



Articulating bargaining theories: movement, chance, and necessity as descriptive principles

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Abstract

The Nash Demand Game (NDG) has been one of the first models (Nash in *Econometrica* 21(1):128–140, 1953. <https://doi.org/10.2307/1906951>) that has tried to describe the process of negotiation, competition, and cooperation. This model has had enormous repercussions and has leveraged basic and applied research on bargaining processes. Therefore, we wonder whether it is possible to articulate extensive and multiple developments into a single unifying framework. The Viability Theory has this inclusive approach. Thus, we investigate the NDG under this point of view, and, carrying out this work, we find that the answer is not only affirmative but that we also advance in characterising viable NDGs. In particular, we found foundations describe the distributive Bargaining Theory: the principle of movement and the principle of chance and necessity. Finally, this initial work has many interesting perspectives. The probably most important idea is to integrate developments of the Bargaining Theory and thus capture the complexity of the real world in an articulated way.

Keywords Bargaining Theory · Viability Theory · Nash Demand Game

1 Introduction

The seminal work of Nash (1953) posed three important questions that remain open in distributive Bargaining Theory: How to reach an equitable agreement? How to discriminate between multiple equilibria? How do bargainers come to a break in a negotiation process? Indeed, in the original game proposed by Nash (the so-called

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Nash Demand Game), there is a multiplicity of equilibria that implies an infinite continuum of agreements.

1.1 The classic Nash demand game

The Nash Demand Game is a simultaneous static non-cooperative game, in which the parties have complete knowledge and do not communicate. The Nash Demand Game is a static bargaining process that can be described as follows. Suppose two negotiators are trying to distribute a pie of size 1. Both negotiators observe the current state of the negotiation represented by fractions of the cake demanded by them. If the total amount requested by the bargainers is less or equal than 1, both negotiators get their demand. If the total quantity is greater than 1, neither player gets their request, and the game ends.

We propose in this article to consider a repeated version of this game as follows: at each step, players replay simultaneously under the same conditions until they reach an agreement or a breakdown. After observation, both offer to increase or decrease their demand. This dynamic version can be considered a simplified version of the bargaining model proposed by Rubinstein (1982) in which players play alternately. In fact, in the *Rubinstein bargaining model*, the first player makes an offer, if the second player rejects, the game continues to the next period in which the second player makes an offer, if the first rejects, the bargainers move to the third period, and so on. Additionally, delays are costly in the Rubinstein bargaining model.

More formally, consider two bargainers, Emile and Frances, want to distribute a pie of size 1. Each player simultaneously chooses a portion of the pie, say x (for Emile) and y (for Frances), such that $x, y \in [0, 1]$. Let us define $\mathcal{H} \doteq \mathbb{R}^2$ the event space and the subset of possible agreements by

$$C \doteq \{(x, y) \in \mathcal{H} : x, y \in [0, 1] \text{ and } x + y \leq 1\}.$$

The loss functions of Emile and Frances are given by

$$f_E : \mathcal{H} \rightarrow]-\infty, +\infty] : (x, y) \mapsto f_E(x, y) \doteq -x + \iota_C(x, y)$$

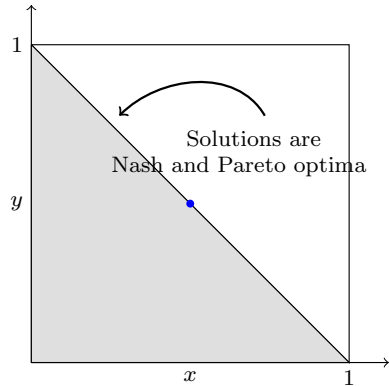
and

$$f_F : \mathcal{H} \rightarrow]-\infty, +\infty] : (x, y) \mapsto f_F(x, y) \doteq -y + \iota_C(x, y),$$

respectively, where ι_C is the indicator function.¹ Each player's problem must satisfy two classes of constraints: a *feasibility* constraint $x, y \in \mathbb{R}_+$ and an *agreement* constraint $x + y \leq 1$. The intersection of both subset will be called the *set of possible agreements*. Observe that given these constraints, the strategies must belong to the interval $[0, 1]$, i.e., $x, y \in [0, 1]$. For instance, since Frances' strategy is $y \in \mathbb{R}_+$, a

¹ $\iota_C(x, y)$ takes the value 0 if $(x, y) \in C$ and $+\infty$ otherwise. We have included all mathematical definitions in the "Appendix" for completeness and consistency.

Fig. 1 The classic Nash Demand Game poses three main questions: How to reach an equitable solution? How to discriminate between multiple equilibria? How do you get to a negotiation breakthrough? The coloured area represents the set of possible agreements while the blue point represents the unique equitable solution



feasible strategy for Emile satisfying the constraint $x + y \leq 1$ necessarily implies that $x \leq 1$.

According to the classical Nash Demand Game, each player tries to minimise her own loss function until an equilibrium is reached, if it exists. Let $(\bar{x}, \bar{y}) \in C$ be a possible Nash equilibrium bi-strategy, then it must satisfy $f_E(\bar{x}, \bar{y}) \leq f_E(x, \bar{y})$ and $f_F(\bar{x}, \bar{y}) \leq f_F(\bar{x}, y)$ for any $(x, y) \in C$. Additionally, we say that the bi-strategy $(\tilde{x}, \tilde{y}) \in C$ is a Pareto optimum if there not exist another bi-strategy $(x, y) \in C$ such that $f_E(x, y) < f_E(\tilde{x}, \tilde{y})$ and $f_F(x, y) < f_F(\tilde{x}, \tilde{y})$. It is not difficult to demonstrate the following Proposition.

Proposition 1 *Let us define the classical Nash Demand Game by the loss functions f_E, f_F and by the subset of possible agreements C as before. Then the set of Nash equilibrium strategies is given by $\{(x, y) \in C : x + y = 1\}$, which coincides with the set of Pareto optimal bi-strategies.*

Proof See Laengle and Loyola (2010).

Figure 1 illustrates the main questions poses the classic Nash Demand Game: How to reach an equitable solution? How to discriminate between multiple equilibria? How do you get to a negotiation breakthrough? The coloured area represents the set of possible agreements while the blue point represents the unique equitable solution.

Previous articles also describe the Nash Demand Game with a unique equilibrium but achieve this either by introducing uncertainty into the game’s information structure, extending the game to a multi-stage framework with a schedule of offers and counteroffers (Rubinstein 1982), or formulating an evolutionary version of the game (Young 1993). Alternatively, the equilibrium problem presented here can also be generalised and formulated using variational inequality models (see for instance Chinchuluun et al. 2008; Giannessi et al. 2004 and Konnov 2007).

To address the three fundamental questions underlying the Nash Demand Game, we have proposed in previous works a model with externalities. In this model, the parties select a smaller set of equilibria and, eventually, the equitable solution. It is a model in which agents suffer from externalities that can represent envy or resentment. Thus, the agents not only prefer a larger pie part but also feel repulsion of the counterpart pie part.

The base model can be found in Laengle and Loyola (2010), it has been applied to the ultimatum game (Laengle and Loyola 2015) and extended to asymmetric information (Laengle and Loyola 2014). Probably the most interesting interpretation of this model is that of a highly polarised environment (see the case of negotiations with the FARC in Colombia under this perspective, Laengle et al. 2020).

In the following subsections, we will present a summary of the axiomatic proposals underpinning the Nash Demand Game and then describe the Nash Demand Game's relationship with the research in distributive Bargaining Theory.

1.2 Explanatory axioms of the Nash demand game

To explain how agents reach a fair settlement, Nash (1953) introduced the now-known Nash Bargaining Solution. The starting point of this solution assumes that the final agreement solution should meet some desirable properties such as symmetry and Pareto optimality. Inspired by this work, other solutions have been proposed based on slightly different axioms concerning the properties of the final agreement point. Here are the main ideas.

The Nash Bargaining Solution is a final agreement that assumes a set of desirable axioms or properties of the solution and the utility functions of the parties. Supposing such requirements, we get an expression of a unique utility function that the parties should agree to reach that solution.

The first axiom refers exclusively to utility functions and is called *invariance*. A utility function is said to be invariant if maintains, after a transformation, the same ordering over preferences and so should not alter the outcome of the bargaining process. In particular, Nash's original work requires invariance of utility functions for linear transformations.

The second axiom refers to utility functions and the set of possible agreements. This property is called *Pareto optimality*. It consists of that that a solution to a bargaining game should come up with an agreement which is desirable for both players. Once an agreement has been reached, an arbitrator, therefore, will not propose an alternative that improves the loss function of one player and not improve the of other players.

The third axiom is called *independence of irrelevant alternatives*. It can be expressed as follows: if A is preferred to B , the introduction a third option C , expanding the choice set to $\{A, B, C\}$, do not make B preferable to A . This property also refers to utility functions through preferences.

The fourth axiom refers to the need for justice and is called *symmetry*. A reasonable arbitrator will require fairness: if the game is symmetric, i.e., both negotiators have the same bargaining power, then the solution proposed to each player should not favour a party. This property, like the one above, is also related to the preferences that the parties have individually.

From the previous axioms and assuming the parties cooperate and communicate, Nash proved that the utility function satisfying these four axioms is the function $f(x, y) \doteq xy$. That is to say, both sides agree to maximise this common function, which has at most the equitable solution.

This set of axioms has inspired further work by slightly modifying the original set. The first proposed axiom is called *resource monotonicity*, a fair division principle. It states that, if there are more resources to distribute, then all agents should be weakly better off, i.e., no player should lose from the increase in resources. This principle has been studied extensively in various distribution problems (Moulin 2003).

E. Kalai and M. Smorodinsky proposed an alternative axiom to the monotonicity of resources called the *independence of irrelevant alternatives*. Such assumptions lead to the so-called Kalai–Smorodinsky bargaining solution. It can be stated as follows: if alternatives of a set were not selected when the set was available and if they are not available anymore, the solution should have the same outcome.

Kalai (1977) also subsequently introduced a third solution, which drops the condition of invariance while including both the axiom of independence of irrelevant alternatives and the axiom of resource monotonicity. This solution attempts to grant equal benefit to the parties. In other words, it is the point which maximises the minimum payoff among players. Kalai notes that this solution is related to the ideas of John Rawls expressed in Rawls (2009).

Additionally to the formulation of a set of axioms that explain the desired behaviour of negotiators, the Nash Demand Game has also stimulated the development of applications of Bargaining Theory that we summarise below.

1.3 The Nash demand game and research in Bargaining Theory

The Nash Demand Game raises fundamental questions that have prompted a variety of research agendas, including the axiomatic proposals we summarised in the previous subsection. The Bargaining Theory has also resulted in a huge number of applications, expansions, and generalisations. We can find in the literature various ways of grouping the concepts that define certain archetypes of problems allowing to synthesise and organise concepts in a unified way. In particular, we can cite the work of Muthoo (1999) and Vetschera (2013).

Muthoo (1999) proposes in his book a typology based on the forces that drive the strategic interaction of negotiators. In words of the author:

The chapters are organised around the main forces that determine the bargaining outcome. I not only analyse the impact on the bargaining outcome of each force but I also often analyse the relative impacts of two or more forces. And, secondly, from an applied perspective, I show how the theory can be fruitfully applied to a variety of economic problems (Muthoo 1999, xiii).

A. Muthoo proposes a typology based on the relative power that negotiators and on applications in the field of economic theory. This classification is visible after a quick review of his book, which, according to us, can be summarised as follows:

1. The first group of forces drives the negotiation process to equitable solutions. The Nash Bargaining Solution (Nash 1953), and the extended model of the Rubinstein (1982), are particularly important in this group.

2. A second group has to do with forces leading to the negotiation breakdown. Phenomena as inside and outside opportunities, asymmetric information, commitment tactics, and repeated bargaining are relevant in this group.
3. Both groups are intertwined and articulated with a significant number of applications such as the bribery and the control of crime, union-firm negotiations, moral hazards in teams, sovereign debt negotiations, and the game of wars of attrition, just to quote a few.

On the other hand, the proposed archetypes of R. Vetschera, published in Vetschera (2013), is also of great interest and is different from that of A. Muthoo. In words of the author:

[It] provides a comprehensive survey of process models of negotiations. We consider models of the substantive process of offer exchange, as well as models focusing on communication content in negotiations, both at the level of individual actions and interactions, and at more aggregate levels. These different models are integrated into a comprehensive framework, and open areas for research are identified (Vetschera 2013, 135).

R. Vetschera structures negotiations as collective decision processes or as an outcome of individual decision processes. So it is possible to look the entire negotiation process as one decision process reflects a macro or global perspective (Koeszegi and Vetschera 2010; Olekalns et al. 2003), while individual decisions or interactions reflect a micro- or local perspective.

Taking into account both perspectives, we postulate that it is also possible to study negotiation processes by using the intuitions of the mathematical Viability Theory. This theory is based on the fact, that many systems and organisations that arise in biology and social sciences, evolve in a Darwinian way, subject to random fluctuations and restricted to remain in a viable environment (Aubin and Cellina 1984; Aubin 2009; Aubin et al. 2011).

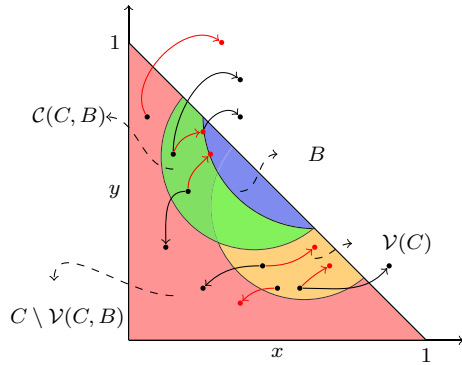
In the next section, we introduce mathematical elements of the Viability Theory that apply to the Bargaining Theory. Then, in Sect. 3, we apply the theory to the problem of the Nash Demand Game and Nash Bargaining Solution. In this way, we are trying to discover how the Viability Theory can help us answer fundamental questions regarding the behaviour of agents in a bargaining process.

2 A unifying perspective of bargaining processes

Inspired by the Viability Theory (Aubin and Cellina 1984; Aubin 2009; Aubin et al. 2011), we postulate in this article two unifying principles of the players behaviour in a distributive negotiation environment. The main results of the Viability Theory provide us with a rich analogy with negotiation processes. In particular, it allows us to define a negotiation as the strategic interaction between the parties that drives bargaining trajectories.

The first principle is *movement*. It means that negotiating paths move between regions, sets of agreements, or eventually drift to disagreement zones. The minimum

Fig. 2 Movements occur between zones. There are one or more zones that are the most preferred as the equilibria zone (target in blue). Then a viable capture basin of the target zone that coloured green. Coloured yellow the starting points of at least one trajectory will remain viable. Finally, in red are represented all trajectories that lead to disagreement (colour figure online)



set of desirable states is called the set of possible agreements or set of viable states. More precisely, this viable set contains initial states of at least one viable trajectory forever. This set may contain, in particular, trajectories that eventually lead to equilibrium or socially optimal states. A non-viable set, on the other hand, is one that contains disagreement states or contains start points that only lead to a breakdown.

The second principle assumes that agents move guided by *chance and necessity*. The *necessity* requirement says that at each instant, a trajectory of possible agreements needs to remain viable. On the other hand, negotiation processes require options, forces or mechanisms that do not violate restrictions on possible agreements. They constitute the *chance* for agents to guide the negotiation process to the desired zone.

In the following subsections, we will describe how negotiation processes can be interpreted in the light of these principles.

2.1 First principle: movement to desirable zones

Thus, negotiation processes are dynamic systems whose trajectories move between more or less desirable areas and are guided by agents according to chance and necessity. Such zones or sets of states can be mathematically characterised in the Viability Theory, in fact, Aubin (2001) proposes a typology of viable sets that we will use in this section.

There are one or more zones that are the most preferred, for example, the Nash or Pareto bi-strategy equilibrium zone. We call this zone target area (coloured in blue in Fig. 2). For instance, a Pareto area is very desirable if it is also equilibrium (not necessarily Nash equilibrium). In addition to the target zone, there is a region of possible agreements that contains starting points such that have at least a trajectory that leads to a target zone. It is a viable capture basin of the target zone and is coloured green in Fig. 2. Thirdly, there is a desirable base or minimum area that we call the viable zone of all possible agreements. This third region ensures that, if you start in it, there will be a trajectory that remains in the negotiation even if it does not reach the final solution (yellow zone in Fig. 2). Finally, there is a red region that we will call death zone, so-called, because all the trajectories that start in that area lead to a disagreement.

Each area of possible agreements will be mathematically defined and characterised by applying the Viability Theory in Sect. 3. Table 1 introduces definitions and notation we will use.

There are movements between sets that are more desirable than others. In fact, among all these sets there are clear preferences. It is clear, for example, that the target zone B (blue zone) is the set *most preferred*, and $C \setminus \mathcal{V}(C, B)$ (red zone) is the set *least preferred*. Indeed, the red zone acts as the *repellent* of the negotiation process, that is: a process that begins in that area ends irretrievably in a break. Also, the capture basin of B that we have denoted $\mathcal{C}(C, B)$ (green zone) is preferred to the purely viable area $\mathcal{V}(C)$ (yellow zone). Additionally, the set of trajectories that are viable in C forever or until they reach the target B in fine time is denoted by $\mathcal{V}(C, B)$ and is equal to $\mathcal{V}(C) \cup \mathcal{C}(C, B)$. Let us then assume that we define an order \preceq on the set 2^C , then this order should satisfy at least the following relationships

$$(C \setminus \mathcal{V}(C, B)) \preceq \mathcal{V}(C) \preceq \mathcal{C}(C, B) \preceq B.$$

We think that this approach, based on the principles of movement, and chance and necessity, offers a very suggestive way of unifying and understanding the negotiation processes. In the following subsection, we introduce the principle of chance and necessity.

2.2 Second principle: chance and necessity

To explain the second principle we will resort to concepts of the Viability Theory making use of differential inclusions and convex analysis in metric spaces. Thus, we will model a bargaining trajectory as a sequence of states $\{x_n\}_{n \in \mathbb{N}}$ that are guided by a collective bargaining rule φ as follows

$$x_{n+1} - x_n \in \varphi(x_{n+1}).$$

The basic problem of viability can be posed as follows: Given a restriction of possible agreements C find the set of initial states $x \in C$, such that there is at least one bargaining trajectory $\{x_n\}_{n \in \mathbb{N}}$ guided by φ to remain viable forever, i.e.,

$$x_0 \doteq x \text{ and } (\forall n \in \mathbb{N}) x_n \in C.$$

With this formulation we can interpret clearly the principles of the Bargaining Theory and express them as follows: Determine conditions and find negotiating trajectories $\{x_n\}_{n \in \mathbb{N}}$ that

$$(\forall n \in \mathbb{N}) \underbrace{x_{n+1} - x_n}_{\text{trajectory}} \in \underbrace{\varphi(x_{n+1})}_{\text{chance}} \subset \underbrace{C}_{\text{necessity}}.$$

Conditions for viability is one of the most important concepts of the Viability Theory and Nonlinear Analysis. Given a bargaining state x , the Viability Theory provides a

Table 1 The first principle has to do with movement. Bargaining trajectories move between zones that are ordered by preferences. Indeed, $(C \setminus \mathcal{V}(C, B)) \preceq \mathcal{V}(C) \preceq \mathcal{C}(C, B) \preceq B$

Zone	Color	Name	In symbols	Explanation
0	—	Set of possible agreements	C	All possible agreements, including viable, equilibrium, or those that only lead to a breakdown
1	Blue	Target set of C	$B, B \subset C$	Possible agreements in equilibrium considered the best
2	Green	Capture basin of target B from C	$\mathcal{C}(C, B)$	Subset of possible viable agreements that can at least lead to B
3	Yellow	Viability kernel of C	$\mathcal{V}(C)$	Subset of viable possible agreements in C
4	Blue or Green or Yellow	Viable subset of C with target B	$\mathcal{V}(C, B)$	Subset of possible agreements that are viable in C forever or until they reach the target B in finite time
5	Red	Death zone	$C \setminus \mathcal{V}(C, B)$	Subset of possible agreements that only conduct to a disagreement

condition (1) on the set of possible agreements C and (2) on the possible collective chances φ for satisfying the *necessity* to remain viable. This condition is called the tangential condition formulated as follows

$$(\forall x \in C) \varphi x \cap T_C x \neq \emptyset,$$

where $T_C x$ represents the set of possible viable states that start from an initial state $x \in C$. If there is at least one element of $T_C x$ that is possible to obtain from the collective bargaining rule φx , then we say that the *chance* exists for satisfying the *necessity* C .

Below we introduce the mathematical foundations of the Viability Theory that we will apply to the Bargaining Theory. We present our proposal in the context of Hilbert spaces. Thus, it is possible to represent in our formulation a wide range of stochastic processes (e.g., random variables with finite second moment) as well as cases in continuous or discrete time.

2.3 Adding formal concepts from the Viability Theory

The negotiating study involves describing and investigating how negotiators select alternatives to reach agreements, solutions, optima, equilibria or, eventually, negotiating breaks. From the simple illustration in the previous section, we will add formal concepts of the Viability Theory.

Let us consider \mathcal{H} the **event space**, i.e., the universe of all values states of negotiation, agreements, solutions, and, eventually, random variables. The event space \mathcal{H} is a Hilbert space.² The **possible agreements set** is a subset set C of \mathcal{H} . We will consider C a nonempty compact convex subset of \mathcal{H} . There are possible agreements that only arrive at a disagreement, that are therefore not viable. We are interested in finding a subset of possible agreements C , from where at least one negotiating trajectory that remains viable begins. This set is called *viability kernel*. There are also viable points of agreements, from which it is possible to reach a subset of desired agreements (such as the fair solution) or *target set*. This subset will also be important to us we call it *viable capture basin* of some target set. We will mathematically define both sets and others in this subsection.

Now, let us consider again our negotiators Emil and Frances. Emil's task is to select a **strategy** x in the set E and that of Frances is to select a decision y in the set F . We will consider that E and F are nonempty compact convex subsets of \mathcal{H} . The pair $(x, y) \in E \times F$ is called the bi-strategy. The selection (x, y) carried out by Emil and Frances is conducted by decision rules. That is, Emil's **decision rule** is a set-valued map $y \mapsto \varphi_E y$, which associates each state $y \in F$ of Frances with the state $x \in \varphi_E y$, which may be offered by Emil (by selecting $x \in E$) when he knows that Frances is demanding y . Similarly, a decision rule of Frances is a set-valued map $x \mapsto \varphi_F x$, which associates each demand $x \in E$ of Emil with possible demands $y \in \varphi_F x$, which may be offering by Frances (by selecting $y \in F$) when she knows

² In order not to lose the fluency of the reading we have left in the "Appendix" definitions of greater mathematical rigour.

that Emil is demanding x . Let $\varphi \doteq \varphi_E \times \varphi_F$ be the set-valued map defined pointwise by $(x, y) \mapsto \varphi(x, y) \doteq \varphi_E y \times \varphi_F x$. Thus, given the bargaining rules φ_E and φ_F that describe the behaviour of Emil and Frances respectively, the set-valued φ describes the behaviour of collective bargaining trajectories. By abusing the notation, we will call the composed decision rule φ as a decision rule. Mathematically, $\varphi : C \rightarrow 2^{\mathcal{H}}$ will be an upper hemicontinuous set-valued operator with nonempty closed convex values.

Moreover, we will define a **bargaining trajectory** (o simply trajectory) as a sequence of states of negotiation $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ defined in \mathcal{H} driven by φ , i.e., such that

$$(\forall n \in \mathbb{N}) (x_n, y_n) \in (x_{n+1}, y_{n+1}) - \varphi(x_{n+1}, y_{n+1}).$$

In particular, we are interested in trajectories that remain on the bargaining set. We call this type of trajectories a viable trajectory, while the set of viable trajectories is called a viable set. That is, we define a bargaining set as viable if for every starting state $(x, y) \in C$ there is at least a sequence driven by φ such that the trajectory remains at C . More formally, let us remember that C is a compact convex subset of \mathcal{H} , so we define $\mathcal{V}(C)$ the **viability kernel of C** as the set of initial states $(x, y) \in C$ such that there exists a trajectory $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ driven by φ with $(x_0, y_0) \doteq (x, y)$ such that

$$(\forall n \in \mathbb{N}) (x_n, y_n) \in C.$$

Let $B \subset C$ a **target set**. Let $\mathcal{C}(C, B)$ the **viable capture basin** of C with target B is the set of initial states $(x, y) \in C$ such that there exists a trajectory $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ driven by φ with $(x_0, y_0) \doteq (x, y)$ and $k \in \mathbb{N}$ such that

$$(\forall n \in \{0, \dots, k\}) (x_n, y_n) \in C \text{ and } (x_k, y_k) \in B.$$

Finally, the subset of possible agreements that are viable in C forever or until they reach the target B in finite time is called the **viability kernel with target B** and is given by

$$\mathcal{V}(C, B) \doteq \mathcal{V}(C) \cup \mathcal{C}(C, B).$$

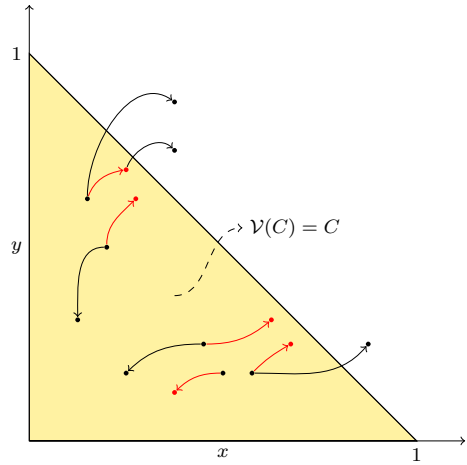
Let us observe that if $B \doteq \emptyset$, i.e., players do not have a target set, then $\mathcal{C}(C, \emptyset) = \emptyset$ and $\mathcal{V}(C, \emptyset) = \mathcal{V}(C)$.

In this context, we will describe in the next section the fundamental questions of mathematical Bargaining Theory.

3 Applying the Viability Theory to the Nash demand game

In this section, we will try to describe trajectories or movements of the distributive Bargaining Theory. The first problem tries to answer the following question: Given a bargaining set C of an event space \mathcal{H} , what characteristics should a decision rule have

Fig. 3 The set of possible agreements is represented by the triangle C . The first question we try to answer is: What are the values μ such that the set C remains viable, i.e., for each start point in C there exists a bargaining trajectory remains in C ? (colour figure online)



of φ such that it is able to drive a trajectory that remains in the bargaining set? The second problem is describing the Nash Demand Game’s movements or negotiating trajectories. Finally, we will describe the movements of the trajectories proposed by the Nash Demand Game in a more general context of polarised negotiation.

3.1 Modelling the viability principles

Let us consider $\mathcal{H} \doteq \mathbb{R}^2$, the subset $C \doteq \{(x, y) \in \mathcal{H} : x, y \in [0, 1] \text{ and } x + y \leq 1\}$. Let $\alpha, \beta \in \mathbb{R}$ and $\mu \in \mathbb{R}_+$, we will also consider the set-valued operator

$$\varphi : C \rightarrow 2^{\mathcal{H}} : (x, y) \mapsto \varphi(x, y) \doteq \{(\alpha x + \beta y + p, \beta x + \alpha y + q) : p, q \in [-\mu, +\mu]\}.$$

The question we will try in this subsection is related to the Fig. 3. The first question we try to answer is: What are the values μ such that the set C remains viable, i.e., for each start point in C there exists a bargaining trajectory remains in C ? The following Theorem answers the question and its proof is based on the dual tangential condition.

Theorem 1 *Let C be the bargaining subset and φ the bargaining rule defined as before. Define $B \doteq \emptyset$. A sufficient condition for the viability of C under φ is that*

$$\mu \geq \max\{|\alpha|, |\beta|, (\alpha + \beta)/2\}.$$

Furthermore, $\mathcal{V}(C) = C$, $\mathcal{C}(C, \emptyset) = \emptyset$, and $\mathcal{V}(C, \emptyset) = C$.

Proof First, let us observe that C is a bargaining set of the event space \mathcal{H} and φ a legal bargaining rule. We will try to find the value μ which assure that the entire C is the kernel of viability of C . For doing this, we will use the dual tangential condition, which is equivalent to the tangential condition (see Proposition 5 in “Appendix”). The dual tangential condition says

$$(\forall (x, y) \in C)(\forall (u, v) \in N_C(x, y)) \sup\langle \varphi(x, y) | - (u, v) \rangle \geq 0. \tag{1}$$

Since for all $(x, y) \in \text{int } C$, we have $N_C(x, y) = (0, 0)$, then the inequality is trivially satisfied. Thus, we will continue assuming that $(x, y) \notin \text{int } C$.

Second, let us prove that the dual tangential condition (1) is equivalent to the following expression

$$(\forall \theta \in [0, 2\pi[)(\forall (x, y) \in S(\theta)) \sup\langle \varphi(x, y) | -(\cos \theta, \sin \theta) \rangle \geq 0, \tag{2}$$

where $S(\theta) \doteq \{(x, y) \in \mathcal{H} : \sup\langle C | (\cos \theta, \sin \theta) \rangle = x \cos \theta + y \sin \theta\}$. Indeed, let us assume the expression (1) is true and fix $\theta \in [0, 2\pi[$ and $(x, y) \in S(\theta)$, thus $\sup\langle C | (\cos \theta, \sin \theta) \rangle = x \cos \theta + y \sin \theta$. Now, replace u, v for $u \doteq r \cos \theta$ and $v \doteq r \sin \theta$ with $\|(u, v)\| = r^2 \neq 0$, whence $\sup\langle C | (u, v) \rangle = xu + yv$, i.e., $(u, v) \in N_C(x, y)$, therefore $\sup\langle \varphi(x, y) | -(\cos \theta, \sin \theta) \rangle \geq 0$. The inverse implication (from expression (2) to (1)) follows the same argumentation.

Third, we need to find $\mu \in \mathbb{R}_+$ such that for all $\theta \in [0, 2\pi[$ and for all $(x, y) \in S(\theta)$ the following dual tangential condition is satisfied

$$\sup\{(\alpha x + \beta y + p, \beta x + \alpha y + q) | -(\cos \theta, \sin \theta)\} : p, q \in [-\mu, +\mu] \geq 0,$$

or, equivalently, we need to find $\mu \in \mathbb{R}_+$ such that for all $\theta \in [0, 2\pi[$

$$\begin{aligned} &\sup\{(\alpha x + \beta y) \cos \theta + (\beta x + \alpha y) \sin \theta : (x, y) \in S(\theta)\} \\ &+ \inf\{p \cos \theta + q \sin \theta : p, q \in [-\mu, +\mu]\} \leq 0. \end{aligned} \tag{3}$$

We will proceed case by case:

(i) The first case is $\theta \in]3\pi/2, 2\pi[$, whence $(x, y) = (1, 0)$. Let us observe that $\cos \theta > 0$ and $\sin \theta < 0$. The inequality (3) becomes for all $\theta \in]3\pi/2, 2\pi[$

$$\alpha \cos \theta + \beta \sin \theta - \mu \cos \theta + \mu \sin \theta \leq 0,$$

which becomes $(\alpha - \mu) \cos \theta + (\beta + \mu) \sin \theta \leq 0$, which is satisfied if $\mu \geq \max\{|\alpha|, |\beta|\}$.

(ii) The second case is $\theta \in [0, \pi/4[$, whence $(x, y) = (1, 0)$. Let us observe that $\cos \theta > 0$ and $\sin \theta > 0$, then the inequality (3) becomes for all $\theta \in]0, \pi/4[$

$$(\alpha - \mu) \cos \theta + (\beta - \mu) \sin \theta \leq 0,$$

which is satisfied if $\mu \geq \max\{|\alpha|, |\beta|\}$.

(iii) The third case is $\theta \doteq \pi/4$, whence $S(\pi/4) = \{(x, y) \in C : x + y = 1\}$. Because $\cos \theta = \sin \theta = \frac{\sqrt{2}}{2}$, then inequality (3) becomes for all $(x, y) \in S(\pi/4)$

$$(\alpha x + \beta y) \frac{\sqrt{2}}{2} + (\beta x + \alpha y) \frac{\sqrt{2}}{2} - \mu \sqrt{2} \leq 0,$$

that is

$$\frac{\sqrt{2}\alpha(x + y) + \sqrt{2}\beta(x + y) - 2\sqrt{2}\mu}{2} \leq 0$$

which is satisfied if $\mu \geq (\alpha + \beta)/2$.

(iv) The fourth case is $\theta \in]\pi/4, \pi/2]$, whence $(x, y) = (0, 1)$. Let us observe that $\cos \theta \geq 0$ and $\sin \theta > 0$, then the inequality (3) becomes for all $\theta \in]\pi/4, \pi/2]$

$$(\beta - \mu) \cos \theta + (\alpha - \mu) \sin \theta \leq 0,$$

which is satisfied if $\mu \geq \max\{|\alpha|, |\beta|\}$.

(v) Now for the case $\theta \in]\pi/2, \pi[$, where $\cos \theta < 0$ and $\sin \theta > 0$, whence $(x, y) = (0, 1)$, then the inequality (3) becomes for all $\theta \in]\pi/2, \pi[$

$$(\beta + \mu) \cos \theta + (\alpha - \mu) \sin \theta \leq 0,$$

which is again satisfied if $\mu \geq \max\{|\alpha|, |\beta|\}$.

(vi) For the case $\theta \doteq \pi$, whence $S(\pi) = \{(0, y) \in C\}$. Since $\cos \theta = -1$ and $\sin \theta = 0$, then inequality (3) becomes for all $(x, y) \in S(\pi)$

$$-\beta y - \mu \leq 0,$$

which is satisfied if $\mu \geq |\beta|$.

(vii) Now for the case $\theta \in]\pi, 3\pi/2[$, where $\cos \theta < 0$ and $\sin \theta < 0$, whence $(x, y) = (0, 0)$, then the inequality (3) becomes for all $\theta \in]\pi, 3\pi/2[$

$$+\mu \cos \theta + \mu \sin \theta \leq 0,$$

which is trivially satisfied if $\mu \geq 0$ is selected.

(viii) Finally, the case $\theta \doteq 3\pi/2$, whence $S(3\pi/2) = \{(x, 0) \in C\}$. Since $\cos \theta = 0$ and $\sin \theta = -1$, then inequality (3) becomes for all $(x, y) \in S(3\pi/2)$

$$-\beta x - \mu \leq 0,$$

which is satisfied if $\mu \geq |\beta|$.

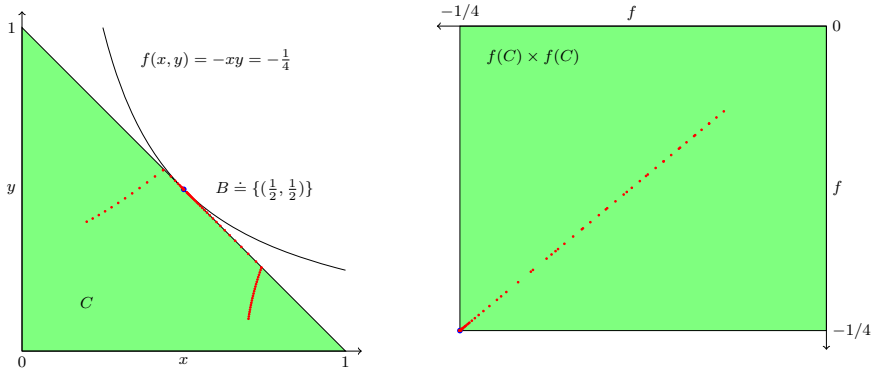
Therefore, we have demonstrated that if $\mu \geq \max\{|\alpha|, |\beta|, (\alpha + \beta)/2\}$, the entire set C is viable and the viability kernel of C is naturally $\mathcal{V}(C) = C$. \square

3.2 The Nash bargaining solution

Let be \mathcal{H} and C as in the previous subsection. The negotiators cooperate and agree to have the same loss function $f : \mathcal{H} \rightarrow]-\infty, +\infty] : (x, y) \mapsto -xy + \iota_C(x, y)$, thus the *linear* bargaining rule will be

$$\varphi : C \rightarrow 2^{\mathcal{H}} : (x, y) \mapsto \varphi(x, y) \doteq (y, x) - N_C(x, y),$$

where $N_C : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is the normal cone to C at $x \in C$ defined as in the ‘‘Appendix’’, and ι_C is the indicator function of the set C . Let us define the target set is $B \doteq \{(1/2, 1/2)\}$, we will try to find the sets $\mathcal{V}(C, B)$ and $\mathcal{V}(C)$.



(a) Two trajectories are shown with start points $(2/10, 4/10)$ and $(7/10, 1/10)$. (b) The set of Pareto optima is represented by the blue point.

Fig. 4 Nash bargaining solution. The triangle C represents the set of possible agreements. The question is about the trajectories movements between the sets $\mathcal{V}(C, B) = \mathcal{V}(C) = C = \mathcal{C}(C, B)$. All start points in C conducts to the equitable solution. Two trajectories are shown with start points $(2/10, 4/10)$ and $(7/10, 1/10)$ (colour figure online)

Proposition 2 *Let the bargaining problem be defined as before, then*

$$\mathcal{V}(C, B) = \mathcal{V}(C) = C = \mathcal{C}(C, B).$$

Proof Firstly, let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}_{++} such that $\sum_{n \in \mathbb{N}} \gamma_n = \infty$. It is easy to check that the bargaining rule defined before coincides with $-\partial f$, thus we can define the bargaining trajectory for all $(x, y) \in C$ by

$$(x_0, y_0) \doteq (x, y) \text{ and } (\forall n \in \mathbb{N}) (x_n, y_n) - (x_{n+1}, y_{n+1}) \in \gamma_n \partial f(x_{n+1}, y_{n+1}).$$

Since $f \in \Gamma_0(\mathcal{H})$ and $\text{Argmin } f \neq \emptyset$, then the trajectory is Fejér monotone with respect to $\text{Argmin } f$ according to the Theorem 5 in “Appendix”. Additionally, according to the Theorem 4, the trajectory converges strongly to a point in $\text{Argmin } f$.

Secondly, regarding that φ is a legal bargaining rule and satisfies the tangential condition on C , there exists a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ such that the trajectory $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ driven by $-\partial f$ is viable. Since the trajectory is strongly convergent to a point in $(x, y) \in \text{Argmin } f$, then $(x, y) \in \text{zer } \varphi$ according the Theorem 3.

Finally, since the former result is valid for all start point $(x, y) \in C$ for a bargaining trajectory, we obtain $\mathcal{V}(C, B) = \mathcal{V}(C) = C = \mathcal{C}(C, B)$. □

Figure 4 illustrates the result of Proposition 2 for the case of two start points: Two trajectories are shown with start points: $(2/10, 4/10)$ and $(7/10, 1/10)$.³

³ The reader can download the file in Notebook of Jupyter Lab from the link: <https://www.dropbox.com/s/0rzcjks6lehwhc/nbs.html?dl=0>.

3.3 A polarised Nash demand game

Let be \mathcal{H} and C as in the Sect. 3.1. The Polarised Nash Demand Game is an extension of the classical Nash Demand Game (Laengle and Loyola 2010). It considers that negotiators do not cooperate and they have envy modelled by an *envy factor* $\lambda \in [0, 1]$. They have the following loss functions:

$$f_E(x, y) : \mathcal{H} \rightarrow]-\infty, +\infty] : (x, y) \mapsto -x + \lambda y + \iota_C(x, y),$$

and

$$f_F(x, y) : \mathcal{H} \rightarrow]-\infty, +\infty] : (x, y) \mapsto -y + \lambda x + \iota_C(x, y).$$

Define the set

$$A \doteq \{(x, y) \in C : x, y \in [\lambda(1 + \lambda)^{-1}, (1 + \lambda)^{-1}]\}.$$

Thus the *linear* bargaining rule will be $\varphi : C \rightarrow 2^{\mathcal{H}}$ given by

$$\varphi(x, y) \doteq \begin{cases} (1, 1) - N_C(x, y) & \text{if } (x, y) \in A, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $N_C : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is the normal cone to C at $x \in C$ defined as in the ‘‘Appendix’’, and ι_C is the indicator function of the set C . Set the target set

$$B \doteq \{(x, y) \in C : x, y \in [\lambda(1 + \lambda)^{-1}, (1 + \lambda)^{-1}] \text{ and } x + y = 1\}.$$

Proposition 3 *Let the bargaining problem be defined as before, then the following holds*

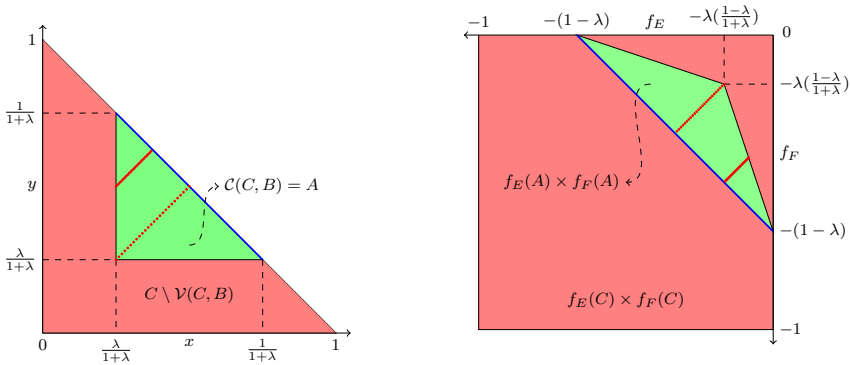
$$B \subset A = \mathcal{V}(C, B) = \mathcal{V}(C) = \mathcal{C}(C, B) \subset C.$$

Proof The proof follows the argument of Proposition 2 above, but we need to define carefully the bargaining rule φ for both negotiators. Firstly, let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}_{++} such that $\sum_{n \in \mathbb{N}} \gamma_n = \infty$. Let us suppose that $\text{zer } \varphi \neq \emptyset$ and $(x, y) \in \text{zer } \varphi$. Since $f_E, f_F \in \Gamma_0(\mathcal{H})$, it is legal to define the trajectories

$$x_0 \doteq x \text{ and } (\forall n \in \mathbb{N}) x_n - x_{n+1} \in \gamma_n \partial f_E(x_{n+1}, y)$$

and

$$y_0 \doteq y \text{ and } (\forall n \in \mathbb{N}) y_n - y_{n+1} \in \gamma_n \partial f_F(x, y_{n+1}),$$



(a) Two trajectories are shown with start points $(1/4, 1/4)$ and $(1/4, 1/2)$.

(b) The set of Pareto optima is represented by the blue line.

Fig. 5 A Polarised Nash demand game. The set of possible agreements is represented by the triangle C . The question is about the trajectories movements between the sets $B \subset A = \mathcal{V}(C, B) = \mathcal{V}(C) = \mathcal{C}(C, B) \subset C$. All start points in A conducts to equilibria solutions (colour figure online)

where \mathcal{E} and \mathcal{F} are two Hilbert spaces such that $\mathcal{E} \oplus \mathcal{F} = \mathcal{H}$ and for all $n \in \mathbb{N}$ $(x_n, y_n) \in \mathcal{E} \times \mathcal{F}$ holds. We define the bargaining rule as

$$\varphi : C \rightarrow 2^{\mathcal{H}} : \varphi(x, y) \doteq \begin{cases} \varphi_E(y) \times \varphi_F(x) & \text{if } (x, y) \in A, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\varphi_E : \mathcal{F} \rightarrow 2^{\mathcal{E}} : y \mapsto \varphi_E(y) \doteq -\partial f_E(\cdot, y)$ and $\varphi_F : \mathcal{E} \rightarrow 2^{\mathcal{F}} : x \mapsto \varphi_F(x) \doteq -\partial f_F(x, \cdot)$, which coincides with that one defined in the Proposition statement (see Laengle and Loyola 2010 for details).

With these definitions, by following the same arguments of Proposition 2, $\{(x_n, y)\}_{n \in \mathbb{N}}$ is Fejér monotone strongly convergent to (x, y) and likewise $\{(x, y_n)\}_{n \in \mathbb{N}}$ is Fejér monotone strongly convergent to (x, y) , therefore $\{(x_n, y)\}_{n \in \mathbb{N}}$ is strongly convergent to (x, y) and $(x, y) \in \text{zer } \varphi$.

Finally, in the article Laengle and Loyola (2010), it is demonstrated that B is the set of (Nash) equilibria, i.e., $B \subset \text{zer } \varphi$, and since the above result holds for all start point $(x, y) \in A$, we obtain that

$$B \subset A = \mathcal{V}(C, B) = \mathcal{V}(C) = \mathcal{C}(C, B) \subset C.$$

Thus, the Proposition is demonstrated. □

Figure 5 illustrates the result of Proposition 3 for the case of two start points: Two trajectories are shown with start points: $(1/4, 1/4)$ and $(1/4, 1/2)$.⁴

In the next section we will conclude our work.

⁴ The reader can download the file in Notebook of Jupyter Lab from the link: <https://www.dropbox.com/s/6e1jf3e66ee5u1p/polarised%20game.html?dl=0>.

4 Conclusions

The Nash Demand Game (Nash 1953) posed three important questions that remain open in distributive Bargaining Theory: How to reach an equitable agreement? How to discriminate between multiple equilibria? How to a negotiation breakdown is reached? The present work has been motivated to answer these questions. Nash's seminal model has stimulated and leveraged extensive and deep fields of research since 1953. On the one hand, it has motivated the formulation of a set of axioms that provide desired solutions and, on the other hand, has driven the Bargaining Theory and its applications. The question we ask ourselves here is, whether it is possible to articulate these developments into a single unifying framework that allows us to understand developments, theories, applications, and proposals in an integrated way.

To answer this question, we investigate the Viability Theory and apply it to three fundamental problems arising from the Nash Demand Game. This was done in three steps. First, we identify basic concepts of The Viability Theory and express them in terms of the Bargaining Theory. Second, we identify two principles that can explain, from a different or complementary perspective, the three basic questions of the distributive Bargaining Theory. Finally, we illustrated the application to three basic problems of the Bargaining Theory.

As a result of this work, we would like to highlight the following findings:

1. We think that the mathematical objects that study the Viability Theory fit correctly with the Bargaining Theory. Moreover, in applying the theory, we find two fundamental principles that offer a different or complementary look or perspective of the Bargaining Theory.
2. The basic principles governing every negotiation process are the principle of movement and the principle of chance and necessity. The *principle of movement* means that bargaining trajectories move between regions, sets of agreements, or eventually drift to break zones. The minimum set of desirable states is called the set of possible agreements or set of viable states. More precisely, this viable set contains initial states whose negotiating trajectory is at least viable forever. This set may contain, in particular, trajectories that eventually lead to equilibrium or socially optimal states. A non-viable set, on the other hand, is one that contains states of disagreement or contains starting points that only lead to a break.
3. The second principle assumes that agents move guided by chance and necessity. *Necessity* is the requirement that at each instant, a trajectory of possible agreements to remain at least a possible agreement. We say that the trajectory that satisfies this principle is a viable trajectory. On the other hand, negotiation processes require options, forces, or mechanisms, such that restrictions on possible agreements are not violated. In other words, options, forces, or mechanisms give the *chance* for agents to conduct the negotiation process into a desirable area.

Finally, we think that this work has interesting perspectives that stimulate subsequent developments integrating and articulating bargaining theories. This includes dynamic, eventually stochastic models, with or without strategic, and cooperative or non-cooperative interaction. More importantly, it allows us to model optimising versus

satisfying approaches in a unified way by trying to capture real-world complexity in an integrated form.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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Appendix: Mathematical definitions and main theorems

Throughout the paper \mathbb{N} denotes the set of nonnegative integers and \mathcal{H} is a real Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and associated norm $\| \cdot \|$. Let \mathcal{E} and \mathcal{F} be two Hilbert spaces such that $\mathcal{E} \oplus \mathcal{F} = \mathcal{H}$. Id denotes the identity operator. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in \mathcal{H} converges strongly to a point $x \in \mathcal{H}$ if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Let $f : \mathcal{H} \rightarrow [-\infty, +\infty]$ be a function and C be a subset of \mathcal{H} . We set $\text{dom } f \doteq \{x \in \mathcal{H} : f(x) \neq \emptyset\}$. The infimum of f over C is denoted by $\inf f(C)$, likewise the supremum of f over C by $\sup f(C)$. $\text{Argmin } f$ is defined by the set $\{x \in \mathcal{H} : f(x) = \inf f(\mathcal{H})\}$. The function f is proper if $-\infty \notin f(\mathcal{H})$ and $\text{dom } f \neq \emptyset$. The class of proper lower semicontinuous convex functions to $] - \infty, +\infty]$ is denoted by $\Gamma_0(\mathcal{H})$. The interior of a subset C is denoted by $\text{int } C$, and by ι_C its indicator function, which takes the value 0 on C and $+\infty$ on its complement.

Let C be a subset of \mathcal{H} . Let $\varphi : C \rightarrow 2^{\mathcal{H}}$ be a set-valued operator. The family of all neighbourhoods of $x \in \mathcal{H}$ is denoted by $\mathcal{V}(x)$. In the following, B will denote the closed unit ball of \mathcal{H} centred at the origin. We shall say that a set-valued operator φ is **upper hemicontinuous** at $x \in C$ if for all $u \in \mathcal{H}$, the function $x \mapsto \sup \langle \varphi x | u \rangle$ is upper semi-continuous at x . It is upper hemicontinuous if it is upper hemicontinuous at all points $x \in C$. Additionally, we shall say that a set-valued operator φ is upper semicontinuous at $x \in C$ if

$$(\forall \epsilon \in \mathbb{R}_{++})(\exists V \in \mathcal{V}(x))(\forall y \in V) \varphi y \subset \varphi x + \epsilon B.$$

It is upper semi-continuous if it is upper semi-continuous at all points $x \in C$. Any upper semicontinuous set-valued operator is upper hemicontinuous.

Let C and D be subsets of \mathcal{H} . Then C and D are **separated** if

$$(\exists u \in \mathcal{H} \setminus \{0\}) \sup \langle C | u \rangle \leq \inf \langle D | u \rangle$$

and strong separated if the above inequality is strict. Moreover, a point $x \in \mathcal{H}$ is separated from D if the sets $\{x\}$ and D are separated. Likewise, x is strongly separated from D if $\{x\}$ and D are strongly separated.

Proposition 4 (Corollary from the Separation Theorem)⁵ *Let C and D be nonempty closed subsets such that $C \cap D \neq \emptyset$ and D bounded. Then C and D are strongly separated.*

Let C be a subset of \mathcal{H} . The polar cone of C is

$$C^\ominus \doteq \{u \in \mathcal{H} : \sup\langle C|u \rangle \leq 0\}.$$

C^\ominus is always a nonempty closed cone. Let C be a nonempty convex subset of \mathcal{H} and let $x \in \mathcal{H}$. The **tangent cone** to C at x is defined by

$$T_C x \doteq \begin{cases} \overline{\text{cone}(C - x)} = \overline{\cup_{\lambda \in \mathbb{R}_{++}} \lambda(C - x)}, & \text{if } x \in C; \\ \emptyset, & \text{otherwise} \end{cases},$$

where \overline{C} denotes the closure of the subset C with respect to \mathcal{H} , and the **normal cone** to C at x is given by

$$N_C x \doteq \begin{cases} (C - x)^\ominus = \{u \in \mathcal{H} : \sup\langle C - x|u \rangle\} & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

Recalling that C is a nonempty convex subset, then for all $x \in C$ the following hold: $T_C x$ and $N_C x$ are nonempty closed convex cones, $T_C^\ominus x = N_C x$ and $N_C^\ominus x = T_C x$. Additionally, in case that \mathcal{H} is finite dimensional, for all $x \in \text{int } C$ we have $T_C x = \mathcal{H}$ and $N_C(x) = \{0\}$.

Let C be a nonempty compact convex subset of \mathcal{H} , φ upper hemicontinuous with nonempty closed and convex values. We shall say that the set-valued operator φ satisfies the **tangential condition** on C if

$$(\forall x \in C) \varphi x \cap T_C x \neq \emptyset.$$

If two set-valued operators φ_1, φ_2 satisfy the tangential condition on C , so do the set-valued operator $\alpha_1 \varphi_1 + \alpha_2 \varphi_2$, where α_1, α_2 are in \mathbb{R}_{++} .

Proposition 5 (Dual Tangential Condition) *Let C be a nonempty compact convex subset of \mathcal{H} , φ upper hemicontinuous with nonempty closed and convex values. If φ satisfies the tangential condition on C , then φ satisfies the **dual tangential condition** on C*

$$(\forall x \in C)(\forall u \in N_C x) \sup\langle \varphi x | -u \rangle \geq 0.$$

If, in addition, the values of φ are bounded, when φ satisfies the dual tangential condition on C , then it satisfies the tangential condition too.

⁵ See Bauschke and Combettes (2011), [56].

Proof Let us suppose that φ satisfies the tangential condition on C . Set $x \in C$ and $u \in N_C x$. Since $\varphi x \cap T_C x \neq \emptyset$, there exists $y \in C$ such that $y \in \varphi x \cap T_C x$ and

$$\inf \langle \varphi x | u \rangle \leq \langle y | u \rangle \leq 0,$$

because $N_C x = T_C^\ominus x$. Thus $\inf \langle \varphi x | u \rangle = -\sup \langle -\varphi x | u \rangle = -\sup \langle \varphi x | -u \rangle \leq 0$. Now, let us suppose that the φ satisfies the dual tangential condition on C and assume it does not satisfy the tangential condition. Then for all $x \in C$, we have $0 \notin \varphi x - T_C x$. Since $T_C x$ and φx are nonempty closed convex subsets of \mathcal{H} and φx is bounded by supposition, then φx and $T_C x$ are strongly separated by the Proposition 4, i.e.,

$$(\exists u \in \mathcal{H} \setminus \{0\}) \sup \langle T_C x | u \rangle < \inf \langle \varphi x | u \rangle,$$

which is a contradiction, because $\sup \langle T_C x | u \rangle = 0$. Therefore, φ satisfies the tangential condition on C . □

Set a sequence $\{x_n\}_{n \in \mathbb{N}}$ in \mathcal{H} and some sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in \mathbb{R}_{++} . We define the sequence $\{x_n\}_{n \in \mathbb{N}}$ driven φ as following

$$(\forall n \in \mathbb{N}) x_n \in x_{n+1} - \gamma_n \varphi x_{n+1}.$$

We say that the nonempty subset C is **viable** with respect to the set-valued operator φ if for all $x \in C$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in \mathcal{H} driven by φ such that

$$(\forall n \in \mathbb{N}) x_n \in C \text{ and } x_0 = x.$$

The set of zeros (or stationary points or **equilibria**) of φ is $\text{zer } \varphi \doteq \{x \in \mathcal{H} : 0 \in \varphi x\}$. Now we pose the more important Theorem that we shall use in this paper.

Theorem 2 (Zeros of set-valued operators) *Let C be a nonempty compact convex subset of \mathcal{H} , φ upper hemicontinuous with nonempty closed and convex values. If φ satisfies the tangential condition on C , then $\text{zer } \varphi \neq \emptyset$.*

Corollary 1 (Viability Theorem) *Let C be a nonempty compact convex subset of \mathcal{H} , φ upper hemicontinuous with nonempty closed and convex values. If φ satisfies the tangential condition on C , then C is viable with respect to φ .*

Proof Assume that φ satisfies the tangential condition on C . For demonstrating that C is viable with respect to φ , it is sufficient to show that

$$(\forall y \in C) \text{zer } (\varphi - \text{Id} + y) \neq \emptyset.$$

Indeed, for all $y \in C$, the set-valued operator $x \mapsto \{y - x\}$ satisfies the tangential condition on C , therefore the sum of both operators $x \mapsto \varphi - x + y$ also satisfies the tangential condition on C . By applying again the main Theorem 2 to the combined operator, we get the required expression. □

Theorem 3 (Convergence to an Equilibrium) *Let C be a nonempty compact convex subset of \mathcal{H} and $\varphi : C \rightarrow 2^{\mathcal{H}}$ an upper semicontinuous set-valued operator with nonempty closed convex values. Let us suppose a trajectory $\{x_n\}_{n \in \mathbb{N}}$ in C driven by φ . If $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $x \in C$, then $x \in \text{zer } \varphi$, i.e., x is an equilibrium.*

Proof See Theorem 6.5.2 of Aubin and Cellina (1984, 310). \square

Let C be a nonempty subset of \mathcal{H} and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{H} . Then $\{x_n\}_{n \in \mathbb{N}}$ is **Fejér monotone** with respect to C if

$$(\forall x \in C)(\forall n \in \mathbb{N}) \|x_{n+1} - x\| \leq \|x_n - x\|.$$

Theorem 4 *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{H} and let C be a nonempty closed convex subset of \mathcal{H} . Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to C , then $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to a point in C .*

Proof See Theorem 5.11 of Bauschke and Combettes (2011, 78). \square

Let $f : \mathcal{H} \rightarrow]-\infty, +\infty]$ be proper. The subdifferential of f is the set-valued operator

$$\partial f : \mathcal{H} \rightarrow 2^{\mathcal{H}} : x \mapsto \{u \in \mathcal{H} : (\forall y \in \mathcal{H}) \langle y - x | u \rangle + f(x) \leq f(y)\}.$$

If $x \in \text{dom } f$, then $\partial f(x)$ is closed and convex. Let $f : \mathcal{H} \rightarrow]-\infty, +\infty]$ be convex, and let $x \in \mathcal{H}$, and suppose that f is Gâteaux differentiable⁶ at x , then

$$(\forall y \in \mathcal{H}) \langle y - x | \nabla f(x) \rangle + f(x) \leq f(y).$$

The normal cone operator of a nonempty closed convex set $C \subset \mathcal{H}$ is $N_C = \partial \iota_C$.

Theorem 5 (Proximal-point Algorithm) *Let $f \in \Gamma_0(\mathcal{H})$ be such that $\text{Argmin } f \neq \emptyset$, let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}_{++} such that $\sum_{n \in \mathbb{N}} \gamma_n = +\infty$, and let $x_0 \in \mathcal{H}$. Set*

$$(\forall n \in \mathbb{N}) x_n - x_{n+1} \in \gamma_n \partial f(x_{n+1}),$$

then $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Argmin } f$

Proof Let $x \in \text{Argmin } f$. We derive from the subdifferential definition that

$$(\forall n \in \mathbb{N}) \langle x - x_{n+1} | x_n - x_{n+1} \rangle / \gamma_n \leq f(x) - f(x_{n+1})$$

and

$$(\forall n \in \mathbb{N}) 0 \leq \langle x_n - x_{n+1} | x_n - x_{n+1} \rangle / \gamma_n \leq f(x_n) - f(x_{n+1}).$$

Thus, for every $n \in \mathbb{N}$, we obtain

$$\|x_{n+1} - x\|^2 = \|x_n - x\|^2 + 2\langle x_n - x | x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2$$

⁶ See Definition in Bauschke and Combettes (2011) (p. 37).

$$\begin{aligned}
&= \|x_n - x\|^2 - \|x_{n+1} - x_n\|^2 + 2\langle x_{n+1} - x | x_{n+1} - x_n \rangle \\
&\leq \|x_n - x\|^2 - 2\gamma_n (f(x_{n+1}) - \inf f(\mathcal{H})).
\end{aligned}$$

This shows that $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Argmin } f$. □

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