FIBRATIONS ASSOCIATED TO SMOOTH QUOTIENTS OF ABELIAN VARIETIES

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ABSTRACT. Let A be an abelian variety and G a finite group of automorphisms of A fixing the origin such that A/G is smooth. The quotient A/G can be seen as a fibration over an abelian variety whose fibers are isomorphic to a product of projective spaces. We classify how the fibers are glued in the case when the fibers are isomorphic to a projective space and we prove that, in general, the quotient A/G is a fibered product of such fibrations.

Keywords: Abelian varieties, fibrations, groups action

1. Introduction

In [ALA20a] R. Auffarth and G. Lucchini Arteche study smooth quotients of abelian varieties by finite groups whose action fixes the origin. Let A be an abelian variety and G a finite group of automorphisms of A fixing the origin such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and P_G be the complementary abelian subvariety with respect to a G-invariant polarization. The action of G on A induces an action of G on P_G. There exists a fibration[1] $A/G \to A/P_G$ and the fibers are isomorphic to P_G/G and smooth (Proposition 2.9 [ALA20a]).

If $A_0 = 0$, or equivalently $A = P_G$, the fibration $A/G \to A/P_G \cong 0$ has just one fiber and this corresponds to A/G. This case was studied in [ALA20a] for dimension ≥ 3, and the case of dimension 2 was studied in [ALAQ19] and can be summarized in the following two results:

Theorem 1.1 (Theorem 1.1 [ALA20a] - Theorem 1.1 [ALAQ19]). *Let A be an abelian variety and let G be a (non trivial) finite group of automorphisms of A that fix the origin. Then the following conditions are equivalent:*

(1) A/G is smooth and the analytic representation of G is irreducible.

(2) $A/G \cong \mathbb{P}^n$.

(3) There exists an elliptic curve E such that $A \cong E^n$ and (A, G) satisfies exactly one of the following:

a) $G \cong C^n \rtimes S_n$ where C is a non-trivial (cyclic) subgroup of automorphims of E that fix the origin; here the action of C^n is coordinatewise and S_n permutes the coordinates.

b) $G \cong S_{n+1}$ and acts on

$$A \cong \{(x_0, x_1, \ldots, x_n) \in E^{n+1} \mid x_0 + x_1 + \cdots + x_n = 0\},$$

by permutations.

c) $E \cong \mathbb{C}/\mathbb{Z}[i]$ and G is the order 16 subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by:

$$\left\{ \begin{pmatrix} -1 & i+1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix} \right\},$$

acting on $A \cong E^2$ in the obvious way.

Even if $A_0 = 0$, the analytic representation of G on A might not always be irreducible. However, in this case the abelian variety A and the group G can be conveniently decomposed as in the next result:

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[1]The quotient A/P_G is isomorphic to $A_0/(A_0 \cap P_G)$, the base of the fibration in Proposition 2.9 [ALA20a]

Theorem 1.2 (Theorem 1.3 [ALA20a]). *Let A be an abelian variety and let G be a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Assume that $A_0 = 0$. There exist $A_1, \ldots, A_r$ abelian subvarieties of A and $G_i \in \mathrm{Aut}_0(A_i)$ such that $G \cong G_1 \times \cdots \times G_r$, $A \cong A_1 \times \cdots \times A_r$ and the pairs (A_i, G_i) satisfy the conditions of Theorem 1.1. Moreover,*

$$A/G \cong A_1/G_1 \times \cdots A_r/G_r.$$

Thus, if $A_0 = 0$, the quotient A/G is a product of projective spaces. In the case when $A_0 \neq 0$ the fibers of the fibration $A/G \to A/P_G$ are isomorphic to P_G/G. The group G acts naturally on P_G and gives a faithful representation of G on $T_0(P_G)$, the analytic representation of G on P_G. Since $\dim(P_G^G) = 0$, by Theorem 1.1 and Theorem 1.2 the fibers P_G/G are isomorphic to a product of projective spaces.

In this article we study how the fibers are glued in the case when $A_0 \neq 0$, and therefore complete the classification of smooth quotients of abelian varieties by finite groups that fix the origin given by Auffarth and Lucchini Arteche.

Smooth quotients of abelian varieties by finite groups whose action fixes the origin have appeared in other contexts, for example in the study of elliptic algebras introduced by Feigin and Odesskii in [OF89]. Much of the representation theory of these algebras is controlled by its characteristic variety and A. Chirvasitu, R. Kanda and S. P. Smith prove in [CKS20a] that the characteristic variety of an elliptic algebra is isomorphic to E^g/S, with E^g identified with the points of E^{g+1} whose coordinates add up to 0 and S is a subgroup of the permutation group S_{g+1} generated by simple reflections. In particular, it is isomorphic to a smooth quotient of an abelian variety by a finite group whose action fixes the origin. In [CKS20b], the authors study more in detail the structure of the quotient variety E^g/S, which corresponds to a particular case of our Theorem 1.4 below.

Remark 1.3. Having completed such a classification a natural question that arises would be *what happens in the case when G does not fix a point of A?*. This question has been studied by the authors of [ALA20a] in [ALA20b].

Main results

If A_0 is trivial, the fibration has a trivial base and the quotient A/G is isomorphic to a product of projective spaces, because of Theorem 1.1 and Theorem 1.2. However, if $A_0 \neq 0$ the base is non-trivial and the fibers are isomorphic to a product of projective spaces. In the case when the analytic representation of G on P_G is irreducible, i.e. when the fibers are isomorphic to a single projective space, we get the first result of this work.

Theorem 1.4. *Let A be an abelian variety and G a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and let P_G be the complementary abelian subvariety with respect to a G-invariant polarization. If the analytic representation of G on P_G is irreducible, there exists a trivialization*

$$
\begin{array}{ccc}
A_0 \times \mathbb{P}^n & \xrightarrow{\ /\Delta\ } & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\ /\Delta\ } & A/P_G,
\end{array}
$$

of the fibration $A/G \to A/P_G$, with $\Delta \cong A_0 \cap P_G$, where Δ acts on A_0 by translation and on the fibers as one of the following:

(1) $\Delta \cong \{0\}$ and therefore acts trivially.
(2) $\Delta \cong \mathbb{Z}/2\mathbb{Z}$ and acts with generator $\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : -z_1 : \cdots : (-1)^n z_n]$.
(3) $\Delta \cong \mathbb{Z}/3\mathbb{Z}$ and acts with generator $\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_3 z_1 : \cdots : \zeta_3^n z_n]$.

(4) $\Delta \cong (\mathbb{Z}/2\mathbb{Z})^2$ and there exists a set of generators that act as

$$\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : (-1)^n z_n],$$
$$\delta' : [z_0 : \cdots : z_n] \mapsto [z_n : z_{n-1} : \cdots : z_1 : z_0].$$

(5) Δ is isomorphic to a subgroup of $(\mathbb{Z}/(n+1)\mathbb{Z})^2$ and there exists a set of generators that act as

$$\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_{n+1}^a z_1 : \cdots : \zeta_{n+1}^{an} z_n],$$
$$\delta' : [z_0 : \cdots : z_n] \mapsto [z_{n+1-b} : \cdots : z_{n-1} : z_n : z_0 : \cdots : z_{n-b}],$$

with a and b positive integers, $b|(n+1)$ and $a|b$, where $z_{-1} := z_n$.

In the case when the analytic representation of G on P_G is not irreducible, we prove that the quotient A/G is isomorphic to a fibered product of fibrations as in Theorem 1.4.

Theorem 1.5. *Let A be an abelian variety and G a finite non trivial subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and let P_G be the complementary abelian subvariety with respect to a G-invariant polarization. There exist pairs $(B_1, G_1), \ldots, (B_r, G_r)$ of abelian varieties B_i and subgroups G_i of $\mathrm{Aut}_0(B_i)$ such that*

- $G \cong G_1 \times \cdots \times G_r$.
- *The pairs (B_i, G_i) satisfy the conditions of Theorem 1.4. In particular, if we denote by $B_{i,0}$ the connected component of $B_i^{G_i}$ that contains 0 and P_{G_i} the complementary abelian subvariety with respect to a G_i-invariant polarization, there exists a fibration $B_i/G_i \to B_i/P_{G_i}$.*
- *For all $i \in \{1, \ldots, r\}$ $B_i/P_{G_i} \cong A/P_G$.*
- *There exists an isomorphism*

$$A/G \cong B_1/G_1 \underset{A/P_G}{\times} \cdots \underset{A/P_G}{\times} B_r/G_r.$$

Structure of the article

In Section 2 we talk about general aspects of this classification and we prove Theorem 1.5.

In Section 3 we give a proof of Theorem 1.4. In this section, the pairs (P_G, G) are separated into pairs of type α, β and γ, corresponding to the three examples in Theorem 1.1 (cf. Definition 3.1). In Section 3.1 we study the pairs of type α and obtain the main result corresponding to points 2, 3 and 4 of Theorem 1.4. In Section 3.2 we study the pairs of type β and prove the main result corresponding to point 5 of Theorem 1.4. The pairs of type γ are studied in Section 3.3, the main result proving that the fibers are glued in the same way as the pairs of type α.

Acknowledgements

2. Reduction to the irreducible case

Let A be an abelian variety and G be a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and P_G be the complementary abelian subvariety with respect to a G-invariant polarization. As stated above, there exists a fibration $A/G \to A/P_G$ whose fibers are smooth and isomorphic to P_G/G. Consider the isogeny

$$A_0 \times P_G \to A;$$
$$(a, b) \mapsto a - b,$$

with kernel

$$\Delta := \{(a, a) \in A_0 \times P_G \mid a \in A_0 \cap P_G\}.$$

The subgroup Δ acts by translations coordinatewise on $A_0 \times P_G$. The group G acts on $A_0 \times P_G$ coordinatewise and this action commutes with the action of Δ because the action of Δ is given by translations by G-invariant elements. We therefore obtain a commutative diagram

(1)
$$
\begin{array}{ccc}
A_0 \times P_G & \xrightarrow{\;/\Delta\;} & A \\
{\scriptstyle /G}\downarrow & & \downarrow{\scriptstyle /G} \\
A_0 \times (P_G/G) & \xrightarrow{\;/\Delta\;} & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\;/\Delta\;} & A/P_G.
\end{array}
$$

In the bottom square of the diagram $A_0 \times (P_G/G) \to A_0$ is the projection onto the first coordinate, and therefore that part of the diagram is a trivialization of the fibration $A/G \to A/P_G$ given by Proposition 2.9 in [ALA20a]. The action of Δ passes to $A_0 \times (P_G/G)$ and acts on A_0 by translation. Thus, giving a full classification of these fibrations is reduced to studying the actions of Δ on P_G/G which are induced by translation by elements of $A_0 \cap P_G$ on P_G. In other words, we need to understand the commutative diagram

(2)
$$
\begin{array}{ccc}
P_G & \xrightarrow{\;t\;} & P_G \\
{\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\
P_G/G & \xrightarrow{\;t_G\;} & P_G/G,
\end{array}
$$

where t is any translation by a non zero element of $A_0 \cap P_G$ and t_G is the induced morphism on the quotient. Notice that $A_0 \cap P_G$ acts faithfully on P_G/G, and particularly, if $A_0 \cap P_G \neq \{0\}$, the fibration $A/G \to A/P_G$ is not trivial. Then, giving a classification of the fibration $A/G \to A/P_G$ is reduced to studying faithful representations of Δ in $\mathrm{Aut}(P_G/G)$, where P_G/G is a product of projective spaces. In the case when the analytic representation of G on P_G is irreducible the problem is reduced to studying faithful representations of Δ in $\mathrm{PGL}_{n+1}(\mathbb{C})$, which are easier to understand than the representations of Δ in $\mathrm{Aut}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r})$. In Section 3 we study the case when the analytic representation of G on P_G is irreducible. Now, let us prove that the general case can be reduced to the irreducible case. As $\dim(P_G^G) = 0$ and P_G/G is smooth, by Theorem 1.2, the abelian subvariety P_G and the group G can be decomposed. Then, there exist abelian subvarieties $P_1, \dots P_r \subset P_G$ and subgroups $G_1, \dots, G_r \leq \mathrm{Aut}_0(A)$ such that $P_G \cong P_1 \times \cdots \times P_r$, $G \cong G_1 \times \cdots \times G_r$ and

$$
P_G/G \cong P_1/G_1 \times \cdots \times P_r/G_r.
$$

Furthermore, the analytic representation of G_i on P_i is irreducible and, by Theorem 1.1, $P_i/G_i \cong \mathbb{P}^{n_i}$. Thus, the morphism

$$
\begin{aligned}
A_0 \times P_1 \times \cdots \times P_r &\to A; \\
(a, b_1, \dots, b_r) &\mapsto a - (b_1 + \cdots + b_r).
\end{aligned}
$$

is an isogeny of kernel Δ. Let us fix an inclusion $\iota : \Delta \to A_0$ given by $(a, b_1, \dots, b_r) \mapsto a$ and denote by Δ_0 the image $\iota(\Delta)$. This morphism is well defined and is an inclusion, because for all $a \in A_0 \cap P_G$ there are unique $b_i \in P_i$ such that $a = b_1 + \cdots + b_r$. Moreover, we have $\Delta_0 = A_0 \cap P_G$.

The fibration $A/G \to A/P_G$ is trivialized by $A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to A_0$ as in the following diagram

$$
\begin{array}{ccc}
A_0 \times P_1 \times \cdots \times P_r & \xrightarrow{/\Delta} & A \\
{\scriptstyle /G}\downarrow & & \downarrow{\scriptstyle /G} \\
A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} & \xrightarrow{/\Delta} & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{/\Delta_0} & A/P_G.
\end{array}
$$

Our purpose is understand to the morphism $A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to A/G$ in order to prove Theorem 1.5:

Proof of Theorem 1.5. Let $\mathcal{P}_i := \prod_{k \in \{1,\ldots,r\}-\{i\}} P_k$ be the product of the P_k with $k \neq i$, which can be considered as an abelian subvariety of P_G, hence of A. Denote $B_i := A/\mathcal{P}_i$. The abelian variety $\mathcal{P}_i$ acts on $A_0 \times P_1 \times \cdots \times P_r$ by translation on each coordinate. This action commutes with the action of Δ. Hence, we have the commutative diagram

$$
\begin{array}{ccc}
A_0 \times P_1 \times \cdots \times P_r & \xrightarrow{/\Delta} & A \\
{\scriptstyle /\mathcal{P}_i}\downarrow & & \downarrow{\scriptstyle /\mathcal{P}_i} \\
A_0 \times P_i & \longrightarrow & B_i.
\end{array}
$$

Denote by Δ_i the kernel of the morphism $A_0 \times P_i \to B_i$. As $\Delta \cap \mathcal{P}_i = 0$, we have $\Delta_i \cong \Delta$. Therefore, the morphism $A_0 \times P_i \to B_i$ is an isogeny. Moreover, the restriction to P_i is an injection, because $P_i \cap \mathcal{P}_i = 0$. The group G_i acts on B_i and this action is faithful. Denote by $B_{i,0}$ the connected component of $B_i^{G_i}$ that contains 0 and P_{G_i} the complementary abelian subvariety by a G_i-invariant polarization. By Proposition 2.9 of [ALA20a] there exists a fibration $B_i/G_i \to B_i/P_{G_i}$. On the one hand, the image of $A_0 \times \{0\}$ in B_i is contained in $B_{i,0}$, and in fact the image is equal to $B_{i,0}$. This gives a surjective morphism $A_0 \to B_{i,0}/(B_{i,0} \cap P_{G_i}) \cong B_i/P_{G_i}$. On the other hand, the image of $\{0\} \times P_i$ is contained in P_{G_i}. Since P_i is injected into P_{G_i}, both are connected and have the same dimension, they are isomorphic. Hence, $P_{G_i}/G_i \cong P_i/G_i \cong \mathbb{P}^{n_i}$ is smooth and, by Proposition 2.9 in [ALA20a], the quotient B_i/G_i is smooth. Then, we have the commutative diagram

(3)
$$
\begin{array}{ccc}
A_0 \times P_i & \xrightarrow{/\Delta_i} & B_i \\
{\scriptstyle /G_i}\downarrow & & \downarrow{\scriptstyle /G_i} \\
A_0 \times \mathbb{P}^{n_i} & \xrightarrow{/\Delta_i} & B_i/G_i \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & B_i/P_{G_i},
\end{array}
$$

where all the morphisms are surjective. We claim that the kernel of the morphism $A_0 \to B_i/P_{G_i}$ is Δ_0. On the one hand, let $f : A_0 \times P_i \to B_i$ be the morphism given by diagram (3) and $a \in A_0$ such that $f(a,0) = \overline{0} \in B_{i,0}/(B_{i,0} \cap P_i)$. Then, $f(a,0) \in B_{i,0} \cap P_i$ and there exist $(0,-b_i) \in \{0\} \times P_i$ such that $f(0,-b_i) = f(a,0)$, so $f(a,b_i) = 0$. This implies that $(a,b_i) \in \Delta_i$, so $a \in \Delta_0$. On the other hand, if $a \in \Delta_0$ there exits $(b_1, b_2, \ldots, b_r) \in \prod_{k=1}^{n} P_k$ such that $a = b_1 + b_2 + \cdots + b_r$. In particular, $a - b_i \in \mathcal{P}_i$, hence $f(a,b_i) = 0$ in B_i and $a \in \ker(f)$. Then, the kernel of the morphism $A_0 \to B_i/P_{G_i}$ is Δ_0 and

$$
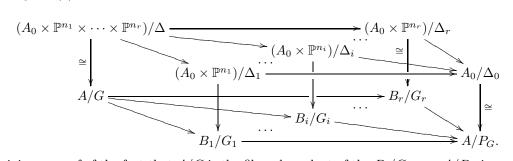B_{i,0}/(B_{i,0} \cap P_i) \cong A_0/\Delta_0,
$$

hence we can see B_i/G_i as a fibration over A_0/Δ_0 and therefore, since $\Delta_0 = A_0 \cap P_G$, as a fibration over A/P_G and, in particular,
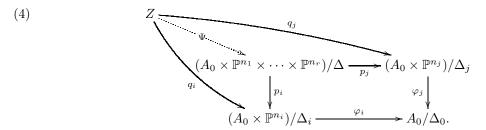
$$
B_i/P_{G_i} \cong A/P_G.
$$

So far, we have proved the three first statements of Theorem 1.5. We need to prove the existence of the isomorphism

$$A/G \cong B_1/G_1 \underset{A/P_G}{\times} \cdots \underset{A/P_G}{\times} B_r/G_r.$$

The morphisms $A \to B_i$ induce morphisms $A/G \to B_i/G$. By Theorem 1.2, the group G_j acts trivially over B_i when $j \neq 1$, therefore $B_i/G = B_i/G_i$. The quotient B_i/G_i is smooth because its fibers are isomorphic to $\mathbb{P}^{n_i}$, then the pairs (B_i, G_i) satisfy the conditions of Theorem 1.4. Therefore, we have the commutative diagram given by the morphisms $A/G \to B_i/G_i$ and the vertical isomorphisms given by the diagram (3)

Thus, giving a proof of the fact that A/G is the fibered product of the B_i/G_i over A/P_G is equivalent to proving that $(A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r})/\Delta$ is the fibered product of the $(A_0 \times \mathbb{P}^{n_i})/\Delta_i$ over A_0/Δ_0. Let us prove the second assertion. Let Z be a variety with morphisms $q_i : Z \to (A_0 \times \mathbb{P}^{n_i})/\Delta_i$ such that the following diagram commutes for all $i, j \in \{1, \ldots, r\}$:

(4)

We need to prove the existence and the uniqueness of Ψ. Let us prove first the existence. Let $z \in Z$ and $q_i(z) = [(x_i, \alpha_i)]_{\Delta_i}$ with $\alpha_i \in \mathbb{P}^{n_i}$, where $[(x_i, \alpha_i)]_{\Delta_i}$ is the class of (x_i, α_i) modulo Δ_i. Since for all $i, j \in \{1, \ldots, r\}$

$$\varphi_i q_i(z) = \varphi_j q_j(z),$$

we have for all $i \in \{2, \ldots, r\}$

$$\varphi_1 q_1(z) = \varphi_1 \left([(x_1, \alpha_1)]_{\Delta_1}\right) = [x_1]_{\Delta_0} = [x_i]_{\Delta_0} = \varphi_i \left([(x_i, \alpha_i)]_{\Delta_i}\right) = \varphi_i q_i(z).$$

This implies that there exists $a_i \in \Delta_0$ such that $a_i * x_1 = x_i$. Since $a_i \in \Delta_0 = A_0 \cap P_G$, there exists a unique $b_{ii} \in P_i$ such that $(a_i, b_{ii}) \in \Delta_i$. Then, there exists $\alpha_i' \in \mathbb{P}^{n_i}$ such that $(a_i, b_{i,i}) * [(x_1, \alpha_i')]_{\Delta_i} = [(x_i, \alpha_i)]_{\Delta_i}$. Then, we have for all $i \in \{2, \ldots, r\}$

$$q_i(z) = [(x_1, \alpha_i')]_{\Delta_i}.$$

This allows us to define the morphism $\Psi : Z \to (A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r})/\Delta$ given by

$$z \mapsto [(x_1, \alpha_1', \ldots, \alpha_r')]_\Delta,$$

where $\alpha_1 = \alpha_1'$, which makes diagram (4) commute for all $i, j \in \{1, \ldots, r\}$. Let us prove the uniqueness. Let $\overline{\Psi} : Z \to (A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r})/\Delta$ be another morphism that makes diagram (4) commute for all

$i, j \in \{1, \ldots, r\}$. Let $z \in Z$ and $\overline{\Psi}(z) = [(x, \beta_1, \ldots, \beta_r)]_\Delta$ for any $(x, \beta_1, \ldots, \beta_r) \in A_0 \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. Then,

$$p_i \circ \overline{\Psi}(z) = p_i([(x, \beta_1, \ldots, \beta_r)]_\Delta) = [(x, \beta_i)]_{\Delta_i} = [(x_1, \alpha'_i)]_{\Delta_i} = q_i(z),$$
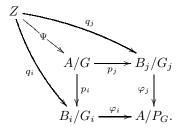
because $p_i \circ \overline{\Psi} = q_i$. This implies that there exists $(a_i, b_i) \in \Delta_i$ such that $(a_i, b_i) * (x_1, \alpha'_i) = (x, \beta_i)$. Since the action of Δ_i on the first coordianate is by translations, we have for all $i \in \{1, \ldots, r\}$ that $x = x_1 + a_i$ and hence there exists $a \in A_0 \cap P_G$ such that $a = a_i$ for all $i \in \{1, \ldots, r\}$. Then, as the decomposition of a in P_G is unique, we have $a = b_1 + \cdots + b_r$. Then, we have

$$(a, b_1, \ldots, b_r) * (x_1, \alpha'_1, \ldots, \alpha'_r) = (x, \beta_1, \ldots, \beta_r),$$

which implies that $[(x_1, \alpha'_1, \ldots, \alpha'_r)]_\Delta = [(x, \beta_1, \ldots, \beta_r)]_\Delta$, and hence the uniqueness of Ψ because

$$\Psi(z) = [(x_1, \alpha'_1, \ldots, \alpha'_r)]_\Delta = [(x, \beta_1, \ldots, \beta_r)]_\Delta = \overline{\Psi}(z).$$

Therefore, there exists a unique morphism $\Psi : Z \to A/G$ such that for all $i, j \in \{1, \ldots, r\}$ the following diagram commutes

This proves that

$$A/G \cong B_1/G_1 \underset{A/P_G}{\times} \cdots \underset{A/P_G}{\times} B_r/G_r.$$

Finally, we have proved Theorem 1.5.

$\square$

Remark 2.1. The trivialization $A_0 \times \mathbb{P}^{n_i} \to A_0$ of $B_i/G_i \to A/P_G$ given by diagram (3) does not correspond to the one given in Theorem 1.4. That trivialization corresponds to the right square of the commutative diagram.
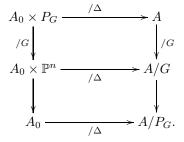

3. Irreducible case

As in the last section, A will be an abelian variety, G a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth, A_0 the connected component of A^G that contains 0 and P_G the complementary abelian subvariety of A_0 with respect to a G-invariant polarization. In this section we will assume that the analytic representation of G in P_G is irreducible. In this case, by Theorem 1.1, we know that $P_G/G \cong \mathbb{P}^n$. Diagram (1), then becomes

And the study of the relation between the fibers is reduced to diagram (2), which becomes

(5)
$$
\begin{array}{ccc}
P_G & \xrightarrow{\ t\ } & P_G \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \xrightarrow{\ t_G\ } & \mathbb{P}^n,
\end{array}
$$

where t is a translation by an element of $A_0 \cap P_G$ and t_G is the induced function on the fibers that makes the diagram commute. The subvariety P_G and the group G, by Theorem 1.1, are classified as pairs (P_G, G).

Definition 3.1. Let P_G and G be as in Theorem 1.1. The pairs (P_G, G) satisfying 3a) will be called of *type α*, those satisfaying 3b) will be called of *type β* and those satisfying 3c) will be called of *type γ*.

In what follows we treat the three cases separately.

3.1. Classification of the fibers with (P_G, G) of type α

If (P_G, G) is a pair of type α, the abelian variety P_G is isomorphic to the self-product of an elliptic curve E and G is isomorphic to $C^n \rtimes S_n$, with $n = \dim(P_G)$ and C a non-trivial subgroup of $\mathrm{Aut}(E)$, and therefore a cyclic group of order $2, 3, 4$ or 6. The action of C^n is coordinatewise and the action of S_n permutes them.

The main result of this section is the following.

Proposition 3.2. *Let A be an abelian variety and G be a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and P_G be the complementary abelian subvariety with respect to a G-invariant polarization. If the analytic representation of G on P_G is irreducible and (P_G, G) is a pair of type α, there exists a trivialization*

$$
\begin{array}{ccc}
A_0 \times \mathbb{P}^n & \xrightarrow{\ /\Delta\ } & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\ /\Delta\ } & A/P_G,
\end{array}
$$

of the fibration $A/G \to A/P_G$, with $\Delta \cong A_0 \cap P_G$, where Δ acts on A_0 by translations and on the fibers as one of the following:

  (1) $\Delta \cong \{0\}$ and therefore acts trivially.
  (2) $\Delta \cong \mathbb{Z}/2\mathbb{Z}$ and acts with generator $\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : -z_1 : \cdots : (-1)^n z_n]$.
  (3) $\Delta \cong \mathbb{Z}/3\mathbb{Z}$ and acts with generator $\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_3 z_1 : \cdots : \zeta_3^n z_n]$.
  (4) $\Delta \cong (\mathbb{Z}/2\mathbb{Z})^2$ and there exists a set of generators that act as

$$
\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : (-1)^n z_n],
$$
$$
\delta' : [z_0 : \cdots : z_n] \mapsto [z_n : z_{n-1} : \cdots : z_1 : z_0].
$$

Proof. The fixed points of P_G by the action of G are

$$
P_G^G \cong \{(x, \ldots, x) \in E^n \mid x \in E^C\} \cong E^C.
$$

Let Δ be a non trivial subgroup of P_G^G. Recall that we want to study the induced morphism t_G between the fibers as in diagram (5)

$$
\begin{array}{ccc}
P_G & \xrightarrow{\ t\ } & P_G \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \xrightarrow{\ t_G\ } & \mathbb{P}^n
\end{array}
$$

where t is a translation by an element of Δ. The morphism $P_G \to \mathbb{P}^n$ can be decomposed as

$$P_G \xrightarrow{/C^n} (\mathbb{P}^1)^n \xrightarrow{/S_n} \mathbb{P}^n \ ,$$

where the function $(\mathbb{P}^1)^n \mapsto \mathbb{P}^n$ is given by

$$g : (\mathbb{P}^1)^n \mapsto \mathbb{P}^n;$$

$$([x_1 : y_1], \dots, [x_n : y_n]) \to [Y^{n-i} p_i(x_1 Y_1, \dots, x_n Y_n)]_i,$$

with $Y = y_1 \cdots y_n$, $Y_i = Y/y_i$, $p_0 \equiv 1$ and

$$p_i(z_1, \dots, z_n) = \sum_{k_1 < \cdots < k_i} z_{k_1} \cdots z_{k_i}.$$

Since the action of Δ on $\mathbb{P}^n$ is faithful, the action induced on $(\mathbb{P}^1)^n$ is too. This reduces the problem to studying the commutative diagram

$$
\begin{array}{ccc}
(\mathbb{P}^1)^n & \xrightarrow{\ \bar{t}\ } & (\mathbb{P}^1)^n \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\
\mathbb{P}^n & \xrightarrow[t_G]{} & \mathbb{P}^n,
\end{array}
$$

where $\bar{t}$ is the morphism induced by t over $(\mathbb{P}^1)^n$. Since t translates by the same element in each coordinate, we can see $\bar{t}$ as an element of $\mathrm{Aut}(\mathbb{P}^1)$ acting in a diagonal way on $(\mathbb{P}^1)^n$. Since the action of Δ on $(\mathbb{P}^1)^n$ is faithful and diagonal, the induced action of Δ on $\mathbb{P}^1$ is faithful. Hence, the group Δ is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$. By Proposition 4.1 in [Bea10], two isomorphic subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are conjugate. Therefore, it is enough to study one arbitrary representation of Δ in $\mathrm{PGL}_2(\mathbb{C})$.

When Δ has order 1, the fibration is the trivial one. Then, we assume Δ non trivial.

If the order of C is 2, then $P_G^G \cong E[2]$ and $E[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, Δ has order 2 or 4. When the order of Δ is 2, we can consider

$$\Delta \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

in $\mathrm{PGL}_2(\mathbb{C})$. Let $\bar{t}$ be the non trivial translation of Δ. We are looking for $t_G \in \mathrm{Aut}(\mathbb{P}^n)$ such that $t_G g = g\bar{t}$. A direct computation gives $t_G : \mathbb{P}^n \to \mathbb{P}^n$ given by $[z_0 : \cdots : z_n] \to [z_0 : -z_1 : \cdots : (-1)^n z_n]$. Then, if Δ has order 2

$$\Delta \cong \langle \delta : [z_0 : z_1 : \cdots : z_n] \to [z_0 : -z_1 : \cdots : (-1)^n z_n] \rangle.$$

This is point 2 of Proposition 3.2.

If Δ has order 4, and hence is isomorphic to the Klein group, we can consider

$$\Delta \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Let $\bar{t}, \bar{u} : P_G \to P_G$ be two translations by elements of Δ such that $\langle t, u \rangle \cong \Delta$. Then, up to a base change, we can set

$$t \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{y} \quad u \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the work done above we know that $t_G : \mathbb{P}^n \to \mathbb{P}^n$ is given by

$$[z_0 : \cdots : z_n] \to [z_0 : -z_1 : \cdots : (-1)^n z_n].$$

Then, we just need to determine $u_G : \mathbb{P}^n \to \mathbb{P}^n$ such that $u_G g = g u$. A direct computation gives $u_G : \mathbb{P}^n \to \mathbb{P}^n$ given by $[z_0 : \cdots : z_n] \to [z_n : z_{n-1} : \cdots : z_1 : z_0]$. Then, up to a change of basis, the action of Δ on $\mathbb{P}^n$ satisfies

$$\Delta \cong \langle \delta : [z_0 : \cdots : z_n] \to [z_0 : -z_1 : \cdots : (-1)^n z_n], \ \delta' : [z_0 : \cdots : z_n] \to [z_n : z_{n-1} : \cdots : z_1 : z_0] \rangle.$$

This is point 4 of Proposition 3.2.

When the order of C is 3 there is only one possible elliptic curve, the one with an action of $\mathbb{Z}/6\mathbb{Z}$. In this case $E^C \cong \mathbb{Z}/3\mathbb{Z}$, because there are only three fixed points by the action of C, hence $\Delta \cong \mathbb{Z}/3\mathbb{Z}$. Then, we can consider

$$\Delta \cong \left\langle \begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Let t be a translation induced by any non trivial element of Δ and $\bar{t} : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$ the morphism induced by t. Then, up to a base change, we can set

$$t_G \to \begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix}.$$

We are looking for t_G such that $t_G \circ g = g \circ \bar{t}$. A direct computation gives $t_G : \mathbb{P}^n \to \mathbb{P}^n$ given by $[z_0 : \cdots : z_n] \to [z_0 : \zeta_3 z_1 : \cdots : \zeta_3^n z_n]$. Then,

$$\Delta \cong \langle \delta : [z_0 : \cdots : z_n] \to [z_0 : \zeta_3 z_1 : \cdots : \zeta_3^n z_n] \rangle.$$

This is point 3 of Proposition 3.2.

If C has order 6, the elliptic curve E is the same as in the last case. But in this case there is only one fixed point, hence $\Delta \cong \{\mathrm{id}\}$ and we are done.

If the order of C is 4, there are only two points on E fixed by the action of G, hence $\Delta \cong \mathbb{Z}/2\mathbb{Z}$. This case has already been studied in the case when C has order 2.

All these results prove Proposition 3.2, which corresponds to points 1, 2, 3 and 4 of Theorem 1.4.

$\square$

3.2. Classification of the fibers with (P_G, G) of type β

The pairs (P_G, G) of type β are composed of the abelian variety P_G isomorphic to

$$\{(x_0, \ldots, x_n) \in E^{n+1} \mid x_0 + \cdots + x_n = 0\},$$

and the group G isomorphic to S_{n+1}.

In this case the next proposition holds.

Proposition 3.3. *Let A be an abelian variety and G a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and P_G be the complementary abelian subvariety with respect to a G-invariant polarization. If the analytic representation of G on P_G is irreducible and (P_G, G) is a pair of type β, there exists a trivialization*

$$
\begin{array}{ccc}
A_0 \times \mathbb{P}^n & \xrightarrow{/\Delta} & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow[/\Delta]{} & A/P_G,
\end{array}
$$

of the fibration $A/G \to A/P_G$, with $\Delta \cong A_0 \cap P_G$, where Δ acts on A_0 by translations and there exists a set of generators such that Δ acts on the fibers as

$$\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_{n+1}^a z_1 : \cdots : \zeta_{n+1}^{an} z_n],$$

$$\delta' : [z_0 : \cdots : z_n] \mapsto [z_{n+1-b} : \cdots : z_{n-1} : z_n : z_0 : \cdots : z_{n-b}],$$

with a and b positive integers, $b \mid (n+1)$ and $a \mid b$, where $z_{-1} := z_n$.

In [Auf17], R. Auffarth proves that we can see the quotient P_G/G as

$$\{D \in \mathrm{Div}(E) \mid D \sim (n+1)[0], \, D \geq 0\},$$

which is the complete linear system of $(n+1)[0]$. Then, the morphism $P_G \to \mathbb{P}^n \cong P_G/G$ is given by

$$P_G \to |(n+1)[0]|;$$

$$(x_0, \ldots, x_n) \mapsto [x_0] + \cdots + [x_n].$$

On the other hand, the fixed points of P_G by the action of G are

$$P_G^G = \{(x,\ldots,x) \in E^{n+1} \mid x \in E[n+1]\} \cong E[n+1].$$

Let $(x,\ldots,x),(y,\ldots,y) \in P_G^G$ be two generators of P_G^G. Denote by t and u the translations by $(x,\ldots,x)$ and $(y,\ldots,y)$, respectively, in P_G. Let t_G and u_G be the induced morphism by t and u on the fibers. We can see these morphisms as automorphisms of $\mathbb{P}^n$, as in the diagram

$$\begin{array}{ccc}
P_G & \xrightarrow{\;\;t,u\;\;} & P_G \\
\downarrow & & \downarrow \\
|(n+1)[0]| & \xrightarrow{\;\;t_G,u_G\;\;} & |(n+1)[0]|.
\end{array}$$

The following results will tell us that there is only one representation of $\Delta = P_G^G$ in $\mathrm{PGL}_{n+1}(\mathbb{C})$, up to a base change.

Lemma 3.4. *Let $(x,\ldots,x),(y,\ldots,y) \in P_G^G$ be generators of P_G^G. Let t and u be the translations by $(x,\ldots,x)$ and $(y,\ldots,y)$, respectively. If t_G and u_G are the induced morphisms on the quotient P_G/G, there exist $x_t, y_u \in E$ such that*

$$\mathrm{Fix}(t_G) = \{[x_t + my] + [x_t + my + x] + \cdots + [x_t + my + nx] \mid m \in \{0,\ldots,n\}\},$$

and

$$\mathrm{Fix}(u_G) = \{[y_u + mx] + [y_u + mx + y] + \cdots + [y_u + mx + ny] \mid m \in \{0,\ldots,n\}\}.$$

Moreover, $\mathrm{Fix}(t_G) \cap \mathrm{Fix}(u_G) = \emptyset$.

Proof. The image of any point of P_G in $|(n+1)[0]|$ is fixed by t_G if and only if $t(x_0,\ldots,x_n) = (x_{\sigma(0)},\ldots,x_{\sigma(n)})$ for some $\sigma \in S_{n+1}$. Suppose that $\sigma = \sigma_1 \sigma_2$ with σ_1 and σ_2 disjoint cycles and σ_1 non trivial. Then, we have $x_{\sigma_1^i(k)} + x = x_{\sigma_1^{i+1}(k)}$ for all $i \in \{0,\ldots,\mathrm{ord}(\sigma_1)-1\}$, where k is the start of σ_1. This implies that $\mathrm{ord}(\sigma_1)x = 0$ and therefore $\mathrm{ord}(x)|\mathrm{ord}(\sigma_1)$. Since the order of x is $n+1$, we have $\sigma = \sigma_1$. In this case, $x_{\sigma^i(0)} = x_0 + ix$ and $(x_0, x_1, \ldots, x_n) \in P_G^G$, then

$$0 = \sum_{i=0}^n x_{\sigma^i(0)} = \sum_{i=0}^n (x_0 + ix) = (n+1)x_0 + \frac{n(n+1)}{2}x.$$

This equation in the variable x_0 has a solution in the elliptic curve and the difference between two of them is an element of $(n+1)$-torsion of the elliptic curve, so there are exactly $(n+1)^2$ different solutions. Let x_t be a solution of the last equation. The divisors

$$D_m := [x_t + my] + [x_t + my + x] + \cdots + [x_t + my + nx],$$

with $m \in \{0,\ldots,n\}$, are all fixed by the action of t_G and they are all different because if $D_m = D_{\tilde{m}}$, there would exist $k \in \{0,\ldots,n\}$ such that

$$x_t + \tilde{m}y = x_t + my + kx$$

and hence $kx + (m - \tilde{m})y = 0$. Since x and y generate the $(n+1)$-torsion of E, it follows that $k = 0$ and $m = \tilde{m}$. Let $x' \in E$ be another solution and $D := [x'] + [x' + x] + \cdots + [x' + nx]$ the divisor associated to this solution. Since the difference between two solutions is an element of $(n+1)$-torsion, there exist $m, k \in \{0,\ldots,n\}$ such that $x' = x_t + my + kx$, hence

$$\begin{aligned}
D &= [x'] + [x' + x] + \cdots + [x' + nx] \\
&= [x_t + my + kx] + [x_t + my + (k+1)x] + \cdots + [x_t + my + (k+n)x] \\
&= [x_t + my] + [x_t + my + x] + \cdots + [x_t + my + nx] = D_m.
\end{aligned}$$

Then, the set of fixed points by t_G is finite and moreover

$$\mathrm{Fix}(t_G) = \{[x_t + my] + [x_t + my + x] + \cdots + [x_t + my + nx] \mid m \in \{0,\ldots,n\}\}.$$

The proof is analogous for $\mathrm{Fix}(u_G)$.

Suppose that $\mathrm{Fix}(t_G) \cap \mathrm{Fix}(u_G) \neq \emptyset$. Then, there exists $m \in \{0, \ldots, n\}$ such that $[x_t + my] + [x_t + my + x] + \cdots + [x_t + my + nx] \in \mathrm{Fix}(u_G)$, therefore $u(x_t + my, x_t + my + x, \ldots, x_t + my + nx)$ is a permutation of $(x_t + my, x_t + my + x, \ldots, x_t + my + nx)$, so there exists $k \in \{0, \ldots, n\}$ such that $x_t + my = x_t + (m+1)y + kx$. This occurs only if $y + kx = 0$, but x and y generate P_G^G. Then, $\mathrm{Fix}(t_G) \cap \mathrm{Fix}(u_G) = \emptyset$.

$\square$

By Lemma 3.4 we have that t_G and u_G have $n + 1$ fixed points and these sets are disjoint. The following proposition allows us to conclude that the faithful representations of $\Delta = P_G^G$ in $\mathrm{PGL}_{n+1}(\mathbb{C})$ induced by the action of Δ on P_G are unique up to conjugation.

In what follows, for any two elements M and N of $\mathrm{GL}_n(\mathbb{C})$, $M \equiv N$ denotes that they have the same image in $\mathrm{PGL}_n(\mathbb{C})$ by the projection.

Proposition 3.5. *Let $\phi, \psi \in \mathrm{PGL}_n(\mathbb{C})$ be two classes such that*
- *both of them have order n and commute,*
- *$|\mathrm{Fix}(\phi)| = |\mathrm{Fix}(\psi)| = n$,*
- *$\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \emptyset$.*

There exists $\xi \in \mathrm{PGL}_n(\mathbb{C})$ such that

$$\xi\phi\xi^{-1} \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_n & 0 & \cdots & 0 \\ 0 & 0 & \zeta_n^2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \zeta_n^{n-1} \end{pmatrix} \quad and \quad \xi\psi\xi^{-1} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

First we will prove the following lemma.

Lemma 3.6. *Let*

$$M \equiv \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & v_{n-1} \\ v_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad and \quad N \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

be two classes in $\mathrm{PGL}_n(\mathbb{C})$ with $v_0 \cdots v_{n-1} = 1$. There exists a diagonal matrix $D \in \mathrm{GL}_n(\mathbb{C})$ such that $D^{-1}MD \equiv N$.

Proof. The matrix given by

$$D = \begin{pmatrix} v_0 & 0 & \cdots & 0 & 0 \\ 0 & v_0 v_1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & v_0 v_1 \cdots v_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

works.

$\square$

Proof of Proposition 3.5. Let $p_1, \ldots, p_n \in \mathrm{Fix}(\phi)$ and let $v_1, \ldots, v_n \in \mathbb{C}^n$ be a lifting of p_i for $i \in \{1, \ldots, n\}$. Let Φ be a lifting of ϕ to $\mathrm{GL}_n(\mathbb{C})$. Since ϕ has exactly n fixed points, we have that Φ has n different eigenvalues and $\{v_1, \ldots, v_n\}$ is a base of $\mathbb{C}^n$. Then, up to a base change,

$$\Phi \equiv \begin{pmatrix} \phi_1 & 0 & 0 & \cdots & 0 \\ 0 & \phi_2 & 0 & \cdots & 0 \\ 0 & 0 & \phi_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \phi_n \end{pmatrix},$$

where the ϕ_i are the eigenvalues of Φ.

Let us prove that ψ acts on $\mathrm{Fix}(\phi)$. Let $p \in \mathrm{Fix}(\phi)$. Since ψ and ϕ commute, we have

$$\phi\psi(p) = \psi\phi(p) = \psi(p).$$

This implies that ψ acts on $\mathrm{Fix}(\phi)$. This allows us to see ψ as an element of the permutation group of $\mathrm{Fix}(\phi)$. We claim that ψ is an n-cycle. Otherwise, we can write ψ as a product of disjoint cyles and let $\sigma := (p_{a_1}, \ldots, p_{a_m})$ and $\tau := (p_{b_1}, \ldots, p_{b_l})$ be two of these cycles. Let $\Psi \in \mathrm{GL}_n(\mathbb{C})$ be a lifting of ψ written in the basis $\{v_1, \ldots, v_n\}$. For some $\psi_{a_1}, \ldots, \psi_{a_m}, \psi_{b_1}, \ldots, \psi_{b_l} \in \mathbb{C}^*$, we have

$$\Psi(v_{a_i}) = \begin{cases} \psi_{a_i} v_{a_{i+1}} & \text{if } 1 \le i \le m-1, \\ \psi_{a_m} v_{a_1} & \text{if } i = m, \end{cases}$$

and

$$\Psi(v_{b_i}) = \begin{cases} \psi_{b_i} v_{b_{i+1}} & \text{if } 1 \le i \le l-1, \\ \psi_{b_l} v_{b_1} & \text{if } i = l. \end{cases}$$

Let $\Sigma \in \mathrm{GL}_n(\mathbb{C})$ be a lifting of σ and $T \in \mathrm{GL}_n(\mathbb{C})$ be a lifting of τ. Since σ and τ are in $\mathrm{PGL}_n(\mathbb{C})$ we can assume that $\det(\Sigma) = \det(T) = 1$, hence $\psi_{a_1} \cdots \psi_{a_m} = 1$ and $\psi_{b_1} \cdots \psi_{b_l} = 1$. Now, define

$$v_a := v_{a_1} + \psi_{a_1} v_{a_2} + \psi_{a_1}\psi_{a_2} v_{a_3} + \cdots + \psi_{a_1} \cdots \psi_{a_{m-1}} v_{a_m},$$

and

$$v_b := v_{b_1} + \psi_{b_1} v_{b_2} + \psi_{b_1}\psi_{b_2} v_{b_3} + \cdots + \psi_{b_1} \cdots \psi_{b_{l-1}} v_{b_l}.$$

These vectors are linearly independent, because they are in two different subspaces with trivial intersection and none is null. Moreover,

$$\Psi(v_a) = v_a \quad \text{and} \quad \Psi(v_b) = v_b.$$

Therefore, the projections of v_a and v_b in $\mathbb{P}^{n-1}$ are different, fixed by ψ and have the same eigenvalue. This is a contradiction, because ψ has n fixed points and any lifting of ψ has n different eigenvalues, one for each fixed point.

Since ψ is a n-cycle, up to reordering the basis $\{v_1, \ldots, v_n\}$, we have that

$$\Psi \equiv \begin{pmatrix} 0 & \psi_2 & 0 & \cdots & 0 \\ 0 & 0 & \psi_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \psi_n \\ \psi_1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \Phi \equiv \begin{pmatrix} \phi_1 & 0 & 0 & \cdots & 0 \\ 0 & \phi_2 & 0 & \cdots & 0 \\ 0 & 0 & \phi_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \phi_n \end{pmatrix}.$$

By Lemma 3.6, up to a base change by a diagonal matrix, we have that $\psi_i = 1$ for all $i \in \{1, \ldots, n\}$. Since $\phi \in \mathrm{PGL}_n(\mathbb{C})$ we can put $\phi_1 = 1$ and by the commutativity of ψ and ϕ whe have that

$$\begin{pmatrix} 0 & \phi_2 & 0 & \cdots & 0 \\ 0 & 0 & \phi_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \phi_n \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \phi_1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \phi_{n-1} \\ \phi_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

therefore there exists $\lambda \in \mathbb{C}^*$ such that $\phi_2 = \lambda$, $1 = \lambda\phi_n$ and $\phi_i = \lambda\phi_{i-1}$ for all $i \in \{3, \ldots, n\}$. Then $\phi_i = \lambda^{i-1}$. Since all ϕ_i are different pairwise, we have that λ is a primitive n-th root of unity. This proves the proposition.

$\square$

Proof of Proposition 3.3. Recall that we want to classify all the trivializations of the form

$$
\begin{array}{ccc}
A_0 \times \mathbb{P}^n & \xrightarrow{\ /\Delta\ } & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\ /\Delta\ } & A/P_G,
\end{array}
$$

with $\Delta \cong A_0 \cap P_G$, where $A_0 \cap P_G \subset P_G^G$. It depends on how much the abelian varieties A_0 and P_G intersect each other. We will assume $A_0 \cap P_G = P_G^G$. This allows us to study the abstract group Δ as $\langle t_G, u_G \rangle$, because Δ acts faithfully on P_G/G, and take $\langle t_G, u_G \rangle \cong P_G^G$. The morphisms t_G, u_G are automorphisms of the linear system $|(n+1)[0]|$ which can be identified with $\mathbb{P}^n$, so we can see them as commuting elements of $\mathrm{PGL}_{n+1}(\mathbb{C})$. By Lemma 3.4 we have that $\mathrm{Fix}(t_G) = n+1$, $\mathrm{Fix}(u_G) = n+1$ and $\mathrm{Fix}(t_G) \cap \mathrm{Fix}(u_G) = \emptyset$. Then, by Proposition 3.5 we have that, up to a base change,

$$
\phi_{t_G} \equiv \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta_{n+1} & 0 & \cdots & 0 \\
0 & 0 & \zeta_{n+1}^2 & \cdots & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \cdots & \zeta_{n+1}^n
\end{pmatrix}
\text{ and } \phi_{u_G} \equiv \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

Hence Δ, up to a base change, is generated by the automorphisms

$$
\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_{n+1} z_1 : \cdots : \zeta_{n+1}^n z_n],
$$
$$
\delta' : [z_0 : \cdots : z_n] \mapsto [z_n : z_0 : z_1 : \cdots : z_{n-1}].
$$

This representation does not depend on the set of generators of $\{t_G, u_G\}$, because if we take another pair of generators t_G' and u_G', everything done above remains the same.

All the subgroups of $(\mathbb{Z}/(n+1)\mathbb{Z})^2$ have the form $\langle t_G^a, u_G^b \rangle$ in some $(\mathbb{Z}/(n+1)\mathbb{Z})$-basis and for some $a, b \in \mathbb{Z}$ with $b \mid (n+1)$ and $a \mid b$. Thus, if Δ is isomorphic to any proper subgroup of P_G^G the action of Δ on P_G/G is generated by

$$
\delta : [z_0 : \cdots : z_n] \mapsto [z_0 : \zeta_{n+1}^a z_1 : \cdots : \zeta_{n+1}^{an} z_n],
$$
$$
\delta' : [z_0 : \cdots : z_n] \mapsto [z_{n+1-b} : \cdots : z_{n-1} : z_n : z_0 : \cdots : z_{n-b}].
$$

This proves Proposition 3.3, which corresponds to the case 5 of Theorem 1.4. $\qquad\square$

3.3. Classification of the fibers with (P_G, G) of type γ

There is just one pair (P_G, G), when the abelian variety P_G is isomorphic to E^2, with E the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$, and G is the group generated by the set

$$
\left\{ \begin{pmatrix} -1 & 1+i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix} \right\}.
$$

This case can be reduced to the one studied in Section 3.1 and we get the following result.

Proposition 3.7. *Let A be an abelian variety and G a finite subgroup of $\mathrm{Aut}_0(A)$ such that A/G is smooth. Let A_0 be the connected component of A^G that contains 0 and P_G be the complementary abelian subvariety with respect to a G-invariant polarization. If the analytic representation of G on P_G is irreducible and (P_G, G) is a pair of type γ, there exists a trivialization*

$$
\begin{array}{ccc}
A_0 \times \mathbb{P}^2 & \xrightarrow{\ /\Delta\ } & A/G \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\ /\Delta\ } & A/P_G,
\end{array}
$$

of the fibration $A/G \to A/P_G$, with $\Delta \cong A_0 \cap P_G$, where Δ acts on A_0 by translations and on the fibers as one of the following:

(1) $\Delta \cong \{0\}$ and therefore acts trivially.
(2) $\Delta \cong \mathbb{Z}/2\mathbb{Z}$ and acts with generator $\delta : [z_0 : z_1 : z_2] \mapsto [z_0 : -z_1 : z_2]$.
(3) $\Delta \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and there exists a set of generators that act as

$$\delta : [z_0 : z_1 : z_n] \mapsto [z_0 : -z_1 : z_2],$$
$$\delta' : [z_0 : z_1 : z_n] \mapsto [z_2 : z_1 : z_0].$$

Proof. The fixed points by the action of G are the points of the form $(a + bi, c + di) \in E^2$ such that

$$\begin{pmatrix} -1 & 1+i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} \equiv_{\mathbb{Z}[i]} \begin{pmatrix} a+bi \\ c+di \end{pmatrix},$$

$$\begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} \equiv_{\mathbb{Z}[i]} \begin{pmatrix} a+bi \\ c+di \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} \equiv_{\mathbb{Z}[i]} \begin{pmatrix} a+bi \\ c+di \end{pmatrix}.$$

From these relations we have that in the elliptic curve

$$i(c + di) \equiv c + di \quad \text{and} \quad -i(a + bi) + (i - 1)(c + di) \equiv a + bi.$$

From the first identity follows $(i - 1)(c + di) \equiv 0 \in E$ and hence $-i(a + bi) \equiv a + bi$. Then, $(c + d) + (d - c)i \equiv (a - b) + (a + b)i \equiv 0 \in E$. This implies that $a + b, a - b, c + d, d - c \in \mathbb{Z}$, thus $2a, 2b, 2c, 2d \in \mathbb{Z}$. Then, $a, b, c, d \in \left\{ 0, \pm\dfrac{1}{2} \right\}$, i.e.,

$$P_G^G \subset \left\langle \left(\frac{1}{2} + \frac{i}{2}, 0 \right), \left(0, \frac{1}{2} + \frac{i}{2} \right) \right\rangle.$$

Since the generators are solutions of the system of equations, the inclusion is actually an equality, and therefore P_G^G is isomorphic to the Klein group. If Δ is isomorphic to any non trivial subgroup of P_G^G then the order of Δ is 2 or 4. We know that the action of Δ on $\mathbb{P}^2$ induced by the action of Δ on P_G is faithful. Thus, we have to determine all the faithful representations of Δ in $\mathrm{PGL}_3(\mathbb{C})$. Let Γ be any representation of Δ in $\mathrm{PGL}_3(\mathbb{C})$ and $H \in \mathrm{SL}_3(\mathbb{C})$ be any lifting of Γ by $\pi : \mathrm{SL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$. We know that the order of H is $3|\Gamma|$. Let us take H' a 2-Sylow of H. Since $H' \cap \ker(\pi) = \{id\}$, it follows that $H' \cong \Gamma$. Then, we can study the representations of Δ in $\mathrm{PGL}_3(\mathbb{C})$ by studying the representations of Δ in $\mathrm{SL}_3(\mathbb{C})$. Since Δ is an abelian group, of order 2 or 4 and exponent 2, there is only one conjugacy class of representations of Δ in $\mathrm{SL}_3(\mathbb{C})$, because there are only 3 diagonal matrices of order 2. Hence there is just a single conjugacy class in $\mathrm{PGL}_3(\mathbb{C})$, which corresponds to the one studied in Proposition 3.2, more precisely, in points 2 and 4. This proves Proposition 3.7.

$\square$

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