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*Research Article*

# Remotely Almost Periodic Solutions of Ordinary Differential Equations

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In this paper, we analyze the existence and uniqueness of remotely almost periodic solutions for systems of ordinary differential equations. The existence and uniqueness of remotely almost periodic solutions are achieved by using the results about the exponential dichotomy and the Bi-almost remotely almost periodicity of the homogeneous part of the corresponding systems of ordinary differential equations under our consideration.

## 1. Introduction and Preliminaries

The notion of an almost periodic function was introduced by a Danish mathematician H. Bohr around 1925 and later generalized by many others. Let $I = \mathbb{R}$ or $I = [0, \infty)$, let $X$ be a complex Banach space, and let $f\colon I \longrightarrow X$ be continuous. Given $\varepsilon > 0$, we call $\tau > 0$ an $\varepsilon$-period for $f(\cdot)$ if and only if

$$\|f(t+\tau) - f(t)\| \leq \varepsilon, \quad t \in I. \tag{1}$$

by $\vartheta(f, \varepsilon)$ we denote the set of all $\epsilon$-periods for $f(\cdot)$. We say that $f(\cdot)$ is almost periodic if and only if, for each $\epsilon > 0$, the set $\vartheta(f, \varepsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length $l$ meets $\vartheta(f, \varepsilon)$. For further information about almost periodic functions and their applications, see [1–10].

It is well known that Sarason defined the notion of a scalar-valued remotely almost periodic function in [11]. The class of vector-valued remotely almost periodic functions defined on $\mathbb{R}^n$ was introduced by Yang and Zhang in [12], where the authors have provided several applications in the study of existence and uniqueness of remotely almost periodic solutions for parabolic boundary value problems (for some results about parabolic boundary value problems, one may refer to [13–15] and references cited therein). In Propositions 2.4–2.6 in [16], the authors have examined the existence and uniqueness of remotely almost periodic solutions of multidimensional heat equations, while the main results of Section 3 are concerned with the existence and uniqueness of remotely almost periodic type solutions of the certain types of parabolic boundary value problems (see also [17, 18], where the authors have investigated almost periodic type solutions and slowly oscillating type solutions for various classes of parabolic Cauchy inverse problems). Concerning applications of remotely almost periodic functions, research articles [19] by Zhang and Piao, where the authors have investigated the time remotely almost periodic viscosity solutions of Hamilton–Jacobi equations, and [20] by Zhang and Jiang, where the authors have investigated remotely almost periodic solutions for a class systems of differential equations with piecewise constant argument, should be mentioned, see [21] and the research articles [22–29], for more details about the subject.

The problem of finding (pseudo) almost periodic solutions for certain classes of ordinary differential equations has been treated by many authors (see e.g., [5, 30–32]). In the existing literature, we can find numerous results about the existence, uniqueness, stability, applications in biology, etc. (concerning the last issue, see, e.g., the research articles by Xu et al. [16, 33–38] as well as the references cited therein).

The strong motivational point for the genesis of this paper lies in the fact that, with the exception of [20] by Zhang and Liang, no one else has applied remotely almost periodic functions in the theory of ordinary differential equations. The class of remotely almost periodic functions is enormously larger than the usually considered class of almost periodic functions, and the interest for studying remotely almost periodic solutions of ordinary differential equations exists. Concerning some practical applications of our theoretical results obtained, we would like to note that we specifically analyze here (see Section 3.1) the Chapman-Richards type equations with external perturbations. It is well known that the Chapman-Richards functions and equations have an important role in the mathematical biology. The Chapman-Richards functions generalize commonly used growth functions as monomolecular functions and Gompertz functions, while the Chapman-Richards equations generalize the logistic equations. The Chapman-Richards model has been widely applied in forestry, thanks to its flexibility and many important analytical features.

The organization and main ideas of this paper can be briefly described as follows. We consider the following systems of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t), \tag{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + f(t), \tag{3}$$

where $A(t)$ is a complex-valued matrix of format $n \times n$ for all $t \in \mathbb{R}$. After repeating some necessary facts about remotely almost periodic functions, we consider the notion of $(\alpha, K, P)$-exponential dichotomy (see Definition 2) for equation (2) as well as the notion of exponential bi-almost periodicity and the notion of integro bi-almost periodicity of the associated Green's function $G(t,s)$ of (2) (see Definition 3 and Definition 4). After that, we introduce the notion of $\alpha$-exponentially bi-remotely almost periodicity and the notion of integro bi-remotely almost periodicity of the associated Green's function $G(t,s)$ of (2) in Definition 5 and Definition 6, respectively. The main results of Section 2, which also contains several important lemmas needed for our further investigations, are Theorem 1 and Theorem 2. In Section 3, we investigate the existence and uniqueness of remotely almost periodic solutions to (2) and (3). We open this section with an important theoretical result, Theorem 3, in which we clarify that, under certain conditions, a unique bounded solution of (3) is remotely almost periodic; see also Theorems 4 and 5. Before we proceed to Section 3.1, in which we analyze the existence and uniqueness of positive remotely almost periodic solutions to the Chapman-Richards equation with external perturbations, we clarify some corollaries, examples, and technical lemmas. The main result of Section 3.1 is Theorem 6, where we show that, under hypotheses (H1)-(H3) clarified below, equation (41) has a unique positive remotely almost periodic solution for small values of nonnegative real parameter $\mu$.

Regarding the previous works of authors in this field, we would like to emphasize that the techniques applied here were born of the classical monographs on this field [5, 8]. However, we deal with the inherent new problems of the remotely almost periodic functions, and some of these problems can be found in [11, 20]. For example, the well-known notion of bi-almost periodicity of the Green function for almost periodic system [5] inspired us to introduce and analyze here the definition of bi-remotely almost periodic function in the remotely almost periodic systems. Furthermore, we give certain conditions such that the Green functions satisfy the bi-remotely almost periodic property.

We use the standard notation throughout the paper. By $BUC(\mathbb{R}: \mathbb{C}^n)$, we denote the Banach space of bounded and uniformly continuous functions $f: \mathbb{R} \longrightarrow \mathbb{C}^n$, equipped with the sup-norm $\|\cdot\|_\infty$; let $\|\cdot\|$ be a fixed norm in $\mathbb{C}^n$. We set $\mathbb{N}_n := \{1, \ldots, n\}$.

To better understand the space of remotely almost periodic functions, denoted by $RAP(\mathbb{R}: \mathbb{C}^n)$, we will recall the notion of a slowly oscillating function (the corresponding space is denoted by $SO(\mathbb{R}: \mathbb{C}^n)$ henceforth). A function $f \in BUC(\mathbb{R}: \mathbb{C}^n)$ is called slowly oscillating if and only if, for every $a \in \mathbb{R}$, we have that

$$\lim_{|t| \longrightarrow +\infty} \|f(t+a) - f(t)\| = 0. \tag{4}$$

Now, we recall the notion of a remotely almost periodic function (see, e.g., [12]).

*Definition 1.* A function $f \in BUC(\mathbb{R}: \mathbb{C}^n)$ is called remotely almost periodic if and only if $\varepsilon > 0$ we have that the set

$$T(f,\varepsilon) := \left\{ \tau \in \mathbb{R}: \limsup_{|t| \longrightarrow +\infty} \|f(t+\tau) - f(t)\| < \varepsilon \right\}, \tag{5}$$

which is relatively dense in $\mathbb{R}$.

Any number $\tau \in T(f,\varepsilon)$ is called an $\epsilon$-remote-translation vector of $f(\cdot)$. We know that $RAP(\mathbb{R}: \mathbb{C}^n)$ is a closed subspace of $BUC(\mathbb{R}: \mathbb{C}^n)$ and, therefore, the Banach space itself. If the functions $F_1(\cdot), \ldots, F_k(\cdot)$ are remotely almost periodic ($k \in \mathbb{N}$), then, for each $\varepsilon > 0$, the set of their common $\epsilon$-remote-translation vectors $s$ is relatively dense in $\mathbb{R}$; see, e.g., Proposition 2.3 in [16].

Furthermore, we know that any remotely almost periodic function $f: \mathbb{R} \longrightarrow \mathbb{R}$ possesses the mean value

$$\mathscr{M}(f) := \lim_{t \longrightarrow +\infty} \frac{1}{t} \int_0^t f(s)\mathrm{d}s, \tag{6}$$

see e.g., Proposition 2.4 in [39]. A similar statement holds for vector-valued remotely almost periodic functions $F: \mathbb{R}^n \longrightarrow X$, but we will not use this fact here.

## 2. Preliminaries on Exponential Dichotomies

The property of exponential dichotomy will be primordial in this section, where we are going to give its definition by considering equations (2) and (3). For the following definitions and for more details about the subject, we refer to the research [40] by Coppel.

*Definition 2.* Let $\Phi(\cdot)$ be a fundamental matrix of equation (2). Then, we say that equation (2) has an $(\alpha, K, P)$-exponential dichotomy if and only if there exist positive constants $\alpha$, $K > 0$ and a projection $P$ $(P^2 = P)$ such that

$$\|G(t,s)\| \le K e^{-\alpha|t-s|}, \quad t, s \in \mathbb{R}, \tag{7}$$

where the Green function $G(t,s)$ of (2) is given by $G(t,s) := \Phi(t)P\Phi^{-1}(s)$ for $t \ge s$ and $G(t,s) := -\Phi(t)[I - P]\Phi^{-1}(s)$ for $t < s$.

The notion of bi-almost periodicity of the Green function, which has been omitted or less considered for a long time, plays a crucial role in our study:

*Definition 3.* We say that the Green function $G(t,s)$ of (2) is exponentially by-almost periodic if and only if, for all $\epsilon > 0$, there exist positive real constants $\alpha\prime > 0$ and $c > 0$ and a relatively dense set $T(G,\varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G,\varepsilon)$, we have

$$\|G(t+\tau, s+\tau) - G(t,s)\| \le \varepsilon c e^{-\alpha\prime|t-s|}, \quad t, s \in \mathbb{R}. \tag{8}$$

*Definition 4.* We say that the Green function $G(t,s)$ of (2) is integro bi-almost periodic if and only if, for all $\varepsilon > 0$, there exist a positive real constant $c > 0$ and a relatively dense set $T(G,\varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G,\varepsilon)$, we have

$$\int_{-\infty}^{+\infty} \|G(t+\tau, s+\tau) - G(t,s)\| \mathrm{d}s \le \varepsilon c, \quad t \in \mathbb{R}. \tag{9}$$

It is worth noting that the Green function is not immediately integro bi-almost periodic.

*Example 1.* The next differential equation has an exponential dichotomy:

$$x\prime = -(1 + b(t))x + 1; \quad b(t) > 0. \tag{10}$$

However, the Green function associated to this system is not bi-almost periodic. The bounded solution, given by

$$x(t) = \int_{-\infty}^{t} e^{-\int_s^t (1+b(r))\mathrm{d}r} \mathrm{d}s, \tag{11}$$

is not almost periodic in general if $b(\cdot)$ is not almost periodic (for example, this can occur if $b(\cdot)$ is almost automorphic but not almost periodic; see [7], for the notion).

Now, we would like to introduce the following notion:

*Definition 5.* Let $\alpha > 0$. Then, we say that the Green function $G(t,s)$ of (2) is $\alpha$-exponentially bi-remotely almost periodic if and only if, for every $\varepsilon > 0$, there exist a positive real constant $c > 0$ and a relatively dense set $T(G, \varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G, \varepsilon)$, we have

$$\begin{aligned} &\limsup_{|t|\longrightarrow\infty} \left\| e^{\alpha(t-s)} [G(t+\tau, s+\tau) - G(t,s)] \right\| \le \varepsilon c, \quad t, s \in \mathbb{R}, t \ge s, \\ &\text{and}, \limsup_{|t|\longrightarrow\infty} \left\| e^{\alpha(s-t)} [G(t+\tau, s+\tau) - G(t,s)] \right\| \le \varepsilon c, \quad t, s \in \mathbb{R}, t < s. \end{aligned} \tag{12}$$

*Definition 6.* Let $\alpha > 0$. Then, we say that the Green function $G(t,s)$ of (2) is integro bi-remotely almost periodic if and only if, for every $\varepsilon > 0$, there exist a positive real constant $c > 0$ and a relatively dense set $T(G,\varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G,\varepsilon)$, we have

$$\limsup_{|t|\longrightarrow\infty} \int_{-\infty}^{+\infty} \|G(t+\tau, s+\tau) - G(t,s)\| \mathrm{d}s \le \varepsilon c, \quad t \in \mathbb{R}. \tag{13}$$

Let us consider now the scalar differential equation $x\prime(t) = a(t)x(t)$. We have the following.

**Theorem 1.** *If $a(\cdot)$ is a remotely almost periodic function with $\mathscr{M}(a) \ne 0$, then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\tau \in T(a,\delta)$, we have*

$$\limsup_{|t|\longrightarrow\infty}\int_{-\infty}^{t}\left|e^{\int_{s+\tau}^{t+\tau}a(r)\mathrm{d}r-\int_{s}^{t}a(r)\mathrm{d}r}\right|\mathrm{d}s<\varepsilon,\quad \text{provided } t<s \text{ and } \mathscr{M}(a)<0,$$
$$\limsup_{|t|\longrightarrow\infty}\int_{t}^{+\infty}\left|e^{\int_{s+\tau}^{t+\tau}a(r)\mathrm{d}r-\int_{s}^{t}a(r)\mathrm{d}r}\right|\mathrm{d}s<\varepsilon,\quad \text{provided } t\geq s \text{ and } \mathscr{M}(a)>0. \tag{14}$$

*proof.* Let $\mathscr{M}(a)<-\gamma<0$. Then, it is not difficult to verify that $|\exp(\int_s^t a(r)\mathrm{d}r)|\leq Ke^{-\gamma(t-s)}$ for $t\geq s$, as well as

$$\left|e^{\int_{s+\tau}^{t+\tau}a(r)\mathrm{d}r-\int_{s}^{t}a(r)\mathrm{d}r}\right|\leq K^2(t-s)e^{-\gamma(t-s)}\sup_{u\in(s,t)}|a(u+\tau)-a(u)|. \tag{15}$$

Therefore,

$$\int_{-\infty}^{t}\left|e^{\int_{s+\tau}^{t+\tau}a(r)\mathrm{d}r-\int_{s}^{t}a(r)\mathrm{d}r}\right|\mathrm{d}s\leq K^2\int_0^\infty xe^{-\gamma x}\sup_{u\in(t,\infty)}|a(u-x+\tau)-a(u-x)|\mathrm{d}x. \tag{16}$$

For every $\varepsilon>0$, we set $\delta:=K^2\gamma^{-1}\varepsilon$. Let us consider first case $t\longrightarrow+\infty$. Given any sequence $(x_n)$ tending to plus infinity, we have

$$\lim_{t\longrightarrow+\infty}\sup_{u\in(t,\infty)}|a(u-x+\tau)-a(u-x)|=\lim_{n\longrightarrow\infty}\sup_{u\in(x_n,\infty)}|a(u-x+\tau)-a(u-x)|. \tag{17}$$

Using the reverse Fatou lemma and (17), we obtain that

$$\limsup_{t\longrightarrow\infty}\int_{-\infty}^{t}\left|e^{\int_{s+\tau}^{t+\tau}a(r)\mathrm{d}r-\int_{s}^{t}a(r)\mathrm{d}r}\right|\mathrm{d}s\leq K^2\int_0^\infty xe^{-\gamma x}\limsup_{t\longrightarrow\infty}\sup_{u\in(t,\infty)}|a(u-x+\tau)-a(u-x)|\mathrm{d}x<\varepsilon. \tag{18}$$

The proof for case $t\longrightarrow-\infty$ can be given analogously. Case $\mathscr{M}(a)>0$ can be considered analogously as well.

This result can be extended to system (2), where the matrix $A(t)$ is diagonal $A(t)=\mathrm{diag}\{a_i(t)\}$ and $\Re(\mathscr{M}(a_i))\neq 0$, for all $i\in\mathbb{N}_n$.

For the sequel, we need the following auxiliary lemma: □

**Lemma 1.** *Let $A(t)$ be the complex-valued matrix of format $n\times n$ for all $t\in\mathbb{R}$, and let $\Phi(\cdot)$ be the fundamental matrix of (2). The transition matrix $\Phi(t,s)\equiv\Phi(t)\Phi^{-1}(s)$ satisfies*

$$\Phi(t+\tau,s+\tau)-\Phi(t,s)=\int_s^t \Phi(t,u)(A(u+\tau)-A(u))\Phi(u+\tau,s+\tau)\,du,\quad \text{provided}\, t>s,$$
$$\Phi(t+\tau,s+\tau)-\Phi(t,s)=\int_t^s \Phi(t,u)(A(u+\tau)-A(u))\Phi(u+\tau,s+\tau)\,du,\quad \text{provided}\, t<s. \tag{19}$$

*Proof.* We will consider case $t>s$ only. Set $V(t,s):=\Phi(t+\tau,s+\tau)-\Phi(t,s)$. Then, we have

$$V_t=A(t+\tau)\Phi(t+\tau,s+\tau)-A(t)\Phi(t,s),$$
$$V_s=A(t)V(t,s)+(A(t+\tau)-A(t))\Phi(t+\tau,s+\tau). \tag{20}$$

This simply implies the required equality.

Suppose now that the matrix $A(t)$ is diagonal by blocks $A_+(t)$ and $A_-(t)$ so that system (2) can be written as the system $z\prime(t)=A_+(t)z$ and $y\prime(t)=A_-(t)y$. By $\Phi_+(t,s)$ and $\Phi_-(t,s)$, we denote the fundamental matrices associated to the equations for $z$ and $y$, respectively; then, we have the following estimates $\|\Phi_+(t-s)\|\le Ke^{-\gamma(t-s)}$ for $t\ge s$ and $\|\Phi_-(t,s)\|\le Ke^{\gamma(t,s)}$ for $t\le s$, where $\gamma>0$. Define $G(t,s):=\operatorname{diag}(\Phi_+(t,s),0)$ for $t\ge s$ and $G(t,s):=\operatorname{diag}(0,\Phi_-(t,s))$ for $t<s$. Hence, $\|G(t,s)\|\le Ke^{-\gamma|t-s|}$ for all $t,\ s\in\mathbb{R}$.

As a straightforward consequence of the previous lemma, the following holds for the above Green function. □

**Lemma 2.** *We have*

$$\|G(t+\tau,s+\tau)-G(t,s)\|\le K^2e^{-\gamma(t,s)}\int_s^t\left\|A_+(u+\tau)-A_+(u)\right\|du,\quad t\ge s,$$
$$\|G(t+\tau,s+\tau)-G(t,s)\|\le K^2e^{-\gamma(s-t)}\int_t^s\left\|A_-(u+\tau)-A_-(u)\right\|du,\quad t\le s. \tag{21}$$

*Now, we are able to prove some important results of this section. We start by stating the following theorem regarding the diagonalization of $A(t)$ into blocks $A_+(t)$ and $A_-(t)$, where we assume that all the above estimates are satisfied.*

**Theorem 2.** *Let $A_+$ and $A_-$ be remotely almost periodic functions, and let the estimate $\|G(t,s)\|\le Ke^{-\gamma|t-s|}$, $t,\ s\in\mathbb{R}$, hold for the associated Green function. Then, for every $\varepsilon>0$, there exists $\delta>0$ such that, for every $\tau\in T(A_+,\delta)\cap T(A_-,\delta)$, we have*

$$\limsup_{|t|\longrightarrow\infty}\int_{-\infty}^{+\infty}\|G(t+\tau,s+\tau)-G(t,s)\|ds<\varepsilon, \tag{22}$$

*and in other words, $G(\cdot,\cdot)$ is integro bi-remotely almost periodic.*

*Proof.* Applying Lemma 2, we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\|G(t+\tau,s+\tau)-G(t,s)\|ds\\
&\le\int_{-\infty}^{t}K^2e^{-\gamma(t-s)}\int_s^t\left\|A_+(u+\tau)-A_+(u)\right\|duds\\
&\quad+\int_t^{+\infty}K^2e^{-\gamma(s-t)}\int_t^s\left\|A_-(u+\tau)-A_-(u)\right\|duds\\
&=\int_0^\infty K^2e^{-\gamma x}\int_{t-x}^t\left\|A_+(u+\tau)-A_+(u)\right\|duds\\
&\quad+\int_0^{+\infty}K^2e^{-\gamma y}\int_t^{t+y}\left\|A_-(u+\tau)-A_-(u)\right\|duds\\
&:=K^2[I_1+I_2].
\end{aligned} \tag{23}$$

It is clear that

$$I_1\le\int_0^\infty e^{-\gamma x}x\sup_{u\in(t-x,\infty)}\left\|A_+(u+\tau)-A_+(u)\right\|dx. \tag{24}$$

Since $A_\pm(\cdot)$ are globally bounded, we get the existence of a finite real constant $M_1>0$ such that $I_1\le M_1$. Set $\delta:=(\varepsilon/(2K^2M_1))$. Taking into account the reverse Fatou lemma and the fact that, for every increasing sequence $(s_n)$ tending to plus infinity, we have

$$\begin{aligned}
&\lim_{s\longrightarrow+\infty}\sup_{u\in(s,\infty)}\left\|A_+(u+\tau)-A_+(u)\right\|\\
&=\lim_{n\longrightarrow+\infty}\sup_{u\in(s_n,\infty)}\left\|A_+(u+\tau)-A_+(u)\right\|,
\end{aligned} \tag{25}$$

and the above simply implies $\limsup_{t\longrightarrow+\infty}I_1(t)\le(\varepsilon/2)$. For the asymptotic behaviour, when $t\longrightarrow-\infty$, we can use the estimate

$$I_1(t)\le\int_0^\infty e^{-\gamma x}x\sup_{u\in(-\infty,t)}\left\|A_+(u+\tau)-A_+(u)\right\|dx \tag{26}$$

and a similar argumentation in order to show that $\limsup_{t\longrightarrow-\infty}I_1(t)\le(\varepsilon/2)$. The calculations and argumentation used for $I_1$ are similar for $I_2$, which completes the proof of theorem. □

*Remark 1.* Suppose that system (2) has an $(\alpha,K,P)$-exponential dichotomy. If $P$ commutes with the fundamental matrix $\Phi(t)$ of this system, then it is possible to diagonalise this system and conclude that the hypothesis of the above theorem are satisfied; in other words, the associated Green function will be integro bi-remotely almost periodic. Also, if system (2) is remotely almost periodic (it means that all coefficients of (2) are remotely almost periodic) and exponentially stable at infinity (or at minus infinity, respectively); then, the associated Green function is integro bi-remotely almost periodic. As easily proven, this also happens in the case that there exists an invertible remotely almost periodic transformation $x = S(t)\omega$, $\omega = (z, y)$ under which the remotely almost periodic linear system (2) admits a diagonalization into blocks $A_+(t)$ and $A_-(t)$ such that the associated Green function satisfies the already used condition of exponentially decaying.

## 3. The Existence and Uniqueness of Remotely Almost Periodic Solutions to (2) and (3)

We start this section by stating the following result.

**Theorem 3.** *Suppose that $f \in RAP(\mathbb{R}: \mathbb{C}^n)$ and the homogeneous system (2) has an $(\alpha, K, P)$-exponential dichotomy and the associated Green function is integro bi-remotely almost periodic. Then, the unique bounded solution of (3) is remotely almost periodic.*

*Proof.* Without loss of generality, we may assume that $f \neq 0$. By the variation of parameters formula, the unique bounded solution of (3) is given by

$$x(t) = \int_{-\infty}^{\infty} G(t,s) f(s) \mathrm{d}s, \quad t \in \mathbb{R}. \tag{27}$$

Let us show that $x(\cdot)$ is remotely almost periodic. Indeed, we have

$$\begin{aligned} \|x(t+\tau) - x(t)\| &\leq \left\| \int_{-\infty}^{\infty} [G(t+\tau, s+\tau) - G(t,s)] f(s+\tau) \mathrm{d}s \right\| \\ &\quad + \left\| \int_{-\infty}^{\infty} G(t,s)[f(s+\tau) - f(s)] \mathrm{d}s \right\| \\ &\leq \|f\|_\infty \cdot \int_{-\infty}^{\infty} \|G(t+\tau, s+\tau) - G(t,s)\| \mathrm{d}s \\ &\quad + \int_{-\infty}^{\infty} K e^{-\alpha|t-s|} \|f(s \end{aligned} \tag{28}$$

Let $\varepsilon > 0$ be given. Since the corresponding Green function is integro bi-remotely almost periodic, we know that there exists $\delta_1 > 0$ such that, for every $\tau \in T(G, \delta_1)$, we have

$$\limsup_{|t| \longrightarrow \infty} \int_{-\infty}^{\infty} \|G(t+\tau, s+\tau) - G(t,s)\| \mathrm{d}s < \frac{\varepsilon}{2} \|f\|_\infty. \tag{29}$$

We also have the existence of a real number $\delta_2 > 0$ such that, for every $\tau \in T(f, \delta_2)$, we have

$$\limsup_{|t| \longrightarrow \infty} \int_{-\infty}^{\infty} e^{-\alpha|t-s|} \|f(s+\tau) - f(s)\| \mathrm{d}s < \frac{\varepsilon}{2K}. \tag{30}$$

Since the operation $\limsup_{|t| \longrightarrow \infty} \cdot$ is subadditive, this simply completes the proof with $\delta = \min(\delta_1, \delta_2)$.

We also have the following result, whose proof can be omitted. □

**Theorem 4.** *Suppose that $f \in RAP(\mathbb{R}: \mathbb{C}^n)$ and $A(t)$ is a triangular matrix for all $t \in \mathbb{R}$ such that $\mathfrak{R}(\mathscr{M}(a_{ii})) \neq 0$ for all $i \in \mathbb{N}_n$. Then, system (3) has a unique bounded solution which is remotely almost periodic.*

*Now, we consider system (3) in which the square matrix $A \equiv A(t)$ is independent of the time variable $t$:*

$$x\prime(t) = Ax(t) + f(t), \tag{31}$$

*where and $f \in RAP(\mathbb{R}: \mathbb{C})$ for all $i \in \mathbb{N}_n$.*

*We need the following lemma from [39].*

**Lemma 3.** *Given a square matrix A, there exists a regular matrix $\alpha$ having the same order as A such that the matrix $B = \alpha^{-1} A \alpha$ is triangular with the diagonal elements being the eigenvalues of A.*

**Corollary 1.** *We have that every bounded solution of system (31) is remotely almost periodic.*

*Proof.* Keeping in mind Lemma 3, we may assume that $A$ is a triangular superior matrix. Applying Theorem 4, we get that the associated solution is remotely almost periodic. For the initial solution, we have $x(t) = \alpha y(t) \in RAP(\mathbb{R}: \mathbb{C}^n)$. This ends the proof. □

*Example 2.* Consider $\lambda = iv \in i(\mathbb{R} \setminus \{0\})$ and the linear equation $y\prime(t) = ivy(t) + g(t)$, where $g(t) = e^{iv(t+t^{(1/3)})}$ is a remotely almost periodic function. The solution is given by

$$y(t) = ce^{itv} - 3iv^{-1} t^{(2/3)} e^{iv\left(t+t^{(1/3)}\right)} + 6v^{-2} t^{(1/3)} e^{iv\left(t+t^{(1/3)}\right)} + 6iv^{-3} e^{iv\left(t+t^{(1/3)}\right)}. \tag{32}$$

So, this equation does not have a bounded solution on the real line.

Consider now the scalar linear differential equation:

$$x\prime(t) = a(t)x(t) + f(t). \tag{33}$$

For the sequel, we need the following technical lemma which follows from our foregoing arguments.

**Lemma 4.** *Let $a(t)$ and $f(t)$ be remotely almost periodic functions such that $\Re(\mathscr{M}(a)) \neq 0$. Then, equation (33) has a unique remotely almost periodic solution $x(t)$ given by*

$$x(t) = -\int_t^{\infty} e^{-\int_s^t a(r)\mathrm{d}r} f(s)\mathrm{d}s, \quad \text{provided } \Re(\mathscr{M}(a)) > 0,$$
$$x(t) = \int_{-\infty}^{t} e^{-\int_t^s a(r)\mathrm{d}r} f(s)\mathrm{d}s, \quad \text{provided } \Re(\mathscr{M}(a)) < 0. \tag{34}$$

*Now, let us consider the equation:*

$$z\prime(t) = A(t)z(t) + f(t) + \mu g(t, z(t)). \tag{35}$$

*We have the following result.*

**Theorem 5.** *Let $f \in RAP(\mathbb{R}: \mathbb{R}^n)$ and let $g(\cdot)$ be remotely almost periodic in the first variable and locally Lipschitz in the second variable. Suppose, further, that the homogeneous system (2) has an $(\alpha, K, P)$-exponential dichotomy and the associated Green function is integro bi-remotely almost periodic. Then, there exists a positive constant $\mu_0$ such that the assumption $\mu \in [0, \mu_0)$ implies that the differential equation (35) has a unique bounded solution which is remotely almost periodic.*

*Proof.* Consider a unique remotely almost periodic solution $\varphi(t)$ of (3). Let $r \in (0, \infty)$ be such that $\|\varphi\| \leq r$, and let $L > 0$ denote the corresponding Lipschitz constant. If $z(t)$ solves (35), then we set $x(t) := z(t) - \varphi(t)$, $t \in \mathbb{R}$. It is clear that

$$x\prime(t) = A(t)x(t) + \mu g(t, x(t) + \varphi(t)), \quad t \in \mathbb{R}. \tag{36}$$

Let the Green function of the homogeneous part satisfy $\|G(t,s)\| \leq K e^{-\alpha|t-s|}$. By the variation of parameters' formula, we have

$$x(t) = \int_{-\infty}^{\infty} G(t,s)\mu g(s, x(s) + \varphi(s))\mathrm{d}s, \quad t \in \mathbb{R}. \tag{37}$$

Define $B(r, 0)$ to be the closed ball of diameter $r$ and the center 0 in the space of remotely almost periodic functions; then, $B(r, 0)$ is a complete metric space with the induced metric. Define the mapping

$$T\psi(t) := \int_{-\infty}^{\infty} G(t,s)\mu g(s, \psi(s) + \varphi(s))\mathrm{d}s, \quad t \in \mathbb{R}\,(\psi \in B(r,0)). \tag{38}$$

We claim that the mapping $T: B(r, 0) \longrightarrow B(r, 0)$ is well-defined and contracted. It is clear that the mapping $T\psi$ is remotely almost periodic for any $\psi \in B(r, 0)$. Furthermore, we have

$$\begin{aligned} \|T\psi\|_{\infty} &\leq 2K\mu\alpha^{-1}\|g\|_{\infty} \\ &\leq 2K\mu\alpha^{-1}\left[\|\psi + \varphi\|_{\infty} + \sup_{s\in\mathbb{R}}\|g(s,0)\|\right] \\ &\leq 2Kr\mu\alpha^{-1}\left(2r + \sup_{s\in\mathbb{R}}\|g(s,0)\|\right) < 1, \end{aligned} \tag{39}$$

provided that $\mu \in [0, \mu_0)$ and $2Kr\mu_0\alpha^{-1}(2r + \sup_{s\in\mathbb{R}}\|g(s,0)\|) < 1$. For the contraction, we can use the following calculation:

$$\|T\psi_1 - T\psi_2\|_{\infty} \leq \mu KL \int_{-\infty}^{\infty} e^{-\alpha|t-s|}\|\psi_1(s) - \psi_2(s)\|\mathrm{d}s \leq \mu 2KL\alpha^{-1}\|\psi_1 - \psi_2\|_{\infty}. \tag{40}$$

Therefore, the mapping $T: B(r,0) \longrightarrow B(r, 0)$ has a unique fixed point, which simply finishes the proof. □

*3.1. The Existence and Uniqueness of Positive Remotely Almost Periodic Solutions.* In this section, we analyze the Chapman-Richards equation with an external perturbation $f(\cdot)$:

$$x\prime(t) = x(t)\left[a(t) - b(t)x^{\theta}(t)\right] + f(t), \tag{41}$$

where $\theta \geq 0$. Consider the following hypotheses:

(H1) $a(t)$, $b(t)$, and $f(t)$ are remotely almost periodic functions

(H2) $0 < \alpha \leq a(t) \leq A$, $0 < \beta \leq b(t) \leq B$, $0 < f(t) < F$

(H3) With $\omega = A^{-1}[\beta - \gamma^{((1+\theta)/\theta)}F]$ and $\gamma = (B/\alpha)$, we have $(1+\theta)F\gamma^{(1/\theta)}\theta^{-1}\alpha^{-1} < 1$ and $\beta(1+\theta)B\theta^{-1} < 1$

*Remark 2.* Suppose that $f(t) \geq 0$ for all $t \in \mathbb{R}$. Then, we have

$$x\prime(t) \geq x(t)\left[a(t) - b(t)x^{\theta}(t)\right], \quad t \in \mathbb{R}. \tag{42}$$

This implies that, for each $t_0 \in \mathbb{R}$, we have

$$x(t) \geq x(t_0)e^{\int_{t_0}^{t}\left[a(s) - b(s)x^{\theta}(s)\right]\mathrm{d}s}, \quad t \in \mathbb{R}. \tag{43}$$

Now, we will state the main result of this section.

**Theorem 6.** *Suppose that hypotheses (H1)-(H3) hold. Then, equation (41) has a unique remotely almost periodic solution $\phi^*(t)$ satisfying $\gamma^{-(1/\theta)} \leq \phi^* \leq \omega^{-(1/\theta)}$ for all $t \in \mathbb{R}$.*

*Proof.* Let $u(t) = x^{-\theta}(t)$. We only consider the positive solutions of (41), by rewriting this system as follows:

$$u\prime(t) = -\theta a(t)u(t) + \theta b(t) - \theta u^{((1+\theta)/\theta)}(t)f(t). \tag{44}$$

Let $\widetilde{B}$ denote the complete metric space consisting of all remotely almost periodic functions whose sup-norm belongs to the interval $[\omega, \gamma]$. Given $\varphi \in \widetilde{B}$, we consider the following equation:

$$u\prime(t) = -\theta a(t)u(t) + \theta b(t) - \theta \varphi^{((1+\theta)/\theta)}(t) f(t). \tag{45}$$

By Lemma 4, this equation has a unique remotely almost periodic solution $\mu(t)$, given by

$$u(t) := T\varphi(t) := \theta \int_{-\infty}^{t} e^{-\theta \int_s^t a(r)\mathrm{d}r} \left[b(s) - \varphi^{((1+\theta)/\theta)}(s) f(s)\right]\mathrm{d}s. \tag{46}$$

It can be simply shown that $\|T\varphi\|_\infty \leq (B/\alpha) = \gamma$. Furthermore, we have

$$\theta \int_{-\infty}^{t} e^{-\theta \int_s^t a(u)\mathrm{d}u} \left[b(s) - \varphi^{((1+\theta)/\theta)}(s) f(s)\right]\mathrm{d}s \leq \theta \int_{-\infty}^{t} e^{-\theta \int_s^t a(u)\mathrm{d}u} \left[\beta - \varphi^{((1+\theta)/\theta)}(s) f(s)\right]\mathrm{d}s, \tag{47}$$

which is always strictly greater or equal than $A^{-1}(\beta - \gamma^{((1+\theta)/\theta)})F = \omega$. Hence, the mapping $T: \widetilde{B} \longrightarrow \widetilde{B}$ is well-defined. To see that this mapping is a contraction, we use the following consequence of the mean value theorem applied to the function $x^{((1+\theta)/\theta)}$, and by the definition of $\widetilde{B}$, one obtains

$$\left\|\psi^{((1+\theta)/\theta)} - \varphi^{((1+\theta)/\theta)}\right\|_\infty \leq \frac{1+\theta}{\theta} \max\left\{\gamma^{(1/\theta)}, \omega^{(1/\theta)}\right\} \|\psi - \varphi\|_\infty, \tag{48}$$

and a simple computation yielding that

$$\|T\psi - T\varphi\|_\infty \leq \frac{F(1+\theta)}{\alpha\theta} \gamma^{(1/\theta)} \|\psi - \varphi\|_\infty. \tag{49}$$

Therefore, $T$ is a contraction mapping $\widetilde{B}$ in $\widetilde{B}$ so that $T$ has a unique fixed point in $\widetilde{B}$, and this point is a unique remotely almost periodic positive solution of (41). This simply completes the proof because the unique solution of our problem is given by $\phi^*(t) = [\phi(t)]^{-(1/\theta)}$. □

## 4. Conclusions

This paper investigates the existence and uniqueness of remotely almost periodic solutions for systems of ordinary differential equations. Our main contributions are achieved by using the results about the exponential dichotomy and the bi-almost remotely almost periodicity of the homogeneous part of the corresponding systems of ordinary differential equations. We particularly analyze the Chapman-Richards equation with external perturbations.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.


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