space $(\mathbb{R}^{1,\mathrm{d}}, \eta)$ to a Riemannian manifold (M, g) with metric g. From a geometrical point of view, the associated Lagrangian is the trace of the pull-back of the metric g under the map ψ. A current choice is $M = \mathbb{S}^{\mathrm{d}}$ with g the associated metric and for $\mathrm{d} = 3$, one obtains the classical $SU(2)$ sigma model. The Euler-Lagrange equation corresponding to the action S is called the wave maps equation. Unfortunately, the $SU(2)$ sigma model does not admit solitons and it develops singularities in finite time [7, 23, 50]. To avoid these inconveniences and to prevent the possible breakdown of the system in finite time, Skyrme [51] modified the associated Lagrangian to (1.19) by adding higher-order terms such that breaks the scaling invariance of the initial model (making it more rigid), which in spherical coordinates (t, r, θ, φ) on $\mathbb{R}^{1,3}$, and co-rotational maps $\psi(t, r, \theta, \varphi) = (u(t, r), \theta, \varphi)$, the Skyrme model leads to the scalar quasilinear wave equation satisfied by the angular variable u, asit will be shown in (1.20).

This equation has a *unique* static solution with boundary values $u(0) = 0$ and $\lim_{r\to\infty} u(r) = \pi$, and which is currently known as *Skyrmion* [53]. This existence was proved in [41] and [53] by using variational methods and ODE techniques respectively. As far as we know, the Skyrmion is not known in a closed form.

In this paper, we are interested in the long time asymptotics of two relevant mathematical physics models. Firstly, as we already mentioned above for the Skyrme model is

$$\left(1 + \frac{2\alpha^2\sin^2 u}{r^2}\right)(u_{tt} - u_{rr}) - \frac{2}{r}u_r + \frac{\sin 2u}{r^2}\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2 u}{r^2}\right)\right] = 0, \tag{1.20}$$

and the second model is a short of generalization of supercritical wave maps as it was presented by Adkins and Nappi [1]. This is a simplified version of the Skyrme model (1.20) and it is currently known as Adkins-Nappi model

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin 2u}{r^2} + \frac{(u - \sin u\cos u)(1 - \cos 2u)}{r^4} = 0. \tag{1.21}$$

These two models have the following low order conserved quantities (subindices "S" and "AN" for Skyrme and Adkins-Nappi models respectively)

$$E_S[u](t) = \int_0^\infty r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2 u}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2 u}{r^2} + \frac{\alpha^2 \sin^4 u}{r^4} \right] \mathrm{d}r, \tag{1.22}$$

$$E_{AN}[u](t) = \int_0^\infty r^2 \left[u_t^2 + u_r^2 + 2\frac{\sin^2 u}{r^2} + \frac{(u - \sin u \cos u)^2}{r^4} \right] \mathrm{d}r. \tag{1.23}$$

Respecting to the Cauchy problem, (1.20) is globally well-posed for small data in $\dot{H}^{5/2}(\mathbb{R}^3)$ (see [26]), and the corresponding global result for the Adkins-Nappi equation (1.21) holds in $\dot{H}^2(\mathbb{R}^3)$. For large-data global well-posedness results, [40] showed that it holds in $H^4(\mathbb{R}^3)$ for Skyrme (1.20).

1.4 The Virial Technique

We describe here one of the main techniques that we will use in this thesis, the Virial technique.

The virial identities are somehow related with Noether's Theorem [52]. In Physics, the Virial Theorem gives a relation between the average total kinetic energy and the total potential energy of the system. Moreover, in elliptic PDEs it is known as the Pokhozhaev's identity, which is applicable to localized solutions to the stationary nonlinear Schrödinger equation.

The Virial identities in its modern form were introduced by Glassey [30] to show blow up for certain focusing nonlinear Schrödinger equation (NLS). In general, these identities are used to show that a positive quantity involving the solution u has a monotonic behavior in time.

Monotonic quantities recently have been used in a powerful way in the context of dispersive equations, see [2, 3, 20, 30, 33, 34, 35, 57, 58, 56, 62]. It has allowed to describe the behavior of several equations in a wide variety of properties, from decay to blow-up, and asymptotically stability.

We describe in simple words how Virial works. The base of the argument is the election of a conserved quantity. A modification of this conservation law is likely to give a virial identity via monotonicity. This monotonicity relation is narrowly related to the behavior in time of some particular norm of the solution. For example, for the Improved Boussinesq equation

$$\partial_t^2 \phi - \partial_x^2 \partial_t^2 \phi - \partial_x^2 \phi - \partial_x^2 (|\phi|^{p-1}\phi) = 0, \quad \text{for } p > 1,\ (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.24}$$

one has the equivalent system

$$\text{(gIB)} \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = (1 - \partial_x^2)^{-1} \partial_x (u + |u|^{p-1} u), \end{cases} \tag{1.25}$$

which is Hamiltonian. Its Hamiltonian character leads to the conservation of energy and momentum, given by

$$
\begin{aligned}
H(u,v) &= \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1}\mathrm{d}x,\\
P(u,v) &= \int_{\mathbb{R}} (uv + \partial_x u \partial_x v)\mathrm{d}x,
\end{aligned}
$$

which, are well defined in $L^2 \times H^1$ and $L^2 \times H^2$, respectively. Now, let the following functional

$$
\mathcal{I}(t; L, \sigma) = \mathcal{I}(t) = \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{x+\sigma t}{L}\right)\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)(t,x)\mathrm{d}x,
$$

where φ is a weight function, which is measuring a particular dynamics in our system. For example, the localization of the mass. Then, for (u, v) global solutions of the equation (1.25), in Chapter 2 we will prove that the following relation is satisfied:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = {} & \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)\mathrm{d}x\\
& - \frac{1}{L}\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x.
\end{aligned}
$$

This identity has the good sign property (for small solutions), and will be useful in our proof. The election of the functional is not an easy task, as we will see in Chapter 2, 4 and 5. However, the technique is very powerful and adaptable.

1.5 Results in this thesis

This thesis contains essentially four results, which are part of the following four articles:

1. C. Maulén, and C. Muñoz, *Decay in the one dimensional generalized Improved Boussinesq equation*, published in SN Partial Differential Equations and Applications (Chapter 2).

2. C. Kwak, and C. Maulén, *Ill-posedness issues on the* (*abc*d)-*Boussinesq system*, preprint arXiv:2102.01248, (Chapter 3).

3. C. Maulén, *Asymptotic stability manifolds for solitons in the generalized Good- Boussinesq equation*, preprint arXiv:2102.01151, (Chapter 4).

4. M. A. Alejo, and C. Maulén, *Decay properties in the Skyrme and Adkins-Nappi equations*, preprint (Chapter 5).

1.5.1 Asymptotic dynamics of small solutions in Improved- Boussinesq

In Chapter 2, we are motivated by the decay problem of solutions to gIB (1.5). This interesting question has attracted the attention of several people before us. Liu [47] showed decay of solutions to (1.5) obtained from initial data satisfying e.g. (u_0, v_0) in $H^1 \times H^2$, $u_0 \in L^1$ and $(1-\partial_x^2)^{1/2} v_0 \in L^1$, all of them small enough. In particular, he showed that for $p > 12$,

$$\sup_{t \geqslant 0} \Big((1+t)^{\frac{1}{10}} \|u(t)\|_{L^\infty} + \|(u,v)(t)\|_{H^1 \times H^2} \Big) < +\infty.$$

He also showed that p can be taken greater than 8 if $s > \frac{3}{2}$ and (u_0, v_0) in $H^s \times H^{s+1}$. Next, in [70], Wang and Chen extended this result to higher dimensions.

The exponent p in (1.5) was recently improved by Cho and Ozawa [21], who showed using modified scattering techniques that p can be taken greater than $9/2$ if $u_0 \in H^s$, $s > \frac{8}{5}$. The solution global in this case satisfies $\|u(t)\|_{L^\infty} = O(t^{-2/5})$ as $t \to +\infty$. Additionally, the same authors showed that the asymptotics as $t \to +\infty$ cannot be the linear one if $1 < p \leqslant 2$ and near zero frequencies vanish at infinity.

Lowering the exponent p for which there is decay seems a complicated problem, due to the quasilinear behavior of gIB. Since there should be modified dynamics, we believe that we need different tools to attack this problem.

Our first result deals with the exterior light-cone decay problem. More precisely, we consider the interval depending on time

$$I(t) = \big(-\infty, -(1+a)t\big) \cup \big((1+b)t, \infty\big), \qquad t > 0, \tag{1.26}$$

where $a, b > 0$ are arbitrary positive numbers. Now, we show that, regardless the power $p > 1$, any global solution (u, v) to (1.5) which is sufficiently small and regular must concentrate inside the light cone.

Theorem 1.1 (Decay in exterior light cones). *Let $(u,v) \in C(\mathbb{R}, H^1 \times H^2)$ be a global small solution of (1.5) such that, for some $\epsilon(a,b) > 0$ small, one has*

$$\sup_{t \in \mathbb{R}} \|(u(t), v(t))\|_{H^1 \times H^2} < \epsilon. \tag{1.27}$$

Then, for $I(t)$ as in (1.26), there is strong decay to zero in the energy space:

$$\lim_{t \to \infty} \|(u(t), v(t))\|_{(L^2 \times H^1)(I(t))} = 0. \tag{1.28}$$

Additionally, one has the mild rate of decay for $|\sigma| > 1$:

$$\int_2^\infty \int_{\mathbb{R}} \mathrm{e}^{-c_0|x+\sigma t|} (u^2 + v^2 + (\partial_x v)^2) \mathrm{d}x \mathrm{d}t \lesssim_{c_0} \epsilon^2. \tag{1.29}$$

Having described the small data behavior in exterior light cones, we concentrate now in the interior light cone behavior. Here things are much more complicated, since the energy (1.6) is no more useful to describe the dynamics. Instead, we shall use a suitable modification of the momentum (1.7).

Theorem 1.2 (Full decay in interior regions). *Let (u, v) be a global solution of* (1.5) *in the class $C(\mathbb{R}, H^1 \times H^2) \cap L^\infty(\mathbb{R}, H^1 \times H^2)$, not necessarily small in norm. Then for any $L \gg 1$ we have*

$$\int_2^\infty \int_{-L}^{L} \left(v^2 + u(1-\partial_x^2)^{-1}u + |u|^{p+1}\right)(t,x)\mathrm{d}x\mathrm{d}t \lesssim 1. \tag{1.30}$$

Moreover, we have strong decay to zero in the energy space $(L^2 \times H^1)(I)$, for any I bounded interval in space:

$$\lim_{t\to\infty} \|(u,v)(t)\|_{(L^2\times H^1)(I)} = 0. \tag{1.31}$$

Estimate (1.30) shows that the local L^2 norm of v is integrable in time, and some mixed norms of u. Note that u seems not locally L^2 integrable in time. However, (1.31) shows that this norm indeed decays to zero in time (even if it is not integrable in time). Also, Theorem 1.2 can be read as "boundedness in time in $H^1 \times H^2$ implies $L^2 \times H^1$ time decay in compact sets of space".

The proof of Theorem 1.1 follows the introduction of a new virial identity, in the spirit of the previous results by Martel and Merle [55, 56] in the gKdV case, and [38, 2] in the BBM case. Note however that in those cases the functional involved is related to the mass (L^2 norm) of the solution. Here, we use instead a modification of the energy (1.6) of the solution.

The techniques that we use to prove Theorem 1.2 are not new, and have been used to show decay for the Born-Infeld equation [3], the good Boussinesq system [62], the Benjamin-Bona-Mahony (BBM) equation [38], and more recently in the more complex *abc*d Boussinesq system [39, 37]. In all these works, suitable virial functionals were constructed to show decay to zero in compact/not compact regions of space.

1.5.2 Ill-posedness in *abc*d–Boussinesq model

In Chapter 3, we are motivated by the ill-posedness of Cauchy problem for (1.12). To describe the ill-posedness it is necessary to understand the previous well-posedness results to the Cauchy problem. But before presenting our results, we clarify what we mean "ill-posedness". To do this, we first define "well-posedness". As the author's best knowledge, the French mathematician Jacques Hadamard initially proposed the concept of well-posed problems as

Definition 1.3 (Well-posedness). *We say that a time-dependent PDE problem is well-posed if*

- *there exists a solution,*
- *the solution is unique,*
- *the solution behaves continuously with the initial condition.*

Obviously, problems that are not well-posed in the sense of Hadamard are termed ill-posed, in other words, the invalidity of one of above properties makes the problem to be ill-posed. In this work, in order to obtain ill-posedness results, we will attack the third property in Definition 1.3.

Some words about the Caucht theory of $abc\mathrm{d}$ systems. These models have been extensively studied (in various perspective) in the literature, see e.g. [11, 12, 16, 24, 46, 66, 64, 17, 10, 65, 39, 60, 37, 67, 68]. Among other them, we focus on Cauchy problems for these systems. In [11, 12], Bona, Chen and Saut first studied local and global well-posedness of linear and nonlinear problems, and established the following results (the following results only exhibit the case when $\mathcal{H}$ (see (3.12)) has order 0):

1. the generic regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s \geqslant 0$.

2. the BBM-BBM regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s \geqslant 0$.

3. the KdV-KdV regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s > 3/4$.

In [24], Dougalis, Mitsotakis, and Saut proved that two-dimensional ($abc\mathrm{d}$)–Boussinesq system under the generic regime is locally well-posed in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for $s > 0$. Note that this local result is indeed valid in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ by improving Grisvard's bilinear estimate [29], see Appendix 3.A (Lemma 3.16). In [46], Linares, Pilod, and Saut focused on the strongly dispersive (KdV-KdV system) regime, and established local well-posedness result in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for $s > 3/2$. Previously, Schonbek [69] and Amick [6] considered a version of the original Boussinesq system ($a = c = b = 0, b = 1/3$), and proved global well-posedness under a non-cavitation condition via parabolic regularization. Later, Burteau [16] improved it without a non-cavitation condition. Studies on long time existence of solutions have been done in, for instance, [66, 61, 65, 67, 68]. In these works, the authors established the well-posedness for large time with appropriate time scales.

As far as we know, there is only few results for ill-posedness issues. Chen and Liu [17] established the (mild) ill-posedness result for one-dimensional system under the weakly dispersive regime (1D BBM-BBM system) below L^2. The main idea follows the abstract theory developed by Bejenaru-Tao [14]. The authors also discussed the formation of singularities and provided blow-up criteria. Recently, [5] Ambrose, Bona and Milgrom have established the ill-posedness of the one-dimensional periodic Kaup system ($a = 1/3$ and $b = c = \mathrm{d} = 0$) is ill-posed in any positive regularity Sobolev space, in the sense that the flow map is discontinuous at the origin. They also concerned with the case that the generic condition (1.32) is negated.

In contrast with results mentioned above, Chapter 3 concerns with the ill-posed issues on one- and two-dimensional ($abc\mathrm{d}$)-Boussinesq systems in the following cases:

1. Generic regime
$$a, c < 0, \quad b, \mathrm{d} > 0, \tag{1.32}$$

2. KdV-KdV regime
$$a = c = \frac{1}{6}, \quad b = \mathrm{d} = 0, \tag{1.33}$$

3. BBM-BBM regime
$$a = c = 0, \quad b = \mathrm{d} = \frac{1}{6}. \tag{1.34}$$

Now, we are ready to present our main theorem.

Theorem 1.4 ([36]). *The 1D- and 2D-abc*d *system* (1.12) *are ill-posed in* $H^s(\mathbb{R})^2$ *or* $H^s(\mathbb{R})^3$*, respectively, for*

1. $s < -\frac{1}{2}$ *in the generic regime.*

2. $s < -\frac{3}{2}$ *in the KdV-KdV regime.*

*In addition, the 2D-(abc*d*)-Boussinesq system* (1.12) *is ill-posed in* $H^s(\mathbb{R})^3$ *for* $s < 0$ *in the BBM-BBM regime.*

The BBM-BMM case of the one-dimensional (*abc*d)-Boussinesq system has been dealt with by Chen and Liu [17]. However, the two-dimensional BBM-BBM system is considered here for the first time, and together with Appendix 3.A, we completely resolve Cauchy problem for it.

The proof of above results follows the same idea developed by Bejenaru and Tao [14], and motivated by an observation as follows: All nonlinear interactions are quadratic, thus *high* $\times$ *high* interaction components over an appropriate short time depending on the frequency cause *resonances* near the origin of the resulting frequency. For this reason, the flow cannot disperse the high-frequency energy for this time so that the smoothness of the flow breaks below certain regularity. Note that this observation is simply applied to a one-dimensional problem, but it is non-trivial to construct initial data that can cause *resonance* in two-dimensional case.

1.5.3 Asymptotic manifolds around the good-Boussinesq standing wave

In Chapter 4, we are motivated by the long time behavior problem for solitary waves of the gGB (1.14) in the case where $f(s) = |s|^{p-1}s$ for $p > 1$. This interesting question has attracted the attention of several authors before us, showing that the behavior of solitary waves in the standard energy space $H^1 \times L^2$ is not an easy problem. Bona and Sachs [15], applying the theory developed by Grillakis, Shatath and Strauss (see [28]), proved that solitary waves are stable if the speed c obeys the condition $(p-1)/4 < c^2 < 1$ and $p > 4$. Li, Ohta, Wu and Xue [42] proved the orbital instability in the degenerate case $1 < p < 5$ and speed $c = (p-1)/4$. Additionally, Kalantarov and Ladyzhenkaya in [31] proved that solutions associated to initial data with nonpositive energy may blow up in some sense. Inspired by this work, Liu [48] showed that there are solutions with initial data arbitrarily near the ground state ($c = 0$) that blow up in finite time.

It is not difficult to realize that (1.15) preserves the even-odd parity in its variables (u, v). In this Chapter, we will prove that any even-odd small perturbation of the static soliton ($c = 0$) in the energy space, under certain orthogonality condition, is orbitally stable and in fact, it is (locally) asymptotically stable. Furthermore, we will construct a manifold of initial data such that the associated solutions are orbitally stable in $H^1 \times L^2$, and locally asymptotically stable in the space $L^2 \cap L^\infty$. Our first result is:

Theorem 1.5. *Let $p \geqslant 2$. There exists $\delta > 0$ such that if a global even-odd solution $(\phi, \partial_t\partial_x^{-1}\phi)$ of* (1.15) *satisfies for all $t \geqslant 0$,*

$$\|(\phi, \partial_t\partial_x^{-1}\phi)(t) - (Q,0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \delta, \tag{1.35}$$

then, for any $\gamma > 0$ small enough and any compact interval I of $\mathbb{R}$,

$$\lim_{t\to+\infty} \left(\|\phi(t) - Q\|_{L^2(I)\cap L^\infty(I)} + \|(1-\gamma\partial_x^2)^{-1}\partial_t\phi(t)\|_{L^2(I)}\right) = 0. \tag{1.36}$$

This is, as far as we understand, the first description of the standing wave dynamics in the Good Boussinesq model, which is unstable by nature. Clearly the data under which (1.35) is satisfied is not empty, the soliton $(Q,0)$ being its most important representative. However, (1.35) cannot define an open set in the energy space as simple as in some stable, subcritical dynamics, such as KdV. Our second result will describe the manifold of initial data leading to (1.35). The following result provides a description of the manifold of initial data leading to global solutions for which (1.35) holds.

Let $\delta_0 > 0$, and let $\mathcal{A}_0$ be the manifold given by

$$\mathcal{A}_0 = \left\{\boldsymbol{\epsilon} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) |\ \boldsymbol{\epsilon} \text{ is even-odd },\|\boldsymbol{\epsilon}\|_{H^1\times L^2} < \delta_0 \text{ and } \langle \boldsymbol{\epsilon}, \boldsymbol{Z}_+\rangle = 0\right\}. \tag{1.37}$$

Theorem 1.6. *Let $p \geqslant 2$. There exist $C, \delta_0 > 0$ and a Lipschitz function $h : \mathcal{A}_0 \to \mathbb{R}$ with $h(0) = 0$ and $|h(\epsilon)| \leqslant C\|\epsilon\|_{H^1\times L^2}^{3/2}$ such that, denoting*

$$\mathcal{M} = \{(Q,0) + \epsilon + h(\epsilon)Y_+ \text{ with } \epsilon \in \mathcal{A}_0\}, \tag{1.38}$$

the following holds:

1. *If $\boldsymbol{\phi}_0 \in \mathcal{M}$ then the solution of* (1.15) *with initial data $\boldsymbol{\phi}_0$ is global and satisfies, for all $t \geqslant 0$,*
$$\|\boldsymbol{\phi}(t) - (Q,0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant C\|\boldsymbol{\phi}_0 - (Q,0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}. \tag{1.39}$$

2. *If a global even-odd solution $\boldsymbol{\phi}$ of* (1.15) *satisfies, for all $t \geqslant 0$,*
$$\|\boldsymbol{\phi}(t) - (Q,0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant \frac{1}{2}\delta_0, \tag{1.40}$$
then for all $t \geqslant 0$, $\phi(t) \in \mathcal{M}$.

The proofs of this results follow the lines of the ideas used recently by Kowalczyk, Martel and Muñoz in [34] to understand the unstable soliton dynamics in the nonlinear Klein-Gordon equation, and by Kowalczyk, Martel, Muñoz and Van Den Bosch [35] to study the stability properties of kinks for (1+1)-dimensional nonlinear scalar field theories.

More precisely, the proofs are based in a series of localized virial type arguments, similar to the ones used in [2, 3, 34, 35, 33, 58, 56]. In our case, we will use a combination of virials to obtain the integrability in time of the $L^2 \times L^2$-norm of $(\phi(t) - Q, (1-\gamma\partial_x^2)^{-1}\partial_t\phi(t))$, for any $\gamma > 0$ small enough, and in any compact interval I, i.e.,

$$\int_0^\infty \left(\|\phi(t) - Q\|_{L^2(I)}^2 + \|(1-\gamma\partial_x^2)^{-1}\partial_t\phi(t)\|_{L^2(I)}^2 \right) \mathrm{d}t < \infty.$$

1.5.4 Decay in Skyrme and Adkin-Nappi field theories

In Chapter 5, we were interested in the long time asymptotics of two relevant mathematical physics models.

Before introducing the main results of this chapter, it is needed to introduce some notations. In that follows, we use the subindices and the superindexes "S" and "AN" to reference the Skyrme and Adkins-Nappi models respectively. Firstly, we defined the main spaces where the energies are bounded. Let $\mathcal{E}_n^X$ the space of all finite energy data of degree n, namely

$$\mathcal{E}_n^X = \left\{ (u, u_t) \middle| E_X[u](t) < \infty, u_0(0) = 0, \lim_{r\to\infty} u_0(r) = n\pi \right\}, \tag{1.41}$$

where here $X = S$ refers to the Skyrme model or when $X = AN$ to the Adkins-Nappi model. In what follows, we consider $(u, u_t) \in \mathcal{E}_0^X$ and such that is a solution of (1.20) or (1.21), respectively.

The main goal of this work is to prove that small global solutions with enough regularity of Skyrme (1.20) and Adkins-Nappi (1.21) equations decay to zero in a certain region of the light cone. Furthermore, we also study the decay of an associated weighted energy for both equations, and which we need them for analyzing their corresponding long time behavior.

More precisely let $b > 0$ and consider the following subset depending on time

$$R(t) = \{x \in \mathbb{R}^3 \mid |x| > (1+b)t\} \subset \mathbb{R}^3. \tag{1.42}$$

We will show that any global solution u to (1.20) (or (1.21)), which is sufficiently regular and without a previous smallness condition, must be concentrated inside the light cone.

Theorem 1.7 (Decay in exterior light cones for the Skyrme and Adkins-Nappi models)**.**
Let $(u_0, u_1) \in \mathcal{E}_0^X$, defined in (1.41), such that u is a global solution, for (1.20) when $X = S$, or (1.21) when $X = AN$, respectively. Then, for $R(t)$ as in (1.42), there is strong decay to zero of the energy E_X, in particular:

$$\lim_{t\to\infty} \|(u_t(t), u_r(t))\|_{L^2\times L^2(\mathbb{R}^3\cap R(t))} = 0. \tag{1.43}$$

Additionally, one has the mild rate of decay for $|\sigma| > 1$:

$$\int_2^\infty \int_0^\infty \mathrm{e}^{-c_0|r+\sigma t|} r^2 (u_t^2 + u_r^2) \mathrm{d}r\mathrm{d}t \lesssim_{c_0} 1. \tag{1.44}$$

For the next results, we have to introduce a weighted version of the spaces (1.41). Let $\mathcal{E}_n^{X,\phi}$ the space of all finite ϕ-weighted energy data of degree n

$$\mathcal{E}_n^{X,\phi} = \{(u, u_t)\, |E_{X,\phi}[u](t) < \infty, u_0(0) = 0, \quad u_0(\infty) = n\pi\}\,, \tag{1.45}$$

where $E_{X,\phi}$ is written for the Skyrme model as

$$E_{S,\phi}[u](t) = \int_0^\infty \phi(r)\left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4}\right], \tag{1.46}$$

and for the Adkins-Nappi model as

$$E_{AN,\phi}[u](t) = \int_0^\infty \phi(r)\left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4}\right]. \tag{1.47}$$

In fact, one can see, that if $E_{X,r^2}[u](t) = E_X[u](t)$, then $\mathcal{E}_n^{X,r^2} = \mathcal{E}_n^X$, for $X \in \{S, AN\}$.

Our second result shows that the energy E_X associated to any global solution $(u, u_t) \in \mathcal{E}_0^{X,r^n} \cap \mathcal{E}_0^{X,r^{n-1}}$ of (1.20) or (1.21), decays to zero when t goes to infinity. This means that for any global solution u which is sufficiently regular and it satisfies a weighted integrability on r, its energy E_{X,r^n} decays to zero when t goes to infinity for both $X = S$ or $X = AN$ cases.

Theorem 1.8 (Decay of weighted energies)**.** *Let $\delta > 0$ small enough. Let $(u, u_t) \in \mathcal{E}_0^{X,r^n} \cap \mathcal{E}_0^{X,r^{n-1}}$ a global solution of (1.20) or (1.21) such that*

$$\sup_{t\in\mathbb{R}} E_X[u](t) < \delta, \quad \text{for } X = AN, S. \tag{1.48}$$

Then, the modified energy $E_{X,\varphi}[u](t)$ with $\varphi(r) = r^n$ decays to zero, for $n > 7$ ($X = S$ case) or for $n \in \left[\frac{3+\sqrt{41}}{2}, 10\right]$ ($X = AN$ case), respectively. In particular,

$$\lim_{t\to\infty} \|r^{\frac{n-2}{2}}(u_t, u_r)(t)\|_{L^2\times L^2(\mathbb{R}^3)} = \lim_{t\to\infty} E_{X,r^n}(t) = 0. \tag{1.49}$$

In order to prove Theorem 1.7, we follow some ideas appeared in [2, 3, 59], where decay for Camassa-Holm, Born-Infeld and Improved-Boussinesq models were considered. The main tool in these works was a suitable virial functional for which the dynamic of solutions is converging to zero when it is integrated in time.

In Chapter 5, the new virial functionals give us relevant information about the dynamics of global solutions of Skyrme and Adkins-Nappi equations. Using a proper virial estimate, we prove that the corresponding energies associated to Skyrme and Adkins-Nappi equations decay to zero in the subset $R(t)$

$$R(t) = \{x \in \mathbb{R}^3 | x > (1 + b)t\} \subset \mathbb{R}^3,$$

which is the complement of the ball of radius $(1+b)t$, for $b > 0$.

Furthermore, to prove Theorem 1.8, we will study the growth rate of polynomial weight energies of the Skyrme and Adkins-Nappi equations. After that, assuming that their growth is bounded, we will prove that this growth decays zero as t tends to infinity. To prove this result, we introduce a functional associated with a sort of weighted momentum. It happens that the virial identity associated to this functional shows no evidence of good sign conditions, i.e. that the derivative of the functional be negative. Therefore, we have to introduce a new functional as a linear combination of these two viral identities and for which there is a good sign property. This ensures the integrability in time of polynomial weighted energies of degree n. Moreover, it also guarantees the decay of polynomial weighted energy of degree $n+1$ over a subsequence of times. Combining these two facts, we conclude that the polynomial weighted energies, which are bounded, decay to zero as t tends to infinity (over $\mathbb{R}^3$).

Bibliography

[1] G. S. Adkins and C. R. Nappi. Stabilization of chiral solitons via vector mesons. Physics Letters B, 137(3-4): 251-256, 1984.

[2] M. A. Alejo, F. Cortez, C. Kwak and C. Muñoz, *On the dynamics of zero-speed solutions for Camassa-Holm type equations*, International Mathematics Research Notices, , rnz038, https://doi.org/10.1093/imrn/rnz038.

[3] M. A. Alejo, and C. Muñoz, *Almost sharp nonlinear scattering in one-dimensional Born-Infeld equations arising in nonlinear Electrodynamics*, Proc. AMS 146 (2018), no. 5, 2225–2237.

[4] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. math. 171, 485–541 (2008).

[5] D. M. Ambrose, J. L. Bona and T. Milgrom, *Global solutions and ill-posedness for the Kaup system and related Boussinesq systems*, Indiana Univ. Math. J. 68 (2019), no. 4, 1173–1198.

[6] C. J. Amick, *Regularity and uniqueness of solutions to the Boussinesq system of equations*, Journal of Differential Equations 54, 231–247 (1984).

[7] P. Bizoń, Tadeusz Chmaj, and Zbis law Tabor. *Dispersion and collapse of wave maps.* Nonlinearity, 13(4):1411-1423, 2000.

[8] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pure Appl. (2) 17 (1872), 55–108.

[9] L.V. Bogdanov and V.E. Zakharov, *The Boussinesq equation revisited*, Physica D: Nonlinear Phenomena 165 (2002), 137–162.

[10] E.S. Bao, R.M. Chen and Q. Liu, *Existence and Symmetry of Ground States to the Boussinesq abcd Systems.* Arch Rational Mech Anal 216, 569?591 (2015) https://doi.org/10.1007/s00205-014-0814-1.

[11] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear. Sci. Vol. 12: pp. 283–318 (2002).

[12] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory*, Nonlinearity 17 (2004) 925–952.

[13] J. L. Bona, T. Colin and D. Lannes, *Long wave approximations for water waves*, Arch. Ration. Mech. Anal. 178 (2005), no. 3, 373–410.

[14] I. Bejenaru and T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. 233 (2006) 228–259.

[15] J. L. Bona, and R. L. Sachs, *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation.* Comm. Math. Phys. 118 (1988), no. 1, 15–29.

[16] C. Burtea, *New long time existence results for a class of Boussinesq-type systems*, Journal des Mathematiques Pures et Appliquees, 106 (2) , pp. 203–236, (2016).

[17] R.M. Chen, Y. Liu, *On the ill-posedness of a weakly dispersive one-dimensional Boussinesq system*, JAMA 121, 299–316 (2013).

[18] E. Cerpa, and E. Crépeau, *On the control of the improved Boussinesq equation*, SIAM J. Control Optim., Vol. 56, No. 4, 2018, pp. 3035–3049.

[19] C. Charlier and J. Lennells, *The "good" Boussinesq equation: A Riemann-Hilbert approach*, https://arxiv.org/abs/2003.02777.

[20] R. Côte, C. Muñoz, D. Pilod, and G. Simpson, *Asymptotic stability of high-dimensional Zakharov–Kuznetsov solitons*, Arch Rational Mech Anal 220, 639–710 (2016).

[21] Y. Cho, T. Ozawa, *Remarks on Modified Improved Boussinesq Equations in One Space Dimension*, Proceedings: Mathematical, Physical and Engineering Sciences, Vol. 462, No. 2071 (2006), pp. 1949–1963.

[22] Chree, C. *Longitudinal Vibrations of a Circular Bar*. The Quarterly Journal of Pure and Applied Mathematics, 21, 287-298,(1886).

[23] R. Donninger, *On stable self-similar blow-up for equivariant wave maps.* Comm. Pure Appl. Math., 64(8):1095-1147, 2011.

[24] V. A. Dougalis, D. E. Mitsotakis and J.-C. Saut, *On some Boussinesq systems in two space dimensions: Theory and numerical analysis*, ESAIM: Mathematical Modelling and Numerical Analysis Vol. 41, No. 5. 2007, pp. 825–854.

[25] G.E. Fal'kovich, M.D. Spector, S.K. Turitsyn, *Destruction of stationary solutions and collapse in the nonlinear string equation*, Physics Letters A, 99, Issues 6-7, (1983), 271–274.

[26] Dan-Andrei Geba, K. Nakanishi and S. G. Rajeev. *Global well-posedness and scattering for Skyrme wave maps.* Commun. Pure Appl. Anal., 11(5) (2012), 1923-1933.

[27] M. Gell-Mann and M. Levy. *The axial vector current in beta decay.* Nuovo Cimento (10), 16:705-726, 1960.

[28] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, Journal of Functional Analysis, 74, 160-197 (1987).

[29] P. Grisvard, *Quelques propriétés des espaces de Sobolev, utiles dans l'étude des équations de Navier-Stokes (1)* Problèmes d'évolution, non linéaires, Séminaire de Nice (1974-1976).

[30] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations.* Journal of Mathematical Physics, 18(9), 1794-1797 (1977).

[31] V.K. Kalantarov, O.A. Ladyzhenskaya , *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types*, J. Sov. Math.10, 53–70 (1978).

[32] N. Kishimoto, *Sharp local well-posedness for the "good" Boussinesq equation*, J. Differential Equations, 254 (2013), 2393–2433.

[33] M. Kowalczyk, Y. Martel, and C. Muñoz, *Kink dynamics in the* ϕ^4 *model: Asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769-798.

[34] M. Kowalczyk, Y. Martel, and C. Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, to appear in J. Eur. Math. Soc.

[35] M. Kowalczyk, Y. Martel, C. Muñoz, and Hanne Van Den Bosch, *A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models*, `https://arxiv.org/abs/2008.01276`.

[36] C. Kwak, C. Maulén, *Ill-posedness issues on the* (*abcd*)*-Boussinesq system*, preprints.

[37] C. Kwak and C. Muñoz, *Asymptotic dynamics for the small data weakly dispersive one-dimensional Hamiltonian ABCD system*, Trans. Amer. Math. Soc. 373 (2020), 1043–1107.

[38] C. Kwak, C. Muñoz, *Extended Decay Properties for Generalized BBM Equations*, to appear in Fields Institute Comm. preprint 2018.

[39] C. Kwak, C. Muñoz, F. Poblete, and J.C. Pozo, *The scattering problem for the Hamiltonian abcd Boussinesq system in the energy space*, J. Math. Pures Appl. (9) 127 (2019), 121–159.

[40] D. Li. *Global wellposedness of hedgehog solutions for the (3+1) Skyrme model.* Preprint arXiv:1208.4977, 2012.

[41] L. V. Kapitanskii and O. A. Ladyzhenskaya. *The Coleman principle for finding stationary points of invariant functionals.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 127:84-102,1983. *Boundary value problems of mathematical physics*

and related questions in the theory of functions, 15.

[42] B. Li, M. Ohta, Y. Wu, and J. Xue, *Instability of the solitary waves for the generalized Boussinesq equations*, SIAM J. Math. Anal., 52(4), 3192–3221.

[43] F. Linares, *Global existence of small solutions for a generalized Boussinesq equation*, J. Diff. Eqns. 106 (1993), 257–293.

[44] F. Linares and G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Springer, Universitext, 2nd ed. 2015 Edition.

[45] F. Linares, *Notes on Boussinesq Equation*, available at `http://preprint.impa.br/FullText/Linares__Fri_Dec_23_09_48_59_BRDT_2005/beq.pdf`, 71pp. (2005).

[46] F. Linares, D. Pilod, and J.-C. Saut, *Well-Posedness of Strongly Dispersive Two-Dimensional Surface Wave Boussinesq Systems*, SIAM Journal on Mathematical Analysis 2012 44:6, 4195–4221.

[47] Y. Liu, *Existence and Blow up of Solutions of a Nonlinear Pochhammer-Chree Equation*, Indiana University Mathematics Journal Vol. 45, No. 3 (Fall, 1996), pp. 797–816.

[48] Y. Liu, *Instability and blow-up of solutions to a generalized Boussinesq equation*, SIAM J. Math. Anal., 26(6), 1527–1546.

[49] Scott Russell, *Report of waves*, 1844.

[50] J. Shatah. *Weak solutions and development of singularities of the SU(2) σ-model.* Comm. Pure Appl.Math., 41(4):459-469, 1988.

[51] T. H. R. Skyrme. *A non-linear field theory.* Proc. Roy. Soc. London Ser. A, 260:127-138, 1961.

[52] T. Tao, *Nonlinear Dispersive Equations: local and global analysis*, CBMS, 106 (2006).

[53] J. B. McLeod and W. C. Troy. *The Skyrme model for nucleons under spherical symmetry.* Proc. Roy.Soc. Edinburgh Sect. A, 118(3-4):271-288, 1991.

[54] V.G. Makhankov, *Dynamics of classical soliton*, Phys. Rep. C 35 (1) (1978) 1–128.

[55] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), no. 3, 219–254.

[56] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical gKdV equations revisited*, Nonlinearity, **18** (2005), no. 1, 55–80.

[57] Y. Martel, *Linear problems related to asymptotic stability of solitons of the generalized KdV equations*, SIAM J. Math. Anal., 38(3), 759-781.

[58] Y. Martel, and F. Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation,* J. Math. Pures Appl. (9) 79 (2000), no. 4, 339–425.

[59] C. Maulén, and C. Muñoz, *Decay in the one dimensional generalized Improved Boussinesq equation*, SN Partial Differential Equations and Applications volume 1, Article number: 1 (2020).

[60] S. Micu and A.F. Pazoto. *Stabilization of a Boussinesq system with localized damping*, JAMA 137, 291–337 (2019). https://doi.org/10.1007/s11854-018-0074-3.

[61] M. Ming, J. C. Saut and P. Zhang, *Long-Time Existence of Solutions to Boussinesq Systems*, SIAM J. Math. Anal., 44(6), 4078–4100.

[62] C. Muñoz, F. Poblete and J.C. Pozo, *Scattering in the energy space for Boussinesq equations*. Commun. Math. Phys. 361, 127–141 (2018).

[63] L. Pochhammer, *Ueber die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder*. Journal für die Reine und Angewandte Mathematik 81 (1876): 324-336.

[64] J. C. Saut, *Asymptotic models for surface and internal waves*, 29th. Coloquio Brasileiro de Matemática, Publicações Matemáticas IMPA, 2013.

[65] J.-C. Saut, C. Wang, and L. Xu *The Cauhy problem on large time for surface waves Bousinesq systems II*, `https://arxiv.org/abs/1511.08824v1`.

[66] J.-C. Saut and L. Xu, *The Cauchy problem on large time for surface waves Boussinesq systems*, J. Math. Pures Appl. 97(2012) 635–662.

[67] J.-C. Saut and L. Xu, *Long Time Existence for a Strongly Dispersive Boussinesq System*, SIAM Journal on Mathematical Analysis 2020 52:3, 2803–2848.

[68] J.-C. Saut and L. Xu, *Long time existence for the Boussinesq-Full dispersion systems*, Journal of Differential Equations, (2020) 269:3, 2627–2663.

[69] M. E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, Journal of Differential Equations 42, 325–352 (1981)

[70] S. Wang, G. Chen. *Small amplitude solutions of the generalized IMBq equation*. J. Math. Anal. Appl. 274(2002) 846–866.

[71] G. Whitham, *Linear and nonlinear waves*, Pure and Applied Mathematics, John Wiley, 1974, 636pp.

[72] V.E Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillations*, Soviet Phys. JEPT 38 (1974), 108-110.

Part II

Boussinesq Models

Chapter 2

Decay in the one dimensional generalized Improved Boussinesq equation

Abstract. We consider the decay problem for the generalized improved (or regularized) Boussinesq model with power type nonlinearity, a modification of the originally ill-posed shallow water waves model derived by Boussinesq. This equation has been extensively studied in the literature, describing plenty of interesting behavior, such as global existence in the space $H^1 \times H^2$, existence of super luminal solitons, and lack of a standard stability method to describe perturbations of solitons. The associated decay problem has been studied by Liu, and more recently by Cho-Ozawa, showing scattering in weighted spaces provided the power of the nonlinearity p is sufficiently large. In this paper we remove that condition on the power p and prove decay to zero in terms of the energy space norm $L^2 \times H^1$, for any $p > 1$, in two almost complementary regimes: (i) outside the light cone for all small, bounded in time $H^1 \times H^2$ solutions, and (ii) decay on compact sets of arbitrarily large bounded in time $H^1 \times H^2$ solutions. The proof consists in finding two new virial type estimates, one for the exterior cone problem based in the energy of the solution, and a more subtle virial identity for the interior cone problem, based in a modification of the momentum.

Contents

2.1 Introduction 28

2.1.1 Setting 28

2.1.2 Main results 30

2.1.3 More about solitons 32

2.2 Virial identities 34

2.3 Decay in exterior light cones. Proof of Theorem 2.1 37

2.3.1 End of proof of Theorem 2.1 41

2.4 Decay in compact sets: Proof of Theorem 2.2 42

2.4.1 End of proof of Theorem 2.2 44

Bibliography 46

This work has been recently published and can be found at SN Partial Differential Equations and Applications .

2.1 Introduction

2.1.1 Setting

This paper is concerned with the so-called generalized *Improved Boussinesq* equation (gIB) [25, 6]

$$\partial_t^2 u - \partial_x^2 \partial_t^2 u - \partial_x^2 u - \partial_x^2(|u|^{p-1}u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2.1}$$

where $u = u(t,x)$ is a real-valued function, and $p > 1$. Sometimes referred as the Pochhammer-Chree equation [19], this model was first introduced by Pochhammer [25] in its linear version in 1876, and in its complete nonlinear form by Chree [6], in 1886. It was derived as a model of the longitudinal vibration of an elastic rod, as well as a model of nonlinear waves in weakly dispersive media, and shallow water waves.

The model gIB (2.1) shares plenty of similarities with the so called *generalized good and bad Boussinesq* models [4]

$$\partial_t^2 u \pm \partial_x^4 u - \partial_x^2 u - \partial_x^2(|u|^{p-1}u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2.2}$$

Here the plus sign denotes the good Boussinesq system, which is locally and globally well-posed in standard Sobolev spaces [8, 9], and the *minus sign* represents the "bad" equation originally derived by Boussinesq [4], which is strongly linearly ill-posed. Precisely, motivated by the similar order of magnitude of ∂_x and ∂_t in shallow water waves, the linearized gIB model (2.1) was discussed by Whitham [29, p. 462]. By doing the "Boussinesq trick" (changing two ∂_x by two ∂_t) in the bad Boussinesq equation, one arrives to (2.1) and ill-posedness is no longer present. This regularization process leads to gIB (2.1), also known as the *regularized Boussinesq* equation.

Although (2.1) is no longer strongly ill-posed as bad Boussinesq (2.2), it is still shares some of its unpleasant behavior, but also some nice surprising properties. In order to explain this in detail, we write (2.1) as the system

$$\text{(gIB)} \quad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = (1-\partial_x^2)^{-1}\partial_x(u + |u|^{p-1}u). \end{cases} \tag{2.3}$$

This 2×2 system is Hamiltonian, but as far as we understand, not integrable. Its Hamiltonian character leads to the conservation of energy and momentum, given by

$$H(u,v) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1}\mathrm{d}x, \tag{2.4}$$

$$P(u,v) = \int_{\mathbb{R}} (uv + \partial_x u \partial_x v)\mathrm{d}x. \tag{2.5}$$

Note in particular the complex character of energy and momentum for gIB: the energy is always nonnegative, and makes sense e.g. for $u \in L^2 \cap L^{p+1}$, and $v \in H^1$. On the other hand, the momentum needs even more regularity than expected, and it is only well-defined for $(u,v) \in H^1 \times H^1$ (or $L^2 \times H^2$). Given this lack of concordance, completely contrary to classical linear waves, understanding the well-posedness problem in gIB is far from trivial.

Indeed, it turns out that $L^2 \times H^1$ seems not well suited to have a well-defined energy, so in this work we shall work in the proper subspace $H^1 \times H^2$, for the reasons explained below.

The pioneering work by Liu [19] showed local and global well-posedness for (2.1) for data $(u_0, v_0) \in H^s \times H^{s+1}$ and $s \geqslant 1$. In addition, the energy and momentum (2.4)-(2.5) are conserved by the flow, or in other words, the $L^2 \times H^1$ norm of the solution remains bounded in time. Note however that the $H^1 \times H^2$ norm of the solution need not be globally bounded in time. The method employed by Liu was essentially based in the Sobolev inclusion H^1 into L^∞ in one dimension, since no useful dispersive decay estimates are available for gIB. The fact that the solvability space differs from the energy space is a property standard in quasilinear models, and gIB has the flavor of a standard one. Consequently, we believe that this weakly ill-posed behavior in gIB is deeply motivated and inherited by the original strongly ill-posed bad Boussinesq equation (2.2). Additionally, Liu also showed blow up of negative energy solutions of (2.3) but with focusing nonlinearities (minus sign in $|u|^{p-1}u$ instead of plus sign). Finally, the controllability problem for gIB in a finite interval (2.3) has been recently studied by Cerpa and Crépeau [5].

However, the gIB system (2.3) also enjoys some nice properties. Indeed, this model is also characterized by the existence of *super-luminal* solitary waves, or just naively solitons, of the form

$$(u, v) = (Q_c, -cQ_c)(x - ct - x_0), \quad x_0 \in \mathbb{R}, \quad |c| > 1. \tag{2.6}$$

The super-luminal character is represented by the condition $|c| > 1$ on the speed. Here, the scaled soliton is slightly different from generalized Korteweg-de Vries (gKdV): $Q_c(s) = (c^2 - 1)^{1/(p-1)} Q\left(\sqrt{\frac{c^2-1}{c^2}}s\right)$, and

$$Q(s) = \left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)s}{2}\right)}\right)^{\frac{1}{p-1}} > 0 \tag{2.7}$$

is the soliton that solves $Q'' - Q + Q^p = 0$, $Q \in H^1(\mathbb{R})$. Note that Q_c must solve the modified elliptic equation

$$c^2 Q_c'' - (c^2 - 1)Q_c + Q_c^p = 0. \tag{2.8}$$

Since the speed of solitons can be arbitrarily large, it clearly implies that (2.3) *possesses infinite speed* of propagation, a fact not present in standard wave-like equations. Note also that solitons with speeds $|c| \downarrow 1$ are small in $L^\infty \cap H^1$, but they do not decay to zero as time evolves, in any standard norm.

In this paper, we are motivated by the decay problem of solutions to gIB (2.3). This interesting question has attracted the attention of several people before us. Liu [19] showed decay of solutions to (2.3) obtained from initial data satisfying e.g. (u_0, v_0) in $H^1 \times H^2$, $u_0 \in L^1$ and $(1 - \partial_x^2)^{1/2} v_0 \in L^1$, all of them small enough. In particular, he showed that for $p > 12$,

$$\sup_{t \geqslant 0} \left((1+t)^{\frac{1}{10}} \|u(t)\|_{L^\infty} + \|(u,v)(t)\|_{H^1 \times H^2}\right) < +\infty.$$

He also showed that p can be taken greater than 8 if $s > \frac{3}{2}$ and (u_0, v_0) in $H^s \times H^{s+1}$. Next, in [10], Wang and Chen extended this result to higher dimensions.

The exponent p in (2.3) was recently improved by Cho and Ozawa [11], who showed using modified scattering techniques that p can be taken greater than 9/2 if $u_0 \in H^s$, $s > \frac{8}{5}$. The solution global in this case satisfies $\|u(t)\|_{L^\infty} = O(t^{-2/5})$ as $t \to +\infty$. Additionally, the same authors showed that the asymptotics as $t \to +\infty$ cannot be the linear one if $1 < p \leqslant 2$ and near zero frequencies vanish at infinity. Lowering the exponent p for which there is decay seems a complicated problem, due to the quasilinear behavior of gIB.

2.1.2 Main results

In this paper, we are interested in the asymptotics of gIB solutions in the lower p case, namely any possible $p > 1$. Since there should be modified dynamics, we believe that we need different tools to attack this problem.

Our first result deals with the exterior light-cone decay problem. More precisely let $a, b > 0$ be arbitrary positive numbers. We consider the interval depending on time

$$I(t) = \big(-\infty, -(1+a)t\big) \cup \big((1+b)t, \infty\big), \qquad t > 0. \tag{2.9}$$

Our first result shows that, regardless the power $p > 1$, any global solution (u, v) to (2.3) which is sufficiently small and regular must concentrate inside the light cone.

Theorem 2.1 (Decay in exterior light cones)**.** *Let $(u, v) \in C(\mathbb{R}, H^1 \times H^2)$ be a global small solution of* (2.3) *such that, for some $\epsilon(a, b) > 0$ small, one has*

$$\sup_{t\in\mathbb{R}} \|(u(t), v(t))\|_{H^1\times H^2} < \epsilon. \tag{2.10}$$

Then, for $I(t)$ as in (2.9)*, there is strong decay to zero in the energy space:*

$$\lim_{t\to\infty} \|(u(t), v(t))\|_{(L^2\times H^1)(I(t))} = 0. \tag{2.11}$$

Additionally, one has the mild rate of decay for $|\sigma| > 1$:

$$\int_2^\infty \int_{\mathbb{R}} \mathrm{e}^{-c_0|x+\sigma t|}(u^2 + v^2 + (\partial_x v)^2)\mathrm{d}x\mathrm{d}t \lesssim_{c_0} \epsilon^2. \tag{2.12}$$

Remark 2.1. Note that Theorem 2.1 is sharp, since it does not persist in the large data case. Indeed, solitons (2.6) can be arbitrarily large and do not decay in the energy norm inside $I(t)$ as time tends to infinity.

Remark 2.2. The smallness condition (2.10) is needed in the proof to get a well-defined flow and good boundedness properties of the $L^\infty_{t,x}$ norm of u, and we do not know if it can be improved to $\|(u_0, v_0)\|_{L^2\times H^1} < \epsilon$ only. Note also that only the conditions $\|(u_0, v_0)\|_{L^2\times H^1} < \epsilon$ and $\sup_{t\in\mathbb{R}} \|(u(t), v(t))\|_{\dot H^1\times \dot H^2} < \epsilon$ are essentially needed in the proofs here.

The proof of Theorem 2.1 follows the introduction of a new virial identity, in the spirit of the previous results by Martel and Merle [20, 21] in the gKdV case, and [12, 1] in the BBM case. Note however that in those cases the functional involved is related to the mass (L^2 norm) of the solution. Here, we use instead a modification of the energy (2.4) of the solution.

Having described the small data behavior in exterior light cones, we concentrate now in the interior light cone behavior. Here things are much more complicated, since the energy (2.4) is no more useful to describe the dynamics. Instead, we shall use a suitable modification of the momentum (2.5).

Theorem 2.2 (Full decay in interior regions)**.** *Let (u, v) be a global solution of (2.3) in the class $C(\mathbb{R}, H^1 \times H^2) \cap L^\infty(\mathbb{R}, H^1 \times H^2)$, not necessarily small in norm. Then for any $L \gg 1$ we have*

$$\int_2^\infty \int_{-L}^{L} \left(v^2 + u(1-\partial_x^2)^{-1}u + |u|^{p+1}\right)(t,x)\mathrm{d}x\mathrm{d}t \lesssim 1. \tag{2.13}$$

Moreover, we have strong decay to zero in the energy space $(L^2 \times H^1)(I)$, for any I bounded interval in space:

$$\lim_{t\to\infty} \|(u,v)(t)\|_{(L^2\times H^1)(I)} = 0. \tag{2.14}$$

Remark 2.3. Estimate (2.13) shows that the local L^2 norm of v is integrable in time, and some mixed norms of u. Note however that u seems not locally L^2 integrable in time. However, (2.14) shows that this norm indeed decays to zero in time (even if it is not integrable in time).

Remark 2.4. The fact that explicit higher regularity is needed in Theorem 2.2 is certainly a consequence of a sort of quasilinear behavior of gIB. See also [2] for a similar behavior in a fully quasilinear model, the 1+1 dimensional Born-Infeld. In some sense, although gIB is well-posed, still shares some bad behavior coming from the originally ill-posed Bad Boussinesq model (2.2). Note that this phenomenon does not occur in the Good Boussinesq case [24]. Also, Theorem 2.2 can be read as "boundedness in time in $H^1 \times H^2$ implies $L^2 \times H^1$ time decay in compact sets of space".

Remark 2.5. Note that arbitrary size solitons (2.6)-(2.7) satisfy the hypotheses in Theorem 2.2. Hence, (2.14) it is also true for large solutions.

The techniques that we use to prove Theorem 2.2 are not new, and have been used to show decay for the Born-Infeld equation [2], the good Boussinesq system [24], the Benjamin-Bona-Mahony (BBM) equation [12], and more recently in the more complex *abc*d Boussinesq system [14, 13]. In all these works, suitable virial functionals were constructed to show decay to zero in compact/not compact regions of space.

However, the case of gIB is different by several reasons: first of all, the small data long time dynamics in all the aforementioned models is *not singular*, in the sense that using wellcooked virial identities, one always gets integrability in time of the whole associated energy norm. This is a nice property present in plenty of Hamiltonian models so far. The gIB case is different because this last property is not true at all: we only gets integrability in time of very particular portions of the $L^2 \times H^1$ norm (see (2.13)). This fact complicates matters, since proving (2.14) will require to prove additional estimates, not coming from the virial itself, but instead coming from tricky bounds and preservation of sign conditions under the nonlocal operator $(1-\partial_x^2)^{-1}$, namely the maximum principle. Second, finding the right virial identity for gIB was a very complicate process, since no clear notion of decay is shown by computing variations of energy and momentum. For instance, the derivation of a localized version of the momentum law (see (2.16))

$$\mathcal{J}(t) = \int_{\mathbb{R}} \varphi\left(\frac{x}{L}\right)(uv + \partial_x u \partial_x v)(t,x)\mathrm{d}x, \qquad L \gg 1,$$

leads to the badly behaved identity (see (2.39))

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) = &\ \frac{1}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + \frac{2}{p+1}|u|^{p+1} - v^2 - (\partial_x v)^2\right)\mathrm{d}x \\ &- \frac{1}{L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x.\end{aligned}$$

No evidence of good sign conditions is clearly shown here. This identity, valid only for the gIB case, is far from being useful (actually, it is the first case among the above mentioned equations where it fails to give decay information). The key to prove decay is an additional term in the virial, called $\mathcal{N}(t)$ (see (2.19)-(2.20)), that allows us to recover the positivity of the virial in (2.40):

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{J}(t) + \mathcal{N}(t)) = &\ \frac{1}{2L}\int_{\mathbb{R}} \varphi' v^2 \mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}u\mathrm{d}x \\ &+ \frac{p-1}{(p+1)L}\int_{\mathbb{R}} \varphi'|u|^{p+1}\mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x,\end{aligned}$$

and therefore the decay property. In that sense, we believe that estimate (2.40) is a true finding, since it is the only positivity property that we have found so far in gIB around compact sets. Note also that at this point we are not able to fully recover the decay of the L^2 norm of u locally in space, but instead only a portion of it, expressed in terms of the estimate $\int_1^\infty \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}u(t,x)\,\mathrm{d}x\mathrm{d}t < +\infty$. In order to prove the more demanding decay property (2.14), we need additional estimates where the hypothesis $(u,v) \in L^\infty(\mathbb{R}, H^1 \times H^2)$ seems to be essential. We have no direct clue about whether or not this condition is also necessary, but it seems to appear in several quasilinear models [2].

2.1.3 More about solitons

One important question left open in this paper is the stability/instability of solitons (2.6)-(2.7). But, as we shall explain below, this question is far from being trivial. However, we believe that part of the techniques introduced in this work could be useful to show a certain degree of (asymptotic) stability of the IB soliton.

Let us be more precise. In an influential work, Grillakis, Shatah, Strauss [7] (GSS) obtained sharp conditions for the orbital stability/instability of ground state solutions for a class of abstract Hamiltonian systems. This result was extended to another class of Hamiltonians of KdV type by Bona, Souganidis and Strauss [3]. Hamiltonian systems as the ones considered in [7] allow us to introduce the Lyapunov functional $F := H - cI$, where H is the Hamiltonian and I is functional generated by the translation invariance of the equation (usually, mass or momentum). Here, c is the corresponding speed of the solitary wave. The stability of the solitary wave is then reduced to the understanding of the second variation of F, in the sense that $\partial^2 F > 0$ leads to stability. Also, if the former positive condition is not satisfied, but the corresponding nonpositive manifold is spanned by two elements (directions) which are associated to the two degrees of freedom of the solitary waves (scaling and shifts), it is still is possible prove stability using $\partial^2 F$, but it is also necessary to restrict the class of perturbations to those which are orthogonal to the nonpositive directions.

Smereka in [28] studied the soliton of IB (2.1) and observed that this soliton fits into the class of abstract Hamiltonian system studied by GSS. However, it is not possible to apply the GSS method since an important hypothesis is not satisfied. In fact, he observed that $\partial^2 F$ is nonpositive on an infinite number of directions, where two of them can be associated to the point spectrum, and the remaining with the continuous spectrum. Therefore, GSS is useless in this case. However, the same author showed numerical evidence that if $\mathrm{d}I(Q_c)/\mathrm{d}c < 0$, then the solitary waves are stable, and if $\mathrm{d}I(Q_c)/\mathrm{d}c > 0$ the solitary waves seem to be unstable.

In a very important paper, Pego and Weinstein [26] proved (among other things) that Q_c is linearly exponentially unstable in H^1 when

$$1 < c^2 < \left(\frac{3(p-1)}{4 + 2(p-1)} \right)^2, \quad \text{with} \ \ p > 5.$$

Their method combines the use of the Evans function as well as ODE techniques. They also showed [27] that the linear equation around Q_c for $c \sim 1$ satisfies a **convective stability** property, based on the similarity of IB with KdV for small speeds. This result has been successfully adapted to a more general setting by Mizumachi in a series of works [22, 23]. Whether or not the asymptotic stability results by Martel and Merle [20, 21] can be applied to this case, is a challenging problem. An interesting result in this direction can be found in the recent work [18].

Organization of this chapter

This paper is organized as follows: Section 2.2 deals with two new virial identities introduced in this paper. Section 2.3 is devoted to the proof of Theorem 2.1. In Section 2.4 we prove Theorem 2.2.

Acknowledgments

We thank C. Kwak and M. A. Alejo for several important remarks that helped to improve a first draft of this paper.

2.2 Virial identities

In this section we present two new virial identities for the gIB equation (2.3). One is related with the exterior light cone behavior (Theorem 2.1), and the other is useful for understanding the compact in space region (Theorem 2.2).

Let $L > 0$ be a large parameter, and $\varphi = \varphi(x)$ be a smooth, bounded weight function, to be chosen later. For each $t, \sigma \in \mathbb{R}$ and $L > 0$, we consider the following functionals.

$$\mathcal{I}(t; L, \sigma) = \mathcal{I}(t) = \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{x+\sigma t}{L}\right)\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)(t,x)\mathrm{d}x, \tag{2.15}$$

$$\mathcal{J}(t; L, \sigma) = \mathcal{J}(t) = \int_{\mathbb{R}} \varphi\left(\frac{x+\sigma t}{L}\right)(uv + \partial_x u \partial_x v)(t,x)\mathrm{d}x. \tag{2.16}$$

Note that both functionals are generalizations of the energy and momentum introduced in (2.4)-(2.5), and they are well-defined if $(u, v) \in H^1 \times H^2$. This fact is essential for the proofs, and it is the key ingredient in both Theorems 2.1-2.2. The following result describes the time variation of both functionals.

Lemma 2.3 (Energy and Momentum local variations). *For any $t \in \mathbb{R}$, one has*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = &\ \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)\mathrm{d}x \\ &- \frac{1}{L}\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x,\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) = &\ \frac{\sigma}{L}\int_{\mathbb{R}} \varphi'(uv + \partial_x u \partial_x v)\mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + \frac{2}{p+1}|u|^{p+1} - v^2 - (\partial_x v)^2\right)\mathrm{d}x \\ &- \frac{1}{L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x.\end{aligned} \tag{2.18}$$

Proof of Lemma 2.3. We have two identities to prove.

Proof of (2.17). We compute using (2.3):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = &\ \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)\mathrm{d}x \\ &+ \underbrace{\int_{\mathbb{R}} \varphi\left(u\partial_t u + v\partial_t v + \partial_x v \partial_{xt}^2 v + |u|^{p-1}u\partial_t u\right)\mathrm{d}x}_{\mathcal{I}_1(t)}.\end{aligned}$$

We deal first with the term $\mathcal{I}_1$. We have from (2.3)

$$\begin{aligned}
\mathcal{I}_1(t) &= \int_{\mathbb{R}} \varphi \left(u\partial_t u + v\partial_t v + \partial_x v \partial_{xt}^2 v + |u|^{p-1}u\partial_t u \right) \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi \left(u + \partial_{xt}^2 v + |u|^{p-1}u \right) \partial_t u \mathrm{d}x + \int_{\mathbb{R}} \varphi v \partial_t v \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi \partial_t u (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x + \int_{\mathbb{R}} \varphi v (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)_x \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi \partial_t u (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x - \int_{\mathbb{R}} \partial_x(\varphi v) (1-\partial_x^2)^{-1}(u + |u|^{p-1}u) \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\varphi \partial_t u - \frac{\varphi' v}{L} - \varphi \partial_x v \right) (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x \\
&= -\frac{1}{L} \int_{\mathbb{R}} \varphi' v (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x.
\end{aligned}$$

Finally, using this last identity, and replacing in the derivative of $\mathcal{I}(t)$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = \frac{\sigma}{2L} \int_{\mathbb{R}} \varphi' \left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1} \right) \mathrm{d}x \\
- \frac{1}{L} \int_{\mathbb{R}} \varphi' v (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x,
\end{aligned}$$

as desired.

Proof of (2.18). The proof here is similar to the previous one. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) &= \frac{\sigma}{L} \int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v \right) \mathrm{d}x + \int_{\mathbb{R}} \varphi \left(\partial_t u v + u \partial_t v + \partial_{xt} u \partial_x v + \partial_x u \partial_{xt}^2 v \right) \mathrm{d}x \\
&= \frac{\sigma}{L} \int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v \right) \mathrm{d}x + \int_{\mathbb{R}} \varphi \left(\partial_x v v + u \partial_t v + \partial_x^2 v \partial_x v + \partial_x u \partial_{xt}^2 v \right) \mathrm{d}x \\
&= \frac{\sigma}{L} \int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v \right) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} \varphi \partial_x \left(v^2 + (\partial_x v)^2 \right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \varphi \left(u \partial_t v + \partial_x u \partial_{xt}^2 v \right) \mathrm{d}x \\
&= \frac{\sigma}{L} \int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v \right) \mathrm{d}x - \frac{1}{2L} \int_{\mathbb{R}} \varphi' \left(v^2 + (\partial_x v)^2 \right) \mathrm{d}x \\
&\quad + \underbrace{\int_{\mathbb{R}} \varphi \left(u \partial_t v + \partial_x u \partial_{xt}^2 v \right) \mathrm{d}x}_{\mathcal{J}_2(t)}.
\end{aligned}$$

Now, integrating by parts,

$$
\begin{aligned}
\mathcal{J}_2(t) &= \int_{\mathbb{R}} \varphi u \partial_t v \mathrm{d}x - \int_{\mathbb{R}} \partial_x(\varphi \partial_{xt}^2 v) u \mathrm{d}x = \int_{\mathbb{R}} \varphi u \partial_t v \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' \partial_{xt}^2 v u \mathrm{d}x - \int_{\mathbb{R}} \varphi \partial_{txx}^3 v u \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi u (1-\partial_x^2)\partial_t v \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' \partial_{xt}^2 v u \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi u (1-\partial_x^2)(1-\partial_x^2)^{-1}\partial_x(u+|u|^{p-1}u) \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' \partial_{xt}^2 v u \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi u \partial_x(u+|u|^{p-1}u) \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' \partial_{xt}^2 v u \mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi \partial_x \left(\frac{1}{2}u^2 + \frac{p}{p+1}|u|^{p+1}\right) \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' u \partial_{xt}^2 v \mathrm{d}x \\
&= -\frac{1}{L}\int_{\mathbb{R}} \varphi' \left(\frac{1}{2}u^2 + \frac{p}{p+1}|u|^{p+1}\right) \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}\partial_x^2(u+|u|^{p-1}u) \mathrm{d}x \\
&= -\frac{1}{L}\int_{\mathbb{R}} \varphi' \left(\frac{1}{2}u^2 + \frac{p}{p+1}|u|^{p+1}\right) \mathrm{d}x - \frac{1}{L}\underbrace{\int_{\mathbb{R}} \partial_x^2(\varphi' u)(1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x}_{\mathcal{J}_3(t)}.
\end{aligned}
$$

We consider now the term $\mathcal{J}_3(t)$:

$$
\begin{aligned}
\mathcal{J}_3(t) &= \int_{\mathbb{R}} ((\varphi' u)_{xx} - \varphi' u + \varphi' u)(1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x \\
&= -\int_{\mathbb{R}} (1-\partial_x^2)(\varphi' u)(1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x + \int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x \\
&= -\int_{\mathbb{R}} \varphi'(u^2+|u|^{p+1}) \mathrm{d}x + \int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) = &\frac{\sigma}{L}\int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v\right) \mathrm{d}x - \frac{1}{2L}\int_{\mathbb{R}} \varphi' \left(v^2 + (\partial_x v)^2\right) \mathrm{d}x \\
&- \frac{1}{L}\int_{\mathbb{R}} \varphi' \left(\frac{1}{2}u^2 + \frac{p}{p+1}|u|^{p+1}\right) \mathrm{d}x + \frac{1}{L}\int_{\mathbb{R}} \varphi'(u^2+|u|^{p+1}) \mathrm{d}x \\
&- \frac{1}{L}\int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x \\
=&\frac{\sigma}{L}\int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v\right) \mathrm{d}x - \frac{1}{2L}\int_{\mathbb{R}} \varphi' \left(v^2 + (\partial_x v)^2\right) \mathrm{d}x \\
&+ \frac{1}{L}\int_{\mathbb{R}} \varphi' \left(\frac{1}{2}u^2 + \frac{1}{p+1}|u|^{p+1}\right) \mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x \\
=&\frac{\sigma}{L}\int_{\mathbb{R}} \varphi' \left(uv + \partial_x u \partial_x v\right) \mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' \left(u^2 + \frac{2}{p+1}|u|^{p+1} - v^2 - (\partial_x v)^2\right) \mathrm{d}x \\
&- \frac{1}{L}\int_{\mathbb{R}} \varphi' u (1-\partial_x^2)^{-1}(u+|u|^{p-1}u) \mathrm{d}x.
\end{aligned}
$$

■

We introduce now a second functional. Let

$$\mathcal{N}(t) := \frac{1}{2L}\int_{\mathbb{R}} \varphi'\left(\frac{x}{L}\right) u\partial_x v \mathrm{d}x. \tag{2.19}$$

Lemma 2.4. *We have for* $\varphi' = \varphi'\left(\frac{x}{L}\right)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t) &= \frac{1}{2L}\int_{\mathbb{R}} \varphi'\Big((\partial_x v)^2 - u^2 + u(1-\partial_x^2)^{-1}u\Big)\mathrm{d}x \\ &\quad + \frac{1}{2L}\int_{\mathbb{R}} \varphi'\Big(-|u|^{p+1} + u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\Big)\mathrm{d}x. \end{aligned} \tag{2.20}$$

Proof. We compute using (2.3)

$$\begin{aligned} 2L\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varphi' u\partial_x v \mathrm{d}x = \int_{\mathbb{R}} \varphi'(\partial_t u \partial_x v + u\partial_{xt} v)\mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi'((\partial_x v)^2 + u(1-\partial_x^2)^{-1}\partial_x^2(u + |u|^{p-1}u))\mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi'(\partial_x v)^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(\partial_x^2 u - u + u + \partial_x^2(|u|^{p-1}u))\mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi'(\partial_x v)^2\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}u\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(\partial_x^2 - 1 + 1)(|u|^{p-1}u)\mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi'(\partial_x v)^2\mathrm{d}x - \int_{\mathbb{R}} \varphi' u^2\mathrm{d}x + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}u\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \varphi'|u|^{p+1}\mathrm{d}x + \int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x. \end{aligned}$$

The final result arrives after multiplication by $\frac{1}{2L}$. ■

2.3 Decay in exterior light cones. Proof of Theorem 2.1

In this Section we prove Theorem 2.1. Recall that we have from (2.17):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) &= \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi'\left(u^2 + v^2 + (\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)\mathrm{d}x \\ &\quad - \frac{1}{L}\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x. \end{aligned} \tag{2.21}$$

In what follows, fix $\sigma \in \mathbb{R}$ such that $|\sigma| > 1$. Controlling this last term requires some work. Indeed, we shall need the following definition (see [15, 14, 12] and references therein for more details)

Definition 2.5 (Canonical variable). *Let $u \in L^2$ be a fixed function. We say that f is canonical variable for u if f uniquely solves the equation*

$$(1-\partial_x^2)f = u, \quad f \in H^2(\mathbb{R}). \tag{2.22}$$

In this case, we denote $f = (1-\partial_x^2)^{-1}u$.

Using f as canonical variable for u, we obtain the following result:

Lemma 2.6. *One has*

$$\int_{\mathbb{R}} \varphi' u^2 \mathrm{d}x = \int_{\mathbb{R}} \varphi'(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2)\mathrm{d}x - \frac{1}{L^2}\int_{\mathbb{R}} \varphi''' f^2 \mathrm{d}x, \tag{2.23}$$

and

$$\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x = \int_{\mathbb{R}} \varphi' v(f + (1-\partial_x^2)^{-1}|u|^{p-1}u)\mathrm{d}x. \tag{2.24}$$

Proof. Computing,

$$\int_{\mathbb{R}} \varphi' u^2 \mathrm{d}x = \int_{\mathbb{R}} \varphi'(f - \partial_x^2 f)^2 \mathrm{d}x = \int_{\mathbb{R}} \varphi'(f^2 + (\partial_x^2 f)^2 - 2f\partial_x^2 f)\mathrm{d}x.$$

Integrating by parts, we have

$$\int_{\mathbb{R}} \varphi' f \partial_x^2 f \mathrm{d}x = -\int_{\mathbb{R}} \varphi'(\partial_x f)^2 \mathrm{d}x + \frac{1}{2L^2}\int_{\mathbb{R}} \varphi''' f^2 \mathrm{d}x.$$

Therefore,

$$\int_{\mathbb{R}} \varphi' u^2 \mathrm{d}x = \int_{\mathbb{R}} \varphi'(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2)\mathrm{d}x - \frac{1}{L^2}\int_{\mathbb{R}} \varphi''' f^2 \mathrm{d}x.$$

This proves (2.23). The proof of (2.24) is direct. ■

Using Lemma 2.6 we can rewrite (2.21) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = \mathcal{Q}(t) + \mathcal{SQ}(t) + \mathcal{PQ}(t),$$

where

$$\mathcal{Q}(t) := \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi'\left(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2 + v^2 + (\partial_x v)^2\right)\mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' v f \mathrm{d}x, \tag{2.25}$$

$$\mathcal{SQ}(t) := -\frac{\sigma}{2L^3}\int_{\mathbb{R}} \varphi''' f^2 \mathrm{d}x, \tag{2.26}$$

$$\mathcal{PQ}(t) := \frac{\sigma}{L(p+1)}\int_{\mathbb{R}} \varphi' |u|^{p+1}\mathrm{d}x - \frac{1}{L}\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}|u|^{p-1}u \mathrm{d}x. \tag{2.27}$$

Now we are ready to prove a first virial estimate.

Lemma 2.7. *Assume $\sigma = -(1+b) < -1$ and $\varphi = \tanh$. Then one has*

$$\mathcal{Q}(t) \lesssim_{L,b} -\int_{\mathbb{R}} \varphi' \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x. \tag{2.28}$$

Similarly, assume now $\sigma = 1 + a > 1$ and $\varphi = -\tanh$. Then one has

$$\mathcal{Q}(t) \lesssim_{L,a} -\int_{\mathbb{R}} |\varphi'| \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x. \tag{2.29}$$

Proof. First we prove (2.28). We concentrate on $\mathcal{Q}(t)$ in (2.25). Note that, if $\varphi' > 0$, we have

$$\left|\int_{\mathbb{R}} \varphi' v f \mathrm{d}x\right| \leqslant \frac{1}{2}\int_{\mathbb{R}} \varphi' v^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \varphi' f^2 \mathrm{d}x.$$

Consequently, if $b > 0$, $\sigma := -(1+b) < -1$, and $\varphi = \tanh$, we have in (2.25)

$$\begin{aligned}
\mathcal{Q}(t) \leqslant{}& \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi' \left(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x \\
&+ \frac{1}{2L}\int_{\mathbb{R}} \varphi' v^2 \mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' f^2 \mathrm{d}x \\
={}& \frac{\sigma+1}{2L}\int_{\mathbb{R}} \varphi' \left(f^2 + v^2\right) \mathrm{d}x + \frac{\sigma}{2L}\int_{\mathbb{R}} \varphi' \left(2(\partial_x f)^2 + (\partial_x^2 f)^2 + (\partial_x v)^2\right) \mathrm{d}x \\
={}& \frac{-b}{2L}\int_{\mathbb{R}} \varphi' \left(f^2 + v^2\right) \mathrm{d}x - \frac{(1+b)}{2L}\int_{\mathbb{R}} \varphi' \left(2(\partial_x f)^2 + (\partial_x^2 f)^2 + (\partial_x v)^2\right) \mathrm{d}x.
\end{aligned}$$

Now we need the following result about equivalence of norms in terms of f and u.

Lemma 2.8 ([12, 14])**.** *Let f be as in (2.22). Let $\tilde{\varphi}$ be a positive, smooth, bounded weight function satisfying $|\tilde{\varphi}'| \leqslant \lambda\tilde{\varphi}$ for some small but fixed $0 < \lambda \ll 1$. Then, for any $a_1, a_2, a_3 > 0$, there exist $c_1, C_1 > 0$, depending of a_j and $\lambda > 0$, such that*

$$c_1 \int_{\mathbb{R}} \tilde{\varphi} u^2 \mathrm{d}x \leqslant \int_{\mathbb{R}} \tilde{\varphi}\left(a_1 f^2 + a_2(\partial_x f)^2 + a_3(\partial_x^2 f)^2\right) \mathrm{d}x \leqslant C_1 \int_{\mathbb{R}} \tilde{\varphi} u^2 \mathrm{d}x. \tag{2.30}$$

Using Lemma 2.8 with $\lambda = L^{-1} \ll 1$ and $\tilde{\varphi} = \varphi'$, we conclude

$$\begin{aligned}
\mathcal{Q}(t) \lesssim_{L,b} & -\int_{\mathbb{R}} \varphi'(f^2 + (\partial_x f)^2 + (\partial_x^2 f)^2 + v^2 + (\partial_x v)^2) \\
\sim & -\int_{\mathbb{R}} \varphi' \left(u^2 + v^2 + (\partial_x v)^2\right).
\end{aligned} \tag{2.31}$$

This proves (2.28). Now we sketch the proof of (2.29), which is similar to the previous case. Set $\sigma = 1 + a$, $a > 0$. Choosing $\varphi = -\tanh$ it is clear that $\varphi' = -\operatorname{sech}^2 < 0$. From (2.25) we have

$$\begin{aligned}
\mathcal{Q}(t) &= -\frac{|\sigma|}{2L}\int_{\mathbb{R}} |\varphi'| \left(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x + \frac{1}{L}\int_{\mathbb{R}} |\varphi'| v f \mathrm{d}x \\
&\leqslant -\frac{|\sigma|}{2L}\int_{\mathbb{R}} |\varphi'| \left(f^2 + 2(\partial_x f)^2 + (\partial_x^2 f)^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x \\
&\quad + \frac{1}{2L}\int_{\mathbb{R}} |\varphi'| v^2 \mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} |\varphi'| f^2 \mathrm{d}x \\
&= -\frac{a}{2L}\int_{\mathbb{R}} |\varphi'| \left(f^2 + v^2\right) \mathrm{d}x - \frac{|\sigma|}{2L}\int_{\mathbb{R}} |\varphi'| \left(2(\partial_x f)^2 + (\partial_x^2 f)^2 + (\partial_x v)^2\right) \mathrm{d}x.
\end{aligned}$$

Therefore, by Lemma 2.8 again,

$$\mathcal{Q}(t) \lesssim_{L,a} -\int_{\mathbb{R}} |\varphi'| \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x.$$

This ends the proof of (2.29). ■

Now we consider the two terms in (2.26) and (2.27). First of all, note that in both cases (2.28) and (2.29),

$$\left|\frac{\sigma}{2L^3}\int_{\mathbb{R}} \varphi''' f^2 \mathrm{d}x\right| \lesssim \frac{1}{L^3}\int_{\mathbb{R}} |\varphi'| f^2 \mathrm{d}x \lesssim \frac{1}{L^3}\int_{\mathbb{R}} |\varphi'| u^2 \mathrm{d}x.$$

Therefore, for L large enough,

$$\mathcal{Q}(t) + \mathcal{SQ}(t) \lesssim_{L,a,b} -\int_{\mathbb{R}} |\varphi'| \left(u^2 + v^2 + (\partial_x v)^2\right) \mathrm{d}x. \tag{2.32}$$

Finally, note that in both cases (2.28) and (2.29),

$$\frac{\sigma}{L(p+1)}\int_{\mathbb{R}} \varphi' |u|^{p+1} \mathrm{d}x = -\frac{|\sigma|}{L(p+1)}\int_{\mathbb{R}} |\varphi'| |u|^{p+1} \mathrm{d}x \leqslant 0.$$

Finally, we deal with the last term in (2.27):

$$\begin{aligned}
\left|\frac{1}{L}\int_{\mathbb{R}} \varphi' v (1-\partial_x^2)^{-1} |u|^{p-1} u \mathrm{d}x\right| \leqslant{}& \frac{\max\{a,b\}}{8L}\int_{\mathbb{R}} |\varphi'| v^2 \\
&+ \frac{C}{\max\{a,b\} L}\int_{\mathbb{R}} |\varphi'| \left((1-\partial_x^2)^{-1} |u|^{p-1} u\right)^2.
\end{aligned} \tag{2.33}$$

The first term on the RHS can be absorbed by $\mathcal{Q}(t)$ in (2.32). In what follows, we need the following auxiliary result.

Lemma 2.9 ([15], see also [14]). *The operator $(1-\partial_x^2)^{-1}$ satisfies the comparison priciple: for any $u, v \in H^1$*

$$v \leqslant w \implies (1-\partial_x^2)^{-1} v \leqslant (1-\partial_x^2)^{-1} w. \tag{2.34}$$

Now, coming back to (2.33), suppose $u \geqslant 0$. Then $0 \leqslant |u|^{p-1}u \leqslant \|u\|_{L^\infty}^{p-1} u$, so that using (2.10) (this is the only place where we use this hypothesis)

$$0 \leqslant (1-\partial_x^2)^{-1}(|u|^{p-1}u) \leqslant \|u\|_{L^\infty}^{p-1}(1-\partial_x^2)^{-1}u \lesssim_{a,b} \varepsilon^{p-1} f.$$

(Note that ε depends on a, b.) Consequently, in this region

$$\begin{aligned} |\varphi'|((1-\partial_x^2)^{-1}(|u|^{p-1}u))^2 &= |\varphi'|((1-\partial_x^2)^{-1}(|u|^{p-1}u))((1-\partial_x^2)^{-1}(|u|^{p-1}u)) \\ &\lesssim \varepsilon^{2(p-1)}|\varphi'|f^2. \end{aligned}$$

If now $u < 0$, just note that

$$|\varphi'|((1-\partial_x^2)^{-1}(|u|^{p-1}u))^2 = |\varphi'|((1-\partial_x^2)^{-1}(|-u|^{p-1}(-u)))^2,$$

which leads to the previous case. Finally, we conclude that the second term on the RHS of (2.33) is bounded by

$$\frac{C}{\max\{a,b\}L}\int_{\mathbb{R}} |\varphi'|\left((1-\partial_x^2)^{-1}|u|^{p-1}u\right)^2 \lesssim \frac{\varepsilon^{2(p-1)}}{L}\int_{\mathbb{R}} |\varphi'|f^2 \mathrm{d}x$$

Consequently, for ε small we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = \mathcal{Q}(t) + \mathcal{SQ}(t) + \mathcal{PQ}(t) \lesssim_{L,a,b} -\int_{\mathbb{R}} |\varphi'|\left(u^2+v^2+(\partial_x v)^2\right)\mathrm{d}x. \tag{2.35}$$

Integrating in time, we have proved (2.12) in Theorem 2.1.

2.3.1 End of proof of Theorem 2.1

Now we conclude the proof of Theorem 2.1. It only remains to prove (2.11). First, we prove decay in the right hand side region, namely $((1+b)t, +\infty)$, $b > 0$. Now we choose $\varphi(x) = \frac{1}{2}(1+\tanh(x))$, $\sigma = -(1+b)$, $\tilde{\sigma} = -(1+\tilde{b})$ with $b > 0$ and $\tilde{b} = b/2$. Consider the modified energy functional, for $t \in [2, t_0]$:

$$\mathcal{I}_{t_0}(t) := \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{x+\sigma t_0 - \tilde{\sigma}(t_0-t)}{L}\right)\left(u^2+v^2+(\partial_x v)^2 + \frac{2}{p+1}|u|^{p+1}\right)\mathrm{d}x. \tag{2.36}$$

Note that $\sigma < \tilde{\sigma} < 0$. From Lemma 2.3 and proceeding exactly as in (2.35), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{t_0}(t) \lesssim_{b,L} -\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x+\sigma t_0 - \tilde{\sigma}(t_0-t)}{L}\right)\left(u^2+v^2+(\partial_x v)^2\right)\mathrm{d}x \leqslant 0 \tag{2.37}$$

what it means that the new functional $\mathcal{I}_{t_0}$ is decreasing in $[2, t_0]$. Therefore, we have

$$\int_2^{t_0} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{t_0}(t)\mathrm{d}t = \mathcal{I}_{t_0}(t_0) - \mathcal{I}_{t_0}(2) \leqslant 0 \implies \mathcal{I}_{t_0}(t_0) \leqslant \mathcal{I}_{t_0}(2).$$

On the other hand, since $\lim_{x\to-\infty}\varphi(x) = 0$, we have

$$\limsup_{t\to\infty}\int_{\mathbb{R}} \varphi\left(\frac{x-\beta t-\gamma}{L}\right)\left(u^2+v^2+(\partial_x v)^2\right)(\delta, x)\mathrm{d}x = 0, \tag{2.38}$$

for $\beta, \gamma, \delta > 0$ fixed. This yields

$$
\begin{aligned}
0 \leqslant \int_{\mathbb{R}} \varphi\left(\frac{x-(1+b)t_0}{L}\right)\left(u^2+v^2+(\partial_x v)^2\right)(t_0,x)\mathrm{d}x \\
\leqslant \int_{\mathbb{R}} \varphi\left(\frac{x-\frac{b}{2}t_0-(2+b)}{L}\right)\left(u^2+v^2+(\partial_x v)^2\right)(2,x)\mathrm{d}x,
\end{aligned}
$$

which implies,

$$
\limsup_{t\to\infty} \int_{\mathbb{R}} \varphi\left(\frac{x-(1+b)t}{L}\right)\left(u^2+v^2+(\partial_x v)^2\right)(t,x)\mathrm{d}x = 0.
$$

In view of (2.35), an analogous argument can be applied for the left side, i.e $(-\infty, -(1+a)t)$, but in this case we choose $\varphi(x) = \frac{1}{2}(1-\tanh(x))$. The proof of (2.11) is complete.

2.4 Decay in compact sets: Proof of Theorem 2.2

Let us find the key virial estimate to understand the dynamics on compact sets in space. Recall that from (2.18), if $\sigma = 0$, we have the identity on $\mathcal{J}$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) = \frac{1}{2L}\int_{\mathbb{R}} \varphi'\left(u^2+\frac{2}{p+1}|u|^{p+1}-v^2-(\partial_x v)^2\right)\mathrm{d}x \\
-\frac{1}{L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(u+|u|^{p-1}u)\mathrm{d}x.
\end{aligned}
\tag{2.39}
$$

Assume that $\varphi' > 0$. From (2.39) and (2.20) we obtain the positivity estimate

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{J}(t)+\mathcal{N}(t)) = \frac{1}{2L}\int_{\mathbb{R}} \varphi' v^2\mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}u\mathrm{d}x \\
+\frac{p-1}{(p+1)L}\int_{\mathbb{R}} \varphi'|u|^{p+1}\mathrm{d}x + \frac{1}{2L}\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x.
\end{aligned}
\tag{2.40}
$$

Note the surprising fact that each term in the RHS above is *nonnegative*.

Lemma 2.10. *For any φ bounded smooth and such that $\varphi' > 0$, and for any $u \in H^1(\mathbb{R})$,*

$$
\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x \geqslant 0. \tag{2.41}
$$

Remark 2.6. Note that this result is independent of the size of u. Note also that in order to prove this lemma, we need at least $u \in H^{1/2+}(\mathbb{R})$. Therefore, we see that $(u,v) \in L^2 \times H^1$ (the energy space) seems not sufficient for our purposes.

Proof of Lemma 2.10. Suppose $\varphi' > 0$. If $u \geqslant 0$, we have $|u|^{p-1}u \geqslant 0$. From Lemma 2.9 we conclude

$$
\int_{\mathbb{R}} \varphi' u(1-\partial_x^2)^{-1}(|u|^{p-1}u)\mathrm{d}x \geqslant 0.
$$

In a similar way, if the case $u \leqslant 0$ follows. ■

From the previous Lemma and (2.41) we obtain

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{J}(t)+\mathcal{N}(t))\\
&\qquad \geqslant \frac{1}{2L}\int_{\mathbb{R}}\varphi' v^2\mathrm{d}x+\frac{1}{2L}\int_{\mathbb{R}}\varphi' u(1-\partial_x^2)^{-1}u\mathrm{d}x+\frac{p-1}{(p+1)L}\int_{\mathbb{R}}\varphi'|u|^{p+1}\mathrm{d}x.
\end{aligned}
\tag{2.42}
$$

This last estimate tells us exactly what are the quantities in gIB which integrate in time. As far as we could understand, it was not possible to get integrability in time of the L^2 norm of u, nor $\partial_x v$. A corollary from this last estimate is the following result.

Corollary 2.11. *Let (u,v) be a global solution of* (2.3) *in the class $(C\cap L^\infty)(\mathbb{R},H^1\times H^2)$, with initial data $(u,v)(t=0)=(u_0,v_0)\in H^1\times H^2$. Let $\varphi(x):=\tanh(x)$ in* (2.40)*, such that $\varphi'=\operatorname{sech}^2>0$. Then we have the following consequences of* (2.42)*:*

1. *Integrability in time:*

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(v^2+u(1-\partial_x^2)^{-1}u+|u|^{p+1})\mathrm{d}x\mathrm{d}t\lesssim_{u_0,v_0,L}1. \tag{2.43}
$$

2. *Sequential decay to zero: there exists $t_n\uparrow\infty$ such that*

$$
\lim_{n\to\infty}\mathcal{I}(t_n)=0. \tag{2.44}
$$

Remark 2.7. Note that the smallness condition on (u,v) is not needed here; only boundedness in time of the $H^1\times H^2$ norm. Also, property (2.44) is not trivially obtained from (2.43) as in previous works; some additional estimates are needed in order to ensure full decay along a subsequence of the local energy norm present in $\mathcal{I}(t_n)$.

Remark 2.8 (About the equivalence of norms for canonical variables)**.** Note that if $f=(1-\partial_x^2)^{-1}u$, then for L large,

$$
\begin{aligned}
\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)u(1-\partial_x^2)^{-1}u\mathrm{d}x &= \int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)f(f-\partial_x^2 f)\mathrm{d}x\\
&= \int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(f^2+(\partial_x f)^2)\mathrm{d}x+\frac{1}{L}\int_{\mathbb{R}}(\operatorname{sech}^2)'\left(\frac{x}{L}\right)f\partial_x f\mathrm{d}x\\
&\gtrsim \frac{1}{2}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(f^2+(\partial_x f)^2)\mathrm{d}x.
\end{aligned}
$$

Consequently, from (2.43),

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(f^2+(\partial_x f)^2)\mathrm{d}x\mathrm{d}t\lesssim_{u_0,v_0,L}1.
$$

This information will be useful in what follows.

Proof of Corollary 2.11. Estimate (2.43) is direct from (2.42). On the other hand, from (2.43) we clearly have the existence of an increasing sequence $t_n \uparrow \infty$ such that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{L}\right)(v^2 + u(1-\partial_x^2)^{-1}u + |u|^{p+1})(t_n, x)\mathrm{d}x\mathrm{d}t = 0.$$

From this fact and the L^∞ boundedness in time of u we easily have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{L}\right) u^2(t_n, x)\mathrm{d}x = 0.$$

Indeed, from Remark 2.8 we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{L}\right)(f^2 + (\partial_x f)^2)(t_n, x)\mathrm{d}x = 0.$$

Hence, using interpolation, and $L \gg 1$,

$$\begin{aligned}\left\|\operatorname{sech}^2\left(\frac{x}{L}\right)\partial_x^2 f(t_n)\right\|_{L^2}^2 &\lesssim \left\|\operatorname{sech}^2\left(\frac{x}{L}\right)\partial_x f(t_n)\right\|_{L^2}\left\|\partial_x^3\left(\operatorname{sech}^2\left(\frac{x}{L}\right)f(t_n)\right)\right\|_{L^2}\\ &\lesssim \left\|\operatorname{sech}^2\left(\frac{x}{L}\right)\partial_x f(t_n)\right\|_{L^2},\end{aligned}$$

therefore we obtain the desired result. Finally, again from interpolation, the boundedness of $v(t_n)$ in H^2, $L \gg 1$, we have the estimate

$$\begin{aligned}\left\|\operatorname{sech}^2\left(\frac{x}{L}\right)\partial_x v(t_n)\right\|_{L^2}^2 &\lesssim \left\|\operatorname{sech}^2\left(\frac{x}{L}\right)v(t_n)\right\|_{L^2}\left\|\partial_x^2\left(\operatorname{sech}^2\left(\frac{x}{L}\right)v(t_n)\right)\right\|_{L^2}\\ &\lesssim \left\|\operatorname{sech}^2\left(\frac{x}{L}\right)v(t_n)\right\|_{L^2},\end{aligned}$$

so $\|\operatorname{sech}^2\left(\frac{x}{L}\right)\partial_x v(t_n)\|_{L^2} \to 0$ as $n \to +\infty$. This proves (2.44). ■

2.4.1 End of proof of Theorem 2.2

Consider $\mathcal{I}(t)$ in (2.15) with $\sigma = 0$, $\varphi = \operatorname{sech}^2$. From (2.17) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = -\frac{1}{L}\int_{\mathbb{R}} \varphi' v(1-\partial_x^2)^{-1}(u + |u|^{p-1}u)\mathrm{d}x.$$

Let $g := (1-\partial_x^2)^{-1}(u + |u|^{p-1}u)$, so that $g = f + (1-\partial_x^2)^{-1}(|u|^{p-1}u)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = -\frac{1}{L}\int_{\mathbb{R}} \varphi' v g\mathrm{d}x = -\frac{1}{L}\int_{\mathbb{R}} \varphi' v\left(f + (1-\partial_x^2)^{-1}(|u|^{p-1}u)\right)\mathrm{d}x.$$

Therefore,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t)\right| \lesssim \frac{1}{L}\int_{\mathbb{R}} \operatorname{sech}^2\left(\frac{x}{L}\right)(v^2 + f^2 + ((1-\partial_x^2)^{-1}(|u|^{p-1}u))^2)\mathrm{d}x. \tag{2.45}$$

Suppose $u \geqslant 0$. Then $0 \leqslant |u|^{p-1}u \leqslant \|u\|_{L^\infty}^{p-1}u$, so that from Lemma 2.9

$$0 \leqslant (1-\partial_x^2)^{-1}(|u|^{p-1}u) \leqslant \|u\|_{L^\infty}^{p-1}(1-\partial_x^2)^{-1}u \lesssim f.$$

(Here we use the boundedness character of u in L^∞.) Consequently, in this region

$$\begin{aligned}\operatorname{sech}^2\left(\frac{x}{L}\right)((1-\partial_x^2)^{-1}(|u|^{p-1}u))^2 =& \operatorname{sech}^2\left(\frac{x}{L}\right)((1-\partial_x^2)^{-1}(|u|^{p-1}u))((1-\partial_x^2)^{-1}(|u|^{p-1}u))\\ \lesssim& \operatorname{sech}^2\left(\frac{x}{L}\right)f^2.\end{aligned}$$

If now $u<0$, just note that

$$\operatorname{sech}^2\left(\frac{x}{L}\right)((1-\partial_x^2)^{-1}(|u|^{p-1}u))^2 = \operatorname{sech}^2\left(\frac{x}{L}\right)((1-\partial_x^2)^{-1}(|-u|^{p-1}(-u)))^2,$$

which leads to the previous case. Finally, we conclude that (2.45) is bounded by

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t)\right| \lesssim \frac{1}{L}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(v^2+f^2)\mathrm{d}x.$$

Integrating in $[t,t_n]$, we have (using Remark 2.8)

$$|\mathcal{I}(t)-\mathcal{I}(t_n)| \lesssim \int_t^{t_n}\frac{1}{L}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(v^2+f^2)\mathrm{d}x\mathrm{d}t.$$

Sending n to infinity, we have from (2.44) that $\mathcal{I}(t_n)\to 0$ and

$$|\mathcal{I}(t)| \lesssim \int_t^{\infty}\frac{1}{L}\int_{\mathbb{R}}\operatorname{sech}^2\left(\frac{x}{L}\right)(v^2+f^2)\mathrm{d}x\mathrm{d}t.$$

Finally, if $t\to\infty$, we conclude. Since for $L^2\times H^1$ data $\mathcal{I}(t)\gtrsim \|(u,v)(t)\|^2_{L^2\times H^1}$, this proves Theorem 2.2.

Bibliography

[1] M. A. Alejo, F. Cortez, C. Kwak and C. Muñoz, *On the dynamics of zero-speed solutions for Camassa-Holm type equations*, preprint arXiv:1810.09594, to appear in IMRN.

[2] M. A. Alejo, and C. Muñoz, *Almost sharp nonlinear scattering in one-dimensional Born-Infeld equations arising in nonlinear Electrodynamics*, Proc. AMS 146 (2018), no. 5, 2225–2237.

[3] J. Bona, P. Souganidis, and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. R. Soc. Lond. A 1987 411, 395–412.

[4] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pure Appl. (2) 17 (1872), 55–108.

[5] E. Cerpa, and E. Crépeau, *On the control of the improved Boussinesq equation*, SIAM J. Control Optim., Vol. 56, No. 4, 2018, pp. 3035–3049.

[6] Chree, C. *Longitudinal Vibrations of a Circular Bar*. The Quarterly Journal of Pure and Applied Mathematics, 21, 287-298,(1886).

[7] M. Grillakis, J. Shatah, W. Strauss, *Stability theory of solitary waves in the presence of symmetry*. I. J. Funct. Anal. 74 (1987), no. 1, 160-197.

[8] F. Linares, *Notes on Boussinesq Equation*, available at `http://preprint.impa.br/FullText/Linares__Fri_Dec_23_09_48_59_BRDT_2005/beq.pdf`, 71pp. (2005).

[9] N. Kishimoto, *Sharp local well-posedness for the "good" Boussinesq equation*. J. Differ. Eqns. 254, 2393–2433 (2013).

[10] S. Wang, G. Chen. *Small amplitude solutions of the generalized IMBq equation*. J. Math. Anal. Appl. 274(2002) 846–866.

[11] Y. Cho, T. Ozawa, *Remarks on Modified Improved Boussinesq Equations in One Space Dimension*, Proceedings: Mathematical, Physical and Engineering Sciences, Vol. 462, No. 2071 (2006), pp. 1949–1963.

[12] C. Kwak, C. Muñoz, *Extended Decay Properties for Generalized BBM Equations*, to appear in Fields Institute Comm. preprint 2018.

[13] C. Kwak, and C. Muñoz, *Asymptotic dynamics for the small data weakly dispersive one-dimensional hamiltonian ABCD system*, preprint arXiv:1902.00454. To appear in T. of the AMS.

[14] C. Kwak, C. Muñoz, F. Poblete, and J.C. Pozo, *The scattering problem for the Hamiltonian abcd Boussinesq system in the energy space*, J. Math. Pures Appl. (9) 127 (2019), 121–159.

[15] K. El Dika, *Smoothing effect of the generalized BBM equation for locelized solutions moving to the right*. Discrete and Contin. Dyn. Syst. 12(2005), no 5, 973–982.

[16] M. Kowalczyk, Y. Martel, and C Muñoz, *Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space*, J. of the AMS 30 (2017), 769–798.

[17] M. Kowalczyk, Y. Martel, and C Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, L. Math. Phys. May 2017, Vol. 107, 5, pp. 921–931.

[18] M. Kowalczyk, Y. Martel, and C Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, preprint arXiv:1903.12460.

[19] Y. Liu, *Existence and Blow up of Solutions of a Nonlinear Pochhammer-Chree Equation*, Indiana University Mathematics Journal Vol. 45, No. 3 (Fall, 1996), pp. 797–816.

[20] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), no. 3, 219–254.

[21] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical gKdV equations revisited*, Nonlinearity, **18** (2005), no. 1, 55–80.

[22] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$, Mem. Amer. Math. Soc. 238 (2015), no. 1125, vii+95 pp.

[23] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$*. II.* Proc. Roy. Soc. Edinburgh Sect. A 148 (2018), no. 1, 149–198.

[24] C. Muñoz, F. Poblete, and J.C. Pozo, *Scattering in the energy space for Boussinesq equations*, Comm. Math. Phys. 361 (2018), no. 1, 127–141.

[25] L. Pochhammer, *Ueber die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder*. Journal für die Reine und Angewandte Mathematik 81 (1876): 324-336.

[26] R. Pego, M. Weinstein, *Eigenvalues, and instabilities of solitary waves*. Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[27] R. Pego, M. Weinstein, *Convective Linear Stability of Solitary Waves for Boussinesq Equations*, Studies in Applied Mathematics, 99, pp. 311-375 (1997).

[28] P. Smereka, *A remark on the solitary wave stability for a Boussinesq equation*. Nonlinear

dispersive wave systems (Orlando, FL, 1991), 255-263, World Sci. Publ., River Edge, NJ, 1992.

[29] Whitham, *Linear and nonlinear waves*, Pure and Applied Mathematics, John Wiley, 1974, 636pp.

Chapter 3

Ill-posedness issues on $(abc\mathrm{d})$-Boussinesq system

Abstract. In this paper, we consider the Cauchy problem for $(abc\mathrm{d})$-Boussinesq system posed on one- and two-dimensional Euclidean spaces. This model, initially introduced by Bona, Chen, and Saut [5, 6], describes a small-amplitude waves on the surface of an inviscid fluid, and is derived as a first order approximation of incompressible, irrotational Euler equations. We mainly establish the ill-posedness of the system under various parameter regimes, which generalize the result of one-dimensional BBM-BBM case by Chen and Liu [14]. Among results established here, we emphasize that the ill-posedness result for two-dimensional BBM-BBM system is optimal. The proof follows from an observation of the *high to low frequency cascade* present in nonlinearity, motivated by Bejenaru and Tao [10].

Contents

3.1 Introduction 50

3.1.1 Setting 50

3.1.2 Main results 52

3.2 Preliminaries 55

3.2.1 Bejenaru-Tao's abstract theory 55

3.2.2 Equivalent representation of $abc\mathrm{d}$ systems 56

3.3 Proof of Theorem 3.2 62

3.3.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$ 62

3.3.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$ 66

3.4 Proof of Theorem 3.3 67

3.4.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$ 67

3.4.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$ 70

3.4.3 BBM-BBM regime: $a = c = 0$ and $b = \mathrm{d} = 1/6$ 71

Appendices 73

3.A Local Well-Posedness 73

3.A.1 Notations 73

3.A.2 Littlewood-Paley Decomposition 73
3.A.3 Bilinear estimates 74
3.B Decomposition of (Q_1, Q_2) on the generic regime: one-dimensional case 76
3.C Decomposition of (Q_1, Q_2, Q_3): generic case 77
3.D Decomposition of (Q_1, Q_2, Q_3): $a = c$ and $b = \mathrm{d} \geqslant 0$ 78
Bibliography 80

This paper was done in collaboration with Chulkwang Kwak (Ewha Womans University), who was partially supported by project France-Chile ECOS-Sud C18E06, and is partially supported by the Ewha Womans University Research Grant of 2020 and the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2020R1F1A1A0106876811). This work has been recently submitted and can be found at arXiv:2102.01248.

3.1 Introduction

3.1.1 Setting

As a rigorous derivation from the free Eulerian formulation of water waves, Bona, Chen, and Saut [6] proposed the model called one-dimensional (abcd)-Boussinesq, as

$$1D\ (abc\mathrm{d}) \quad \begin{cases} (1 - b\partial_x^2)\partial_t \eta + \partial_x(a\partial_x^2 u + u + u\eta) = 0, \\ (1 - \mathrm{d}\partial_x^2)\partial_t u + \partial_x(c\partial_x^2 \eta + \eta + \frac{1}{2}u^2) = 0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{3.1}$$

As two-dimensional model, Bona, Colin and Lannes [7], formulated 2D (abcd) as

$$2D\ (abc\mathrm{d}) \quad \begin{cases} (1 - b\,\Delta)\partial_t \eta + \nabla \cdot (a\,\Delta \vec{u} + \vec{u} + \vec{u}\eta) = 0, \\ (1 - \mathrm{d}\,\Delta)\partial_t \vec{u} + \nabla \left(c\,\Delta\eta + \eta + \frac{1}{2}|\vec{u}|^2 \right) = 0, \end{cases} \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^2. \tag{3.2}$$

Here, unknowns η and u (also $\vec{u}$) describe the free surface and the horizontal velocity of fluid, respectively. Both systems (3.1) and (3.2) are all first-order approximations of the incompressible and irrotational Euler equations assuming the small parameters defined by

$$\alpha = \frac{A}{h} \ll 1, \quad \beta = \frac{h^2}{\ell^2} \ll 1, \quad \alpha \sim \beta,$$

where A and ℓ are typical wave amplitude and wavelength, and h is the constant depth. Such assumptions sometimes referred to as small-amplitude long waves or Boussinesq or simply shallow water waves regimes (see [9]). In the two-dimensional case, the irrotational hypothesis can be (mathematically) characterized as

$$\nabla \wedge \vec{u} = 0, \tag{3.3}$$

which is preserved by the evolution. Note that the condition (3.3) is not necessary in the one-dimensional case since there is a single horizontal direction. See also [1] for relevant result.

The parameters (a, b, c, d) in both (3.1) and (3.2) are not arbitrary and hold the relations (see [6])

$$a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\nu, \quad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\nu),$$
$$c = \frac{1}{2}\left(1-\theta^2\right)\nu - \tau, \quad \mathrm{d} = \frac{1}{2}(1-\theta^2)(1-\mu),$$

where $\theta \in [0,1]$ appears in the change of scaled horizontal velocity corresponding to the depth $(1-\theta)h$ below the undisturbed surface, τ is the surface tension ($\tau \geqslant 0$), and ν, μ are arbitrary real numbers ensuring

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + \mathrm{d} = \frac{1}{2}(1-\theta^2) - \tau, \quad a + b + c + \mathrm{d} = \frac{1}{3} - \tau.$$

The dispersive properties of the systems depend on the choice of the parameters. Precisely, the pair (a, c) enhances the dispersion, while the pair (b, d) weakens it (see [14]). This versatility makes the (abcd)-Boussinesq model interesting and challenging.

Two systems 1D (abcd) and $2D$(abcd) allow the following energies

$$E_{1D}[u,\eta](t) = \frac{1}{2}\int_{\mathbb{R}}(-au_x^2 - c\eta_x^2 + u^2(1+\eta) + \eta^2)(t,x)\mathrm{d}x,$$

and

$$E_{2D}[\vec{u},\eta](t) = \frac{1}{2}\int_{\mathbb{R}^2}(-a|\nabla\vec{u}|^2 - c|\nabla\eta|^2 + |\vec{u}|^2(1+\eta) + \eta^2)(t,x)\mathrm{d}x,$$

respectively, that both are conserved in time when $b = \mathrm{d}$ and $a, c < 0$. Thus local well-posedness in H^1-level space is immediately extended to the global one at least for small data. Note that Sobolev embedding in two-dimensional case is not enough to control L^∞ norm of η, but Gagliardo-Nirenberg interpolation inequality can control $\eta|\vec{u}|^2$.

These models have been extensively studied (in various perspective) in the literature, see e.g. [5, 6, 11, 15, 22, 29, 27, 14, 4, 28, 21, 23, 20, 30, 31]. Among other them, we focus on Cauchy problems for these systems. In [5, 6], Bona, Chen and Saut first studied local and global well-posedness of linear and nonlinear problems, and established the following results (the following results only exhibit the case when $\mathcal{H}$ (see (3.12)) has order 0):

1. the generic regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s \geqslant 0$.

2. the BBM-BBM regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s \geqslant 0$.

3. the KdV-KdV regime in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s > 3/4$.

In [15], Dougalis, Mitsotakis, and Saut proved that two-dimensional (abcd) Boussinesq system under the generic regime is locally well-posed in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for $s > 0$. Note that this local result is indeed valid in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ by improving Grisvard's bilinear estimate [16], see Appendix 3.A (Lemma 3.16). In [22], Linares, Pilod, and Saut focused on the strongly dispersive (KdV-KdV system) regime, and established local well-posedness result in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for $s > 3/2$. Previously, Schonbek [32] and Amick [2] considered a version of the original Boussinesq system ($a = c = b = 0, b = 1/3$), and proved global well-posedness under a non-cavitation condition via parabolic regularization. Later, Burteau [11] improved it without a non-cavitation condition. Studies on long time existence of solutions have been done in, for instance, [29, 24, 28, 30, 31]. In these works, the authors established the well-posedness for large time with appropriate time scales.

In contrast with results mentioned above, this paper concerns with the ill-posed issues on one- and two-dimensional (abcd)-Boussinesq systems in the following cases:

1. Generic regime
$$a, c < 0, \quad b, \mathrm{d} > 0, \tag{3.4}$$

2. KdV-KdV regime
$$a = c = \frac{1}{6}, \quad b = \mathrm{d} = 0, \tag{3.5}$$

3. BBM-BBM regime
$$a = c = 0, \quad b = \mathrm{d} = \frac{1}{6}. \tag{3.6}$$

As far as the authors know, there is only few results for ill-posedness issues. Chen and Liu [14] established the (mild) ill-posedness result for one-dimensional system under the weakly dispersive regime (1D BBM-BBM system) below L^2. The main idea follows the abstract theory developed by Bejenaru-Tao [10]. The authors also discussed the formation of singularities and provided blow-up criteria. Recently, [3] Ambrose, Bona and Milgrom have established the ill-posedness of the one-dimensional periodic Kaup system ($a = 1/3$ and $b = c = \mathrm{d} = 0$) is ill-posed in any positive regularity Sobolev space, in the sense that the flow map is discontinuous at the origin. They also concerned with the case that the generic condition (3.4) is negated.

3.1.2 Main results

Before presenting our results, we clarify what we mean "ill-posedness". To do this, we first define "well-posedness" of Partial Differential equations problems. As the author's best knowledge, The French mathematician Jacques Hadamard initially proposed the concept of well-posed problems as

Definition 3.1 (Well-posedness). *The mathematical models of physical phenomena should have the following properties:*

- *there exists a solution,*

- *the solution is unique,*
- *the solution behaves continuously with the initial condition.*

Obviously, problems that are not well-posed in the sense of Hadamard are termed ill-posed, in other words, invalidity of one of above properties makes problems be ill-posed. In this paper, in order to obtain ill-posedness results, we attack the third property in Definition 3.1. A precise strategy follows the negation of Proposition 3.5.

We are now ready to present our main theorems.

Theorem 3.2. *The 1D-abcd system* (3.1) *is ill-posed in the sense that the flow map from initial data to solutions is discontinuous at the origin in* $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, *where*

1. $s < -\frac{1}{2}$ *for the generic case (see* (3.4)*).*
2. $s < -\frac{3}{2}$ *for the KdV-KdV case (see* (3.5)*).*

Analogously,

Theorem 3.3. *The 2D-abcd system* (3.2) *is ill-posed in the sense that the flow map from initial data to solutions is discontinuous at the origin in* $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, *where*

1. $s < -\frac{1}{2}$ *for the generic case (see* (3.4)*).*
2. $s < -\frac{3}{2}$ *for the KdV-KdV case (see* (3.5)*) .*
3. $s < 0$ *for the BBM-BBM case (see* (3.6)*).*

Remark 3.1. The BBM-BMM case of the one-dimensional (*abc*d)-Boussinesq system has been dealt with by Chen and Liu [14]. However, the two-dimensional BBM-BBM system is considered here for the first time, and together with Appendix 3.A, we completely resolve Cauchy problem for it.

The proofs of theorems follows the same idea developed by Bejenaru and Tao [10], and motivated by an observation as follows: All nonlinear interactions are quadratic, thus *high* $\times$ *high* interaction components over an appropriate short time depending on the frequency cause *resonances* near the origin of the resulting frequency. For this reason, the flow cannot disperse the high-frequency energy for this time so that the smoothness of the flow breaks below certain regularity. Note that this observation is simply applied to a one-dimensional problem, but it is non-trivial to construct initial data that can cause *resonance* in two-dimensional case.

It is easy to see that (*abc*d) systems are completely coupled systems, thus an attempt at decoupling of (at least) the linear system must take precedence in order to observe its propagators. Under generic regime, standard transforms (see (3.11) and (3.26) for one- and two-dimensional cases, respectively) diagonalize the linear operator with eigenvalues σ (see (3.13)) for one-dimensional case and ρ (3.22) for two-dimensional case of order 0, while those under BBM-BBM and KdV-KdV regimes have order -2 and 2, respectively. The difference of orders of eigenvalues directly affect the dispersive properties of solutions, thus so flow maps of stronger dispersive systems can take rougher initial data. Such observations can be seen in, for instance, Lemma 3.8, and relevant lemmas.

Our results are coherent with one-dimensional BBM-BBM system, generalized BBM equation and KdV equation. On one hand, In [8], the authors established that the flow map is not of class C^2, and warned that their result is not suitable to assert that the BBM-equation is ill-posed in H^s for negative values of s. Conversely, in [12], the authors proved the discontinuity of the flow map at the origin in H^s for $s < 0$. On the other hand, our result for the KdV-KdV system differs from the ill-posedness result of the original KdV equation established by Molinet [25]. The proof follows from an argument of functional analysis together with the discontinuity Miura transform and the validity of Kato smoothing effect of mKdV solutions. However, the same argument may not apply to the KdV-KdV system, since it has no such good structure. We also refer to, e.g., [17, 18, 19, 13, 26, 14] for relevant ill-posedness problems of single equations.

In Appendix 3.A, we give a bilinear estimates of $(1-\Delta)\nabla(fg)$, which slightly improve Grisvard's result [16]. This improvement enable us to obtain the local well-posedness of two-dimensional (abcd) system under generic and BBM-BBM regimes in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. As mentioned in Remark 3.1, the well-posedness result for two-dimensional BBM-BBM system, in addition to Theorem 3.3 (3), asserts the completion of Cauchy problem for it. The proof is based on Littlewood-Paley theory, and is completed by a delicate observation of frequency interactions.

Organization of this chapter

This paper is organized as follows: Section 3.2 devotes to introducing abstract and general well- and ill-posedness arguments developed by Bejenaru and Tao [10], and to representing Boussinesq equations as linearly decoupled forms. In Sections 3.3 and 3.4 we prove Theorems 3.2 and 3.3, respectively. In Appendix 3.A we briefly provide a refined bilinear estimate to establish the well-posedness of some classes of systems. In Appendices 3.B, 3.C and 3.D, we give precise computations for decomposition of quadratic terms in the nonlinearities.

Notations

For $x, y \in \mathbb{R}_+$ $(= \mathbb{R} \cap (0, \infty))$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leqslant Cy$, and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. For a Schwartz function f in $x \in \mathbb{R}^{\mathrm{d}}$, we denote the Fourier transform of f by $\mathcal{F}(f)$ or $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^{\mathrm{d}},$$

and $\check{f}$ denotes the inverse Fourier transform of f defined by

$$\check{f}(\xi) = \frac{1}{(2\pi)^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^{\mathrm{d}}.$$

In the rest of sections, the following properties of the Fourier transform among others will be used frequently:

$$\mathcal{F}(f * g)(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) \quad \text{and} \quad \mathcal{F}(\partial_{x_{\rm i}} f) = {\rm i}\xi_{\rm i}\widehat{f}(\xi), \;\; 1 \leqslant {\rm i} \leqslant {\rm d}.$$

3.2 Preliminaries

3.2.1 Bejenaru-Tao's abstract theory

Here, we briefly present the abstract well- and ill-posed theory initially introduced in [10]. Consider the abstract equation

$$\vec{v} = \mathcal{L}(\vec{v}_0) + \mathcal{N}_2(\vec{v}, \vec{v}), \tag{3.7}$$

where $\vec{v}_0 \in D$ is the initial data, and $\vec{v} \in S$ is a solution of abstract equation. Here, $\mathcal{L}(\vec{v}_0)$ and $\mathcal{N}_2(\vec{v}, \vec{v})$ are the linear and bilinear part of Duhamel's formula, respectively.

In the context of this work, $\vec{v}$ is a solution to the (abcd) system (3.1) ($\vec{v} = (\eta, u)$) or (3.2) ($\vec{v} = (\eta, \vec{u}) = (\eta, u_1, u_2)$). First, we introduce the definition of quantitative well-posedness introduced in [10].

Definition 3.4 (Quantitative well-posedness, [10]). *Let $(D, \|\cdot\|_D)$ be a Banach space of initial data, and $(S, \|\cdot\|_S)$ be a Banach space of space-time functions. We say that* (3.7) *is quantitatively well-posed in D, S if one has the estimate of the form*

$$\|\mathcal{L}(\vec{v}_0)\|_S \lesssim \|\vec{v}_0\|_D$$

and

$$\|\mathcal{N}_2(\vec{v}, \vec{v})\|_S \lesssim \|\vec{v}\|_S^2, \tag{3.8}$$

for all $\vec{v}_0 \in D$ and $\vec{v} \in S$.

As we can see in the introduction, it is known that abcd-system (in one- or two-dimensional case) is locally well-posed in the sense of Definition 3.4 (see [6]).

The following theorem asserts that the quantitative well-posedness indeed guarantees the analytic well-posedness, that is, the flow from given initial data to a solution is represented as a power series expansion of continuous nonlinear maps.

Theorem 3.5 (Theorem 3 in [10]). *Suppose that* (3.7) *is quantitatively well-posed in the space D, S. Then, there exist constants $C_0, \varepsilon_0 > 0$ such that for all $\vec{v}_0 \in B_D(0, \varepsilon_0)$, there exists a unique solution $\vec{v}[\vec{v}_0] \in B_S(0, C_0\varepsilon_0)$ to* (3.7). *More specifically, if we define the non-linear maps $A_n : D \to S$ for $n \in \mathbb{Z}_{>0}$ by the recursive formula*

$$\begin{aligned} A_1(\vec{v}_0) &:= \mathcal{L}(\vec{v}_0) \\ A_n(\vec{v}_0) &:= \sum_{\substack{n_1, n_2 \geqslant 1 \\ n_1 + n_2 = n}} \mathcal{N}_2(A_{n_1}(\vec{v}_0), A_{n_2}(\vec{v}_0)), \quad n > 1, \end{aligned}$$

then we have the absolutely convergent (in S) power series expansion

$$\vec{v}[\vec{v}_0] = \sum_{n=1}^{\infty} A_n(\vec{v}_0),$$

for all $\vec{v}_0 \in B_D(0, \varepsilon_0)$.

On the other hand, Theorem 3.5 alternatively says that one can prove ill-posedness of (3.7), once showing discontinuity of A_n, for some n, i.e., A_n does not satisfy (3.8). This observation can be precisely stated as follows:

Proposition 3.6 (Proposition 1 in [10])**.** *Suppose that* (3.7) *is quantitatively well-posed in the Banach spaces D and S, with a solution map $f \mapsto u[f]$ from a ball B_D in D to a ball B_S in S. Suppose that these spaces are then given other norms D' and S', which are weaker than D and S in the sense that*

$$\|\vec{v}_0\|_{D'} \lesssim \|\vec{v}_0\|_D, \quad \|\vec{v}\|_{S'} \lesssim \|\vec{v}\|_S.$$

Suppose that the solution map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous from $(B_D, \| \, \|_{D'})$ to $(B_S, \| \, \|_{S'})$. Then for each n, the non-linear operator $A_n : D \to S$ is continuous from from $(B_D, \| \, \|_{D'})$ to $(S, \| \, \|_{S'})$.

3.2.2 Equivalent representation of abcd systems

This subsection devotes to rewriting (abcd) systems (3.1) and (3.2) in the form of (3.7). We follow the arguments in [6] and [15] for the one- and two-dimensional cases, respectively.

One-dimensional case.

We first deal with one-dimensional case. Applying the Fourier transform to the linear abcd system ((3.1) without $u\eta$ and $\frac{1}{2}u^2$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widehat{\eta}\\ \widehat{u}\end{pmatrix} + \mathrm{i}\xi\begin{pmatrix}0 & \omega_1(\xi)\\ \omega_2(\xi) & 0\end{pmatrix}\begin{pmatrix}\widehat{\eta}\\ \widehat{u}\end{pmatrix} = 0, \tag{3.9}$$

where

$$\omega_1(\xi) := \frac{1 - a\xi^2}{1 + b\xi^2} \quad \text{and} \quad \omega_2(\xi) := \frac{1 - c\xi^2}{1 + \mathrm{d}\xi^2}. \tag{3.10}$$

Using the transform

$$\begin{pmatrix}\eta\\ u\end{pmatrix} = \begin{pmatrix}\mathcal{H} & \mathcal{H}\\ 1 & -1\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix}, \tag{3.11}$$

where $\mathcal{H}$ is the Fourier multiplier defined by

$$\widehat{\mathcal{H}f}(\xi) = h(\xi)\widehat{f}(\xi), \quad \text{with} \quad h(\xi) := \left(\frac{\omega_1(\xi)}{\omega_2(\xi)}\right)^{\frac{1}{2}} = \left(\frac{(1 - a\xi^2)(1 + \mathrm{d}\xi^2)}{(1 - c\xi^2)(1 + b\xi^2)}\right)^{\frac{1}{2}}, \tag{3.12}$$

the system (3.9) becomes a symmetric form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widehat{v}\\ \widehat{w}\end{pmatrix} + \mathrm{i}\xi\begin{pmatrix}\sigma(\xi) & 0\\ 0 & -\sigma(\xi)\end{pmatrix}\begin{pmatrix}\widehat{v}\\ \widehat{w}\end{pmatrix} = 0,$$

where

$$\sigma(\xi) := (\omega_1(\xi)\omega_2(\xi))^{\frac{1}{2}} = \left(\frac{(1 - a\xi^2)(1 - c\xi^2)}{(1 + b\xi^2)(1 + \mathrm{d}\xi^2)}\right)^{\frac{1}{2}}. \tag{3.13}$$

Note that (3.10) and (3.12) are well-defined whenever the parameters a, b, c and d satisfy the generic or KdV-KdV cases. Therefore, the system is written in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} + B\begin{pmatrix} v \\ w \end{pmatrix} = 0,$$

where

$$B\begin{pmatrix} v \\ w \end{pmatrix} := \left(\mathrm{i}\xi\begin{pmatrix} \sigma(\xi) & 0 \\ 0 & -\sigma(\xi) \end{pmatrix}\begin{pmatrix} \widehat{v} \\ \widehat{w} \end{pmatrix}\right)^{\vee},$$

Coming back to the original variables η and u, we write the linear system as

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta \\ u \end{pmatrix} + A\begin{pmatrix} \eta \\ u \end{pmatrix} = 0,$$

where an operator A is determined by (also explicitly computed by taking the Fourier transform)

$$A = \begin{pmatrix} \mathcal{H} & \mathcal{H} \\ 1 & -1 \end{pmatrix} B \begin{pmatrix} \mathcal{H} & \mathcal{H} \\ 1 & -1 \end{pmatrix}^{-1},$$

and thus the solutions to the linear abcd system are of the form

$$\begin{pmatrix} \eta \\ u \end{pmatrix}(t, x) = S(t)\begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix}(x),$$

where $S(t)$ is associated to the linear flow of the system generated by A. It is clear that $S(t)$ is a **unitary group** on $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. When

$$\omega_1(\xi)\omega_2(\xi) > 0,$$

the linear flow $S(t)$ can be expressed

$$\mathcal{F}\left(S(t)\begin{pmatrix} f \\ g \end{pmatrix}\right) := \begin{pmatrix} \cos(\xi\sigma(\xi)t) & -\mathrm{i}\sin(\xi\sigma(\xi)t)\dfrac{\omega_1(\xi)}{\sigma(\xi)} \\ -\mathrm{i}\sin(\xi\sigma(\xi)t)\dfrac{\omega_2(\xi)}{\sigma(\xi)} & \cos(\xi\sigma(\xi)t) \end{pmatrix}\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}. \tag{3.14}$$

Note that

$$\frac{\omega_1(\xi)}{\sigma(\xi)} = \left(\frac{\omega_1(\xi)}{\omega_2(\xi)}\right)^{\frac{1}{2}} = h(\xi) \quad \text{and} \quad \frac{\omega_2(\xi)}{\sigma(\xi)} = \left(\frac{\omega_2(\xi)}{\omega_1(\xi)}\right)^{\frac{1}{2}} = \frac{1}{h(\xi)},$$

where h is as in (3.12). Let

$$\widehat{L}_1(t, \xi) = \cos(\xi\sigma(\xi)t) \quad \text{and} \quad \widehat{L}_2(t, \xi) = \mathrm{i}\sin(\xi\sigma(\xi)t). \tag{3.15}$$

Then we rewrite (3.14) as

$$\mathcal{F}\left(S(t)\begin{pmatrix} f \\ g \end{pmatrix}\right) := \begin{pmatrix} \widehat{L}_1(t, \xi) & -h(\xi)\widehat{L}_2(t, \xi) \\ -(h(\xi))^{-1}\widehat{L}_2(t, \xi) & \widehat{L}_1(t, \xi) \end{pmatrix}\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}. \tag{3.16}$$

Duhamel's principle for the nonlinear system (3.1) yields

$$\begin{aligned}\begin{pmatrix}\eta\\u\end{pmatrix} &= S(t)\begin{pmatrix}\eta_0\\u_0\end{pmatrix} - \int_0^t S(t-s)\partial_x\begin{pmatrix}(1-b\partial_x^2)^{-1}(\eta u)\\(1-\mathrm{d}\partial_x^2)^{-1}(\frac{1}{2}u^2)\end{pmatrix}(s)\,\mathrm{d}s\\ &=: S(t)\begin{pmatrix}\eta_0\\u_0\end{pmatrix} + \mathcal{N}_2\left(\begin{pmatrix}\eta\\u\end{pmatrix},\begin{pmatrix}\eta\\u\end{pmatrix}\right).\end{aligned}\tag{3.17}$$

KdV-KdV regime. Making a simple additional scaling, one may assume that $a = c = 1$, and obtain that the linear system of (3.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widehat{\eta}\\\widehat{u}\end{pmatrix} + \mathrm{i}\xi(1-\xi^2)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\widehat{\eta}\\\widehat{u}\end{pmatrix} = 0.$$

Analogously, we obtain

that the linear propagator $S_K(t)$ is represented as

$$\mathcal{F}\left(S_K(t)\begin{pmatrix}f\\g\end{pmatrix}\right) := \begin{pmatrix}\widehat{L_1^K}(t,\xi) & -\widehat{L_2^K}(t,\xi)\\-\widehat{L_2^K}(t,\xi) & \widehat{L_1^K}(t,\xi)\end{pmatrix}\begin{pmatrix}\widehat{f}\\\widehat{g}\end{pmatrix},\tag{3.18}$$

where

$$\widehat{L_1^K}(t,\xi) = \cos(\xi\sigma_K(\xi)t),\quad \widehat{L_2^K}(t,\xi) = \mathrm{i}\sin(\xi\sigma_K(\xi)t)\quad\text{and}\quad \sigma_K(\xi) = 1-\xi^2.\tag{3.19}$$

Two-dimensional case.

We write the two-dimensional linear *abc*d system ((3.2) without $\vec{u}\eta$ and $\frac{1}{2}|\vec{u}|^2$) in the equivalent form (in the Fourier space, for fixed $\xi \in \mathbb{R}^2\backslash\{\mathbf{0}\}$)

$$\partial_t\begin{pmatrix}\widehat{\eta}\\\widehat{u}_1\\\widehat{u}_2\end{pmatrix} + \mathrm{i}|\xi|\mathcal{A}(\xi)\begin{pmatrix}\widehat{\eta}\\\widehat{u}_1\\\widehat{u}_2\end{pmatrix} = 0,\tag{3.20}$$

where

$$\mathcal{A}(\xi) = \begin{pmatrix}0 & \frac{\xi_1}{|\xi|}\left(\frac{1-a|\xi|^2}{1+b|\xi|^2}\right) & \frac{\xi_2}{|\xi|}\left(\frac{1-a|\xi|^2}{1+b|\xi|^2}\right)\\ \frac{\xi_1}{|\xi|}\left(\frac{1-c|\xi|^2}{1+\mathrm{d}|\xi|^2}\right) & 0 & 0\\ \frac{\xi_2}{|\xi|}\left(\frac{1-c|\xi|^2}{1+\mathrm{d}|\xi|^2}\right) & 0 & 0\end{pmatrix}.\tag{3.21}$$

Define the Fourier symbol $\varrho(|\xi|)$ by

$$\varrho(|\xi|) = \left(\frac{(1-a|\xi|^2)(1-c|\xi|^2)}{(1+b|\xi|^2)(1+\mathrm{d}|\xi|^2)}\right)^{\frac{1}{2}}.\tag{3.22}$$

A straightforward computation yields that the matrix $\mathcal{A}(\xi)$ has three eigenvalues 0, $\varrho(\xi)$ and $-\varrho(\xi)$, thus the matrix $\mathcal{A}(\xi)$ is diagonalized as follows:

$$\mathcal{P}^{-1}(\xi)\mathcal{A}(\xi)\mathcal{P}(\xi) = \begin{pmatrix}0&0&0\\0&\varrho(|\xi|)&0\\0&0&-\varrho(|\xi|)\end{pmatrix},$$

where the *block matrix* and its inverse are given by

$$\mathcal{P}(\xi) = \begin{pmatrix} 0 & \varsigma(|\xi|) & -\varsigma(|\xi|) \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} \\ \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_2}{|\xi|} \end{pmatrix} \tag{3.23}$$

and

$$\mathcal{P}^{-1}(\xi) = \frac{1}{2\varsigma(|\xi|)} \begin{pmatrix} 0 & -2\varsigma(|\xi|)\frac{\xi_2}{|\xi|} & 2\varsigma(|\xi|)\frac{\xi_1}{|\xi|} \\ 1 & \varsigma(|\xi|)\frac{\xi_1}{|\xi|} & \varsigma(|\xi|)\frac{\xi_2}{|\xi|} \\ -1 & \varsigma(|\xi|)\frac{\xi_1}{|\xi|} & \varsigma(|\xi|)\frac{\xi_2}{|\xi|} \end{pmatrix}, \tag{3.24}$$

respectively, for

$$\varsigma(|\xi|) = \left(\frac{(1-a|\xi|^2)(1+\mathrm{d}|\xi|^2)}{(1-c|\xi|^2)(1+b|\xi|^2)} \right)^{\frac{1}{2}}. \tag{3.25}$$

We consider the following change of variables:

$$\begin{pmatrix} \widehat{\mu} \\ \widehat{\nu}_1 \\ \widehat{\nu}_2 \end{pmatrix} = \mathcal{P}^{-1}(\xi) \begin{pmatrix} \widehat{\eta} \\ \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{\xi_2}{|\xi|}\widehat{u}_1 + \frac{\xi_1}{|\xi|}\widehat{u}_2 \\ \frac{\widehat{\eta}}{2\varsigma(|\xi|)} + \frac{\xi_1}{2|\xi|}\widehat{u}_1 + \frac{\xi_2}{2|\xi|}\widehat{u}_2 \\ -\frac{\widehat{\eta}}{2\varsigma(|\xi|)} + \frac{\xi_1}{2|\xi|}\widehat{u}_1 + \frac{\xi_2}{2|\xi|}\widehat{u}_2 \end{pmatrix}. \tag{3.26}$$

Note that $\widehat{\mu} = 0$, since the fluid is *irrotational* (equivalently, (3.3)). Thus, new variables (ν_1, ν_2) finally determine an equivalent expression of the system (3.20), and for (3.2), as

$$\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + \mathcal{B}(-\mathrm{i}\nabla) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = 0, \quad \begin{pmatrix} \nu_1(t=0) \\ \nu_2(t=0) \end{pmatrix} = \begin{pmatrix} \nu_{1,0} \\ \nu_{2,0} \end{pmatrix},$$

where $\mathcal{B}(-\mathrm{i}\nabla)$ is the 2×2 matrix operator whose entries are pseudo-differential operators, with symbol

$$\mathrm{i}|\xi| \begin{pmatrix} \varrho(|\xi|) & 0 \\ 0 & -\varrho(|\xi|) \end{pmatrix},$$

which is the skew-Hermitian matrix.

Coming back to the original variables η, u_1 and u_2, we write the linear system as

$$\frac{\partial}{\partial t} \begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix} + \mathbf{A} \begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix} = 0,$$

where the linear operator $\mathbf{A}$ is indeed given by the inverse Fourier transform of $\mathrm{i}|\xi|\mathcal{A}(\xi)$ as in (3.20). Thus, the solutions to the linear *abc*d system are of the form

$$\begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix}(t,x) = \mathbf{S}(t) \begin{pmatrix} \eta_0 \\ u_{1,0} \\ u_{2,0} \end{pmatrix}(x),$$

where $\mathbf{S}(t)$ is associated to the linear flow of the system, generated by $\mathbf{A}$. When, a, b, c and d satisfy the generic condition, the linear flow $\mathbf{S}(t)$ can be expressed as,

$$\begin{aligned}
&\mathcal{F}\left(\mathbf{S}(t)\begin{pmatrix} f\\ g\\ h\end{pmatrix}\right)\\
&:= \mathcal{P}(\xi)\begin{pmatrix} 0 & 0 & 0\\ 0 & \mathrm{e}^{\mathrm{i}|\xi|\varrho(|\xi|)t} & 0\\ 0 & 0 & \mathrm{e}^{-\mathrm{i}|\xi|\varrho(|\xi|)t}\end{pmatrix}\mathcal{P}^{-1}(\xi)\begin{pmatrix} \widehat{f}\\ \widehat{g}\\ \widehat{h}\end{pmatrix}\\
&= \begin{pmatrix} \widehat{J}_1(t,\xi) & \varsigma(|\xi|)\frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2(t,\xi) & \varsigma(|\xi|)\frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2(t,\xi)\\ \frac{\mathrm{i}\xi_1}{\varsigma(|\xi|)|\xi|}\widehat{J}_2(t,\xi) & \frac{\xi_1^2}{|\xi|^2}\widehat{J}_1(t,\xi) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1(t,\xi)\\ \frac{\mathrm{i}\xi_2}{\varsigma(|\xi|)|\xi|}\widehat{J}_2(t,\xi) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1(t,\xi) & \frac{\xi_2^2}{|\xi|^2}\widehat{J}_1(t,\xi)\end{pmatrix}\begin{pmatrix} \widehat{f}\\ \widehat{g}\\ \widehat{h}\end{pmatrix},
\end{aligned}\tag{3.27}$$

where

$$\widehat{J}_1(t,\xi) = \cos(|\xi|\varrho(|\xi|)t), \quad \widehat{J}_2(t,\xi) = \sin(|\xi|\varrho(|\xi|)t), \tag{3.28}$$

and $\varrho(|\xi|)$ is defined in (3.22).

Duhamel's principle for the nonlinear system (3.2) yields

$$\begin{aligned}
\begin{pmatrix} \eta\\ u_1\\ u_2\end{pmatrix} &= \mathbf{S}(t)\begin{pmatrix} \eta_0\\ u_{1,0}\\ u_{2,0}\end{pmatrix} - \int_0^t \mathbf{S}(t-s)\begin{pmatrix} (1-b\,\Delta)^{-1}(\nabla\cdot(\eta\vec{u}))\\ \frac{1}{2}(1-\mathrm{d}\,\Delta)^{-1}\partial_{x_1}|\vec{u}|^2\\ \frac{1}{2}(1-\mathrm{d}\,\Delta)^{-1}\partial_{x_2}|\vec{u}|^2\end{pmatrix}(s)\,\mathrm{d}s\\
&=: \mathbf{S}(t)\begin{pmatrix} \eta_0\\ u_{1,0}\\ u_{2,0}\end{pmatrix} + \mathcal{N}_2\left(\begin{pmatrix} \eta\\ u_1\\ u_2\end{pmatrix}, \begin{pmatrix} \eta\\ u_1\\ u_2\end{pmatrix}\right).
\end{aligned}\tag{3.29}$$

Remark 3.2. Notice that the eigenvalues σ and ϱ (see (3.13) and (3.22), respectively) have the same radial behavior regardless of the dimension.

Case $a = c$ and $b = \mathrm{d} \geqslant 0$.

In this case, we insert the conditions $a = c$ and $b = \mathrm{d}$ into (3.21), we then have

$$\partial_t\begin{pmatrix} \widehat{\eta}\\ \widehat{u}_1\\ \widehat{u}_2\end{pmatrix} + \mathrm{i}|\xi|\varrho_{ab}(|\xi|)\mathcal{A}_{ab}(\xi)\begin{pmatrix} \widehat{\eta}\\ \widehat{u}_1\\ \widehat{u}_2\end{pmatrix} = 0, \quad \xi \in \mathbb{R}^2\backslash\{\mathbf{0}\} \tag{3.30}$$

where

$$\mathcal{A}_{ab}(\xi) = \begin{pmatrix} 0 & \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|}\\ \frac{\xi_1}{|\xi|} & 0 & 0\\ \frac{\xi_2}{|\xi|} & 0 & 0\end{pmatrix} \quad\text{and}\quad \varrho_{ab}(|\xi|) = \frac{1-a|\xi|^2}{1+b|\xi|^2}. \tag{3.31}$$

Analogously as above (for the generic case), we can find three eigenvalues 0, 1 and -1 of the matrix $\mathcal{A}_{ab}(\xi)$. Thus the matrix $\mathcal{A}_{ab}(\xi)$ is diagonalized as follows:

$$\mathcal{P}_{ab}^{-1}(\xi)\mathcal{A}_{ab}(\xi)\mathcal{P}_{ab}(\xi) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},$$

where the *block matrix* $\mathcal{P}_{ab}$ and its inverse are represented as in (3.23) and (3.24), respectively, with $\varsigma(|\xi|) = 1$. Change the variables analogous to (3.26), then we have

$$\begin{pmatrix} \widehat{\mu} \\ \widehat{\nu}_1 \\ \widehat{\nu}_2 \end{pmatrix} = \mathcal{P}_{ab}^{-1}(\xi) \begin{pmatrix} \widehat{\eta} \\ \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{\xi_2}{|\xi|}\widehat{u}_1 + \frac{\xi_1}{|\xi|}\widehat{u}_2 \\ \frac{\widehat{\eta}}{2} + \frac{\xi_1}{2|\xi|}\widehat{u}_1 + \frac{\xi_2}{2|\xi|}\widehat{u}_2 \\ -\frac{\widehat{\eta}}{2} + \frac{\xi_1}{2|\xi|}\widehat{u}_1 + \frac{\xi_2}{2|\xi|}\widehat{u}_2 \end{pmatrix}.$$

As same as before, $\widehat{\mu} = 0$, and new variables (ν_1, ν_2) determine an equivalent expression of the system (3.30) as

$$\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + \mathcal{B}_{ab}(-\mathrm{i}\nabla) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = 0, \quad \begin{pmatrix} \nu_1(t=0) \\ \nu_2(t=0) \end{pmatrix} = \begin{pmatrix} \nu_{1,0} \\ \nu_{2,0} \end{pmatrix},$$

where $\mathcal{B}_{ab}(-\mathrm{i}\nabla)$ is the 2×2 matrix operator whose entries are pseudo-differential operators, with symbol

$$\mathrm{i}|\xi|\varrho_{ab}(|\xi|) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is the skew-Hermitian matrix.

Solutions to the linear (*abc*d) system are of the form

$$\begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix}(t,x) = \mathbf{S}_{ab}(t) \begin{pmatrix} \eta_0 \\ u_{1,0} \\ u_{2,0} \end{pmatrix}(x),$$

where $\mathbf{S}_{ab}(t)$ is associated to the linear flow of the system, precisely expressed as,

$$\mathcal{F}\left(\mathbf{S}_{ab}(t)\begin{pmatrix} f \\ g \\ h \end{pmatrix}\right) = \begin{pmatrix} \widehat{J}_1^{ab}(t,\xi) & \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^{ab}(t,\xi) & \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^{ab}(t,\xi) \\ \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^{ab}(t,\xi) & \frac{\xi_1^2}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi) \\ \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^{ab}(t,\xi) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi) & \frac{\xi_2^2}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi) \end{pmatrix} \begin{pmatrix} \widehat{f} \\ \widehat{g} \\ \widehat{h} \end{pmatrix}, \tag{3.32}$$

where

$$\widehat{J}_1^{ab}(t,\xi) = \cos(|\xi|\varrho_{ab}(|\xi|)t), \quad \widehat{J}_2^{ab}(t,\xi) = \sin(|\xi|\varrho_{ab}(|\xi|)t), \tag{3.33}$$

and $\varrho_{ab}(|\xi|)$ is defined in (3.31).

Duhamel's principle for the nonlinear system (3.2) yields

$$\begin{aligned} \begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix} &= \mathbf{S}_{ab}(t) \begin{pmatrix} \eta_0 \\ u_{1,0} \\ u_{2,0} \end{pmatrix} - \int_0^t \mathbf{S}_{ab}(t-s) \begin{pmatrix} (1 - b\,\Delta)^{-1}(\nabla \cdot (\eta\vec{u})) \\ \frac{1}{2}(1 - b\,\Delta)^{-1}\partial_{x_1}|\vec{u}|^2 \\ \frac{1}{2}(1 - b\,\Delta)^{-1}\partial_{x_2}|\vec{u}|^2 \end{pmatrix}(s)\,\mathrm{d}s \\ &=: \mathbf{S}_{ab}(t) \begin{pmatrix} \eta_0 \\ u_{1,0} \\ u_{2,0} \end{pmatrix} + \mathcal{N}_2\left(\begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix}\right). \end{aligned} \tag{3.34}$$

The above case covers KdV-KdV and BBM-BBM regimes, but we distinguish them below for simplicity.

KdV-KdV regime. As same as the one-dimensional case, one may assume that $a = c = 1$. Then, the system (3.2) has the form

$$\begin{cases} \partial_t \eta + \nabla \cdot (\Delta \vec{u} + \vec{u} + \vec{u}\eta) = 0, \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^2, \\ \partial_t \vec{u} + \nabla \left(\Delta \eta + \eta + \frac{1}{2}|\vec{u}|^2 \right) = 0. \end{cases}$$

Analogously, we have

$$\varrho_K(|\xi|) = 1 - |\xi|^2. \tag{3.35}$$

Due to $\varrho_K(|\xi|) \in \mathbb{R}$, the semigroup has the form

$$\mathcal{F}(\mathbf{S}_K(t)) = \begin{pmatrix} \widehat{J}_1^K(\xi,t) & \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^K(\xi,t) & \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^K(\xi,t) \\ \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^K(\xi,t) & \frac{\xi_1^2}{|\xi|^2}\widehat{J}_1^K(\xi,t) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^K(\xi,t) \\ \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^K(\xi,t) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^K(\xi,t) & \frac{\xi_2^2}{|\xi|^2}\widehat{J}_1^K(\xi,t) \end{pmatrix}.$$

where

$$\widehat{J}_1^K(\xi,t) = \cos(|\xi|\varrho_K(|\xi|)t), \quad \widehat{J}_2^K(\xi,t) = \sin(|\xi|\varrho_K(|\xi|)t). \tag{3.36}$$

BBM-BBM regime Let $a = c = 0$ and $b = \mathrm{d} = 1/6$. Then, the system (3.2) has the form

$$\begin{cases} \left(1 - \frac{1}{6}\Delta\right)\partial_t \eta + \nabla \cdot (\vec{u} + \vec{u}\eta) = 0, \\ \left(1 - \frac{1}{6}\Delta\right)\partial_t \vec{u} + \nabla \left(\eta + \frac{1}{2}|\vec{u}|^2\right) = 0, \end{cases} \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^2.$$

Analogously, we have

$$\varrho_B(|\xi|) = \left(1 + \frac{1}{6}|\xi|^2\right)^{-1}.$$

Due to $\varrho_B \in \mathbb{R}$ and $\varsigma_B(|\xi|) = 1$, the semigroup is

$$\mathcal{F}(\mathbf{S}_B(t)) = \begin{pmatrix} \widehat{J}_1^B(\xi,t) & \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^B(\xi,t) & \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^B(\xi,t) \\ \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^B(\xi,t) & \frac{\xi_1^2}{|\xi|^2}\widehat{J}_1^B(\xi,t) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^B(\xi,t) \\ \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^B(\xi,t) & \frac{\xi_1\xi_2}{|\xi|^2}\widehat{J}_1^B(\xi,t) & \frac{\xi_2^2}{|\xi|^2}\widehat{J}_1^B(\xi,t) \end{pmatrix},$$

where

$$\widehat{J}_1^B(\xi,t) = \cos(|\xi|\varrho_B(|\xi|)t), \quad \widehat{J}_2^B(\xi,t) = \sin(|\xi|\varrho_B(|\xi|)t). \tag{3.37}$$

3.3 Proof of Theorem 3.2

3.3.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$

Before proving Theorem 1, we address the following two lemmas, which play key roles in our proof.

Lemma 3.7. *Let $\sigma(\xi)$ be as in (3.13). For $|\xi| \sim N$, we have the following relation:*

$$\sigma(\xi) = \sqrt{\frac{ac}{b\mathrm{d}}} + \tilde{\sigma}(\xi),$$

where N is sufficiently large and $\tilde{\sigma}(\xi) = O(\xi^{-2})$ as $|\xi| \to \infty$.

Proof. A straightforward computation gives

$$(\sigma(\xi))^2 = \frac{(1-a\xi^2)(1-c\xi^2)}{(1+b\xi^2)(1+\mathrm{d}\xi^2)} = \frac{ac}{b\mathrm{d}}\left(1 + \frac{\alpha\xi^2+\beta}{(1+b\xi^2)(1+\mathrm{d}\xi^2)}\right),$$

where

$$\alpha = -(b+\mathrm{d}) - \frac{b\mathrm{d}(a+c)}{ac} \quad \text{and} \quad \beta = \frac{b\mathrm{d}}{ac} - 1,$$

which implies

$$\sigma(\xi) = \sqrt{\frac{ac}{b\mathrm{d}}}\sqrt{1 + \frac{\alpha\xi^2+\beta}{(1+b\xi^2)(1+\mathrm{d}\xi^2)}}.$$

Using the binomial series expansion, we know

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{1}{2}x + O(x^2) \quad \text{for} \quad |x| < 1.$$

Thus, we can write

$$\sigma(\xi) = \sqrt{\frac{ac}{b\mathrm{d}}} + \tilde{\sigma}(\xi),$$

where

$$\tilde{\sigma}(\xi) = \sqrt{\frac{ac}{b\mathrm{d}}} \cdot \frac{\alpha\xi^2+\beta}{2(1+b\xi^2)(1+\mathrm{d}\xi^2)} + O(\xi^{-4}) \ \text{ as } |\xi| \gg 1,$$

since

$$\left|\frac{\alpha\xi^2+\beta}{(1+b\xi^2)(1+\mathrm{d}\xi^2)}\right| \ll 1 \ \text{ for } |\xi| \gg 1.$$

This concludes the proof. ■

Lemma 3.7 enables us to capture a specific nonlinear interaction among other interactions which makes non-smoothness of the flow, see Lemma 3.8 below.

Lemma 3.8. *Let $N \gg 1$ be sufficiently large, $T = \frac{1}{100N}$ and $0 \leqslant s \leqslant t \leqslant T$. If $|\xi_1|, |\xi - \xi_1| \sim N$ and $|\xi| \sim 1$, then*

$$\left(\frac{1+\mathrm{d}\xi^2}{1+b\xi^2}\right)\frac{h(\xi_1)}{h(\xi)}\widehat{L}_2(t-s,\xi)\widehat{L}_2(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1) + \frac{1}{2}\widehat{L}_1(t-s,\xi)\widehat{L}_1(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1) \geqslant \frac{1}{32},$$

where h and $\widehat{L}_j$, $j = 1, 2$ are as in (3.12) and (3.15), respectively.

Proof. A direct observation gives

$$\left|\frac{1+\mathrm{d}\xi^2}{1+b\xi^2}\right| \leqslant \max\left(1, \frac{\mathrm{d}}{b}\right).$$

From the definition of h and the sizes of ξ and ξ_1, we immediately know

$$\left|\frac{h(\xi_1)}{h(\xi)}\right| \lesssim \max\left(1, \frac{ac}{b\mathrm{d}}\right).$$

Moreover, we also know from Lemma 3.7 that

$$\xi\sigma(\xi) = \sqrt{\frac{ac}{b\mathrm{d}}}\xi + O(\xi^{-1}),$$

as $|\xi| \to \infty$. On one hand, the conditions $|\xi| \sim 1$ and $0 \leqslant s \leqslant t \leqslant T$ with $T = \frac{1}{100N}$ yield

$$|\sin(\xi\sigma(\xi)(t-s))| \lesssim \frac{1}{N},$$

which implies

$$\frac{h(\xi_1)}{h(\xi)}\left|\widehat{L}_2(t-s,\xi)\widehat{L}_2(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\right| \leqslant \frac{1}{32},$$

for sufficiently large N. On the other hand, since

$$|\xi|\sigma(\xi)|t-s|, |\xi_1|\sigma(\xi_1)|s|, |\xi-\xi_1|\sigma(\xi-\xi_1)|s| \leqslant \frac{\pi}{3},$$

for sufficiently large N, we obtain

$$\frac{1}{2}\widehat{L}_1(t-s,\xi)\widehat{L}_1(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1) \geqslant \frac{1}{16}.$$

Collecting all, we complete the proof of Lemma 3.8. ■

***Proof of Theorem** 3.2* (1). We use a contradiction argument. Suppose that the flow map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous in H^s, $s < -\frac{1}{2}$. Then, from Proposition 3.6, the map $\vec{v}_0 \mapsto A_2(\vec{v}_0)$ is also continuous, where

$$A_2(\vec{v}_0) = \mathcal{N}_2(A_1(\vec{v}_0), A_1(\vec{v}_0)).$$

In what follows, we are going to prove that the map $\vec{v}_0 \mapsto A_2(\vec{v}_0)$ violates the following inequality:

$$\|A_2(\vec{v}_0)\|_{X_T^{s'}} \lesssim \|\vec{v}_0\|_s^2,$$

for $s < -\frac{1}{2}$ and $s' \in \mathbb{R}$, which completes the proof.

Let $\vec{v}_0$ be an initial datum, which will be chosen later. Using (3.16), we write

$$\vec{v}_1 = \begin{pmatrix}\eta_1\\ u_1\end{pmatrix} = S(t)\begin{pmatrix}\eta_0\\ u_0\end{pmatrix} = \begin{pmatrix}L_1\eta_0 - h(-\mathrm{i}\partial_x)L_2u_0\\ -(h(-\mathrm{i}\partial_x))^{-1}L_2\eta_0 + L_1u_0\end{pmatrix}, \tag{3.38}$$

where h and L_j, $j = 1, 2$, are as in (3.12) and (3.15), respectively. Let

$$A_2(\vec{v_0}) = \int_0^t S(t-s)\partial_x \begin{pmatrix} (1-b\partial_x^2)^{-1}(\eta_1 u_1) \\ (1-\mathrm{d}\partial_x^2)^{-1}\dfrac{u_1^2}{2} \end{pmatrix}(s)\,\mathrm{d}s =: \int_0^t \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix}\,\mathrm{d}s, \tag{3.39}$$

as in (3.17).

For N large enough, we choose the initial data η_0 as the zero function and u_0 as a large frequency localized function, more precisely, $\vec{v_0} = (\eta_0, u_0)$ so that

$$\widehat{\eta_0} = 0 \quad \text{and} \quad \widehat{u}_0(\xi) = N^{-s}\chi_{\mathcal{A}_N}(\xi), \tag{3.40}$$

where χ is the characteristic function and the set $\mathcal{A}_N$ is given by

$$\mathcal{A}_N = \left\{\xi \in \mathbb{R} : N - \frac{1}{2} \leqslant |\xi| \leqslant N + \frac{1}{2}\right\}.$$

Note that $\|u_0\|_{H^s} \sim 1$. Inserting the initial data (3.40) into (3.38), thus (3.39), we obtain $Q_2 = Q_{21} + Q_{22}$, where

$$\widehat{Q}_{21} = \frac{\mathrm{i}\xi}{h(\xi)(1+b\xi^2)}\int_{\mathbb{R}} h(\xi_1)\widehat{L}_2(t-s,\xi)\widehat{L}_2(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1$$

and

$$\widehat{Q}_{22} = \frac{\mathrm{i}\xi}{2(1+\mathrm{d}\xi^2)}\int_{\mathbb{R}} \widehat{L}_1(t-s,\xi)\widehat{L}_1(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1.$$

A precise computation is given in Appendix 3.B. From the supports of $\widehat{u}_0(\xi_1)$ and $\widehat{u}_0(\xi-\xi_1)$, the possible values of ξ satisfy

$$2N-1 \leqslant |\xi| \leqslant 2N+1 \quad \text{or} \quad |\xi| \leqslant 1.$$

Moreover, by Lemma 3.7, we have

$$|\xi_1\sigma(\xi_1)|, |(\xi-\xi_1)\sigma(\xi-\xi_1)| \sim \sqrt{\frac{ac}{b\mathrm{d}}}N + O(N^{-1}),$$

for sufficiently large N. Set $t := \frac{1}{100N}$. Then, from Lemma 3.8, we obtain

$$\begin{aligned}
\|A_2(\vec{v_0})\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})} &\geqslant \left\| \langle\xi\rangle^{s'} \int_0^t \begin{pmatrix} \widehat{Q}_1 \\ \widehat{Q}_2 \end{pmatrix}\,\mathrm{d}s \right\|_{(L^2\times L^2)(|\xi|\leqslant 1)} \\
&\geqslant \left\| \langle\xi\rangle^{s'} \int_0^t \widehat{Q}_2\mathrm{d}s \right\|_{L^2(|\xi|\leqslant 1)} \\
&\geqslant \frac{1}{32}\left\| \langle\xi\rangle^{s'} \frac{\mathrm{i}\xi}{1+\mathrm{d}\xi^2}\int_0^t\int_{\mathbb{R}} \widehat{u}_0(\xi_1)\widehat{u}_0(\xi-\xi_1)\,\mathrm{d}\xi_1\mathrm{d}s \right\|_{L^2(|\xi|\leqslant 1)} \\
&\gtrsim N^{-2s-1}\|\xi\|_{L^2(|\xi|\leqslant 1)},
\end{aligned}$$

which does not guarantee the uniform boundedness of $\|A_2(\vec{v_0})\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})}$ for $s < -\frac{1}{2}$. This complete the proof. ∎

3.3.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$

In order to prove Theorem 3.2 (2), we need a modification of Lemma 3.8, which was essential to prove the generic case. We have

Lemma 3.9. *Let $N \gg 1$ be sufficiently large and $T = \frac{1}{100N^3}$ and $0 \leqslant s \leqslant t \leqslant T$. If $|\xi_1|, |\xi - \xi_1| \sim N$ and $|\xi| \sim 1$, then we have*

$$\widehat{L_2^K}(t-s,\xi)\widehat{L_2^K}(s,\xi_1)\widehat{L_1^K}(s,\xi-\xi_1) + \frac{1}{2}\widehat{L_1^K}(t-s,\xi)\widehat{L_1^K}(s,\xi_1)\widehat{L_1^K}(s,\xi-\xi_1) \geqslant \frac{1}{32},$$

where σ_K and $\widehat{L_j^K}$, $j = 1, 2$ are as in (3.19), respectively.

Proof. The proof is analogous to the proof of Lemma 3.8, thus we omit the details. ■

***Proof of Theorem** 3.2* (2). The proof follows the proof of Theorem 3.2 (1). Suppose that the flow map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous in H^s, $s < -3/2$.

Let $\vec{v}_0 = (\eta_0, u_0)$ be initial data given by (3.40), and

$$\vec{v}_1 = \begin{pmatrix} \eta_1 \\ u_1 \end{pmatrix} = S_K(t) \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix},$$

where $S_K(t)$ is given by (3.18). Then, we write

$$A_2(\vec{v_0}) = \int_0^t S_K(t-s)\partial_x \begin{pmatrix} \eta_1 u_1 \\ \dfrac{u_1^2}{2} \end{pmatrix}(s)\, \mathrm{d}s =: \int_0^t \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} \mathrm{d}s.$$

A straightforward computation enables us to decompose Q_1 and Q_2 as $Q_1 = Q_{11} + Q_{12}$ and $Q_2 = Q_{21} + Q_{22}$, where

$$\widehat{Q}_{11} = -\mathrm{i}\xi \int_{\mathbb{R}} \widehat{L}_1^K(t-s,\xi)\widehat{L}_2^K(s,\xi_1)\widehat{L}_1^K(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1,$$

$$\widehat{Q}_{12} = -\frac{\mathrm{i}\xi}{2} \int_{\mathbb{R}} \widehat{L}_2^K(t-s,\xi)\widehat{L}_1^K(s,\xi_1)\widehat{L}_1^K(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1,$$

$$\widehat{Q}_{21} = \mathrm{i}\xi \int_{\mathbb{R}} \widehat{L}_2^K(t-s,\xi)\widehat{L}_2^K(s,\xi_1)\widehat{L}_1^K(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1$$

and

$$\widehat{Q}_{22} = \frac{\mathrm{i}\xi}{2} \int_{\mathbb{R}} \widehat{L}_1^K(t-s,\xi)\widehat{L}_1^K(s,\xi_1)\widehat{L}_1^K(s,\xi-\xi_1)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\mathrm{d}\xi_1,$$

where L_j^K, $j = 1, 2$, are given in (3.19). On the supports of $\widehat{u}_0(\xi_1)$ and $\widehat{u}_0(\xi-\xi_1)$, the resulting frequency ξ possibly lies in the regions

$$2N - 1 \leqslant |\xi| \leqslant 2N + 1 \quad \text{or} \quad |\xi| \leqslant 1.$$

Moreover, for sufficiently large N, we observe

$$|\xi_1\sigma_K(\xi_1)|, |(\xi-\xi_1)\sigma_K(\xi-\xi_1)| \sim N^3.$$

Set $t := \frac{1}{100N^3}$. From Lemma 3.9, we conclude

$$\begin{aligned}\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})} &\geqslant \left\|\int_0^t \widehat{Q}_2 \mathrm{d}s\right\|_{L^2(|\xi|\leqslant 1)} \\ &\geqslant \frac{1}{32}\left\|\langle\xi\rangle^{s'} \mathrm{i}\xi \int_0^t \int_{\mathbb{R}} \widehat{u}_0(\xi_1)\widehat{u}_0(\xi-\xi_1)\ \mathrm{d}\xi_1 \mathrm{d}s\right\|_{L^2_\xi(|\xi|\leqslant 1))} \\ &\gtrsim N^{-2s-3}\|\xi\|_{L^2(|\xi|\leqslant 1)},\end{aligned}$$

which does not guarantee the uniform boundedness of $\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})}$ for $s < -\frac{3}{2}$ and this completes the proof. ∎

3.4 Proof of Theorem 3.3

3.4.1 Generic regime: $a,\ c < 0$ and $b,\ \mathrm{d} > 0$

We first address the following lemma, which plays a similar role as Lemma 3.8.

Lemma 3.10. *Let $N \gg 1$ be sufficiently large, and $T = \frac{1}{100N}$ and $0 \leqslant s \leqslant t \leqslant T$. If $|\kappa|, |\xi-\kappa| \sim N$, $|\xi| \sim 1$ and $(\kappa-\xi)\cdot\kappa > 0$, then we have*

$$\begin{aligned}&\frac{(1+\mathrm{d}|\xi|^2)}{(1+b|\xi|^2)}\frac{\xi\cdot\kappa}{|\xi||\kappa|}\frac{\varsigma(|\xi-\kappa|)}{\varsigma(|\xi|)}\widehat{J}_2(t,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa) \\ &\qquad + \frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|}\frac{1}{2}\widehat{J}_1(t,\xi)\widehat{J}_1(s,\xi-\kappa)\widehat{J}_1(s,\kappa) \geqslant \frac{1}{16}\left(\frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|}-\frac{1}{2}\right)\end{aligned}$$

where $\cdot$ denotes the standard inner product in Euclidean space, and ς, $\widehat{J}_j$, $j = 1, 2$ are as in (3.22), (3.28), *respectively.*

Proof. A direct computation yields

$$\left|\frac{\varsigma(|\xi-\kappa|)}{\varsigma(|\xi|)}\right| \leqslant \max\left(1, \frac{ab}{cb}\right) \quad \text{and} \quad \left|\frac{\xi\cdot\kappa}{|\xi||\kappa|}\frac{(1+\mathrm{d}|\xi|^2)}{(1+b|\xi|^2)}\right| \leqslant \max\left(1, \frac{\mathrm{d}}{b}\right).$$

Note that Lemma 3.7 is valid for ϱ, thus

$$|\kappa|\varrho(|\kappa|) = \sqrt{\frac{ac}{b\mathrm{d}}}|\kappa| + O(|\kappa|^{-1}),$$

as $|\kappa| \to \infty$. On one hand, since $|\xi| \sim 1$ and $0 \leqslant s \leqslant t \leqslant T$ with $T = \frac{1}{100N}$, we know $|\sin(|\xi|\varrho(|\xi|)(t-s))| \lesssim \frac{1}{N}$, hence

$$\left|\frac{\xi\cdot\kappa}{|\xi||\kappa|}\frac{\varsigma(|\xi-\kappa|)}{\varsigma(|\xi|)}\frac{(1+\mathrm{d}|\xi|^2)}{(1+b|\xi|^2)}\widehat{J}_2(t-s,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\right| \leqslant \frac{1}{32},$$

for N large enough. On the other hand, since

$$|\xi|\varrho(|\xi|)|t-s|, |\kappa|\varrho(|\kappa|)|s|, |\xi-\kappa|\varrho(|\xi-\kappa|)|s| \leqslant \frac{\pi}{3},$$

we have

$$\frac{1}{2}\widehat{J}_1(t-s,\xi)\widehat{J}_1(s,\kappa)\widehat{J}_1(s,\xi-\kappa) \geqslant \frac{1}{16}.$$

We complete the proof from the last frequency condition

$$\frac{(\kappa-\xi)\cdot\kappa}{|\kappa-\xi||\xi|} > 0.$$

■

Remark 3.3. The last condition $(\kappa-\xi)\cdot\kappa > 0$ in Lemma 3.10 is not artificial under the rest conditions $|\kappa|, |\xi-\kappa| \sim N$ and $|\xi| \sim 1$, since the low resulting frequency from two high frequencies interaction occurs only when two high frequencies lie in the opposite side around the origin. A precise observation will be seen in the proof of Theorem 3.3 below.

Proof of Theorem *3.3* (1). Analogously to Theorem 3.2 (1), suppose that the flow map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous in H^s, $s < -\frac{1}{2}$.

Let $\vec{v}_0 = (\eta_0, u_{01}, u_{02})$ be an initial data to be chosen later. Let

$$\vec{v}_1 = \begin{pmatrix} \eta_1 \\ u_{11} \\ u_{12} \end{pmatrix} = \mathbf{S}(t)\begin{pmatrix} \eta_0 \\ u_{01} \\ u_{02} \end{pmatrix} = \begin{pmatrix} J_1\eta_0 + \varsigma(|\nabla|)\frac{\partial_{x_1}}{|\nabla|}J_2u_{01} + \varsigma(|\nabla|)\frac{\partial_{x_2}}{|\nabla|}J_2u_{02} \\ \frac{\partial_{x_1}}{\varsigma(|\nabla|)|\nabla|}J_2\eta_0 + \frac{-\partial_{x_1}^2}{|\nabla|^2}J_1u_{01} + \frac{-\partial_{x_1}\partial_{x_2}}{|\nabla|^2}J_1u_{02} \\ \frac{\partial_{x_2}}{\varsigma(|\nabla|)|\nabla|}J_2\eta_0 + \frac{-\partial_{x_1}\partial_{x_2}}{|\nabla|^2}J_1u_{01} + \frac{-\partial_{x_2}^2}{|\nabla|^2}J_1u_{02} \end{pmatrix}, \tag{3.41}$$

where J_1, J_2 are defined in (3.28), and $\varsigma(|\xi|)$ is given in (3.25).

For N large enough, set

$$\begin{aligned}\mathcal{S}_N := &\left\{\kappa\in\mathbb{R}^2 : N-\frac{1}{2}\leqslant\kappa_1\leqslant N+\frac{1}{2} \text{ and } |\kappa_2|\leqslant 1\right\} \\ &\cup\left\{\kappa\in\mathbb{R}^2 : -N-\frac{1}{2}\leqslant\kappa_1\leqslant -N+\frac{1}{2} \text{ and } |\kappa_2|\leqslant 1\right\} =: \mathcal{S}_N^+\cup\mathcal{S}_N^-.\end{aligned}$$

We choose the initial data $\vec{v}_0 = (\eta_0, u_{01}, u_{02})$ as

$$\eta_0 = 0, \quad \text{and} \quad u_{01} = u_{02} = \psi_N, \tag{3.42}$$

where $\widehat{\psi}_N(\xi) = N^{-s}\chi_{\mathcal{S}_N}(\xi)$. Note that $\|\psi_N\|_{H^s} \sim 1$. Moreover, on the supports of $\widehat{\psi}_N(\xi-\kappa)$ and $\widehat{\psi}_N(\kappa)$, the resulting frequency ξ belongs to

$$S_L := \{\xi\in\mathbb{R}^2 : |\xi_j| \leqslant 2, j = 1,2\}$$

or

$$S_H := \left\{\xi \in \mathbb{R}^2 : 2N-1 \leqslant |\xi_1| \leqslant 2N+1, \quad |\xi_2| \leqslant 2\right\}.$$

From (3.29), we have

$$A_2(\vec{v_1}) = \int_0^t \mathbf{S}(t-s)\begin{pmatrix} (1-b\Delta)^{-1}[\partial_{x_1}(\eta_1 u_{11}) + \partial_{x_2}(\eta_1 u_{12})] \\ 2^{-1}(1-\mathrm{d}\Delta)^{-1}\partial_{x_1}(u_{11}^2 + u_{12}^2) \\ 2^{-1}(1-\mathrm{d}\Delta)^{-1}\partial_{x_2}(u_{11}^2 + u_{12}^2) \end{pmatrix} =: \int_0^t \begin{pmatrix} Q_1(s) \\ Q_2(s) \\ Q_3(s) \end{pmatrix} \mathrm{d}s. \quad (3.43)$$

Inserting (3.42) into (3.41), thus (3.43), we obtain $\widehat{Q}_2$ as $Q_2 = Q_{21} + Q_{22}$, where

$$\widehat{Q}_{21} = -\,\mathrm{i}\xi_1 \int_{\mathbb{R}^2} \frac{p(\xi,\kappa)}{(1+b|\xi|^2)} \frac{\xi\cdot\kappa}{|\xi||\kappa|} \frac{\varsigma(|\xi-\kappa|)}{\varsigma(|\xi|)} \widehat{J}_2(t,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa,$$

$$\widehat{Q}_{22} = -\,\mathrm{i}\xi_1 \int_{\mathbb{R}^2} \frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|} \frac{p(\xi,\kappa)}{(1+\mathrm{d}|\xi|^2)} \frac{1}{2} \widehat{J}_1(t,\xi)\widehat{J}_1(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa,$$

for

$$p(\xi,\kappa) = \frac{(\xi_1+\xi_2-\kappa_1-\kappa_2)}{|\xi-\kappa|}\frac{(\kappa_1+\kappa_2)}{|\kappa|}. \quad (3.44)$$

See Appendix 3.C for precise computations of Q_j, $j = 1,2,3$.

For $\xi-\kappa,\ \kappa \in\subset \mathcal{S}_N$, the resulting frequency ξ lies in $\mathcal{S}_L$ only when the vectors $\xi-\kappa$ and κ are located in the opposite side around the origin, that is, $\xi-\kappa \in \mathcal{S}_N^+$ and $\kappa \in \mathcal{S}_N^-$ or $\kappa \in \mathcal{S}_N^+$ and $\xi-\kappa \in \mathcal{S}_N^-$. Then, the angle β between two vectors $\xi-\kappa$ and κ satisfies

$$\pi - \tan^{-1}\left(\frac{1}{N-\frac{1}{2}}\right) < \beta < \pi + \tan^{-1}\left(\frac{1}{N-\frac{1}{2}}\right).$$

Then, by taking sufficiently large N, we make

$$-\frac{3}{4} \geqslant \cos(\beta) = \frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}. \quad (3.45)$$

Moreover, on each support of $\phi_N(\xi-\kappa)$ and $\phi_N(\kappa)$, we have

$$-p(\xi,\kappa) = \frac{(\kappa_1-\xi_1+\kappa_2-\xi_2)}{|\xi-\kappa|}\frac{(\kappa_1+\kappa_2)}{|\kappa|} \geqslant \frac{\left(N-\frac{3}{2}\right)^2}{1+(\frac{1}{2}+N)^2} \geqslant \frac{3}{4}, \quad (3.46)$$

provided that $N > 16$. Thus, for $t := \frac{1}{100N}$, by Lemma 3.10, (3.45) and (3.46), we get

$$\begin{aligned} \|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})} &\geqslant \left\| \langle\xi\rangle^{s'} \int_0^t \begin{pmatrix} \widehat{Q}_1 \\ \widehat{Q}_2 \\ \widehat{Q}_3 \end{pmatrix} \mathrm{d}s \right\|_{(L^2\times L^2\times L^2)(\mathbb{R}^2)} \\ &\geqslant \left\| \langle\xi\rangle^{s'} \int_0^t \widehat{Q}_2 \mathrm{d}s \right\|_{L^2(\mathbb{R}^2)} \\ &\geqslant \frac{1}{64}\cdot\frac{3}{4} \left\| \langle\xi\rangle^{s'} \frac{\xi_1}{(1+\mathrm{d}|\xi|^2)} \int_0^t \int_{\mathbb{R}^2} \widehat{\psi}_N(\kappa)\widehat{\psi}_N(\xi-\kappa)\mathrm{d}\kappa\mathrm{d}s \right\|_{L^2_\xi(S_L)} \\ &\gtrsim N^{-2s-1} \left\| \langle\xi\rangle^{s'} \frac{\xi_1}{(1+\mathrm{d}|\xi|^2)} \right\|_{L^2_\xi(S_L)} \end{aligned}$$

which does not guarantee the uniform boundedness of $\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R}^2)\times H^{s'}(\mathbb{R}^2)}$ for $s < -\frac{1}{2}$. This completes the proof. ∎

Remark 3.4. Thanks to the symmetric structure of Q_2 and Q_3 (see Appendix 3.C for more details), the same conclusion is obtained by taking Q_3 as a target instead of Q_2 in the proof of Theorem 3.3 (1).

3.4.2 KdV-KdV regime: $b = \mathrm{d} = 0$ and $a = c = \frac{1}{6}$

[1] For ϱ_K is given in (3.35), from the observation below

$$||\xi|\varrho_K(|\xi|)| = |\xi||1 - |\xi|^2| \sim |\xi|^3,$$

for $|\xi| \gg 1$, we have

Lemma 3.11. *Let $N \gg 1$ be sufficiently large, $T = \frac{1}{100N^3}$, and $0 \leqslant s \leqslant t \leqslant T$. If $|\kappa|, |\xi-\kappa| \sim N$, $|\xi| \sim 1$ and $(\kappa - \xi)\cdot\kappa > 0$, then we have*

$$\begin{aligned}\frac{\xi\cdot\kappa}{|\xi||\kappa|}\widehat{J}_2^K(t,\xi)\widehat{J}_2^K(s,\xi-\kappa)\widehat{J}_1^K(s,\kappa) + \frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|}\frac{1}{2}\widehat{J}_1^K(t,\xi)\widehat{J}_1^K(s,\xi-\kappa)\widehat{J}_1^K(s,\kappa)\\ \geqslant \frac{1}{16}\left(\frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|} - \frac{1}{2}\right)\end{aligned}$$

where $\widehat{J_j^K}$, $j = 1, 2$ are as in (3.36).

Proof. The proof is analogous to the proof of Lemma 3.10, thus we omit the details. ■

Proof of Theorem 3.3 (2)

Suppose that the flow map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous in H^s, $s < -\frac{3}{2}$. Taking the initial data $\vec{v}_0 = (\eta_0, u_{01}, u_{02})$ as in (3.40), we obtain

$$\vec{v}_1 = \begin{pmatrix}\eta_1\\ u_{11}\\ u_{12}\end{pmatrix} = \mathbf{S}_K(t)\begin{pmatrix}\eta_0\\ u_{01}\\ u_{02}\end{pmatrix} = \begin{pmatrix} J_1^K\eta_0 + \frac{\partial_{x_1}}{|\nabla|}J_2^K u_{01} + \frac{\partial_{x_2}}{|\nabla|}J_2^K u_{02}\\ \frac{\partial_{x_1}}{|\nabla|}J_2^K\eta_0 + \frac{-\partial_{x_1}^2}{|\nabla|^2}J_1^K u_{01} + \frac{-\partial x_1\partial_{x_2}}{|\nabla|^2}J_1^K u_{02}\\ \frac{\partial_{x_2}}{|\nabla|}J_2^K\eta_0 + \frac{-\partial_{x_1}\partial_{x_2}}{|\nabla|^2}J_1^K u_{01} + \frac{-\partial_{x_2}^2}{|\nabla|^2}J_1^K u_{02}\end{pmatrix},$$

thus so

$$A_2(\vec{v_1}) = \int_0^t \mathbf{S}_K(t-s)\begin{pmatrix}\partial_{x_1}(\eta_1 u_{11}) + \partial_{x_2}(\eta_1 u_{12})\\ 2^{-1}\partial_{x_1}(u_{11}^2 + u_{12}^2)\\ 2^{-1}\partial_{x_2}(u_{11}^2 + u_{12}^2)\end{pmatrix} =: \int_0^t \begin{pmatrix}Q_1(s)\\ Q_2(s)\\ Q_3(s)\end{pmatrix}\mathrm{d}s.$$

Analogously to the proof of Theorem 3.3 (1), we focus on Q_2. A direct computation yields that Q_2 is decomposed as $Q_2 = Q_{21} + Q_{22}$, where

$$\begin{aligned}\widehat{Q}_{21} =& -\mathrm{i}\xi_1\int_{\mathbb{R}^2}\frac{\xi\cdot\kappa}{|\xi||\kappa|}p(\xi,\kappa)\widehat{J}_2^K(t,\xi)\widehat{J}_2^K(s,\xi-\kappa)\widehat{J}_1^K(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa\\ \widehat{Q}_{22} =& \mathrm{i}\xi_1\int_{\mathbb{R}^2}\frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}p(\xi,\kappa)\frac{1}{2}\widehat{J}_1^K(t,\xi)\widehat{J}_1^K(s,\xi-\kappa)\widehat{J}_1^K(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa,\end{aligned}$$

[1]By scaling, we make $a = c = 1$ as one dimensional case.

for p given by (3.44). Note that all computations in Appendix 3.D are available for KdV-KdV case (by putting $b = \mathrm{d} = 0$ and $a = c = 1$). Thus, for $t := \frac{1}{100N^3}$, by Lemma 3.11 , (3.45) and (3.46), we get

$$\begin{aligned}\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})} &\geqslant \frac{1}{64}\cdot\frac{3}{4}\left\|\langle\xi\rangle^{s'}\xi_1\int_0^t\int_{\mathbb{R}^2}\widehat{\psi}_N(\kappa)\widehat{\psi}_N(\xi-\kappa)\mathrm{d}\kappa\mathrm{d}s\right\|_{L^2_\xi(S_L)}\\ &\gtrsim N^{-2s-3}\left\|\langle\xi\rangle^{s'}\xi_1\right\|_{L^2_\xi(S_L)}\end{aligned}$$

which does not guarantee the uniform boundedness of $\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R}^2)\times H^{s'}(\mathbb{R}^2)}$ for $s < -\frac{3}{2}$. This completes the proof.

3.4.3 BBM-BBM regime: $a = c = 0$ and $b = \mathrm{d} = 1/6$

We have

Lemma 3.12. *Let $N \gg 1$ be sufficiently large, and $T = \frac{1}{1000}$ and $0 \leqslant s \leqslant t \leqslant T$. If $|\kappa|, |\xi-\kappa| \sim N$ and $|\xi| \sim 1$, then we have*

$$\begin{aligned}\frac{\xi\cdot\kappa}{|\xi||\kappa|}\widehat{J}_2^B(t,\xi)\widehat{J}_2^B(s,\xi-\kappa)\widehat{J}_1^B(s,\kappa)+&\frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|}\frac{1}{2}\widehat{J}_1^B(t,\xi)\widehat{J}_1^B(s,\xi-\kappa)\widehat{J}_1^B(s,\kappa)\\ &\geqslant \frac{1}{16}\left(\frac{(\kappa-\xi)\cdot\kappa}{|\xi-\kappa||\kappa|}-\frac{1}{2}\right),\end{aligned}$$

where ς and $\widehat{J_j^B}$ with $j = 1, 2$ are as in (3.22), (3.37) and respectively.

Proof. The proof is analogous to the proof of Lemma 3.10 with

$$|\xi|\varrho_B(|\xi|) = \frac{|\xi|}{(1+\frac{1}{6}|\xi|^2)} \sim |\xi|^{-1},$$

for $|\xi| \gg 1$. ■

Proof of Theorem 3.3 (3)

Suppose that the flow map $\vec{v}_0 \mapsto \vec{v}[\vec{v}_0]$ is continuous in H^s, $s < 0$. Taking the same initial data $\vec{v}_0 = (\eta_0, u_{01}, u_{02})$ as in (3.40), we compute

$$\vec{v}_1 = \begin{pmatrix}\eta_1\\ u_{11}\\ u_{12}\end{pmatrix} = \mathbf{S}_B(t)\begin{pmatrix}\eta_0\\ u_{01}\\ u_{02}\end{pmatrix} = \begin{pmatrix} J_1^B\eta_0 + \frac{\partial_{x_1}}{|\nabla|}J_2^Bu_{01} + \frac{\partial_{x_2}}{|\nabla|}J_2^Bu_{02}\\ \frac{\partial_{x_1}}{|\nabla|}J_2^B\eta_0 + \frac{-\partial_{x_1}^2}{|\nabla|^2}J_1^Bu_{01} + \frac{-\partial x_1\partial_{x_2}}{|\nabla|^2}J_1^Bu_{02}\\ \frac{\partial_{x_2}}{|\nabla|}J_2^B\eta_0 + \frac{-\partial_{x_1}\partial_{x_2}}{|\nabla|^2}J_1^Bu_{01} + \frac{-\partial_{x_2}^2}{|\nabla|^2}J_1^Bu_{02}\end{pmatrix},$$

where J_1^B and J_2^B defined by (3.37), thus so

$$A_2(\vec{v}_1) = \int_0^t \mathbf{S}_B(t-s)\begin{pmatrix}(1-\frac{1}{6}\Delta)^{-1}[\partial_{x_1}(\eta_1u_{11}) + \partial_{x_2}(\eta_1u_{12})]\\ 2^{-1}(1-\frac{1}{6}\Delta)^{-1}\partial_{x_1}(u_{11}^2+u_{12}^2)\\ 2^{-1}(1-\frac{1}{6}\Delta)^{-1}\partial_{x_2}(u_{11}^2+u_{12}^2)\end{pmatrix} =: \int_0^t\begin{pmatrix}Q_1(s)\\ Q_2(s)\\ Q_3(s)\end{pmatrix}\mathrm{d}s.$$

Note also that all computations in Appendix 3.D are available for KdV-KdV case (by putting $a = c = 0$ and $b = \mathrm{d} = 1/6$).Thus, Q_2 is decomposed as $Q_2 = Q_{21} + Q_{22}$, where

$$\widehat{Q}_{21} = -\mathrm{i}\xi_1 \int_{\mathbb{R}^2} \frac{\xi \cdot \kappa}{|\xi||\kappa|} \frac{p(\xi,\kappa)}{(1+\frac{1}{6}|\xi|^2)} \widehat{J}_2^B(t,\xi)\widehat{J}_2^B(s,\xi-\kappa)\widehat{J}_1^B(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa,$$

$$\widehat{Q}_{22} = \mathrm{i}\xi_1 \int_{\mathbb{R}^2} \frac{(\xi-\kappa)\cdot \kappa}{|\xi-\kappa||\kappa|} \frac{p(\xi,\kappa)}{(1+\frac{1}{6}|\xi|^2)} \frac{1}{2} \widehat{J}_1^B(t,\xi)\widehat{J}_1^B(s,\xi-\kappa)\widehat{J}_1^B(s,\kappa)\widehat{\psi}_N(\xi-\kappa)\widehat{\psi}_N(\kappa)\mathrm{d}\kappa,$$

for p given by (3.44). Thus, for $t := \frac{1}{1000}$, by Lemma 3.12, (3.45) and (3.46), we get

$$\begin{aligned} \|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})\times H^{s'}(\mathbb{R})} &\geqslant \frac{1}{64}\cdot\frac{3}{4}\left\|\langle\xi\rangle^{s'}\xi_1 \int_0^t \int_{\mathbb{R}^2} \widehat{\psi}_N(\kappa)\widehat{\psi}_N(\xi-\kappa)\mathrm{d}\kappa\mathrm{d}s\right\|_{L^2_\xi(S_L)} \\ &\gtrsim N^{-2s}\left\|\langle\xi\rangle^{s'}\xi_1\right\|_{L^2_\xi(S_L)}, \end{aligned}$$

which does not guarantee the uniform boundedness of $\|A_2(\vec{v}_0)\|_{H^{s'}(\mathbb{R}^2)\times H^{s'}(\mathbb{R}^2)}$ for $s < 0$, this ends the proof.

■

Appendices

3.A Local Well-Posedness

This section briefly shows the local well-posedness of (3.1) and (3.2) including BBM-BBM case, but not KdV-KdV case. This result may not be optimal except for BBM-BBM case. The well-known well-posedness theorem is given by

Theorem 3.13. *Let $n = 1, 2$. Fix $s \geqslant 0$. For any $(u_0, v_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, there exists a $T(u_0, v_0) > 0$ and a unique solution $(u, v) \in X_T^s$ (for a suitable solution space X_T^s) of the initial value problem* (3.1). *The maximal existence time $T = T_s$ for the solution has the property that*

$$T_s \geqslant \frac{C_s}{\|(u_0, v_0)\|_{H^s(\mathbb{R}) \times H^s(\mathbb{R})}}$$

where the positive constant C_s depends only on s.

It is well-known that Theorem 3.13 immediately follows from multilinear estimates, thus in what follows, we only focus on bilinear estimate (see Section 3.A.3 below).

3.A.1 Notations

We define Bessel and Riesz potentials (J^s and D^s, respectively) of order $-s$, $s \in \mathbb{R}$, as Fourier multipliers by

$$J^s f(x) := \mathcal{F}^{-1}\left((1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}\right) \quad \text{and} \quad D^s f(x) := \mathcal{F}^{-1}\left(|\xi|^s \widehat{f}\right).$$

In particular, $\sqrt{-\Delta} = D^1 = D$ is the Fourier multiplier of the symbol $|\xi|$.

3.A.2 Littlewood-Paley Decomposition

This section devotes to explaining the Littlewood-Paley decomposition, which is an useful way to improve the bilinear estimate for the local well-posedness theory. As well-known, the Littlewood-Paley decomposition is a particular way to write a single function as a superposition of a countably infinite family of functions of varying frequencies.

Let $\varphi(\xi)$ be a real-valued radially symmetric bump function on $\mathbb{R}^n$ with the support $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ which is identical to 1 on the set $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$ and is decreasing on $\{\xi \in \mathbb{R}^n : 1 \leqslant |\xi| \leqslant 2\}$. Define a dyadic number $N \in 2^{\mathbb{Z}_{\geqslant 0}}$ of the form $N = 2^k$, $k \in \mathbb{Z}_{\geqslant 0}$. Let denote $\varphi_1 = \varphi$ and define

$$\varphi_N(\xi) = \varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right), \quad N \geqslant 2.$$

By construction, the sequence of φ_N satisfies

$$\sum_{N \geqslant 1 \,:\, \text{dyadic}} \varphi_N(\xi) \equiv 1.$$

We simply write $\sum_{N\geqslant 1}$ by dropping "dyadic". This provides a typical *partition of unity* which allows to define the projection operator (one of the so-called *Littlewood-Paley* projection operator) on $L^2(\mathbb{R}^n)$ by $P_N f(x) = \mathcal{F}^{-1}\left(\varphi_N(\xi)\widehat{f}\right)(x)$. Using the projection operators, one decompose any function f in $L^2(\mathbb{R})$ as

$$f = \sum_{N\geqslant 1} P_N f.$$

We sometimes denote $P_N f$ by simply f_N. Note that f_N belongs to any Sobolev space $H^s(\mathbb{R}^n)$, $s \geqslant 0$ (or smooth), whenever $f \in L^2(\mathbb{R}^n)$. The following lemma is well-known *Bernstein inequality*, which is to upgrade low Lebesgue integrability to high Lebesgue integrability with the price of some powers of N:

Lemma 3.14 (Bernstein's inequalities)**.** *Let $f \in L^2(\mathbb{R}^n)$, $1 \leqslant p \leqslant q \leqslant \infty$, and $s \geqslant 0$. Then, we have*

$$\left\|D^{\pm s} f_N\right\|_{L^p} \sim N^{\pm s} \left\|f_N\right\|_{L^p} \tag{3.47}$$

and

$$\left\|f_N\right\|_{L^q} \lesssim N^{\frac{n}{p}-\frac{n}{q}} \left\|f_N\right\|_{L^p}. \tag{3.48}$$

The implicit constants in both (3.47) *and* (3.48) *depend only on s, n, p and q.*

3.A.3 Bilinear estimates

Lemma 3.15. *Let $f, g \in L^2(\mathbb{R}^n)$. Then, we have*

$$\left\|P_1(fg)\right\|_{L^2} \lesssim \left\|f\right\|_{L^2} \left\|g\right\|_{L^2}.$$

Proof. The proof follows from $|(\widehat{f} * \widehat{g})(\xi)| \lesssim \|f\|_{L^2} \|g\|_{L^2}$ and $\int_{|\xi|\leqslant 1} \mathrm{d}\xi \lesssim 1$. ∎

Lemma 3.16 (Refined bilinear estimate)**.** *Let $s \geqslant \frac{n-2}{2}$ and $f, g \in H^s(\mathbb{R}^n)$. Then, we have*

$$\left\|J^{-1}D(fg)\right\|_{H^s} \lesssim \left\|f\right\|_{H^s} \left\|g\right\|_{H^s}. \tag{3.49}$$

Proof. From the duality argument, it suffices for the left-hand side of (3.49) to estimate

$$\int_{\mathbb{R}^2} \left(J^{s-1}D(fg)\right) w \,\mathrm{d}x,$$

where $w \in L^2$ with $\|w\|_{L^2} = 1$. We make the Littlewood-Paley decomposition of f, g and w as

$$f = \sum_{N_1\geqslant 1} f_{N_1}, \quad g = \sum_{N_2\geqslant 1} g_{N_2} \quad \text{and} \quad w = \sum_{N\geqslant 1} w_N,$$

respectively. Without loss of generality, we may assume $N_1 \leqslant N_2$. By Lemma 3.15, we are now reduced to establishing

$$\sum_{N>1} \sum_{N_1,N_2\geqslant 1} N^{-1+s} \int f_{N_1} g_{N_2} w_N \,\mathrm{d}x \lesssim \|f\|_{H^s} \|g\|_{H^s}. \tag{3.50}$$

We separate (3.50) into two cases : $N \sim N_2 \gtrsim N_1$ and $N_2 \sim N_1 \gg N$.

(Case I.) $N \sim N_2 \gtrsim N_1$. Using Hölder's and Bernstein's (3.48) inequalities, one has

$$\int f_{N_1} g_{N_2} w_N \, \mathrm{d}x \leqslant \|f_{N_1}\|_{L^\infty} \|g_{N_2}\|_{L^2} \|w_N\|_{L^2} \lesssim N_1^{\frac{n}{2}} \|f_{N_1}\|_{L^2} \|g_{N_2}\|_{L^2} \|w_N\|_{L^2}.$$

With this, we further reduce (3.50) to

$$\sum_{N>1} \sum_{N_2 \sim N} \sum_{N_1 \lesssim N_2} N^{-1+s} N_2^{-s} N_1^{\frac{n}{2}-s} \|f_{N_1}\|_{H^s} \|g_{N_2}\|_{H^s} \|w_N\|_{L^2} \lesssim \|f\|_{H^s} \|g\|_{H^s}, \tag{3.51}$$

thanks to (3.47). We denote the multiplier in the left-hand side of (3.51) by $m(N, N_1, N_2)$, namely, $m(N, N_1, N_2) = N^{-1+s} N_2^{-s} N_1^{1-s}$. Note that the number of N_2 is finite, and then

$$\sum_N \sum_{N_2 \sim N} \|g_{N_2}\|_{L^2}^2 \sim \sum_N \|g_N\|_{L^2}^2.$$

With this observation, Cauchy-Schwarz inequality yields

$$\text{LHS of (3.51)} \lesssim \left(\sup_{N>1} \sum_{\substack{1 \leqslant N_1 \lesssim N_2 \\ N_2 \sim N}} m^2(N, N_1, N_2) \right)^{\frac{1}{2}} \|f\|_{H^s} \|g\|_{H^s} \|w\|_{L^2}.$$

Hence, it remains to prove

$$\sup_{N>1} \sum_{N_2 \sim N} \sum_{1 \leqslant N_1 \lesssim N_2} N^{-2+2s} N_2^{-2s} N_1^{n-2s} \lesssim 1. \tag{3.52}$$

This is sometimes referred as *Schur's test*, see, for instance, [33, Lemma 3.11] for more details. Then, one gets if $s > \frac{n}{2}$

$$\sum_{N_2 \sim N} \sum_{1 \leqslant N_1 \lesssim N_2} N^{-2+2s} N_2^{-2s} N_1^{n-2s} \lesssim \sum_{N_2 \sim N} N^{-2+2s} N_2^{-2s} \lesssim N^{-2},$$

otherwise ($s \leqslant \frac{n}{2}$),

$$\sum_{N_2 \sim N} \sum_{1 \leqslant N_1 \lesssim N_2} N^{-2+2s} N_2^{-2s} N_1^{n-2s} \lesssim \sum_{N_2 \sim N} N^{-2+2s} N_2^{n-4s} \lesssim N^{n-2-2s}.$$

Thus, (3.52) holds true if $s \geqslant \frac{n-2}{2}$.

(Case II.) $N_2 \sim N_1 \gg N$. Analogously, using Hölder's and Bernstein's (3.48) inequalities, one obtains

$$\int f_{N_1} g_{N_2} w_N \, \mathrm{d}x \leqslant \|f_{N_1}\|_{L^2} \|g_{N_2}\|_{L^2} \|w_N\|_{L^\infty} \lesssim N^{\frac{n}{2}} \|f_{N_1}\|_{L^2} \|g_{N_2}\|_{L^2} \|w_N\|_{L^2},$$

which ensures (3.49) provided that

$$\sup_{N_2>1} \sum_{N_1 \sim N_2} \sum_{1 < N \lesssim N_1} N_2^{-2s} N_1^{-2s} N^{n-2+2s} \lesssim 1. \tag{3.53}$$

One immediately obtains (3.53) for $s \geqslant \frac{n-2}{2}$, we thus complete the proof. ■

Remark 3.5. Lemma 3.16 slightly improves the bilinear estimates by Grisvard [16], particularly, validity of the bilinear estimates in $H^{\frac{n-2}{2}}$. This seems to facilitate the global well-posedness in the energy space for 4-dimensional problem.

Remark 3.6. Replacing the Bessel potential J^{-1} by $(1-b\Delta)^{-1}$ or $(1-\mathrm{d}\Delta)^{-1}$ in Lemma 3.16 affects only the constant in the left-hand side, precisely, the implicit constant should depend on b or d.

Remark 3.7. The standard Picard iteration method immediately assures the (smooth) local well-posedness in $H^s(\mathbb{R}^n)$, $s = \max(0, \frac{n-2}{2})$. Thus, BBM-BBM case is optimal in the sense that the flow map is analytic.

3.B Decomposition of (Q_1, Q_2) on the generic regime: one-dimensional case

This section provides a precise computation of Q_j, $j = 1, 2$, for one-dimensional case. Recall (3.39)

$$A_2(\vec{v_1}) = \int_0^t S(t-s) \begin{pmatrix} (1-b\partial_x^2)^{-1}\partial_x(\eta_1 u_1) \\ 2^{-1}(1-\mathrm{d}\partial_x^2)^{-1}\partial_x(u_1^2) \end{pmatrix} =: \int_0^t \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \mathrm{d}s,$$

where the linear propagator S is given in (3.16). A direct computation yields

$$\begin{aligned} \widehat{Q}_1 &= \mathrm{i}\xi \left[\frac{1}{(1+b\xi^2)} \widehat{L}_1(t-s,\xi)\widehat{\eta_1 u_1}(s,\xi) - \frac{h(\xi)}{2(1+\mathrm{d}\xi^2)} \widehat{L}_2(t-s,\xi)\widehat{u_1^2}(s,\xi) \right] \\ \widehat{Q}_2 &= \mathrm{i}\xi \left[-\frac{1}{h(\xi)(1+b\xi^2)} \widehat{L}_2(t-s,\xi)\widehat{\eta_1 u_1}(s,\xi) + \frac{1}{2(1+\mathrm{d}\xi^2)} \widehat{L}_1(t-s,\xi)\widehat{u_1^2}(s,\xi) \right] \end{aligned} \tag{3.54}$$

Inserting the initial data $\vec{v}_0 = (\eta_0, u_0) = (0, \phi)$ into (3.38), we have

$$\mathcal{F}\begin{pmatrix} \eta_1 \\ u_1 \end{pmatrix} = \mathcal{F}\left(S(t) \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} \right) = \begin{pmatrix} -h(\xi)\widehat{L}_2(t,\xi)\widehat{\phi} \\ \widehat{L}_1(t,\xi)\widehat{\phi} \end{pmatrix}.$$

With this, a direct computation gives

$$\widehat{\eta_1 u_1}(\xi) = \int_{\mathbb{R}} \widehat{\eta_1}(\xi_1)\widehat{u_1}(\xi-\xi_1)\mathrm{d}\xi_1 = -\int_{\mathbb{R}} h(\xi_1)\widehat{L}_2(t,\xi_1)\widehat{L}_1(t,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)$$

and

$$\widehat{u_1 u_1}(\xi) = \int_{\mathbb{R}} \widehat{u_1}(\xi-\xi_1)\widehat{u_1}(\xi_1)\mathrm{d}\xi_1 = \int_{\mathbb{R}} \widehat{L}_1(t,\xi_1)\widehat{L}_1(t,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\mathrm{d}\xi_1.$$

Thus, by inserting them into (3.54), we conclude that

$$\begin{aligned} \widehat{Q}_1 = -\mathrm{i}\xi \Bigg[&\frac{1}{(1+b\xi^2)} \int_{\mathbb{R}} h(\xi_1)\widehat{L}_1(t-s,\xi)\widehat{L}_2(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\mathrm{d}\xi_1 \\ &+ \frac{h(\xi)}{2(1+\mathrm{d}\xi^2)} \int_{\mathbb{R}} \widehat{L}_2(t-s,\xi)\widehat{L}_1(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\mathrm{d}\xi_1 \Bigg] \end{aligned}$$

and

$$\widehat{Q}_2 = \mathrm{i}\xi\Bigg[\frac{1}{h(\xi)(1+b\xi^2)}\int_{\mathbb{R}} h(\xi_1)\widehat{L}_2(t-s,\xi)\widehat{L}_2(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\mathrm{d}\xi_1 + \frac{1}{2(1+\mathrm{d}\xi^2)}\int_{\mathbb{R}} \widehat{L}_1(t-s,\xi)\widehat{L}_1(s,\xi_1)\widehat{L}_1(s,\xi-\xi_1)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1)\mathrm{d}\xi_1\Bigg].$$

3.C Decomposition of (Q_1, Q_2, Q_3): generic case

This section provides a precise computation of Q_j, $j = 1, 2, 3$, for two-dimensional case. Recall (3.27)

$$\mathcal{F}\left(\mathbf{S}(t)\begin{pmatrix} f \\ g \\ h \end{pmatrix}\right) = \begin{pmatrix} \widehat{J}_1(t,\xi)\widehat{f} + \varsigma(|\xi|)\frac{\mathrm{i}}{|\xi|}\widehat{J}_2(t,\xi)\big[\xi_1\widehat{g} + \xi_2\widehat{h}\big] \\ \frac{\mathrm{i}\xi_1}{\varsigma(|\xi|)|\xi|}\widehat{J}_2(t,\xi)\widehat{f} + \frac{\xi_1}{|\xi|^2}\widehat{J}_1(t,\xi)\big[\xi_1\widehat{g} + \xi_2\widehat{h}\big] \\ \frac{\mathrm{i}\xi_2}{\varsigma(|\xi|)|\xi|}\widehat{J}_2(t,\xi)\widehat{f} + \frac{\xi_2}{|\xi|^2}\widehat{J}_1(t,\xi)\big[\xi_1\widehat{g} + \xi_2\widehat{h}\big] \end{pmatrix}, \tag{3.55}$$

where J_j, $j = 1, 2$, are given in (3.28). Inserting the initial data $\vec{v}_0 = (\eta_0, u_{01}, u_{02}) = (0, \phi, \phi) = (f, g, h)$ into (3.55), we have

$$\mathcal{F}\begin{pmatrix} \eta_1 \\ u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} \varsigma(|\xi|)\frac{\mathrm{i}(\xi_1+\xi_2)}{|\xi|}\widehat{J}_2(t,\xi)\widehat{\phi} \\ \frac{\xi_1(\xi_1+\xi_2)}{|\xi|^2}\widehat{J}_1(t,\xi)\widehat{\phi} \\ \frac{\xi_2(\xi_1+\xi_2)}{|\xi|^2}\widehat{J}_1(t,\xi)\widehat{\phi} \end{pmatrix}. \tag{3.56}$$

Recall the integral part in (3.29)

$$A_2(\vec{v_1}) = \int_0^t \mathbf{S}(t-s)\begin{pmatrix} (1-b\Delta)^{-1}[\partial_{x_1}(\eta_1 u_{11}) + \partial_{x_2}(\eta_1 u_{12})] \\ 2^{-1}(1-\mathrm{d}\Delta)^{-1}\partial_{x_1}(u_{11}^2 + u_{12}^2) \\ 2^{-1}(1-\mathrm{d}\Delta)^{-1}\partial_{x_2}(u_{11}^2 + u_{12}^2) \end{pmatrix} =: \int_0^t \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}\mathrm{d}s.$$

A direct computation on Q_j, $j = 1, 2, 3$, yields

$$\begin{aligned} \widehat{Q}_1 =& \frac{\mathrm{i}}{(1+b|\xi|^2)}\widehat{J}_1(t,\xi)\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] - \frac{\varsigma(|\xi|)|\xi|}{2(1+\mathrm{d}|\xi|^2)}\widehat{J}_2(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right), \\ \widehat{Q}_2 =& -\frac{\xi_1}{\varsigma(|\xi|)|\xi|}\frac{\widehat{J}_2(t,\xi)}{(1+b|\xi|^2)}\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] + \frac{\mathrm{i}\xi_1}{2(1+\mathrm{d}|\xi|^2)}\widehat{J}_1(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right), \\ \widehat{Q}_3 =& -\frac{\xi_2}{\varsigma(|\xi|)|\xi|}\frac{\widehat{J}_2(t,\xi)}{(1+b|\xi|^2)}\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] + \frac{\mathrm{i}\xi_2}{2(1+\mathrm{d}|\xi|^2)}\widehat{J}_1(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right). \end{aligned} \tag{3.57}$$

With (3.56), we first compute nonlinear interactions

$$\widehat{\eta_1 u_{11}}(\xi) = \mathrm{i}\int_{\mathbb{R}^2} \varsigma(|\kappa|)p(\xi,k)\frac{(\xi_1-\kappa_1)}{|\xi-\kappa|}\widehat{J}_2(t,\kappa)\widehat{J}_1(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

$$\widehat{\eta_1 u_{12}}(\xi) = \mathrm{i}\int_{\mathbb{R}^2} \varsigma(|\kappa|)p(\xi,k)\frac{(\xi_2-\kappa_2)}{|\xi-\kappa|}\widehat{J}_2(t,\kappa)\widehat{J}_1(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

$$\widehat{u_{11}^2}(\xi) = \int_{\mathbb{R}^2} p(\xi,k)\frac{\kappa_1}{|\kappa|}\frac{(\xi_1-\kappa_1)}{|\xi-\kappa|}\widehat{J}_1(t,\kappa)\widehat{J}_1(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

and

$$\widehat{u_{12}^2}(\xi) = \int_{\mathbb{R}^2} p(\xi,k)\frac{\kappa_2}{|\kappa|}\frac{(\xi_2-\kappa_2)}{|\xi-\kappa|}\widehat{J}_1(t,\kappa)\widehat{J}_1(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

where

$$p(\xi,\kappa) = \frac{(\xi_1+\xi_2-\kappa_1-\kappa_2)}{|\xi-\kappa|}\frac{(\kappa_1+\kappa_2)}{|\kappa|}.$$

Inserting them into (3.57), we obtain

$$\begin{aligned}\widehat{Q}_1 =& -\frac{1}{(1+b|\xi|^2)}\int_{\mathbb{R}^2}\varsigma(|\kappa|)p(\xi,\kappa)\frac{\xi\cdot(\xi-\kappa)}{|\xi-\kappa|}\widehat{J}_1(t,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\ &-\frac{\varsigma(|\xi|)|\xi|}{2(1+\mathrm{d}|\xi|^2)}\int_{\mathbb{R}^2}p(\xi,\kappa)\frac{\kappa\cdot(\xi-\kappa)}{|\xi-\kappa||\kappa|}\widehat{J}_2(t,\xi)\widehat{J}_1(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa,\end{aligned}$$

$$\begin{aligned}\widehat{Q}_2 =& -\frac{\mathrm{i}\xi_1}{(1+b|\xi|^2)}\int_{\mathbb{R}^2}p(\xi,\kappa)\frac{\varsigma(|\kappa|)}{\varsigma(|\xi|)}\frac{\xi\cdot(\xi-\kappa)}{|\xi||\xi-\kappa|}\widehat{J}_2(t,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\ &+\frac{\mathrm{i}\xi_1}{2(1+\mathrm{d}|\xi|^2)}\int_{\mathbb{R}^2}\frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}p(\xi,\kappa)\widehat{J}_1(t,\xi)\widehat{J}_1(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\end{aligned}$$

and

$$\begin{aligned}\widehat{Q}_3 =& -\frac{\mathrm{i}\xi_2}{(1+b|\xi|^2)}\int_{\mathbb{R}^2}p(\xi,\kappa)\frac{\varsigma(|\kappa|)}{\varsigma(|\xi|)}\frac{\xi\cdot(\xi-\kappa)}{|\xi||\xi-\kappa|}\widehat{J}_2(t,\xi)\widehat{J}_2(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\ &+\frac{\mathrm{i}\xi_2}{2(1+\mathrm{d}|\xi|^2)}\int_{\mathbb{R}^2}\frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}p(\xi,\kappa)\widehat{J}_1(t,\xi)\widehat{J}_1(s,\xi-\kappa)\widehat{J}_1(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa.\end{aligned}$$

3.D Decomposition of (Q_1, Q_2, Q_3): $a = c$ and $b = \mathrm{d} \geqslant 0$

This section provides a precise computation of Q_j, $j = 1, 2, 3$, for two-dimensional case when $a = c$ and $b = \mathrm{d} \geqslant 0$. Note that the decomposition here is valid for both KdV-KdV and BBM-BBM cases. Recall (3.32)

$$\mathcal{F}\left(\mathbf{S}_{ab}(t)\begin{pmatrix} f\\ g\\ h\end{pmatrix}\right) = \begin{pmatrix} \widehat{J}_1^{ab}(t,\xi)\widehat{f} + \frac{\mathrm{i}}{|\xi|}\widehat{J}_2^{ab}(t,\xi)\big[\xi_1\widehat{g}+\xi_2\widehat{h}\big]\\ \frac{\mathrm{i}\xi_1}{|\xi|}\widehat{J}_2^{ab}(t,\xi)\widehat{f} + \frac{\xi_1}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi)\big[\xi_1\widehat{g}+\xi_2\widehat{h}\big]\\ \frac{\mathrm{i}\xi_2}{|\xi|}\widehat{J}_2^{ab}(t,\xi)\widehat{f} + \frac{\xi_2}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi)\big[\xi_1\widehat{g}+\xi_2\widehat{h}\big]\end{pmatrix},$$

where J_j^{ab}, $j = 1, 2$, are given in (3.33). Inserting the initial data $\vec{v}_0 = (\eta_0, u_{01}, u_{02}) = (0, \phi, \phi) = (f, g, h)$ into $\vec{v}_1 = \mathbf{S}_{ab}\vec{v}_0$, we have

$$\mathcal{F}\begin{pmatrix}\eta_1\\ u_{11}\\ u_{12}\end{pmatrix} = \begin{pmatrix}\frac{\mathrm{i}(\xi_1+\xi_2)}{|\xi|}\widehat{J}_2^{ab}(t,\xi)\widehat{\phi}\\ \frac{\xi_1(\xi_1+\xi_2)}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi)\widehat{\phi}\\ \frac{\xi_2(\xi_1+\xi_2)}{|\xi|^2}\widehat{J}_1^{ab}(t,\xi)\widehat{\phi}\end{pmatrix}. \tag{3.58}$$

Recall the integral part in (3.34)

$$A_2(\vec{v_1}) = \int_0^t \mathbf{S}_{ab}(t-s)\begin{pmatrix} (1-b\Delta)^{-1}[\partial_{x_1}(\eta_1 u_{11}) + \partial_{x_2}(\eta_1 u_{12})] \\ 2^{-1}(1-b\Delta)^{-1}\partial_{x_1}(u_{11}^2+u_{12}^2) \\ 2^{-1}(1-b\Delta)^{-1}\partial_{x_2}(u_{11}^2+u_{12}^2)\end{pmatrix} =: \int_0^t \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \mathrm{d}s.$$

A direct computation on Q_j, $j=1,2,3$, yields

$$\begin{aligned}
\widehat{Q}_1 =& \frac{\mathrm{i}}{(1+b|\xi|^2)}\widehat{J}_1^{ab}(t,\xi)\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] - \frac{|\xi|}{2(1+b|\xi|^2)}\widehat{J}_2^{ab}(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right),\\
\widehat{Q}_2 =& -\frac{\xi_1}{|\xi|}\frac{\widehat{J}_2^{ab}(t,\xi)}{(1+b|\xi|^2)}\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] + \frac{\mathrm{i}\xi_1}{2(1+b|\xi|^2)}\widehat{J}_1^{ab}(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right),\\
\widehat{Q}_3 =& -\frac{\xi_2}{|\xi|}\frac{\widehat{J}_2^{ab}(t,\xi)}{(1+b|\xi|^2)}\left[\xi_1(\widehat{\eta_1 u_{11}}) + \xi_2(\widehat{\eta_1 u_{12}})\right] + \frac{\mathrm{i}\xi_2}{2(1+b|\xi|^2)}\widehat{J}_1^{ab}(t,\xi)\left(\widehat{u_{11}^2} + \widehat{u_{12}^2}\right).
\end{aligned} \tag{3.59}$$

With (3.58), we first compute nonlinear interactions

$$\widehat{\eta_1 u_{11}}(\xi) = \mathrm{i}\int_{\mathbb{R}^2} p(\xi,k)\frac{(\xi_1-\kappa_1)}{|\xi-\kappa|}\widehat{J}_2^{ab}(t,\kappa)\widehat{J}_1^{ab}(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

$$\widehat{\eta_1 u_{12}}(\xi) = \mathrm{i}\int_{\mathbb{R}^2} p(\xi,k)\frac{(\xi_2-\kappa_2)}{|\xi-\kappa|}\widehat{J}_2^{ab}(t,\kappa)\widehat{J}_1^{ab}(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

$$\widehat{u_{11}^2}(\xi) = \int_{\mathbb{R}^2} p(\xi,k)\frac{\kappa_1}{|\kappa|}\frac{(\xi_1-\kappa_1)}{|\xi-\kappa|}\widehat{J}_1^{ab}(t,\kappa)\widehat{J}_1^{ab}(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

and

$$\widehat{u_{12}^2}(\xi) = \int_{\mathbb{R}^2} p(\xi,k)\frac{\kappa_2}{|\kappa|}\frac{(\xi_2-\kappa_2)}{|\xi-\kappa|}\widehat{J}_1^{ab}(t,\kappa)\widehat{J}_1^{ab}(t,\xi-\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\,\mathrm{d}\kappa,$$

where

$$p(\xi,\kappa) = \frac{(\xi_1+\xi_2-\kappa_1-\kappa_2)}{|\xi-\kappa|}\frac{(\kappa_1+\kappa_2)}{|\kappa|}.$$

Inserting them into (3.59), we obtain

$$\begin{aligned}
\widehat{Q}_1 =& -\frac{1}{(1+b|\xi|^2)}\int_{\mathbb{R}^2} p(\xi,\kappa)\frac{\xi\cdot(\xi-\kappa)}{|\xi-\kappa|}\widehat{J}_1^{ab}(t,\xi)\widehat{J}_2^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\
&-\frac{|\xi|}{2(1+b|\xi|^2)}\int_{\mathbb{R}^2} p(\xi,\kappa)\frac{\kappa\cdot(\xi-\kappa)}{|\xi-\kappa||\kappa|}\widehat{J}_2^{ab}(t,\xi)\widehat{J}_1^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa,\\
\widehat{Q}_2 =& -\frac{\mathrm{i}\xi_1}{(1+b|\xi|^2)}\int_{\mathbb{R}^2} p(\xi,\kappa)\frac{\xi\cdot(\xi-\kappa)}{|\xi||\xi-\kappa|}\widehat{J}_2^{ab}(t,\xi)\widehat{J}_2^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\
&+\frac{\mathrm{i}\xi_1}{2(1+b|\xi|^2)}\int_{\mathbb{R}^2}\frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}p(\xi,\kappa)\widehat{J}_1^{ab}(t,\xi)\widehat{J}_1^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa
\end{aligned}$$

and

$$\begin{aligned}
\widehat{Q}_3 =& -\frac{\mathrm{i}\xi_2}{(1+b|\xi|^2)}\int_{\mathbb{R}^2} p(\xi,\kappa)\frac{\xi\cdot(\xi-\kappa)}{|\xi||\xi-\kappa|}\widehat{J}_2^{ab}(t,\xi)\widehat{J}_2^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa\\
&+\frac{\mathrm{i}\xi_2}{2(1+b|\xi|^2)}\int_{\mathbb{R}^2} p(\xi,\kappa)\frac{(\xi-\kappa)\cdot\kappa}{|\xi-\kappa||\kappa|}\widehat{J}_1^{ab}(t,\xi)\widehat{J}_1^{ab}(s,\xi-\kappa)\widehat{J}_1^{ab}(s,\kappa)\widehat{\phi}(\kappa)\widehat{\phi}(\xi-\kappa)\mathrm{d}\kappa.
\end{aligned}$$

Bibliography

[1] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. math. 171, 485–541 (2008).

[2] C. J. Amick, *Regularity and uniqueness of solutions to the Boussinesq system of equations*, Journal of Differential Equations 54, 231–247 (1984).

[3] D. M. Ambrose, J. L. Bona and T. Milgrom, *Global solutions and ill-posedness for the Kaup system and related Boussinesq systems*, Indiana Univ. Math. J. 68 (2019), no. 4, 1173–1198.

[4] E.S. Bao, R.M. Chen and Q. Liu, *Existence and Symmetry of Ground States to the Boussinesq abcd Systems*, Arch Rational Mech Anal 216, 569–591 (2015). https://doi.org/10.1007/s00205-014-0814-1.

[5] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear. Sci. Vol. 12: pp. 283–318 (2002).

[6] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory*, Nonlinearity 17 (2004) 925–952.

[7] J. L. Bona, T. Colin and D. Lannes, *Long wave approximations for water waves*, Arch. Ration. Mech. Anal. 178 (2005), no. 3, 373–410.

[8] J. Bona and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*, Discrete & Continuous Dynamical Systems - A, 2009, 23 (4): 1241–1252. doi: 10.3934/dcds.2009.23.1241.

[9] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pure Appl. (2) 17 (1872), 55–108.

[10] I. Bejenaru and T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. 233 (2006) 228–259.

[11] C. Burtea, *New long time existence results for a class of Boussinesq-type systems*, Journal des Mathematiques Pures et Appliquees, 106 (2) , pp. 203–236, (2016).

[12] X. Carvajal and M. Panthee, *On ill-posedness for the generalized BBM equation*, Discrete & Continuous Dynamical Systems - A, 2014, 34 (11): 4565–4576. doi: 10.3934/dcds.2014.34.4565.

[13] X. Carvajal and M. Panthee, *On sharp global well-posedness and ill-posedness for a fifth-order KdV-BBM type equation*, Journal of Mathematical Analysis and Applications, 479(1), 688–702(2019). doi:10.1016/j.jmaa.2019.06.045.

[14] Chen, R. M. and Liu, Y., *On the ill-posedness of a weakly dispersive one-dimensional Boussinesq system*, J. Anal. Math. 121 (2013), 299–316.

[15] V. A. Dougalis, D. E. Mitsotakis and J.-C. Saut, *On some Boussinesq systems in two space dimensions: Theory and numerical analysis*, ESAIM: Mathematical Modelling and Numerical Analysis Vol. 41, No. 5. 2007, pp. 825–854.

[16] P. Grisvard, *Quelques propriétés des espaces de Sobolev, utiles dans l'étude des équations de Navier-Stokes (1)* Problèmes d'évolution, non linéaires, Séminaire de Nice (1974-1976).

[17] M. Hieber and S. Monniaux, *Well-posedness results for the Navier-Stokes equation in the rotational framework*, Discrete and Continuous Dynamical Systems- Series A, Volume 33, Issue 11-12, November 2013, Pages 5143-5151.

[18] N. Kishimoto, *Low-regularity bilinear estimates for a quadratic nonlinear Schrödinger equation*, J. Differential Equations 247 (2009), no. 5, 1397–1439.

[19] N. Kishimoto and K. Tsugawa, *Local well-posedness for quadratic nonlinear Schrödinger equations and the "Good" Boussinesq equation*, Differential Integral Equations 23 (2010), no. 5-6, 463–493.

[20] C. Kwak and C. Muñoz, *Asymptotic dynamics for the small data weakly dispersive one-dimensional Hamiltonian ABCD system*, Trans. Amer. Math. Soc. 373 (2020), 1043–1107.

[21] C. Kwak, C. Muñoz, F. Poblete and J.C. Pozo, *The scattering problem for Hamiltonian ABCD Boussinesq systems in the energy space*, Journal de Mathématiques Pures et Appliquées, Volume 127, July 2019, Pages 121–159.

[22] F. Linares, D. Pilod, and J.-C. Saut, *Well-Posedness of Strongly Dispersive Two-Dimensional Surface Wave Boussinesq Systems*, SIAM Journal on Mathematical Analysis 2012 44:6, 4195–4221.

[23] S. Micu and A.F. Pazoto. *Stabilization of a Boussinesq system with localized damping*, JAMA 137, 291–337 (2019). https://doi.org/10.1007/s11854-018-0074-3.

[24] M. Ming, J. C. Saut and P. Zhang, *Long-Time Existence of Solutions to Boussinesq Systems*, SIAM J. Math. Anal., 44(6), 4078–4100.

[25] L. Molinet, *A note on ill posedness for the KdV equation* , Differential Integral Equations,

Volume 24, Number 7/8 (2011), 759–765.

[26] M. Panthee. *On the ill-posedness result for the BBM equation*, Discrete & Continuous Dynamical Systems - A, 2011, 30 (1) : 253-259. doi: 10.3934/dcds.2011.30.253.

[27] J. C. Saut, *Asymptotic models for surface and internal waves*, 29th. Coloquio Brasileiro de Matemática, Publicações Matemáticas IMPA, 2013.

[28] J.-C. Saut, C. Wang, and L. Xu *The Cauhy problem on large time for surface waves Bousinesq systems II*, `https://arxiv.org/abs/1511.08824v1`.

[29] J.-C. Saut and L. Xu, *The Cauchy problem on large time for surface waves Boussinesq systems*, J. Math. Pures Appl. 97(2012) 635–662.

[30] J.-C. Saut and L. Xu, *Long Time Existence for a Strongly Dispersive Boussinesq System*, SIAM Journal on Mathematical Analysis 2020 52:3, 2803–2848.

[31] J.-C. Saut and L. Xu, *Long time existence for the Boussinesq-Full dispersion systems*, Journal of Differential Equations, (2020) 269:3, 2627–2663.

[32] M. E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, Journal of Differential Equations 42, 325–352 (1981)

[33] T. Tao, *Multilinear weighted convolution of L^2 functions and applications to nonlinear dispersive equations*, Amer. J. Math. 123(5) (2001) 839–908.

Chapter 4

Asymptotic stability manifolds for solitons in the generalized Good Boussinesq equation

Abstract. We consider the generalized Good-Boussinesq model in one dimension, with power nonlinearity and data in the energy space $H^1 \times L^2$. This model has solitary waves with speeds $-1 < c < 1$. When $|c|$ approaches 1, Bona and Sachs showed orbital stability of such waves. It is well-known from a work of Liu that for small speeds solitary waves are unstable. In this paper we consider in more detail the long time behavior of zero speed solitary waves, or standing waves. By using virial identities, in the spirit of Kowalczyk, Martel and Muñoz, we construct and characterize a manifold of even-odd initial data around the standing wave for which there is asymptotic stability in the energy space.

Contents

4.1 Introduction . . . 84

4.1.1 Setting . . . 84

4.1.2 Standing waves . . . 86

4.1.3 Main results . . . 87

4.1.4 Idea of the proof . . . 89

4.2 A virial identity for the (gGB) system . . . 91

4.2.1 Decomposition of the solution in a vicinity of the soliton . . . 91

4.2.2 Notation for virial argument . . . 92

4.2.3 Virial estimate . . . 93

4.2.4 End of Proposition 4.3 . . . 99

4.3 Transformed problem and second virial estimates . . . 99

4.3.1 The transformed problem . . . 99

4.3.2 Virial functional for the transformed problem . . . 99

4.3.3 Proof of Proposition 4.8: first computations . . . 100

4.3.4 First technical estimates . . . 102

4.3.5 Controlling error and nonlinear terms . . . 106
4.3.6 End of proof of Proposition 4.8 . . . 108
4.4 Gain of derivatives via transfer estimates . . . 109
4.4.1 A virial estimate related to $\mathcal{M}(t)$ . . . 110
4.4.2 Second set of technical estimates . . . 111
4.4.3 Start of proof of Proposition 4.19 . . . 114
4.4.4 Controlling error terms . . . 116
4.4.5 Controlling nonlinear terms . . . 118
4.4.6 End of proof Proposition 4.19 . . . 119
4.5 A second transfer estimate . . . 120
4.5.1 A virial identity for $\mathcal{N}(t)$ . . . 120
4.5.2 Start of proof of Proposition 4.25 . . . 121
4.5.3 Control of nonlinear terms . . . 124
4.5.4 End of proof of Proposition 4.25 . . . 124
4.6 Proof of Theorem 4.1 . . . 125
4.6.1 Coercivity . . . 125
4.6.2 Proof of Theorem 4.1 . . . 129
4.7 Proof of Theorem 4.2 . . . 133
4.7.1 Conservation of Energy . . . 133
4.7.2 Construction of the graph . . . 134
4.7.3 Uniqueness and Lipschitz regularity . . . 136
Appendices . . . 140
4.A Linear spectral theory for $-\partial_x^2\mathcal{L}$ . . . 140
4.A.1 Asymptotic behavior of fundamental solutions of $-\partial_x^2\mathcal{L}(u) = 0$ . . . 144
4.B Proof of Claims 4.21 and 4.22 . . . 145
4.B.1 Relation between $\partial_x z_{\rm i}$ and $\partial_x v_{\rm i}$ . . . 145
4.B.2 Relation between $\partial_x^2 z_{\rm i}$ and $\partial_x^2 v_{\rm i}$ . . . 147
4.C Proof of Lemma 4.27 . . . 149
Bibliography . . . 150

This work has been recently submitted and can be found at arXiv:2102.01151.

4.1 Introduction

4.1.1 Setting

In the 1870's, J. Boussinesq [7] deduced a system of equations to describe two-dimensional irrotational and inviscid fluids in a uniform rectangular channel with flat bottom. He was the first to give a favorable explanation to the traveling-waves, solitons, or solitary waves solutions discovered by Scott Rusell thirty years earlier [31], which remained in their form and travelled with constant velocity.

In a first order approximation, Boussinesq's matrix model reduces to a scalar, fourth order model

$$\partial_t^2\phi - \partial_x^4\phi - \partial_x^2\phi + \partial_x^2(f(\phi)) = 0, \tag{4.1}$$

However, this model, known as the bad Boussinesq equation, is strongly linearly ill-posed. Consequently, in order to repair this problem, the following equation was proposed [36, 28]:

$$\partial_t^2\phi + \partial_x^4\phi - \partial_x^2\phi + \partial_x^2(f(\phi)) = 0. \tag{4.2}$$

Here the physical model considers the nonlinearity as quadratic, i.e. $f(\phi) = \phi^2$ and $\phi(t,x)$ is a real-valued function. This model is called good Boussinesq, and if formally $u = \phi$ and $v = \partial_x^{-1}\partial_t\phi$, this model has the following representation as 2×2 system:

$$(g\text{GB}) \qquad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x(-\partial_x^2 u + u - f(u)). \end{cases} \tag{4.3}$$

This will be the exact model worked in this paper, which is Hamiltonian, and has the following associated conserved quantities:

$$\begin{aligned} E[u,v] &= \frac{1}{2}\int \left[v^2 + u^2 + (\partial_x u)^2 - 2F(u)\right] && \text{(Energy)}, \\ P[u,v] &= \int uv && \text{(Momentum)}. \end{aligned} \tag{4.4}$$

(Here $\int$ means $\int_{\mathbb{R}} \mathrm{d}x$.) These laws define a standard energy space $(u,v) \in H^1 \times L^2$. As well as the Korteweg-de Vries (KdV) equation, (gGB) is considered as a canonical model of shallow water waves, see [35]. In addition, (gGB) arises in the so-called "nonlinear string equation" describing small nonlinear oscillations in an elastic beam (see [11]).

The study of the Boussinesq-type equations has increased recently, mainly due to the versatility of these models when describing nonlinear phenomena. There are several authors that focus on the good Boussinesq equation. The fundamental works Bona and Sachs [6], using abstract techniques of Kato, proved that the Cauchy problem is locally and globally well-posed for small data, and showed the existence of solitary waves for velocities $c^2 < 1$. Linares [22, 14], using Stricharz estimates, proved that the Cauchy problem is globally well-posed in the energy space in the case of small data. Kishimoto [16], in the case of a quadratic nonlinearity, proved that the Cauchy problem is globally well-posed in $H^s(\mathbb{R})$, for $s \geqslant -1/2$, and ill-posed for $s < -1/2$. In [30], it was proved that small solutions in the energy space must decay to zero as time tends to infinity in proper subsets of space. Recently, Charlier and Lenells [9] developed the inverse scattering transform and a Riemann-Hilbert approach for the quadratic (gGB), which is integrable. In general, solitons (solitary waves in integrable equations) are stable objects. However, this is not the case of good Boussinesq (similar to Klein-Gordon). Indeed, small perturbations of solitons may decay or form singularities in finite time, see [11, 23, 3, 36].

In this paper, we are motivated by the long time behavior problem for solitary waves of the gGB (4.2) in the case where $f(s) = |s|^{p-1}s$ for $p > 1$. A solitary wave is a solution to (4.2) of the form

$$(u, v) = (Q_c, -cQ_c)(x - ct - x_0), \quad |c| < 1, \quad x_0 \in \mathbb{R},$$

with Q_c solving $(c^2-1)Q_c + Q_c'' + f(Q_c) = 0$ in $H^1(\mathbb{R})$. This interesting question has attracted the attention of several authors before us, showing that the behavior of solitary waves in the standard energy space $H^1 \times L^2$ is not an easy problem. Bona and Sachs [6], applying the theory developed by Grillakis, Shatath and Strauss (see [13]), proved that solitary waves are stable if the speed c obeys the condition $(p-1)/4 < c^2 < 1$ and $p > 4$. Li, Ohta, Wu and Xue [21] proved the orbital instability in the degenerate case $1 < p < 5$ and speed $c = (p-1)/4$. Additionally, Kalantarov and Ladyzhenkaya in [15] proved that solutions associated to initial data with nonpositive energy may blow up in some sense. Inspired by this work, Liu [23] showed that there are solutions with initial data arbitrarily near the ground state ($c = 0$) that blow up in finite time.

4.1.2 Standing waves

In the case that f is a pure power nonlinearity of the form $f(s) = |s|^{p-1}s$ for $p > 1$, it is well-known that (up to shifts) standing solitary waves have the form

$$u(t, x) = Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)}, \quad v(t, x) = 0. \tag{4.5}$$

Here, Q satisfies the equation

$$Q''(x) - Q(x) + f(Q(x)) = 0. \tag{4.6}$$

Let us consider a perturbation in (4.3) of Q of the form

$$u(t, x) = Q(x) + w(t, x), \quad v(t, x) = z(t, x).$$

Then one can see that this perturbation satisfies the following linear system at first order:

$$\begin{cases} \partial_t w = \partial_x z \\ \partial_t z = \partial_x \mathcal{L}w, \end{cases} \tag{4.7}$$

where

$$\mathcal{L}(w) = -\partial_x^2 w + V_0(x)w, \quad \text{with} \quad V_0(x) = 1 - f'(Q). \tag{4.8}$$

$\mathcal{L}$ is the classical Schrödinger operator associated to the soliton Q. This operator has been extensively studied in [8] for instance.

Therefore, from (4.7) one has $\partial_t^2 w = \partial_x^2 \mathcal{L}w$. Consequently, for the well-understanding of the problem we require to study the fourth order operator $-\partial_x^2\mathcal{L}$, much in the spirit of the fundamental works by Pego and Weinstein results [32, 33]. In Appendix 4.A, we will prove the following: for any $p > 1$, the linear operator

$$- \partial_x^2 \mathcal{L}(u) = \partial_x^4 u - \partial_x^2 u + \partial_x^2 (pQ^{p-1}u), \tag{4.9}$$

has a unique eigenfunction $\phi_0(x)$ associated to a negative first eigenvalue $-\nu_0^2 < 0$, satisfying

$$\langle \partial_x^{-1}\phi_0, \partial_x^{-1}\phi_0 \rangle = 1, \quad -\partial_x^2 \mathcal{L}(\phi_0) = -\nu_0^2 \phi_0, \quad |\phi_0(x)| \lesssim \mathrm{e}^{-1^-|x|}. \tag{4.10}$$

Note that we also have $\partial_x^{-1}\phi_0$ well-defined, exponentially decreasing and part of L^2. Here $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$, and 1^- is a number slightly below 1. The second eigenvalue of $-\partial_x^2\mathcal{L}$ is 0 but it is also a resonance in the classical sense (in $L^\infty \backslash L^2$), but the unique L^2 eigenvalue is $\phi_1(x) = c_1 Q'(x)$. Therefore, by the Spectral Theorem, orthogonal to ϕ_0 the operator $-\partial_x^2 \mathcal{L}$ is nonnegative. See Appendix 4.A for more details and full proofs of all the previous statements.

Let

$$\boldsymbol{Y}_\pm = \begin{pmatrix} \phi_0 \\ \pm\nu_0 \partial_x^{-1}\phi_0 \end{pmatrix}, \quad \boldsymbol{Z}_\pm = \begin{pmatrix} \partial_x^{-2}\phi_0 \\ \pm\nu_0^{-1}\partial_x^{-1}\phi_0 \end{pmatrix}. \tag{4.11}$$

These are even-odd functions, i.e. the first coordinate is even and the second odd (see Appendix 4.41). The functions $\mathbf{u}_\pm(t,x) = \mathrm{e}^{\pm\nu_0 t}\mathbf{Y}_\pm(x)$ are solutions of the linearized problem (4.7), showing the presence of exponentially stable and unstable linear manifolds relevant for the dynamics of nonlinear solutions in a neighborhood of the soliton.

In that follows, we refers to *global solution* of (4.3) to a function $C([0,\infty), H^1 \times L^2)$ that satisfies (4.3) for all $t \geqslant 0$.

4.1.3 Main results

It is not difficult to realize that (4.3) preserves the even-odd parity in its variables (u, v). In this paper, we will prove that any even-odd small perturbation of the static soliton ($c = 0$) in the energy space, under certain orthogonality condition, is orbitally stable and in fact, it is (locally) asymptotically stable. Furthermore, we will construct a manifold of initial data such that the associated solutions are orbitally stable in $H^1 \times L^2$, and locally asymptotically stable in the space $L^2 \cap L^\infty$. Our first result is:

Theorem 4.1. *Let $p \geqslant 2$. There exists $\delta > 0$ such that if a global even-odd solution $(\phi, \partial_t\partial_x^{-1}\phi)$ of (4.3) satisfies for all $t \geqslant 0$,*

$$\|(\phi, \partial_t\partial_x^{-1}\phi)(t) - (Q, 0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \delta, \tag{4.12}$$

then, for any $\gamma > 0$ small enough and any compact interval I of $\mathbb{R}$,

$$\lim_{t\to+\infty} \left(\|\phi(t) - Q\|_{L^2(I)\cap L^\infty(I)} + \|(1-\gamma\partial_x^2)^{-1}\partial_t\phi(t)\|_{L^2(I)} \right) = 0. \tag{4.13}$$

This is, as far as we understand, the first description of the standing wave dynamics in the Good Boussinesq model, which is unstable by nature. Clearly the data under which (4.12) is satisfied is not empty, the soliton $(Q,0)$ being its most important representative. However, (4.12) cannot define an open set in the energy space as simple as in some stable, subcritical dynamics, such as KdV. Our second result will describe the manifold of initial data leading to (4.12), but first we need to clarify some remarks.

Remark 4.1 (On the lack of decay of derivatives)**.** Estimate (4.13) provides a clean and clear description of the local decay of $\phi(t)$ in the Lebesgue spaces $L^2 \cap L^\infty$. However, no clear description of the derivative $\partial_x \phi(t)$ has been found, which remains an interesting open problem.

Remark 4.2 (On the $\partial_t \partial_x^{-1} \phi$ term)**.** We have been unable to provide a clean description of decay for the second component of the Good Boussinesq system. This is due to some deep problems present at the level of the dynamics. However, (4.13) provides additional information on the decay of a suitable modification of the second variable. The constant γ depends on δ, but it can be taken arbitrarily small if needed.

Remark 4.3 (About general data)**.** The construction performed in this paper uses in several steps the parity of the data. Extending our results to general data is a challenging problem, mainly because one needs to introduce shifts that may affect in a strong fashion the dynamics. We hope to consider this problem in a forthcoming publication.

Remark 4.4 (About the condition $p \geqslant 2$)**.** The condition $p \geqslant 2$ is of technical type, and it is needed to ensure a control on the unstable direction, sufficiently good for our purposes. We believe that the situation for p close to 1 may be very complicated because of the weak decay of the amplitude associated to the unstable direction.

The following result provides a description of the manifold of initial data leading to global solutions for which (4.12) holds.

Let $\delta_0 > 0$, and let $\mathcal{A}_0$ be the manifold given by

$$\mathcal{A}_0 = \left\{ \boldsymbol{\epsilon} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) | \ \boldsymbol{\epsilon} \text{ is even-odd }, \|\boldsymbol{\epsilon}\|_{H^1 \times L^2} < \delta_0 \text{ and } \langle \boldsymbol{\epsilon}, \boldsymbol{Z}_+ \rangle = 0 \right\}. \tag{4.14}$$

Theorem 4.2. *Let $p \geqslant 2$. There exist $C, \delta_0 > 0$ and a Lipschitz function $h : \mathcal{A}_0 \to \mathbb{R}$ with $h(0) = 0$ and $|h(\epsilon)| \leqslant C\|\epsilon\|_{H^1 \times L^2}^{3/2}$ such that, denoting*

$$\mathcal{M} = \{(Q,0) + \epsilon + h(\epsilon) Y_+ \ \textit{with} \ \epsilon \in \mathcal{A}_0\}, \tag{4.15}$$

the following holds:

1. *If $\boldsymbol{\phi}_0 \in \mathcal{M}$ then the solution of* (4.3) *with initial data $\boldsymbol{\phi}_0$ is global and satisfies, for all $t \geqslant 0$,*

$$\|\boldsymbol{\phi}(t) - (Q,0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} \leqslant C \|\boldsymbol{\phi}_0 - (Q,0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}. \tag{4.16}$$

2. *If a global even-odd solution* $\boldsymbol{\phi}$ *of* (4.3) *satisfies, for all* $t \geqslant 0$,

$$\|\boldsymbol{\phi}(t) - (Q,0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leqslant \frac{1}{2}\delta_0, \tag{4.17}$$

then for all $t \geqslant 0$, $\phi(t) \in \mathcal{M}$.

Remark 4.5 (About blow-up)**.** Liu [23] showed that initial data (u_0, v_0) for which $E(u_0, v_0) < 0$, or $E(u_0, v_0) \geqslant 0$ and less than a particular function of $\operatorname{Im}\int \partial_x^{-1} u_0 v_0$ (which is zero in our case), lead to blow up solutions in finite time. In our case, we work with perturbation of the soliton $(Q, 0)$. One can easily check that $E(Q,0) = \frac{p-1}{2(p+1)}\int Q^{p+1} > 0$, therefore we are not in the blow-up regime determined by Liu.

Remark 4.6 (Extension to other models)**.** We believe that our results open the door to the understanding of long time solitary wave dynamics in several other Boussinesq models. We mention for instance the asymptotic stability of *abc*d solitary waves, at least in the zero speed even data case [4, 5], and the more involved case of the Improved Boussinesq solitary wave; see [29] for further details on this challenging problem.

4.1.4 Idea of the proof

The proofs in this paper follow the lines of the ideas used recently by Kowalczyk, Martel and Muñoz in [18] to understand the unstable soliton dynamics in the nonlinear Klein-Gordon equation, and by Kowalczyk, Martel, Muñoz and Van Den Bosch [19] to study the stability properties of kinks for (1+1)-dimensional nonlinear scalar field theories.

More precisely, the proofs are based in a series of localized virial type arguments, similar to the ones used in [1, 2, 18, 19, 17, 25, 27]. In our case, we will use a combination of virials to obtain the integrability in time of the $L^2 \times L^2$-norm of $(\phi(t) - Q, (1-\gamma\partial_x^2)^{-1}\partial_t\phi(t))$, for any $\gamma > 0$ small enough, and in any compact interval I, i.e.,

$$\int_0^\infty \left(\|\phi(t) - Q\|_{L^2(I)}^2 + \|(1-\gamma\partial_x^2)^{-1}\partial_t\phi(t)\|_{L^2(I)}^2\right) \mathrm{d}t < \infty.$$

However, some important issues, not present in the previously mentioned works [18, 19] will appear along the proofs. The beginning of the proof is similar to [18]: The first step is to decompose the solution close to the solitary waves in an adequate way. We will consider $(u_1, u_2) \in H^1 \times L^2$ be an even-odd perturbation of the solitary waves, which are in some sense orthogonal to $\boldsymbol{Y}_+$ and $\boldsymbol{Y}_-$, and the flow on these directions: for a_1, a_2 unique,

$$\begin{cases} u(t,x) = Q(x) + a_1(t)\phi_0(x) + u_1(t,x), \\ v(t,x) = a_2(t)\nu_0\partial_x^{-1}\phi_0(x) + u_2(t,x). \end{cases}$$

Then, we will focus on $(u_1, u_2) \in H^1 \times L^2$, which satisfy the linearized equation (4.7). Following [30], for an adequate weight function φ_A placed at scale A large, we obtain the virial estimate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi_A(x)u_1u_2 \leqslant &-\frac{1}{2}\int \left[w_2^2+2(\partial_x w_1)^2+\left(1-C_1A^{-1}\right)w_1^2\right] \\
&+C_1a_1^4+C_1\int \operatorname{sech}(x)\, u_1^2,
\end{aligned} \tag{4.18}
$$

where (w_1,w_2) is localized version of (u_1,u_2) at A scale, and C_1 denotes a fixed constant. This virial estimate has no good sign because of the term $C_1\int \operatorname{sech}(x)\, u_1^2$. Then we require to transform the system to a new one which has better virial estimates, in the spirit of Martel [24]. For any $\gamma>0$ small enough, we define new variables $(v_1,v_2)\in H^1\times H^2$ by

$$
\begin{cases} v_1=(1-\gamma\partial_x^2)^{-1}\mathcal{L}u_1,\\ v_2=(1-\gamma\partial_x^2)^{-1}u_2. \end{cases}
$$

(see (4.56)). Note that $(v_1,v_2)\in H^1\times H^2$, which is bad news because of the lack of a correct regularity order in the variables. This will cause problems later on. However, the new system for (v_1,v_2) (see (4.57)) satisfies, for an adequate weight function $\psi_{A,B}$, $B\ll A$, the virial estimate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \psi_{A,B}v_1v_2 \leqslant &-\frac{1}{2}\int \left[z_1^2+(V_0(x)-C_2B^{-1})z_2^2+2(\partial_x z_2)^2\right] \\
&+B^{-1}C_2\left(\|w_1\|_{L^2}^2+\|w_2\|_{L^2}^2\right)+C_2|a_1|^3,
\end{aligned} \tag{4.19}
$$

where (z_1,z_2) is a lozalized version of (v_1,v_2), at the smaller scale B, V_0 given by (4.8), and C_2 denotes a fixed constant.

Following [18], in order to combine estimates (4.18) and (4.19) we need an estimate for the last term in (4.18). However, unlike previous works, here we have the following coercivity estimate in terms of the variables (w_1,w_2) and (z_1,z_2):

$$
\int \operatorname{sech}(x)u_1^2 \lesssim B^{-1/2}\left(\|w_1\|_{L^2}+\|\partial_x w_1\|_{L^2}^2\right)+B^{1/2}\|z_1\|_{L^2}^2+B^{-4}\|\partial_x z_1\|_{L^2}^2. \tag{4.20}
$$

We can directly observe that the term $\partial_x z_1$ does not appears in (4.19), leading to the main obstruction present in this paper. This problem is deeply related to the fact that $(v_1,v_2)\in H^1\times H^2$, i.e., the new variables are in opposed order of regularity.

In order to overcome this problem, we introduce a series of modifications that will allow us to close estimates (4.18) and (4.19) properly. First, we must gain derivates. In a new virial estimate for the system of $(\partial_x v_1,\partial_x v_2)$ (see (4.98)), we obtain the third virial estimate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \psi_{A,B}\partial_x v_1\partial_x v_2 \leqslant &-\frac{1}{2}\int \left((\partial_x z_1)^2+\left(V_0(x)-C_3B^{-1}\right)(\partial_x z_2)^2+2(\partial_x^2 z_2)^2\right) \\
&+C_3\|z_2\|_{L^2}^2+C_3B^{-1}\|z_1\|_{L^2}^2 \\
&+C_3B^{-1}\left(\|\partial_x w_1\|_{L^2}^2+\|w_1\|_{L^2}^2+\|w_2\|_{L^2}^2\right)+C_3|a_1|^3,
\end{aligned} \tag{4.21}
$$

with $C_3>0$ fixed. This new estimate give us local L^2 control on $\partial_x z_1$ and $\partial_x^2 z_2$, which was not present before. Finally, our last contribution is a transfer virial estimate that exchanges information between $\partial_x z_1$, $\partial_x z_2$ and $\partial_x^2 z_2$, in the form of

$$\begin{aligned}\frac{1}{2}\int(\partial_x z_1)^2 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \rho_{A,B}\partial_x v_1 v_2 + C_4\int\left[(\partial_x^2 z_2)^2+(\partial_x z_2)^2+z_2^2+z_1^2\right]\\ + C_4 B^{-3}\left(\|w_1\|_{L^2}^2+\|w_2\|_{L^2}^2\right)+C_4|a_1|^3.\end{aligned} \tag{4.22}$$

Here $C_4 > 0$ is fixed and $\rho_{A,B}$ is a suitable weight function. Finally, we consider a functional $\mathcal{H}$ being a well-chosen linear combination of (4.18), (4.19), (4.21), (4.20) and (4.22). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \leqslant -C_2 B^{-1}\left(\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2+\|w_2\|_{L^2}^2\right)+C_5|a_1|^3, \quad \text{for all } t\geqslant 0.$$

This final estimate allows us to close estimates, and prove local decay for u_1 after some standard change of variables from w_j to u_j.

Organization of this chapter

This paper is organized as follows. Section 4.2 deals with a first virial estimate for a decomposition system, namely (4.23). In Section 4.3 we introduce the transformed problem and prove first virial estimates on that system. In Section 4.4 we obtain virial estimates for higher order derivatives of the transformed problem. Section 4.5 is devoted to a technical transfer estimate dealing with higher order transformed variables. Finally, in Section 4.6 we prove Theorem 4.1, and in Section 4.7 we prove Theorem 4.2.

Acknowledgments

I deeply thank professors Didier Pilod (U. Bergen), Juan Soler (U. Granada), Francisco Gancedo (U. Sevilla) and Miguel A. Alejo (U. Córdoba) for the funding and their hospitality during the research stays where this work was completed.

4.2 A virial identity for the (gGB) system

Recall the (gGB) system (4.3). The first step in our proof is to consider a small **even-odd** perturbation of soliton $(Q,0)$. In what follows we will describe this decomposition, introduce some notation, and develop a virial estimate for the good Boussinesq system.

4.2.1 Decomposition of the solution in a vicinity of the soliton

Let $(u,v)=(\phi,\partial_t\partial_x^{-1}\phi)$ be a solution of (4.3) satisfying (4.12) for some small $\delta>0$. Using $\boldsymbol{Y}_+$ as in (4.11), we decompose (u,v) as follows

$$\begin{cases} u(t,x) = Q(x)+a_1(t)\phi_0(x)+u_1(t,x),\\ v(t,x) = a_2(t)\nu_0\partial_x^{-1}\phi_0(x)+u_2(t,x),\end{cases} \tag{4.23}$$

where (see (4.10))

$$\begin{aligned} a_1(t) &= \langle u(t)-Q, \nu_0^{-2}\mathcal{L}\phi_0\rangle = \langle u(t)-Q,\partial_x^{-2}\phi_0\rangle,\\ a_2(t) &= \frac{1}{\nu_0}\langle \partial_x v, \nu_0^{-2}\partial_x\mathcal{L}\phi_0\rangle = \frac{1}{\nu_0}\langle v,\partial_x^{-1}\phi_0\rangle,\end{aligned}$$

such that

$$\langle u_1(t), \partial_x^{-2}\phi_0\rangle = 0 = \langle u_2(t), \partial_x^{-1}\phi_0\rangle, \tag{4.24}$$

or equivalently,

$$\langle u_1(t), \mathcal{L}\phi_0\rangle = 0 = \langle u_2(t), \partial_x\mathcal{L}\phi_0\rangle. \tag{4.25}$$

Orthogonalities (4.24) are nonstandard particular choices motivated by key cancelation properties. See Appendix 4.A for a detailed construction of $\partial_x^{-1}\phi_0$ and $\partial_x^{-2}\phi_0$. Setting

$$b_+ = \frac{1}{2}(a_1 + a_2), \quad b_- = \frac{1}{2}(a_1 - a_2), \tag{4.26}$$

from (4.12), we have for all $t \in \mathbb{R}_+$

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| + |b_+(t)| + |b_-(t)| \leqslant C_0\delta. \tag{4.27}$$

Moreover, using (4.6), (4.10) and (4.24), (a_1, a_2) satisfies the following differential system

$$\begin{cases} \dot{a}_1 = \nu_0 a_2 \\ \dot{a}_2 = \nu_0 a_1 + \dfrac{N_0}{\nu_0}, \end{cases} \quad \text{or equivalently} \quad \begin{cases} \dot{b}_+ = \nu_0 b_+ + \dfrac{N_0}{2\nu_0} \\ \dot{b}_- = -\nu_0 b_- - \dfrac{N_0}{2\nu_0}. \end{cases} \tag{4.28}$$

where

$$\begin{aligned} N &= \partial_x \left(f(Q) + f'(Q)(a_1\phi_0 + u_1) - f(Q + a_1\phi_0 + u_1)\right), \\ N^{\perp} &= N - N_0\partial_x^{-1}\phi_0, \quad \text{and} \quad N_0 = \langle N, \partial_x^{-1}\phi_0\rangle. \end{aligned} \tag{4.29}$$

Then, (u_1, u_2) satisfies the system

$$\begin{cases} \dot{u}_1 = \partial_x u_2 \\ \dot{u}_2 = \partial_x\mathcal{L}(u_1) + N^{\perp}, \end{cases} \tag{4.30}$$

with u_1 even and u_2 odd.

4.2.2 Notation for virial argument

We consider a smooth even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi = 1 \text{ on } [-1, 1], \quad \chi = 0 \text{ on } (-\infty, 2] \cup [2, \infty), \quad \chi' \leqslant 0 \text{ on } [0, \infty). \tag{4.31}$$

For $A > 0$, we define the functions ζ_A and φ_A as follows

$$\zeta_A(x) = \exp\left(-\frac{1}{A}(1 - \chi(x))|x|\right), \quad \varphi_A(x) = \int_0^x \zeta_A^2(y)\mathrm{d}y, \quad x \in \mathbb{R}. \tag{4.32}$$

For $B > 0$, we also define

$$\zeta_B(x) = \exp\left(-\frac{1}{B}(1 - \chi(x))|x|\right), \quad \varphi_B(x) = \int_0^x \zeta_B^2(y)\mathrm{d}y, \quad x \in \mathbb{R}. \tag{4.33}$$

We consider the function $\psi_{A,B}$ defined as

$$\psi_{A,B}(x) = \chi_A^2(x)\varphi_B(x) \text{ where } \quad \chi_A(x) = \chi\left(\frac{x}{A}\right), \quad x \in \mathbb{R}. \tag{4.34}$$

These functions will be used in two distinct virial arguments with different scales

$$1 \ll B \ll B^{10} \ll A. \tag{4.35}$$

The following remark will be essential for the well-boundedness of some nonlinear terms in what follow.

Remark 4.7. One can see that for each function v

$$\int \chi_A^2 v^2 \leqslant \int_{|x|\leqslant 2A} v^2 \leqslant C \int_{|x|\leqslant 2A} \mathrm{e}^{-4|x|/A} v^2 \lesssim \int v^2 \zeta_A^4 \leqslant \|\zeta_A^2 v\|_{L^2}^2.$$

This estimate will be useful later on (see Subsections 4.3.5 and 4.3.5).

4.2.3 Virial estimate

Set

$$\mathcal{I}(t) = \int_{\mathbb{R}} \varphi_A(x) u_1 u_2, \tag{4.36}$$

and

$$w_{\mathrm{i}} = \zeta_A u_{\mathrm{i}}, \quad \mathrm{i} = 1, 2. \tag{4.37}$$

Here, (w_1, w_2) represents a localized version of (u_1, u_2) at scale A. The following virial argument has been used in [18, 19] in a similar context.

Proposition 4.3. *There exist $C_1 > 0$ and $\delta_1 > 0$ such that for any $0 < \delta \leqslant \delta_1$, the following holds. Fix $A = \delta^{-1}$. Assume that for all $t \geqslant 0$, (4.27) holds. Then for all $t \geqslant 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) \leqslant -\frac{1}{2}\int \left[w_2^2 + 2(\partial_x w_1)^2 + \left(1 - C_1 A^{-1}\right) w_1^2\right] + C_1 a_1^4 + C_1 \int \operatorname{sech}\left(\frac{x}{2}\right) w_1^2. \tag{4.38}$$

Some remarks are in order.

Remark 4.8. This virial has several similarities with the developed in [18] for nonlinear Klein-Gordon equation. In that paper, the main part of the virial is composed by the $\dot{H}^1$-norm of w_1. In our case, this main part is similar to the $H^1 \times L^2$-norm of (w_1, w_2), and the rest of the terms are the same. Unlike [18], we did not use a correction term since the momentum of the equation (4.4) works well in this case. This virial was already used in [30] in a different context (small solutions around zero).

The proof of Proposition 4.3 follows after the next intermediate lemma.

Lemma 4.4. *Let $(u_1, u_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ a solution of (4.30). Consider $\varphi_A = \varphi_A(x)$ a smooth bounded function to be chosen later. Then*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) =& -\frac{1}{2}\int \varphi_A' \left(u_2^2 + u_1^2 + 3(\partial_x u_1)^2\right) + \frac{1}{2}\int \varphi_A''' u_1^2 \\
&+ \int (\varphi_A' u_1 + \varphi_A \partial_x u_1)\left(f(Q) + f'(Q)a_1\phi_0 - f(Q + a_1\phi_0 + u_1) - N_0 \partial_x^{-2}\phi_0\right).
\end{aligned}
\tag{4.39}
$$

Proof. Taking derivative in (4.36) and using (4.30),

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) &= \int \varphi_A(\dot{u}_1 u_2 + u_1 \dot{u}_2) = \int \varphi_A(\partial_x u_2 u_2 + u_1(\partial_x \mathcal{L}(u_1) + N^\perp)) \\
&= -\frac{1}{2}\int \varphi_A' u_2^2 + \int \varphi_A u_1 \partial_x \mathcal{L}(u_1) + \int \varphi_A u_1 N^\perp.
\end{aligned}
\tag{4.40}
$$

For the second integral in the RHS of the above equation, we have

$$
\begin{aligned}
\int \varphi_A u_1 \partial_x \mathcal{L}(u_1) &= \int \varphi_A u_1(-\partial_x^3 u_1 + \partial_x u_1 - \partial_x(f'(Q)u_1)) \\
&= -\int \varphi_A u_1 \partial_x^3 u_1 + \frac{1}{2}\int \varphi_A \partial_x(u_1^2) - \int \varphi_A u_1 \partial_x(f'(Q)u_1).
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
\int \varphi_A u_1 \partial_x \mathcal{L}(u_1) &= -\frac{1}{2}\int \varphi_A' u_1^2 + \int (\varphi_A' u_1 + \varphi_A \partial_x u_1)\partial_x^2 u_1 - \int \varphi_A u_1 \partial_x(f'(Q)u_1) \\
&= -\frac{1}{2}\int \varphi_A' \left[u_1^2 + (\partial_x u_1)^2\right] + \int \varphi_A' u_1 \partial_x^2 u_1 - \int \varphi_A u_1 \partial_x(f'(Q)u_1).
\end{aligned}
\tag{4.41}
$$

Integrating by parts in the second integral in the RHS of the above equation, we get

$$
\begin{aligned}
\int \varphi_A' u_1 \partial_x^2 u_1 &= -\int (\varphi_A'' u_1 + \varphi_A' \partial_x u_1)\partial_x u_1 \\
&= -\int \left(\varphi_A'' \frac{\partial_x(u_1^2)}{2} + \varphi_A'(\partial_x u_1)^2\right) = -\int \varphi_A'(\partial_x u_1)^2 + \int \varphi_A''' \frac{u_1^2}{2}.
\end{aligned}
$$

For the last integral in the RHS of (4.40), separating terms and integrating by parts we obtain

$$
\begin{aligned}
\int \varphi_A u_1 N^\perp =& \int \varphi_A u_1 \left(\partial_x \left(f(Q) + f'(Q)(a_1\phi_0 + u_1) - f(Q + a_1\phi_0 + u_1)\right) - N_0 \partial_x^{-1}\phi_0\right) \\
=& -\int (\varphi_A' u_1 + \varphi_A \partial_x u_1)\left(f(Q) + f'(Q)a_1\phi_0 - f(Q + a_1\phi_0 + u_1)\right) \\
&+ \int \varphi_A u_1 \partial_x(f'(Q)u_1) - N_0 \int \varphi_A u_1 \partial_x^{-1}\phi_0.
\end{aligned}
$$

Cancelling terms, we finally obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = & -\int \varphi_A' \left(\frac{u_2^2}{2} + \frac{u_1^2}{2} + (\partial_x u_1)^2 + \frac{(\partial_x u_1)^2}{2} \right) \\
& + \int \varphi_A''' \frac{u_1^2}{2} - \int \varphi_A u_1 \partial_x (f'(Q) u_1) \\
& + \int (\varphi_A' u_1 + \varphi_A \partial_x u_1) \left(f(Q) + f'(Q) a_1 \phi_0 - f(Q + a_1\phi_0 + u_1) \right) \\
& + \int \varphi_A u_1 \partial_x (f'(Q) u_1) - N_0 \int \varphi_A u_1 \partial_x^{-1} \phi_0 \\
= & -\frac{1}{2} \int \varphi_A' \left(u_2^2 + u_1^2 + 3(\partial_x u_1)^2 \right) + \frac{1}{2} \int \varphi_A''' u_1^2 - N_0 \int \varphi_A u_1 \partial_x^{-1} \phi_0. \\
& + \int (\varphi_A' u_1 + \varphi_A \partial_x u_1) \left(f(Q) + f'(Q) a_1 \phi_0 - f(Q + a_1\phi_0 + u_1) \right).
\end{aligned}
\tag{4.42}
$$

This concludes the proof. ■

Now we rewrite the main part of the virial identity using the new variables (w_1, w_2).

Lemma 4.5. *It holds*

$$
\int \varphi_A' \left(u_2^2 + u_1^2 + 3(\partial_x u_1)^2 \right) - \int \varphi_A''' u_1^2 = \int \left(w_2^2 + 3(\partial_x w_1)^2 + \left(1 + \frac{\zeta_A''}{\zeta_A} - 2\frac{(\zeta_A')^2}{\zeta_A^2} \right) w_1^2 \right),
$$

with

$$
\left| \frac{\zeta_A''}{\zeta_A} - 2\frac{(\zeta_A')^2}{\zeta_A^2} \right| \lesssim \frac{1}{A}. \tag{4.43}
$$

Proof. Considering $w_{\mathrm{i}} = \zeta_A u_{\mathrm{i}}$, $\mathrm{i} = 1, 2$, and $\varphi_A' = \zeta_A^2$, we have

$$
\int \varphi_A' \left(u_2^2 + u_1^2 \right) = \int \zeta_A^2 \left(u_2^2 + u_1^2 \right) = \int \left(w_2^2 + w_1^2 \right). \tag{4.44}
$$

Also,

$$
\int \varphi_A' (\partial_x u_1)^2 = \int (\partial_x w_1)^2 + \int w_1^2 \frac{\zeta_A''}{\zeta_A}. \tag{4.45}
$$

In the case of the last terms we have,

$$
\int \varphi_A''' u_1^2 = \int \frac{(\zeta_A^2)''}{\zeta_A^2} w_1^2 = 2 \int \left(\frac{\zeta_A''}{\zeta_A} + \frac{(\zeta_A')^2}{\zeta_A^2} \right) w_1^2.
$$

By (4.32), we have

$$
\begin{aligned}
\frac{\zeta_A'}{\zeta_A} &= -\frac{1}{A} \left[-\chi'(x)|x| + (1 - \chi(x)\mathrm{sgn}(x)) \right], \\
\frac{\zeta_A''}{\zeta_A} &= \left(\frac{\zeta_A'}{\zeta_A} \right)^2 + \frac{1}{A} \left[\chi''(x)|x| + 2\chi'(x)\mathrm{sgn}(x) \right].
\end{aligned}
\tag{4.46}
$$

Then, substracting

$$\frac{\zeta_A''}{\zeta_A} - 2\left(\frac{\zeta_A'}{\zeta_A}\right)^2 = -\frac{1}{A^2}\left[-\chi'(x)|x| + (1-\chi(x))\text{sgn}(x)\right]^2 + \frac{1}{A}\left[\chi''(x)|x| + 2\chi'(x)\text{sgn}(x)\right].$$

For $1 \leqslant |x| \leqslant 2$, one can see that

$$\left|\frac{\zeta_A''}{\zeta_A} - 2\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| \lesssim \frac{1}{A} + \frac{1}{A^2}.$$

For $|x| \geqslant 2$, we have that

$$\left|\frac{\zeta_A'}{\zeta_A} - 2\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| = \frac{1}{A^2}.$$

Then,

$$\left|\frac{\zeta_A''}{\zeta_A} - 2\left(\frac{\zeta_A'}{\zeta_A}\right)^2\right| \lesssim \left(\frac{1}{A} + \frac{1}{A^2}\right)\mathbf{1}_{\{|x|\geqslant 1\}} \lesssim \frac{1}{A}.$$

This ends the proof. ■

Next, we deal with the nonlinear terms.

Lemma 4.6.

$$\begin{aligned}\Bigg|\int (\varphi_A' u_1 + \varphi_A \partial_x u_1)\left(f(Q + a_1\phi_0 + u_1) - f(Q) - f'(Q)a_1\phi_0\right) - N_0 \int \varphi_A u_1 \partial_x^{-1}\phi_0\Bigg| \\ \lesssim a_1^4 + \int \text{sech}\left(\frac{x}{2}\right) w_1^2 + A^2 \|u_1\|_{L^\infty}^{p-1} \int |\partial_x w_1|^2.\end{aligned} \tag{4.47}$$

Proof. First, we treat the term $N_0 \int \varphi_A u_1 \partial_x^{-1}\phi_0$. Noticing that

$$N_0 = \langle N, \partial_x^{-1}\phi_0\rangle = -\langle f(Q) + f'(Q)(a_1\phi_0 + u_1) - f(Q + a_1\phi_0 + u_1), \phi_0\rangle, \tag{4.48}$$

and by Taylor's expansion, one has

$$|f(Q+a_1\phi_0+u_1) - f(Q) - f'(Q)(a_1\phi_0+u_1)| \lesssim a_1^2 f''(Q)\phi_0^2 + f''(Q)u_1^2 + |a_1|^p\phi_0^p + |u_1|^p. \tag{4.49}$$

Thus, by exponential decay estimates on Q and ϕ_0 (see Appendix 4.A), and by (4.27), $|a_1| \lesssim 1, \|u_1\|_{L^\infty} \leqslant \|u_1\|_{H^1} \lesssim 1$, it holds

$$|N_0| \lesssim a_1^2 + \int \text{sech}(9x/10)u_1^2, \tag{4.50}$$

taking $A \geqslant 4$, we have

$$|N_0| \lesssim a_1^2 + \int \text{sech}\left(\frac{x}{2}\right) w_1^2. \tag{4.51}$$

Noticing that for all $x \in \mathbb{R}$, $|\varphi_A| \leqslant |x|$,

$$|\varphi_A \partial_x^{-1}\phi_0| \lesssim |x\,\text{sech}(9x/10)| \leqslant \left|\text{sech}\left(\frac{3}{4}x\right)\right|,$$

and using Hölder inequality, we have

$$\begin{aligned}\left|\int u_1\varphi_A\partial_x^{-1}\phi_0\right| &\lesssim \left|\int w_1 \operatorname{sech}\left(\frac{x}{2}\right)\right| \\ &\lesssim \left|\int w_1^2 \operatorname{sech}\left(\frac{x}{2}\right)\right|^{1/2}\left|\int \operatorname{sech}\left(\frac{x}{2}\right)\right|^{1/2} \lesssim \left|\int w_1^2 \operatorname{sech}\left(\frac{x}{2}\right)\right|^{1/2}.\end{aligned} \tag{4.52}$$

We conclude using Cauchy-Schwarz inequality

$$\left|N_0\int \varphi_A u_1\partial_x^{-1}\phi_0\right| \lesssim |N_0|^2 + \left|\int u_1\varphi_A\partial_x^{-1}\phi_0\right|^2 \lesssim a_1^4 + \int \operatorname{sech}\left(\frac{x}{2}\right) w_1^2.$$

For the remaining terms, we consider the following decomposition

$$\begin{aligned}&\int (\varphi_A\partial_x u_1 + \varphi_A' u_1)\left[f(Q+a_1\phi_0+u_1) - f(Q) - f'(Q)a_1\phi_0\right] \\ &= \int \varphi_A\partial_x\left[F(Q+a_1\phi_0+u_1) - F(Q+a_1\phi_0) - (f(Q)+f'(Q)a_1\phi_0)u_1\right] \\ &\quad - \int \varphi_A Q'\left[f(Q+a_1\phi_0+u_1) - f(Q+a_1\phi_0) - (f'(Q)+f''(Q)a_1\phi_0)u_1\right] \\ &\quad - a_1\int \varphi_A\partial_x\phi_0\left[f(Q+a_1\phi_0+u_1) - f(Q+a_1\phi_0) - f'(Q)u_1\right] \\ &\quad + \int \varphi_A' u_1[f(Q+a_1\phi_0+u_1) - f(Q) - f'(Q)a_1\phi_0] \\ &=: I_1 + I_2 + I_3 + I_4,\end{aligned}$$

and rewriting as

$$\begin{aligned}I_1 =& -\int \varphi_A'\left[F(Q+a_1\phi_0+u_1) - F(Q+a_1\phi_0) - F'(Q+a_1\phi_0)u_1 - F(u_1)\right] \\ & -\int \varphi_A'\left[f(Q+a_1\phi_0) - f(Q) + f'(Q)a_1\phi_0\right]u_1 - \int \varphi_A' F(u_1),\end{aligned}$$

$$\begin{aligned}I_2 =& -\int \varphi_A Q'\left[f(Q+a_1\phi_0+u_1) - f(Q+a_1\phi_0) - f'(Q+a_1\phi_0)u_1\right]\right] \\ & -\int \varphi_A Q'\left(f'(Q+a_1\phi_0) - f'(Q) + f''(Q)a_1\phi_0\right)u_1,\end{aligned}$$

$$\begin{aligned}I_3 =& -a_1\int \varphi_A\partial_x\phi_0\left[f(Q+a_1\phi_0+u_1) - f(Q+a_1\phi_0) - f'(Q+a_1\phi_0)u_1\right] \\ & -a_1\int \varphi_A\partial_x\phi_0\left(f'(Q+a_1\phi_0) - f'(Q)\right)u_1,\end{aligned}$$

and

$$\begin{aligned}I_4 =& \int \varphi_A' u_1[f(Q+a_1\phi_0+u_1) - f(Q+a_1\phi_0) - f(u_1)] \\ & + \int \varphi_A' u_1[f(Q+a_1\phi_0) - f(Q) - f'(Q)a_1\phi_0] + \int \varphi_A' u_1 f(u_1).\end{aligned}$$

By Taylor expansion, $p \geqslant 1$, $|a_1|, \|u_1\|_{L^\infty} \lesssim 1$, we have

$$\begin{aligned}
|F(Q+&a_1\phi_0 + u_1) - F(Q + a_1\phi_0) - F'(Q + a_1\phi_0)u_1 - F(u_1)|,\\
&\lesssim |Q + a_1\phi_0|^{p-1}u_1^2 + |Q + a_1\phi_0||u_1|^p,\\
&\lesssim |Q + a_1\phi_0|^{p-1}u_1^2 + |Q + a_1\phi_0||u_1|^2 \lesssim \operatorname{sech}(9x/10)u_1^2 \lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2.
\end{aligned}$$

Similarly, using (4.32) and $A \geqslant 4$, we find the following estimates

$$\begin{aligned}
|\varphi_A Q' \left[f(Q + a_1\phi_0 + u_1) - f(Q + a_1\phi_0) - f'(Q + a_1\phi_0)u_1\right]| &\lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2,\\
|a_1\varphi_A\partial_x\phi_0 \left[f(Q + a_1\phi_0 + u_1) - f(Q + a_1\phi_0) - f'(Q + a_1\phi_0)u_1\right]| &\lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2,\\
|\varphi_A' u_1 \left[f(Q + a_1\phi_0 + u_1) - f(Q + a_1\phi_0) - f(u_1)\right]| &\lesssim \operatorname{sech}\left(\frac{x}{2}\right) w_1^2.
\end{aligned}$$

Furthermore, once again by Taylor expansion, we have

$$\begin{aligned}
&|\varphi_A' \left[f(Q + a_1\phi_0) - f(Q) + f'(Q)a_1\phi_0)\right] u_1|\\
&\quad + |\varphi_A Q' \left[f'(Q + a_1\phi_0) - f'(Q) + f''(Q)a_1\phi_0\right] u_1|\\
&\quad + |a_1\varphi_A\partial_x\phi_0 \left[f'(Q + a_1\phi_0) - f'(Q)\right] u_1|\\
&\quad + |\varphi_A' u_1 \left[f(Q + a_1\phi_0) - f(Q) - f'(Q)a_1\phi_0\right]|\\
&\lesssim \operatorname{sech}\left(\frac{x}{2}\right) |a_1|^2|u_1| \lesssim \operatorname{sech}\left(\frac{x}{2}\right) |w_1|^2 + \operatorname{sech}\left(\frac{x}{4}\right) |a_1|^4.
\end{aligned} \tag{4.53}$$

For the last step, we need the following claim proved in [18].

Claim 4.7. *It holds*

$$\int \zeta_A^2 |u_1|^{p+1} = \int \zeta_A^{-p+1}|w_1|^{p+1} \lesssim A^2\|u_1\|_{L^\infty}^{p-1} \int |\partial_x w_1|^2.$$

Using this claim, we have

$$\left|\int \varphi_A' F(u_1)\right| + \left|\int \varphi_A' u_1 f(u_1)\right| \lesssim \int \zeta_A^2|u_1|^{p+1} \lesssim A^2\|u_1\|_{L^\infty}^{p-1} \int |\partial_x w_1|^2.$$

Finally, we get

$$\begin{aligned}
&\left|\int (\partial_x\varphi_A u_1 + \varphi_A\partial_x u_1)\left(f(Q + a_1\phi_0 + u_1) - f(Q) - f'(Q)a_1\phi_0 + N_0\nu_0^{-2}\mathcal{L}\phi_0\right)\right|\\
&\quad\lesssim a_1^4 + \int \operatorname{sech}\left(\frac{x}{2}\right) w_1^2 + A^2\|u_1\|_{L^\infty}^{p-1} \int |\partial_x w_1|^2.
\end{aligned} \tag{4.54}$$

This ends the proof of Lemma 4.6. ■

4.2.4 End of Proposition 4.3

Applying Lemmas 4.5 and 4.6, and taking $\|u_1\|_{L^\infty} \leqslant \delta_A$, for δ_A small enough, we have proved

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) \leqslant & -\frac{1}{2}\int\left[w_2^2 + 3(\partial_x w_1)^2 + \left(1 - \left(\frac{1}{A} + \frac{1}{A^2}\right)\mathbf{1}_{\{|x|\geqslant 1\}}\right)w_1^2\right] \\ & + C_1 a_1^4 + C_1\int \operatorname{sech}\left(\frac{x}{2}\right)w_1^2 + A^2\|u_1\|_{L^\infty}^{p-1}\int|\partial_x w_1|^2 \\ \leqslant & -\frac{1}{2}\int\left[w_2^2 + 2(\partial_x w_1)^2 + \left(1 - \frac{C_1}{A}\right)w_1^2\right] + C_1 a_1^4 + C_1\int \operatorname{sech}\left(\frac{x}{2}\right)w_1^2.\end{aligned} \tag{4.55}$$

This concludes the proof.

4.3 Transformed problem and second virial estimates

Following the idea of Martel [24], we will consider the function $v_1 = \mathcal{L}u_1$ instead u_1 to obtain a transformed problem with better virial properties. However, we must be careful since our original variables (u_1, u_2) belong to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and by using $\mathcal{L}$, the new variables are not well-defined. Therefore, we need a regularization procedure, as in [18].

4.3.1 The transformed problem

Let $\gamma > 0$ small, to be determined later, set

$$\begin{cases} v_1 = (1-\gamma\partial_x^2)^{-1}\mathcal{L}u_1, \\ v_2 = (1-\gamma\partial_x^2)^{-1}u_2. \end{cases} \tag{4.56}$$

From the system (4.30), follows that $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$, and satisfies the system

$$\begin{cases} \dot{v}_1 = \mathcal{L}(\partial_x v_2) + G(x), \\ \dot{v}_2 = \partial_x v_1 + H(x), \end{cases} \tag{4.57}$$

where

$$\begin{aligned} H(x) &= (1-\gamma\partial_x^2)^{-1}N^{\perp}, \\ G(x) &= \gamma(1-\gamma\partial_x^2)^{-1}\left[\partial_x^2(f'(Q))\partial_x v_2 + 2\partial_x(f'(Q))\partial_x^2 v_2\right]. \end{aligned} \tag{4.58}$$

Now we compute a second virial estimate, this time on (v_1, v_2).

4.3.2 Virial functional for the transformed problem

Set now

$$\mathcal{J}(t) = \int \psi_{A,B} v_1 v_2, \tag{4.59}$$

with

$$\psi_{A,B} = \chi_A^2\varphi_B, \quad z_{\mathrm{i}} = \chi_A\zeta_B v_{\mathrm{i}}, \quad \mathrm{i} = 1, 2. \tag{4.60}$$

Here, (z_1, z_2) represents a localized version of the variables (v_1, v_2) at the scale B. This scale is intermediate, and $\mathcal{J}(t)$ involves a cut-off at scale A, which is needed to bound some bad error and nonlinear terms; see [19] for a similar procedure.

Proposition 4.8. *There exist $C_2 > 0$ and $\delta_2 > 0$ such that for $\gamma = B^{-4}$ and for any $0 < \delta \leqslant \delta_2$, the following holds. Fix $B = \delta^{-1/8}$. Assume that for all $t \geqslant 0$, (4.27) holds. Then for all $t \geqslant 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) \leqslant & -\frac{1}{2}\int \left[z_1^2 + (V_0(x) - C_2B^{-1})z_2^2 + 2(\partial_x z_2)^2\right] \\ & + C_2B^{-1}\left(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_2|a_1|^3,\end{aligned} \tag{4.61}$$

where $V_0(x)$ is given by (4.8).

The rest of this section is devoted to the proof of this proposition, which has been divided in several subsections.

4.3.3 Proof of Proposition 4.8: first computations

We have from (4.59) and (4.57),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) &= \int \psi_{A,B}\left[(\mathcal{L}\partial_x v_2)v_2 + \frac{1}{2}\partial_x(v_1^2) + H(x)v_1 + G(x)v_2\right] \\ &= \int \psi_{A,B}(\mathcal{L}\partial_x v_2)v_2 - \frac{1}{2}\int \psi'_{A,B}v_1^2 + \int \psi_{A,B}\left[G(x)v_2 + H(x)v_1\right].\end{aligned} \tag{4.62}$$

In a similar way to the computation in (4.41), we have

$$\int \psi_{A,B}(\mathcal{L}\partial_x v_2)v_2 = -\frac{1}{2}\int \psi'_{A,B}\left(v_2^2 + 3(\partial_x v_2)^2\right) + \frac{1}{2}\int \psi'''_{A,B}v_2^2 - \frac{1}{2}\int \psi_{A,B}f'(Q)\partial_x(v_2^2).$$

We consider now the following decomposition

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) =& -\frac{1}{2}\int \psi'_{A,B}\left(v_1^2 + v_2^2 + 3(\partial_x v_2)^2\right) + \frac{1}{2}\int \psi'''_{A,B}v_2^2 \\ & - \frac{1}{2}\int \psi_{A,B}f'(Q)\partial_x(v_2^2) + \int \psi_{A,B}G(x)v_2 + \int \psi_{A,B}H(x)v_1 \\ =&: (J_1 + J_2) + (J_3 + J_4 + J_5).\end{aligned} \tag{4.63}$$

By definition of $\psi_{A,B}$ (see (4.60)), it follows that

$$\begin{aligned}\psi'_{A,B} &= \chi_A^2\zeta_B^2 + (\chi_A^2)'\varphi_B, \\ \psi''_{A,B} &= \chi_A^2(\zeta_B^2)' + 2(\chi_A^2)'\zeta_B^2 + (\chi_A^2)''\varphi_B, \\ \psi'''_{A,B} &= \chi_A^2(\zeta_B^2)'' + 3(\chi_A^2)'(\zeta_B^2)' + 3(\chi_A^2)''\zeta_B^2 + (\chi_A^2)'''\varphi_B.\end{aligned} \tag{4.64}$$

Also, by the definition of z_{i} in (4.60), we have:

$$\begin{aligned}-2J_1 &= \int \psi'_{A,B}\left(v_1^2 + v_2^2 + 3(\partial_x v_2)^2\right) \\ &= \int (\chi_A^2\zeta_B^2 + (\chi_A^2)'\varphi_B)\left(v_1^2 + v_2^2 + 3(\partial_x v_2)^2\right) \\ &= \int (z_1^2 + z_2^2) + \int (\chi_A^2)'\varphi_B\left(v_1^2 + v_2^2 + 3(\partial_x v_2)^2\right) + 3\int \chi_A^2\zeta_B^2(\partial_x v_2)^2.\end{aligned} \tag{4.65}$$

Derivating $z_2 = \zeta_B \chi_A v_2$, replacing and integrating by parts, we obtain

$$\int \chi_A^2 \zeta_B^2 (\partial_x v_2)^2 = \int \frac{\zeta_B''}{\zeta_B} z_2^2 + \int (\partial_x z_2)^2 + \int \left[\chi_A'' + 2\chi_A' \frac{\zeta_B'}{\zeta_B} \right] \chi_A \zeta_B^2 v_2^2. \tag{4.66}$$

Then, for J_1 we obtain

$$\begin{aligned} -2J_1 = & \int (z_1^2 + z_2^2) + 3 \int \frac{\zeta_B''}{\zeta_B} z_2^2 + 3 \int (\partial_x z_2)^2 \\ & + 3 \int \left[\chi_A'' + 2\chi_A' \frac{\zeta_B'}{\zeta_B} \right] \chi_A \zeta_B^2 v_2^2 + \int (\chi_A^2)' \varphi_B \left(v_1^2 + v_2^2 + 3(\partial_x v_2)^2 \right). \end{aligned} \tag{4.67}$$

Now we turn into J_2. By (4.64), J_2 satisfies the following decomposition

$$\begin{aligned} 2J_2 = & \int (\chi_A^2 (\zeta_B^2)'' + 3(\chi_A^2)'(\zeta_B^2)' + 3(\chi_A^2)'' \zeta_B^2 + (\chi_A^2)''' \varphi_B) v_2^2 \\ = & \, 2 \int \left[\left(\frac{\zeta_B'}{\zeta_B} \right)^2 + \frac{\zeta_B''}{\zeta_B} \right] z_2^2 + \int (3(\chi_A^2)'(\zeta_B^2)' + 3(\chi_A^2)'' \zeta_B^2 + (\chi_A^2)''' \varphi_B) v_2^2. \end{aligned}$$

For J_3, integrating by parts and using the definition of z_2, we obtain

$$\begin{aligned} -2J_3 = & - \int \partial_x (\psi_{A,B} f'(Q)) v_2^2 = - \int \left[(\chi_A^2 \zeta_B^2 + (\chi_A^2)' \varphi_B) f'(Q) + \chi_A^2 \varphi_B \partial_x (f'(Q)) \right] v_2^2 \\ = & - \int \left[f'(Q) + \partial_x (f'(Q)) \frac{\varphi_B}{\zeta_B^2} \right] z_2^2 - \int (\chi_A^2)' \varphi_B f'(Q) v_2^2. \end{aligned}$$

Finally, we obtain that the main part of the virial can be write as

$$J_1 + J_2 + J_3 = -\frac{1}{2} \int \left[z_1^2 + V(x) z_2^2 + 3(\partial_x z_2)^2 \right] + \tilde{J}_1,$$

where

$$V(x) = 1 + \frac{\zeta_B''}{\zeta_B} - 2\frac{(\zeta_B')^2}{\zeta_B^2} - f'(Q) - \partial_x (f'(Q)) \frac{\varphi_B}{\zeta_B^2},$$

and the error term is given by

$$\begin{aligned} \tilde{J}_1 = & -\frac{1}{2} \int \left[3\big(\chi_A'' \chi_A - (\chi_A^2)''\big) \zeta_B^2 - \frac{3}{2} (\chi_A^2)'(\zeta_B^2)' + \big((\chi_A^2)' - (\chi_A^2)'''\big) \varphi_B \right] v_2^2 \\ & + \frac{1}{2} \int (\chi_A^2)' \varphi_B f'(Q) v_2^2 - \frac{1}{2} \int (\chi_A^2)' \varphi_B v_1^2 - \frac{3}{2} \int (\chi_A^2)' \varphi_B (\partial_x v_2)^2. \end{aligned} \tag{4.68}$$

To control the main part of the virial is necessary a lower bound for the potential $V(x)$. We have the following result:

Lemma 4.9. *There are $C > 0$ and $B_0 > 0$ such that for all $B \geqslant B_0$, one has*

$$V(x) \geqslant V_0(x) - CB^{-1}, \quad \text{where } \; V_0(x) = 1 - f'(Q). \tag{4.69}$$

Proof. First, recalling (4.43) and changing the scale, we have

$$\left|\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2\right| \lesssim \frac{1}{B}. \tag{4.70}$$

Using that for $x \in [0,\infty) \mapsto \zeta_B(x)$ is a non-increasing function, we have for $x > 0$,

$$\frac{\varphi_B}{\zeta_B^2} = \frac{\int_0^x \zeta_B^2}{\zeta_B^2} > 0,$$

and $\partial_x(f'(Q)) < 0$ for $x > 0$. Then,

$$V(x) \geqslant 1 - CB^{-1} - f'(Q) + |\partial_x(f'(Q))x| \geqslant 1 - CB^{-1} - f'(Q) = V_0(x) - CB^{-1}. \tag{4.71}$$

The case $x \leqslant 0$ is similar. These estimates hold for any $x \in \mathbb{R}$. This concludes the proof. ∎

First conclusion. Using this lemma, and the above definition of $\tilde{J}_1$, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) \leqslant -\frac{1}{2}\int \left[z_1^2 + (V_0 - CB^{-1})z_2^2 + 3(\partial_x z_2)^2\right] + \tilde{J}_1 + J_4 + J_5, \tag{4.72}$$

where J_4 and J_5 are related with the nonlinear term in (4.63). To control the terms $\tilde{J}_1, J_4, J_5$, and the terms that will appear in the sections below, some technical estimates will be needed.

4.3.4 First technical estimates

For $\gamma > 0$, let $(1-\gamma\partial_x^2)^{-1}$ be the bounded operator from L^2 to H^2 defined by its Fourier transform as

$$(\widehat{(1-\gamma\partial_x^2)^{-1}g})(\xi) = \frac{\widehat{g}(\xi)}{1+\gamma\xi^2}, \quad \text{for any } g \in L^2.$$

We start with a basic but essential result, in the spirit of [19].

Lemma 4.10. *Let $f \in L^2(\mathbb{R})$ and $0 < \gamma < 1$, we have the following estimates*

(i) $\|(1-\gamma\partial_x^2)^{-1}f\|_{L^2(\mathbb{R})} \leqslant \|f\|_{L^2(\mathbb{R})}$,

(ii) $\|(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2(\mathbb{R})} \leqslant \gamma^{-1/2}\|f\|_{L^2(\mathbb{R})}$,

(iii) $\|(1-\gamma\partial_x^2)^{-1}f\|_{H^2(\mathbb{R})} \leqslant \gamma^{-1}\|f\|_{L^2(\mathbb{R})}$.

We also enunciate the following result that appears in [19, 18]:

Lemma 4.11. *There exist $\gamma_1 > 0$ and $C > 0$ such that for any $\gamma \in (0,\gamma_1)$, $0 < K \leqslant 1$ and $g \in L^2$, the following estimates holds*

$$\left\|\operatorname{sech}(Kx)\,(1-\gamma\partial_x^2)^{-1}g\right\|_{L^2} \leqslant C\left\|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}(Kx)\,g\right]\right\|_{L^2}, \tag{4.73}$$

and

$$\left\|\cosh(Kx)\,(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}(Kx)\,g\right]\right\|_{L^2} \leqslant C\left\|(1-\gamma\partial_x^2)^{-1}g\right\|_{L^2}.$$

From this lemma, we obtain the following result.

Corollary 4.12. *For any $0 < K \leqslant 1$ and $\gamma > 0$ small enough, for any $f \in L^2$,*

$$\| \operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2} \lesssim \gamma^{-1/2}\| \operatorname{sech}(Kx) f\|_{L^2}, \tag{4.74}$$

where the implicit constant is independent of γ and K.

Proof. Using (4.73) and rewriting, we have

$$\begin{aligned}\| \operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2} &\lesssim \|(1-\gamma\partial_x^2)^{-1}\left[\operatorname{sech}(Kx)\partial_x f\right]\|_{L^2}\\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}\partial_x\left[\operatorname{sech}(Kx) f\right]\|_{L^2}\\ &\quad + \|(1-\gamma\partial_x^2)^{-1}(\partial_x \operatorname{sech}(Kx)) f\|_{L^2}.\end{aligned}$$

The proof concludes applying Lemma 4.10. ■

Following the spirit of Lemma 4.11, we obtain

Lemma 4.13. *For any $0 < K \leqslant 1$ and $\gamma > 0$ small enough, for any $f \in L^2$,*

$$\| \operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1}(1-\partial_x^2) f\|_{L^2} \lesssim \gamma^{-1}\| \operatorname{sech}(Kx) f\|_{L^2}, \tag{4.75}$$

where the implicit constant is independent of γ and K.

Proof. Set $h = \operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1}(1-\partial_x^2)f$ and $k = \operatorname{sech}(Kx) f$. We have

$$\cosh(Kx)\, h = (1-\gamma\partial_x^2)^{-1}(1-\partial_x^2)\left[\cosh(Kx)k\right]. \tag{4.76}$$

Thus, we obtain

$$\begin{aligned}\cosh(Kx)\, h &= (1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right] - \partial_x^2(1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right]\\ &= (1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right] + \gamma^{-1}(1-\gamma\partial_x^2-1)(1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right]\\ &= (1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right] + \gamma^{-1}\cosh(Kx)k - \gamma^{-1}(1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right].\end{aligned}$$

Thus,

$$\gamma h = (\gamma-1)\operatorname{sech}(Kx)(1-\gamma\partial_x^2)^{-1}\left[\cosh(Kx)k\right] + k,$$

using Lemma 4.11 and dividing by γ, we obtain

$$\|h\|_{L^2} \lesssim \gamma^{-1}\|k\|_{L^2}.$$

This concludes the proof. ■

We need some additional auxiliary estimates to related the several variables defined.

Lemma 4.14. *One has:*

(a) Estimates on v_1:

$$\begin{aligned} \|v_1\| &\lesssim \gamma^{-1}\|u_1\|_{L^2}, \\ \|\partial_x v_1\| &\lesssim \gamma^{-1/2}\|f'(Q)u_1\|_{L^2} + \gamma^{-1}\|\partial_x u_1\|_{L^2}^2. \end{aligned} \tag{4.77}$$

(b) Estimates on v_2:

$$\begin{aligned} \|v_2\|_{L^2} \lesssim \|u_2\|_{L^2}, \quad \|\partial_x v_2\|_{L^2} \lesssim \gamma^{-1/2}\|u_2\|_{L^2}, \\ \|\partial_x^2 v_2\|_{L^2} \lesssim \gamma^{-1}\|u_2\|_{L^2}. \end{aligned} \tag{4.78}$$

The proof of the above results are a direct application of Lemma 4.10.

Lemma 4.15. *Let $1 \leqslant K \leqslant A$ fixed and ζ_K as in* (4.33)*. Then*

(a) Estimates on v_1:

$$\begin{aligned} \|\zeta_K v_1\| &\lesssim \gamma^{-1}\|w_1\|_{L^2}, \\ \|\zeta_K \partial_x v_1\| &\lesssim \gamma^{-1}\|\partial_x w_1\|_{L^2} + \gamma^{-1}\|w_1\|_{L^2}. \end{aligned} \tag{4.79}$$

(b) Estimates on v_2:

$$\begin{aligned} \|\zeta_K v_2\|_{L^2} &\lesssim \|w_2\|_{L^2}, \\ \|\zeta_K \partial_x v_2\|_{L^2} &\lesssim \gamma^{-1/2}\|w_2\|_{L^2}, \\ \|\zeta_K \partial_x^2 v_2\|_{L^2} &\lesssim \gamma^{-1}\|w_2\|_{L^2}. \end{aligned} \tag{4.80}$$

Proof. Proof of (4.79). (i) Applying the definition of v_1 (4.56), we have

$$\|\zeta_K v_1\|_{L^2} \lesssim \|\zeta_K (1-\gamma\partial_x^2)^{-1}\left[(1-\partial_x^2)(u_1) - f'(Q)u_1\right]\|_{L^2}.$$

Using Lemma 4.13 and Lemma 4.10, (4.32) and $1 \leqslant K \leqslant A$, we conclude

$$\begin{aligned} \|\zeta_K v_1\|_{L^2} &\lesssim \gamma^{-1}\|\zeta_K u_1\|_{L^2} + \|\zeta_K f'(Q)u_1\|_{L^2} \\ &\lesssim \gamma^{-1}\|\zeta_K\zeta_A^{-1}w_1\|_{L^2} + \|\zeta_K\zeta_A^{-1}f'(Q)w_1\|_{L^2} \lesssim \gamma^{-1}\|w_1\|_{L^2}. \end{aligned}$$

Proof of (ii). First, by the definition of w_1 (4.37), we get

$$\zeta_K \partial_x u_1 = \zeta_K\zeta_A^{-1}\left(\partial_x w_1 - \frac{\zeta_A'}{\zeta_A}w_1\right). \tag{4.81}$$

Then, by definition of v_1 in (4.56),

$$\|\zeta_K \partial_x v_1\|_{L^2} \lesssim \|\zeta_K(1-\gamma\partial_x^2)^{-1}(1-\partial_x^2)\partial_x u_1\|_{L^2} + \|\zeta_K(1-\gamma\partial_x^2)^{-1}\partial_x[f'(Q)u_1]\|_{L^2}.$$

and using Lemma 4.13,

$$\begin{aligned} \|\zeta_K \partial_x v_1\|_{L^2} &\lesssim \gamma^{-1}\|\zeta_K\partial_x u_1\|_{L^2} + \|\zeta_K f'(Q)u_1\|_{L^2} \\ &\lesssim \gamma^{-1}(\|\partial_x w_1\|_{L^2} + A^{-1}\|w_1\|_{L^2}) + \|w_1\|_{L^2} \\ &\lesssim \gamma^{-1}(\|\partial_x w_1\|_{L^2} + \|w_1\|_{L^2}). \end{aligned}$$

This ends the proof of (4.79). Following the preceding steps for v_2, the proof concludes. ∎

Now we perform some technical estimates on the variable z_{i}.

Corollary 4.16. *One has:*

(a) Estimates on z_1:

$$\begin{aligned} \|z_1\| &\lesssim \gamma^{-1}\|w_1\|_{L^2}, \\ \|\partial_x z_1\| &\lesssim \gamma^{-1}\|\partial_x w_1\|_{L^2} + \gamma^{-1}\|w_1\|_{L^2}. \end{aligned} \tag{4.82}$$

(b) Estimates on z_2:

$$\begin{aligned} \|z_2\|_{L^2} &\lesssim \|w_2\|_{L^2}, \\ \|\partial_x z_2\|_{L^2} &\lesssim \gamma^{-1/2}\|w_2\|_{L^2}, \\ \|\partial_x^2 z_2\|_{L^2} &\lesssim \gamma^{-1}\|w_2\|_{L^2}. \end{aligned} \tag{4.83}$$

Proof. Proof of (4.82). For (i), from definition of $z_1 = \chi_A \zeta_B v_1$, we have

$$\|z_1\|_{L^2} \lesssim \|\zeta_B v_1\|_{L^2},$$

and using Lemma 4.15, we conclude

$$\|z_1\|_{L^2} \lesssim \gamma^{-1}\|w_1\|_{L^2}.$$

For (ii), derivating z_1, we obtain

$$\begin{aligned} \partial_x z_1 &= \chi_A' \zeta_B v_1 + \chi_A \zeta_B' v_1 + \chi_A \zeta_B \partial_x v_1 \\ &= \chi_A' \zeta_B v_1 + \frac{\zeta_B'}{\zeta_B} z_1 + \chi_A \zeta_B \partial_x v_1. \end{aligned}$$

Then, by Lemma 4.15 we have

$$\|\partial_x z_1\| \leqslant \gamma^{-1}\big(\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}\big).$$

For z_2 we use the same strategy, and we skip the details. This ends the proof. ■

Lemma 4.17. *One has:*

(a) Estimates on u_1:

$$\begin{aligned} \left\|\operatorname{sech}^{1/2}(x) u_1\right\|_{L^2} &\lesssim \|w_1\|_{L^2}, \\ \left\|\operatorname{sech}^{1/2}(x) \partial_x u_1\right\|_{L^2} &\lesssim \|\partial_x w_1\|_{L^2} + \|w_1\|_{L^2}. \end{aligned} \tag{4.84}$$

(b) Estimates on u_2:

$$\left\|\operatorname{sech}^{1/2}(x) u_2\right\|_{L^2} \lesssim \|w_2\|_{L^2}. \tag{4.85}$$

Proof. Proof of (4.84). Recalling definition of $w_{\mathrm{i}} = \zeta_A u_{\mathrm{i}}$ for $\mathrm{i} = 1, 2.$. We have

$$\|\operatorname{sech}^{1/2}(x) u_{\mathrm{i}}\|_{L^2} \lesssim \|\operatorname{sech}^{1/2}(x)\zeta_A^{-1} w_{\mathrm{i}}\|_{L^2} \leqslant \|w_{\mathrm{i}}\|_{L^2}.$$

Furthermore, derivating w_1, we have

$$\zeta_A \partial_x u_1 = \partial_x w_1 - \frac{\zeta_A'}{\zeta_A} w_1.$$

Then,

$$\| \operatorname{sech}^{1/2}(x) \partial_x u_1 \|_{L^2} = \left\| \operatorname{sech}^{1/2}(x) \zeta_A^{-1} \left(\partial_x w_1 - \frac{\zeta_A'}{\zeta_A} w_1 \right) \right\| \leqslant \|\partial_x w_1\|_{L^2} + A^{-1} \|w_1\|_{L^2}.$$

This concludes the proof. ■

4.3.5 Controlling error and nonlinear terms

By the definition of ζ_B and χ_A in (4.33) and (4.34), it holds

$$\begin{aligned} &\zeta_B(x) \leqslant \mathrm{e}^{-\frac{|x|}{B}}, \quad |\zeta_B'(x)| \lesssim \frac{1}{B} \mathrm{e}^{-\frac{|x|}{B}}, \quad |\varphi_B| \lesssim B, \\ &|(\chi_A^2)'| \lesssim A^{-1}, \ |(\chi_A^2)''| \lesssim A^{-2}, \quad |(\chi_A^2)'''| \lesssim A^{-3}. \end{aligned} \tag{4.86}$$

Control of $\tilde{J}_1$.

Considering the following decomposition $\tilde{J}_1$:

$$\begin{aligned} \tilde{J}_1 = &- \frac{1}{2} \int (\chi_A^2)' \varphi_B [v_1^2 + 3(\partial_x v_2)^2] + \frac{1}{2} \int \left[(\chi_A^2)' f'(Q) + \left((\chi_A^2)''' - (\chi_A^2)' \right) \right] \varphi_B v_2^2 \\ &- \frac{3}{2} \int \left[\left(\chi_A'' \chi_A - (\chi_A^2)'' \right) - (\chi_A^2)' \frac{\zeta_B'}{\zeta_B} \right] \zeta_B^2 v_2^2 =: H_1 + H_2 + H_3. \end{aligned}$$

For H_1 and H_2, using $|(\chi_A^2)' \varphi_B| \lesssim A^{-1} B$ and Remark 4.7, we obtain

$$|H_1| \lesssim A^{-1} B (\|v_1\|_{L^2(|x| \leqslant 2A)}^2 + \|\partial_x v_2\|_{L^2(|x| \leqslant 2A)}^2) \lesssim A^{-1} B (\|\zeta_A^2 v_1\|_{L^2}^2 + \|\zeta_A^2 \partial_x v_2\|_{L^2}^2),$$

and

$$|H_2| \lesssim A^{-1} B \|v_2\|_{L^2(A \leqslant |x| \leqslant 2A)}^2 \lesssim A^{-1} B \|v_2\|_{L^2(|x| \leqslant 2A)}^2 \lesssim A^{-1} B \|\zeta_A^2 v_2\|_{L^2}^2.$$

For H_3, using (4.86), we have

$$\begin{aligned} |H_3| &\leqslant \frac{3}{2} \int \left| \left(\chi_A'' \chi_A - (\chi_A^2)'' \right) - (\chi_A^2)' \frac{\zeta_B'}{\zeta_B} \right| \zeta_B^2 v_2^2 \\ &\lesssim (AB)^{-1} \|\zeta_B v_2\|_{L^2(|x| \leqslant 2A)}^2 \lesssim (AB)^{-1} \|\zeta_B v_2\|_{L^2}^2. \end{aligned}$$

Finally, we get

$$|\tilde{J}_1| \lesssim A^{-1} B \left(\|\zeta_A v_1\|_{L^2}^2 + \|\zeta_A v_2\|_{L^2}^2 + \|\zeta_A \partial_x v_2\|_{L^2}^2 \right), \tag{4.87}$$

since $\zeta_B \lesssim \zeta_A$.

Control of J_4.

Recall that

$$J_4 = \gamma \int \psi_{A,B} v_2 (1-\gamma\partial_x^2)^{-1} \left[2\partial_x(\partial_x(f'(Q))\partial_x v_2) - \partial_x^2(f'(Q))\partial_x v_2\right].$$

Using Hölder's inequality

$$|J_4| \lesssim \gamma \|\psi_{A,B} v_2\|_{L^2} \underbrace{\|(1-\gamma\partial_x^2)^{-1}[2\partial_x(\partial_x(f'(Q))\partial_x v_2) - \partial_x^2(f'(Q))\partial_x v_2]\|_{L^2}}_{J_4^1}.$$

First we focus on J_4^1. Using (4.10),

$$J_4^1 \lesssim \gamma^{-1/2} \|(\partial_x(f'(Q))\partial_x v_2\|_{L^2} + \|\partial_x^2(f'(Q))\partial_x v_2\|_{L^2}.$$

Recall that $|\partial_x(f'(Q))|, |\partial_x^2(f'(Q))| \sim Q^{p-2}|Q'| \lesssim \mathrm{e}^{-(p-1)|x|}$. Therefore, we are led to the estimate of

$$\|\mathrm{e}^{-(p-1)|x|}\partial_x v_2\|_{L^2}.$$

Differentiating $z_2 = \chi_A \zeta_B v_2$, we obtain

$$\chi_A \zeta_B \partial_x v_2 = \partial_x z_2 - \frac{\zeta_B'}{\zeta_B} z_2 - \chi_A' \zeta_B v_2,$$

we get

$$\begin{aligned}
&\mathrm{e}^{-2(p-1)|x|}(\partial_x v_2)^2 \\
&= \mathrm{e}^{-2(p-1)|x|}\chi_A^2(\partial_x v_2)^2 + \mathrm{e}^{-2(p-1)|x|}(1-\chi_A^2)(\partial_x v_2)^2 \\
&\lesssim \mathrm{e}^{-(p-1)|x|}\left[(\partial_x z_2)^2 + \left(\frac{\zeta_B'}{\zeta_B}\right)^2 z_2^2 + (\chi_A'\zeta_B v_2)^2\right] + \mathrm{e}^{-(p-1)A}\mathrm{e}^{-(p-1)|x|}(\partial_x v_2)^2 \\
&\lesssim \mathrm{e}^{-(p-1)|x|}\left[(\partial_x z_2)^2 + \frac{1}{B^2} z_2^2\right] + \mathrm{e}^{-(p-1)|x|}(\chi_A'\zeta_B v_2)^2 + \mathrm{e}^{-(p-1)A}\mathrm{e}^{-(p-1)|x|}(\partial_x v_2)^2.
\end{aligned}$$

Hence,

$$\|\mathrm{e}^{-|x|}\partial_x v_2\|_{L^2} \lesssim \|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + \mathrm{e}^{-(p-1)A}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B \partial_x v_2\|_{L^2}).$$

By the above inequality, we have

$$J_4^1 \lesssim \gamma^{-1/2}\Big(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + \mathrm{e}^{-(p-1)A}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B \partial_x v_2\|_{L^2})\Big). \tag{4.88}$$

Second, using $\psi_{A,B} = \chi_A^2 \varphi_B$ in (4.60), and Remark 4.7, one can see that

$$\|\psi_{A,B} v_2\|_{L^2} \lesssim B\|\chi_A v_2\|_{L^2} \lesssim B\|v_2\|_{L^2(|x|<2A)} \lesssim B\|\zeta_A^2 v_2\|_{L^2}.$$

We conclude

$$|J_4| \lesssim \gamma^{1/2} B \|\zeta_A^2 v_2\|_{L^2}\Big(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + \mathrm{e}^{-(p-1)A}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B \partial_x v_2\|_{L^2})\Big). \tag{4.89}$$

Control of J_5.

Recalling that $\psi_{A,B} = \chi_A^2\varphi_B$, using the Hölder inequality and Remark 4.7, we get

$$\begin{aligned}|J_5| = \left|\int \psi_{A,B} H(x) v_1\right| &\lesssim \|\chi_A\varphi_B v_1\|_{L^2}\|\chi_A(1-\gamma\partial_x^2)^{-1}N^\perp\|_{L^2}\\ &\lesssim \|\chi_A\varphi_B v_1\|_{L^2}\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}N^\perp\|_{L^2}.\end{aligned}$$

By the definition of $N^\perp$ (see (4.29)), it follows that

$$\begin{aligned}\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}N^\perp\|_{L^2} &\leqslant \|\zeta_A^2(1-\gamma\partial_x^2)^{-1}N\|_{L^2} + |N_0|\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}\partial_x^{-1}\phi_0\|_{L^2}\\ &\lesssim \|\zeta_A^2(1-\gamma\partial_x^2)^{-1}N\|_{L^2} + |N_0|,\end{aligned}$$

since $\partial_x^{-1}\phi_0 \in L^2$ and $0 \leqslant \zeta_A \lesssim 1$.
Furthermore, by definition of N in (4.29), and using Corollary 4.12, (4.49) and Lemma 4.10, we have

$$\begin{aligned}&\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}N\|_{L^2}\\ &\leqslant \gamma^{-1/2}\|\zeta_A^2[f(Q)+f'(Q)(a_1\phi_0+u_1)-f(Q+a_1\phi_0+u_1)]\|_{L^2}\\ &\leqslant \gamma^{-1/2}\|\zeta_A^2[a_1^2 f''(Q)\phi_0^2 + f''(Q)u_1^2 + |a_1|^p\phi_0^p + |u_1|^p]\|_{L^2}\\ &\leqslant \gamma^{-1/2}\left(a_1^2\|f''(Q)\zeta_A^2\phi_0^2\|_{L^2} + \|f''(Q)\zeta_A w_1^2\|_{L^2} + |a_1|^p\|\zeta_A^2\phi_0^p\|_{L^2} + \|\zeta_A|u_1|^{p-1}w_1\|_{L^2}\right)\\ &\lesssim \gamma^{-1/2}\left(a_1^2 + \|u_1\|_{L^\infty}\|f''(Q)w_1\|_{L^2} + |a_1|^p + \|u_1\|_{L^\infty}^{p-1}\|w_1\|_{L^2}\right)\\ &\lesssim \gamma^{-1/2}\left(a_1^2 + \|u_1\|_{L^\infty}\|w_1\|_{L^2}\right).\end{aligned} \tag{4.90}$$

Note that we have used that $p \geqslant 2$. Since $|\chi_A\varphi_B| \lesssim B$, we have

$$\|\chi_A\varphi_B v_1\|_{L^2} \lesssim B\|\chi_A v_1\|_{L^2} \lesssim B\|v_1\|_{L^2(|x|<2A)} \lesssim B\|\zeta_A^2 v_1\|_{L^2}. \tag{4.91}$$

Finally, by (4.51), (4.90) and (4.91) , we conclude

$$|J_5| \lesssim B\gamma^{-1/2}\|\zeta_A^2 v_1\|_{L^2}\big(a_1^2 + \|u_1\|_{L^\infty}\|w_1\|_{L^2}\big). \tag{4.92}$$

4.3.6 End of proof of Proposition 4.8

From (4.87), (4.89), (4.92), and choosing

$$0 < \gamma = B^{-4}, \tag{4.93}$$

it follows

$$\begin{aligned}|\tilde{J}_1 + J_4 + J_5| \lesssim{}& A^{-1}B\left(\|\zeta_A v_1\|_{L^2}^2 + \|\zeta_A v_2\|_{L^2}^2 + \|\zeta_A\partial_x v_2\|_{L^2}^2\right)\\ &+ \gamma^{1/2}B\|\zeta_A v_2\|_{L^2}\left(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + \mathrm{e}^{-(p-1)A}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B\partial_x v_2\|_{L^2})\right)\\ &+ B\gamma^{-1/2}\|\zeta_A v_1\|_{L^2}\left(a_1^2 + \|u_1\|_{L^\infty}\|w_1\|_{L^2}\right).\end{aligned} \tag{4.94}$$

Applying Lemma 4.15-(4.79) and (4.35), we obtain

$$\begin{aligned}|\tilde{J}_1 + J_4 + J_5| \lesssim B^{-1}&\bigg(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 \bigg)\\ &+ B^8\bigg(a_1^4 + \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2 \bigg).\end{aligned} \tag{4.95}$$

Choosing

$$B \leqslant \delta^{-1/8}, \tag{4.96}$$

(to be fixed later) and using (4.27), we arrive to

$$B^8(a_1^4 + \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2) \lesssim \delta^{-1}(a_1^4 + \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2) \lesssim |a_1|^3 + \delta \|w_1\|_{L^2}^2.$$

Then, using the above estimates, we obtain that the error term and the associated to the nonlinear part are bounded as follows:

$$|\tilde{J}_1 + J_4 + J_5| \lesssim B^{-1} \left(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 \right) + |a_1|^3.$$

Finally, the virial estimate is concluded as follows: for some $C_2 > 0$ independent of B large,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) \leqslant & -\frac{1}{2}\int \left[z_1^2 + (V_0(x) - CB^{-1})z_2^2 + 3(\partial_x z_2)^2 \right]\\ &+ CB^{-1}\bigg(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 \bigg) + C|a_1|^3.\\ \leqslant & -\frac{1}{2}\int \left[z_1^2 + (V_0(x) - C_2B^{-1})z_2^2 + 2(\partial_x z_2)^2 \right]\\ &+ C_2B^{-1}\bigg(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 \bigg) + C_2|a_1|^3.\end{aligned} \tag{4.97}$$

This ends the proof of Proposition 4.8.

4.4 Gain of derivatives via transfer estimates

We must note that in (4.38) the last term is a localized one, which in the language of estimate (4.97) will correspond to a term of type $\partial_x z_1$, not appearing in this last estimate. However, this new term will be well-defined by the regularity of the original variables (u_1, u_2). We think that this problem appears as a product of the lack of balance in the regularity of (v_1, v_2) (see Subsection 4.56). Therefore, we need new estimates to control $\partial_x z_1$.

To solve this new problem, we will focus on a new virial obtained for a new system of equations involving the variables $\tilde{v}_{\mathrm{i}} = \partial_x v_{\mathrm{i}}$, for $\mathrm{i} = 1, 2$. Formally taking derivatives in (4.57), we have

$$\begin{cases} \dot{\tilde{v}}_1 = \mathcal{L}(\partial_x \tilde{v}_2) - \partial_x(f'(Q))v_2 + \tilde{G}(x), \qquad \tilde{G}(x) = \partial_x G(x), \\ \dot{\tilde{v}}_2 = \partial_x \tilde{v}_1 + \tilde{H}(x), \qquad \tilde{H}(x) = \partial_x H(x), \end{cases} \tag{4.98}$$

where G and H are given in (4.58).
For this new system, we consider the virial

$$\mathcal{M}(t) = \int \phi_{A,B}\tilde{v}_1\tilde{v}_2 = \int \phi_{A,B}\partial_x v_1 \partial_x v_2. \tag{4.99}$$

Later we will choose $\phi_{A,B} = \psi_{A,B} = \chi_A^2 \varphi_B$ (see (4.60)).

4.4.1 A virial estimate related to $\mathcal{M}(t)$

Lemma 4.18. *Let $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ a solution of (4.57). Consider $\phi_{A,B}$ an odd smooth bounded function to be a choose later. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = &-\frac{1}{2}\int \phi'_{A,B}\left((\partial_x v_1)^2 + (\partial_x v_2)^2 + 3(\partial_x^2 v_2)^2\right) + \frac{1}{2}\int \phi'''_{A,B}(\partial_x v_2)^2 \\ &-\frac{1}{2}\int \phi_{A,B} f'(Q)\partial_x((\partial_x v_2)^2) + \int \phi_{A,B}\tilde{G}(x)\partial_x v_2 + \int \phi_{A,B}\tilde{H}(x)\partial_x v_1.\end{aligned} \tag{4.100}$$

The identity (4.100) is interesting because it has exactly the same structure that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t)$ in (4.63). This holds despite the new derivative terms appearing in (4.98). To obtain this we will benefit from a cancellation given by the parity of the data.

Proof of Lemma 4.18. From (4.57), (4.98) and (4.63), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = &-\frac{1}{2}\int \phi'_{A,B}\left(\tilde{v}_1^2 + \tilde{v}_2^2 + 3(\partial_x \tilde{v}_2)^2\right) + \frac{1}{2}\int \phi'''_{A,B}\tilde{v}_2^2 \\ &-\frac{1}{2}\int \phi_{A,B} f'(Q)\partial_x(\tilde{v}_2^2) + \int \phi_{A,B}\big[-\partial_x(f'(Q))v_2 + \tilde{G}(x)\big]\tilde{v}_2 + \int \phi_{A,B}\tilde{H}(x)\tilde{v}_1.\end{aligned} \tag{4.101}$$

Rewriting the above identity in term of the variables (v_1, v_2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = &-\frac{1}{2}\int \phi'_{A,B}\left((\partial_x v_1)^2 + (\partial_x v_2)^2 + 3(\partial_x^2 v_2)^2\right) + \frac{1}{2}\int \phi'''_{A,B}(\partial_x v_2)^2 \\ &-\frac{1}{2}\int \phi_{A,B} f'(Q)\partial_x((\partial_x v_2)^2) - \int \phi_{A,B}\partial_x(f'(Q))v_2\partial_x v_2 \\ &+\int \phi_{A,B}\tilde{G}(x)\partial_x v_2 + \int \phi_{A,B}\tilde{H}(x)\partial_x v_1.\end{aligned} \tag{4.102}$$

Noticing that $v_2\partial_x v_2$, $\partial_x(f'(Q))$ and $\phi_{A,B}$ are odd functions (see (4.56) and (4.30)), we have

$$\int \phi_{A,B}\partial_x(f'(Q))v_2\partial_x v_2 = 0.$$

This ends the proof of Lemma 4.18. ■

The following proposition connects two virial identities in the variable (z_1, z_2). Recall that from (4.93) and (4.96), $\gamma = B^{-4}$, $B \leqslant \delta^{-1/8}$.

Proposition 4.19. *There exist $C_3 > 0$ and $\delta_3 > 0$ such that for any $0 < \delta \leqslant \delta_3$, the following holds. Fix $B = \delta^{-1/19} \leqslant \delta^{-1/8}$. Assume that for all $t \geqslant 0$, (4.27) holds. Then for all $t \geqslant 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) \leqslant &-\frac{1}{2}\int \left((\partial_x z_1)^2 + \left(V_0(x) - C_3 B^{-1}\right)(\partial_x z_2)^2 + 2(\partial_x^2 z_2)^2\right) \\ &+ C_3\|z_2\|_{L^2}^2 + C_3 B^{-1}\|z_1\|_{L^2}^2 \\ &+ C_3 B^{-1}\left(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_3|a_1|^3.\end{aligned} \tag{4.103}$$

The proof of the above result requires some technical estimates. We first state them, to then prove Proposition 4.19 (Subsection 4.4.3).

4.4.2 Second set of technical estimates

Now, we recall the following technical estimates on the variables ζ_B and other related error terms. These estimates are similar to the ones obtained in (4.43), therefore we only prove the new ones.

Lemma 4.20. *Let ζ_B and χ be defined by* (4.33) *and* (4.31)*, respectively. Then*

$$\frac{\zeta_B'}{\zeta_B} = -\frac{1}{B}[-\chi'(x)|x| + (1-\chi(x))sgn(x)], \quad \frac{\zeta_B''}{\zeta_B} = \left(\frac{\zeta_B'}{\zeta_B}\right)^2 + \frac{1}{B}[\chi''(x)|x| + 2\chi'(x)sgn(x)], \tag{4.104}$$

and

$$\begin{aligned} \frac{\zeta_B'''}{\zeta_B} &= 3\frac{\zeta_B''}{\zeta_B}\frac{\zeta_B'}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^3 + B^{-1}\left[\chi'''(x)|x| + 3\chi''(x)sgn(x)\right], \\ \frac{\zeta_B^{(4)}}{\zeta_B} &= 4\frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B} + 3\left(\frac{\zeta_B''}{\zeta_B}\right)^2 - 12\frac{\zeta_B''}{\zeta_B}\left(\frac{\zeta_B'}{\zeta_B}\right)^2 + 6\left(\frac{\zeta_B'}{\zeta_B}\right)^4 + \frac{1}{B}\left[\chi^{(4)}(x)|x| + 4\chi'''(x)sgn(x)\right]. \end{aligned} \tag{4.105}$$

Proof. Direct. ∎

Remark 4.9. From the previous lemma we observe that

$$\left|\frac{\zeta_B'}{\zeta_B}\right| \lesssim B^{-1}\mathbf{1}_{\{|x|>1\}}(x),$$

$$\left|\frac{\zeta_B''}{\zeta_B}\right| \lesssim B^{-2}\mathbf{1}_{\{|x|>1\}}(x) + B^{-1}\mathbf{1}_{\{1<|x|<2\}}(x) \lesssim B^{-2} + B^{-1}\operatorname{sech}(x),$$

$$\left|\frac{\zeta_B'''}{\zeta_B}\right| \lesssim B^{-3} + B^{-1}\operatorname{sech}(x), \quad \left|\frac{\zeta_B^{(4)}}{\zeta_B}\right| \lesssim B^{-4} + B^{-1}\operatorname{sech}(x).$$

In particular, for A large enough, the following estimates hold:

$$\begin{aligned} \left|\frac{\zeta_B'}{\zeta_B}\right| &\lesssim B^{-1}, \quad \left|\mathbf{1}_{\{A<|x|<2A\}}\frac{\zeta_B''}{\zeta_B}\right| \lesssim B^{-2}, \\ \left|\mathbf{1}_{\{A<|x|<2A\}}\frac{\zeta_B'''}{\zeta_B}\right| &\lesssim B^{-3}, \quad \left|\mathbf{1}_{\{A<|x|<2A\}}\frac{\zeta_B^{(4)}}{\zeta_B}\right| \lesssim B^{-4}. \end{aligned} \tag{4.106}$$

Finally,

$$\left|\frac{\zeta_B''}{\zeta_B}\right| + \left|\frac{\zeta_B'''}{\zeta_B}\right| + \left|\frac{\zeta_B^{(4)}}{\zeta_B}\right| \lesssim B^{-1}. \tag{4.107}$$

These estimates will be useful in Claim 4.22. Now we prove a formula for changing variables.

Claim 4.21. *Let $P \in W^{1,\infty}(\mathbb{R})$, $v_{\rm i}$ be as in (4.56), and $z_{\rm i}$ be as in (4.60). Then*

$$\int P(x)\chi_A^2\zeta_B^2(\partial_x v_{\rm i})^2 = \int P(x)(\partial_x z_{\rm i})^2 + \int \left[P'(x)\frac{\zeta_B'}{\zeta_B} + P(x)\frac{\zeta_B''}{\zeta_B}\right] z_{\rm i}^2 + \int \mathcal{E}_1(P(x),x)\zeta_B^2 v_{\rm i}^2, \tag{4.108}$$

where

$$\mathcal{E}_1(P(x),x) = P(x)\left[\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + \frac{1}{2}P'(x)(\chi_A^2)', \tag{4.109}$$

and

$$|\mathcal{E}_1(P(x),x)| \lesssim A^{-1}\|P'\|_{L^\infty} + (AB)^{-1}\|P\|_{L^\infty}. \tag{4.110}$$

For the proof of these results, see Appendix 4.B.1.

Remark 4.10. For $P = 1$, we get

$$\int \chi_A^2\zeta_B^2(\partial_x v_{\rm i})^2 = \int (\partial_x z_{\rm i})^2 + \int \frac{\zeta_B''}{\zeta_B} z_{\rm i}^2 + \int \mathcal{E}_1(1,x)\zeta_B^2 v_{\rm i}^2, \tag{4.111}$$

where

$$\mathcal{E}_1(1,x) = \frac{1}{2}\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}. \tag{4.112}$$

Finally, one has the following estimate:

$$\|\chi_A\zeta_B\partial_x v_{\rm i}\|^2 \lesssim \|\partial_x z_{\rm i}\|_{L^2}^2 + B^{-1}\|z_{\rm i}\|_{L^2}^2 + (AB)^{-1}\|w_{\rm i}\|_{L^2}^2.$$

We need a second claim on the second derivative of $v_{\rm i}$.

Claim 4.22. *Let R be a $W^{2,\infty}(\mathbb{R})$ function, $v_{\rm i}$ be as in (4.56), and $z_{\rm i}$ be as in (4.60). Then*

$$\begin{aligned}\int R(x)\chi_A^2\zeta_B^2(\partial_x^2 v_{\rm i})^2 = &\int R(x)(\partial_x^2 z_{\rm i})^2 + \int \tilde{R}(x) z_{\rm i}^2 + \int P_R(x)(\partial_x z_{\rm i})^2\\ &+ \int \left[P_R'(x)\frac{\zeta_B'}{\zeta_B} + P_R(x)\frac{\zeta_B''}{\zeta_B}\right] z_{\rm i}^2 + \int \mathcal{E}_2(R(x),x)\zeta_B^2 v_{\rm i}^2\\ &+ \int \mathcal{E}_1(P_R(x),x)\zeta_B^2 v_{\rm i}^2 + \int \mathcal{E}_3(R(x),x)\zeta_B^2(\partial_x v_{\rm i})^2,\end{aligned}$$

where

$$\tilde{R}(x) = -2R(x)\left[\frac{\zeta_B^{(4)}}{\zeta_B} + \frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B}\right] - 2R'(x)\frac{\zeta_B'''}{\zeta_B} - R''(x)\frac{\zeta_B''}{\zeta_B}, \tag{4.113}$$

$$P_R(x) = R(x)\left[4\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2\right] + 2R'(x)\frac{\zeta_B'}{\zeta_B}, \tag{4.114}$$

$\mathcal{E}_1$ is defined in (4.109),

$$\begin{aligned}\mathcal{E}_2(R(x),x) = &- R(x)\left(\chi_A^{(4)}\chi_A + 4\chi_A'''\chi_A\frac{\zeta_B'}{\zeta_B^2} + 6\chi_A''\chi_A\frac{\zeta_B''}{\zeta_B} + 2(\chi_A^2)'\frac{\zeta_B'''}{\zeta_B}\right)\\ &- R'(x)\left(2\chi_A'''\chi_A + 6\chi_A''\chi_A\frac{\zeta_B'}{\zeta_B} + 6\chi_A'\chi_A\frac{\zeta_B''}{\zeta_B}\right)\\ &- R''(x)\left(\chi_A''\chi_A + \frac{1}{2}(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right),\end{aligned} \tag{4.115}$$

and

$$\mathcal{E}_3(R(x),x) = R(x)\left[4\chi_A''\chi_A - 2(\chi_A')^2 + 2\frac{\zeta_B'}{\zeta_B}(\chi_A^2)'\right] + R'(x)(\chi_A^2)'. \tag{4.116}$$

Finally, P_R, $\mathcal{E}_2$ and $\mathcal{E}_3$ satisfy the following inequalities

$$\begin{aligned}|P_R| &\lesssim B^{-1}\|R'\|_{L^\infty} + B^{-1}\|R\|_{L^\infty},\\ |P_R'| &\lesssim B^{-1}\|R''\|_{L^\infty} + B^{-1}\|R'\|_{L^\infty} + B^{-1}\|R\|_{L^\infty},\\ |\mathcal{E}_2| &\lesssim (AB)^{-1}\|R''\|_{L^\infty} + (AB^2)^{-1}\|R'\|_{L^\infty} + (AB^3)^{-1}\|R\|_{L^\infty},\\ |\mathcal{E}_3| &\lesssim A^{-1}\|R'\|_{L^\infty} + (AB)^{-1}\|R\|_{L^\infty}.\end{aligned} \tag{4.117}$$

For the proof of these results, see Appendix 4.B.2.

Remark 4.11. For $R = 1$, we obtain

$$\begin{aligned}\int \chi_A^2\zeta_B^2(\partial_x^2 v_{\rm i})^2 = &\int(\partial_x^2 z_{\rm i})^2 + \int \tilde{R}_1(x)z_{\rm i}^2 + \int P_1(x)(\partial_x z_{\rm i})^2 + \int\left[P_1'(x)\frac{\zeta_B'}{\zeta_B} + P_1(x)\frac{\zeta_B''}{\zeta_B}\right]z_{\rm i}^2\\ &+ \int \mathcal{E}_2(1,x)\zeta_B^2 v_{\rm i}^2 + \int \mathcal{E}_1(P_1(x),x)\zeta_B^2 v_{\rm i}^2 + \int \mathcal{E}_3(1,x)\zeta_B^2(\partial_x v_{\rm i})^2,\end{aligned} \tag{4.118}$$

where,

$$\tilde{R}_1(x) = -2\left[\frac{\zeta_B^{(4)}}{\zeta_B} + \frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B}\right], \qquad P_1(x) = 4\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2, \tag{4.119}$$

$\mathcal{E}_1$ is defined in (4.109),

$$\mathcal{E}_2(1,x) = -\left(\chi_A^{(4)}\chi_A + 4\chi_A'''\chi_A\frac{\zeta_B'}{\zeta_B^2} + 6\chi_A''\chi_A\frac{\zeta_B''}{\zeta_B} + 2(\chi_A^2)'\frac{\zeta_B'''}{\zeta_B}\right), \tag{4.120}$$

and

$$\mathcal{E}_3(1,x) = 4\chi_A''\chi_A - 2(\chi_A')^2 + 2\frac{\zeta_B'}{\zeta_B}(\chi_A^2)'. \tag{4.121}$$

By Lemma 4.15, we obtain the estimate:

$$\|\chi_A\zeta_B\partial_x^2 v_{\rm i}\| \lesssim \|\partial_x^2 z_{\rm i}\|_{L^2}^2 + B^{-1}\|\partial_x z_{\rm i}\|_{L^2}^2 + B^{-1}\|z_{\rm i}\|_{L^2}^2 + A^{-1}B^3\|w_{\rm i}\|_{L^2}^2.$$

4.4.3 Start of proof of Proposition 4.19

The proof of this result is based in the following computation:

Lemma 4.23. *Let* $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ *a solution of* (4.57). *Consider* $\phi_{A,B} = \psi_{A,B} = \chi_A^2 \varphi_B$. *Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} = &-\frac{1}{2}\int \left((\partial_x z_1)^2 + \left(V_0(x) - \frac{\varphi_B}{\zeta_B^2}\partial_x (f'(Q)) \right)(\partial_x z_2)^2 + 3(\partial_x^2 z_2)^2 \right) \\ &+ \frac{1}{2}\int \frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x^2 (f'(Q)) z_2^2 + \mathcal{R}_z(t) + \mathcal{R}_v(t) + \mathcal{D}\mathcal{R}_v(t) \\ &+ \int \phi_{A,B}\tilde{G}(x)\partial_x v_2 + \int \phi_{A,B}\tilde{H}(x)\partial_x v_1,\end{aligned} \tag{4.122}$$

where $\mathcal{R}_z(t)$, $\mathcal{R}_v(t)$ *and* $\mathcal{D}\mathcal{R}_v(t)$ *are error terms that satisfy the following bounds*

$$\begin{aligned}|\mathcal{R}_z(t)| + |\mathcal{R}_v(t)| + |\mathcal{D}\mathcal{R}_v(t)| \lesssim\ & B^{-1}\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 \right) \\ &+ B^{-1}\left(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 \right),\end{aligned} \tag{4.123}$$

valid for B *sufficiently large.*

Proof. First, we recall that $z_i = \chi_A \zeta_B v_i$, and by (4.64) and Claim 4.21

$$\begin{aligned}\int \phi_{A,B}' \left[(\partial_x v_1)^2 + (\partial_x v_2)^2 \right] = &\int (\partial_x z_1)^2 + (\partial_x z_2)^2 + \int \frac{\zeta_B''}{\zeta_B}\left(z_1^2 + z_2^2 \right) \\ &+ \int \mathcal{E}_1(1,x)\zeta_B^2\left(v_1^2 + v_2^2 \right) + \int (\chi_A^2)'\varphi_B \left[(\partial_x v_1)^2 + (\partial_x v_2)^2 \right],\end{aligned} \tag{4.124}$$

where $\mathcal{E}_1(1,x)$ is given by (4.112). Now, using Remark 4.11 (4.118), we get

$$\begin{aligned}\int \phi_{A,B}'(\partial_x^2 v_2)^2 = &\int (\partial_x^2 z_2)^2 + \int P_1(x)(\partial_x z_2)^2 + \int \left[\tilde{R}_1(x) + P_1'(x)\frac{\zeta_B'}{\zeta_B} + P_1(x)\frac{\zeta_B''}{\zeta_B} \right] z_2^2 \\ &+ \int \mathcal{E}_2(1,x)\zeta_B^2 v_2^2 + \int \mathcal{E}_1(P_1(x),x)\zeta_B^2 v_2^2 + \int \mathcal{E}_3(1,x)\zeta_B^2(\partial_x v_2)^2 \\ &+ \int (\chi_A^2)'\varphi_B(\partial_x^2 v_2)^2,\end{aligned} \tag{4.125}$$

where $\tilde{R}_1(x), P_1(x), \mathcal{E}_2(1,x), \mathcal{E}_3(1,x)$ are given by (4.119), (4.120), (4.121), and $\mathcal{E}_1$ is gyven by (4.109).
Now, continuing with the second integral in the RHS of (4.100), we have

$$\int \phi_{A,B}'''(\partial_x v_2)^2 = \int \frac{(\zeta_B^2)''}{\zeta_B^2}\chi_A^2\zeta_B^2(\partial_x v_2)^2 + \int \left[6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2} \right]\zeta_B^2(\partial_x v_2)^2,$$

and using Claim 4.21,

$$\begin{aligned}\int \phi'''_{A,B}(\partial_x v_2)^2 =& \int \frac{(\zeta_B^2)''}{\zeta_B^2}(\partial_x z_2)^2 + \int \frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}z_2^2 + \int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B}z_2^2\\ &+ \frac{1}{2}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\left[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2 + \frac{1}{2}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'(\chi_A^2)'\zeta_B^2 v_2^2 \\ &+ \int \left[6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2}\right]\zeta_B^2(\partial_x v_2)^2.\end{aligned} \tag{4.126}$$

For the third integral in the RHS of (4.100), integrating by parts

$$\begin{aligned}\int \phi_{A,B} f'(Q)\partial_x((\partial_x v_2)^2) =& -\int \left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)\chi_A^2\zeta_B^2(\partial_x v_2)^2\\ &- \int (\chi_A^2)'\varphi_B f'(Q)(\partial_x v_2)^2.\end{aligned}$$

By the extended version of Claim 4.21 and expanding the derivates in terms of z_2, we have

$$\begin{aligned}&\int \phi_{A,B} f'(Q)\partial_x((\partial_x v_2)^2)\\ &= -\int \left[\left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)\frac{\zeta_B''}{\zeta_B} - \left(2\partial_x(f'(Q)) - \frac{\varphi_B(\zeta_B^2)'}{\zeta_B^4}\partial_x(f'(Q)) + \frac{\varphi_B}{\zeta_B^2}\partial_x^2(f'(Q))\right)\frac{\zeta_B'}{\zeta_B}\right]z_2^2\\ &\quad - \int \left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\partial_x z_2)^2 - \frac{1}{2}\int \left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)\left[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2\\ &\quad - \frac{1}{2}\int \partial_x\left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\chi_A^2)'\zeta_B^2 v_2^2 - \int (\chi_A^2)'\varphi_B f'(Q)(\partial_x v_2)^2.\end{aligned} \tag{4.127}$$

Collecting (4.124), (4.125),(4.126) and (4.127), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} =& -\frac{1}{2}\int (\partial_x z_1)^2 + \left(V_0(x) - \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\partial_x z_2)^2 + 3(\partial_x^2 z_2)^2\\ &+ \frac{1}{2}\int \left[\frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x^2(f'(Q)) + \frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}\partial_x(f'(Q))\right]z_2^2 + \mathcal{R}_z(t) + \mathcal{R}_v(t) + \mathcal{DR}_v(t)\\ &+ \int \phi_{A,B}\tilde{G}(x)\partial_x v_2 + \int \phi_{A,B}\tilde{H}(x)\partial_x v_1,\end{aligned}$$

where the error terms are the following: associated to (z_1, z_2) is

$$\begin{aligned}\mathcal{R}_z(t) =& -\frac{1}{2}\int \frac{\zeta_B''}{\zeta_B}(z_1^2 + z_2^2) - \frac{3}{2}\int \left[\tilde{R}_1(x) + P_1'(x)\frac{\zeta_B'}{\zeta_B} + P_1(x)\frac{\zeta_B''}{\zeta_B}\right]z_2^2\\ &+ \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B} + \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B}\right]z_2^2\\ &+ \frac{1}{2}\int \left[f'(Q)\frac{\zeta_B''}{\zeta_B} + \left(2\partial_x(f'(Q)) - 2\frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x(f'(Q))\right)\frac{\zeta_B'}{\zeta_B}\right]z_2^2\\ &+ \frac{1}{2}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2} - 3P_1(x)\right)(\partial_x z_2)^2,\end{aligned} \tag{4.128}$$

associated to (v_1, v_2) is

$$\begin{aligned}\mathcal{R}_v(t) = &-\frac{1}{2}\int \mathcal{E}_1(1,x)\zeta_B^2\big(v_1^2+v_2^2\big) - \frac{3}{2}\int \Big[\mathcal{E}_2(1,x)+\mathcal{E}_1(P_1(x),x)\Big]\zeta_B^2 v_2^2\\ &+\frac{1}{4}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\Big[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\Big]\zeta_B^2 v_2^2 + \frac{1}{4}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'(\chi_A^2)'\zeta_B^2 v_2^2\\ &+\frac{1}{4}\int \left(f'(Q)+\frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)\Big[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\Big]\zeta_B^2 v_2^2\\ &+\frac{1}{4}\int \partial_x\left(f'(Q)+\frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\chi_A^2)'\zeta_B^2 v_2^2,\end{aligned} \tag{4.129}$$

and associated to $(\partial_x v_1, \partial_x v_2)$ is

$$\begin{aligned}\mathcal{D}\mathcal{R}_v(t) = &-\frac{1}{2}\int (\chi_A^2)'\varphi_B\left[(\partial_x v_1)^2+(\partial_x v_2)^2+3(\partial_x^2 v_2)^2\right] - \frac{3}{2}\int \mathcal{E}_3(1,x)\zeta_B^2(\partial_x v_2)^2\\ &+\frac{1}{2}\int \left[3(\chi_A^2)'(\zeta_B^2)' + 3(\chi_A^2)''\zeta_B^2 + (\chi_A^2)'''\varphi_B + (\chi_A^2)'\varphi_B f'(Q)\right](\partial_x v_2)^2.\end{aligned} \tag{4.130}$$

We have obtained the identity (4.122). To conclude the proof of Lemma 4.23, we must estimate the error terms.

4.4.4 Controlling error terms

We consider the following decomposition for $\mathcal{R}_z(t)$ from (4.128),

$$\mathcal{R}_z(t) = \mathcal{R}_z^1(t) + \mathcal{R}_z^2(t) + \mathcal{R}_z^3(t),$$

where

$$\begin{aligned}\mathcal{R}_z^1(t) = &-\frac{1}{2}\int \frac{\zeta_B''}{\zeta_B}\big(z_1^2+z_2^2\big) + \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B} + \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B} - 3\tilde{R}_1(x)\right]z_2^2,\\ \mathcal{R}_z^2(t) = &-\frac{3}{2}\int \left[P_1'(x)\frac{\zeta_B'}{\zeta_B} + P_1(x)\frac{\zeta_B''}{\zeta_B}\right]z_2^2 + \frac{1}{2}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2} - 3P_1(x)\right)(\partial_x z_2)^2,\\ \mathcal{R}_z^3(t) = &\ \frac{1}{2}\int \left[f'(Q)\frac{\zeta_B''}{\zeta_B} + \left(2\partial_x(f'(Q)) - 2\frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x(f'(Q))\right)\frac{\zeta_B'}{\zeta_B}\right]z_2^2.\end{aligned}$$

For $\mathcal{R}_z^1(t)$, recalling estimate (4.107) and $\tilde{R}_1$ (see (4.119)), and we obtain

$$|\mathcal{R}_z^1(t)| \leqslant B^{-1}\|z_1\|_{L^2}^2 + B^{-1}\|z_2\|_{L^2}^2. \tag{4.131}$$

For $\mathcal{R}_z^2(t)$, we recall the form of P_1 (see (4.119)) and by (4.107), we conclude

$$|\mathcal{R}_z^2(t)| \lesssim B^{-1}\|z_2\|_{L^2}^2 + B^{-1}\|\partial_x z_2\|_{L^2}^2. \tag{4.132}$$

For $\mathcal{R}_z^3(t)$, first we note

$$\left|\frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right| \lesssim B,$$

and by (4.106), we obtain

$$|\mathcal{R}_z^3(t)| \lesssim B^{-1}\|z_2\|_{L^2}^2. \tag{4.133}$$

Collecting (4.131),(4.132) and (4.133), we have

$$|\mathcal{R}_z(t)| \leqslant B^{-1}\Big(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2\Big). \tag{4.134}$$

For $\mathcal{R}_v(t)$, given by (4.129), we consider the following decomposition

$$\begin{aligned}
\mathcal{R}_v^1(t) =& -\frac{1}{2}\int \mathcal{E}_1(1,x)\zeta_B^2\big(v_1^2+v_2^2\big) - \frac{3}{2}\int \Big[\mathcal{E}_2(1,x) + \mathcal{E}_1(P_1(x),x)\Big]\zeta_B^2 v_2^2 \\
&+ \frac{1}{4}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\Big[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\Big]\zeta_B^2 v_2^2 + \frac{1}{4}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'(\chi_A^2)'\zeta_B^2 v_2^2,\\
\mathcal{R}_v^2(t) =& \frac{1}{4}\int \partial_x\left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\chi_A^2)'\zeta_B^2 v_2^2\\
&+ \frac{1}{4}\int \left(f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)\left[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2.
\end{aligned}$$

We note that the terms $\mathcal{E}_1(P_1(x),x)$ and $\mathcal{E}_2(1,x)$ (see (4.109), (4.119) and (4.120)), by (4.106), are bounded and satisfy the following estimates:

$$|\mathcal{E}_2(1,x)| \lesssim (AB^3)^{-1}, \text{ and } |\mathcal{E}_1(P_1(x),x)| \lesssim (AB^3)^{-1},$$

and for $\mathcal{E}_1(1,x)$ in (4.112),

$$|\mathcal{E}_1(1,x)| \lesssim (AB)^{-1}.$$

Then, we have

$$|\mathcal{R}_v^1(t)| \lesssim (AB)^{-1}\left(\|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2\right).$$

For $\mathcal{R}_v^2(t)$, expanding the derivative and using (4.86), we obtain

$$\begin{aligned}
\left|\mathcal{R}_v^2(t)\right| \lesssim& A^{-1}\int\left|\frac{\varphi_B}{\zeta_B^2}\partial_x^2(f'(Q)) + 2\left(1 - \frac{\zeta_B'}{\zeta_B}\frac{\varphi_B}{\zeta_B^2}\right)\partial_x(f'(Q))\right|\zeta_B^2 v_2^2\\
&+ A^{-1}\int\left|f'(Q) + \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right|\zeta_B^2 v_2^2 \lesssim A^{-1}B\|\zeta_B v_2\|_{L^2}^2.
\end{aligned}$$

Then,

$$|\mathcal{R}_v(t)| \lesssim A^{-1}B\left(\|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2\right). \tag{4.135}$$

For $\mathcal{D}\mathcal{R}_v$, given by (4.130), computing directly and using Remark 4.7, we have

$$|\mathcal{D}\mathcal{R}_v(t)| \lesssim BA^{-1}\left(\|\zeta_A^2\partial_x v_1\|_{L^2}^2 + \|\zeta_A^2\partial_x v_2\|_{L^2}^2 + \|\zeta_A^2\partial_x^2 v_2\|_{L^2}^2\right). \tag{4.136}$$

And, by (4.134), (4.135) and (4.136), we obtain

$$\begin{aligned}
&|\mathcal{R}_z(t)| + |\mathcal{R}_v(t)| + |\mathcal{D}\mathcal{R}_v(t)|\\
&\lesssim BA^{-1}\big(\|\zeta_A^2\partial_x v_1\|_{L^2}^2 + \|\zeta_A^2\partial_x v_2\|_{L^2}^2 + \|\zeta_A^2\partial_x^2 v_2\|_{L^2}^2 + \|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2\big)\\
&\quad + B^{-1}\left(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2\right).
\end{aligned}$$

Using Lemma 4.15, we conclude

$$\begin{aligned}
|\mathcal{R}_z(t)| + |\mathcal{R}_v(t)| + |\mathcal{D}\mathcal{R}_v(t)| \lesssim& A^{-1}B^9\big(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\big)\\
&+ B^{-1}\big(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2\big).
\end{aligned} \tag{4.137}$$

This ends the proof of Lemma 4.23. ■

4.4.5 Controlling nonlinear terms

Recall (4.122). We set

$$\mathcal{M}_2 = \int \phi_{A,B}\tilde{G}(x)\partial_x v_2, \qquad \mathcal{M}_1 = \int \phi_{A,B}\tilde{H}(x)\partial_x v_1.$$

These are the two remaining terms in (4.122) to be controlled.

Control of $\mathcal{M}_2$

Recalling that $\tilde{G}(x) = \partial_x G(x)$ and G is given by (4.58), we have

$$\begin{aligned}\mathcal{M}_2 =& -\gamma\int((\chi_A^2)'\zeta_B^2 + \chi_A^2(\zeta_B^2)')\partial_x v_2(1-\gamma\partial_x^2)^{-1}\left[\partial_x^2(f'(Q))\partial_x v_2 + 2\partial_x(f'(Q))\partial_x^2 v_2\right]\\ &-\gamma\int \chi_A^2\varphi_B\partial_x^2 v_2(1-\gamma\partial_x^2)^{-1}\left[\partial_x^2(f'(Q))\partial_x v_2 + 2\partial_x(f'(Q))\partial_x^2 v_2\right]\\ =:& \, M_{21} + M_{22}.\end{aligned}$$

First, we focus on M_{21}. Using Remark 4.10 and Lemma 4.15, we have

$$\begin{aligned}\|((\chi_A^2)'\zeta_B^2 + \chi_A^2(\zeta_B^2)')\partial_x v_2\|_{L^2} &\lesssim A^{-1}\|\zeta_B^2\partial_x v_2\|_{L^2} + B^{-1}\|\chi_A\zeta_B\partial_x v_2\|_{L^2}\\ &\lesssim A^{-1}B^2\|w_2\|_{L^2} + B^{-1}\left[\|\partial_x z_2\|_{L^2} + B^{-1}\|z_2\|_{L^2} + (AB)^{-1/2}\|w_2\|_{L^2}\right]\\ &\lesssim B^{-1}\Big[\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2}\Big] + B^{-5}\|w_2\|_{L^2},\end{aligned}$$

and by (4.88), we conclude

$$|M_{21}| \lesssim B^{-3}\left[\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|w_2\|_{L^2}^2\right]. \tag{4.138}$$

Secondly, for M_{22}. Set $\rho(x) = \operatorname{sech}(x/10)$, making the following separation

$$\begin{aligned}|M_{22}| &\lesssim \gamma\|\chi_A^2\varphi_B\rho(x)\partial_x^2 v_2\|_{L^2}\left\|(\rho(x))^{-1}(1-\gamma\partial_x^2)^{-1}\left[\partial_x^2(f'(Q))\partial_x v_2 + 2\partial_x(f'(Q))\partial_x^2 v_2\right]\right\|_{L^2}\\ &\lesssim \gamma B\|\chi_A\zeta_B\partial_x^2 v_2\|_{L^2}\underbrace{\left\|(\rho(x))^{-1}(1-\gamma\partial_x^2)^{-1}\left[\partial_x^2(f'(Q))\partial_x v_2 + 2\partial_x(f'(Q))\partial_x^2 v_2\right]\right\|_{L^2}}_{M_{23}}.\end{aligned}$$

Using Lemma 4.11 in M_{23}, we obtain

$$\begin{aligned}M_{23} &= \left\|(\rho(x))^{-1}(1-\gamma\partial_x^2)^{-1}\left[\rho(x)\left\{\partial_x^2(f'(Q))(\rho(x))^{-1}\partial_x v_2 + 2\partial_x(f'(Q))(\rho(x))^{-1}\partial_x^2 v_2\right\}\right]\right\|_{L^2}\\ &\leqslant \left\|(1-\gamma\partial_x^2)^{-1}\left[(\partial_x^2(f'(Q))(\rho(x))^{-1}\partial_x v_2 + 2\partial_x(f'(Q))(\rho(x))^{-1}\partial_x^2 v_2)\right]\right\|_{L^2}\\ &\lesssim \|(\partial_x^2(f'(Q))(\rho(x))^{-1}\partial_x v_2\|_{L^2} + \|\partial_x(f'(Q))(\rho(x))^{-1}\partial_x^2 v_2)\|_{L^2}.\end{aligned} \tag{4.139}$$

Since $\partial_x^2(f'(Q))\rho^{-1} \sim \mathrm{e}^{-4|x|/5}$ and making the following decomposition, we have

$$\mathrm{e}^{-8|x|/5}(\partial_x v_2)^2 = \mathrm{e}^{-8|x|/5}\zeta_B^{-2}(\chi_A\zeta_B\partial_x v_2)^2 + \mathrm{e}^{-8|x|/5}\zeta_A^{-2}(1-\chi_A^2)(\zeta_A\partial_x v_2)^2.$$

Since $\mathrm{e}^{-4|x|/5}\zeta_B^{-1} \leqslant 1$ and $\mathrm{e}^{-8A/5}\zeta_A^{-2}(A) \sim \mathrm{e}^{-2A/5}$, using Remark 4.10 and Lemma 4.15, we conclude

$$\begin{aligned}\|\mathrm{e}^{-4|x|/5}\partial_x v_2\|_{L^2} &\lesssim \left[\|\partial_x z_2\|_{L^2} + B^{-1}\|z_2\|_{L^2} + A^{-1}\|\zeta_B v_2\|_{L^2}\right] + \mathrm{e}^{-A/5}\|\zeta_A \partial_x v_2\|_{L^2}\\ &\lesssim \|\partial_x z_2\|_{L^2} + B^{-1}\|z_2\|_{L^2} + A^{-1}\|w_2\|_{L^2}.\end{aligned}$$

And, for the second term in the RHS of (4.139). We note $\partial_x(f'(Q))\rho^{-1} \sim \mathrm{e}^{-4|x|/5}$ and repeating the decomposition, we have

$$\mathrm{e}^{-8|x|/5}(\partial_x^2 v_2)^2 = \mathrm{e}^{-8|x|/5}\zeta_B^{-2}\chi_A^2\zeta_B^2(\partial_x^2 v_2)^2 + \mathrm{e}^{-8|x|/5}\zeta_A^{-2}(1-\chi_A^2)\zeta_A^2(\partial_x^2 v_2)^2.$$

By a similar argument as before, $\mathrm{e}^{-4|x|/5}\zeta_B^{-1} \leqslant 1$ and $\mathrm{e}^{-8A/5}\zeta_A^{-2}(A) \sim \mathrm{e}^{-2A/5}$, applying Remark 4.11 and Lemma 4.15, we have

$$\begin{aligned}\|\mathrm{e}^{-4|x|/5}\partial_x^2 v_2\|_{L^2} &\lesssim \|\chi_A\zeta_B\partial_x^2 v_2\|_{L^2} + \mathrm{e}^{-A/5}\|\zeta_A\partial_x^2 v_2\|_{L^2}\\ &\lesssim \|\partial_x^2 z_2\|_{L^2} + B^{-1}\|\partial_x z_2\|_{L^2} + B^{-2}\|z_2\|_{L^2} + (A^{-1}B^3)^{1/2}\|w_2\|_{L^2}.\end{aligned}$$

Finally, for M_{22} we have

$$|M_{22}| \lesssim B^{-3}\left(\|\partial_x^2 z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + B^{-2}\|z_2\|_{L^2}^2 + A^{-1}B^3\|w_2\|_{L^2}^2\right). \tag{4.140}$$

Collecting (4.138) and (4.140), we conclude

$$\mathcal{M}_2 \lesssim B^{-3}\left[\|\partial_x^2 z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|w_2\|_{L^2}^2\right]. \tag{4.141}$$

Control of $\mathcal{M}_1$.

Recalling that $\tilde{H} = \partial_x H$, H is given by (4.58) and using Lemma 4.10, we obtain

$$\begin{aligned}|\mathcal{M}_1| &\lesssim \|\chi_A\varphi_B\partial_x v_1\|_{L^2}\|\chi_A(1-\partial_x^2)^{-1}\partial_x N^\perp\|_{L^2}\\ &\lesssim \|\chi_A\varphi_B\partial_x v_1\|_{L^2}\|\zeta_A^2(1-\partial_x^2)^{-1}\partial_x N^\perp\|_{L^2}.\end{aligned}$$

Now, by a similar computation that (4.90), we have

$$\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}\partial_x(N^\perp)\|_{L^2} \leqslant \gamma^{-1}(a_1^2 + \|u_1\|_{L^\infty}\|w_1\|_{L^2}). \tag{4.142}$$

Then, by (4.91), Lemma 4.15, (4.51) and the above estimates, we conclude

$$\begin{aligned}|\mathcal{M}_1| &\lesssim \gamma^{-2}B\|w_1\|_{L^2}(a_1^2 + \|u_1\|_{L^\infty}\|w_1\|_{L^2})^2\\ &\lesssim B^{-1}\|w_1\|_{L^2}^2 + \gamma^{-4}B^3(a_1^4 + \|u_1\|_{L^\infty}^4\|w_1\|_{L^2}^2).\end{aligned} \tag{4.143}$$

4.4.6 End of proof Proposition 4.19

Using a similar computation that Lemma 4.9, we are able to estimates $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}$. Set $B = \delta^{-1/19}$, and considering (4.137), (4.143), and (4.141), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leqslant &-\frac{1}{2}\int\left((\partial_x z_1)^2 + \left(V_0(x) - \frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q))\right)(\partial_x z_2)^2 + 2(\partial_x^2 z_2)^2\right)\\ &+\frac{1}{2}\int\left(\frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x^2(f'(Q)) + \frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}\partial_x(f'(Q))\right)z_2^2\\ &+C\max\{B^9A^{-1}, B^{-1}, \delta\}\left(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right)\\ &+CB^{-1}\left[\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|z_1\|_{L^2}^2\right] + C|a_1|^3.\end{aligned}$$

Since

$$\frac{\varphi_B}{\zeta_B^2}\partial_x(f'(Q)) < 0, \text{ and } \qquad \left|\frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B'}{\zeta_B}\partial_x^2(f'(Q)) + \frac{\varphi_B}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}\partial_x(f'(Q))\right| \lesssim 1,$$

we conclude

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} \leqslant & -\frac{1}{2}\int(\partial_x z_1)^2 + \left(V_0(x) - CB^{-1}\right)(\partial_x z_2)^2 + 2(\partial_x^2 z_2)^2 \\ & + C\|z_2\|_{L^2}^2 + CB^{-1}\|z_1\|_{L^2}^2 \\ & + C\max\{B^9A^{-1}, B^{-1}, \delta\}\left(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C|a_1|^3.\end{aligned}$$

Calling $C_3 = C$, and using (4.35), the proof is completed.

4.5 A second transfer estimate

The variation of the virial $\mathcal{M}(t)$ involve the terms $\partial_x z_1$ and $\partial_x^2 z_2$, these terms do not appear in the variation of the virial related to the dual problem. Hence, we need to find a way to transfer information between the terms $\partial_x z_1$ to $\partial_x^2 z_2$. The virial $\mathcal{N}$, defined as

$$\mathcal{N} = \int \rho_{A,B}\tilde{v}_1 v_2 = \int \rho_{A,B}\partial_x v_1 v_2, \tag{4.144}$$

where $\rho_{A,B}$ is a well-chosen localized weight depending on A and B, its variation will give us that relation. A similar quantity was considered in [19]. Note that the virial $\mathcal{N}$ considers the dynamics in (4.57) and (4.98).

4.5.1 A virial identity for $\mathcal{N}(t)$

Lemma 4.24. *Let $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ a solution of (4.57). Consider $\rho_{A,B}$ an even smooth bounded function to be a choose later. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = & \, 2\int \rho_{A,B}''(\partial_x v_2)^2 - \int \rho_{A,B}\left[(\partial_x^2 v_2)^2 + V_0(x)(\partial_x v_2)^2\right] + \int \rho_{A,B}(\partial_x v_1)^2 \\ & + \frac{1}{2}\int \partial_x^2[\rho_{A,B}V_0(x)]v_2^2 - \frac{1}{2}\int \rho_{A,B}^{(4)}v_2^2 + \int \rho_{A,B}v_2\tilde{G}(x) + \int \rho_{A,B}\partial_x v_1 H(x).\end{aligned} \tag{4.145}$$

Proof. Computing the variation of $\mathcal{N}$, using (4.57) and (4.98), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = \int \rho_{A,B}v_2\mathcal{L}\partial_x\tilde{v}_2 + \int \rho_{A,B}\tilde{v}_1^2 + \int \rho_{A,B}v_2(\tilde{G}(x) - \partial_x(f'(Q))v_2) + \int \rho_{A,B}\tilde{v}_1 H(x).$$

Integrating by parts the first integral of the RHS, we have

$$\begin{aligned}\int \rho_{A,B}v_2\mathcal{L}\partial_x\tilde{v}_2 = & \int \rho_{A,B}v_2(-\partial_x^3\tilde{v}_2 + V_0(x)\partial_x\tilde{v}_2) \\ = & -\int \partial_x^2(\rho_{A,B}v_2)\partial_x\tilde{v}_2 - \int\left(\partial_x[\rho_{A,B}V_0(x)]v_2\tilde{v}_2 + \rho_{A,B}V_0(x)\tilde{v}_2^2\right) \\ = & \int \rho_{A,B}'''v_2\tilde{v}_2 + 2\int \rho_{A,B}''\tilde{v}_2^2 - \int \rho_{A,B}\left((\partial_x\tilde{v}_2)^2 + V_0(x)\tilde{v}_2^2\right) - \int \partial_x[\rho_{A,B}V_0(x)]v_2\tilde{v}_2.\end{aligned}$$

Then, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = -\int \rho_{A,B}\left((\partial_x \tilde{v}_2)^2 + V_0(x)\tilde{v}_2^2\right) + \int \rho_{A,B}\tilde{v}_1^2 + 2\int \rho''_{A,B}\tilde{v}_2^2 - \int \partial_x[\rho_{A,B}V_0(x)]v_2\tilde{v}_2 + \int \rho'''_{A,B}v_2\tilde{v}_2 + \int \rho_{A,B}v_2(\tilde{G}(x) - \partial_x(f'(Q))v_2) + \int \rho_{A,B}\tilde{v}_1 H(x).$$

Rewriting the last expression in terms of (v_1, v_2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = -\int \rho_{A,B}\left((\partial_x^2 v_2)^2 + V_0(x)(\partial_x v_2)^2\right) + \int \rho_{A,B}(\partial_x v_1)^2 + 2\int \rho''_{A,B}(\partial_x v_2)^2 - \frac{1}{2}\int \rho^{(4)}_{A,B}v_2^2 + \frac{1}{2}\int \partial_x^2[\rho_{A,B}V_0(x)]v_2^2 + \int \rho_{A,B}v_2(\tilde{G}(x) - \partial_x(f'(Q))v_2) + \int \rho_{A,B}\partial_x v_1 H(x).$$

The proof concludes from the fact that $\rho_{A,B}v_2^2$ is even and $\partial_x(f'(Q))$ is an odd function. ■

Now we choose the weight function $\rho_{A,B}$. As in [19], let

$$\rho_{A,B}(x) = \chi_A^2\zeta_B^2, \tag{4.146}$$

with χ_A and ζ_B introduced in (4.34) and (4.33).

We will make the connection between (4.145) and the variables (z_1, z_2), through the following result. Recall that from (4.93) and Proposition 4.19, $\gamma = B^{-4}$, $B = \delta^{-1/19}$.

Proposition 4.25. *Under (4.146), the following holds. There exist C_4 and $\delta_4 > 0$ such that for $\gamma = B^{-4}$ and for any $0 < \delta \leqslant \delta_4$, the following holds. Fix $B = \delta^{-1/5}$ holds. Assume that for all $t \geqslant 0$, (4.27) holds. Then, for all $t \geqslant 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t) \geqslant \frac{1}{2}\int (\partial_x z_1)^2 - C_4\int \left[(\partial_x^2 z_2)^2 + (\partial_x z_2)^2 + z_2^2 + z_1^2\right] \\ - C_4\delta^{14/19}\left(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + |a_1|^3\right).\end{aligned} \tag{4.147}$$

4.5.2 Start of proof of Proposition 4.25

The proof of this result is based in the following result, which relates Lemma 4.24 and the variables z_i.

Lemma 4.26. *Let $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ a solution of (4.57). Consider $\rho_{A,B} = \chi_A^2\zeta_B^2$, then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = -\int \left[(\partial_x^2 z_2)^2 + (1 - f'(Q))(\partial_x z_2)^2 + \frac{1}{2}\partial_x^2(f'(Q))z_2^2\right] + \int (\partial_x z_1)^2 \\ + \mathcal{RZ}(t) + \mathcal{RV}(t) + \mathcal{RDV}(t) + \int \rho_{A,B}v_2\tilde{G}(x) + \int \rho_{A,B}\partial_x v_1 H(x),\end{aligned} \tag{4.148}$$

where $\mathcal{RZ}(t)$, $\mathcal{RV}(t)$ and $\mathcal{RDV}(t)$ are error term that satisfy the following estimates

$$\begin{aligned} |\mathcal{RZ}(t)| &\lesssim B^{-2}\|z_1\|_{L^2}^2 + B^{-1}\|z_2\|_{L^2}^2 + B^{-2}\|\partial_x z_2\|_{L^2}^2, \\ |\mathcal{RV}(t)| &\lesssim (AB)^{-1}\gamma^{-2}\|w_1\|_{L^2}^2 + A^{-1}\|w_2\|_{L^2}^2, \\ |\mathcal{RDV}(t)| &\lesssim (AB)^{-1}\gamma^{-1}\|w_2\|_{L^2}^2. \end{aligned}$$

Proof. First, we consider the following decomposition from (4.145):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} =& \int \rho_{A,B}(\partial_x v_1)^2 - \int \rho_{A,B}(\partial_x^2 v_2)^2 - \int \rho_{A,B}V_0(x)(\partial_x v_2)^2 + 2\int \rho''_{A,B}(\partial_x v_2)^2 \\ &+ \frac{1}{2}\int \partial_x^2[\rho_{A,B}V_0(x)]v_2^2 - \frac{1}{2}\int \rho_{A,B}^{(4)}v_2^2 + \int \rho_{A,B}\partial_x v_1 H(x) + \int \rho_{A,B}v_2\tilde{G}(x) \\ =&: (N_1+N_2+N_3+N_4) + (N_5+N_6) + (\mathcal{N}_1+\mathcal{N}_2). \end{aligned} \tag{4.149}$$

Secondly, from the definition of $\rho_{A,B}$ (4.146)

$$\begin{aligned} \rho'_{A,B} =& (\chi_A^2)'\zeta_B^2 + \chi_A^2(\zeta_B^2)', \\ \rho''_{A,B} =& (\chi_A^2)''\zeta_B^2 + 2(\chi_A^2)'(\zeta_B^2)' + \chi_A^2(\zeta_B^2)'', \\ \rho'''_{A,B} =& (\chi_A^2)'''\zeta_B^2 + 3(\chi_A^2)''(\zeta_B^2)' + 3(\chi_A^2)'(\zeta_B^2)'' + \chi_A^2(\zeta_B^2)''', \\ \rho^{(4)}_{A,B} =& (\chi_A^2)^{(4)}\zeta_B^2 + 4(\chi_A^2)'''(\zeta_B^2)' + 6(\chi_A^2)''(\zeta_B^2)'' + 4(\chi_A^2)'(\zeta_B^2)''' + \chi_A^2(\zeta_B^2)^{(4)}. \end{aligned} \tag{4.150}$$

For N_3, applying Claim 4.21 with i = 2 and $P(x) = V_0(x)$, we have

$$-N_3 = \int V_0(x)(\partial_x z_2)^2 + \int \left[V_0(x)\frac{\zeta''_B}{\zeta_B} + \partial_x(V_0(x))\frac{\zeta'_B}{\zeta_B}\right] z_2^2 + \int \mathcal{E}_1(V_0(x),x)\zeta_B^2 v_2^2, \tag{4.151}$$

where $\mathcal{E}_1$ is given by (4.109) For N_4, by (4.150), we have

$$\frac{1}{2}N_4 = \int [(\chi_A^2)''\zeta_B^2 + (\chi_A^2)'(\zeta_B^2)'](\partial_x v_2)^2 + \int \frac{(\zeta_B^2)''}{\zeta_B^2}\chi_A^2\zeta_B^2(\partial_x v_2)^2, \tag{4.152}$$

and using Claim 4.21, with i = 2 and $P(x) = (\zeta_B^2)''/\zeta_B^2$, we get

$$\begin{aligned} \frac{1}{2}N_4 =& \int \frac{(\zeta_B^2)''}{\zeta_B^2}(\partial_x z_2)^2 + \int \left[\left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta'_B}{\zeta_B} + \frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta''_B}{\zeta_B}\right] z_2^2 + \int \mathcal{E}_1\left(\frac{(\zeta_B^2)''}{\zeta_B^2},x\right)\zeta_B^2 v_2^2 \\ &+ \int [(\chi_A^2)'' + 2(\chi_A^2)'\frac{\zeta'_B}{\zeta_B}]\zeta_B^2(\partial_x v_2)^2. \end{aligned} \tag{4.153}$$

Now, for N_5, expanding the derivative, replacing (4.150) and using definition of z_2, we have

$$\begin{aligned} N_5 =& -\int \left[-\partial_x^2(V_0(x)) - \frac{(\zeta_B^2)''}{\zeta_B^2}V_0(x) - 2\frac{(\zeta_B^2)'}{\zeta_B^2}\partial_x(V_0(x))\right] z_2^2 \\ &+ \int \left(\left[2(\chi_A^2)'\frac{\zeta'_B}{\zeta_B} + (\chi_A^2)''\right] V_0(x) + 2(\chi_A^2)'\partial_x(V_0(x))\right)\zeta_B^2 v_2^2. \end{aligned} \tag{4.154}$$

Finally, for N_6, reeplacing (4.150), we have

$$N_6 = \int \frac{(\zeta_B^2)^{(4)}}{\zeta_B^2}z_2^2 + \int \left[4(\chi_A^2)'\frac{(\zeta_B^2)'''}{\zeta_B^2} + 6(\chi_A^2)''\frac{(\zeta_B^2)''}{\zeta_B^2} + 8(\chi_A^2)'''\frac{\zeta'_B}{\zeta_B} + (\chi_A^2)^{(4)}\right]\zeta_B^2 v_2^2. \tag{4.155}$$

Therefore, collecting (4.118), (4.111), (4.151), (4.153), (4.154) and (4.155) (and also for N_1 and N_2 we use the relations in Remarks 4.10 and 4.11), we conclude

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N} = & -\int\left[(\partial_x^2 z_2)^2 + V_0(x)(\partial_x z_2)^2 + \frac{1}{2}\partial_x^2(f'(Q))z_2^2\right] + \int(\partial_x z_1)^2 \\
& + \mathcal{RZ}(t) + \mathcal{RV}(t) + \mathcal{RDV}(t) + \int \rho_{A,B} v_2 \tilde{G}(x) + \int \rho_{A,B}\partial_x v_1 H(x),
\end{aligned}
$$

where the error term related to $z = (z_1, z_2)$ is

$$
\begin{aligned}
\mathcal{RZ}(t) = & \int \frac{\zeta_B''}{\zeta_B} z_1^2 - \int\left[V_0(x)\frac{\zeta_B''}{\zeta_B} + \partial_x(V_0(x))\frac{\zeta_B'}{\zeta_B}\right] z_2^2 \\
& - \frac{1}{2}\int\left[-\frac{(\zeta_B^2)''}{\zeta_B^2}V_0(x) - 2\frac{(\zeta_B^2)'}{\zeta_B^2}\partial_x(V_0(x)) + \frac{(\zeta_B^2)^{(4)}}{\zeta_B^2}\right] z_2^2 \\
& - \int\left[\tilde{R}_1(x) + P_1'(x)\frac{\zeta_B'}{\zeta_B} + P_1(x)\frac{\zeta_B''}{\zeta_B}\right] z_2^2 \\
& + 2\int\left[\left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B} + \frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}\right] z_2^2 + \int\left[2\frac{(\zeta_B^2)''}{\zeta_B^2} - P_1(x)\right](\partial_x z_2)^2,
\end{aligned}
$$

the related to (v_1, v_2) is

$$
\begin{aligned}
\mathcal{RV}(t) = & \int \mathcal{E}_1(1,x)\zeta_B^2 v_1^2 \\
& - \int\left[\mathcal{E}_1(V_0(x),x) + \mathcal{E}_2(1,x) + \mathcal{E}_1(P_1(x),x) - 2\mathcal{E}_1\left(\frac{(\zeta_B^2)''}{\zeta_B^2}, x\right)\right]\zeta_B^2 v_2^2 \\
& + \frac{1}{2}\int\left[\left(2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + (\chi_A^2)''\right)V_0(x) + 2(\chi_A^2)'\partial_x(V_0(x))\right]\zeta_B^2 v_2^2 \\
& - \frac{1}{2}\int\left[4(\chi_A^2)'\frac{(\zeta_B^2)'''}{\zeta_B^2} + 6(\chi_A^2)''\frac{(\zeta_B^2)''}{\zeta_B^2} + 8(\chi_A^2)'''\frac{\zeta_B'}{\zeta_B} + (\chi_A^2)^{(4)}\right]\zeta_B^2 v_2^2,
\end{aligned}
$$

and the related to $\partial_x v_2$ is

$$
\mathcal{RDV}(t) = 2\int\left[(\chi_A^2)'' + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B} - \frac{1}{2}\mathcal{E}_3(1,x)\right]\zeta_B^2(\partial_x v_2)^2.
$$

It is clear, from (4.107), that the error terms satisfies the following estimates

$$
|\mathcal{RZ}(t)| \lesssim B^{-1}\left(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2\right), \tag{4.156}
$$

and

$$
\begin{aligned}
|\mathcal{RV}(t)| &\lesssim (AB)^{-1}\|\zeta_B v_1\|_{L^2}^2 + A^{-1}\|\zeta_B v_2\|_{L^2}^2, \\
|\mathcal{RDV}(t)| &\lesssim (AB)^{-1}\|\zeta_B \partial_x v_2\|_{L^2}^2.
\end{aligned}
$$

Recalling that $\gamma = B^{-4}$ and applying Lemma 4.15 , we conclude

$$
\begin{aligned}
|\mathcal{RV}(t)| &\lesssim A^{-1}B^7\|w_1\|_{L^2}^2 + A^{-1}\|w_2\|_{L^2}^2, \\
|\mathcal{RDV}(t)| &\lesssim A^{-1}B^7\|w_2\|_{L^2}^2.
\end{aligned} \tag{4.157}
$$

This concludes the proof of the Lemma 4.26. ■

4.5.3 Control of nonlinear terms

The nonlinear terms in (4.148) are denoted

$$\mathcal{N}_2 = \int \rho_{A,B} v_2 \tilde{G}(x), \qquad \mathcal{N}_1 = \int \rho_{A,B} \partial_x v_1 H(x).$$

Control of $\mathcal{N}_2$. Recalling that $\tilde{G} = \partial_x G$ and G is given by (4.58), using definition of z_2 and (4.146), we have

$$|\mathcal{N}_2| = \left| \int ((\chi_A \zeta_B)' z_2 + \chi_A \zeta_B \partial_x z_2) G(x) \right| \lesssim \gamma (\|z_2\|_{L^2} + \|\partial_x z_2\|_{L^2}) \|G\|_{L^2}.$$

By Cauchy-Schwarz inequality, (4.88) and Lemma 4.15, we conclude

$$\begin{aligned}
|\mathcal{N}_2| &\lesssim \gamma^{1/2} (\|z_2\|_{L^2} + \|\partial_x z_2\|_{L^2}) \Big(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + \mathrm{e}^{-(p-1)A} (\|\zeta_B v_2\|_{L^2} + \|\zeta_B \partial_x v_2\|_{L^2}) \Big) \\
&\lesssim \gamma^{1/2} \Big(\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \mathrm{e}^{-2(p-1)A} (\|\zeta_B v_2\|_{L^2}^2 + \|\zeta_B \partial_x v_2\|_{L^2}^2) \Big) \\
&\lesssim \gamma^{1/2} \Big(\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \mathrm{e}^{-2(p-1)A} \gamma^{-1} \|w_2\|_{L^2}^2 \Big).
\end{aligned} \tag{4.158}$$

Control of $\mathcal{N}_1$. We observe that

$$(\chi_A \zeta_B)^2 \partial_x v_1 = \chi_A \zeta_B \partial_x z_1 - (\chi_A \zeta_B)' z_1,$$

then, we have

$$\mathcal{N}_1 = \int [\chi_A \zeta_B \partial_x z_1 - (\chi_A \zeta_B)' z_1] H(x).$$

Recalling that $H(x)$ is given by (4.58). Moreover, using (4.90), (4.51) and (4.27) we have

$$\begin{aligned}
|\mathcal{N}_1| &\lesssim B^{-1} \left(\|\partial_x z_1\|_{L^2}^2 + \|z_1\|_{L^2}^2 \right) + B^5 \left(a_1^4 + \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2 \right) \\
&\lesssim B^{-1} \left(\|\partial_x z_1\|_{L^2}^2 + \|z_1\|_{L^2}^2 \right) + B^5 \delta \left(|a_1|^3 + \|w_1\|_{L^2}^2 \right).
\end{aligned} \tag{4.159}$$

4.5.4 End of proof of Proposition 4.25

Since $\gamma = B^{-4}$ and $B = \delta^{-1/19}$, collecting (4.156), (4.157), (4.158) and (4.159), we obtain for some $C_4 > 0$ fixed in (4.148)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{N}(t) \geqslant &\int (\partial_x z_1)^2 - \int \left[(\partial_x^2 z_2)^2 + (1 - f'(Q)) (\partial_x z_2)^2 + \frac{1}{2} \partial_x^2 (f'(Q)) z_2^2 \right] \\
&- |\mathcal{RZ}(t)| - |\mathcal{RV}(t)| - |\mathcal{RDV}(t)| - |\mathcal{N}_1| - |\mathcal{N}_2| \\
\geqslant &\frac{1}{2} \int (\partial_x z_1)^2 - C_4 \int \left[(\partial_x^2 z_2)^2 + (\partial_x z_2)^2 + z_2^2 + z_1^2 \right] \\
&- C_4 \max\{A^{-1} B^7, \delta^{14/19}\} \Big(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + |a_1|^3 \Big).
\end{aligned}$$

Using (4.35), $A^{-1} B^7 \ll B^{-3} \ll \delta^{14/19}$. This ends the proof.

4.6 Proof of Theorem 4.1

Before starting the proof of Theorem 4.1, we need a coercivity result to deal with the term

$$\int \operatorname{sech}(x) w_1^2$$

that appears in the virial estimates of $\mathcal{I}(t)$ (see (4.38)). We will decompose this term in terms of the variables (w_1, w_2) and (z_1, z_2). The last ones involve the variables (v_1, v_2); then we should be able to reconstruct the operator $\mathcal{L}$ from our computations.

4.6.1 Coercivity

We shall prove a coercivity result adapted to the orthogonality conditions $\langle u, Q'\rangle = \langle u, \mathcal{L}(\phi_0)\rangle = 0$ in (4.24), where ϕ_0 was introduced in (4.10). The idea is to follow the strategy used in [34] and [10]. Recently, in [18] the operator $\mathcal{L}$ was appeared in a similar setting. It has a unique negative single eigenvalue $\tau_0 = -(p+1)(p+3)/4$, associated to an L^2 eigenfunction denoted Y_0.

Our first result is a coercivity property for $\mathcal{L}$ whenever the first eigenfunction Y_0 is changed by $\mathcal{L}(\phi_0)$.

Lemma 4.27 (Coercivity lemma). *Consider the bilinear form*

$$H(u,v) = \langle \mathcal{L}(u), v\rangle = \int (\partial_x u \partial_x v + uv - f'(Q)uv).$$

Then, there exists $\lambda > 0$ such that

$$H(v,v) \geqslant \lambda \|v\|_{H^1}^2, \tag{4.160}$$

for all $v \in H^1(\mathbb{R})$ satisfying $\langle v, Q'\rangle = \langle v, \mathcal{L}(\phi_0)\rangle = 0$.

Proof. See Appendix 4.C. ■

We will need a weighted version of the previous result. See e.g. Côte-Muñoz-Pilod-Simpson [10] for a very similar proof of this result.

Lemma 4.28 (Coercivity with weight function). *Consider the bilinear form*

$$H_{\phi_\ell}(u,v) = \langle \sqrt{\phi_\ell}\mathcal{L}(u), \sqrt{\phi_\ell} v\rangle = \int \phi_\ell(\partial_x u \partial_x v + uv - f'(Q)uv).$$

for ϕ_ℓ smooth and bounded and such that $0 < \phi_\ell' \leqslant C\ell\phi_\ell$, where C is independent from ℓ. Then, there exists $\lambda > 0$ independent of ℓ small such that

$$H_{\phi_\ell}(v,v) \geqslant \lambda \int \phi_\ell((\partial_x v)^2 + v^2),$$

for all $v \in H^1(\mathbb{R})$ satisfying $\langle v, Q'\rangle = \langle v, \mathcal{L}(\phi_0)\rangle = 0$, and provided ℓ is taken small enough.

The key element of the proof of Theorem 4.1 is the following transfer estimate.

Lemma 4.29. *Let u_1 be even and satisfying* (4.25)*, (w_1, w_2) be as in* (4.37)*, and (z_1, z_2) as in* (4.60)*. Then, for any B large enough, it holds*

$$\int \operatorname{sech}(x) u_1^2 \lesssim B^{-1/2} \left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 \right) + B^{1/2} \|z_1\|_{L^2}^2 + \gamma \|\partial_x z_1\|_{L^2}^2. \tag{4.161}$$

Proof. Set $\frac{2}{B} < \ell < \min\{\frac{1}{2}, \frac{1}{4}\sqrt{\lambda}\} \leqslant \frac{1}{2}$. We note that

$$\int \operatorname{sech}(x) u_1^2 \lesssim \int \operatorname{sech}^2(\ell x)\, u_1^2.$$

Now, we focus on the term on the RHS of the last equation. Applying Lemma 4.28 for $\phi = \operatorname{sech}^2(\ell x)$, since $|\phi'| \leqslant C\ell\phi$. We obtain

$$\begin{aligned} \int \operatorname{sech}^2(\ell x)\, u_1^2 &\leqslant \int \operatorname{sech}^2(\ell x) \left[u_1^2 + (\partial_x u_1)^2 \right] \\ &\leqslant \frac{1}{\lambda} \int \operatorname{sech}^2(\ell x) \left[u_1^2 + (\partial_x u_1)^2 - f'(Q) u_1^2 \right]. \end{aligned}$$

Now, integrating by parts

$$\int \operatorname{sech}(\ell x)(\partial_x u_1)^2 = -\int \operatorname{sech}^2(\ell x) u_1 \partial_x^2 u_1 + \frac{1}{2} \int (\operatorname{sech}^2(\ell x))'' u_1^2,$$

and by

$$|(\operatorname{sech}^2(\ell x))''| \leqslant \ell^2 \operatorname{sech}^2(\ell x).$$

Choosing ℓ small enough ($0 < \ell \leqslant \frac{\sqrt{\lambda}}{4}$), we obtain

$$\int \operatorname{sech}^2(\ell x) u_1^2 \lesssim \int \operatorname{sech}^2(\ell x)\, \mathcal{L}(u_1) u_1.$$

Now, using definition of v_1, we obtain

$$\int \operatorname{sech}^2(\ell x)\, \mathcal{L}(u_1) u_1 \lesssim \int \operatorname{sech}^2(\ell x)\, u_1 v_1 - \gamma \int \operatorname{sech}^2(\ell x)\, u_1 \partial_x^2 v_1. \tag{4.162}$$

For the first integral in RHS of (4.162), using definition of z_1 and w_1, one can see that

$$\begin{aligned} \int \operatorname{sech}^2(\ell x)\, u_1 v_1 &= \int \chi_A^3 \operatorname{sech}^2(\ell x)\, u_1 v_1 + \int (1 - \chi_A^3) \operatorname{sech}^2(\ell x)\, u_1 v_1 \\ &= \int \chi_A^2 \operatorname{sech}^2(\ell x)\, (\zeta_A \zeta_B)^{-1} w_1 z_1 + \int (1 - \chi_A^3) \operatorname{sech}^2(\ell x)\, \zeta_A^{-2} w_1 (\zeta_A v_1) \\ &\lesssim \max_{|x|<2A} \{\operatorname{sech}^2(\ell x)\, (\zeta_A \zeta_B)^{-1}\} \|w_1\|_{L^2} \|z_1\|_{L^2} + \max_{|x|>A} \{\operatorname{sech}^2(\ell x)\, \zeta_A^{-2}\} \|w_1\|_{L^2} \|\zeta_A v_1\|_{L^2} \\ &\lesssim \max_{|x|<2A} \{\operatorname{sech}^2(\ell x)\, (\zeta_A \zeta_B)^{-1}\} \|w_1\|_{L^2} \|z_1\|_{L^2} + \gamma^{-1} \max_{|x|>A} \{\operatorname{sech}^2(\ell x)\, \zeta_A^{-2}\} \|w_1\|_{L^2}^2 \\ &\lesssim \epsilon \|w_1\|_{L^2}^2 + \epsilon^{-1} \|z_1\|_{L^2}^2 + \gamma^{-1} \mathrm{e}^{-\frac{A}{4B}} \|w_1\|_{L^2}^2. \end{aligned} \tag{4.163}$$

Note that the last inequality holds if $2B^{-1} < \ell$.
Now, for the second integral on the RHS of (4.162), integrating by parts we obtain the following expression

$$\begin{aligned}
\int \partial_x\big[\operatorname{sech}^2(\ell x)\, u_1\big]\partial_x v_1 \\
&= \int \big[(\operatorname{sech}^2(\ell x))' u_1 + \operatorname{sech}^2(\ell x)\,\partial_x u_1\big]\partial_x v_1 \\
&= \int (\operatorname{sech}^2(\ell x))'\chi_A^2 u_1 \partial_x v_1 + \int (1-\chi_A^2)(\operatorname{sech}^2(\ell x))' u_1\partial_x v_1 \\
&\quad + \int \operatorname{sech}^2(\ell x)\,\chi_A^2 \partial_x u_1 \partial_x v_1 + \int (1-\chi_A^2)\operatorname{sech}^2(\ell x)\,\partial_x u_1\partial_x v_1.
\end{aligned} \tag{4.164}$$

Using the following decomposition and by Hölder inequality, we get

$$\begin{aligned}
\left|\int (\operatorname{sech}^2(\ell x))' u_1\partial_x v_1\right| &\lesssim \left|\int (\operatorname{sech}^2(\ell x))'\chi_A^3 u_1\partial_x v_1\right| + \left|\int (\operatorname{sech}^2(\ell x))'(1-\chi_A^3)u_1\partial_x v_1\right| \\
&\lesssim \left|\int (\operatorname{sech}^2(\ell x))'\chi_A^3 u_1\partial_x v_1\right| + \left|\int (\operatorname{sech}^2(\ell x))'\zeta_A^{-2}(1-\chi_A^3)w_1(\zeta_A\partial_x v_1)\right| \\
&\lesssim \ell\|\chi_A u_1\|_{L^2}\|\zeta_B\chi_A^2\partial_x v_1\|_{L^2} + \ell \max_{|x|>A}\{\operatorname{sech}^2(\ell x)\zeta_A^{-2}\}\|w_1\|_{L^2}\|\zeta_A\partial_x v_1\|_{L^2}.
\end{aligned}$$

Furthermore, by the definition of z_1, we can check

$$\chi_A^2\zeta_B\partial_x v_1 = \chi_A\partial_x z_1 - \chi_A\frac{\zeta_B'}{\zeta_B}z_1 - \chi_A' z_1; \tag{4.165}$$

and by Lemma 4.15 and Remark 4.7, we obtain

$$\begin{aligned}
\left|\int (\operatorname{sech}^2(\ell x))' u_1\partial_x v_1\right| &\lesssim \ell\|w_1\|_{L^2}(\|\partial_x z_1\|_{L^2} + B^{-1}\|z_1\|_{L^2}) \\
&\quad + \ell\gamma^{-1}\max_{|x|>A}\{\operatorname{sech}^2(\ell x)\zeta_A^{-2}\}(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2).
\end{aligned} \tag{4.166}$$

In similar way, we obtain

$$\begin{aligned}
\left|\int \operatorname{sech}^2(\ell x)\,\partial_x u_1\partial_x v_1\right| &\lesssim \left|\int \operatorname{sech}^2(\ell x)\,\chi_A^3\partial_x u_1\partial_x v_1\right| + \left|\int \operatorname{sech}^2(\ell x)\,(1-\chi_A^3)\partial_x u_1\partial_x v_1\right| \\
&\lesssim \|\chi_A\partial_x u_1\|_{L^2}\|\zeta_B\chi_A^2\partial_x v_1\|_{L^2} + \max_{|x|>A}\{\operatorname{sech}^2(\ell x)\zeta_A^{-2}\}\|\zeta_A\partial_x u_1\|_{L^2}\|\zeta_A\partial_x v_1\|_{L^2}.
\end{aligned}$$

By (4.165), Lemma 4.15 and Remark 4.7, we get

$$\left|\int \operatorname{sech}^2(\ell x)\,\partial_x u_1\partial_x v_1\right| \lesssim \|\zeta_A\partial_x u_1\|_{L^2}(\|\partial_x z_1\|_{L^2} + \|z_1\|_{L^2}) + \gamma^{-1}\max_{|x|>A}\{\operatorname{sech}^2(\ell x)\zeta_A^{-2}\}\|\zeta_A\partial_x u_1\|_{L^2}^2.$$

We conclude using (4.81) with $K = A$, we have

$$\begin{aligned}
\left|\int \operatorname{sech}^2(\ell x)\,\partial_x u_1\partial_x v_1\right| &\lesssim (\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2})(\|\partial_x z_1\|_{L^2} + \|z_1\|_{L^2}) \\
&\quad + \gamma^{-1}\max_{|x|>A}\{\operatorname{sech}^2(\ell x)\zeta_A^{-2}\}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2).
\end{aligned} \tag{4.167}$$

Collecting (4.163), (4.166), (4.167) and by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int \operatorname{sech}(x)u_1^2 \lesssim{}& \epsilon\|w_1\|_{L^2} + \frac{1}{\epsilon}\|z_1\|_{L^2}\\ &+ \gamma\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2\right) + \gamma\left(\|\partial_x z_1\|_{L^2}^2 + B^{-2}\|z_1\|_{L^2}^2\right)\\ \lesssim{}& \max\{\epsilon, A^{-1}, \gamma\}\|w_1\|_{L^2} + \gamma\|\partial_x w_1\|_{L^2}^2\\ &+ \max\{\epsilon^{-1}, B^{-2}\}\|z_1\|_{L^2}^2 + \gamma\|\partial_x z_1\|_{L^2}^2.\end{aligned}$$

Finally, choosing $\epsilon = B^{-1/2}$, we conclude

$$\int \operatorname{sech}(x)u_1^2 \lesssim B^{-1/2}\left(\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}^2\right) + B^{1/2}\|z_1\|_{L^2}^2 + \gamma\|\partial_x z_1\|_{L^2}^2.$$

This ends the proof of Lemma 4.29. ■

We will need a third coercivity estimate, related to the function z_2 in (4.60).

Lemma 4.30. *Recall $\mathcal{L} = -\partial_x^2 + V_0(x)$, with V_0 defined in* (4.8). *Assume that $\int Q\phi_0 \neq 0$. Then there exists $m_0 > 0$ fixed such that*

$$\langle \mathcal{L}(u), u\rangle \geqslant m_0\|u\|_{H^1}^2 - \frac{1}{m_0}\left|\langle u, \partial_x^{-1}\phi_0\rangle\right|^2,$$

for any $u \in H^1(\mathbb{R})$ odd.

Proof. Since u is odd, one clearly has $\langle \mathcal{L}(u), u\rangle \geqslant 0$. Since $\ker \mathcal{L} = \operatorname{span}\{Q'\}$, we only need to check that

$$\langle \mathcal{L}(u), u\rangle \geqslant m_0\|u\|_{L^2}^2, \tag{4.168}$$

for any $u \in H^1(\mathbb{R})$ odd, and provided $\langle u, \partial_x^{-1}\phi_0\rangle = 0$. First of all, it is not difficult to check that for some $m_0 > 0$,

$$\langle \mathcal{L}(u), u\rangle \geqslant m_0\left(\|u\|_{L^2}^2 - \|Q'\|_{L^2}^{-2}\langle u, Q'\rangle^2\right).$$

Assume that $\|u\|_{L^2} = 1$. The term on the right hand side is zero only if u is parallel to Q', which is not possible since $\int Q\phi_0 \neq 0$. Therefore, after rescaling, (4.168) is proved. ■

Remark 4.12. Lemma 4.30 will be used in the following way: from (4.24) we have $\langle u_2, \partial_x^{-1}\phi_0\rangle = 0$, and from (4.56), we have

$$\langle v_2, (1-\gamma\partial_x^2)\partial_x^{-1}\phi_0\rangle = 0.$$

Using (4.60) and (4.37), and the exponential decay of $\partial_x^{-1}\phi_0$ we obtain

$$\begin{aligned}|\langle z_2, \partial_x^{-1}\phi_0\rangle| &\leqslant |\langle z_2, (1-\gamma\partial_x^2)\partial_x^{-1}\phi_0\rangle| + \gamma|\langle z_2, \partial_x\phi_0\rangle|\\ &\lesssim |\langle v_2\chi_A\zeta_B, (1-\gamma\partial_x^2)\partial_x^{-1}\phi_0\rangle| + \gamma\|z_2\|_{L^2}\\ &\lesssim |\langle v_2, (1-\chi_A\zeta_B)(1-\gamma\partial_x^2)\partial_x^{-1}\phi_0\rangle| + \gamma\|z_2\|_{L^2}\\ &\lesssim |\langle u_2\zeta_A, \zeta_A^{-1}(1-\gamma\partial_x^2)^{-1}(1-\chi_A\zeta_B)(1-\gamma\partial_x^2)\partial_x^{-1}\phi_0\rangle| + \gamma\|z_2\|_{L^2}\\ &\lesssim e^{-\frac{1}{2}B}\|w_2\|_{L^2} + \gamma\|z_2\|_{L^2}.\end{aligned}$$

Finally, we prove that

Lemma 4.31. $\int Q\phi_0 \neq 0$.

Proof. If $\int Q\phi_0 = 0$, from (4.10) one has $\langle \partial_x^2 \mathcal{L}Q, Q\rangle \leqslant 0$. However

$$0 \geqslant \langle \partial_x^2 \mathcal{L}Q, Q\rangle = -(p-1)\langle Q^p, Q''\rangle = -(p-1)\langle Q^p, Q - Q^p\rangle = -(p-1)\int_{\mathbb{R}} (Q^{p+1} - Q^{2p}).$$

Finally, from the equation $Q'' = Q - Q^p$ and multiplying by Q^p and integrating by parts, we get

$$-p\int Q^{p-1}Q'^2 = \int Q^{p+1} - \int Q^{2p}.$$

Finally, using that $Q'^2 = Q^2 - \frac{2}{p+1}Q^{p+1}$, we get $\int Q^{p+1} = \frac{3p+1}{(p+1)^2}\int Q^{2p}$, and replacing,

$$0 \geqslant \langle \partial_x^2 \mathcal{L}Q, Q\rangle = -(p-1)\int_{\mathbb{R}} (Q^{p+1} - Q^{2p}) = \frac{p(p-1)^2}{(p+1)^2}\int_{\mathbb{R}} Q^{2p} > 0,$$

a contradiction. ■

Now we are ready to conclude the proof of Theorem 4.1.

4.6.2 Proof of Theorem 4.1

Recalling that the constants γ, C_{i} and $\delta_{\text{i}} > 0$ for i $= 1, \dots, 4$ were defined in Propositions 4.3 4.8, 4.19, 4.25.

Proposition 4.32. *There exist C_5 and $0 < \delta_5 \leqslant \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ such that for any $0 < \delta \leqslant \delta_5$, the following holds. Fix $A = \delta^{-1}$, $B = \delta^{-1/19}$ and $\gamma = B^{-4}$. Assume that for all $t \geqslant 0$, (4.27) holds. Let*

$$\mathcal{H} = \mathcal{J} + 16C_2B^{-1}\mathcal{I} + B^{-1}\mathcal{M} - 16B^{-5}C_1C_2\mathcal{N}.$$

Then, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \leqslant -C_2B^{-1}\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_5|a_1|^3. \tag{4.169}$$

Proof. First, from (4.38) and (4.161) we obtain for some $C_1 > 0$ fixed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) \leqslant &-\frac{1}{2}\left[\|w_2\|_{L^2}^2 + 2\|\partial_x w_1\|_{L^2}^2 + \frac{1}{2}\|w_1^2\|_{L^2}\right] \\ &+ C_1a_1^4 + C_1B^{-1/2}\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2\right) + C_1B^{1/2}\|z_1\|_{L^2}^2 + C_1\gamma\|\partial_x z_1\|_{L^2}^2.\end{aligned}$$

Using (4.147) and $\gamma = B^{-4}$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) \leqslant &-\frac{1}{4}\left(\|w_2\|_{L^2}^2 + 2\|\partial_x w_1\|_{L^2}^2 + \frac{1}{2}\|w_1\|_{L^2}^2\right) + C_1|a_1|^3 + B^{-4}C_1\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t) \\ &+ B^{1/2}C_1\|z_1\|_{L^2}^2 + B^{-4}C_1\left[\|\partial_x^2 z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|z_1\|_{L^2}^2\right].\end{aligned}$$

Secondly, for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}$, using (4.61), Lemma 4.30, and Remark 4.12,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) \leqslant -\frac{1}{4}m_0\bigg(\|z_1\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2\bigg) + C_2B^{-1}\bigg(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\bigg) + C_2|a_1|^3.$$

We conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(t) + 16C_2B^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) \leqslant\ & -\frac{1}{8}m_0\bigg(\|z_1\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2\bigg)\\ & - 4C_2B^{-1}\left(\|w_2\|_{L^2}^2 + 2\|\partial_x w_1\|_{L^2}^2 + \frac{1}{2}\|w_1\|_{L^2}^2\right)\\ & + 16B^{-5}C_1C_2\|\partial_x^2 z_2\|_{L^2}^2 + 2C_2|a_1|^3 + 16B^{-5}C_1C_2\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}(t).\end{aligned}$$

Thirdly, using (4.103) for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) \lesssim\ & -\frac{1}{2}\left(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2\right) + C_3\|\partial_x z_2\|_{L^2}^2 + \frac{1}{2}C_3\|z_2\|_{L^2}^2 + C_3B^{-1}\|z_1\|_{L^2}^2\\ & + C_3B^{-1}\left(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_3|a_1|^3.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{J}(t) + 16C_2B^{-1}\mathcal{I}(t) + B^{-1}\mathcal{M}(t) - 16B^{-5}C_1C_2\mathcal{N}(t)\right)\\ &\quad\leqslant -\frac{1}{16}m_0\bigg(\|z_1\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2\bigg) - \frac{1}{4}B^{-1}\left(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2\right)\\ &\quad\quad - C_2B^{-1}\left(\|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right) + 3C_2|a_1|^3.\end{aligned}$$

Setting $\mathcal{H} = \mathcal{J} + 16C_2B^{-1}\mathcal{I} + B^{-1}\mathcal{M} - 16B^{-5}C_1C_2\mathcal{N}$, and $C_5 = 3C_2$, we obtain the desired property. ■

We define now

$$\mathcal{B} = b_+^2 - b_-^2,$$

where b_+, b_- are given in (4.26).

Lemma 4.33. *There exist C_6 and $\delta_6 > 0$, such that for any $0 < \delta \leqslant \delta_6$, the following holds. Assume that for all $t \geqslant 0$ (4.27) holds. Then, for all $t \geqslant 0$,*

$$|\dot{b}_+ - \nu_0 b_+| + |\dot{b}_- + \nu_0 b_-| \leqslant C_6\left(b_+^2 + b_-^2 + \left\|\operatorname{sech}^{1/2}(x/2)w_1\right\|_{L^2}^2\right), \tag{4.170}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}b_+^2 - 2\nu_0 b_+^2\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}b_-^2 + 2\nu_0 b_-^2\right| \leqslant C_6\left(b_+^2 + b_-^2 + \left\|\operatorname{sech}^{1/2}(x/2)w_1\right\|_{L^2}^2\right)^{3/2}. \tag{4.171}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} \geqslant \frac{\nu_0}{2}(a_1^2 + a_2^2) - C_6\left\|\operatorname{sech}^{1/2}(x/2)w_1\right\|_{L^2}^2. \tag{4.172}$$

Proof. From (4.51) and (4.26), it holds

$$|N_0| \lesssim b_+^2 + b_-^2 + \int \operatorname{sech}\left(\frac{x}{2}\right) w_1^2.$$

From (4.28) we conclude the estimates (4.170) and (4.171). Finally, (4.172) follows directly from (4.171) and taking $\delta_6 > 0$ small enough. ■

Combining (4.169) and (4.172), it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{B} - 2B\frac{C_6}{C_2}\mathcal{H}\right) \geqslant \frac{\nu_0}{4}(a_1^2 + a_2^2) + C_6\left(\|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right). \tag{4.173}$$

By the choice of $A = \delta^{-1}$, the bound $|\varphi_A| \lesssim A$, (4.36) and (4.27), we have for all $t \geqslant 0$

$$|\mathcal{I}(t)| \lesssim A\|u_1\|_{L^2}\|u_2\|_{L^2} \lesssim \delta.$$

Analogously, using Lemma 4.14, we have

$$\begin{aligned}|\mathcal{J}(t)| &\lesssim B\|v_1\|_{L^2}\|v_2\|_{L^2} \lesssim B\gamma^{-1}\|u_1\|_{L^2}\|u_2\|_{L^2}\\ &\lesssim B^5\|u_1\|_{L^2}\|u_2\|_{L^2} \lesssim B^5\delta^2 \lesssim \delta,\end{aligned}$$

$$\begin{aligned}|\mathcal{M}(t)| &\lesssim B\|\partial_x v_1\|_{L^2}\|\partial_x v_2\|_{L^2} \lesssim B\gamma^{-3/2}\|u_1\|_{L^2}\|u_2\|_{L^2}\\ &\lesssim B^7\|u_1\|_{L^2}\|u_2\|_{L^2} \lesssim B^7\delta^2 \lesssim \delta,\end{aligned}$$

and

$$|\mathcal{N}(t)| \lesssim \|\partial_x v_1\|_{L^2}\|v_2\|_{L^2} \lesssim \gamma^{-1}\|u_1\|_{H^1}\|u_2\|_{L^2} \lesssim B^4\|u_1\|_{H^1}\|u_2\|_{L^2} \lesssim B^4\delta^2 \lesssim \delta.$$

Then, we have

$$|\mathcal{H}| \leqslant \delta.$$

Estimate $|\mathcal{B}| \leqslant \delta^2$ is also clear from (4.27). Therefore, integrating estimates (4.173) on $[0, t]$ and passing the limit as $t \to \infty$, we have

$$\int_0^\infty \left[a_1^2 + a_2^2 + \|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right] \mathrm{d}t \lesssim \delta.$$

By Lemma 4.17 one can see

$$\int_0^\infty \left(a_1^2 + a_2^2 + \int (u_1^2 + (\partial_x u_1)^2 + u_2^2)\operatorname{sech}(x)\right) \mathrm{d}t \leqslant \delta. \tag{4.174}$$

Using the above equation, we will conclude the proof of Theorem 4.1.

Let

$$\mathcal{K}(t) = \int \operatorname{sech}(x)u_1^2 + \int \operatorname{sech}(x)((1-\gamma\partial_x^2)^{-1}\partial_x u_2)^2 =: \mathcal{K}_1(t) + \mathcal{K}_2(t).$$

For $\mathcal{K}_1$, using (4.30) and integrating by parts, we have

$$\frac{\mathrm{d}\mathcal{K}_1}{\mathrm{d}t} = 2\int \operatorname{sech}(x)(u_1\dot{u}_1) = 2\int \operatorname{sech}(x)(u_1\partial_x u_2) = -2\int (\operatorname{sech}'(x)u_1 + \operatorname{sech}(x)\partial_x u_1)u_2.$$

Then,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_1(t)\right| \leqslant \int \operatorname{sech}(x)(u_1^2 + (\partial_x u_1)^2 + u_2^2).$$

For $\mathcal{K}_2$, passing to the variables (v_1, v_2) (see (4.56))

$$\mathcal{K}_2 = \int \operatorname{sech}(x)(\partial_x v_2)^2,$$

and using (4.57), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_2 = 2\int \operatorname{sech}(x)\partial_x v_2 \partial_x^2 v_1 + 2\int \operatorname{sech}(x)\partial_x v_2 \partial_x H(x) =: K_{21} + K_{22}.$$

Integrating by parts in K_{21}, we have

$$K_{21} = -2\int (\operatorname{sech}'(x)\partial_x v_2 + \operatorname{sech}(x)\partial_x^2 v_2)\partial_x v_1,$$

besides using Cauchy-Schwarz inequality and Lemma 4.14, we obtain

$$\begin{aligned} |K_{21}| &\lesssim \int \operatorname{sech}(x)((\partial_x v_2)^2 + (\partial_x^2 v_2)^2 + (\partial_x v_1)^2) \\ &\lesssim \gamma^{-2}\int \operatorname{sech}(x)(u_2^2 + (\partial_x u_1)^2). \end{aligned}$$

For K_{22}, we use Cauchy-Schwartz inequality, Corollary 4.12 and a similar computation of (4.142), then

$$\begin{aligned} |K_{22}| &\lesssim \int \operatorname{sech}(x)[((1-\gamma\partial_x^2)^{-1}\partial_x u_2)^2 + ((1-\gamma\partial_x^2)^{-1}\partial_x N^{\perp})^2] \\ &\lesssim \int \operatorname{sech}(x)[\gamma^{-1}u_2^2 + ((1-\gamma\partial_x^2)^{-1}\partial_x N^{\perp})^2] \lesssim_\gamma a_1^2 + \int \operatorname{sech}(x)[u_2^2 + u_1^2]. \end{aligned}$$

Then, we conclude

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_2(t)\right| \lesssim_\gamma a_1^2 + \int \operatorname{sech}(x)(u_1^2 + (\partial_x u_1)^2 + u_2^2).$$

By (4.174), there exists and increasing sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty}\left[a_1^2(t_n) + a_2^2(t_n) + \mathcal{K}_1(t_n) + \mathcal{K}_2(t_n)\right] = 0.$$

For $t \geqslant 0$, integrating on $[t, t_n]$, and passing to the limit as $n \to \infty$, we obtain

$$\mathcal{K}(t) \lesssim \int_t^\infty \left[a_1^2 + \int \operatorname{sech}(x)(u_1^2 + (\partial_x u_1)^2 + u_2^2)\right]\mathrm{d}t.$$

By (4.174), we deduce

$$\lim_{t\to\infty}\mathcal{K}(t) = 0.$$

Finally, by (4.28) and (4.51), we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(a_1^2)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}(a_2^2)\right| \lesssim a_1^2 + a_2^2 + \int \operatorname{sech}(x) u_1^2.$$

In a similar way as above, integrating on $[t, t_n]$ and taking $n \to \infty$, we conclude

$$a_1^2(t) + a_2^2(t) \lesssim \int_t^\infty \left[a_1^2 + a_2^2 + \int u_1^2 \operatorname{sech}(x) \right] \mathrm{d}t,$$

which proves $\lim_{t\to\infty} |a_1(t)| + |a_2(t)| = 0$. By the decomposition of solution (4.23) this implies (4.13). This ends the proof of Theorem 4.1.

Remark 4.13. We have not being able to describe the asymptotic behavior of $(\partial_x u_1)^2$ and u_2^2, due to the fact that we are working in the energy space, and any variation of the virial that involves these terms is not well-defined. In fact, the regularity considered for the variation of $\mathcal{K}_1$ and $\mathcal{K}_2$ is sharp, in the sense that we do not have a gap where to include terms with higher-order derivatives. For example, for

$$\mathcal{K}_3 = \int \operatorname{sech}(x) u_2^2,$$

its variation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_3 = 2 \int \operatorname{sech}(x) u_2 (\partial_x \mathcal{L} u_1 + N^{\perp}).$$

One can see that $\mathcal{L}u_1 \in H^{-1}$ and $u_2 \in L^2$. Then, the last estimate may not be well-defined.

4.7 Proof of Theorem 4.2

Now we construct initial data for which Theorem 4.1 remains valid. We follow the ideas in [18], with some particular differences in some estimates.

4.7.1 Conservation of Energy

Using (4.4), (4.23), (4.8), and by the orthogonality condition (4.24), we have

$$\begin{aligned}
2\big[E(u,v) - E(Q,0)\big] &= \int \big[v^2 + u^2 + (\partial_x u)^2 - 2F(u)\big] - 2E(Q,0) \\
&= a_2^2 \int \nu_0^2 (\partial_x^{-1}\phi_0)^2 + a_1^2 \int ((\partial_x \phi_0)^2 + V_0(x)\phi_0^2) + \int f'(Q)(a_1\phi_0 + u_1)^2 \\
&\quad + \int ((\partial_x u_1)^2 + V_0(x) u_1^2 + u_2^2) - 2 \int \big(F(u) - F(Q) - f(Q)(a_1\phi_0 + u_1)\big) \\
&= a_2^2 \nu_0^2 \|\partial_x^{-1}\phi_0\|_{L^2}^2 + a_1^2 \langle \mathcal{L}(\phi_0), \phi_0\rangle + \langle \mathcal{L}(u_1), u_1\rangle + \|u_2\|_{L^2}^2 \\
&\quad - 2 \int \left(F(u) - F(Q) - f(Q)(a_1\phi_0 + u_1) - f'(Q)\frac{(a_1\phi_0 + u_1)^2}{2} \right).
\end{aligned}$$

Using (4.10), we get

$$\langle \mathcal{L}(\phi_0), \phi_0\rangle = \langle \nu_0^2 \partial_x^{-2}\phi_0, \phi_0\rangle = -\nu_0^2 \langle \partial_x^{-1}\phi_0, \partial_x^{-1}\phi_0\rangle = -\nu_0^2,$$

and, by (4.26), we obtain the identity

$$2\big[E(u,v)-E(Q,0)\big] = -4\nu_0^2 b_+ b_- + \langle \mathcal{L}(u_1),u_1\rangle + \|u_2\|_{L^2}^2 - 2\int \left(F(u)-F(Q)-f(Q)(a_1\phi_0+u_1)-f'(Q)\frac{(a_1\phi_0+u_1)^2}{2}\right). \tag{4.175}$$

Let δ_0 be defined by

$$\delta_0^2 = b_+^2(0)+b_-^2(0)+\|u_1(0)\|_{H^1}^2+\|u_2(0)\|_{L^2}^2.$$

Considering (4.175) at $t=0$ follows $|2\big[E(u,v)-E(Q,0)\big]| \lesssim \delta_0^2$. Besides, by the conservation of energy, estimate (4.175) at some $t>0$ gives

$$|-4\nu_0^2 b_+ b_- + \langle \mathcal{L}(u_1),u_1\rangle + \|u_2\|_{L^2}^2 - O(|b_+|^3+|b_-|^3+\|u_1\|_{H^1}^3)| \lesssim \delta_0^2.$$

Considering the orthogonality condition $\langle u_1, Q'\rangle = \langle u_1, \mathcal{L}(\phi_0)\rangle = 0$, the parity of u_1, and using the Lemma 4.27, it follows that for some $\lambda \in (0,1)$,

$$\langle \mathcal{L}(u_1),u_1\rangle \geqslant \lambda \|u_1\|_{H^1}^2.$$

Due to $\|u_1\|_{H^1}+\|u_2\|_{L^2}+|b_+|+|b_-| \lesssim \delta_0$, the following estimate holds

$$\|u_1\|_{H^1}^2+\|u_2\|_{L^2}^2 \lesssim |b_+|^2+|b_-|^2+\delta_0^2. \tag{4.176}$$

4.7.2 Construction of the graph

We will construct initial data that directs to global solutions close to the ground state Q. To accomplish this objective, we use the energy estimate (4.176), Lemma 4.33 and a standard contradiction argument.

Let $\boldsymbol{\epsilon} = (\epsilon_1,\epsilon_2) \in \mathcal{A}_0$. Let $\boldsymbol{Z}_+$ be as in (4.11). Then, the condition $\langle \boldsymbol{\epsilon}, \boldsymbol{Z}_+\rangle = 0$ rewrites

$$\langle \epsilon_1, \partial_x^{-2}\phi_0\rangle + \langle \epsilon_2, \nu_0^{-1}\partial_x^{-1}\phi_0\rangle = 0.$$

Define $b_-(0)$ and $(u_1(0),u_2(0))$ such that

$$b_-(0) = -\langle \epsilon_1, \partial_x^{-2}\phi_0\rangle = \langle \epsilon_2, \nu_0^{-1}\partial_x^{-1}\phi_0\rangle,$$

and

$$\epsilon_1 = b_-(0)\phi_0 + u_1(0), \quad \epsilon_2 = -b_-(0)\nu_0\partial_x^{-1}\phi_0 + u_2(0).$$

Then, it holds

$$\langle u_1(0), \partial_x^{-2}\phi_0\rangle = \langle u_2(0), \partial_x^{-1}\phi_0\rangle = 0.$$

From (4.15) and (4.14), we observe that the initial condition in Theorem 4.2 holds the following decomposition:

$$\boldsymbol{\phi}_0 = \boldsymbol{\phi}(0) = (Q,0) + (u_1,u_2)(0) + b_-(0)\boldsymbol{Y}_- + h(\boldsymbol{\epsilon})\boldsymbol{Y}_+.$$

We will prove that there is a function $h(\boldsymbol{\epsilon})$ such that the corresponding solution $\boldsymbol{\phi}$ is global and satisfies (4.16). We show that at least $h(\boldsymbol{\epsilon}) = b_+(0)$ satisfies this statement.

Let $\delta_0 > 0$ small enough and $K > 1$ large enough to be chosen. Following the scheme of [18], we introduce the following bootstrap estimates

$$\|u_1\|_{H^1} \leqslant K^2\delta_0 \quad \text{and} \quad \|u_2\|_{L^2} \leqslant K^2\delta_0, \tag{4.177}$$

$$|b_-| \leqslant K\delta_0, \tag{4.178}$$

$$|b_+| \leqslant K^5\delta_0^2. \tag{4.179}$$

Given any $(u_1(0), u_2(0))$ and $b_-(0)$ such that

$$\|u_1(0)\|_{H^1} \leqslant \delta_0, \quad \|u_2(0)\|_{L^2} \leqslant \delta_0, \quad |b_-(0)| \leqslant \delta_0 \tag{4.180}$$

and $b_+(0)$ satisfying

$$|b_+(0)| \leqslant K^5\delta_0.$$

Let

$$T = \sup\{t \geqslant 0 \quad \text{such that} \quad (4.177), (4.178), (4.179) \quad \text{hold on } [0,t]\}.$$

Since $K > 1$ follow that T is well-defined in $[0, +\infty]$. Our aim is to prove that there exists at least a value of $b_+(0) \in [-K^5\delta^2, K^5\delta_0^2]$ such that $T = \infty$. To prove this we argue by contradiction: we assume that for all values of $b_+(0) \in [-K^5\delta^2, K^5\delta_0^2]$, one has $T < \infty$.

The first step is improve the estimates (4.177). By (4.177), we have

$$\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \leqslant 2K^4\delta_0^2 \tag{4.181}$$

Otherwise, using the energy estimates (4.176) it holds

$$\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \leqslant C_8(K^2\delta_0^2 + K^{10}\delta^4 + \delta_0^2),$$

for some constant $C_8 > 0$. Thus, using the smallness of δ_0 and largeness of K, it holds

$$C_8 \leqslant \frac{1}{4}K^4, \quad C_8K^{10}\delta_0^2 \leqslant \frac{1}{4}K^4, \quad 1 \leqslant \frac{1}{4}K^4, \tag{4.182}$$

and we obtain

$$\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \leqslant \frac{3}{4}K^4\delta_0^2,$$

that it is a clear improve of the inequality (4.181).

The second step is control b_-. Using (4.171), (4.177), (4.178) and (4.179), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2\nu_0 t}b_-^2)\right| \leqslant C_9(K^{15}\delta_0^6 + K^6\delta_0^3)\mathrm{e}^{2\nu_0 t},$$

for some constant $C_9 > 0$. Therefore, by integration on $[0,t]$ and using (4.180), we obtain

$$b_-^2 \leqslant \frac{C_9}{2\nu_0}(K^{15}\delta_0^6 + K^6\delta_0^3) + \delta_0^2.$$

Under the constraints

$$\frac{C_9}{2\nu_0}K^{15}\delta_0^4 \leqslant \frac{1}{4}K^4, \quad \frac{C_9}{2\nu_0}K^6\delta_0 \leqslant \frac{1}{4}K^4, \quad 1 \leqslant \frac{1}{4}K^4, \tag{4.183}$$

we get

$$b_-^2 \leqslant \frac{3}{4}K^2\delta_0^2,$$

that is an improvement of (4.178).

By the improved estimates (4.177) and (4.178), and a continuity argument, we observe that if $T < +\infty$, then $|b_+(T)| = K^5\delta_0^2$.

The third step is to analyze the growth of b_+. If $t \in [0, T]$ is such that $|b_+(t)| = K^5\delta_0^2$, then follows from (4.170) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} b_+^2 &\geqslant 2\nu_0 b_+^2 - 2C_4|b_+|(b_+^2 + b_-^2 + \|u_1\|_{H^1}^2) \\ &\geqslant 2\nu_0 b_+^2 - 2C_4|b_+|(b_+^2 + K^2\delta_0^2 + K^4\delta_0^2) \\ &\geqslant 2\nu_0 K^{10}\delta_0^4 - C_{10}(K^{15}\delta_0^6 + K^9\delta_0^4), \end{aligned}$$

for some constant $C_{10} > 0$. Under the constraints

$$C_{10}K^{15}\delta_0^2 \leqslant \frac{1}{2}\nu_0 K^{10}, \quad C_{10}K^9 \leqslant \frac{1}{2}\nu_0 K^{10}, \tag{4.184}$$

the following inequality holds

$$\frac{\mathrm{d}}{\mathrm{d}t} b_+^2 \geqslant \nu_0 K^{10}\delta_0^4 > 0.$$

By standard arguments, such transversality condition implies that T is the first time for which $|b_+(t)| = K^5\delta_0^2$ and moreover that T is continuous in the variable $b_+(0)$. The image of the continuous map

$$b_+(0) \in [-K^5\delta_0^2, K^5\delta_0^2] \longmapsto b_+(T) \in \{-K^5\delta_0^2, K^5\delta_0^2\}$$

is exactly $\{-K^5\delta_0^2, K^5\delta_0^2\}$ which is a contradiction. We conclude that there exists at least one value of $b_+(0) \in (-K^5\delta_0^2, K^5\delta_0^2)$ such that $T = \infty$, when constraints in (4.182), (4.183), (4.184) are fulfilled. Finally, to satisfy the conditions (4.182), (4.183), (4.184) it is sufficient first to fix $K > 0$ large enough, depending only on C_8, C_9, C_{10}, and then to choose $\delta_0 > 0$ small enough.

4.7.3 Uniqueness and Lipschitz regularity

To finish the proof of Theorem 2, we will prove the following proposition that implies the uniqueness of the choice of $h(\boldsymbol{\epsilon}) = b_+(0)$, for a given $\boldsymbol{\epsilon} \in \mathcal{A}_0$, as well the Lipschitz regularity of the graph $\mathcal{M}$ (see (4.15))

Proposition 4.34. *There exist $C, \delta > 0$ such if $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ are two even-odd solution of* (4.3) *satisfying*

$$\text{for all } t \geqslant 0, \|\boldsymbol{\phi}(t) - (Q,0)\|_{H^1\times L^2} < \delta, \quad \|\widetilde{\boldsymbol{\phi}}(t) - (Q,0)\|_{H^1\times L^2} < \delta \tag{4.185}$$

then, decomposing

$$\boldsymbol{\phi}(0) = (Q,0) + \boldsymbol{\epsilon} + b_+(0)\boldsymbol{Y}_+, \quad \widetilde{\boldsymbol{\phi}}(0) = (Q,0) + \tilde{\boldsymbol{\epsilon}} + \tilde{b_+}(0)\boldsymbol{Y}_+ \tag{4.186}$$

with $\langle \boldsymbol{\epsilon}, \boldsymbol{Z}_+\rangle = \langle \tilde{\boldsymbol{\epsilon}}, \boldsymbol{Z}_+\rangle = 0$, *it holds*

$$|b_+(0) - \tilde{b}_+(0)| \leqslant C\delta^{1/2}\|\epsilon - \tilde{\epsilon}\|_{H^1\times L^2}. \tag{4.187}$$

Proof. Let $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ solutions of (4.3) likes in the Subsection 4.2.1, i.e., satisfies the decomposition (4.23) and the smallest condition (4.27). Then,

$$\|u_1\|_{H^1} + \|\tilde{u}_1\|_{H^1} + \|u_2\|_{L^2} + \|\tilde{u}_2\|_{L^2} + |b_\pm| + |\tilde{b}_\pm| \leqslant C_0\delta. \tag{4.188}$$

Let

$$\begin{gathered}\breve{a}_1 = a_1 - \tilde{a}_1, \quad \breve{a}_2 = a_2 - \tilde{a}_2, \quad \breve{b}_+ = b_+ - \tilde{b}_+, \quad \breve{b}_- = b_- - \tilde{b}_-, \quad \breve{u}_1 = u_1 - \tilde{u}_1,\\ \breve{u}_2 = u_2 - \tilde{u}_2, \quad \breve{N} = N - \tilde{N}, \quad \breve{N}^\perp = N^\perp - \tilde{N}^\perp, \quad \breve{N}_0 = N_0 - \tilde{N}_0.\end{gathered} \tag{4.189}$$

Then, by (4.28) and (4.30), $(\breve{u}_1, \breve{u}_2)$ and $(\breve{b}_+, \breve{b}_-)$ satisfy the following equations:

$$\begin{cases}\dot{\breve{u}}_1 = \partial_x \breve{u}_2\\ \dot{\breve{u}}_2 = \partial_x \mathcal{L}(\breve{u}_1) + \breve{N}^\perp\end{cases} \quad \text{and} \quad \begin{cases}\dot{\breve{b}}_+ = \nu_0 \breve{b}_+ + \dfrac{\breve{N}_0}{2\nu_0}\\ \dot{\breve{b}}_- = -\nu_0 \breve{b}_- - \dfrac{\breve{N}_0}{2\nu_0}.\end{cases} \tag{4.190}$$

Furthermore, let

$$\beta_+ = \breve{b}_+^2, \quad \beta_- = \breve{b}_-^2, \quad \beta_c = \langle \mathcal{L}\breve{u}_1, \breve{u}_1\rangle + \langle \breve{u}_2, \breve{u}_2\rangle.$$

Computing the variation of β_c, we obtain

$$\dot{\beta}_c = 2\,\langle \breve{N}^\perp, \breve{u}_2\rangle.$$

Now, recalling (4.29) and (4.51), we get

$$\begin{aligned}\breve{N} = {} & (Q' - \tilde{a}_1\partial_x\phi_0 - \partial_x\tilde{u}_1)\Big[f'(Q) + f''(Q)(a_1\phi_0 + u_1) - f'(Q + a_1\phi_0 + u_1)\\ & \qquad - \Big(f'(Q) + f''(Q)(\tilde{a}_1\phi_0 + \tilde{u}_1) - f'(Q + \tilde{a}_1\phi_0 + \tilde{u}_1)\Big)\Big]\\ & + (\breve{a}_1\partial_x\phi_0 + \partial_x\breve{u}_1)(f'(Q) - f'(Q + a_1\phi_0 + u_1))\\ & + (\breve{a}_1\phi_0 + \breve{u}_1)f''(Q)(\tilde{a}_1\partial_x\phi_0 + \partial_x\tilde{u}_1).\end{aligned}$$

By Taylor expansion, for any $v, \tilde{v}$, it holds

$$\begin{aligned}&\big|f'(Q+v) - f'(Q) - f''(Q)v - (f'(Q+\tilde{v}) - f'(Q) - f''(Q)\tilde{v})\big|\\ &\lesssim |v - \tilde{v}|(|v| + |\tilde{v}|)(Q^{p-3} + |v|^{p-3} + |\tilde{v}|^{p-3}) \lesssim |v - \tilde{v}|(|v| + |\tilde{v}|)\end{aligned}$$

Then,

$$\begin{aligned}|\breve{N}| \lesssim {} & |\breve{a}_1\phi_0 + \breve{u}_1||Q' - \tilde{a}_1\partial_x\phi_0 - \partial_x\tilde{u}_1|\big(|\tilde{a}_1\phi_0 + \tilde{u}_1| + |a_1\phi_0 + u_1|\big)\\ & + f''(Q)|\breve{a}_1\partial_x\phi_0 + \partial_x\breve{u}_1||a_1\phi_0 + u_1| + f''(Q)|\breve{a}_1\phi_0 + \breve{u}_1||\tilde{a}_1\partial_x\phi_0 + \partial_x\tilde{u}_1|.\end{aligned}$$

Then, using Sobolev embbeding, L^2- norm of $\check{N}$ is bounded by

$$\begin{aligned}\|\check{N}\|_{L^2} &\lesssim \|\check{a}_1\phi_0+\check{u}_1\|_{L^\infty}\|Q'-\tilde{a}_1\partial_x\phi_0-\partial_x\tilde{u}_1\|_{L^2}\big(\|\tilde{a}_1\phi_0+\tilde{u}_1\|_{L^\infty}+\|a_1\phi_0+u_1\|_{L^\infty}\big)\\ &\quad+\|\check{a}_1\partial_x\phi_0+\partial_x\check{u}_1\|_{L^2}\|a_1\phi_0+u_1\|_{L^\infty}+\|\check{a}_1\phi_0+\check{u}_1\|_{L^\infty}\|\tilde{a}_1\partial_x\phi_0+\partial_x\tilde{u}_1\|_{L^2}\\ &\lesssim (|\check{a}_1|+\|\check{u}_1\|_{H^1})\|Q'-\tilde{a}_1\partial_x\phi_0-\partial_x\tilde{u}_1\|_{L^2}\big(|\tilde{a}_1|+\|\tilde{u}_1\|_{H^1}+|a_1|+\|u_1\|_{H^1}\big)\\ &\quad+(|\check{a}_1|+\|\check{u}_1\|_{H^1})(|a_1|+\|u_1\|_{H^1})+(|\check{a}_1|+\|\check{u}_1\|_{H^1})(|\tilde{a}_1|+\|\tilde{u}_1\|_{H^1})\\ &\lesssim (|\check{a}_1|+\|\check{u}_1\|_{H^1})\big[|a_1|+|\tilde{a}_1|+\|u_1\|_{H^1}+\|\tilde{u}_1\|_{H^1}\big].\end{aligned} \tag{4.191}$$

Then, by (4.190), (4.191), and using $|\check{N}_0|\lesssim\|\check{N}\|_{L^2}\|\partial_x^{-1}\phi_0\|_{L^2}$, we get

$$|\dot{\beta}_c|+|\dot{\beta}_+-2\nu_0\beta_+|+|\dot{\beta}_-+2\nu_0\beta_-|\leqslant K\delta(\beta_c+\beta_++\beta_-) \text{ for some } K>0. \tag{4.192}$$

In order to obtain a contradiction, assume that the following holds

$$0<K\delta(\beta_c(0)+\beta_+(0)+\beta_-(0))<\frac{\nu_0}{10}\beta_+(0). \tag{4.193}$$

Now, we consider the following bootstrap estimate

$$K\delta(\beta_c+\beta_++\beta_-)\leqslant\nu_0\beta_+. \tag{4.194}$$

and let

$$T=\sup\{t>0 \text{ such that (4.194) holds }\}>0.$$

From (4.192) and (4.194), it holds

$$\nu_0\beta_+\leqslant 2\nu_0\beta_+-K\delta(\beta_c+\beta_++\beta_-)\leqslant\dot{\beta}_+,\ \text{for } t\in[0,T]. \tag{4.195}$$

Then, β_+ is positive and increasing function on $[0,T]$.
Now, by (4.192) and (4.194), we get

$$\dot{\beta}_c\leqslant\nu_0\beta_+\leqslant\dot{\beta}_+$$

integrating and using that $\beta_+(0)>0$, we obtain

$$\beta_c(t)\leqslant\beta_c(0)+\beta_+(t)-\beta_+(0)\leqslant\beta_c(0)+\beta_+(t).$$

Furtheremore, by (4.193) and for δ small enough, we get

$$K\delta\beta_c(t)\leqslant K\delta(\beta_c(0)+\beta_+(t))\leqslant\frac{\nu_0}{10}\beta_+(0)+K\delta\beta_+(t)\leqslant\frac{\nu_0}{5}\beta_+(t).$$

For β_-, using (4.192) and (4.194), we get

$$\dot{\beta}_-\leqslant-2\nu_0\beta_-+\nu_0\beta_+,$$

integrating and using (4.193), we have

$$\beta_-(t)\leqslant\mathrm{e}^{-2\nu_0t}\beta_-(0)+\nu_0\beta_+\mathrm{e}^{-2\nu_0t}\int_0^t\mathrm{e}^{2\nu_0s}\mathrm{d}s\leqslant\beta_-(0)+\frac{1}{2}\beta_+(t).$$

For δ small enough, we get

$$K\delta\beta_-(t) \leqslant K\delta(\beta_-(0) + \beta_+(t)) \leqslant \frac{\nu_0}{10}\beta_+(0) + K\delta\beta_+(t) \leqslant \frac{\nu_0}{5}\beta_+(t).$$

For β_+, it is clear that holds $K\delta \leqslant \frac{\nu_0}{5}$ for δ small enough.
We have proved that, for all $t \in [0, T]$,

$$K\delta(\beta_c(t) + \beta_+(t) + \beta_-(t)) \leqslant \frac{3}{5}\nu_0\beta_+(t).$$

By a continuity argument, we get that $T = \infty$. However, by the exponential growth (4.195) and $\beta_+(0) > 0$, we obtain a contradiction with (4.188) on $|b_+|$.

Since it holds

$$\boldsymbol{\epsilon} = \boldsymbol{u}(0) + b_-(0)\boldsymbol{Y}_-, \qquad \tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{u}}(0) + \tilde{b}_-(0)\boldsymbol{Y}_-,$$

with $\langle \boldsymbol{u}(0), \boldsymbol{Y}_- \rangle = \langle \tilde{\boldsymbol{u}}(0), \boldsymbol{Y}_- \rangle = 0$, and estimates (4.193) is contradicted, we have proved (4.187). ∎

Appendices

4.A Linear spectral theory for $-\partial_x^2\mathcal{L}$

In this section we describe the spectral properties of the operator $-\partial_x^2\mathcal{L}$, where $\mathcal{L}$ is introduced in (4.8). Notice that this last operator has been widely studied (see [25, 26]). For the study of the operator $-\partial_x^2\mathcal{L}$ we shall start with the following result.

Lemma 4.35. *Let $p > 1$. The operator $\mathcal{L}$ defined in* (4.8) *satisfies the following properties.*

1. *The continuum spectrum of $\mathcal{L}$ is $[1, \infty)$.*
2. *The kernel of $\mathcal{L}$ is only spanned by the function Q'.*
3. *The generalized kernel of $\mathcal{L}$ is given by span $\left\{Q'(x), Q'(x)\int_\epsilon^x (Q'(r))^{-2}\mathrm{d}r\right\}$, for any $x \geqslant \epsilon > 0$ or $x \leqslant \epsilon < 0$.*

In what follows, and with a slight abuse of notation, we will write

$$\int_0^x (Q'(r))^{-2}\mathrm{d}r$$

instead of $\int_\epsilon^x (Q'(r))^{-2}\mathrm{d}r$; but it is understood that the zero limit of integration corresponds to any ε sufficiently close to zero.

An important remark is the following:

Remark 4.14. Note that

$$\mathcal{L}(fg) = g\mathcal{L}(f) - 2f'g' - fg''. \tag{4.196}$$

This property will be useful in the following computations.

Now, we study the properties of the operator $-\partial_x^2\mathcal{L}$.

Remark 4.15. A direct analysis shows that the null space of $\partial_x^2\mathcal{L}_0 = \partial_x^4 - \partial_x^2$ is spanned by functions of the type

$$\mathrm{e}^x,\ \mathrm{e}^{-x},\ 1,\ x, \quad \text{as} \quad x \to \infty.$$

Note that this set is linearly independent and among these four functions there is only one L^2 integrable in the semi-infinite line $[0, \infty)$. Therefore, since $\partial_x^2\mathcal{L}$ is a compact perturbation of the scalar operator $\partial_x^2\mathcal{L}_0$, the null space of $\partial_x^2\mathcal{L}|_{H^4(\mathbb{R})}$ is spanned by at most one L^2-function.

Lemma 4.36. *Let $p > 1$. The operators $-\partial_x^2\mathcal{L}$ satisfy the following properties.*

1. *The continuum spectrum of $-\partial_x^2\mathcal{L}$ is $[0, \infty)$.*

2. The generalized kernel of $-\partial_x^2\mathcal{L}$ is spanned by

$$\begin{aligned}\textit{span}\Bigg\{&\left[-\frac{2}{p-1}+-\frac{p+1}{p-1}Q'(x)\int_0^x Q^{-1}(r)\mathrm{d}r\right],\\ &Q'(x)\int_0^x \frac{\left(sQ(s)-\int_{-\infty}^s Q(y)\mathrm{d}y\right)}{(Q')^2(s)}\mathrm{d}s,\ Q'(x)\ ,\ Q'(x)\int_0^x (Q'(r))^{-2}\mathrm{d}r\Bigg\}.\end{aligned} \tag{4.197}$$

Proof. The proof of (1) follows directly from the form of the operator.

Proof of (2). Clearly

$$u_1(x):=Q'(x),\qquad u_2(x):=Q'(x)\int_0^x (Q'(r))^{-2}\mathrm{d}r,$$

are solutions to $-\partial_x^2\mathcal{L}(u)=0$. Notice that if $-\partial_x^2\mathcal{L}(u)=0$ is equivalent to $\mathcal{L}(u)=ax+b$ with $a,b\in\mathbb{R}$. Then we should solve this equation. First, we consider the case $a=0$. Without loss of generality, we consider $b=1$. One has $\mathcal{L}(1)=1-pQ^{p-1}$. Computing,

$$\begin{aligned}\mathcal{L}\left(Q'\int_0^x Q^n\right)&=\mathcal{L}(Q')\int_0^x Q^n-Q^n(2Q'')-nQ^{n-1}(Q')^2\\ &=-2Q^n(Q-Q^p)-nQ^{n-1}\left(Q^2-\frac{2}{p+1}Q^{p+1}\right)\\ &=Q^{n+1}(-2-n)+\left(\frac{2n+2p+2}{p+1}\right)Q^{p+n}.\end{aligned}$$

If $n=-1$, we have

$$\mathcal{L}\left(Q'\int_0^x Q^{-1}\right)=-1+\frac{2p}{p+1}Q^{p-1}.$$

Set $u_3(x)=-\frac{2}{p-1}\left[1+\frac{p+1}{2}Q'(x)\int_0^x Q^{-1}(r)\mathrm{d}r\right]$. We observe that $\mathcal{L}\left(u_3(x)\right)=1$. Therefore, up to the generalized kernel of $\mathcal{L}$, u_3 solves the equation $\mathcal{L}(u_3)=1$.

Now, without loss of generality, we consider $a=1$ and $b=0$, then we must solve $\mathcal{L}(u_4)=x$. Using the method of reduction of order with an unknown function ψ, consider $u_4=Q'\psi$. Using (4.196), we have

$$\mathcal{L}(Q'\psi)=-2Q''\psi'-Q'\psi''=x.$$

We obtain that the solution of this equation is

$$\begin{aligned}u_4(x)&=Q'(x)\int_0^x (Q')^{-2}(s)\left(\int_0^s Q'(y)y\mathrm{d}y\right)\mathrm{d}s-\left(\int_{-\infty}^0 Q(y)\mathrm{d}y\right)Q'(x)\int_0^x (Q')^{-2}(s)\mathrm{d}s\\ &=Q'(x)\int_0^x (Q')^{-2}(s)\left(\int_0^s yQ'(y)\mathrm{d}y-\int_{-\infty}^0 Q(y)\mathrm{d}y\right)\mathrm{d}s\\ &=Q'(x)\int_0^x (Q')^{-2}(s)\left(sQ(s)-\int_{-\infty}^s Q(y)\mathrm{d}y\right)\mathrm{d}s.\end{aligned}$$

We finally conclude that the fundamental set of solutions for $\partial_x^2\mathcal{L}(u) = 0$ is given by

$$\{u_1(x), u_2(x), u_3(x), u_4(x)\}.$$

This ends the proof. ■

Corollary 4.37. *There is, up to constant, only one solution of $-\partial_x^2\mathcal{L}(u) = 0$ in $L^2(\mathbb{R})$.*

Now, we focus on describing the eigenfunctions and negative eigenvalues of operator $-\partial_x^2\mathcal{L}$. This analysis will be the main ingredient to describe the stability of the soliton. Our first result establishes the parity of eigenfunctions associated to nonzero eigenvalues.

Lemma 4.38. *If $\phi_0 \in H^4(\mathbb{R})$ is an eigenfunction associated to an eigenvalue $\lambda_0 \neq 0$ of the operator $-\partial_x^2\mathcal{L}$, then $\partial_x^{-1}\phi_0 \in H^3$ and and $\partial_x^{-2}\phi_0 \in H^2$, i.e., are well-defined. Furthermore, if ϕ_0 is an even function then $\partial_x^{-1}\phi_0$ is an odd function and*

$$\int_0^\infty \phi_0(y)\mathrm{d}y = 0.$$

Proof. We have

$$-\partial_x^2\mathcal{L}\phi_0 = \lambda_0\phi_0, \quad \text{with} \quad \lambda_0 \neq 0,$$

this is equivalent to

$$\partial_x^4\phi_0 - \partial_x^2\phi_0 + \partial_x^2(pQ^{p-1}\phi_0) = \lambda_0\phi_0. \tag{4.198}$$

Applying Fourier transform, we have

$$\xi^4\widehat{\phi_0} + \xi^2\widehat{\phi_0} - \xi^2 p(\widehat{Q^{p-1}\phi_0}) = \lambda_0\widehat{\phi_0}.$$

From this identity, and the fact that $\phi_0 \in H^4(\mathbb{R})$, we observe that

$$\lim_{\xi\to 0}\widehat{\phi_0}(\xi) = \lambda_0^{-1}\lim_{\xi\to 0}\left(\xi^4\widehat{\phi_0} + \xi^2\widehat{\phi_0} - \xi^2 p\widehat{Q^{p-1}\phi_0}\right) = 0.$$

Also

$$\lim_{\xi\to 0}\xi^{-1}\widehat{\phi_0}(\xi) = \lambda_0^{-1}\lim_{\xi\to 0}\left(\xi^3\widehat{\phi_0} + \xi\widehat{\phi_0} - \xi p\widehat{Q^{p-1}\phi_0}\right) = 0,$$
$$\lim_{\xi\to 0}\xi^{-2}\widehat{\phi_0}(\xi) = \lambda_0^{-1}\lim_{\xi\to 0}\left(\xi^2\widehat{\phi_0} + \widehat{\phi_0} - p\widehat{Q^{p-1}\phi_0}\right) = -p\lambda_0^{-1}\widehat{Q^{p-1}\phi_0}(0).$$

Then, we obtain that

$$\int \phi_0(x) = 0, \quad \int\int_{-\infty}^x \phi_0(s)\mathrm{d}s = 0.$$

Also, we know that $\widehat{Q^{p-1}\phi_0}$ is well defined (the Fourier transform is an homeomorphism from L^2 into L^2). Then $\partial_x^{-1}\phi_0$ and $\partial_x^{-2}\phi_0$ are well-defined, and exponentially decreasing, provided ϕ_0 and its derivatives are also exponentially decreasing.

Now, suppose that ϕ_0 is an **even** function. Integrating between 0 and x in (4.198), we obtain

$$(\partial_x^3\phi_0 - \partial_x\phi_0 + \partial_x(pQ^{p-1}\phi_0))(x) - [\partial_x^3\phi_0 - \partial_x\phi_0 + \partial_x(pQ^{p-1}\phi_0)]|_{x=0} = \lambda_0 \int_0^x \phi_0.$$

Since Q^{p-1} is an even function and $\partial_x^3\phi_0$, $\partial_x\phi_0$ and $\partial_x(Q^{p-1}\phi_0)$ are odd functions, satisfying $\partial_x^3\phi_0(0) = \partial_x\phi_0(0) = \partial_x(Q^{p-1}\phi_0)(0) = 0$, we conclude

$$\partial_x\mathcal{L}(\phi_0)(x) = (\partial_x^3\phi_0 - \partial_x\phi_0 + \partial_x(pQ^{p-1}\phi_0))(x) = \lambda_0 \int_0^x \phi_0(y)\mathrm{d}y.$$

Now, given that $\phi_0 \in H^4(\mathbb{R})$, one has $\partial_x^3\phi_0(x), \partial_x\phi_0(x), \partial_x(Q^{p-1}\phi_0)(x) \to 0$ as $x \to \pm\infty$. We conclude

$$\int_0^x \phi_0(y)\mathrm{d}y = -\int_0^{-x} \phi_0(y)\mathrm{d}y \quad \text{and} \quad \int_0^\infty \phi_0(y)\mathrm{d}y = 0.$$

This proves the oddness of $\partial_x^{-1}\phi_0$ and concludes the proof. ∎

We observe that $-\partial_x^2\mathcal{L}$ is not a self-adjoint operator. In fact, if $\varphi, \psi \in H^4(\mathbb{R})$,

$$\langle -\partial_x^2\mathcal{L}\varphi, \psi \rangle = \langle \varphi, -\partial_x^2\mathcal{L}(\psi)\rangle + \langle \varphi, f'(Q)\partial_x^2\psi - \partial_x^2(f'(Q)\psi)\rangle,$$

since the operators ∂_x^2 and $\mathcal{L}$ do not commute. For this reason, we need to consider this operator in an appropriate sense. A way to face this problem is to consider the following result.

Lemma 4.39. *The operator $-\partial_x^2\mathcal{L}$ has only real eigenvalues.*

Proof. Given $\varphi_0 \in H^4(\mathbb{R})$ eigenfunction of the operator $-\partial_x^2\mathcal{L}$ with eigenvalue $\lambda_0 \in \mathbb{C}$, we consider $\varphi_0 = \partial_x\psi_0$ or $\psi_0 = \partial_x^{-1}\varphi_0$. We know that this function is well defined by Lemma 4.38. Now, we have

$$-\partial_x^2\mathcal{L}(\partial_x\psi_0) = -\partial_x^2\mathcal{L}(\varphi_0) = \lambda_0\varphi_0 = \lambda_0\partial_x\psi_0.$$

Integrating, we obtain

$$-\partial_x\mathcal{L}(\partial_x\psi_0) = \lambda_0\psi_0.$$

We can easily check that the operator $-\partial_x\mathcal{L}\partial_x$ is self-adjoint with eigenvalue λ_0 and eigenfunction ψ_0. We conclude that λ_0 is real, hence the eigenvalues of $-\partial_x^2\mathcal{L}$ are real. ∎

Therefore, the operator $-\partial_x^2\mathcal{L}$ has a similar structure of a self-adjoint operator. This fact allows to follow the strategy of Greenberg and Maddocks-Sachs [12, 20] for counting the negatives eigenvalues of this operator.

The most important property about $-\partial_x^2\mathcal{L}$ is that it possesses only one negative eigenvalue.

Theorem 4.40. *The operator $-\partial_x^2\mathcal{L}$ has a unique negative eigenvalue $-\nu_0^2 < 0$ of multiplicity one. The associated eigenfunction ϕ_0 satisfies the exponential decay in* (4.10)*, along with its derivatives.*

This is just a consequence of the fact that the only solution of $-\partial_x^2\mathcal{L}(u) = 0$ converging to zero at $-\infty$ is $Q'(x)$, see Claim 4.42. This function has a unique zero. The exponential decay is just consequence of Remark 4.15.

Corollary 4.41. *Given ϕ_0 eigenfunction associated to the unique negative eigenvalue $-\nu_0^2$, then ϕ_0 is an even function and $\partial_x^{-1}\phi_0$ is an odd function.*

Proof. Consider the function $\psi(x) = \phi_0(-x)$, we have

$$-\partial_x^2\mathcal{L}(\psi) = \partial_x^4(\psi) - \partial_x^2\psi + \partial_x^2(pQ^{p-1}\psi).$$

Notice that $Q^{p-1}, \partial_x^2(Q^{p-1})$ are even functions and $\partial_x(Q^{p-1})$ is an odd function, also

$$\partial_x^2(pQ^{p-1}\psi)(x) = \partial_x^2(pQ^{p-1}(x)\phi_0(-x)) = \partial_x^2(pQ^{p-1}\phi_0)(-x).$$

Then, we observe that

$$-\partial_x^2\mathcal{L}(\psi) = -\partial_x^2\mathcal{L}(\phi_0)(-x) = -\nu_0^2\phi_0(-x) = -\nu_0^2\psi(x).$$

Finally, since λ_0 is the unique negative eigenvalue of multiplicity one, we conclude that $\phi_0(x) = \psi(x) = \phi_0(-x)$, i.e., ϕ_0 is an even function. Finally, by Lemma 4.38 we know $\partial_x^{-1}\phi_0$ is an odd function. ■

4.A.1 Asymptotic behavior of fundamental solutions of $-\partial_x^2\mathcal{L}(u) = 0$

The following computations are direct, but we include them by the sake of completeness. They are just simple applications of L'Hôpital's rule.

Claim 4.42. *The functions u_1, u_2, u_3 and u_4 found in Lemma 4.35 and 4.36 satisfy*

$$\lim_{x\to-\infty} u_1(x) = 0, \quad \lim_{x\to-\infty} u_2(x) = +\infty, \quad \lim_{x\to-\infty} u_3(x) = 1, \quad \lim_{x\to-\infty} u_4(x) = -\infty.$$

Proof. One has

1.
$$\lim_{x\to-\infty} u_1(x) = \lim_{x\to-\infty} Q'(x) = 0.$$

2. Second,
$$\begin{aligned}\lim_{x\to-\infty} u_2(x) &= \lim_{x\to-\infty} Q'(x)\int_0^x (Q'(r))^{-2}\mathrm{d}r = \lim_{x\to-\infty} \frac{\int_0^x (Q'(r))^{-2}\mathrm{d}r}{(Q'(x))^{-1}} \\ &= \lim_{x\to-\infty} \frac{(Q'(x))^{-2}}{-(Q'(x))^{-2}Q''(x)} = \lim_{x\to-\infty} \frac{1}{-Q''(x)} = \lim_{x\to-\infty} \frac{1}{Q(x)(Q^{p-1}(x)-1)} = +\infty.\end{aligned}$$

3. Third,

$$\lim_{x\to-\infty} u_3(x) = \lim_{x\to-\infty} -\frac{2}{p-1}\left[1+\frac{p+1}{2}Q'\int_0^x Q^{-1}(r)\mathrm{d}r\right]$$

$$= -\frac{2}{p-1} - \frac{p+1}{p-1}\lim_{x\to-\infty} Q'(x)\int_0^x Q^{-1}(r)\mathrm{d}r = -\frac{2}{p-1} - \frac{p+1}{p-1}\lim_{x\to-\infty}\frac{\int_0^x Q^{-1}(r)\mathrm{d}r}{(Q'(x))^{-1}}$$

$$= -\frac{2}{p-1} - \frac{p+1}{p-1}\lim_{x\to-\infty}\frac{Q^{-1}(x)}{-(Q'(x))^{-2}Q''(x)} = -\frac{2}{p-1} + \frac{p+1}{p-1}\lim_{x\to-\infty}\frac{Q^{-1}(x)(Q^2-\frac{2}{p+1}Q^{p+1})}{Q-Q^p}$$

$$= -\frac{2}{p-1} + \frac{p+1}{p-1}\lim_{x\to-\infty}\frac{(1-\frac{2}{p+1}Q^{p-1})}{1-Q^{p-1}} = 1.$$

4. Finally,

$$\lim_{x\to-\infty} u_4(x) = \lim_{x\to-\infty} Q'(x)\int_0^x (Q')^{-2}\left(sQ(s)-\int_{-\infty}^s Q\right)\mathrm{d}s$$

$$= \lim_{x\to-\infty}\frac{\int_0^x (Q')^{-2}\left(sQ(s)-\int_{-\infty}^s Q\right)\mathrm{d}s}{(Q'(x))^{-1}} = \lim_{x\to-\infty}\frac{(Q'(x))^{-2}\left(xQ(x)-\int_{-\infty}^x Q\right)}{-(Q'(x))^{-2}Q''(x)}$$

$$= \lim_{x\to-\infty}\frac{\left(xQ(x)-\int_{-\infty}^x Q\right)}{-Q''(x)} = \lim_{x\to-\infty}\frac{\left(xQ(x)-\int_{-\infty}^x Q\right)}{Q(x)(Q^{p-1}(x)-1)} = \lim_{x\to-\infty}\frac{(xQ'(x)+Q(x)-Q(x))}{Q'(x)(pQ^{p-1}(x)-1)}$$

$$= \lim_{x\to-\infty}\frac{x}{pQ^{p-1}(x)-1} = -\infty.$$

■

4.B Proof of Claims 4.21 and 4.22

4.B.1 Relation between $\partial_x z_{\mathrm{i}}$ and $\partial_x v_{\mathrm{i}}$

We prove Claim 4.21. First, recall that $z_{\mathrm{i}} = \chi_A\zeta_B v_{\mathrm{i}}$ and

$$\partial_x z_{\mathrm{i}} = (\chi_A\zeta_B)'v_{\mathrm{i}} + \chi_A\zeta_B\partial_x v_{\mathrm{i}}. \tag{4.199}$$

Then

$$(\partial_x z_{\mathrm{i}})^2 = ((\chi_A\zeta_B)'v_{\mathrm{i}})^2 + 2(\chi_A\zeta_B)'\chi_A\zeta_B v_{\mathrm{i}}\partial_x v_{\mathrm{i}} + (\chi_A\zeta_B\partial_x v_{\mathrm{i}})^2. \tag{4.200}$$

For a function $P(x)\in C^1(\mathbb{R})$, we consider

$$\int P(x)\chi_A^2\zeta_B^2(\partial_x v_{\mathrm{i}})^2.$$

Using (4.200), we obtain

$$\begin{aligned}\int P(x)\chi_A^2\zeta_B^2(\partial_x v_{\mathrm{i}})^2 &= \int P(x)(\partial_x z_{\mathrm{i}})^2 - \int P(x)[(\chi_A\zeta_B)']^2v_{\mathrm{i}}^2 - \frac{1}{2}\int P(x)((\chi_A\zeta_B)^2)'\partial_x(v_{\mathrm{i}}^2)\\ &= \int P(x)(\partial_x z_{\mathrm{i}})^2 - \int P(x)[(\chi_A\zeta_B)']^2v_{\mathrm{i}}^2 + \frac{1}{2}\int[P(x)((\chi_A\zeta_B)^2)']'v_{\mathrm{i}}^2.\end{aligned} \tag{4.201}$$

Now

$$P(x)[(\chi_A\zeta_B)']^2 = P(x)\zeta_B^2\left[(\chi_A')^2 + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + P(x)(\chi_A\zeta_B)^2\left(\frac{\zeta_B'}{\zeta_B}\right)^2.$$

Then, we have

$$\int P(x)[(\chi_A\zeta_B)']^2 v_{\rm i}^2 = \int P(x)\left(\frac{\zeta_B'}{\zeta_B}\right)^2 z_{\rm i}^2 + \int P(x)\left[(\chi_A')^2 + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_{\rm i}^2.$$

As for the third integral in the RHS of (4.201), we have

$$\begin{aligned}
[P(x)(\chi_A^2\zeta_B^2)']' &= P'(x)(\chi_A^2\zeta_B^2)' + P(x)(\chi_A^2\zeta_B^2)'' \\
&= P'(x)\zeta_B^2\left[(\chi_A^2)' + 2\chi_A^2\left(\frac{\zeta_B'}{\zeta_B}\right)\right] \\
&\quad + P(x)\zeta_B^2\left[(\chi_A^2)'' + 4(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 2\chi_A^2\left[\left(\frac{\zeta_B'}{\zeta_B}\right)^2 + \frac{\zeta_B''}{\zeta_B}\right]\right] \\
&= 2\chi_A^2\zeta_B^2\left[P'(x)\frac{\zeta_B'}{\zeta_B} + P(x)\left[\left(\frac{\zeta_B'}{\zeta_B}\right)^2 + \frac{\zeta_B''}{\zeta_B}\right]\right] \\
&\quad + P(x)\zeta_B^2\left[(\chi_A^2)'' + 4(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + P'(x)\zeta_B^2(\chi_A^2)'.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\int [P(x)((\chi_A\zeta_B)^2)']' v_{\rm i}^2 &= 2\int\left[P'(x)\frac{\zeta_B'}{\zeta_B} + P(x)\left[\left(\frac{\zeta_B'}{\zeta_B}\right)^2 + \frac{\zeta_B''}{\zeta_B}\right]\right] z_{\rm i}^2 \\
&\quad + \int P(x)\left[(\chi_A^2)'' + 4(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_{\rm i}^2 + \int P'(x)(\chi_A^2)'\zeta_B^2 v_{\rm i}^2.
\end{aligned}$$

We conclude in (4.201):

$$\begin{aligned}
\int P(x)\chi_A^2\zeta_B^2(\partial_x v_{\rm i})^2 &= \int P(x)(\partial_x z_{\rm i})^2 - \int P(x)[(\chi_A\zeta_B)']^2 v_{\rm i}^2 + \frac{1}{2}\int [P(x)((\chi_A\zeta_B)^2)']' v_{\rm i}^2 \\
&= \int P(x)(\partial_x z_{\rm i})^2 - \int P(x)\left(\frac{\zeta_B'}{\zeta_B}\right)^2 z_{\rm i}^2 - \int P(x)\left[(\chi_A')^2 + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_{\rm i}^2 \\
&\quad + \int\left[P'(x)\frac{\zeta_B'}{\zeta_B} + P(x)\left[\left(\frac{\zeta_B'}{\zeta_B}\right)^2 + \frac{\zeta_B''}{\zeta_B}\right]\right] z_{\rm i}^2 \\
&\quad + \frac{1}{2}\int P(x)\left[(\chi_A^2)'' + 4(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_{\rm i}^2 + \frac{1}{2}\int P'(x)(\chi_A^2)'\zeta_B^2 v_{\rm i}^2 \\
&= \int P(x)(\partial_x z_{\rm i})^2 + \int\left[P'(x)\frac{\zeta_B'}{\zeta_B} + P(x)\frac{\zeta_B''}{\zeta_B}\right] z_{\rm i}^2 \\
&\quad + \int \mathcal{E}_1(P(x),x)\zeta_B^2 v_{\rm i}^2,
\end{aligned} \tag{4.202}$$

where

$$\mathcal{E}_1(P(x),x) = P(x)\left[\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + \frac{1}{2}P'(x)(\chi_A^2)'. \tag{4.203}$$

Finally, (4.110) follows directly from the definition of $\mathcal{E}_1(P(x),x)$ and Remark 4.9 replacing B by A. This ends the proof of Claim 4.21.

4.B.2 Relation between $\partial_x^2 z_{\rm i}$ and $\partial_x^2 v_{\rm i}$

Now we prove Claim 4.22. The following relation is obtained from $z_{\rm i}$ in (4.60):

$$\begin{aligned}
\partial_x^2 z_{\rm i} &= (\chi_A\zeta_B)'' v_{\rm i} + 2(\chi_A\zeta_B)'\partial_x v_{\rm i} + \chi_A\zeta_B\partial_x^2 v_{\rm i},\\
(\partial_x^2 z_{\rm i})^2 &= [(\chi_A\zeta_B)'' v_{\rm i}]^2 + 4[(\chi_A\zeta_B)'\partial_x v_{\rm i}]^2 + [\chi_A\zeta_B\partial_x^2 v_{\rm i}]^2\\
&\quad + 4(\chi_A\zeta_B)'' v_{\rm i}(\chi_A\zeta_B)'\partial_x v_{\rm i} + 4(\chi_A\zeta_B)'\partial_x v_{\rm i}\chi_A\zeta_B\partial_x^2 v_{\rm i} + 2(\chi_A\zeta_B)'' v_{\rm i}\chi_A\zeta_B\partial_x^2 v_{\rm i}\\
&= [(\chi_A\zeta_B)'' v_{\rm i}]^2 + 4[(\chi_A\zeta_B)'\partial_x v_{\rm i}]^2 + [\chi_A\zeta_B\partial_x^2 v_{\rm i}]^2\\
&\quad + 2(\chi_A\zeta_B)''(\chi_A\zeta_B)'\partial_x(v_{\rm i}^2) + [\chi_A^2\zeta_B^2]'\partial_x[(\partial_x v_{\rm i})^2] + 2(\chi_A\zeta_B)''\chi_A\zeta_B v_{\rm i}\partial_x^2 v_{\rm i}.
\end{aligned}$$

Then,

$$\begin{aligned}
(\chi_A\zeta_B\partial_x^2 v_{\rm i})^2 &= (\partial_x^2 z_{\rm i})^2 - ((\chi_A\zeta_B)'' v_{\rm i})^2 - 4((\chi_A\zeta_B)'\partial_x v_{\rm i})^2\\
&\quad - 2(\chi_A\zeta_B)''(\chi_A\zeta_B)'\partial_x(v_{\rm i}^2) - 2(\chi_A\zeta_B)''\chi_A\zeta_B v_{\rm i}\partial_x^2 v_{\rm i}\\
&\quad - [\chi_A^2\zeta_B^2]'\partial_x[(\partial_x v_{\rm i})^2].
\end{aligned} \tag{4.204}$$

Now,

$$\begin{aligned}
&\int R(x)(\chi_A\zeta_B\partial_x^2 v_{\rm i})^2\\
&= \int R(x)(\partial_x^2 z_{\rm i})^2 + \int R(x)\left(-[(\chi_A\zeta_B)'']^2 v_{\rm i}^2 - 4[(\chi_A\zeta_B)']^2(\partial_x v_{\rm i})^2\right)\\
&\quad + \int R(x)\left[\; - ([(\chi_A\zeta_B)']^2)'\partial_x(v_{\rm i}^2) - [(\chi_A\zeta_B)^2]'\partial_x[(\partial_x v_{\rm i})^2] - 2(\chi_A\zeta_B)''\chi_A\zeta_B v_{\rm i}\partial_x^2 v_{\rm i}\right]\\
&= \int R(x)(\partial_x^2 z_{\rm i})^2 + \int R(x)(-[(\chi_A\zeta_B)'']^2 v_{\rm i}^2 - 4[(\chi_A\zeta_B)']^2(\partial_x v_{\rm i})^2)\\
&\quad + \int \partial_x[R(x)([(\chi_A\zeta_B)']^2)'] v_{\rm i}^2 + \int \partial_x[R(x)[(\chi_A\zeta_B)^2]'](\partial_x v_{\rm i})^2 - \int 2R(x)(\chi_A\zeta_B)''\chi_A\zeta_B v_{\rm i}\partial_x^2 v_{\rm i}.
\end{aligned} \tag{4.205}$$

Since

$$-\int 2R(x)(\chi_A\zeta_B)''\chi_A\zeta_B v_{\rm i}\partial_x^2 v_{\rm i} = -\int \partial_x^2[R(x)(\chi_A\zeta_B)''\chi_A\zeta_B] v_{\rm i}^2 + 2\int R(x)(\chi_A\zeta_B)''\chi_A\zeta_B(\partial_x v_{\rm i})^2,$$

we get

$$\begin{aligned}
&\int R(x)(\chi_A\zeta_B\partial_x^2 v_{\rm i})^2\\
&= \int R(x)(\partial_x^2 z_{\rm i})^2 + \int R(x)(-[(\chi_A\zeta_B)'']^2 v_{\rm i}^2 - 4[(\chi_A\zeta_B)']^2(\partial_x v_{\rm i})^2)\\
&\quad + \int \partial_x[R(x)([(\chi_A\zeta_B)']^2)'] v_{\rm i}^2 + \int \partial_x[R(x)[(\chi_A\zeta_B)^2]'](\partial_x v_{\rm i})^2\\
&\quad - \int \partial_x^2[R(x)(\chi_A\zeta_B)''\chi_A\zeta_B] v_{\rm i}^2 + 2\int R(x)(\chi_A\zeta_B)''\chi_A\zeta_B(\partial_x v_{\rm i})^2\\
&= \int R(x)(\partial_x^2 z_{\rm i})^2 + \int [-\partial_x^2[R(x)(\chi_A\zeta_B)''\chi_A\zeta_B] + \partial_x[R(x)([(\chi_A\zeta_B)']^2)'] - R(x)[(\chi_A\zeta_B)'']^2] v_{\rm i}^2\\
&\quad + \int \Big[2R(x)(\chi_A\zeta_B)''\chi_A\zeta_B + \partial_x[R(x)[(\chi_A\zeta_B)^2]'] - 4R(x)[(\chi_A\zeta_B)']^2\Big](\partial_x v_{\rm i})^2.
\end{aligned} \tag{4.206}$$

Now we perform the following splitting:

$$\begin{aligned}\int R(x)(\chi_A\zeta_B\partial_x^2 v_{\rm i})^2 =&\int R(x)(\partial_x^2 z_{\rm i})^2\\&+\int\Bigg[\partial_x[R(x)([(\chi_A\zeta_B)']^2)'] - R(x)((\chi_A\zeta_B)'')^2 - \partial_x^2[R(x)(\chi_A\zeta_B)''\chi_A\zeta_B]\Bigg]v_{\rm i}^2\\&+\int\Bigg[\partial_x[R(x)(\chi_A^2\zeta_B^2)'] + 2R(x)(\chi_A\zeta_B)''\chi_A\zeta_B - 4R(x)((\chi_A\zeta_B)')^2\Bigg](\partial_x v_{\rm i})^2\\=:&\ R_1+R_2+R_3.\end{aligned}\tag{4.207}$$

Firstly, we will focus on R_2. The term that accompanies to $v_{\rm i}^2$, holds the following decomposition

$$\begin{aligned}&\partial_x[R(x)([(\chi_A\zeta_B)']^2)'] - R(x)((\chi_A\zeta_B)'')^2 - \partial_x^2[R(x)(\chi_A\zeta_B)''\chi_A\zeta_B]\\&=\chi_A^2\zeta_B^2\tilde{R}(x) + \mathcal{E}_2(R(x),x)\zeta_B^2,\end{aligned}\tag{4.208}$$

where

$$\tilde{R}(x) = -2R(x)\left[\frac{\zeta_B^{(4)}}{\zeta_B} + \frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B}\right] - 2R'(x)\frac{\zeta_B'''}{\zeta_B} - R''(x)\frac{\zeta_B''}{\zeta_B},\tag{4.209}$$

and

$$\begin{aligned}\mathcal{E}_2(R(x),x) =& -R(x)\left(\chi_A^{(4)}\chi_A + 4\chi_A'''\chi_A\frac{\zeta_B'}{\zeta_B^2} + 6\chi_A''\chi_A\frac{\zeta_B''}{\zeta_B} + 2(\chi_A^2)'\frac{\zeta_B'''}{\zeta_B}\right)\\&-R'(x)\left(2\chi_A'''\chi_A + 6\chi_A''\chi_A\frac{\zeta_B'}{\zeta_B} + 6\chi_A'\chi_A\frac{\zeta_B''}{\zeta_B}\right)\\&-R''(x)\left(\chi_A''\chi_A + \frac{1}{2}[\chi_A^2]'\frac{\zeta_B'}{\zeta_B}\right).\end{aligned}\tag{4.210}$$

Rewriting R_2, we obtain

$$R_2 = \int\tilde{R}(x)z_{\rm i}^2 + \int\mathcal{E}_2(R(x),x)\zeta_B^2 v_{\rm i}^2.\tag{4.211}$$

Secondly, for R_3, the term that accompanies to $(\partial_x v_{\rm i})^2$ satisfies the following decomposition

$$\begin{aligned}\partial_x[R(x)(\chi_A^2\zeta_B^2)'] + 2R(x)(\chi_A\zeta_B)''\chi_A\zeta_B - 4R(x)((\chi_A\zeta_B)')^2&\\=P_R(x)\chi_A^2\zeta_B^2 + \mathcal{E}_3(R(x),x)\zeta_B^2&,\end{aligned}\tag{4.212}$$

where

$$P_R(x) = R(x)\left[4\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2\right] + 2R'(x)\frac{\zeta_B'}{\zeta_B},\tag{4.213}$$

and

$$\mathcal{E}_3(R(x),x) =R(x)\left[4\chi_A''\chi_A - 2(\chi_A')^2 + 2\frac{\zeta_B'}{\zeta_B}(\chi_A^2)'\right] + R'(x)(\chi_A^2)'.\tag{4.214}$$

Now, by Claim 4.21, we have

$$\begin{aligned}\int P_R(x)\chi_A^2\zeta_B^2(\partial_x v_{\rm i})^2 =& \int P_R(x)(\partial_x z_{\rm i})^2 + \int\left[P_R'(x)\frac{\zeta_B'}{\zeta_B} + P_R(x)\frac{\zeta_B''}{\zeta_B}\right]z_{\rm i}^2\\&+\frac{1}{2}\int\mathcal{E}_1(P_R(x),x)\zeta_B^2 v_{\rm i}^2,\end{aligned}\tag{4.215}$$

where $\mathcal{E}_1$ is given by (4.203). Finally, we obtain that R_3 has the following decomposition

$$\begin{aligned} R_3 = & \int P_R(x)(\partial_x z_{\rm i})^2 + \int \left[P_R'(x)\frac{\zeta_B'}{\zeta_B} + P_R(x)\frac{\zeta_B''}{\zeta_B}\right] z_{\rm i}^2 \\ & + \frac{1}{2}\int \mathcal{E}_1(P_R(x),x)\zeta_B^2 v_{\rm i}^2 + \int \mathcal{E}_3(R(x),x)\zeta_B^2(\partial_x v_{\rm i})^2. \end{aligned} \tag{4.216}$$

Collecting R_1, (4.211) and (4.216), we obtain

$$\begin{aligned} \int R(x)\chi_A^2\zeta_B^2(\partial_x^2 v_{\rm i})^2 = & \int R(x)(\partial_x^2 z_{\rm i})^2 + \int \tilde{R}(x) z_{\rm i}^2 + \int \mathcal{E}_2(R(x),x)\zeta_B^2 v_{\rm i}^2 \\ & + \int P_R(x)(\partial_x z_{\rm i})^2 + \int \left[P_R'(x)\frac{\zeta_B'}{\zeta_B} + P_R(x)\frac{\zeta_B''}{\zeta_B}\right] z_{\rm i}^2 \\ & + \frac{1}{2}\int \mathcal{E}_1(P_R(x),x)\zeta_B^2 v_{\rm i}^2 + \int \mathcal{E}_3(R(x),x)\zeta_B^2(\partial_x v_{\rm i})^2, \end{aligned} \tag{4.217}$$

where $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ and P_R are given in (4.203), (4.210), (4.214) and (4.213), respectively. Finally, the proof of (4.117) is direct. This concludes the proof of the Claim 4.22.

4.C Proof of Lemma 4.27

Proof. We claim that for all $v \in H^1(\mathbb{R})$ that satisfies $\langle \mathcal{L}\phi_0, v\rangle = 0$, one has

$$\langle \mathcal{L}v, v\rangle \geqslant 0.$$

Then the conclusion is evident since $\langle Q', v\rangle = 0$. Suppose that for some nonzero $u \in H^1(\mathbb{R})$ with $\langle \mathcal{L}\phi_0, u\rangle = 0$, we have $\langle \mathcal{L}u, u\rangle < 0$. Then, since ϕ_0 satisfies (4.10),

$$\langle \mathcal{L}\phi_0, \phi_0\rangle = \nu_0^2\langle \partial_x^{-2}\phi_0, \phi_0\rangle = \nu_0^2\langle \partial_x^{-2}\phi_0, \partial_x\partial_x^{-1}\phi_0\rangle = -\nu_0^2\|\partial_x^{-1}\phi_0\|_{L^2}^2 < 0.$$

Then we observe that the quadratic form $(\mathcal{L}\cdot,\cdot)$ is negative definite in $\text{span}(\phi_0, u)$. Since $\langle \phi_0, Q\rangle \neq 0$ (see Lemma 4.31), there exists $u_0 \in \mathit{span}(\phi_0, u)$ such that $u_0 \perp Q$ and $\langle \mathcal{L}u_0, u_0\rangle < 0$. This is a contradiction with the result

$$\inf_{\langle v, Q\rangle = 0}\langle \mathcal{L}v, v\rangle = 0.$$

(See Proposition 2.9 in [34] for more details.) ■

Bibliography

[1] M. A. Alejo, F. Cortez, C. Kwak and C. Muñoz, *On the dynamics of zero-speed solutions for Camassa-Holm type equations*, International Mathematics Research Notices, , rnz038, `https://doi.org/10.1093/imrn/rnz038`.

[2] M. A. Alejo, and C. Muñoz, *Almost sharp nonlinear scattering in one-dimensional Born-Infeld equations arising in nonlinear Electrodynamics*, Proc. AMS 146 (2018), no. 5, 2225–2237.

[3] L.V. Bogdanov and V.E. Zakharov, *The Boussinesq equation revisited*, Physica D: Nonlinear Phenomena 165 (2002), 137–162.

[4] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear. Sci. Vol. 12: pp. 283–318 (2002).

[5] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory*, Nonlinearity 17 (2004) 925–952.

[6] J. L. Bona, and R. L. Sachs, *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation.* Comm. Math. Phys. 118 (1988), no. 1, 15–29.

[7] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pure Appl. (2) 17 (1872), 55–108.

[8] S.–M. Chang, S. Gustafson, K. Nakanishi, and Tai–Peng Tsai, *Spectra of linearized operators for NLS solitary waves*, SIAM J. Math. Anal., 39(4), 1070–1111.

[9] C. Charlier and J. Lennells, *The "good" Boussinesq equation: A Riemann-Hilbert approach*, `https://arxiv.org/abs/2003.02777`.

[10] R. Côte, C. Muñoz, D. Pilod, and G. Simpson, *Asymptotic stability of high-dimensional Zakharov–Kuznetsov solitons*, Arch Rational Mech Anal 220, 639–710 (2016).

[11] G.E. Fal'kovich, M.D. Spector, S.K. Turitsyn, *Destruction of stationary solutions and collapse in the nonlinear string equation*, Physics Letters A, 99, Issues 6-7, (1983), 271–

274.

[12] L. Greenberg, *An oscillation method for fourth-order, selfadjoint, two-point boundary value problems with nonlinear eigenvalues*, SIAM Journal on Mathematical Analysis 1991 22:4, 1021–1042.

[13] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, Journal of Functional Analysis, 74, 160-197 (1987).

[14] F. Linares, *Notes on Boussinesq Equation*, available at `http://preprint.impa.br/FullText/Linares__Fri_Dec_23_09_48_59_BRDT_2005/beq.pdf`, 71pp. (2005).

[15] V.K. Kalantarov, O.A. Ladyzhenskaya , *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types*, J. Sov. Math.10, 53–70 (1978).

[16] N. Kishimoto, *Sharp local well-posedness for the "good" Boussinesq equation*, J. Differential Equations, 254 (2013), 2393–2433.

[17] M. Kowalczyk, Y. Martel, and C. Muñoz, *Kink dynamics in the ϕ^4 model: Asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. 30 (2017), 769-798.

[18] M. Kowalczyk, Y. Martel, and C. Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, to appear in J. Eur. Math. Soc.

[19] M. Kowalczyk, Y. Martel, C. Muñoz, and Hanne Van Den Bosch, *A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models*, `https://arxiv.org/abs/2008.01276`.

[20] J. Maddocks and R.L Sachs *On the stability of KdV multi–solitons*, Comm. Pure Appl. Math., 46: 867-901.

[21] B. Li, M. Ohta, Y. Wu, and J. Xue, *Instability of the solitary waves for the generalized Boussinesq equations*, SIAM J. Math. Anal., 52(4), 3192–3221.

[22] F. Linares, *Global existence of small solutions for a generalized Boussinesq equation*, J. Diff. Eqns. 106 (1993), 257–293.

[23] Y. Liu, *Instability and blow-up of solutions to a generalized Boussinesq equation*, SIAM J. Math. Anal., 26(6), 1527–1546.

[24] Y. Martel, *Linear problems related to asymptotic stability of solitons of the generalized KdV equations*, SIAM J. Math. Anal., 38(3), 759-781.

[25] Y. Martel, and F. Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation,* J. Math. Pures Appl. (9) 79 (2000), no. 4, 339–425.

[26] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations* Arch. Ration. Mech. Anal. 157 (2001), no. 3, 219–254.

[27] Y. Martel, and F. Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity 18 (2005), no. 1, 55–80.

[28] H. P. McKean, *Boussinesq's equation on the circle*, Comm. Pure Appl. Math., 34 (1981), pp. 599–691.

[29] C. Maulén, and C. Muñoz, *Decay in the one dimensional generalized Improved Boussinesq equation*, SN Partial Differential Equations and Applications volume 1, Article number: 1 (2020).

[30] C. Muñoz, F. Poblete and J.C. Pozo, *Scattering in the energy space for Boussinesq equations*. Commun. Math. Phys. 361, 127–141 (2018).

[31] Scott Russell, *Report of waves*, 1844.

[32] R. Pego, M. Weinstein, *Eigenvalues, and instabilities of solitary waves*. Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[33] R. Pego, M. Weinstein, *Convective Linear Stability of Solitary Waves for Boussinesq Equations*, Studies in Applied Mathematics, 99, pp. 311-375 (1997).

[34] M. I. Weinstein, *Modulational Stability of Ground States of Nonlinear Schrödinger Equations*, SIAM J. Math. Anal., 16(3), 472–491.

[35] G. Whitham, *Linear and nonlinear waves*, Pure and Applied Mathematics, John Wiley, 1974, 636pp.

[36] V.E Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillations*, Soviet Phys. JEPT 38 (1974), 108-110.

Part III

High Energy Models

Chapter 5

On the decay problem for the Skyrme wave maps

Abstract. We consider the decay problem for the Skyrme and Adkins-Nappi equations. We prove that the energy associated to any bounded energy solution of the Skyrme (or Adkins-Nappi) equation decays to zero outside the light cone (in the radial coordinate). Furthermore, we prove that suitable polynomial weighted energies of any small solution decays to zero when these energies are bounded. The proof consists of finding three new virial type estimates, one for the exterior of the light cone, based on the energy of the solution, and a more subtle virial identity for the weighted energies, based on a modification of momentum type quantities.

Contents

5.1 Introduction **155**
- 5.1.1 Main results 155
- 5.1.2 Idea of the proof 158

5.2 Virial Identities **158**
- 5.2.1 Virial identities for the Skyrme Model 159
- 5.2.2 Virial identities for the Adkins-Nappi Model 164

5.3 Decay in exterior light cones for the Skyrme and Adkins-Nappi models **167**
- 5.3.1 Proof of Theorem 5.1: Skyrme and Adkins-Nappi equations 169

5.4 Decay of weighted energies **170**
- 5.4.1 Growth rate for the modified energy in the Skyrme and Adkins-Nappi equations 170
- 5.4.2 Decay to zero for modified Energies: Proof of the Theorem 5.2 . . 171
- 5.4.3 End of proof of Theorem 5.2 181

Bibliography **182**

This paper was done in collaboration with Miguel Angel Alejo (Universidad de Córdoba, Spain), who was partially supported by project supported by CNPq grant no. 305205/2016-1.

5.1 Introduction

This work is concerned with two *nonlinear* quantum field models, also known in the literature as *Skyrme and Adkins-Nappi equations*. Physically these models intend to describe interactions between nucleons and π mesons. Classical nonlinear field theories played an important role in the description of particles as solitonic objects. A well known example of these nonlinear theories is the $SU(2)$ sigma model [9], obtained as a formal critical point from the action

$$S(\psi) = \int_{\mathbb{R}^{1,\mathrm{d}}} \eta^{\mu\nu}(\psi^* g)_{\mu\nu} = \int_{\mathbb{R}^{1,\mathrm{d}}} \eta^{\mu\nu}\partial_\mu\psi^A\partial_\nu\psi^B g_{AB}\circ\psi. \tag{5.1}$$

Here ψ is a map from a $(1+\mathrm{d})$-dimensional Minkowski space $(\mathbb{R}^{1,\mathrm{d}}, \eta)$ to a Riemannian manifold (M, g) with metric g. From a geometrical point of view, the associated Lagrangian is the trace of the pull-back of the metric g under the map ψ. A current choice is $M = \mathbb{S}^{\mathrm{d}}$ with g the associated metric and for $\mathrm{d} = 3$, one obtains the classical $SU(2)$ sigma model. The Euler-Lagrange equation corresponding to the action S is called the wave maps equation. Unfortunately, the $SU(2)$ sigma model does not admit solitons and it develops singularities in finite time [4, 8, 15]. To avoid these inconveniences and to prevent the possible breakdown of the system in finite time, Skyrme [16] modified the associated Lagrangian to (5.1) by adding higher-order terms which break the scaling invariance of the initial model, which in spherical coordinates (t, r, θ, φ) on $\mathbb{R}^{1,3}$, and co-rotational maps $\psi(t, r, \theta, \varphi) = (u(t, r), \theta, \varphi)$, the Skyrme model leads to the scalar quasilinear wave equation satisfied by the angular variable u, as it will be shown in (5.2).

This equation has a *unique* static solution with boundary values $u(0) = 0$ and $\lim_{r\to\infty} u(r) = \pi$, and which is currently known as *Skyrmion* [14]. This existence was proved in [10] and [14] by using variational methods and ODE techniques respectively. As far as we know, the Skyrmion is not known in a closed form.

5.1.1 Main results

In this paper, we are interested in the long time asymptotics of two relevant mathematical physics models. First, the Skyrme model is written as

$$\left(1 + \frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_{tt} - u_{rr}) - \frac{2}{r}u_r + \frac{\sin(2u)}{r^2}\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right] = 0, \tag{5.2}$$

and the second model is a short of generalization of supercritical wave maps as it was presented by Adkins and Nappi [1]. This is a simplified version of the Skyrme model (5.2) and it is currently known as Adkins-Nappi model

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin(2u)}{r^2} + \frac{(u - \sin(u)\cos(u))(1 - \cos(2u))}{r^4} = 0. \tag{5.3}$$

These two models have the following low order conserved quantities (subindices "S" and "AN" for Skyrme and Adkins-Nappi models respectively)

$$E_S[u](t) = \int_0^\infty r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right], \tag{5.4}$$

$$E_{AN}[u](t) = \int_0^\infty r^2 \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4} \right]. \tag{5.5}$$

(Here $\int_0^\infty$ means $\int_0^\infty \mathrm{d}r$.) Respecting to the Cauchy problem, (5.2) is globally well-posed for small data in $\dot{H}^{5/2}(\mathbb{R}^3)$, and the corresponding global result for the Adkins-Nappi equation (5.3) holds in $(\dot{B}^{5/2} \times \dot{B}^{3/2}) \cap (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ (see [7]). For large-data global well-posedness results, Li showed that it holds in $H^4(\mathbb{R}^3)$ for Skyrme (5.2) (see [12]).

We denote by $\mathcal{E}_n^X$ the space of all finite energy data of degree n, namely

$$\mathcal{E}_n^X = \left\{ (u, u_t) \middle| E_X[u](t) < \infty, u_0(0) = 0, \quad \lim_{r\to\infty} u_0(r) = n\pi \right\}, \tag{5.6}$$

where here $X = S$ refers to the Skyrme model or when $X = AN$ to the Adkins-Nappi model. In what follows, we consider $(u, u_t) \in \mathcal{E}_0^X$ and such that is a solution of (5.2) or (5.3), respectively.

The main goal of this work is to prove that small global solutions with enough regularity of Skyrme (5.2) and Adkins-Nappi (5.3) equations decay to zero in a certain region of the light cone. Furthermore, we also study the decay of an associated weighted energy for both equations, and which we need them for analyzing their corresponding long time behavior.

More precisely let $b > 0$ and consider the following subset depending on time

$$R(t) = \{x \in \mathbb{R}^3 \mid |x| > (1+b)t\} \subset \mathbb{R}^3. \tag{5.7}$$

We will show that any global solution u to (5.2) (or (5.3)), which is sufficiently regular and without a previous smallness condition, must be concentrated inside the light cone.

Theorem 5.1 (Decay in exterior light cones for the Skyrme and Adkins-Nappi models)**.** *Let $(u_0, u_1) \in \mathcal{E}_0^X$, defined in (5.6), such that u is a global solution, for (5.2) when $X = S$, or (5.3) when $X = AN$, respectively. Then, for $R(t)$ as in (5.7), there is strong decay to zero of the energy E_X, in particular:*

$$\lim_{t\to\infty} \|(u_t(t), u_r(t))\|_{L^2 \times L^2(\mathbb{R}^3 \cap R(t))} = 0. \tag{5.8}$$

Additionally, one has the mild rate of decay for $|\sigma| > 1$:

$$\int_2^\infty \int_0^\infty \mathrm{e}^{-c_0|r+\sigma t|} r^2 (u_t^2 + u_r^2) \mathrm{d}r\mathrm{d}t \lesssim_{c_0} 1. \tag{5.9}$$

Remark 5.1. The spaces $\mathcal{E}_0^X$ are not empty. In fact, for the Skyrme and Adkins-Nappi equations, the corresponding energy is well-defined in the homogeneous Sobolev space $\dot{H}^{7/4} \cap \dot{H}^1(\mathbb{R}^3)$ and in $\dot{H}^{5/3} \cap \dot{H}^1(\mathbb{R}^3)$ respectively.

For the next results, we have to introduce a weighted version of the spaces (5.6). Let $\mathcal{E}_n^{X,\phi}$ the space of all finite ϕ-weighted energy data of degree n

$$\mathcal{E}_n^{X,\phi} = \{(u, u_t) \,|E_{X,\phi}[u](t) < \infty, u_0(0) = 0, \quad u_0(\infty) = n\pi\}, \tag{5.10}$$

where $E_{X,\phi}$ is written for the Skyrme model as

$$E_{S,\phi}[u](t) = \int_0^\infty \phi(r) \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right], \tag{5.11}$$

and for the Adkins-Nappi model as

$$E_{AN,\phi}[u](t) = \int_0^\infty \phi(r) \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4} \right]. \tag{5.12}$$

In fact, one can see, that if $E_{X,r^2}[u](t) = E_X[u](t)$, then $\mathcal{E}_n^{X,r^2} = \mathcal{E}_n^X$, for $X \in \{S, AN\}$.

Our second result shows that the energy E_X associated to any global solution $(u, u_t) \in \mathcal{E}_0^{X,r^n} \cap \mathcal{E}_0^{X,r^{n-1}}$ of (5.2) or (5.3), decays to zero when t goes to infinity. This means that for any global solution u which is sufficiently regular and it satisfies a weighted integrability on r, its energy E_{X,r^n} decays to zero when t goes to infinity for both $X = S$ or $X = AN$ cases.

Theorem 5.2 (Decay of weighted energies)**.** *Let $\delta > 0$ small enough. Let $(u, u_t) \in \mathcal{E}_0^{X,r^n} \cap \mathcal{E}_0^{X,r^{n-1}}$ a global solution of* (5.2) *or* (5.3) *such that*

$$\sup_{t\in\mathbb{R}} E_X[u](t) < \delta, \quad \text{for } X = AN, S. \tag{5.13}$$

Then, the modified energy $E_{X,\varphi}[u](t)$ with $\varphi(r) = r^n$ decays to zero, for $n > 7$ ($X = S$ case) or for $n \in \left[\frac{3+\sqrt{41}}{2}, 10\right]$ ($X = AN$ case), respectively. In particular,

$$\lim_{t\to\infty} \|r^{\frac{n-2}{2}}(u_t, u_r)(t)\|_{L^2\times L^2(\mathbb{R}^3)} = \lim_{t\to\infty} E_{X,r^n}(t) = 0. \tag{5.14}$$

The next remark will be useful in the proof of Theorem 5.2.

Remark 5.2 ([5, 6, 11])**.** Note that finite energy smooth solutions of Skyrme (5.2) and Adkins-Nappi (5.3) equations are uniformly bounded as follows

$$\|u\|_{L^\infty_{t,x}} \leqslant C(E_X[u](0)), \text{where } X \in \{S, AN\}, \tag{5.15}$$

and $C(s) \to 0$ as $s \to 0$.

5.1.2 Idea of the proof

In order to prove Theorem 5.1, we follow some ideas appeared in [2, 3, 13], where decay for Camassa-Holm, Born-Infeld and Improved-Boussinesq models were considered. The main tool in these works was a suitable virial functional for which the dynamic of solutions is converging to zero when it is integrated in time.

In this paper, the new virial functionals give us relevant information about the dynamics of global solutions of Skyrme and Adkins-Nappi equations. Using a proper virial estimate, we prove that the corresponding energies associated to Skyrme and Adkins-Nappi equations decay to zero in the subset $R(t)$

$$R(t) = \{x \in \mathbb{R}^3 | x > (1+b)t\} \subset \mathbb{R}^3,$$

which is the complement of the ball of radius $(1+b)t$, for $b > 0$.

Furthermore, to prove Theorem 5.2, we will study the growth rate of polynomial weight energies of the Skyrme and Adkins-Nappi equations. After that, assuming that their growth is bounded, we will prove that this growth decays zero as t tends to infinity. To prove this result, we introduce a functional associated with a sort of weighted momentum. It happens that the virial identity associated to this functional shows no evidence of good sign conditions, i.e. that the derivative of the functional be negative. Therefore, we have to introduce a new functional as a linear combination of these two viral identities and for which there is a good sign property. This ensures the integrability in time of polynomial weighted energies of degree n. Moreover, it also guarantees the decay of polynomial weighted energy of degree $n+1$ over a subsequence of times. Combining these two facts, we conclude that the polynomial weighted energies, which are bounded, decay to zero as t tends to infinity (over $\mathbb{R}^3$).

Organization of this chapter

This chapter is organized as follows: Section 5.2 is splitted in two subsections where a series of virial identities are presented: in Subsection 5.2.1 and 5.2.2 we show the virial identities used for to prove the decay of the energy in the Skyrme equation (5.2) and (5.3), respectively. Section 5.3 deals with the proof of Theorem 5.1 for the Skyrme and Adkins-Nappi equations. Finally, Section 5.4 deals with the proof of Theorem 5.2 for the Skyrme equation and Adkins-Nappi.

Acknowledgments

We are indebted to C. Muñoz for stimulating discussions and valuable suggestions that helped to improve a previous version of this work. I am deeply thanks F. Gancedo (U. Sevilla) for his hospitality during some research stays where part of this work was done.

5.2 Virial Identities

In this section three virial identities for the Skyrme and Adkins-Nappi models (5.2)-(5.3) are presented. One of the virial functionals is related with the exterior light cone behavior

(Theorem 5.1), and the other ones are useful for understanding the decay of the weighted energy of Skyrme and Adkins-Nappi models (Theorem 5.2). Moreover, we remark here that the energies $E_S[u]$ and $E_{AN}[u]$, defined in (5.4) and (5.5), are bounded in spaces $\mathcal{E}_0^S$ and $\mathcal{E}_0^{AN}$ respectively. Furthermore, it is well-known that these energies are well defined in the homogeneous Sobolev spaces $\dot{H}^{7/4} \cap \dot{H}^1(\mathbb{R}^3)$ and $\dot{H}^{5/3} \cap \dot{H}^1(\mathbb{R}^3)$ for the Skyrme and Adkins-Nappi equations, respectively.

5.2.1 Virial identities for the Skyrme Model

Let $\varphi = \varphi(t,r)$ be a smooth, bounded weight function, to be chosen later. For each $t \in \mathbb{R}$ we consider the following functional

$$\mathcal{I}_S(t) = \int_0^\infty \varphi r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right], \tag{5.16}$$

which is a generalization of the energy introduced in (5.4), and well-defined for $(u, u_t) \in (\dot{H}^{7/4} \cap \dot{H}^1) \times L^2(\mathbb{R}^3)$. Moreover, if φ only depends on r and it can be written as $\varphi(r) = \phi/r^2$, then we recover $E_{S,\phi}$, which is the weighted energy defined in (5.11). The following identities will be useful for the proof of Theorems 5.1-5.2.

The following result shows the variation of the localized energy for the Skyrme equation:

Lemma 5.3 (Energy local variations: Skyrme Model)**.** *For any $t \in \mathbb{R}$, $\varphi(t,r)$ a smooth function previously defined, and $\mathcal{I}_S(t)$ as in* (5.16)*, we have that*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) = & \int_0^\infty \varphi_t r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] \\ & - \int_0^\infty \varphi_r r^2 \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) 2u_t u_r. \end{aligned} \tag{5.17}$$

Proof of Lemma 5.3. Derivating (5.16) with respect to time, and using a basic trigonometric relation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) = & \int_0^\infty \varphi_t r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] \\ & + 2\int_0^\infty \varphi r^2 u_t \left[\left(\frac{\alpha^2 \sin(2u)}{r^2}\right)(u_t^2 + u_r^2) + \frac{\sin(2u)}{r^2} + \frac{\alpha^2 \sin^2(u)\sin(2u)}{r^4} \right] \\ & + 2\int_0^\infty \varphi r^2 u_t \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) u_{tt} + 2\int_0^\infty \varphi r^2 \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) u_r u_{rt} \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now, using the equation (5.2) in I_3, we have

$$I_3 = 2\int_0^\infty \varphi r^2 u_t \left\{ \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) u_{rr} + \frac{2}{r}u_r - \frac{\sin(2u)}{r^2}\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right] \right\}.$$

And integrating by parts in the last integral I_4, we obtain

$$\begin{aligned}\frac{1}{2}I_4 = &-\int_0^\infty \varphi_r r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)u_r u_t\\ &-\int_0^\infty \varphi r^2 u_t\left(\left(\frac{2}{r}u_r+\frac{2\alpha^2\sin(2u)}{r^2}u_r^2\right)-\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)u_{rr}\right).\end{aligned}$$

Finally, we have

$$I_2+I_3+I_4 = -2\int_0^\infty \varphi_r r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)u_t u_r,$$

and we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) = &\int_0^\infty \varphi_t r^2\left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2)+2\frac{\sin^2(u)}{r^2}+\frac{\alpha^2\sin^4(u)}{r^4}\right]\\ &-2\int_0^\infty \varphi_r r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)u_t u_r.\end{aligned}$$

This concludes the proof. ■

Remark 5.3. With the change of variables $\varphi=\phi/r^2$, we avoid the term r^2 in the weighted function (5.16), and which coming from the dimension of the problem. Then, $E_{S,\phi}$ is recovered from $\mathcal{I}_S(t)$ and applying the Lemma 5.3, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{S,\phi} = &\int_0^\infty \phi_t\left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2)+2\frac{\sin^2(u)}{r^2}+\frac{\alpha^2\sin^4(u)}{r^4}\right]\\ &-2\int_0^\infty\left(\phi'-2\frac{\phi}{r}\right)\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)u_t u_r.\end{aligned}\tag{5.18}$$

This relation will be useful in the proof of Theorem 5.2.

Now, we define two functionals that we will use to prove the decay of the weighted energy $E_{X,\phi}$. Firstly, denote by f the following function

$$f(u)=1+\frac{2\alpha^2\sin^2(u)}{r^2}.\tag{5.19}$$

Now, considering ψ and ϕ smooth weight functions of r, which will be chosen later, we define the functional $\mathcal{K}_S(t)$ associated with a sort of momentum, given by

$$\mathcal{K}_S(t)=\int_0^\infty \psi f(u)u_t u_r,\tag{5.20}$$

and the functional $\mathcal{P}_S(t)$, which corrects the bad sign of the variation in time on the functional $\mathcal{K}_S(t)$, and which is given by

$$\mathcal{P}_S(t)=\int_0^\infty \phi f(u)u_t u.\tag{5.21}$$

Lemma 5.4. *Let $t \in \mathbb{R}$, ψ be a smooth weight function and $\mathcal{K}_S(t)$ defined as in (5.20). If $u \in \mathcal{E}_0^{S,\psi}$ and $p(r) = \left(\frac{\psi'}{r^2} - 4\frac{\psi}{r^3}\right)$, then we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_S(t) = & -\frac{1}{2}\int_0^\infty \frac{\psi'}{r^2} r^2 u_t^2 - \frac{1}{2}\int_0^\infty p(r) r^2 u_r^2 - \frac{\alpha^2}{2}\int_0^\infty p(r)\sin^2(u)(u_t^2 + u_r^2) \\ & + \int_0^\infty \left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\sin^2(u) + \frac{\alpha^2}{2}\int_0^\infty p(r)\frac{\sin^4(u)}{r^2}.\end{aligned} \tag{5.22}$$

Proof. First of all, we notice that the derivates of f (5.19) are

$$(f(u))_t = 2\alpha^2 \frac{\sin(2u)u_t}{r^2}, \quad \text{and} \quad (f(u))_r = 2\alpha^2\frac{\sin(2u)u_r}{r^2} - 4\alpha^2\frac{\sin^2(u)}{r^3}. \tag{5.23}$$

Secondly, derivating the functional (5.20) with respect to time, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_S(t) &= \int_0^\infty \psi\left(f(u)\right)_t u_t u_r + \int_0^\infty \psi f(u)(u_{tt}u_r + u_t u_{rt}) \\ &= \int_0^\infty \psi\left(f(u)\right)_t u_t u_r + \int_0^\infty \psi f(u) u_{tt} u_r + \frac{1}{2}\int_0^\infty \psi f(u)(u_t^2)_r.\end{aligned}$$

Integrating by parts in the last term of RHS, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_S(t) = \int_0^\infty \psi\left(f(u)\right)_t u_t u_r + \int_0^\infty \psi f(u) u_{tt} u_r - \frac{1}{2}\int_0^\infty \psi' f(u) u_t^2 \\ - \frac{1}{2}\int_0^\infty \psi (f(u))_r u_t^2 := K_1 + K_2 + K_3 + K_4.\end{aligned} \tag{5.24}$$

For K_2, using (5.2), we obtain

$$\begin{aligned}K_2 &= \int_0^\infty \psi u_r \left\{ f(u)u_{rr} + \frac{2}{r}u_r - \frac{\sin(2u)}{r^2}\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right]\right\} \\ &= -\int_0^\infty \left(\psi f(u)\right)_r \frac{u_r^2}{2} + 2\int_0^\infty \frac{\psi}{r}u_r^2 - \int_0^\infty \frac{\psi}{r^2}\sin(2u)u_r\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right] \\ &= -\int_0^\infty \left[\psi' f(u) + \psi(f(u))_r\right]\frac{u_r^2}{2} + 2\int_0^\infty \frac{\psi}{r}u_r^2 - \int_0^\infty \frac{\psi}{r^2}\sin(2u)u_r\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right],\end{aligned}$$

and replacing (5.23), we get

$$\begin{aligned}K_2 &= -\int_0^\infty \psi' f(u)\frac{u_r^2}{2} - \alpha^2\int_0^\infty \psi\frac{\sin(2u)}{r^2}u_r^3 + 2\alpha^2\int_0^\infty \psi\frac{\sin^2(u)}{r^3}u_r^2 + 2\int_0^\infty \frac{\psi}{r}u_r^2 \\ &\quad - \int_0^\infty \frac{\psi}{r^2}\sin(2u)u_r\left[1 + \alpha^2\left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right] \\ &= -\int_0^\infty \psi' f(u)\frac{u_r^2}{2} + 2\alpha^2\int_0^\infty \psi\frac{\sin^2(u)}{r^3}u_r^2 + 2\int_0^\infty \frac{\psi}{r}u_r^2 \\ &\quad - \int_0^\infty \frac{\psi}{r^2}\sin(2u)u_r\left[1 + \alpha^2\left(u_t^2 + \frac{\sin^2(u)}{r^2}\right)\right].\end{aligned} \tag{5.25}$$

Using (5.25) and (5.23) in (5.24), one can see

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_S(t) = -\frac{1}{2}\int_0^\infty \psi'\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\alpha^2\int_0^\infty \psi\frac{\sin^2(u)}{r^3}(u_t^2+u_r^2)$$
$$+ 2\int_0^\infty \frac{\psi}{r}u_r^2 - \int_0^\infty \frac{\psi}{r^2}(\sin^2(u))_r - \frac{\alpha^2}{2}\int_0^\infty \frac{\psi}{r^4}(\sin^4(u))_r.$$

Finally, integrating by parts and regrouping terms, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_S(t) =& -\frac{1}{2}\int_0^\infty \psi'\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\alpha^2\int_0^\infty \frac{\psi}{r}\frac{\sin^2(u)}{r^2}(u_t^2+u_r^2) + 2\int_0^\infty \frac{\psi}{r}u_r^2\\
&+ \int_0^\infty \frac{\psi'}{r^2}\sin^2(u)\left(1+\frac{\alpha^2}{2}\frac{\sin^2(u)}{r^2}\right) - 2\int_0^\infty \frac{\psi}{r^3}\sin^2(u)\left(1+\alpha^2\frac{\sin^2(u)}{r^2}\right)\\
=& -\frac{1}{2}\int_0^\infty \frac{\psi'}{r^2}r^2u_t^2 - \frac{1}{2}\int_0^\infty p(r)r^2u_r^2 - \frac{\alpha^2}{2}\int_0^\infty p(r)\sin^2(u)(u_t^2+u_r^2)\\
&+ \int_0^\infty \left(\frac{\psi'}{r^2}-2\frac{\psi}{r^3}\right)\sin^2(u) + \frac{\alpha^2}{2}\int_0^\infty p(r)\frac{\sin^4(u)}{r^2},\end{aligned}$$

and we conclude. ■

Similarly, we have the following result for the correction term $\mathcal{P}_S(t)$ (5.21).

Lemma 5.5. *Let $t \in \mathbb{R}$, ϕ be a smooth weight function and $\mathcal{P}_S(t)$ as in (5.21). Then, if $u \in \mathcal{E}_0^{S,\phi}$, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_S(t) =& \ \alpha^2\int_0^\infty \frac{\phi}{r^2}\left(\sin(2u)u + 2\sin^2(u)\right)(u_t^2-u_r^2) + \int_0^\infty \phi(u_t^2-u_r^2)\\
&+ \alpha^2\int_0^\infty \left(r\phi''-4\phi'+6\frac{\phi}{r}\right)\frac{\sin^2(u)}{r^3}u^2 - \alpha^2\int_0^\infty \frac{\phi}{r}\frac{u\sin(2u)\sin^2(u)}{r^3}\\
&+ \alpha^2\int_0^\infty \left(\phi'-2\frac{\phi}{r}\right)\frac{\sin(2u)}{r^2}u_ru^2 - \int_0^\infty \left[\left(\frac{\phi}{r}\right)_r - \frac{1}{2}\phi''\right]u^2 - \int_0^\infty \phi\frac{\sin(2u)}{r^2}u.\end{aligned} \tag{5.26}$$

Proof. Derivating the functional (5.21) with respect to time, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_S(t) =& \int_0^\infty \phi\,(f(u))_t\,u_tu + \int_0^\infty \phi f(u)(u_{tt}u+u_t^2)\\
=& \ 2\alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}u_t^2u \ + \int_0^\infty \phi f(u)u_t^2 + \int_0^\infty \phi uf(u)u_{tt}u\\
:=& P_1+P_2+P_3.\end{aligned} \tag{5.27}$$

Using (5.2) in P_3, we get

$$P_3 = \int_0^\infty \phi u\left\{f(u)u_{rr} + \frac{2}{r}u_r - \frac{\sin(2u)}{r^2}\left[1+\alpha^2\left(u_t^2-u_r^2+\frac{\sin^2(u)}{r^2}\right)\right]\right\}.$$

Integrating by parts the first term on the RHS, we get

$$\begin{aligned}\int_0^\infty \phi f(u)uu_{rr} &= -\int_0^\infty \phi' f(u)uu_r - \int_0^\infty \phi(f(u))_r uu_r - \int_0^\infty \phi f(u)u_r^2 \\ &= \frac{1}{2}\int_0^\infty (\phi'' f(u) + \phi'(f(u))_r)u^2 - \int_0^\infty \phi(f(u))_r uu_r - \int_0^\infty \phi f(u)u_r^2.\end{aligned}$$

Having in mind derivatives in (5.23), we get

$$\begin{aligned}\int_0^\infty \phi f(u)uu_{rr} &= \frac{1}{2}\int_0^\infty \phi'' f(u)u^2 + \alpha^2\int_0^\infty \phi'\left(\frac{\sin(2u)u_r}{r^2} - 2\frac{\sin^2(u)}{r^3}\right)u^2 \\ &\quad - 2\alpha^2\int_0^\infty \phi\left(\frac{\sin(2u)u_r}{r^2} - 2\frac{\sin^2(u)}{r^3}\right)uu_r - \int_0^\infty \phi f(u)u_r^2 \\ &= \frac{1}{2}\int_0^\infty \phi'' f(u)u^2 + \alpha^2\int_0^\infty \phi'\frac{\sin(2u)}{r^2}u_r u^2 - 2\alpha^2\int_0^\infty \phi'\frac{\sin^2(u)}{r^3}u^2 \\ &\quad - 2\alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}uu_r^2 + 4\alpha^2\int_0^\infty \phi\frac{\sin^2(u)}{r^3}uu_r - \int_0^\infty \phi f(u)u_r^2.\end{aligned} \tag{5.28}$$

Now, integrating by parts the second term in the above line in the RHS, we obtain

$$\begin{aligned}4\alpha^2\int_0^\infty \phi\frac{\sin^2(u)}{r^3}uu_r &= -2\alpha^2\int_0^\infty \left(\phi\frac{\sin^2(u)}{r^3}\right)_r u^2 \\ &= -2\alpha^2\int_0^\infty \phi'\frac{\sin^2(u)}{r^3}u^2 - 2\alpha^2\int_0^\infty \phi\left(\frac{\sin(2u)u_r}{r^3} - 3\frac{\sin^2(u)}{r^4}\right)u^2.\end{aligned} \tag{5.29}$$

Then, substituting into (5.28), we get

$$\begin{aligned}P_3 = {} & \frac{1}{2}\int_0^\infty \phi'' f(u)u^2 + \alpha^2\int_0^\infty \left(-4\phi' + 6\frac{\phi}{r}\right)\frac{\sin^2(u)}{r^3}u^2 - \alpha^2\int_0^\infty \frac{\phi}{r}\frac{u\sin(2u)\sin^2(u)}{r^3} \\ & + \alpha^2\int_0^\infty \left(\phi' - 2\frac{\phi}{r}\right)\frac{\sin(2u)}{r^2}u_r u^2 - \int_0^\infty \phi\left(\alpha^2\frac{\sin(2u)}{r^2}u + f(u)\right)u_r^2 \\ & - \int_0^\infty \left(\frac{\phi}{r}\right)_r u^2 - \int_0^\infty \phi\frac{\sin(2u)}{r^2}u - \alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}uu_t^2.\end{aligned}$$

Replacing (5.19) and regrouping again, we obtain

$$\begin{aligned}P_3 = {} & \alpha^2\int_0^\infty \left(r\phi'' - 4\phi' + 6\frac{\phi}{r}\right)\frac{\sin^2(u)}{r^3}u^2 - \alpha^2\int_0^\infty \frac{\phi}{r}\frac{u\sin(2u)\sin^2(u)}{r^3} \\ & + \alpha^2\int_0^\infty \left(\phi' - 2\frac{\phi}{r}\right)\frac{\sin(2u)}{r^2}u_r u^2 - \alpha^2\int_0^\infty \frac{\phi}{r^2}\left(\sin(2u)u + 2\sin^2(u)\right)u_r^2 \\ & - \int_0^\infty \left(\frac{\phi}{r}\right)_r u^2 + \frac{1}{2}\int_0^\infty \phi'' u^2 - \int_0^\infty \phi\frac{\sin(2u)}{r^2}u - \int_0^\infty \phi u_r^2 - \alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}uu_t^2.\end{aligned} \tag{5.30}$$

Collecting P_3 in (5.30), (5.27), and using (5.19), we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_S(t) =\ & 2\alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}u_t^2 u + \int_0^\infty \phi f(u)u_t^2 + \alpha^2\int_0^\infty \left(r\phi'' - 4\phi' + 6\frac{\phi}{r}\right)\frac{\sin^2(u)}{r^3}u^2 \\
& - \alpha^2\int_0^\infty \frac{\phi}{r}\frac{u\sin(2u)\sin^2(u)}{r^3} \\
& + \alpha^2\int_0^\infty \left(\phi' - 2\frac{\phi}{r}\right)\frac{\sin(2u)}{r^2}u_r u^2 - \alpha^2\int_0^\infty \frac{\phi}{r^2}\left(\sin(2u)u + 2\sin^2(u)\right)u_r^2 \\
& - \int_0^\infty \left(\frac{\phi}{r}\right)_r u^2 + \frac{1}{2}\int_0^\infty \phi'' u^2 - \int_0^\infty \phi\frac{\sin(2u)}{r^2}u - \int_0^\infty \phi u_r^2 - \alpha^2\int_0^\infty \phi\frac{\sin(2u)}{r^2}uu_t^2.
\end{aligned}
$$

Finally, regrouping terms,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_S(t) =\ & \alpha^2\int_0^\infty \frac{\phi}{r^2}\left(\sin(2u)u + 2\sin^2(u)\right)(u_t^2 - u_r^2) + \int_0^\infty \phi(u_t^2 - u_r^2) \\
& + \alpha^2\int_0^\infty \left(r\phi'' - 4\phi' + 6\frac{\phi}{r}\right)\frac{\sin^2(u)}{r^3}u^2 - \alpha^2\int_0^\infty \frac{\phi}{r}\frac{u\sin(2u)\sin^2(u)}{r^3} \\
& + \alpha^2\int_0^\infty \left(\phi' - 2\frac{\phi}{r}\right)\frac{\sin(2u)}{r^2}u_r u^2 - \int_0^\infty \left[\left(\frac{\phi}{r}\right)_r - \frac{1}{2}\phi''\right]u^2 - \int_0^\infty \phi\frac{\sin(2u)}{r^2}u.
\end{aligned}
$$

We conclude the proof. ■

5.2.2 Virial identities for the Adkins-Nappi Model

Let $\rho = \rho(t,r)$ a smooth, weight function, to be chosen later. Similarly to the previous section, for the Adkins-Nappi equation we introduce a suitable functional, as a weighted generalization of the energy (5.5), and given by

$$
\mathcal{I}_{AN}(t) = \int_0^\infty \rho r^2\left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4}\right], \text{ for each } t \in \mathbb{R}. \tag{5.31}
$$

Recalling Remark 5.1, if ρ is a bounded function, the functional $\mathcal{I}_{AN}(t)$ is well-defined for $(u,u_t) \in (\dot{H}^{5/3} \cap \dot{H}^1 \times L^2)(\mathbb{R}^3)$. The following result describes the time variation of (5.31).

Lemma 5.6 (Energy local variations: Adkins-Nappi Model)**.** *For any $t \in \mathbb{R}$, one has*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{AN}(t) = \int_0^\infty \rho_t r^2\left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4}\right] - \int_0^\infty \rho_r 2r^2 u_t u_r. \tag{5.32}
$$

Proof. Derivating the functional (5.31) with respect to time and using basic trigonometric identities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{AN}(t) = \int_0^\infty \rho_t r^2 \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u-\sin(u)\cos(u))^2}{r^4} \right] + 2\int_0^\infty \rho r^2 u_r u_{rt}$$
$$+ \underbrace{\int_0^\infty 2\rho u_t r^2 \left[u_{tt} + \frac{\sin(2u)}{r^2} + \frac{(u-\sin(u)\cos(u))\left(1+\sin^2(u)-\cos^2 u\right)}{r^4} \right]}_{J_1}. \tag{5.33}$$

Now, using the equation (5.3) and integrating by parts in J_1, we have

$$\begin{aligned} J_1 &= \int_0^\infty 2\rho u_t r^2 \left[u_{rr} + \frac{2}{r}u_r \right] \\ &= \int_0^\infty 4\rho r u_t u_r - \int_0^\infty 2\Big(\rho_r r^2 u_t + \rho 2r u_t + \rho r^2 u_{tr}\Big) u_r. \end{aligned}$$

Finally, substituting J_1 in (5.33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{AN}(t) = \int_0^\infty \rho_t r^2 \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u-\sin(u)\cos(u))^2}{r^4} \right] - 2\int_0^\infty \rho_r r^2 u_t u_r. \tag{5.34}$$

This ends the proof of the lemma. ■

Remark 5.4. Similarly to Remark 5.3, using the change of variables $\rho = \phi/r^2$, the term r^2 in $\mathcal{I}_{AN}(t)$ is avoided, and therefore recovering the functional $E_{AN,\phi}[u]$ (5.12). Furthermore, by Lemma 5.6 we have the following identity for the time evolution of $E_{AN,\phi}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{AN,\phi}(t) = &\int_0^\infty \phi_t \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u-\sin(u)\cos(u))^2}{r^4} \right] \\ &- 2\int_0^\infty \left(\phi_r - 2\frac{\phi}{r}\right) u_t u_r. \end{aligned} \tag{5.35}$$

This relation will be useful in the proof of Theorem 5.2.

Now, let ψ and ϕ smooth weight functions of r, which will be chosen later. We define the functional $\mathcal{M}_{AN}(t)$ associated with a sort of momentum, given by

$$\mathcal{M}_{AN}(t) = \int_0^\infty \psi u_t u_r. \tag{5.36}$$

and the functional $\mathcal{R}_{AN}(t)$, which is the term that corrects the bad sign of the variation on the functional $\mathcal{M}_{AN}(t)$, given by

$$\mathcal{R}_{AN}(t) = \int_0^\infty \phi u_t u. \tag{5.37}$$

The following results show the time variation of these functionals, which will be used in the proof of Theorem 5.2.

Lemma 5.7. *Let $t \in \mathbb{R}$, ψ be a smooth weight function and $\mathcal{M}_{AN}(t)$ as in (5.36). Then, if $u \in \mathcal{E}_0^{AN,\psi}$, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{AN}(t) =& -\frac{1}{2}\int_0^\infty \psi' u_t^2 - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2 - \frac{1}{2}\int_0^\infty \left(2\frac{\psi}{r} - \psi'\right)\frac{\sin^2(u)}{r^2} \\ & - \frac{1}{2}\int_0^\infty \left(4\frac{\psi}{r} - \psi'\right)\frac{(u - \sin(u)\cos(u))^2}{r^4}.\end{aligned} \tag{5.38}$$

Proof. Just derivating the functional (5.36) with respect to time, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{AN}(t) =& \int_0^\infty \psi(u_{tt}u_r + u_t u_{rt}) \\ =& -\frac{1}{2}\int_0^\infty \psi' u_t^2 + \int_0^\infty \psi u_{tt} u_r := M_1 + M_2.\end{aligned}$$

For M_2, using (5.3) and integrating by parts, we have

$$\begin{aligned}M_2 =& \int_0^\infty \psi u_r \left(u_{rr} + \frac{2}{r}u_r\right) - \int_0^\infty \psi u_r \left(\frac{\sin(2u)}{r^2} + \frac{(u - \sin(u)\cos(u))\,(1 - \cos(2u))}{r^4}\right) \\ =& -\int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2 - \int_0^\infty \psi u_r \left(\frac{\sin(2u)}{r^2} + \frac{(u - \sin(u)\cos(u))\,(1 - \cos(2u))}{r^4}\right) \\ =& \; M_{21} + M_{22}.\end{aligned}$$

With respect to M_{22}, note that rewriting it and integrating by parts, we obtain

$$\begin{aligned}M_{22} =& -\frac{1}{2}\int_0^\infty \frac{\psi}{r^2}(\sin^2(u))_r - \frac{1}{2}\int_0^\infty \frac{\psi}{r^4}\left((u - \sin(u)\cos(u))^2\right)_r \\ =& -\frac{1}{2}\int_0^\infty \left(2\frac{\psi}{r} - \psi'\right)\frac{\sin^2(u)}{r^2} - \frac{1}{2}\int_0^\infty \left(4\frac{\psi}{r} - \psi'\right)\frac{(u - \sin(u)\cos(u))^2}{r^4}.\end{aligned}$$

Finally, collecting M_1, M_{21}, and M_{22}, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{AN}(t) =& -\frac{1}{2}\int_0^\infty \psi' u_t^2 - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2 - \frac{1}{2}\int_0^\infty \left(2\frac{\psi}{r} - \psi'\right)\frac{\sin^2(u)}{r^2} \\ & - \frac{1}{2}\int_0^\infty \left(4\frac{\psi}{r} - \psi'\right)\frac{(u - \sin(u)\cos(u))^2}{r^4}.\end{aligned}$$

This ends the proof of this lemma. ■

Lemma 5.8. *Let $t \in \mathbb{R}$, ϕ be a smooth weight function and $\mathcal{R}_{AN}(t)$ as in (5.37). Then, if $u \in \mathcal{E}_0^{AN,\phi/r}$, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_{AN}(t) =& \int_0^\infty \phi u_t^2 - \int_0^\infty \left[\phi' r - \phi - \frac{r^2\phi_{rr}}{2}\right]\frac{u^2}{r^2} - \int_0^\infty \phi u_r^2 - \int_0^\infty \frac{\phi}{r^2} u \sin(2u) \\ & - \int_0^\infty \frac{\phi}{r^4} u\,(u - \sin(u)\cos(u))\,(1 - \cos(2u)).\end{aligned} \tag{5.39}$$

Proof. Just derivating the functional (5.37) with respect to time, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_{AN}(t) = \int_0^\infty \phi u_t^2 + \int_0^\infty \phi u_{tt} u := R_1 + R_2. \tag{5.40}$$

For R_2, using (5.3) and integrating by parts, we get

$$\begin{aligned} R_2 &= -\int_0^\infty (\phi' u + \phi u_r) u_r - \int_0^\infty \left(\frac{\phi}{r}\right)_r u^2 \\ &\quad - \int_0^\infty \phi u \left(\frac{\sin(2u)}{r^2} + \frac{(u - \sin(u)\cos(u))(1 - \cos(2u))}{r^4}\right) \\ &= \int_0^\infty \frac{\phi_{rr}}{2} u^2 - \int_0^\infty \phi u_r^2 - \int_0^\infty \left(\frac{\phi}{r}\right)_r u^2 \\ &\quad - \int_0^\infty \phi u \left(\frac{\sin(2u)}{r^2} + \frac{(u - \sin(u)\cos(u))(1 - \cos(2u))}{r^4}\right). \end{aligned}$$

Regrouping the terms, we obtain

$$\begin{aligned} R_2 &= \int_0^\infty \left(\frac{\phi_{rr} r^2}{2} - r\phi' + \phi\right) \frac{u^2}{r^2} - \int_0^\infty \phi u_r^2 \\ &\quad - \int_0^\infty \left(\frac{\phi}{r^2} u \sin(2u) + \frac{\phi}{r^4}(u - \sin(u)\cos(u))(1 - \cos(2u)) u\right). \end{aligned} \tag{5.41}$$

Then, substituting (5.41) in (5.40), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_{AN}(t) &= \int_0^\infty \phi u_t^2 - \int_0^\infty \left[\phi' r - \phi - \frac{r^2 \phi_{rr}}{2}\right] \frac{u^2}{r^2} - \int_0^\infty \phi u_r^2 \\ &\quad - \int_0^\infty \frac{\phi}{r^2} u \sin(2u) - \int_0^\infty \frac{\phi}{r^4} u (u - \sin(u)\cos(u))(1 - \cos(2u)). \end{aligned}$$

This concludes the proof of the lemma. ■

5.3 Decay in exterior light cones for the Skyrme and Adkins-Nappi models

This section deals with the proof of Theorems 5.1 for the Skyrme and Adkins-Nappi equations. In what follows, fix $\sigma \in \mathbb{R}$ such that $|\sigma| > 1$. Recalling the identity (5.17) and using the weight function $\varphi = \varphi\left(\frac{r + \sigma t}{L}\right)$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) &= \frac{\sigma}{L}\int_0^\infty \varphi' r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4}\right] \\ &\quad - \frac{1}{L}\int_0^\infty \varphi' r^2 \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) 2u_t u_r. \end{aligned} \tag{5.42}$$

Furthermore, from Lemma 5.6, we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{AN}(t) &= \frac{\sigma}{L}\int_0^\infty \varphi' r^2 \left[u_t^2 + u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u)\cos(u))^2}{r^4}\right] \\ &\quad - \frac{1}{L}\int_0^\infty \varphi' 2r^2 u_t u_r. \end{aligned} \tag{5.43}$$

Now, we are ready to prove a first virial estimate.

Lemma 5.9. *Let $L > 0$, $\sigma = -(1+b) < -1$, and $\rho = \varphi = \tanh\left(\dfrac{r+\sigma t}{L}\right)$. Then*

1. *for the Skyrme equation, we get*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) \lesssim_{L,b} -\int_0^\infty \varphi' r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right]. \tag{5.44}$$

2. *for the Sgkins-Nappi equation, we get*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{AN}(t) \lesssim_{L,b} -\int_0^\infty \rho' r^2 \left[u_t^2+u_r^2 + 2\frac{\sin^2(u)}{r^2} + \frac{(u-\sin(u)\cos(u))^2}{r^4}\right]. \tag{5.45}$$

Proof. Firstly we prove (5.44). Focusing on the last term in RHS of (5.42), note that, if $\varphi' > 0$, then using a Cauchy-Schwarz inequality, we have

$$\left|\int_0^\infty \varphi' r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)2u_t u_r\right| \leqslant \int_0^\infty \varphi' r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2).$$

Therefore, if $b > 0$, $\sigma = -(1+b) < -1$, and $\varphi = \tanh$, we have from (5.42)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) \leqslant{}& \frac{\sigma}{L}\int_0^\infty \varphi' r^2\left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right] \\ &+ \frac{1}{L}\int_0^\infty \varphi' r^2\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2).\end{aligned}$$

Consequently, we obtain (5.44)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) \lesssim_{L,b} -\int_0^\infty |\varphi'| r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right]. \tag{5.46}$$

The proof of (5.45) proceeds in a similar way. Only note that the last term in (5.43) holds the following inequality

$$\left|\int_0^\infty \rho' r^2 2u_t u_r\right| \leqslant \int_0^\infty \rho' r^2(u_t^2+u_r^2), \text{ for } \rho' > 0,$$

the rest of the proof follows the same lines as in the Skyrme case and hence, for the sake of simplicity, we do not show it here.

Finally, we can observe that integrating in time on (5.44) and (5.45), we have proved (5.9) in Theorem 5.1. ■

5.3.1 Proof of Theorem 5.1: Skyrme and Adkins-Nappi equations

Firstly, we focus on the Skyrme case. It only remains to prove (5.8). We must to show decay in the right hand side region, namely $((1+b)t, +\infty)$, $b > 0$. Now we choose $\varphi(r) = \frac{1}{2}(1+\tanh(r))$, $\sigma = -(1+b)$, $\tilde{\sigma} = -(1+b/2)$ with $b > 0$. Consider the modified energy functional, for $t \in [2, t_0]$:

$$\mathcal{I}_{S,t_0}(t) := \frac{1}{2}\int_0^\infty \varphi\left(\frac{r+\sigma t_0 - \tilde{\sigma}(t_0-t)}{L}\right) r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right].$$

Note that $\sigma < \tilde{\sigma} < 0$. From Lemma 5.3 and proceeding exactly as in (5.44), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{S,t_0}(t) \lesssim_{b,L} -\int_0^\infty \operatorname{sech}^2\left(\frac{r+\sigma t_0 - \tilde{\sigma}(t_0-t)}{L}\right) r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right] \leqslant 0,$$

which means that the new functional $\mathcal{I}_{S,t_0}$ is decreasing in $[2, t_0]$. Therefore, we have

$$\int_2^{t_0} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{S,t_0}(t)\mathrm{d}t = \mathcal{I}_{S,t_0}(t_0) - \mathcal{I}_{S,t_0}(2) \leqslant 0 \implies \mathcal{I}_{S,t_0}(t_0) \leqslant \mathcal{I}_{S,t_0}(2).$$

On the other hand, since $\lim_{x\to-\infty}\varphi(x) = 0$, we have

$$\limsup_{t\to\infty}\int_0^\infty \varphi\left(\frac{r-\beta t-\gamma}{L}\right) r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right](\nu, r) = 0,$$

for $\beta, \gamma, \nu > 0$ fixed. This yields

$$\begin{aligned} 0 &\leqslant \int_0^\infty \varphi\left(\frac{r-(1+b)t_0}{L}\right) r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right](t_0, r) \\ &\leqslant \int_0^\infty \varphi\left(\frac{r-\frac{b}{2}t_0-(2+b)}{L}\right) r^2 \left[\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right](2, r), \end{aligned}$$

which implies,

$$\limsup_{t\to\infty}\int_0^\infty \varphi\left(\frac{r-(1+b)t}{L}\right) r^2 \left(\left(1+\frac{2\alpha^2\sin^2(u)}{r^2}\right)(u_t^2+u_r^2) + 2\frac{\sin^2(u)}{r^2} + \frac{\alpha^2\sin^4(u)}{r^4}\right)(t, r)\mathrm{d}r = 0.$$

This means that the energy over $R(t)$ (see (5.7)) converges to zero, implying (5.8) and then we conclude the Skyrme case.

For the Adkins-Nappi case, the proof is analogous but this time considering the modified energy functional

$$\mathcal{I}_{AN,t_0}(t) := \int_0^\infty \rho\left(\frac{r+\sigma t_0-\tilde{\sigma}(t_0-t)}{L}\right) r^2\left[u_t^2+u_r^2+2\frac{\sin^2(u)}{r^2}+\frac{(u-\sin(u)\cos(u))^2}{r^4}\right],$$

and repeating the same steps as in the Skyrme case. This concludes the proof of Theorem 5.1.

5.4 Decay of weighted energies

Firstly, we study the growth rate of the modified energies introduced in (5.11) and (5.12).

5.4.1 Growth rate for the modified energy in the Skyrme and Adkins-Nappi equations

In this section we study the growth rate for the power type weighted energy of the Skyrme and Adkins-Nappi equations

Proposition 5.10. *Let u a global solution of (5.2) (or (5.3)) such that $u \in \bigcap_{i=2}^{n}\mathcal{E}_0^{X,r^i}$, for $X = S$ or $X = AN$. Then the corresponding weighted energy satisfies*

$$E_{X,r^n}[u](t) = O(t^{n-2}), \tag{5.47}$$

where $E_{X,r^n}[u](t)$ is given in (5.11) and (5.12), respectively .

Proof. Firstly, we consider $X = S$. We note that for $\varphi = \phi/r^2$, we get

$$\mathcal{I}_S(t) = E_{S,\phi}[u](t). \tag{5.48}$$

Then, using (5.35) with $\phi = r^n$, one can see

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_S(t) = -2\mathcal{K}_S(t),$$

where $\mathcal{K}_S(t)$ is given by (5.20) and $\psi = \phi' - 2\frac{\phi}{r} = (n-2)r^{n-1}$. Now using (5.48), we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}E_{S,r^n}[u](t)\right| \lesssim E_{S,r^{n-1}}[u](t), \tag{5.49}$$

and for $n = 3$, we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}E_{S,r^3}[u](t)\right| &\lesssim E_{S,r^2}[u](t) = E_S[u](0),\\ |E_{S,r^3}[u](t)| &\lesssim E_S[u](0)t + |E_{S,r^3}[u](0)|.\end{aligned} \tag{5.50}$$

Similarly, for $n = 4$ and using the last inequality, we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}E_{S,r^4}[u](t)\right| \lesssim E_{S,r^3}[u](t) \lesssim E_S[u](0)t + |E_{S,r^3}[u](0)|. \tag{5.51}$$

Now, integrating with respect of time, we have

$$|E_{S,r^4}[u](t)| \lesssim E_S[u](0)\frac{t^2}{2} + |E_{S,r^3}[u](0)|t + |E_{S,r^4}[u](0)|. \tag{5.52}$$

Repeating this procedure, we conclude

$$|E_{S,r^n}[u](t)| \lesssim E_S[u](0)t^{n-2} + \sum_{j=0}^{n-3} t^j |E_{S,r^{n-j}}[u](0)|. \tag{5.53}$$

This ends the proof for the case $X = S$. Analogously, following the same ideas, it can be proved for the case $X = AN$ case. This completes the proof. ■

5.4.2 Decay to zero for modified Energies: Proof of the Theorem 5.2

In the spirit of [2, 3, 13], we consider a suitable linear combination of virials $\mathcal{K}_S(t)$ and $\mathcal{P}_S(t)$ (see (5.20) and (5.21)), and $\mathcal{M}_{AN}(t)$ $\mathcal{R}_{AN}(t)$ (see (5.36) and (5.37)), for the Skyrme and Adkins-Nappi models. Let

$$\mathcal{H}_S(t) = \mathcal{K}_S(t) + \gamma_S \mathcal{P}_S(t), \tag{5.54}$$

and,

$$\mathcal{H}_{AN}(t) = \mathcal{M}_{AN}(t) + \gamma_{AN}\mathcal{R}_{AN}(t), \tag{5.55}$$

be new virials, where γ_S and γ_{AN} will be chosen later. These new virials introduce u^2 terms, which allow us to simplify the problem considering Taylor expansions for the involved trigonometric functions.

Decay to zero for modified Energy: Proof of the Theorem 5.2 for the Skyrme model

Lemma 5.11. *Let u be a global solution of* (5.2) *such that $\|u\|_{L^\infty} \leqslant \delta$, $u \in \mathcal{E}_0^{S,\psi}$, and $\psi = r\phi$ (where ψ and ϕ are the weight functions presented in* (5.20) *and* (5.21)*). Then, $\mathcal{H}_S(t)$ in* (5.54) *satisfies the following identity*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t) = &-\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_S\frac{\psi}{r}\right)u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + (2\gamma_S - 4)\frac{\psi}{r}\right)u_r^2 \\
&- \int_0^\infty \left((2-\gamma_S)\frac{\psi}{r} + (2\gamma_S - 1)\psi' - \frac{\gamma_S}{2}r\psi''\right)\frac{u^2}{r^2} \\
&- \alpha^2\int_0^\infty \left(\psi' - (2+4\gamma_S)\frac{\psi}{r}\right)\frac{u^2}{r^2}(u_t^2 + u_r^2) \\
&- \alpha^2\int_0^\infty \left(-\frac{1}{2}(1-6\gamma_S)\psi' + (2-4\gamma_S)\frac{\psi}{r} - \frac{\gamma_S}{2}r\psi''\right)\frac{u^4}{r^4} + H_{\mathrm{e}}(t),
\end{aligned}$$

where

$$
\begin{aligned}
&H_{\mathrm{e}}(t)\\
&= \frac{1}{9}\alpha^2 \int_0^\infty \left(-\gamma_S r\psi'' + (-3+6\gamma_S)\,\psi' + (6\gamma_S+12)\,\frac{\psi}{r}\right)\left(\frac{u^6}{r^4} + \frac{O(u^8)}{r^4}\right)\\
&\quad - \frac{1}{3}\int_0^\infty \left(\psi' - 2(1+2\gamma_S)\frac{\psi}{r}\right)\frac{u^4}{r^2} + \frac{2}{45}\int_0^\infty \left(\psi' - 2\,(1+\gamma_S)\,\frac{\psi}{r}\right)\left(\frac{u^6}{r^2} + \frac{O(u^8)}{r^2}\right)\\
&\quad - \alpha^2 \int_0^\infty \left[\frac{1}{3}\left(-\psi' + 2(1+6\gamma_S)\frac{\psi}{r}\right)u^2\right.\\
&\qquad\qquad \left. + \frac{2}{45}\left(\psi' - 2(1+4\gamma_S)\frac{\psi}{r}\right)\left(u^4 + O(u^6)\right)\right]\frac{u^2}{r^2}(u_t^2+u_r^2).
\end{aligned}
$$

Proof. Collecting (5.26) and (5.22) and regrouping terms, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t)\\
&= -\frac{1}{2}\int_0^\infty (\psi' - 2\gamma_S\phi)\,u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + 2\gamma_S\phi - 4\frac{\psi}{r}\right)u_r^2\\
&\quad + \int_0^\infty \left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\sin^2(u) - \gamma_S\int_0^\infty \frac{\phi}{r^2}u\sin(2u) - \gamma_S\int_0^\infty \left(\left(\frac{\phi}{r}\right)_r - \frac{1}{2}\phi''\right)u^2\\
&\quad - \alpha^2\int_0^\infty \left[\frac{\psi'}{r^2}\sin^2(u) - 2\frac{\psi}{r^3}\sin^2(u) - \gamma_S\frac{\phi}{r^2}\left(\sin(2u)u + 2\sin^2(u)\right)\right](u_t^2+u_r^2)\\
&\quad + \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi''}{r^2} - 4\frac{\phi'}{r^3} + 6\frac{\phi}{r^4}\right)\sin^2(u)u^2 - \gamma_S\alpha^2\int_0^\infty \frac{\phi}{r^4}u\sin(2u)\sin^2(u)\\
&\quad + \frac{\alpha^2}{2}\int_0^\infty \left(\frac{\psi'}{r^4} - 4\frac{\psi}{r^5}\right)\sin^4(u) + \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi'}{r^2} - 2\frac{\phi}{r^3}\right)\sin(2u)u_r u^2.
\end{aligned}
$$

Now, let $\delta > 0$ small enough such that $\|u\|_{L^\infty} < \delta$ (by Remark 5.2), we note

$$
\begin{aligned}
\sin^2(u) &= u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8),\\
u\sin(2u) &= 2u^2 - \frac{4}{3}u^4 + \frac{4}{15}u^6 + O(u^8),\\
2\sin^2(u) + u\sin(2u) &= 4u^2 - 2u^4 + \frac{16}{45}u^6 + O(u^8),\\
u\sin^2(u)\sin(2u) &= 2u^4 - 2u^6 + O(u^8),\\
\sin^4(u) &= u^4 - \frac{2}{3}u^6 + O(u^8).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t)\\
&= -\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_S\phi\right)u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + 2\gamma_S\phi - 4\frac{\psi}{r}\right)u_r^2 - \gamma_S\int_0^\infty \left(\left(\frac{\phi}{r}\right)_r - \frac{1}{2}\phi''\right)u^2\\
&\quad + \int_0^\infty \left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\left[u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8)\right] - \gamma_S\int_0^\infty \frac{\phi}{r^2}\left(2u^2 - \frac{4}{3}u^4 + \frac{4}{15}u^6 + O(u^8)\right)\\
&\quad - \alpha^2\int_0^\infty \Bigg[\left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\left(u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8)\right)\\
&\qquad\qquad - \gamma_S\frac{\phi}{r^2}\left(4u^2 - 2u^4 + \frac{16}{45}u^6 + O(u^8)\right)\Bigg](u_t^2 + u_r^2)\\
&\quad + \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi''}{r^2} - 4\frac{\phi'}{r^3} + 6\frac{\phi}{r^4}\right)\left(u^4 - \frac{1}{3}u^6 + O(u^8)\right) - \gamma_S\alpha^2\int_0^\infty \frac{\phi}{r^4}\left(2u^4 - 2u^6 + O(u^8)\right)\\
&\quad + \frac{\alpha^2}{2}\int_0^\infty \left(\frac{\psi'}{r^4} - 4\frac{\psi}{r^5}\right)\left(u^4 - \frac{2}{3}u^6 + O(u^8)\right) + \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi'}{r^2} - 2\frac{\phi}{r^3}\right)\sin(2u)u_r u^2.
\end{aligned}
$$

We now consider the following decomposition

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t) = H_1 + H_2 + H_3 + H_4 + H_5, \tag{5.56}
$$

where

$$
\begin{aligned}
H_1 =& -\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_S\phi\right)u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + 2\gamma_S\phi - 4\frac{\psi}{r}\right)u_r^2,\\
H_2 =& \int_0^\infty \left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\left[u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8)\right] - \gamma_S\int_0^\infty \left(\left(\frac{\phi}{r}\right)_r - \frac{1}{2}\phi''\right)u^2\\
& - \gamma_S\int_0^\infty \frac{\phi}{r^2}\left(2u^2 - \frac{4}{3}u^4 + \frac{4}{15}u^6 + O(u^8)\right),\\
H_3 =& -\alpha^2\int_0^\infty \Bigg[\left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3}\right)\left(u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8)\right)\\
& \qquad - \gamma_S\frac{\phi}{r^2}\left(4u^2 - 2u^4 + \frac{16}{45}u^6 + O(u^8)\right)\Bigg](u_t^2 + u_r^2),\\
H_4 =& \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi''}{r^2} - 4\frac{\phi'}{r^3} + 6\frac{\phi}{r^4}\right)\left(u^4 - \frac{1}{3}u^6 + O(u^8)\right) - \gamma_S\alpha^2\int_0^\infty \frac{\phi}{r^4}\left(2u^4 - 2u^6 + O(u^8)\right)\\
& + \frac{\alpha^2}{2}\int_0^\infty \left(\frac{\psi'}{r^4} - 4\frac{\psi}{r^5}\right)\left(u^4 - \frac{2}{3}u^6 + O(u^8)\right),
\end{aligned}
$$

and

$$
H_5 = \gamma_S\alpha^2\int_0^\infty \left(\frac{\phi'}{r^2} - 2\frac{\phi}{r^3}\right)\sin(2u)u_r u^2. \tag{5.57}
$$

Regrouping terms of the same order, we get

$$
\begin{aligned}
H_2 =& -\int_0^\infty \left(\gamma_S\left(\frac{\phi}{r}\right)_r - \frac{\gamma_S}{2}\phi'' - \frac{\psi'}{r^2} + 2\frac{\psi}{r^3} + 2\gamma_S\frac{\phi}{r^2}\right)u^2 - \int_0^\infty \left(\frac{\psi'}{3r^2} - \frac{2}{3}\frac{\psi}{r^3} - \gamma_S\frac{4}{3}\frac{\phi}{r^2}\right)u^4\\
& + \int_0^\infty \left(\frac{2}{45}\frac{\psi'}{r^2} - \frac{4}{45}\frac{\psi}{r^3} - \gamma_S\frac{4}{15}\frac{\phi}{r^2}\right)\left(u^6 + O(u^8)\right).
\end{aligned}
$$

Similarly, for H_3 we get

$$\begin{aligned} H_3 = &-\alpha^2 \int_0^\infty \left(\frac{\psi'}{r^2} - 2\frac{\psi}{r^3} - 4\gamma_S \frac{\phi}{r^2} \right) u^2 (u_t^2 + u_r^2) \\ &- \alpha^2 \int_0^\infty \Bigg[\left(\frac{2}{3}\frac{\psi}{r^3} - \frac{1}{3}\frac{\psi'}{r^2} + 2\gamma_S \frac{\phi}{r^2} \right) u^4 \\ &\qquad + \frac{1}{45}\left(2\frac{\psi'}{r^2} - 4\frac{\psi}{r^3} - 16\gamma_S \frac{\phi}{r^2} \right) \left(u^6 + O(u^8) \right) \Bigg] (u_t^2 + u_r^2). \end{aligned}$$

For H_4, we have

$$\begin{aligned} H_4 = &\ \alpha^2 \int_0^\infty \left(\frac{1}{2}\frac{\psi'}{r^4} - 2\frac{\psi}{r^5} - 2\gamma_S \frac{\phi}{r^4} + \gamma_S \frac{\phi''}{r^2} - 4\gamma_S \frac{\phi'}{r^3} + 6\gamma_S \frac{\phi}{r^4} \right) u^4 \\ &+ \alpha^2 \int_0^\infty \left(-\frac{1}{3}\gamma_S \frac{\phi''}{r^2} + \frac{4}{3}\gamma_S \frac{\phi'}{r^3} - 2\gamma_S \frac{\phi}{r^4} + 2\gamma_S \frac{\phi}{r^4} - \frac{1}{3}\frac{\psi'}{r^4} + \frac{4}{3}\frac{\psi}{r^5} \right) \left(u^6 + O(u^8) \right). \end{aligned}$$

For H_5, first we note

$$\sin(2u) u_r u^2 = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}r}(2u\sin(2u) - 2u^2\cos(2u) + \cos(2u) - 1).$$

Now, replacing in H_5 and integrating by parts, we get

$$H_5 = -\frac{\gamma_S}{4}\alpha^2 \int_0^\infty \left(\frac{\phi'}{r^2} - 2\frac{\phi}{r^3} \right)_r (2u\sin(2u) - 2u^2\cos(2u) + \cos(2u) - 1),$$

using its Taylor expansion and regrouping the terms, we have

$$\begin{aligned} H_5 &= -\frac{\gamma_S}{4}\alpha^2 \int_0^\infty \left(\frac{\phi''}{r^2} - 4\frac{\phi'}{r^3} + 6\frac{\phi}{r^4} \right) \left(2u^4 - \frac{8}{9}u^6 + O(u^8) \right) \\ &= \alpha^2 \int_0^\infty \left(-\frac{\gamma_S}{2}\frac{\phi''}{r^2} + 2\gamma_S \frac{\phi'}{r^3} - 3\gamma_S \frac{\phi}{r^4} \right) u^4 \\ &\quad + \alpha^2 \int_0^\infty \left(\frac{2}{9}\gamma_S \frac{\phi''}{r^2} - \frac{8}{9}\gamma_S \frac{\phi'}{r^3} + \frac{4}{3}\gamma_S \frac{\phi}{r^4} \right) \left(u^6 + O(u^8) \right). \end{aligned}$$

Collecting the last equation and H_4, we obtain

$$\begin{aligned} H_4 + H_5 = &\ \alpha^2 \int_0^\infty \left(\frac{1}{2}\frac{\psi'}{r^4} - 2\frac{\psi}{r^5} + \gamma_S \frac{\phi}{r^4} - 2\gamma_S \frac{\phi'}{r^3} + \frac{\gamma_S}{2}\frac{\phi''}{r^2} \right) u^4 \\ &+ \alpha^2 \int_0^\infty \left(-\frac{1}{9}\gamma_S \frac{\phi''}{r^2} + \frac{4}{9}\gamma_S \frac{\phi'}{r^3} - \frac{1}{3}\frac{\psi'}{r^4} + \frac{4}{3}\frac{\psi}{r^5} + \frac{4}{3}\gamma_S \frac{\phi}{r^4} \right) \left(u^6 + O(u^8) \right). \end{aligned} \tag{5.58}$$

Having in mind that $\psi = r\phi$, we have

$$\phi' = \frac{\psi'}{r} - \frac{\psi}{r^2} \quad \text{and} \quad \phi'' = \frac{\psi''}{r} - 2\frac{\psi'}{r^2} + 2\frac{\psi}{r^3}.$$

Now, rewriting H_i, for $\mathrm{i} = 1, \dots, 5$, in terms of ψ and its derivatives, we get

$$H_1 = -\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_S \frac{\psi}{r} \right) u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + (2\gamma_S - 4)\frac{\psi}{r} \right) u_r^2, \tag{5.59}$$

$$
\begin{aligned}
H_2 = &- \int_0^\infty \left((2-\gamma_S)\frac{\psi}{r^3} + (2\gamma_S - 1)\frac{\psi'}{r^2} - \frac{\gamma_S}{2}\frac{\psi''}{r} \right) u^2 - \int_0^\infty \left(\frac{\psi'}{3r^2} - \frac{2}{3}(1+2\gamma_S)\frac{\psi}{r^3} \right) u^4 \\
&+ \int_0^\infty \left(\frac{2}{45}\frac{\psi'}{r^2} - \frac{4}{45}\left(1+\gamma_S\right)\frac{\psi}{r^3} \right) \left(u^6 + O(u^8)\right),
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
H_3 = &- \alpha^2 \int_0^\infty \left(\frac{\psi'}{r^2} - (2+4\gamma_S)\frac{\psi}{r^3} \right) u^2(u_t^2 + u_r^2) \\
&- \alpha^2 \int_0^\infty \Bigg[\left(-\frac{1}{3}\frac{\psi'}{r^2} + \frac{2}{3}(1+6\gamma_S)\frac{\psi}{r^3} \right) u^4 \\
&\qquad + \left(\frac{2}{45}\frac{\psi'}{r^2} - \frac{4}{45}(1+4\gamma_S)\frac{\psi}{r^3} \right) \left(u^6 + O(u^8)\right) \Bigg] (u_t^2 + u_r^2),
\end{aligned}
\tag{5.61}
$$

and

$$
\begin{aligned}
H_4 + H_5 = &\ \alpha^2 \int_0^\infty \left(\frac{1}{2}(1-6\gamma_S)\frac{\psi'}{r^4} - (2-4\gamma_S)\frac{\psi}{r^5} + \frac{\gamma_S}{2}\frac{\psi''}{r^3} \right) u^4 \\
&+ \alpha^2 \int_0^\infty \left(-\frac{1}{9}\gamma_S\frac{\psi''}{r^3} - \left(\frac{1}{3} - \frac{6}{9}\gamma_S \right)\frac{\psi'}{r^4} + \left(\frac{6}{9}\gamma_S + \frac{4}{3} \right)\frac{\psi}{r^5} \right) \left(u^6 + O(u^8)\right).
\end{aligned}
\tag{5.62}
$$

Finally, collecting (5.59), (5.60), (5.61), (5.62), and regrouping the terms of the same order, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t) = &- \frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_S\frac{\psi}{r} \right) u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' + (2\gamma_S - 4)\frac{\psi}{r} \right) u_r^2 \\
&- \int_0^\infty \left((2-\gamma_S)\frac{\psi}{r} + (2\gamma_S - 1)\psi' - \frac{\gamma_S}{2} r\psi'' \right) \frac{u^2}{r^2} \\
&- \alpha^2 \int_0^\infty \left(\psi' - (2+4\gamma_S)\frac{\psi}{r} \right) \frac{u^2}{r^2}(u_t^2 + u_r^2) \\
&- \alpha^2 \int_0^\infty \left(-\frac{1}{2}(1-6\gamma_S)\psi' + (2-4\gamma_S)\frac{\psi}{r} - \frac{\gamma_S}{2} r\psi'' \right) \frac{u^4}{r^4} + H_{\mathrm{e}}(t),
\end{aligned}
\tag{5.63}
$$

where

$$
\begin{aligned}
&H_{\mathrm{e}}(t) = \\
&\frac{1}{9}\alpha^2 \int_0^\infty \left(-\gamma_S r\psi'' + \left(-3+6\gamma_S\right)\psi' + \left(6\gamma_S + 12\right)\frac{\psi}{r} \right) \left(\frac{u^6}{r^4} + \frac{O(u^8)}{r^4} \right) \\
&- \frac{1}{3}\int_0^\infty \left(\psi' - 2(1+2\gamma_S)\frac{\psi}{r} \right) \frac{u^4}{r^2} + \frac{2}{45}\int_0^\infty \left(\psi' - 2\left(1+\gamma_S\right)\frac{\psi}{r} \right) \left(\frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right) \\
&- \alpha^2 \int_0^\infty \Bigg[\frac{1}{3}\left(-\psi' + 2(1+6\gamma_S)\frac{\psi}{r} \right) u^2 \\
&\qquad + \frac{2}{45}\left(\psi' - 2(1+4\gamma_S)\frac{\psi}{r} \right) \left(u^4 + O(u^6)\right) \Bigg] \frac{u^2}{r^2}(u_t^2 + u_r^2).
\end{aligned}
\tag{5.64}
$$

This ends the proof. ■

Under the hypothesis of Lemma 5.11 the functionals $\mathcal{K}_S(t)$ and $\mathcal{P}_S(t)$ (see (5.20)–(5.21)) are well-defined. In fact, using the Cauchy-Schwarz inequality, we have

$$\mathcal{K}_S(t) \leqslant \int_0^\infty \psi \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2),$$

and

$$\mathcal{P}_S(t) \leqslant \int_0^\infty r\phi \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + \frac{u^2}{r^2}).$$

Then, assuming $\psi = r\phi$, and $u \in \mathcal{E}_0^{S,\psi}$, we get

$$|\mathcal{K}_S(t)| + |\mathcal{P}_S(t)| \leqslant E_{S,\psi}[u](t),$$

hence concluding that the functionals $\mathcal{K}_S(t)$ and $\mathcal{P}_S(t)$ in (5.20)-(5.21) are well-defined.

Corollary 5.12. *Let $n, \gamma_S \in \mathbb{R}$ and $\psi = r^n\chi$. Then, under the hypothesis of Lemma 5.11, the following identity holds:*

$$\begin{aligned}
\mathcal{H}_S(t) = &-\frac{1}{2}\int_0^\infty \left((n-2\gamma_S)r^{n-1}\chi + r^n\chi'\right)u_t^2 - \frac{1}{2}\int_0^\infty \left((n+2\gamma_S-4)r^{n-1}\chi + r^n\chi'\right)u_r^2 \\
&- \alpha^2\int_0^\infty \left((n-2-4\gamma_S)r^{n-1}\chi + r^n\chi'\right)\frac{u^2}{r^2}(u_t^2+u_r^2) \\
&- \int_0^\infty \Bigg((2-\gamma_S+n(2\gamma_S-1) - \frac{\gamma_S}{2}n(n-1))r^{n-1}\chi \\
&\qquad\qquad + (\gamma_S(2-n)-1)r^n\chi' - \frac{\gamma_S}{2}r^{n+1}\chi''\Bigg)\frac{u^2}{r^2} \\
&- \alpha^2\int_0^\infty \Bigg(\frac{1}{2}(4-n-8\gamma_S-\gamma_S(-7+n)n)r^{n-1}\chi \\
&\qquad\qquad + (\gamma_S(3-n)-\frac{1}{2})r^n\chi' - \frac{\gamma_S}{2}r^{n+1}\chi''\Bigg)\frac{u^4}{r^4} + H_e(t)
\end{aligned} \tag{5.65}$$

and for H_{e} holds

$$
\begin{aligned}
H_{\mathrm{e}}(t) = &- \alpha^2 \int_0^\infty \left[\frac{1}{3}\left((-n+2(1+6\gamma_S))r^{n-1}\chi + r^n\chi'\right)u^2\right]\frac{u^2}{r^2}(u_t^2+u_r^2)\\
&- \frac{1}{3}\int_0^\infty \left((n-2(1+2\gamma_S))r^{n-1}\chi + r^n\chi'\right)\frac{u^4}{r^2}\\
&+ \frac{2}{45}\int_0^\infty \left((n-2\,(1+\gamma_S))r^{n-1}\chi + r^n\chi'\right)\left(\frac{u^6}{r^2}+\frac{O(u^8)}{r^2}\right)\\
&- \frac{2}{45}\alpha^2\int_0^\infty \left((n-2(1+4\gamma_S))r^{n-1\chi} + r^n\chi'\right)\left(u^4+O(u^6)\right)\frac{u^2}{r^2}(u_t^2+u_r^2)\\
&+ \frac{1}{9}\alpha^2\int_0^\infty \Big([\gamma_S n(5-n) - 3n + 6\gamma_S + 12]r^{n-1}\chi'\\
&\qquad + (\gamma_S(6-n)-3)r^n\chi' - \gamma_S r^{n+1}\chi''\Big)\left(\frac{u^6}{r^4}+\frac{O(u^8)}{r^4}\right)
\end{aligned}
\tag{5.66}
$$

Proof. The proof follows directly replacing $\psi = r^n\chi$ in Lemma 5.11. ■

Now, if we set $\chi = 1$, we obtain the following result:

Corollary 5.13. *Let $\delta > 0$ small enough, $\psi = r^n$ and u be a global solution of (5.2) such that $u \in \mathcal{E}_0^{S,r^n}$ and $\|u\|_{L^\infty} < \delta$. Then, for $\gamma_S = -1$ and $n \geqslant 2$, the functional $\mathcal{H}_S(t)$ (5.54) satisfies*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t) = &- \frac{1}{2}\int_0^\infty r^{n-1}\Bigg[(n+2)u_t^2 + (n-6)u_r^2 + (n-6)(n-1)\frac{u^2}{r^2}\\
&+2\alpha^2\,(n+2)\,\frac{u^2}{r^2}(u_t^2+u_r^2) + \alpha^2(n-6)(n-2)\frac{u^4}{r^4}\Bigg] + H_{\mathrm{e}}(t),
\end{aligned}
\tag{5.67}
$$

with $|H_{\mathrm{e}}(t)| \leqslant \delta^2 E_{S,r^{n-1}}(t)$.

Assuming $\delta > 0$ small enough, $n \geqslant 6$ and applying Corollary 5.13, we obtain the following virial inequality

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_S(t) \geqslant &\ \frac{1}{4}\int_0^\infty r^{n-1}\Bigg[(n+2)u_t^2 + (n-6)u_r^2 + (n-6)(n-1)\frac{u^2}{r^2}\\
&+2\alpha^2\,(n+2)\,\frac{u^2}{r^2}(u_t^2+u_r^2) + \alpha^2(n-6)(n-2)\frac{u^4}{r^4}\Bigg] \geqslant 0.
\end{aligned}
\tag{5.68}
$$

In particular, as an application of (5.68), we obtain the following result for $r^{6+\epsilon}$ and $r^{7+\epsilon}$ weighted energies.

Corollary 5.14. *Let $\epsilon > 0$ and u be a global solution of (5.2) in the class $\mathcal{E}_0^{S,r^{6+\epsilon}} \cap \mathcal{E}_0^{S,r^{7+\epsilon}}$. Then,* *1. Integrability in time:*

$$\int_2^\infty \int_0^\infty (r^{6+\epsilon} + r^{7+\epsilon}) \left[u_t^2 + u_r^2 + \frac{u^2}{r^2} + 2\alpha^2 \frac{u^2}{r^2}(u_t^2 + u_r^2) + \alpha^2 \frac{u^4}{r^4} \right] \mathrm{d}r\mathrm{d}t \lesssim_n 1. \tag{5.69}$$

2. Sequential decay to zero: there exists $s_n,\ t_n \uparrow \infty$ such that

$$\lim_{n\to\infty} E_{S,r^{6+\epsilon}}[u](t_n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} E_{S,r^{7+\epsilon}}[u](s_n) = 0. \tag{5.70}$$

Decay to zero for modified Energy : Proof of the Theorem 5.2 for the Adkins-Nappi model

Similarly to the Skyrme equation, we will need the following technical lemmas.

Lemma 5.15. *Let u be a global solution of* (5.3) *such that $u \in \mathcal{E}_0^{AN,\psi}$, and $\psi = r\phi$ (where ψ and ϕ are the weight functions presented in* (5.36) *and* (5.37)*). Then, the functional $\mathcal{H}_{AN}(t)$, defined in* (5.55)*, satisfies the following identity*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t) \\
&\quad = -\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_{AN}\frac{\psi}{r}\right) u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' - 2(\gamma_{AN}-2)\frac{\psi}{r}\right) u_r^2 \\
&\quad - \frac{1}{2}\int_0^\infty \left[2(1-\gamma_{AN})\frac{\psi}{r} - \psi' + \gamma_{AN} r\psi''\right] \frac{u^2}{r^2} - \frac{1}{3}\int_0^\infty \left[-(4\gamma_{AN}+1)\frac{\psi}{r} + \frac{1}{2}\psi'\right] \frac{u^4}{r^2} \\
&\quad - \frac{1}{45}\int_0^\infty \left[2(1+6\gamma_{AN})\frac{\psi}{r} - \psi'\right] \left(\frac{u^6}{r^2} + \frac{O(u^8)}{r^2}\right) \\
&\quad - \frac{1}{9}\int_0^\infty \left[4(2+3\gamma_{AN})\frac{\psi}{r} - 2\psi'\right] \left(\frac{u^6}{r^4} + \frac{O(u^8)}{r^4}\right).
\end{aligned} \tag{5.71}$$

Proof. First, we note that $\mathcal{M}_{AN}(t)$ and $\mathcal{R}_{AN}(t)$ are well defined in $\mathcal{E}_0^{AN,\psi}$. Collecting (5.38) and (5.39), we get that $\mathcal{H}_{AN}(t)$ is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t) = &-\frac{1}{2}\int_0^\infty (\psi' - 2\gamma_{AN}\phi)u_t^2 - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2 - \gamma_{AN}\int_0^\infty \phi u_r^2 \\
&- \frac{1}{2}\int_0^\infty \left(2\frac{\psi}{r} - \psi'\right) \frac{\sin^2(u)}{r^2} - \frac{1}{2}\int_0^\infty \left(4\frac{\psi}{r} - \psi'\right) \frac{(u - \sin(u)\cos(u))^2}{r^4} \\
&- \gamma_{AN}\int_0^\infty \left[\phi' r - \phi - \frac{r^2\phi_{rr}}{2}\right] \frac{u^2}{r^2} - \gamma_{AN}\int_0^\infty \left(\frac{\phi}{r^2}\right) u\sin(2u) \\
&- \gamma_{AN}\int_0^\infty \left(\frac{\phi}{r^4}\right) u\,(u - \sin(u)\cos(u))\,(1 - \cos(2u)).
\end{aligned}$$

Now, let $\delta > 0$ small enough and using the Taylor approximation for $\|u\|_{L^\infty} < \delta$, we have

$$\begin{aligned}
u\sin(2u) &= 2u^2 - \frac{4}{3}u^4 + \frac{4}{15}u^6 + O(u^8),\\
\sin^2(u) &= u^2 - \frac{1}{3}u^4 + \frac{2}{45}u^6 + O(u^8),\\
(u - \sin u \cos u)^2 &= \frac{4}{9}u^6 - \frac{8}{45}u^8 + O(u^{10}),\\
u(u - \sin(u)\cos(u))(1 - \cos(2u)) &= \frac{4}{3}u^6 - \frac{32}{47}u^8 + O(u^{10}).
\end{aligned}$$

Replacing in $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t)$ and regrouping the terms of same order, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t)\\
&= -\frac{1}{2}\int_0^\infty (\psi' - 2\gamma_{AN}\phi)u_t^2 - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r} + \gamma_{AN}\phi\right) u_r^2\\
&\quad - \int_0^\infty \left[\frac{\psi}{r} - \frac{1}{2}\psi' + \gamma_{AN}\phi' r + \gamma_{AN}\phi - \gamma_{AN}\frac{r^2\phi_{rr}}{2}\right]\frac{u^2}{r^2}\\
&\quad - \int_0^\infty \left[-\frac{1}{3}\frac{\psi}{r} + \frac{1}{6}\psi' - \frac{4}{3}\gamma_{AN}\phi\right]\frac{u^4}{r^2} - \int_0^\infty \left[\frac{2}{45}\frac{\psi}{r} - \frac{1}{45}\psi' + \frac{4}{15}\gamma_{AN}\phi\right]\left(\frac{u^6}{r^2} + \frac{O(u^8)}{r^2}\right)\\
&\quad - \int_0^\infty \left[\frac{8}{9}\frac{\psi}{r} - \frac{2}{9}\psi' + \frac{4}{3}\gamma_{AN}\phi\right]\left(\frac{u^6}{r^4} + \frac{O(u^8)}{r^4}\right).
\end{aligned}$$

Since $\psi = r\phi$, we get

$$\phi' = \frac{\psi'}{r} - \frac{\psi}{r^2}, \quad \phi'' = \frac{\psi''}{r} - 2\frac{\psi'}{r^2} + 2\frac{\psi}{r^3}.$$

Then, rewriting $\mathcal{H}_{AN}(t)$ in terms of ψ, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t)\\
&= -\frac{1}{2}\int_0^\infty \left(\psi' - 2\gamma_{AN}\frac{\psi}{r}\right)u_t^2 - \frac{1}{2}\int_0^\infty \left(\psi' - 2(\gamma_{AN} - 2)\frac{\psi}{r}\right)u_r^2\\
&\quad - \frac{1}{2}\int_0^\infty \left[2(1 - \gamma_{AN})\frac{\psi}{r} - \psi' + \gamma_{AN} r\psi''\right]\frac{u^2}{r^2}\\
&\quad - \frac{1}{3}\int_0^\infty \left[-(4\gamma_{AN} + 1)\frac{\psi}{r} + \frac{1}{2}\psi'\right]\frac{u^4}{r^2} - \frac{1}{45}\int_0^\infty \left[2(1 + 6\gamma_{AN})\frac{\psi}{r} - \psi'\right]\left(\frac{u^6}{r^2} + \frac{O(u^8)}{r^2}\right)\\
&\quad - \frac{1}{9}\int_0^\infty \left[4(2 + 3\gamma_{AN})\frac{\psi}{r} - 2\psi'\right]\left(\frac{u^6}{r^4} + \frac{O(u^8)}{r^4}\right).
\end{aligned}$$

This ends the proof. ■

Similarly to Skyrme equation, using $\psi = r\phi$ and the Cauchy-Schwarz inequality, we get

$$|\mathcal{M}_{AN}(t)| + |\mathcal{R}_{AN}(t)| \leqslant E_{AN,\psi}[u](t).$$

Then, the functionals $\mathcal{M}_{AN}(t)$ and $\mathcal{R}_{AN}(t)$ are well-defined if $u \in \mathcal{E}_0^{AN,\psi}$.

Corollary 5.16. *Under the hypothesis of Lemma 5.15 and assuming $n, \gamma_{AN} \in \mathbb{R}$ and $\psi = r^n\chi$, the following holds:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t) = \\
&\quad -\frac{1}{2}\int_0^\infty \left[(n-2\gamma_{AN})\, r^{n-1}\chi + r^n\chi'\right] u_t^2 - \frac{1}{2}\int_0^\infty \left[(n+4-2\gamma_{AN})\, r^{n-1}\chi + r^n\chi'\right] u_r^2 \\
&\quad -\frac{1}{2}\int_0^\infty \left[(n-2)(1-\gamma_{AN}n-\gamma_{AN})r^{n-1}\chi + (\gamma_{AN}-1)r^n\chi' + \gamma_{AN}r^{n+1}\chi''\right]\frac{u^2}{r^2} \\
&\quad -\frac{1}{3}\int_0^\infty \left[\left(-(4\gamma_{AN}+1)+\frac{1}{2}n\right) r^{n-1}\chi + r^n\chi'\right]\frac{u^4}{r^2} \\
&\quad -\frac{1}{45}\int_0^\infty \left[(2(1+6\gamma_{AN})-n)\, r^{n-1}\chi - r^n\chi'\right]\left(\frac{u^6}{r^2}+\frac{O(u^8)}{r^2}\right) \\
&\quad -\frac{1}{9}\int_0^\infty \left[(4(2+3\gamma_{AN})-2n)\, r^{n-1}\chi - 2r^n\chi'\right]\left(\frac{u^6}{r^4}+\frac{O(u^8)}{r^4}\right).
\end{aligned}$$

Proof. The proof follows directly using (5.71) and replacing $\psi = r^n\chi$. ∎

Now, considering that $\chi = 1$, we obtain the following result:

Corollary 5.17. *Let $\psi = r^n$, and u be a global solution of (5.3) such that $u \in \mathcal{E}_0^{AN,r^n}$. Then, for $\gamma_{AN} = (n-2)/8$ and $n \geqslant 2$, the functional $\mathcal{H}_{AN}$, defined in (5.55), satisfies the following identity*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t) \leqslant -\frac{1}{2}\int_0^\infty r^{n-1}\Bigg(&\frac{3n+2}{4}u_t^2 + \frac{3(6+n)}{4}u_r^2 + \frac{(n-2)(n^2-n-10)}{8}\frac{u^2}{r^2} \\
&+\frac{n-2}{45}\left(\frac{u^6}{r^2}+\frac{O(u^8)}{r^2}\right) + \frac{(10-n)}{9}\left(\frac{u^6}{r^4}+\frac{O(u^8)}{r^4}\right)\Bigg).
\end{aligned} \tag{5.72}$$

For $n \in \left[\frac{1+\sqrt{41}}{2}, 10\right]$, by Corollary 5.17, we obtain the following inequality

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{AN}(t) \geqslant \frac{1}{4}\int_0^\infty r^{n-1}\Bigg(&\frac{3n+2}{4}u_t^2 + \frac{3(6+n)}{4}u_r^2 + \frac{(n-2)(n^2-n-10)}{8}\frac{u^2}{r^2} \\
&+\frac{n-2}{45}\frac{u^6}{r^2} + \frac{(10-n)}{9}\frac{u^6}{r^4}\Bigg) \geqslant 0,
\end{aligned} \tag{5.73}$$

which is essential to obtain the integrability property. In particular, we obtain the following result for the $r^{4+\epsilon}$ and $r^{5+\epsilon}$ weighted energies.

Corollary 5.18. *Let u be a global solution of (5.3) in the class $\mathcal{E}_0^{AN,r^{4+\epsilon}} \cap \mathcal{E}_0^{AN,r^{5+\epsilon}}$ for $\epsilon \in [0,4[$. Then,*

1. *Integrability in time:*

$$\int_2^\infty \int_0^\infty (r^{4+\epsilon}+r^{5+\epsilon})\left((u_t^2+u_r^2)+\frac{u^2}{r^2}+\frac{u^6}{r^4}\right)\mathrm{d}r\mathrm{d}t \lesssim_{u_0} 1. \tag{5.74}$$

2. Sequential decay to zero: there exists $s_n,\ t_n \uparrow \infty$ such that

$$\lim_{n\to\infty} E_{AN,r^{5+\epsilon}}[u](t_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} E_{AN,r^{4+\epsilon}}[u](s_n) = 0. \tag{5.75}$$

The proof of above corollary follows directly from (5.73). With these results, we are ready to conclude the proof of Theorem 5.2 for the Adkins-Nappi equation.

Now, we are ready to prove Theorem 5.2 for the Skyrme equation.

5.4.3 End of proof of Theorem 5.2

Consider $E_{S,\varphi}$ as in (5.11) with $\varphi = r^{7+\epsilon}$. From (5.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{S,\varphi}[u](t) = -2\int_0^\infty \left(\varphi' - 2\frac{\varphi}{r}\right)\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right) u_t u_r.$$

Therefore,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} E_{S,\varphi}[u](t)\right| \lesssim \int_0^\infty \left|\varphi' - 2\frac{\varphi}{r}\right| \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2). \tag{5.76}$$

Integrating in $[t, t_n]$, we have

$$|E_{S,\varphi}[u](t) - E_{S,\varphi}[u](t_n)| \lesssim \int_t^{t_n}\int_0^\infty \left|\varphi' - 2\frac{\varphi}{r}\right| \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_r^2 + u_t^2)\mathrm{d}r\mathrm{d}t.$$

Sending n to infinity, we have from (5.70) that $E_{S,\varphi}[u](t_n) \to 0$ and

$$|E_{S,\varphi}[u](t)| \lesssim \int_t^\infty\int_0^\infty \left|\varphi' - 2\frac{\varphi}{r}\right| \left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_r^2 + u_t^2)\mathrm{d}r\mathrm{d}t.$$

Finally, if $t \to \infty$, we conclude. Since $E_{S,\varphi}[u](t) \gtrsim \left\|(r^{\frac{5+\epsilon}{2}} u_t, r^{\frac{5+\epsilon}{2}} u_r)(t)\right\|^2_{L^2\times L^2(\mathbb{R}^3)}$, this proves Theorem 5.2 for the Skyrme equation. The proof in the Adkins-Nappi case is analogous considering $E_{AN,\varphi}$ in (5.12) with $\varphi = r^{5+\epsilon}$.

This concludes the proof of Theorem 5.2.

Bibliography

[1] G. S. Adkins and C. R. Nappi. Stabilization of chiral solitons via vector mesons. Physics Letters B, 137(3-4): 251-256, 1984.

[2] M. A. Alejo, F. Cortez, C. Kwak and C. Muñoz, *On the dynamics of zero-speed solutions for Camassa-Holm type equations*, IMRN (International Mathematics Research Notices), DOI: 10.1093/imrn/rnz038.

[3] M. A. Alejo, and C. Muñoz, *Almost sharp nonlinear scattering in one-dimensional Born-Infeld equations arising in nonlinear Electrodynamics*, Proc. AMS 146 (2018), no. 5, 2225–2237.

[4] P. Bizoń, Tadeusz Chmaj, and Zbis law Tabor. *Dispersion and collapse of wave maps.* Nonlinearity, 13(4):1411-1423, 2000.

[5] D.-A. Geba and S. G. Rajeev, *A continuity argument for a semilinear Skyrme model*, Electron. J. Differential Equations, 2010 (2010), no. 86, 1–9.

[6] D.-A. Geba and S. G. Rajeev, *Energy arguments for the Skyrme model*, preprint, 2011.

[7] Dan-Andrei Geba, K. Nakanishi and S. G. Rajeev. *Global well-posedness and scattering for Skyrme wave maps.* Commun. Pure Appl. Anal., 11(5) (2012), 1923-1933.

[8] R. Donninger, *On stable self-similar blow-up for equivariant wave maps.* Comm. Pure Appl. Math., 64(8):1095-1147, 2011.

[9] M. Gell-Mann and M. Levy. *The axial vector current in beta decay.* Nuovo Cimento (10), 16:705-726, 1960.

[10] L. V. Kapitanskii and O. A. Ladyzhenskaya. *The Coleman principle for finding stationary points of invariant functionals.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 127:84-102,1983. *Boundary value problems of mathematical physics and related questions in the theory of functions,* 15.

[11] A. Lawrie, Conditional Global Existence and Scattering for a Semi-Linear Skyrme Equation with Large Data. Commun. Math. Phys. 334, 1025–1081 (2015).

[12] D. Li. *Global wellposedness of hedgehog solutions for the (3+1) Skyrme model.* Preprint arXiv:1208.4977, 2012.

[13] C. Maulén and C. Muñoz, *Decay in the one dimensional generalized Improved Boussinesq equation*. SN Partial Differ. Equ. Appl. 1, 1 (2020), https://doi.org/10.1007/s42985-019-0002-0.

[14] J. B. McLeod and W. C. Troy. *The Skyrme model for nucleons under spherical symmetry.* Proc. Roy.Soc. Edinburgh Sect. A, 118(3-4):271-288, 1991.

[15] J. Shatah. *Weak solutions and development of singularities of the SU(2) σ-model.* Comm. Pure Appl.Math., 41(4):459-469, 1988.

[16] T. H. R. Skyrme. *A non-linear field theory.* Proc. Roy. Soc. London Ser. A, 260:127-138, 1961.

Part IV

Conclusion

Chapter 6

Conclusions and Perspectives

6.1 Conclusions

This work concerned the study of well-posedness, and long time asymptotics of small or solitonic solutions for five models appearing in Nature: Improved, abcd and Good Boussinesq models, Skyrme and Adkins-Nappi equations.

The results obtained in the Part II of this work essentially consist of a deep analysis of the following topics:

- Decay of solutions of the **Improved-Boussinesq**,

$$\partial_t^2\phi - \partial_x^2\partial_t^2\phi - \partial_x^2\phi - \partial_x^2(|\phi|^{p-1}\phi) = 0, \quad \text{for } p > 1,\ (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{6.1}$$

  for which we improved decay results by Cho-Ozawa.
- Asymptotic stability of standing waves of the **Good-Boussinesq** model,

$$\partial_t^2\phi + \partial_x^4\phi - \partial_x^2\phi - \partial_x^2(\phi^2) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{6.2}$$

  where we constructed a manifold of data around the standing wave $(Q,0)$, and characterize the asymptotic behavior in this set. This work is the first of his type in Boussinesq models, and opens a new area of research for next years.
- Ill-posedness for the $(abc\mathrm{d})$**-Boussinesq** system:

$$(abc\mathrm{d})\begin{cases}(1-b\,\Delta)\partial_t\eta + \nabla\cdot(a\,\Delta\vec{u} + \vec{u} + \vec{u}\eta) = 0,\\ (1-\mathrm{d}\,\Delta)\partial_t\vec{u} + \nabla\left(c\,\Delta\eta + \eta + \dfrac{1}{2}|\vec{u}|^2\right) = 0.\end{cases} \quad (t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^{\mathrm{d}},\ \mathrm{d} = 1,2. \tag{6.3}$$

  Here we improved existing results on well and ill-posedness for 1D and 2D (abcd) Boussinesq.

In Part III of this thesis we studied the following high energy models:

- Decay of solutions of the **Skyrme equation**,

$$\left(1+\frac{2\alpha^2\sin^2 u}{r^2}\right)(u_{tt}-u_{rr})-\frac{2}{r}u_r+\frac{\sin 2u}{r^2}\left[1+\alpha^2\left(u_t^2-u_r^2+\frac{\sin^2 u}{r^2}\right)\right]=0, \quad (6.4)$$

- Decay of solutions of the **Adkins–Nappi equation**,

$$u_{tt}-u_{rr}-\frac{2}{r}u_r+\frac{\sin 2u}{r^2}+\frac{(u-\sin u\cos u)(1-\cos 2u)}{r^4}=0. \quad (6.5)$$

For each of these models, we provided the first decay results known to date.

We provided new ideas (the virial technique for instance) and decay results for Boussinesq, Skyrme and Adkins-Nappi models. However, several questions were left open here, and we believe that these are nice continuations for the future.

6.2 Future Work

We want to focus on the understanding of the long time behavior problem in non-decaying solutions, e.g., solitary waves, standing waves, and kinks. The first step is to focus on the kink's stability and asymptotic stability for the **Wave-Cahn-Hilliard equation**. The second step is to study the stability of standing wave solutions of the (*abc*d)**-Boussinesq** system. Finally, I want to concentrate my efforts on the stability of solitary waves solution in the **Improved-Boussinesq** model.

6.2.1 Stability and Asymptotic Stability of Kinks in Wave-Cahn-Hilliard equation

The well-known Cahn–Hilliard equation given by

$$\partial_t u+\partial_x^4 u+\partial_x^2 u-\partial_x^2(u^3)=0.$$

This model has a huge literature, and has many interesting properties. See [15, 16, 5] for further details. We propose to study the model

$$\partial_t^2 u+\partial_x^4 u+\partial_x^2 u-\partial_x^2(u^3)=0, \quad (6.6)$$

that we call the Wave-Cahn-Hilliard equation. This variation makes the dissipative nature of the Cahn-Hilliard equation change to a wave-like behavior, with a huge flavor to good-Boussinesq. This model is also a generalization of the ϕ^4 model. A kink is a solution to (6.6) of the form

$$u(t,x)=H_c(x-ct-x_0),, \quad c,x_0\in\mathbb{R},$$

with H_c solving $H_c''+(c^2+1)H_c-H_c^3=0$. It is well-known that (up to shifts) standing kink ($c=0$) has the form $H(x)=\tanh\left(\frac{x}{\sqrt{2}}\right)$. Cahn-Hilliard and wave-Cahn-Hilliard models share the same kink solution. However, since our modified equation has not been studied yet, there are many research possibilities for their study, ranging from well-posedness to the classical stability properties. We expect that the study of the elementary properties of the Wave-Cahn-Hilliard equation will open a new field of research.

6.2.2 Asymptotic stability of even data perturbations of solitary waves in the $(abc\mathrm{d})$-Boussinesq system

For $a, c < 0$ and $b = \mathrm{d} > 0$, the system carries a Hamiltonian structure, and are the conserved quantities, given by

$$E[\vec{u}, \eta](t) = \frac{1}{2}\int(-a|\nabla\vec{u}|^2 - c|\nabla\eta|^2 + |\vec{u}|^2(1+\eta) + \eta^2)(t,x)\mathrm{d}x, \quad \mathcal{I}(t) = \int \eta u + b\nabla\eta\cdot\nabla u.$$

These are the energy and impulse functional, respectively. Recently Bao, Chen and Liu [1] considered ground states for 1D abcd of the form:

$$(\eta, u)(x) = (N_\omega, U_\omega)(x - \omega t) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \quad \omega \in \mathbb{R},$$

in the natural energy space. For $a, c < 0$, $b = \mathrm{d}$ they proved the existence of these ground states, such that the traveling speed ω satisfies

$$|\omega| < \begin{cases} \min\left\{1, \dfrac{\sqrt{ac}}{|b|}\right\}, & \text{when } b \neq 0 \\ \qquad 1 & \text{, when } b = 0. \end{cases}$$

In general, (N_ω, U_ω) are not explicit but sometimes they are. In [3, 4, 7] it was given an explicit description of the solitary waves. For instance, for $\omega = 0$ and $a = c < 0$,

$$(N_0, U_0)(x) := \left(-Q\left(x/\sqrt{|a|}\right), \sqrt{2}Q\left(x/\sqrt{|a|}\right)\right)$$

where $Q(x) = \frac{3}{2}\operatorname{sech}^2(\frac{x}{2})$ is the positive solution of $Q'' - Q + Q^2 = 0$. Furthermore, they proved that the solitary waves are spectral stable for all subsonic speeds, i.e. $|\omega| < 1$. In the spirit of my previous work [12], under orthogonality and parity conditions, I want to prove that if the standing wave is orbitally stable then it is asymptotic stable. Once this is done, the natural extension of the problem is to consider the moving solitary wave and its dynamics.

6.2.3 Stability of the solitary wave in the Improved-Boussinesq equation

One important question that remains open is the stability/instability of Improved-Boussinesq solitons. But, as we shall explain below, this question is far from being trivial.

In an influential work, Grillakis, Shatah, Strauss [6] (GSS) obtained sharp conditions for the orbital stability/instability of ground state solutions for a class of abstract Hamiltonian systems. This result was extended to another class of Hamiltonians of KdV type by Bona, Souganidis and Strauss [2]. Hamiltonian systems as the ones considered in [6] allow us to introduce the Lyapunov functional $F := H - cI$, where H is the Hamiltonian and I is a functional generated by the translation invariance of the equation (usually, mass or momentum). Here, c is the corresponding speed of the solitary wave. The stability of the solitary wave is then reduced to the understanding of the second variation of F, in the sense that $\partial^2 F > 0$ leads to stability. Also, if the former positive condition is not satisfied, but the corresponding nonpositive manifold is spanned by two elements (directions) which are associated to the two degrees of freedom of the solitary waves (scaling and shifts), it is still possible prove stability using $\partial^2 F$, but it is also necessary to restrict the class of perturbations to those which are orthogonal to the nonpositive directions.

Smereka in [19] studied the soliton of IB (6.1) and observed that this soliton fits into the class of abstract Hamiltonian system studied by GSS. However, it is not possible to apply the GSS method since an important hypothesis is not satisfied. In fact, he observed that $\partial^2 F$ is nonpositive on an infinite number of directions, where two of them can be associated to the point spectrum, and the remaining with the continuous spectrum. Therefore, GSS is useless in this case. However, the same author showed numerical evidence that if $\mathrm{d}I(Q_c)/\mathrm{d}c < 0$, then the solitary waves are stable, and if $\mathrm{d}I(Q_c)/\mathrm{d}c > 0$ the solitary waves seem to be unstable.

In a very important paper, Pego and Weinstein [17] proved (among other things) that Q_c is linearly exponentially unstable in H^1 when

$$1 < c^2 < \left(\frac{3(p-1)}{4+2(p-1)}\right)^2, \quad \text{with} \;\; p > 5.$$

Their method combines the use of the Evans function as well as ODE techniques. They also showed [18] that the linear equation around Q_c for $c \sim 1$ satisfies a **convective stability** property, based on the similarity of IB with KdV for small speeds. This result has been successfully adapted to a more general setting by Mizumachi in a series of works [13, 14]. Whether the asymptotic stability results by Martel and Merle [10, 11] and the recent work of Kowalczyk, Martel and Muñoz [9] can be applied to this case, **is a challenging problem that I would like to consider**. An interesting result in this direction can be found in the work [8] and my recent work [12].

Bibliography

[1] E.S. Bao, R.M. Chen and Q. Liu, *Existence and Symmetry of Ground States to the Boussinesq abcd Systems*. Arch Rational Mech Anal 216, 569?591 (2015) https://doi.org/10.1007/s00205-014-0814-1.

[2] J. Bona, P. Souganidis, and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. R. Soc. Lond. A 1987 411, 395–412.

[3] M. Chen, N.V Nguyen, and S-M Sun, *Existence of traveling-wave solutions to Boussinesq systems*, Differential Integral Equations 24 (2011), no. 9–10, 895–908

[4] M. Chen, N.V Nguyen, and S-M Sun, *Solitary-wave solutions to Boussinesq systems with large surface tension*, Discrete Contin. Dyn. Syst. 26 (2010), no. 4, 1153–1184.

[5] J. C. Eilbeck, J. E. Furter and M. Grinfeld, *On a stationary state characterization of transition from spinodal decomposition to nucleation behaviour in the Cahn-Hilliard model of phase separation*, Phys. Lett. A 135 (1989), no. 4-5, 272–275.

[6] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, Journal of Functional Analysis, 74, 160-197 (1987).

[7] S. Hakkaev, M. Stanislavova, and A. Stefanov, *Spectral Stability for Subsonic Traveling Pulses of the Boussinesq "abc" System*, SIAM Journal on Applied Dynamical Systems 12(2), pp. 878–898.

[8] M. Kowalczyk, Y. Martel, and C. Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, to appear in J. Eur. Math. Soc.

[9] M. Kowalczyk, Y. Martel, C. Muñoz, and Hanne Van Den Bosch, *A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models*, `https://arxiv.org/abs/2008.01276`.

[10] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), no. 3, 219–254.

[11] Y. Martel, and F. Merle, *Asymptotic stability of solitons for subcritical gKdV equations revisited*, Nonlinearity, **18** (2005), no. 1, 55–80.

[12] C.Maulén, *Asymptotic stability manifolds for solitons in the generalized Good- Boussi-*

nesq equation, preprint.

[13] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$, Mem. Amer. Math. Soc. 238 (2015), no. 1125, vii+95 pp.

[14] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$*. II.* Proc. Roy. Soc. Edinburgh Sect. A 148 (2018), no. 1, 149–198.

[15] A. Novick-Cohen and L. A. Segel, *Nonlinear aspects of the Cahn-Hilliard equation*. Phys. D 10 (1984), no. 3, 277–298.

[16] A. Novick-Cohen, *Energy methods for the Cahn-Hilliard equation*. Quart. Appl. Math. 46 (1988), no. 4, 681–690.

[17] R. Pego, M. Weinstein, *Eigenvalues, and instabilities of solitary waves*. Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[18] R. Pego, M. Weinstein, *Convective Linear Stability of Solitary Waves for Boussinesq Equations*, Studies in Applied Mathematics, 99, pp. 311-375 (1997).

[19] P. Smereka, *A remark on the solitary wave stability for a Boussinesq equation*. Nonlinear dispersive wave systems (Orlando, FL, 1991), 255-263, World Sci. Publ., River Edge, NJ, 1992.