

STABILITY ESTIMATE FOR THE SEMI-DISCRETE LINEARIZED BENJAMIN-BONA-MAHONY EQUATION^{*,**}

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Abstract. In this work we study the semi-discrete linearized Benjamin-Bona-Mahony equation (BBM) which is a model for propagation of one-dimensional, unidirectional, small amplitude long waves in non-linear dispersive media. In particular, we derive a stability estimate which yields a unique continuation property. The proof is based on a Carleman estimate for a finite difference approximation of Laplace operator with boundary observation in which the large parameter is connected to the mesh size.

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Dedicated to Enrique Zuazua on the occasion of his 60th birthday

1. INTRODUCTION AND RESULTS

In this work, we are interested in a linearized version of the Benjamin-Bona-Mahony equation (BBM)

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad (1.1)$$

proposed by T. Benjamin *et al.* [2] as a model for propagation of one-dimensional, unidirectional, small amplitude long waves in non-linear dispersive media.

In the last years, several authors have widely studied dispersive equations in the context of controllability and inverse problems. Nevertheless, the BBM equation presents several particularities due to the structure of the operator. In particular, the infinitesimal generator of the semigroup is given by $(I - \partial_x^2)^{-1} \partial_x$, which is a

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compact operator, in opposition with the most common situation in PDE where the generator is an unbounded operator among others.

We note some interesting results about the unique continuation property (UCP) for BBM for the continuous case. We refer to the reader to those works and their references for a more detailed discussion: L. Rosier and B.-Y. Zhang in [14] developed a UCP for (1.1) on a periodic domain. Moreover, in [6] P. L. da Silva and I. L. Freire give an alternative proof using geometrical arguments for the periodic case, and for the case when (1.1) is solved in \mathbb{R} . In [16], X. Zhang and E. Zuazua considered a linearized BBM equation with space-dependent potential

$$u_t - u_{xxt} = [\alpha(x)u]_x + \beta(x)u, \quad (x, t) \in (0, 1) \times (0, T). \quad (1.2)$$

In that work, the authors established that the only solution of (1.2), such that $u(0, t) = u(1, t) = 0$, is the trivial one $u \equiv 0$ provided that both α and β do not vanish on some open subset of $(0, 1)$. Furthermore, if $\alpha(x) = -1$ and $\beta(x) = 0$ in (1.2), S. Micu proved in [12] a UCP with the additional boundary condition $u_x(1, t) = 0$, and study controllability results. On the other hand, in [15], M. Yamamoto established a UCP for BBM-like equation with time and space dependent potential

$$\partial_t u(x, t) - \partial_x^2 \partial_t u(x, t) = p(x, t) \partial_x u(x, t) + q(x, t) u(x, t), \quad (x, t) \in (0, 1) \times (0, T), \quad (1.3)$$

where $p \in L^\infty((0, T) \times (0, 1))$ and $q \in L^\infty(0, T; L^2(0, 1))$. It was shown that the solution of (1.3) shall vanish in $(0, 1) \times (0, T)$, provided $u(1, t) = \partial_x u(1, t) = 0$ for all $t \in (0, T)$ and $u(x, 0) = 0$ for $x \in (0, 1)$. The main tool to prove this result is a Carleman estimate for the Laplacian operator. Through a more refined version of this Carleman estimate, a stability estimate can be formulated for equation (1.3) (see Sect. 5).

We note that the UCP's have been well studied in several continuous partial differential equations, but it is possible to see that the corresponding discrete case does not hold. For instance, if u is a harmonic function in a domain Ω and $u = \partial_n u = 0$ on $\Gamma \subset \partial\Omega$ then $u \equiv 0$ in Ω , it does not generally hold its discrete formulation. We refer to the counterexample due to O. Kavian, presented by E. Zuazua in [18]. However, for (1.4) it is possible to obtain a quantitative UCP under restriction over the mesh size. This result shall be discussed in more detail in Section 1.2.

In this paper, we are interested whether a unique continuation property, as in the work of Yamamoto [15], still holds for a semi-discrete approximation in space of (1.3). In this sense, for $N \in \mathbb{N}$ given, we set the space discretization parameter $h := 1/(N + 1)$. We consider the pairs (x_i, t) with $t \in (0, T)$, $T > 0$, and $x_i = ih$, for $i = 1, \dots, N$. Thus, the space semi-discrete approximation of equation (1.3) by using the centered finite difference method with respect to the space variable is given by

$$\partial_t u_i(t) - \frac{\partial_t u_{i+1}(t) - 2\partial_t u_i(t) + \partial_t u_{i-1}(t)}{h^2} = p_i(t) \frac{u_{i+1}(t) - u_{i-1}(t)}{2h} + q_i(t) u_i(t), \quad (1.4)$$

for $i \in \{1, 2, \dots, N\}$ and $t \in (0, T)$, where $u_i(t)$ stands for $u(x_i, t)$.

1.1. Notation

We introduce the notation of meshes and operators that shall be used throughout this paper and necessary to state our results. We consider the following regular partition of the interval $[0, 1]$ as

$$\mathcal{M}_h := \{x_i \mid x_i := ih, \ i = 0, 1, \dots, N + 1\},$$

for $N \in \mathbb{N}$ and $h := 1/(N + 1)$. For any sets of points $\mathcal{W}_h \subset \mathcal{M}_h$, we define the following dual meshes \mathcal{W}'_h and \mathcal{W}^*_h as

$$\mathcal{W}'_h := \tau_+(\mathcal{W}_h) \cap \tau_-(\mathcal{W}_h), \quad \mathcal{W}^*_h := \tau_+(\mathcal{W}_h) \cup \tau_-(\mathcal{W}_h), \quad (1.5)$$

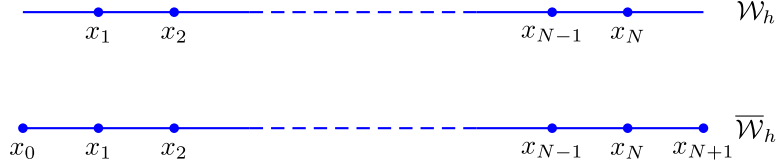


FIGURE 1. Primal meshes, discretization of space variable.

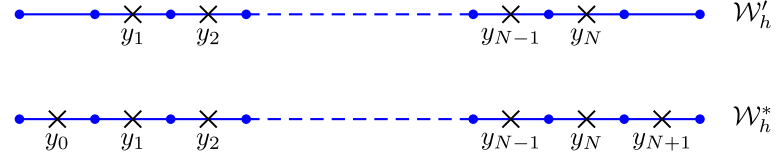


FIGURE 2. Dual meshes, discretization of space variable.

where

$$\tau_{\pm}(\mathcal{W}_h) := \left\{ x_h \pm \frac{h}{2} \mid x_h \in \mathcal{W}_h \right\}.$$

We shall denote $\overline{\mathcal{W}}_h = \mathcal{W}_h^{**} := (\mathcal{W}_h^*)^*$ and $\hat{\mathcal{W}}_h = \mathcal{W}'_h := (\mathcal{W}'_h)'$.

We note that if $\overline{\mathcal{W}}_h = \mathcal{W}_h$, then for two consecutive points $x_i, x_{i+1} \in \mathcal{W}_h$ we have $x_{i+1} - x_i = h$. Thus, any subset $\mathcal{W}_h \subset \mathcal{M}_h$ that verifies $\overline{\mathcal{W}}_h = \mathcal{W}_h$ is called regular mesh. Finally, we define the boundary of a regular mesh \mathcal{W}_h as $\partial\mathcal{W}_h := \overline{\mathcal{W}}_h \setminus \mathcal{W}_h$. These sets defined above can be seen in Figures 1 and 2.

We introduce, using (1.5), the semi-discrete sets. Let us consider $T > 0$, we define $Q_h := \mathcal{W}_h \times (0, T)$. We also define the dual semi-discrete sets by $Q'_h := \mathcal{W}'_h \times (0, T)$ and $Q_h^* := \mathcal{W}_h^* \times (0, T)$. Similarly, the semi-discrete boundary is given by $\partial Q_h = \partial\mathcal{W}_h \times (0, T)$.

We define the average operator A_h and the difference operator D_h by

$$A_h(u_h)(x_h, t) := \frac{\tau_+ u_h(x_h, t) + \tau_- u_h(x_h, t)}{2},$$

$$D_h(u_h)(x_h, t) := \frac{\tau_+ u_h(x_h, t) - \tau_- u_h(x_h, t)}{h},$$

where $\tau_{\pm} u_h(x_h, t) := u_h(x_h \pm \frac{h}{2}, t)$.

We denote by $C(Q_h)$ the set of real-valued functions defined in Q_h , and by $L_h^2(Q_h)$ the set $C(Q_h)$ endowed with the norm

$$\|u_h\|_{L_h^2(Q_h)}^2 := \int_0^T \|u\|_{L_h^2(\mathcal{W}_h)}^2 dt,$$

where $\|u_h\|_{L_h^2(\mathcal{W}_h)}^2$ is induced by the inner product

$$\langle u_h, v_h \rangle_{\mathcal{W}_h} := \int_{\mathcal{W}_h} u_h v_h := h \sum_{x_h \in \mathcal{W}_h} u_h(x_h) v_h(x_h).$$

For $u_h \in C(Q_h)$, we define its $L_h^\infty(Q_h)$ -norm as

$$\|u_h\|_{L_h^\infty(Q_h)} := \max_{(x_h, t) \in Q_h} \{|u_h(x_h, t)|\}.$$

To introduce the boundary conditions, we define the outward normal for $(x_h, t) \in \partial Q_h$ as

$$n_h(x_h, t) := \begin{cases} 1 & (\tau_-(x_h), t) \in Q_h^* \text{ and } (\tau_+(x_h), t) \notin Q_h^*, \\ -1 & (\tau_-(x_h), t) \notin Q_h^* \text{ and } (\tau_+(x_h), t) \in Q_h^*, \\ 0 & \text{otherwise.} \end{cases}$$

We indicate by ∂Q^+ (resp. ∂Q^-) the set of points such that $n_h(x_h, t) = 1$ (resp. $n_h(x_h, t) = -1$). We also introduce the trace operator for $u_h \in C(Q_h^*)$ as

$$\forall (x_h, t) \in \partial Q_h, t_r(u_h) := \begin{cases} \tau_- u_h(x_h, t) & n_h(x_h, t) = 1, \\ \tau_+ u_h(x_h, t) & n_h(x_h, t) = -1, \\ 0 & n_h(x_h, t) = 0. \end{cases}$$

Let us finally introduce the discrete integration on the boundary for $u_h \in C(\partial \mathcal{W}_h)$ as

$$\int_{\partial \mathcal{W}_h} u_h := \sum_{x_h \in \partial \mathcal{W}_h} u_h(x_h).$$

1.2. Discrete Carleman estimates with boundary observation

For the discrete Carleman estimate, we consider the weight function of the form $r(x, t) = e^{s\varphi(x, t)}$ for $s \geq 1$, with $\varphi(x, t) = e^{\lambda\psi(x, t)}$ where ψ is a continuous function whose domain of definition $\bar{\Omega}$ is contained in an enlarged smooth open and connected neighborhood $\tilde{\Omega}$. We also assume $\psi \in C^k(\tilde{\Omega})$ with k large enough such that it satisfies the following property

$$\partial_x \psi(x, t) > 0, \quad (x, t) \in \bar{\Omega} \times (0, T). \quad (1.6)$$

The assumption of the higher-order derivatives is needed to obtain the estimates on the weight function presented in Section 4, in contrast to the continuous case. We shall use the same notation for the sample of the continuous function on the discrete or semi-discrete sets. We now state a uniform Carleman estimate for the operator D_h^2 with boundary observation. Although for the Carleman estimate just the condition (1.6) is needed, to achieve the UCP property for the system (1.4) we also consider the following assumption

$$\partial_t \psi(x, t) < 0, \quad (x, t) \in \bar{\Omega} \times (0, T). \quad (1.7)$$

It is not difficult to find a function that verifies conditions (1.6) and (1.7), for instance the following function,

$$\psi(x, t) := (x - x_0)^2 - t^2, \quad x_0 < 0. \quad (1.8)$$

Theorem 1.1. (Discrete Carleman estimate)

Let ψ be a function verifying (1.6) and $T > 0$. For the parameter $\lambda_0 \geq 1$ sufficiently large, there exist $s_0(\lambda_0) \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$ and $C = C(\varepsilon_0, s_0, \lambda_0)$ independent of $h > 0$ such that

$$\begin{aligned}
C \left(\|e^{s\varphi} D_h^2 v_h\|_{L_h^2(Q_h)}^2 + s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r (e^{2s\varphi}) t_r (|D_h v_h|^2) + s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r (e^{2s\varphi}) t_r (A_h (|v_h|^2)) \right) \\
\geq s^3 \|e^{s\varphi} v_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h v_h\|_{L_h^2(Q_h^*)}^2, \quad (1.9)
\end{aligned}$$

for all $h \in (0, h_0)$, $s \in (s_0, \varepsilon_0/h)$ and v_h defined in $Q_h := \mathring{\mathcal{M}}_h \times (0, T)$.

Several recent works have been concerned with discrete and semi-discrete Carleman estimates for second-order differential operators. The hyperbolic case has been developed for the one-dimensional case by L. Baudouin and S. Ervedoza [1], to study the stability of an inverse problem to recover a potential term in a semi-discrete wave equation. The elliptic case has been developed by F. Boyer *et al.* in [3], for the one-dimensional case, to establish a relaxed observability estimate for the associated semi-discrete parabolic equation. In [7], S. Ervedoza and F. de Gournay study the Laplacian operator in arbitrary dimension to prove the stability for the discrete Calderon problem, with limiting Carleman weight function. Semi-discrete Carleman estimates for parabolic operators have been established by F. Boyer and J. Le Rousseau in [4] for multidimensional Cartesian grids. Moreover, in [13], T. N. T. Nguyen studied in the one-dimensional setting a semi-discrete parabolic operator with discontinuous diffusion coefficient; both of them obtain relaxed controllability results for their respective systems.

In the aforementioned works, the discretization was based on a finite difference scheme. Also, the Carleman parameter cannot be arbitrarily large, which is related to the discretization step size, in contrast to the continuous setting. Let us finally mention that recently in [10], a fully discrete Carleman estimates for parabolic operator have been obtained by V. Hernández-Santamaría and P. González Casanova, where the spatial and time discrete step-size parameters are connected to the Carleman parameter.

We recall that T. Carleman introduced an estimate, known as the Carleman estimate, to prove a UCP for second-order elliptic partial differential equations in [5] when coefficients fail to be analytic. Nowadays, it has become an efficient tool to prove UCP, the study of controllability, observability, and stabilization for PDEs. We refer to the work of X. Fu *et al.* [8], and references therein, where the authors present a unified approach to Carleman estimates for second-order PDEs and their applications to control theory and inverse problems.

One of the main difficulties in the development of discrete or semi-discrete Carleman estimates is to compute multiple discrete operators such as D_h and A_h on the Carleman weight functions. For this reason, we establish Theorem 4.9 (see Sect. 4) to reduce some tedious computation, and it represents an extension of the results presented by F. Boyer *et al.* in [3] related to discrete estimate on the weight function.

1.3. Semi-discrete setting and main results

With the notation we have introduced we can rewrite the semi-discretization (1.4) as

$$\partial_t u_h - D_h^2 \partial_t u_h = p_h D_h A_h u_h + q_h u_h \text{ in } Q_h := \mathring{\mathcal{M}}_h \times (0, T), \quad (1.10)$$

where $p_h, q_h \in L_h^\infty(Q_h)$. In (1.10) $u_h(t)$ provides an approximation of $u(x_h, t)$, u being the solution of the continuous equation (1.3). $\partial_t u_h$ stands for the first order differentiation with respect to t and the operators D_h and D_h^2 are the classical central finite-difference approximation of the space derivatives. We assume that there exists a constant $M > 0$ independent of h such that $\max\{\|p_h\|_{L_h^\infty(Q_h)}, \|q_h\|_{L_h^\infty(Q_h)}\} \leq M$.

Theorem 1.2. (*Stability for space semi-discrete BBM equation*)

Let ψ be a function verifying (1.6) and (1.7), and $T > 0$. For $\lambda_0 \geq 1$ sufficiently large, there exist $s_0(\lambda_0, M) \geq 1$, $h_0 > 0$ depending on M , $\varepsilon_0 > 0$ and a constant $C > 0$ independent of $h > 0$ such that the following estimate

holds

$$\begin{aligned}
s^3 \|e^{s\varphi} u_h\|_{L_h^2(Q_h)}^2 + s^3 \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q_h^*)}^2 \\
\leq C s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r (e^{2s\varphi}) t_r (|D_h \partial_t u_h|^2) \\
+ C s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r (e^{2s\varphi}) t_r (A_h (|\partial_t u_h|^2)),
\end{aligned} \tag{1.11}$$

for $0 < h \leq h_0$ and $s \in (s_0, \varepsilon_0/h)$ and $u_h(x_h, 0) = 0$ in \mathcal{M}_h .

As a consequence of Theorem 1.2, we have the following unique continuation property for semi-discrete BBM equation (1.10).

Corollary 1.3. (*UCP for a semi-discrete BBM equation*)

There exists $h_0 > 0$ depending on M such that if $u_h = 0$ on $\{1\} \times (0, T)$, $D_h u_h = 0$ on $\{1 - h/2\} \times (0, T)$ and $u_h(\cdot, 0) = 0$ in \mathcal{M}_h ; then $u_h(x_h, t) = 0$ in Q_h for all $h \in (0, h_0)$.

The methodology of the proof of Theorem 1.2 is similar in spirit to [15], where it have been obtained a UCP for equation (1.3). However, it cannot be followed straightly from the proof of the continuous case since the parameter s cannot be arbitrarily large. As we mentioned above, this parameter is related to the mesh size. Thus, a semi-discrete version of the Carleman estimate used in [15] is not enough. For this reason, we develop a more refined Carleman estimate, (5.2), and its semi-discrete counterpart, see Theorem 1.1.

The rest of this article is organized as follows. First, in Section 2, we introduce and prove some discrete calculus formulas for uniform meshes. Then, we establish a stability estimate for the semi-discrete numerical approximation based on a uniform spatial discretization of equation (1.3) in Section 3. The last three sections are devoted to the proof of the discrete Carleman estimate for a finite-difference approximation of Laplacian operator with boundary observation. In Section 4, we extend the estimations developed in Section 3 of [3] for arbitrary order, which reduce some computation in the proof of the Carleman estimate presented in Section 5. Some calculations are postponed and computed in Section 6.

2. CALCULUS FORMULAS FOR UNIFORM MESHES

In this section, we state the elementary notions concerning discrete calculus formulas. First, we set some useful identities that shall be used in what follows. Then, we present the integration by parts for the difference and average operator. For the sake of presentation, we set the results in the discrete space variable framework instead of the semi-discrete setting.

The following Lemma gives us a calculus rule for finite difference operators. Its proof can be found in [3].

Lemma 2.1. *Let $\mathcal{W}_h \subset \mathcal{M}_h$ be a regular mesh. For $u_h, v_h \in C(\mathcal{M}_h)$, we have the following identities on \mathcal{W}_h^**

$$\begin{aligned}
D_h(u_h v_h) &= D_h(u_h) A_h(v_h) + A_h(u_h) D_h(v_h), \\
A_h(u_h v_h) &= A_h(u_h) A_h(v_h) + \frac{h^2}{4} D_h(u_h) D_h(v_h).
\end{aligned} \tag{2.1}$$

As a direct consequence of Lemma 2.1, we point out some identities to develop the rest of this paper.

Corollary 2.2. *Let $\mathcal{W}_h \subseteq \mathcal{M}_h$ be a regular mesh.*

– For $u \in C(\mathcal{W}_h)$,

$$A_h(u_h^2) = (A_h u_h)^2 + \frac{h^2}{4} (D_h u_h)^2, \text{ on } \mathcal{W}_h^*. \tag{2.2}$$

In particular, for all $u \in C(\mathcal{W}_h)$,

$$A_h(u_h^2) \geq (A_h u_h)^2, \text{ on } \mathcal{W}_h^*. \quad (2.3)$$

– For $u \in (\mathcal{W}_h)$

$$D_h(u_h^2) = 2D_h u_h A_h u_h. \quad (2.4)$$

Note that as a consequence of the first identity in (2.1) of Lemma 2.1, it can be proved by induction the following identity.

Corollary 2.3. *Suppose that for $n \in \mathbb{N}$ and $h > 0$, the set $(nh/2, 1 - nh/2)$ is not empty. Then, for each $u, v \in C(\mathcal{W}_h)$ we have*

$$D_h^n(uv) = \sum_{k=0}^n \binom{n}{k} D_h^{n-k} A_h^k u A_h^{n-k} D_h^k v \text{ on } \mathcal{X}_h, \quad (2.5)$$

where

$$\mathcal{X}_h = \begin{cases} \mathcal{M}_h^* \cap [nh/2, 1 - nh/2], & \text{if } n \text{ is odd,} \\ \mathcal{M}_h \cap [nh/2, 1 - nh/2], & \text{if } n \text{ is even.} \end{cases}$$

Now, we state the following integration by parts presented for discrete space variable. We note that the same identities could be considered in the semi-discrete setting since the temporal variable does not play any significant role.

Proposition 2.4. *Let $\mathcal{W}_h \subseteq \mathcal{M}_h$ be a regular mesh. For $u_h \in C(\overline{\mathcal{W}_h})$ and $v_h \in C(\mathcal{W}_h^*)$ we have*

$$\int_{\mathcal{W}_h} u_h D_h(v_h) = - \int_{\mathcal{W}_h^*} D_h(u_h) v_h + \int_{\partial \mathcal{W}_h} u_h t_r(v_h) n_h$$

and

$$\int_{\mathcal{W}_h} u_h A_h(v_h) = \int_{\mathcal{W}_h^*} A_h(u_h) v_h - \frac{h}{2} \int_{\partial \mathcal{W}_h} u_h t_r(v_h).$$

Proof. We have

$$\begin{aligned} \int_{\mathcal{W}_h} u_h \tau_+(v_h) &= \int_{\tau_+(\mathcal{W}_h)} \tau_-(u_h) v_h \\ &= \int_{\mathcal{W}_h^*} \tau_-(u_h) v_h - \int_{\mathcal{W}_h^* \setminus \tau_+(\mathcal{W}_h)} \tau_-(u_h) v_h \\ &= \int_{\mathcal{W}_h^*} \tau_-(u_h) v_h - \int_{\tau_-^2(\mathcal{W}_h) \setminus \mathcal{W}_h} u_h \tau_+(v_h). \end{aligned} \quad (2.6)$$

and

$$\begin{aligned}
\int_{\mathcal{W}_h} u_h \tau_-(v_h) &= \int_{\tau_-(\mathcal{W}_h)} \tau_+(u_h) v_h \\
&= \int_{\mathcal{W}_h^*} \tau_+(u_h) v_h - \int_{\mathcal{W}_h^* \setminus \tau_-(\mathcal{W}_h)} \tau_+(u_h) v_h \\
&= \int_{\mathcal{W}_h^*} \tau_+(u_h) v_h - \int_{\tau_+^2 \mathcal{W}_h \setminus \mathcal{W}_h} u_h \tau_-(v_h).
\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) it follows that

$$\begin{aligned}
\int_{\mathcal{W}_h} u_h D_h(v_h) &= - \int_{\mathcal{W}_h^*} D_h(u_h) v_h - \frac{1}{h} \int_{\tau_-^2 \mathcal{W}_h \setminus \mathcal{W}_h} u_h \tau_+(v_h) \\
&\quad + \frac{1}{h} \int_{\tau_+^2 \mathcal{W}_h \setminus \mathcal{W}_h} u_h \tau_-(v_h) \\
&= - \int_{\mathcal{W}_h^*} D_h(u_h) v_h + \int_{\partial \mathcal{W}} u_h t_r(v_h) n_h.
\end{aligned}$$

Similarly, averaging the equations (2.6) and (2.7) we obtain

$$\begin{aligned}
\int_{\mathcal{W}_h} u_h A_h(v_h) &= \int_{\mathcal{W}_h^*} A_h(u_h) v_h - \frac{1}{2} \int_{\tau_-^2(\mathcal{W}_h) \setminus \mathcal{W}_h} u_h \tau_+(v_h) \\
&\quad - \frac{1}{2} \int_{\tau_+^2 \mathcal{W}_h \setminus \mathcal{W}_h} u_h \tau_-(v_h) \\
&= \int_{\mathcal{W}_h^*} A_h(u_h) v_h - \frac{h}{2} \int_{\partial \mathcal{W}_h} u_h t_r(v_h),
\end{aligned}$$

which completes the proof. \square

Remark 2.5. One can consider the previous result for the meshes \mathcal{W}'_h , see for instance a similar result by S. Ervedoza and F. de Gournay in [7]. In that case we should change the definition of the boundary nodes of our mesh. As we are interested in the boundary that kind of integral-by-parts formulas does not fulfill our goal. In contrast with the integration-by-part formulas from F. Boyer *et al.* [3], we do not make any distinction on the difference and average operators, then we need to specify that difference in the respective meshes.

3. STABILITY ESTIMATE FOR SPACE SEMI-DISCRETE BMM EQUATION

This section is devoted to proof Theorem 1.2. We follow as close as possible the ideas from its continuous formulation. For this reason we state a stability estimate for (1.3). The main tool for the proof is a Carleman estimate for Laplacian operator. For sake of exposition we postpone that proof, see Section 5. It is worth to mention that we refined the result presented in [15] (see Sect. 5.1).

3.1. The continuous case

We consider $Q := (0, 1) \times (0, T)$, for $T > 0$, and we define the classical inner product

$$(u, v)_{L^2(Q)} := \int_Q u v \, dx dt$$

and its respective L^2 -norm $\|u\|_{L^2(Q)}^2 = (u, v)_{L^2(Q)}$.

Following the methodology from [15] we can obtain a stability estimate for (1.3). The proof is based on the Carleman estimate (5.2) and Lemma 6.4.2 from V. Isakov [11].

Theorem 3.1. *Let $\partial_x^j \partial_t^k u \in C([0, 1] \times [0, T])$ with $j = 0, 1, 2$ and $k = 0, 1$. For $\lambda_0 > 0$ sufficiently large, there exist constants $s_0 \geq 0$ and $C(s_0, \lambda_0, \psi) > 0$, such that*

$$\begin{aligned} s^3 \|e^{s\varphi} u\|_{L^2(Q)}^2 + s^3 \|e^{s\varphi} \partial_t u\|_{L^2(Q)}^2 + s \|e^{s\varphi} \partial_x \partial_t u\|_{L^2(Q)}^2 &\leq C s^3 \int_0^T ((\partial_x \psi)^3 \varphi^3 e^{2s\varphi} |\partial_t u|^2)(1, t) dt \\ &+ C s \int_0^T (\partial_x \psi \varphi e^{2s\varphi} |\partial_x \partial_t u|^2)(1, t) dt, \end{aligned} \quad (3.1)$$

for all $s \geq s_0$, and u verify (1.3) with $u(x, 0) = 0$ for all $x \in (0, 1)$.

As a Corollary of Theorem 3.1, it follows the main result presented in [15].

Corollary 3.2. *Let $\partial_x^j \partial_t^k u \in C([0, 1] \times [0, T])$ with $0, 1, 2$ and $k = 0, 1$. If u is solution of (1.3) such that $u(1, t) = \partial_x u(1, t) = 0$ for all $t \in (0, T)$ and $u(x, 0) = 0$ in $(0, 1)$, then $u(x, t) = 0$ in $(0, 1) \times (0, T)$.*

3.2. Proof of Theorem 1.2

As we mentioned above the proof of Theorem 1.2 is based on the continuous setting strategy. Then, we write down a space semi-discrete version of Lemma 6.4.2 from [11], which is a Poincaré weighted inequality.

Lemma 3.3. *Let $\varphi \in C^1(\overline{Q_h})$ be such that $\frac{\partial \varphi}{\partial t} \leq 0$, then*

$$\int_{Q_h} \left| \int_0^t u_h(x_h, \sigma) d\sigma \right|^2 e^{2s\varphi(x_h, t)} \leq T^2 \int_{Q_h} (u_h(x_h, t))^2 e^{2s\varphi(x_h, t)},$$

for all $u_h \in L_h^2(Q_h)$.

Proof. We have

$$\begin{aligned} \int_0^T \left| \int_0^t u(x_h, \sigma) d\sigma \right|^2 e^{2s\varphi(x_h, t)} dt &\leq \int_0^T t \int_0^t (u(x_h, \sigma))^2 e^{2s\varphi(x_h, t)} d\sigma dt \\ &= \int_0^T \int_\sigma^T t e^{2s\varphi(x_h, t)} (u(x_h, \sigma))^2 dt d\sigma \\ &\leq T^2 \int_0^T (u(x_h, \sigma))^2 e^{2s\varphi(x_h, \sigma)} d\sigma, \end{aligned}$$

and integrating over \mathcal{M}_h we complete the proof. □

3.2.1. Proof of Theorem 1.2

In this section, we shall give the proof of the Theorem 1.2.

Proof. Note that from (1.10) we have

$$D_h^2 \partial_t u_h = \partial_t u_h - p_h D_h A_h u_h - q_h u_h \quad \text{in } Q_h.$$

Then, applying the Carleman estimate (1.9) to $v_h = \partial_t u_h$ we obtain

$$\begin{aligned}
s^3 \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q_h^*)}^2 &\leq s \int_{\partial \mathcal{M}_h^+} \varphi \partial_x \psi t_r (e^{2s\varphi}) t_r (|D_h \partial_t u_h|^2) \\
&+ s^3 \int_{\partial \mathcal{M}_h^+} (\varphi \partial_x \psi)^3 t_r (e^{2s\varphi}) t_r (A_h (|\partial_t u_h|^2)) \\
&+ \|e^{s\varphi} (\partial_t u_h - p_h D_h A_h u_h - q_h u_h)\|_{L_h^2(Q_h)}^2,
\end{aligned} \tag{3.2}$$

for $0 < h \leq h_0$, $s \geq s_0$ and $sh < \varepsilon_0$.

On the other hand, we note that

$$D_h A_h u_h(x_h, t) = \int_0^t D_h A_h \partial_t u_h(x_h, \sigma) d\sigma$$

and

$$u_h(x_h, t) = \int_0^t \partial_t u_h(x_h, \sigma) d\sigma,$$

since $u_h(x_h, 0) = 0$ in $\mathring{\mathcal{M}}_h$. Thus, by Lemma 3.3 it follows that

$$\begin{aligned}
\|e^{s\varphi} (\partial_t u_h - p_h D_h A_h u_h - q_h u_h)\|_{L_h^2(Q_h)}^2 &\leq CT^2 \|p_h\|_{L_h^\infty(Q_h)}^2 \|e^{s\varphi} D_h A_h \partial_t u_h\|_{L_h^2(Q_h)}^2 \\
&+ C(1 + T^2 \|q_h\|_{L_h^\infty(Q_h)}^2) \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 \\
&\leq CT^2 M^2 \|e^{s\varphi} D_h A_h \partial_t u_h\|_{L_h^2(Q_h)}^2 \\
&+ C(1 + T^2 M^2) \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2.
\end{aligned} \tag{3.3}$$

Now, we focus on the first term of the right-hand side above. Using (2.3) and a discrete integration by parts for the discrete average operator we get

$$\begin{aligned}
\|e^{s\varphi} D_h A_h \partial_t u_h\|_{L_h^2(Q_h)}^2 &= \int_{Q_h} e^{2s\varphi} (D_h A_h \partial_t u_h)^2 \\
&\leq \int_{Q_h} e^{2s\varphi} A_h \left((D_h \partial_t u_h)^2 \right) \\
&= \int_{Q_h^*} A_h (e^{2s\varphi}) (D_h \partial_t u_h)^2 - \frac{h}{2} \int_{\partial Q_h} e^{2s\varphi} t_r \left((D_h \partial_t u_h)^2 \right) \\
&\leq \int_{Q_h^*} A_h (e^{2s\varphi}) (D_h \partial_t u_h)^2.
\end{aligned}$$

From Proposition 4.5 we have $A_h(e^{2s\varphi}) \leq C_\lambda e^{2s\varphi}$, we thus obtain

$$\|e^{s\varphi} D_h A_h \partial_t u_h\|_{L_h^2(Q_h)}^2 \leq C_\lambda \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q^*)}^2. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4) we get

$$\begin{aligned}
s^3 \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q_h^*)}^2 &\leq s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi}) t_r(|D_h \partial_t u_h|^2) \\
&\quad + s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi}) t_r(A_h(|\partial_t u_h|^2)) \\
&\quad + (1 + T^2 M^2) \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 \\
&\quad + C_\lambda T^2 M^2 \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q_h^*)}^2.
\end{aligned} \tag{3.5}$$

We note that by choosing some $s \geq T^{2/3}(1 + M^2)^{1/3} + T^2 M$, the last term on the right-hand side from (3.5) can be absorbed by its left-hand side. Thus, recalling the hypothesis on s from the Carleman estimates, by choosing $s_1 := \max\{s_0, k(M, T)\} \geq s_0$ large enough we obtain

$$\begin{aligned}
s^3 \|e^{s\varphi} \partial_t u_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h \partial_t u_h\|_{L_h^2(Q_h^*)}^2 &\leq s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi}) t_r(|D_h \partial_t u_h|^2) \\
&\quad + s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi}) t_r(A_h(|\partial_t u_h|^2)),
\end{aligned}$$

provided $s \geq s_1$, where $k(M, T) := T^{2/3}(1 + M^2)^{1/3} + T^2 M^2$. Now, we need to connect the condition over the Carleman parameter s with the mesh size h . Defining

$$h_1 := \frac{\varepsilon_0}{s_1},$$

it follows that for $0 < h \leq \min\{h_0, h_1\}$ we have $sh \leq \varepsilon_0$ provided $s \in (s_1, \varepsilon_0/h)$, which concludes the proof. \square

As a consequence, we obtain the UCP presented in Corollary 1.3 for semi-discrete BBM equation (1.10).

4. SOME PRELIMINARY DISCRETE CALCULUS RESULTS

In this section, we establish some previous estimates for the Carleman weight function that shall be used in the next section to obtain the discrete Carleman estimate, see Theorem 1.1. Recall that our weight function is defined as $e^{s\varphi}$ for $s \geq 1$, with $\varphi = e^{\lambda\psi}$, where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. Our goal is to generalize the results presented previously in Section 3, obtained by F. Boyer *et al.* in [3], related to discrete operations performed on the Carleman weight functions, considering estimates and expansions for higher order discrete operators.

For easier comparison, we use the same notation by setting $r = e^{s\varphi}$ and $\rho = r^{-1}$, these positive parameters s and h shall be large and small respectively and limited by the condition $sh \leq 1$. The proofs are similar in spirit to those given in [3].

We denote by $\mathcal{O}_\lambda(sh)$ the functions that verify $\|\mathcal{O}_\lambda(sh)\|_{L^\infty(Q_h)} \leq C_\lambda sh$ for some constant C_λ depending on λ . By $\mathcal{O}(1)$ we denote bounded functions and by $\mathcal{O}_\lambda(1)$ a bounded function once λ is fixed.

We say that α is a multi-index if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ and for $y \in \mathbb{R}^n$ we write:

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n}.$$

Proposition 4.1. *Let us consider $n \in \mathbb{N}$. Let f be a $(n+2)$ -times differentiable and g a twice differentiable functions on \mathbb{R} . Then*

$$\begin{aligned} A_h^n g &= g + R_{A_h^n}(g), \\ D_h^n f &= f^{(n)} + R_{D_h^n}(f). \end{aligned}$$

where $R_{D_h^n}$ and $R_{A_h^n}$ are given by

$$R_{D_h^n}(f) := h^2 \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{n-2k}{2} \right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)}\left(\cdot + \frac{(n-2k)h}{2}\sigma\right) d\sigma$$

and

$$R_{A_h^n}(g) := \frac{h^2}{2^{n+2}} \sum_{k=0}^n \binom{n}{k} (n-2k)^2 \int_0^1 (1-\sigma) g^{(2)}\left(\cdot + \frac{(n-2k)h}{2}\sigma\right) d\sigma.$$

Proof. The proof of this proposition follows from Taylor expansion

$$g(x+y) = \sum_{j=0}^{i-1} \frac{y^j}{j!} h^{(j)}(x) + y^i \int_0^1 \frac{(1-\sigma)^{i-1}}{(i-1)!} g^{(i)}(x+\sigma y) d\sigma. \quad (4.1)$$

First, we use (4.1) with $i=2$ and $y = \frac{(n-2k)h}{2}$ to obtain

$$\tau_+^{n-2k} g = g + \frac{(n-2k)h}{2} g' + \left(\frac{(n-2k)h}{2} \right)^2 \int_0^1 (1-\sigma) g^{(2)}\left(\cdot + \frac{(n-2k)h}{2}\sigma\right) d\sigma.$$

Then, it follows that

$$\begin{aligned} A_h^n g &= \frac{1}{2^n} (\tau_+ + \tau_-)^n g \\ &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \tau_+^{n-k} \tau_-^k g \\ &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \tau_+^{n-2k} g \\ &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \left(g + \frac{(n-2k)h}{2} g' \right) + R_{A_h^n}(g) \end{aligned}$$

Now, using $\sum_{k=0}^n \binom{n}{k} = 2^n$ and $\sum_{k=0}^n \binom{n}{k} k = n2^{n-1}$ we write

$$A_h^n g = g + R_{A_h^n}(g).$$

On the other hand, applying (4.1) for f with $i = n + 2$ and $y = \frac{(n-2k)h}{2}$, we have

$$\tau_+^{n-2k} f = \sum_{j=0}^{n+1} \frac{1}{j!} \left(\frac{(n-2k)h}{2} \right)^j f^{(j)} + \left(\frac{(n-2k)h}{2} \right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)} \left(\cdot + \frac{(n-2k)h}{2} \sigma \right) d\sigma.$$

Thus, for the difference operator we get

$$\begin{aligned} D_h^n f &= \frac{1}{h^n} (\tau_+ - \tau_-)^n f \\ &= \frac{1}{h^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \tau_+^{n-k} \tau_-^k f \\ &= \frac{1}{h^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \tau_+^{n-2k} f \\ &= \frac{1}{h^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{j=0}^{n+1} \frac{1}{j!} \left(\frac{(n-2k)h}{2} \right)^j f^{(j)} \\ &\quad + R_{D_h^n}(f) \\ &= \frac{1}{h^n} \sum_{j=0}^{n+1} \frac{1}{j!} h^j \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{n}{2} - k \right)^j f^{(j)} + R_{D_h^n}(f) \end{aligned}$$

Now, using $\sum_{k=0}^n (-1)^k \binom{n}{k} (x-k)^n = n!$ and $\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{n}{2} - k \right)^{n+1} = 0$, we obtain

$$\begin{aligned} D_h^n f &= \frac{1}{h^n} \frac{1}{(n)!} h^n \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{n}{2} - k \right)^n f^{(n)} \\ &\quad + \frac{1}{h^n} \frac{1}{(n+1)!} h^{n+1} \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{n}{2} - k \right)^{n+1} f^{(n+1)} + R_{D_h^n}(f) \\ &= f^{(n)} + R_{D_h^n}(f), \end{aligned}$$

and the proof is complete. \square

Corollary 4.2. *Let f be a $(n+4)$ -times differentiable function defined on \mathbb{R} and $m, n \in \mathbb{N}$. Then*

$$A_h^m D_h^n f = f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f),$$

where

$$R_{A_h^m D_h^n}(f) := \sum_{k,k'=0}^{n,m} a_{k,k'} \int_0^1 \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} (1-\sigma') f^{(n+4)} \left(x + (n-2k) \frac{h}{2} \sigma + \frac{(m-2k')h}{2} \sigma' \right) d\sigma' d\sigma$$

with

$$a_{k,k'} := \frac{h^4}{2^m} \binom{m}{k'} \binom{n}{k} (-1)^k \left(\frac{(n-2k)}{2} \right)^{n+2} \frac{(m-2k')^2}{4}.$$

Proof. It is enough to see that

$$\begin{aligned} A_h^m(D_h^n f) &= A_h^m f^{(n)} + A_h^m(R_{D_h^n}(f)) \\ &= f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f). \end{aligned}$$

□

Note that $R_{A_h^m D_h^n} = R_{A_h^m} \circ R_{D_h^n} = R_{D_h^n} \circ R_{A_h^m}$. Now, we consider two fundamental estimates for our weight function. The proofs of these results can be found in [3]. We consider $\alpha = (\alpha_t, \alpha_x) \in \mathbb{N}^2$ multi-indices.

Lemma 4.3. *Let α and β be multi-indices. We have*

$$\begin{aligned} \partial^\beta(r\partial^\alpha \rho) &= |\alpha|^{|\beta|} (-s\varphi)^{|\alpha|} \lambda^{|\alpha+\beta|} (\partial_x \psi)^{\alpha+\beta} \\ &\quad + |\alpha||\beta| (s\varphi)^{|\alpha|} \lambda^{|\alpha+\beta|-1} \mathcal{O}(1) + s^{|\alpha|-1} |\alpha| (|\alpha| - 1) \mathcal{O}_\lambda(1) \\ &= \mathcal{O}_\lambda(s^{|\alpha|}). \end{aligned} \tag{4.2}$$

Moreover, let $\sigma \in [0, 1]$ and $sh \leq 1$, then $\partial^\beta(r(x)(\partial^\alpha \rho)(x + \sigma h)) = s^{|\alpha|} \mathcal{O}_\lambda(1)$.

Corollary 4.4. *Let α , β and δ be multi-indices. We have*

$$\begin{aligned} \partial^\delta(r^2(\partial^\alpha \rho)\partial^\beta \rho) &= |\alpha + \beta|^{|\delta|} (-s\varphi)^{|\alpha+\beta|} \lambda^{|\alpha+\beta+\delta|} (\partial_x \psi)^{\alpha+\beta+\delta} \\ &\quad + |\delta||\alpha + \beta| (s\varphi)^{|\alpha+\beta|} \lambda^{|\alpha+\beta+\delta|-1} \mathcal{O}(1) \\ &\quad + s^{|\alpha+\beta|-1} (|\alpha|(|\alpha| - 1) + |\beta|(|\beta| - 1)) \mathcal{O}_\lambda(1) \\ &= \mathcal{O}_\lambda(s^{|\alpha+\beta|}). \end{aligned}$$

Corollary 4.2 and Lemma 4.3 yield.

Proposition 4.5. *Let α be a multi-index and $n, m \in \mathbb{N}$. Provided $sh \leq 1$, we have*

$$rA_h^m D_h^n \partial^\alpha \rho = r\partial_x^n \partial^\alpha \rho + s^{|\alpha|+n} \mathcal{O}_\lambda((sh)^2) = s^{|\alpha|+n} \mathcal{O}_\lambda(1).$$

Proof. From Corollary 4.2 we write

$$rA_h^m D_h^n \partial^\alpha \rho = r\partial_x^n \partial^\alpha \rho + rR_{A_h^m}(\partial_x^n \partial^\alpha \rho) + rR_{D_h^n}(\partial^\alpha \rho) + rR_{A_h^m D_h^n}(\partial^\alpha \rho)$$

By Lemma 4.3 we have

$$r(x)(\partial_x^{n+2} \partial^\alpha \rho)(x + (n - 2k)h\sigma/2) = \mathcal{O}_\lambda(s^{|\alpha|+n+2})$$

and

$$r(x)\partial_x^{n+4} \partial^\alpha \rho(x + (n - 2k)h\sigma/2) = \mathcal{O}_\lambda(s^{|\alpha|+n+4}).$$

Then

$$\begin{aligned} rR_{A_h^m}(\partial_x^n \partial^\alpha \rho) &= s^{|\alpha|+n} \mathcal{O}_\lambda((sh)^2), \\ rR_{D_h^n}(\partial^\alpha \rho) &= s^{|\alpha|+n} \mathcal{O}_\lambda((sh)^2), \\ rR_{A_h^m D_h^n}(\partial^\alpha \rho) &= s^{|\alpha|+n} \mathcal{O}_\lambda((sh)^4), \end{aligned}$$

which yields the result. \square

Lemma 4.6. *Let α and β multi-index and $n \in \mathbb{N}$. Provided $sh \leq 1$, we have*

$$A_h^m D_h^n (\partial^\beta (r \partial^\alpha \rho)) = \partial_x^n \partial^\beta (r \partial^\alpha \rho) + h^2 \mathcal{O}_\lambda(s^{|\alpha|})$$

Let $\sigma \in [0, 1]$, we have $A_h^m D_h^n \partial^\beta (r(x) \partial^\alpha \rho(x + \sigma h)) = \mathcal{O}_\lambda(s^{|\alpha|})$.

Proof. By Corollary 4.2 we write

$$\begin{aligned} A_h^m D_h^n (\partial^\beta (r \partial^\alpha \rho)) &= \partial_x^n \partial^\beta (r \partial^\alpha \rho) + R_{A_h^m} (\partial_x^n \partial^\beta (r \partial^\alpha \rho)) \\ &\quad + R_{D_h^n} (\partial^\beta (r \partial^\alpha \rho)) + R_{A_h^m D_h^n} (\partial^\beta (r \partial^\alpha \rho)). \end{aligned}$$

By Lemma 4.3, it follows that

$$\begin{aligned} (\partial_x^n \partial^\beta (r \partial^\alpha \rho))(x + (n - 2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}), \\ (\partial_x^{n+2} \partial^\beta (r \partial^\alpha \rho))(x + (n - 2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}), \\ (\partial_x^{n+4} \partial^\beta (r \partial^\alpha \rho))(x + (n - 2k)h\sigma/2) &= \mathcal{O}_\lambda(s^{|\alpha|}), \end{aligned}$$

which concludes the proof of the first result.

On the other hand, we set $\nu(x, \sigma h) := r(x)\rho(x + \sigma h)$ and $\mu_\alpha := r \partial^\alpha \rho$. Since $r\rho = 1$ it follows that $r(x)\partial^\alpha \rho(x + \sigma h) = \nu(x, \sigma h)\mu_\alpha(x + \sigma h)$. Note that, by continuous Leibniz rule, $\partial_x^n \partial^\beta (\nu \mu_\alpha)$ is a linear combination of terms of the form $\partial^{\beta'} \nu \partial^{\beta''} \mu_\alpha$, with $\beta' + \beta'' = n + \beta$ and by Lemma 4.3 we write $\partial^{\beta'} \nu = \mathcal{O}_\lambda(1)$ and $\partial^{\beta''} \mu_\alpha = \mathcal{O}_\lambda(s^{|\alpha|})$. Besides, it holds for the terms $\partial_x^{n+2} \partial^\beta (\nu \mu_\alpha)$ and $\partial_x^{n+4} \partial^\beta (\nu \mu_\alpha)$ as well. Therefore, applying Corollary 4.2 to $\nu \mu_\alpha$ we obtain

$$A_h^m D_h^n (\partial^\beta (r(x) \partial^\alpha \rho(x + \sigma h))) = \mathcal{O}_\lambda(s^{|\alpha|}) + h^2 \mathcal{O}_\lambda(s^{|\alpha|}) + h^4 \mathcal{O}_\lambda(s^{|\alpha|}).$$

\square

Lemma 4.7. *Let α, β, δ be multi-indices and $n, m \in \mathbb{N}$. Provided $sh \leq 1$, we have:*

$$\begin{aligned} A_h^m D_h^n \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho) &= \partial_x^n \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho) + h^2 \mathcal{O}_\lambda(s^{|\alpha|+|\beta|}) \\ &= \mathcal{O}_\lambda(s^{|\alpha|+|\beta|}). \end{aligned}$$

Let $\sigma, \sigma' \in [0, 1]$. We have

$$A_h^m D_h^n \partial^\delta (r(x)^2 (\partial^\alpha \rho(x + \sigma h)) \partial^\beta \rho(x + \sigma' h)) = \mathcal{O}_\lambda(s^{|\alpha|+|\beta|}).$$

Proof. Applying Corollary 4.2 to $\partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho)$ we obtain

$$\begin{aligned} A_h^m D_h^n \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho) &= \partial_x^n \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho) + R_{A_h^m} (\partial_x^n \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho)) \\ &\quad + R_{D_h^n} (\partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho)) + R_{A_h^m D_h^n} (\partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho)). \end{aligned}$$

Then, the first result follows from Corollary 4.4. For the second one, we proceed similarly as the proof of the second result of Lemma 4.6, that is, we apply Corollary 4.2 to $\nu^2 \mu_\alpha \mu_\beta$, then we use continuous Leibniz rule and Lemma 4.6 to conclude. \square

Lemma 4.8. *Let α be a multi-index. For $j, k, m, n \in \mathbb{N}$ and for $sh \leq 1$, we have*

$$A_h^j D_h^k \partial^\alpha (r A_h^m D_h^n \rho) = \partial_x^k \partial^\alpha (r \partial_x^n \rho) + s^n \mathcal{O}_\lambda((sh)^2) = s^n \mathcal{O}_\lambda(1).$$

Proof. By Corollary 4.2 we write

$$\partial^\alpha (r A_h^m D_h^n \rho) = \partial^\alpha (r \partial_x^n \rho) + \partial^\alpha (r R_{A_h^m}(\partial_x^n \rho)) + \partial^\alpha (r (R_{D_h^n}(\rho)) + \partial^\alpha (r R_{A_h^m D_h^n}(\rho))).$$

Then, applying again Corollary 4.2 to the first term of the above expression, we have

$$\begin{aligned} A_h^j D_h^k \partial^\alpha (r A_h^m D_h^n \rho) &= A_h^j D_h^k \partial^\alpha (r \partial_x^n \rho) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m}(\partial_x^n \rho)) \\ &\quad + A_h^j D_h^k \partial^\alpha (r (R_{D_h^n}(\rho)) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m D_h^n}(\rho))) \\ &= \partial_x^k \partial^\alpha (r \partial_x^n \rho) + R_{A_h^j}(\partial^\alpha (r \partial_x^n \rho)) + R_{D_h^k}(\partial^\alpha (r \partial_x^n \rho)) \\ &\quad + A_h^j D_h^k \partial^\alpha (r (R_{D_h^n}(\rho)) + A_h^j D_h^k \partial^\alpha (r R_{A_h^m D_h^n}(\rho))). \end{aligned}$$

Thus, by Lemma 4.3, we obtain

$$A_h^j D_h^k \partial^\alpha (r A_h^m D_h^n \rho) = \partial_x^k \partial^\alpha (r \partial_x^n \rho) + s^n \mathcal{O}_\lambda((sh)^2).$$

□

Theorem 4.9. *Let α, β be multi-indices and $j, k, l, m, n, p \in \mathbb{N}$. Provided $sh \leq 1$, we have*

$$\begin{aligned} A_h^p D_h^l \partial^\beta (r^2 A_h^j D_h^k (\partial^\alpha \rho) A_h^m D_h^n (\rho)) &= \partial_x^l \partial^\beta (r^2 \partial_x^k \partial^\alpha \rho \partial_x^n \rho) + s^{n+k+|\alpha|} \mathcal{O}_\lambda((sh)^2) \\ &= s^{n+k+|\alpha|} \mathcal{O}_\lambda(1). \end{aligned}$$

Proof. We have

$$\begin{aligned} A_h^m D_h^n (\rho) &= \partial_x^n \rho + R_{A_h^m}(\partial_x^n \rho) + R_{D_h^n}(\rho) + R_{A_h^m D_h^n}(\rho), \\ A_h^j D_h^k (\partial^\alpha \rho) &= \partial_x^k \partial^\alpha \rho + R_{A_h^j}(\partial_x^k \partial^\alpha \rho) + R_{D_h^k}(\partial^\alpha \rho) + R_{A_h^j D_h^k}(\partial^\alpha \rho). \end{aligned}$$

By Lemma 4.7 we write

$$A_h^p D_h^l \partial^\beta (r^2 A_h^j D_h^k (\partial^\alpha \rho) A_h^m D_h^n (\rho)) = \partial_x^l \partial^\beta (r^2 \partial_x^k \partial^\alpha \rho \partial_x^n \rho) + s^{n+k+|\alpha|} \mathcal{O}_\lambda((sh)^2),$$

which is our claim. □

Let us finally mention that the results of this section can be extended for time-dependent case. For instance, if we consider a weight function of the form $r(x, t) = e^{s\theta(t)\varphi(x)}$ then the condition $sh \leq 1$ must be replaced by $sh(\max_{[0, T]} \theta(t)) \leq 1$ which implies that $s\theta(t)h \leq 1$.

5. DISCRETE CARLEMAN ESTIMATE

In this section, we establish a discrete Carleman estimate with boundary observation for a finite difference approximation of the Laplacian operator in the one-dimensional setting. In order to do so, it is natural to look closer at the continuous version of such estimates. For this purpose, we follow the methodology of A. V. Fursikov and O. Y. Imanuvilov [9] to obtain a Carleman estimate for Laplacian operator in the continuous setting, which is similar to the estimate obtained by Yamamoto in [15]. The main difference with the one in [15]

to our estimate is that we do not consider a density argument, and we thus obtain a Carleman estimate with boundary observation. Then, following the methodology in [3] we establish a discrete Carleman estimate.

5.1. The continuous case

The proof of the following Carleman estimate has two steps. First, we consider the conjugate operator defined by $P_\varphi u := e^{s\varphi} \partial_x^2 (e^{-s\varphi} u)$. In this case, our Carleman weight function is defined as $e^{s\varphi}$ for $s > 0$ with $\varphi = e^{\lambda\psi}$, where $\lambda > 0$, and satisfy

$$\partial_x \psi(x, t) > 0, \quad (x, t) \in \overline{Q}. \quad (5.1)$$

Then, we split P_φ into the operators P_1 and P_2 , and it is estimated the scalar product $(P_1 u, P_2 u)_{L^2(Q)}$.

Theorem 5.1. (*Carleman estimate*) *Let $\psi \in C(\mathbb{R}^2)$, and for any $t \in (0, T)$ let $\psi(\cdot, t) \in C^4(\mathbb{R})$ such that $\partial_x \psi(x, t) > 0$ for $(x, t) \in \overline{Q}$. For the parameter $\lambda_0 > 0$ sufficiently large, there exists $s_0(\lambda_0) \geq 0$, and $C(s_0, \lambda_0, \psi) > 0$, such that*

$$\begin{aligned} & C \int_Q e^{2s\varphi} |\partial_x^2 v|^2 + s^3 \lambda_0^3 \int_0^T ((\partial_x \psi)^3 \varphi^3 e^{2s\varphi} |v|^2)(1, t) + s \lambda_0 \int_0^T (\partial_x \psi \varphi e^{2s\varphi} |\partial_x v|^2)(1, t) \\ & \geq s^3 \lambda_0^4 \int_Q (\partial_x \psi)^4 \varphi^3 e^{2s\varphi} |v|^2 + s \lambda_0^2 \int_Q (\partial_x \psi)^2 \varphi e^{2s\varphi} |\partial_x v|^2, \end{aligned} \quad (5.2)$$

for all $s \geq s_0$.

Proof. We set $u = e^{s\varphi} v$. Then the conjugate operator can be expanded as follows

$$P_\varphi u = e^{s\varphi} \partial_x^2 (e^{-s\varphi} u) + 2e^{s\varphi} \partial_x (e^{-s\varphi}) \partial_x u + \partial_x^2 u. \quad (5.3)$$

Adding $-s \partial_x^2 (\varphi) u$, (5.3) can be written as

$$P_\varphi u - s \partial_x^2 (\varphi) u = P_1 u + P_2 u, \quad (5.4)$$

where $P_1 u := \partial_x^2 u + e^{s\varphi} \partial_x^2 (e^{-s\varphi} u)$ and $P_2 u := 2e^{s\varphi} \partial_x (e^{-s\varphi}) \partial_x u - s \partial_x^2 (\varphi) u$. Besides, from (5.4), we have

$$\|P_\varphi u - s \partial_x^2 (\varphi) u\|_{L^2(Q)}^2 = \|P_1 u\|_{L^2(Q)}^2 + \|P_2 u\|_{L^2(Q)}^2 + 2(P_1 u, P_2 u)_Q. \quad (5.5)$$

Note that

$$\|P_\varphi u - s \partial_x^2 (\varphi) u\|_{L^2(Q)}^2 \leq C_\varphi \left(\|P_\varphi u\|_{L^2(Q)}^2 + s^2 \|u\|_{L^2(Q)}^2 \right), \quad (5.6)$$

since $\partial_x^2 \varphi$ is bounded in Q . On the other hand, defining $C_1 u := \partial_x^2 u$, $C_2 u := e^{s\varphi} \partial_x^2 (e^{-s\varphi} u)$, $B_1 u := 2e^{s\varphi} \partial_x (e^{-s\varphi}) \partial_x u$ and $B_2 u := -s \partial_x^2 (\varphi) u$ we have

$$(P_1 u, P_2 u)_{L^2(Q)} = \sum_{i,j=1}^2 (C_i, B_j)_{L^2(Q)}. \quad (5.7)$$

We note that, integrating by parts in space, (5.7) can be rewritten as

$$\begin{aligned}
(P_1 u, P_2 u)_{L^2(Q)} &= 2s^3 \int_Q (\partial_x \varphi)^2 \partial_x^2 \varphi |u|^2 + \int_Q s^2 ((\partial_x^2 \varphi)^2 - (\partial_x \varphi)^2 - \partial_x \varphi \partial_x^2 \varphi) |u|^2 - \frac{s}{2} \int_Q \partial_x^4(\varphi) |u|^2 \\
&\quad + 2s \int_Q \partial_x^2(\varphi) |\partial_x u|^2 - s \int_0^T \partial_x(\varphi) |\partial_x u|^2 \Big|_0^1 + \frac{s}{2} \int_0^T \partial_x^3(\varphi) |u|^2 \Big|_0^1 \\
&\quad + \int_0^T (-s^3 (\partial_x \varphi)^3 + s^2 \partial_x \varphi \partial_x^2 \varphi) |u|^2 \Big|_0^1 - s \int_0^T u \partial_x u \partial_x^2(\varphi) \Big|_0^1.
\end{aligned}$$

Now, using the Young's inequality on the last integral above, we have

$$\begin{aligned}
(P_1 u, P_2 u)_{L^2(Q)} &\geq 2s^3 \int_Q (\partial_x \varphi)^2 \partial_x^2 \varphi |u|^2 + \int_Q s^2 ((\partial_x^2 \varphi)^2 - (\partial_x \varphi)^2 - \partial_x \varphi \partial_x^2 \varphi) |u|^2 - \frac{s}{2} \int_Q \partial_x^4(\varphi) |u|^2 \\
&\quad + 2s \int_Q \partial_x^2(\varphi) |\partial_x u|^2 - s \int_0^T \partial_x(\varphi) |\partial_x u|^2 \Big|_0^1 + \frac{s}{2} \int_0^T \partial_x^3(\varphi) |u|^2 \Big|_0^1 \\
&\quad - \frac{s}{2} \int_0^T (\partial_x^2 \varphi)^2 |u|^2 \Big|_0^1 - \frac{s}{2} \int_0^T (\partial_x^2 \varphi)^2 |u|^2 \Big|_1^1 - \frac{s}{2} \int_0^T |\partial_x u|^2 \Big|_0^1 - \frac{s}{2} \int_0^T |\partial_x u|^2 \Big|_1^1 \\
&\quad + \int_0^T (-s^3 (\partial_x \varphi)^3 + s^2 \partial_x \varphi \partial_x^2 \varphi) |u|^2 \Big|_0^1.
\end{aligned}$$

For λ large enough, there exist $C_{\lambda_0} > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ we obtain

$$\begin{aligned}
C_{\lambda_0} (P_1 u, P_2 u)_{L^2(Q)} &\geq s^3 \lambda^4 \int_Q (\partial_x \psi)^4 \varphi^3 |u|^2 + s^2 \lambda^4 \int_Q (\partial_x \psi)^4 \varphi^2 |u|^2 - s \lambda^4 \int_Q (\partial_x \psi)^4 \varphi |u|^2 \\
&\quad + s \lambda^2 \int_Q (\partial_x \psi)^2 \varphi |\partial_x u|^2 - s \lambda \int_0^T \partial_x \psi \varphi |\partial_x u|^2 \Big|_0^1 + s \lambda^3 \int_0^T (\partial_x \psi)^3 \varphi |u|^2 \Big|_0^1 \\
&\quad - s^3 \lambda^3 \int_0^T (\partial_x \psi)^3 \varphi^3 |u|^2 \Big|_0^1 + s^2 \lambda^3 \int_0^T (\partial_x \psi)^3 \varphi^2 |u|^2 \Big|_0^1 \\
&\quad - s \int_0^T (\lambda \partial_x \psi)^2 \varphi |u|^2 \Big|_0^1 - s \int_0^T (\lambda \partial_x \psi)^2 \varphi |u|^2 \Big|_1^1.
\end{aligned}$$

Now, if we fix $\lambda = \lambda_0$, there exist $C_{s_0, \lambda_0} > 0$ and $s_0(\lambda_0) > 0$ such that

$$\begin{aligned}
C_{s_0, \lambda_0} (P_1 u, P_2 u)_{L^2(Q)} &\geq s^3 \lambda^4 \int_Q (\partial_x \psi)^4 \varphi^3 |u|^2 + s \lambda^2 \int_Q (\partial_x \psi)^2 \varphi |\partial_x u|^2 \\
&\quad - s^3 \lambda^3 \int_0^T (\partial_x \psi)^3 \varphi^3 |u|^2 \Big|_0^1 - s \lambda \int_0^T \partial_x \psi \varphi |\partial_x u|^2 \Big|_0^1, \tag{5.8}
\end{aligned}$$

for $s \geq s_0$. Note that $|e^{s\varphi} \partial_x v|^2 = |s u \partial_x \varphi + \partial_x u|^2 \leq C_\varphi (s^2 |u|^2 + |\partial_x u|^2)$. Thus, from (5.5), (5.6) and (5.8) we obtain for λ large enough

$$\begin{aligned} C_{\lambda_0, s_0, \varphi} \int_Q e^{2s\varphi} |\partial_x^2 v|^2 &\geq s^3 \lambda^4 \int_Q (\partial_x \psi)^4 \varphi^3 e^{2s\varphi} |v|^2 + s \lambda^2 \int_Q (\partial_x \psi)^2 \varphi |\partial_x v|^2 \\ &\quad - s^3 \lambda^3 \int_0^T (\partial_x \psi)^3 \varphi^3 e^{2s\varphi} |v|^2 \Big|_0^1 - s \lambda \int_0^T \partial_x \psi \varphi e^{2s\varphi} |\partial_x v|^2 \Big|_0^1, \end{aligned}$$

which proves the required result. \square

Remark 5.2. Note that taking $x_0 > 1$, the observation data in (5.2) can be switched to the point $(0, t)$, for $t \in (0, T)$.

5.2. Proof of the discrete Carleman estimate

Now, we establish a discrete Carleman estimate for the discrete operator D_h^2 . Note that this is the discrete Laplacian in one-dimensional setting. There are Carleman estimates for this kind of operator (see F. Boyer *et al.* [3, 4] and S. Ervedoza *et al.* [7]). The main difference respect to our estimate is the fact that we consider boundary observation, due to the choice of the weight function. Indeed, our Carleman weight function is defined as $e^{s\varphi}$ for $s \geq 1$, with $\varphi = e^{\lambda\psi}$ where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. We also assume that

$$\partial_x \psi(x, t) > 0, \quad (x, t) \in Q. \quad (5.9)$$

We follow a classical scheme based on conjugating the original operator with a well chosen exponential function.

5.2.1. Proof Theorem 1.1

We make the change of variable $u_h = e^{s\varphi} v_h$. Our first task is to obtain an expression for $P_{h,\varphi} := e^{s\varphi} D_h^2 (e^{-s\varphi} u_h)$ with the change of variable proposed. By using (2.5), we have

$$P_{h,\varphi} = e^{s\varphi} D_h^2 (e^{-s\varphi}) A_h^2 u + 2e^{s\varphi} A_h D_h (e^{-s\varphi}) D_h A_h u_h + e^{s\varphi} A_h^2 (e^{-s\varphi}) D_h^2 u_h. \quad (5.10)$$

We define the following coefficients $\alpha_1 := e^{s\varphi} A_h^2 (e^{-s\varphi})$, $\alpha_2 := e^{s\varphi} D_h^2 (e^{-s\varphi})$ and $\beta_1 := e^{s\varphi} A_h D_h (e^{-s\varphi})$. On the other hand, we set

$$\begin{aligned} C_1 u_h &:= \alpha_1 D_h^2 u_h, \\ C_2 u_h &:= \alpha_2 A_h^2 u_h, \\ B_1 u_h &:= 2\beta_1 D_h A_h u_h, \\ B_2 u_h &:= -s(\partial_x^2 \varphi) u_h. \end{aligned}$$

Equation (5.10) thus reads $P_{h,\varphi} u_h - s(\partial_x^2 \varphi) u_h = P_1 u_h + P_2 u_h$, where

$$\begin{aligned} P_1 u_h &:= C_1 u_h + C_2 u_h \\ P_2 u_h &:= B_1 u_h + B_2 u_h. \end{aligned}$$

We write

$$\|P_{h,\varphi} - s(\partial_x^2 \varphi) u_h\|_{L_h^2(Q_h)}^2 = \|P_1 u_h\|_{L_h^2(Q_h)}^2 + \|P_2 u_h\|_{L_h^2(Q_h)}^2 + 2\langle P_1 u_h, P_2 u_h \rangle_{Q_h}. \quad (5.11)$$

Since $\partial_x^2 \varphi$ is bounded, we have

$$\|P_{h,\varphi} - s(\partial_x^2 \varphi)u_h\|_{L_h^2(Q_h)}^2 \leq C \left(\|P_\varphi u_h\|_{L_h^2(Q_h)}^2 + s^2 \|u_h\|_{L_h^2(Q_h)}^2 \right). \quad (5.12)$$

Now, we shall estimate the scalar product

$$\langle P_1 u_h, P_2 u_h \rangle_{Q_h} = \sum_{i,j=1}^2 \langle C_i u_h, B_j u_h \rangle_{Q_h}. \quad (5.13)$$

For each term of (5.13), we obtain the following results.

Lemma 5.3. *For $sh \leq 1$, we have*

$$\langle C_1 u_h, B_1 u_h \rangle_{Q_h} = \int_{Q_h^*} s\lambda^2 \varphi (\partial_x \psi)^2 |D_h u_h|^2 + \int_{Q_h^*} s\lambda \varphi \partial_x^2 \psi |D_h u_h|^2 - X_1 + Y_1,$$

where

$$X_1 := \int_{Q_h^*} s\mathcal{O}_\lambda((sh)^2) |D_h u_h|^2$$

and

$$Y_1 := \int_{\partial Q_h} (-s\lambda \varphi \partial_x \psi + s\mathcal{O}_\lambda((sh)^2)) t_r(|Du_h|^2) n_h.$$

Lemma 5.4. *For $sh \leq 1$, we have*

$$\langle C_1 u_h, B_2 u_h \rangle_{Q_h} \geq \int_{Q_h^*} s\lambda^2 (\partial_x \psi)^2 \varphi |D_h u_h|^2 + \int_{Q_h^*} s\lambda \varphi \partial_x^2 \psi |D_h u_h|^2 - X_2 + Y_2,$$

where

$$X_2 := \int_{Q_h} s\mathcal{O}_\lambda(1) |u_h|^2 + \int_{Q_h^*} s\mathcal{O}_\lambda(h^2 + (sh)^2) |D_h u_h|^2$$

and

$$Y_2 := \int_{\partial Q_h} s\mathcal{O}_\lambda(1) |u_h|^2 - \int_{\partial Q_h} s^2 \mathcal{O}_\lambda(1) |u_h|^2 - \int_{\partial Q_h} \mathcal{O}_\lambda(1) t_r(|D_h u_h|^2).$$

Lemma 5.5. *For $sh \leq 1$, we have*

$$\langle C_2 u_h, B_1 u_h \rangle_{Q_h} = 3 \int_{Q_h} s^3 \lambda^4 \varphi^3 (\partial_x \psi)^4 |u_h|^2 + \int_{Q_h} (s\lambda \varphi)^3 \mathcal{O}(1) |u_h|^2 - X_3 + Y_3,$$

where

$$X_3 := \int_{Q_h} s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2) |u_h|^2 - \int_{Q_h^*} s\mathcal{O}_\lambda((sh)^2) |D_h u_h|^2$$

and

$$Y_3 := \int_{\partial Q_h} \left(-(s\lambda\varphi\partial_x\psi)^3 + s^2\mathcal{O}_\lambda(1) + s^3\mathcal{O}_\lambda((sh)^2) \right) t_r(A_h(|u_h|^2))n_h \\ - \int_{\partial Q_h} s\mathcal{O}_\lambda((sh)^2)t_r(|D_h u_h|^2)n_h.$$

Lemma 5.6. *For $sh \leq 1$, we have*

$$\langle C_2 u_h, B_2 u_h \rangle_{Q_h} \geq - \int_{Q_h} s^3 \lambda^4 \varphi^3 (\partial_x \psi)^4 |u_h|^2 + \int_{Q_h} s^3 \lambda^3 \varphi^2 (\partial_x \psi)^2 \partial_x^2 \psi |u_h|^2 - X_4$$

where

$$X_4 := \int_{Q_h} (s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2)) |u_h|^2 + \int_{Q_h} s \mathcal{O}_\lambda(sh) |u_h|^2 + \int_{Q_h^*} s \mathcal{O}_\lambda((sh)^2) |D_h u_h|^2,$$

and

$$Y_4 := \int_{\partial Q_h} s \mathcal{O}_\lambda(1) |u_h|^2 + \int_{\partial Q_h} s \mathcal{O}_\lambda((sh)^2) |u_h|^2 + \int_{\partial Q_h} s \mathcal{O}_\lambda((sh)^2) t_r(|D_h u_h|^2) n_h.$$

The proof of Lemmas 5.3–5.6 can be found in Section 6.

Combining the aforementioned Lemmas, for $sh \leq 1$ there exist $\lambda_1 \geq 1$ and ε small enough such that for $\lambda \geq \lambda_1$ and $0 < sh \leq \min\{\varepsilon_1(\lambda), 1\} = \varepsilon_1(\lambda)$, there exists a constant $C_{\lambda_1, \varepsilon_1} > 0$ such that

$$C_{\lambda_1, \varepsilon_1} \langle P_1 u_h, P_2 u_h \rangle_{Q_h} \geq \int_{Q_h^*} s (\partial_x \psi)^2 \varphi |D_h u_h|^2 + \int_{Q_h} s^3 \varphi^3 (\partial_x \psi)^4 |u_h|^2 + \sum_{i=1}^4 Y_i - X_i. \quad (5.14)$$

Thus, from (5.11), (5.12) and (5.14) we get

$$C_{\lambda_1, \varepsilon_1} \left(\|P_{h, \varphi}\|_{L_h^2(Q_h)}^2 + s^2 \|u_h\|_{L_h^2(Q_h)}^2 \right) + \sum_{i=1}^4 X_i \geq \sum_{i=1}^4 Y_i + s^3 \int_{Q_h} \varphi^3 (\partial_x \psi)^4 |u_h|^2 \\ + s \int_{Q_h^*} (\partial_x \psi)^2 \varphi |D_h u_h|^2.$$

On the other hand, we have to deal with the boundary terms. To do this, we can estimate separately the right and left boundary observation. Indeed, let us denote by Y_i^- and Y_i^+ the left and the right boundary observation of the term Y_i , respectively. Once λ is fixed, for s large enough there exist positive constants C_0 and C_1 such that

$$C_0 s \int_{\partial Q_h^-} \varphi \partial_x \psi t_r(|D_h u_h|^2) n_h + C_0 s^3 \int_{\partial Q_h^-} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) \leq \sum_{i=1}^4 Y_i^-, \\ \sum_{i=1}^4 Y_i^+ \leq C_1 s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) + C_1 s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)).$$

Therefore, if we fix $\lambda = \lambda_1$, we can choose ε_0 and h_0 sufficiently small, with $0 < \varepsilon_0 \leq \varepsilon_1(\lambda_1)$, and $s_0 \geq 1$ sufficiently large, such that for $s \geq s_0$, $0 < h \leq h_0$, and $sh \leq \varepsilon_0$ we obtain

$$\begin{aligned} C_{\lambda_1, \varepsilon_0, s_0} \left(\|P_{h, \varphi}\|_{L_h^2(Q_h)}^2 \right) &\geq s^3 \|u_h\|_{L_h^2(Q_h)}^2 + s \|D_h u_h\|_{L_h^2(Q_h^*)}^2 \\ &\quad - C_1 s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) n_h - C_1 s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) \\ &\quad + C_0 s \int_{\partial Q_h^-} \varphi \partial_x \psi t_r(|D_h u_h|^2) - C_0 s^3 \int_{\partial Q_h^-} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)). \end{aligned} \quad (5.15)$$

Finally, we return to the variable v_h . To this end, we need the following Lemma.

Lemma 5.7. *For $sh \leq 1$, we have*

$$s \|e^{s\varphi} D_h v_h\|_{L_h^2(Q_h^*)}^2 \leq C \left(s \|D_h u_h\|_{L_h^2(Q_h^*)}^2 + s^3 \|u_h\|_{L_h^2(Q_h)}^2 \right) + s^2 \mathcal{O}(sh) \int_{\partial Q_h} |u_h|^2, \quad (5.16)$$

$$\begin{aligned} s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(|D_h u_h|^2) &\leq C s^3 \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi} A_h(|v_h|^2)) \\ &\quad + C s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi} |D_h v_h|^2), \end{aligned} \quad (5.17)$$

$$\begin{aligned} s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(A_h(|u_h|^2)) &\leq s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi} A_h(|v_h|^2)) \\ &\quad + s \mathcal{O}((sh)^2) \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi} |D_h v_h|^2). \end{aligned} \quad (5.18)$$

For a proof see Section 6.

Combining (5.15) with Lemma 5.7, we can choose $\tilde{\varepsilon} > 0$ and $\tilde{h} > 0$ sufficiently small, with $0 < \tilde{h} \leq h_0$, $0 < \tilde{\varepsilon} \leq \varepsilon_0$, and \tilde{s} sufficiently large, such that for $s \geq \tilde{s}$, $0 < h \leq \tilde{h}$, and $sh \leq \tilde{\varepsilon}$, we obtain

$$\begin{aligned} s^3 \|e^{s\varphi} v_h\|_{L_h^2(Q_h)}^2 + s \|e^{s\varphi} D_h v_h\|_{L_h^2(Q_h^*)}^2 &\leq C_{\tilde{\varepsilon}, \tilde{s}} \left(\|e^{s\varphi} D_h^2 v_h\|_{L_h^2(Q_h)}^2 + s \int_{\partial Q_h^+} \varphi \partial_x \psi t_r(e^{2s\varphi}) t_r(|D_h v_h|^2) \right. \\ &\quad \left. + s^3 \int_{\partial Q_h^+} (\varphi \partial_x \psi)^3 t_r(e^{2s\varphi}) t_r(A_h(|v_h|^2)) \right), \end{aligned}$$

where we have dropped the left boundary observation, and the proof is complete.

6. PROOF OF INTERMEDIATE RESULTS

In this section, we shall prove some technical results used in the development of the discrete Carleman estimate. We consider $sh \leq 1$ in the following Lemmas in order to ensure that every Lemma from Section 4 holds. Recall that our Carleman weight function defined as $r(x) := e^{s\varphi(x)}$ for $s \geq 1$, with $\varphi(x) = e^{\lambda\psi(x)}$ where $\psi \in C^k$ for k sufficiently large and $\lambda \geq 1$. We denote $\rho := r^{-1}$ and ψ verifies $\partial_x \psi > 0$ in Q_h . The proof we develop in each Lemma is standard in the following sense. We begin rewritten the semi-discrete integral, if necessary, using some identity related to the discrete operators from Corollary 2.2. Then we apply a semi-discrete integration by parts from Proposition 2.4 to identify the leader terms of the Carleman estimate. Finally, thanks to Theorem 4.9, we can obtain the estimate claimed in each Lemma.

6.1. Proof of Lemma 5.3

Recalling the definition of C_1 and B_1 , and setting $\gamma_{11} := \beta_1 \alpha_1$ and $I_{11} := \langle C_1 u_h, B_1 u_h \rangle_{Q_h}$, we write

$$I_{11} := \int_{Q_h} 2\gamma_{11} D_h^2 u_h D_h A_h u_h.$$

From Corollary 2.2 the semi-discrete integral I_{11} can be rewritten as

$$I_{11} = \int_{Q_h} \gamma_1 D_h (|D_h u_h|^2).$$

Using Proposition 2.4, for I_{11} we obtain

$$I_{11} = - \int_{Q_h^*} D_h(\gamma_1) |D_h u_h|^2 + \int_{\partial Q_h} \gamma_1 t_r (|D_h u_h|^2) n_h.$$

The proof is completed by showing that

$$\begin{aligned} D_h(\gamma_1) &= -s\varphi\lambda^2(\partial_x \psi)^2 - s\lambda\varphi\partial_x^2 \psi + s\mathcal{O}_\lambda((sh)^2), \\ \gamma_1 &= -s\lambda\varphi\partial_x \psi + s\mathcal{O}_\lambda((sh)^2), \end{aligned}$$

which follows from Theorem 4.9 and Corollary 4.4.

6.2. Proof of Lemma 5.4

Set $I_{12} := \langle C_1 u_h, B_2 u_h \rangle_{Q_h}$. From the definition of the operators C_1 and B_2 , we have

$$I_{12} := -s \int_{Q_h} \partial_x^2 \varphi \alpha_1 u_h D_h^2 u_h.$$

A semi-discrete integration by parts, Proposition 2.4, yields

$$I_{12} = s \int_{Q_h^*} D_h(\partial_x^2 \varphi \alpha_1 u_h) D_h u_h - s \int_{\partial Q_h} \partial_x^2 \varphi \alpha_1 u_h t_r (D_h u_h) n_h := I_{12}^{(a)} - I_{12}^{(b)}.$$

Let us focus on $I_{12}^{(a)}$. We note that thanks to Lemma 2.1, $I_{12}^{(a)}$ can be rewritten as

$$I_{12}^{(a)} = s \int_{Q_h^*} D_h(\partial_x^2 \varphi \alpha_1) A_h u_h D_h u_h + s \int_{Q_h^*} A_h(\partial_x^2 \varphi \alpha_1) |D_h u_h|^2 := I_{12}^{(a_1)} + I_{12}^{(a_2)}.$$

To estimate the term $I_{12}^{(a_2)}$, due to Lemma 2.1, we write

$$A_h(\alpha_1 \partial_x^2 \varphi) = A_h(\alpha_1) A_h(\partial_x^2 \varphi) + \frac{h^2}{4} D_h(\alpha_1) D_h(\partial_x^2 \varphi). \quad (6.1)$$

By using Proposition 4.1 we obtain the following estimates

$$\begin{aligned} A_h(\partial_x^2 \varphi) &= \partial_x^2 \varphi + \mathcal{O}_\lambda(h^2), \\ D_h(\partial_x^2 \varphi) &= \partial_x^3 \varphi + \mathcal{O}_\lambda(h^2). \end{aligned}$$

Moreover, Lemma 4.8 leads to

$$\begin{aligned} A_h(\alpha_1) &= 1 + \mathcal{O}_\lambda((sh)^2), \\ D_h(\alpha_1) &= \mathcal{O}_\lambda((sh)^2). \end{aligned}$$

The previous estimates enables us to write (6.1) as

$$A_h(\alpha_1 \partial_x^2 \varphi) = \partial_x^2 \varphi + \mathcal{O}_\lambda(h^2 + (sh)^2) = \lambda^2 (\partial_x \psi)^2 \varphi + \lambda \varphi \partial_x^2 \psi + \mathcal{O}_\lambda(h^2 + (sh)^2).$$

Therefore, $I_{12}^{(a_2)}$ can be estimated as

$$I_{12}^{(a_2)} = \int_{Q_h^*} s \lambda^2 (\partial_x \psi)^2 \varphi |D_h u_h|^2 + \int_{Q_h^*} s \lambda \varphi \partial_x^2 \psi |D_h u_h|^2 + \int_{Q_h^*} s \mathcal{O}_\lambda(h^2 + (sh)^2) |D_h u_h|^2. \quad (6.2)$$

On the other hand, by using (2.3), $I_{12}^{(a_1)}$ can be rewritten as

$$I_{12}^{(a_1)} = \frac{s}{2} \int_{Q_h^*} D_h (\alpha_1 \partial_x^2 \varphi) D_h (|u_h|^2).$$

A semi-discrete integration by parts with respect to the difference operator D_h leads to

$$I_{12}^{(a_1)} = -\frac{s}{2} \int_{Q_h} D_h^2 (\partial_x^2 \varphi \alpha_1) |u_h|^2 + \frac{s}{2} \int_{\partial Q_h} t_r(D_h(\partial_x^2 \varphi \alpha_1)) |u_h|^2 n_h.$$

By using (2.5), it follows that

$$D_h^2 (\partial_x^2 \varphi \alpha_1) = D_h^2 (\partial_x^2 \varphi) A_h^2 (\alpha_1) + 2D_h A_h (\partial_x^2 \varphi) A_h D_h (\alpha_1) + A_h^2 (\partial_x^2 \varphi) D_h^2 (\alpha_1). \quad (6.3)$$

Now, applying Lemma 4.8 to $\alpha_1 := e^{s\varphi} A_h^2 (e^{-s\varphi})$, we have

$$\begin{aligned} A_h^2 (\alpha_1) &= \mathcal{O}_\lambda(1), \\ A_h D_h (\alpha_1) &= \mathcal{O}_\lambda(1), \\ D_h^2 (\alpha_1) &= \mathcal{O}_\lambda(1). \end{aligned}$$

Moreover, applying Proposition 4.1 to $\partial_x^2 \varphi$, we get

$$\begin{aligned} D_h^2 (\partial_x^2 \varphi) &= \partial_x^4 \varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1), \\ D_h A_h (\partial_x^2 \varphi) &= \partial_x^3 \varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1), \\ A_h^2 (\partial_x^2 \varphi) &= \partial_x^2 \varphi + \mathcal{O}_\lambda(h^2) = \mathcal{O}_\lambda(1). \end{aligned}$$

Thus, (6.3) can be estimated as

$$D_h^2 (\alpha_1 \partial_x^2 \varphi) = \mathcal{O}_\lambda(1).$$

Similarly, we get

$$D_h (\alpha_1 \partial_x^2 \varphi) = \mathcal{O}_\lambda(1).$$

Hence, for $I_{12}^{(a_1)}$ we obtain

$$I_{12}^{(a_1)} = -s \int_{Q_h} \mathcal{O}_\lambda(1) |u_h|^2 + s \int_{\partial Q_h} \mathcal{O}_\lambda(1) |u_h|^2. \quad (6.4)$$

Finally, by using the Young's inequality, $I_{12}^{(b)}$ can be bounded as

$$|I_{12}^{(b)}| \leq s^2 \int_{\partial Q_h} |\mathcal{O}_\lambda(1)| |u_h|^2 + \int_{\partial Q_h} |\mathcal{O}_\lambda(1)| t_r (|D_h u_h|^2). \quad (6.5)$$

Therefore, collecting the estimates (6.2), (6.4) and (6.5), I_{12} can be estimated as

$$I_{12} \geq \int_{Q_h^*} s \lambda^2 (\partial_x \psi)^2 \varphi |D_h u_h|^2 + \int_{Q_h^*} s \lambda \varphi \partial_x^2 \psi |D_h u_h|^2 - X_2 + Y_2,$$

where X_2 and Y_2 are given by

$$X_2 := \int_{Q_h} s \mathcal{O}_\lambda(1) |u_h|^2 + \int_{Q_h^*} s \mathcal{O}_\lambda(h^2 + (sh)^2) |D_h u_h|^2$$

and

$$Y_2 := \int_{\partial Q_h} s \mathcal{O}_\lambda(1) |u_h|^2 - \int_{\partial Q_h} s^2 \mathcal{O}_\lambda(1) |u_h|^2 - \int_{\partial Q_h} \mathcal{O}_\lambda(1) t_r (|D_h u_h|^2),$$

which is our claim.

6.3. Proof of Lemma 5.5

Setting $\gamma_{21} := \alpha_2 \beta_1$ and $I_{21} := \langle C_2 u_h, B_1 u_h \rangle_{Q_h}$. Let us compute

$$I_{21} = \int_{Q_h} 2\gamma_{21} A_h^2 u_h D_h A_h u_h.$$

By using Lemma 2.1 the above semi-discrete integral can be rewritten as

$$I_{21} = \int_{Q_h} \gamma_{21} D_h ((A_h u_h)^2).$$

A semi-discrete integration by parts with respect to the difference operator yields

$$\begin{aligned} I_{21} &= - \int_{Q_h^*} D_h(\gamma_{21})(A_h u_h)^2 + \int_{\partial Q_h} \gamma_{21} t_r ((A_h u_h)^2) n_h \\ &:= I_{21}^{(a)} + I_{21}^{(b)}. \end{aligned} \quad (6.6)$$

Let us first estimate $I_{21}^{(a)}$. Note that (2.2) leads to

$$I_{21}^{(a)} = - \int_{Q_h^*} D_h(\gamma_{21}) A_h (u_h^2) + \frac{h^2}{4} \int_{Q_h^*} D_h(\gamma_{21}) |D_h u_h|^2.$$

Then, by Proposition 2.4 we obtain

$$I_{21}^{(a)} = - \int_{Q_h} A_h D_h(\gamma_{21}) |u_h|^2 - \frac{h}{2} \int_{\partial Q} t_r(D_h(\gamma_{21})) |u_h|^2 + \frac{h^2}{4} \int_{Q_h^*} D_h(\gamma_{21}) |D_h u_h|^2.$$

Recalling that $\alpha_2 := e^{s\varphi} D_h^2(e^{-s\varphi})$ and $\beta_1 := e^{s\varphi} A_h D_h(e^{-s\varphi})$ we have

$$A_h D_h(\gamma_{21}) = -3s^3 \lambda^4 \varphi^3 (\partial_x \psi)^4 + (s\lambda\varphi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2)$$

and $D_h(\gamma_{21}) = s^3 \mathcal{O}_\lambda(1)$, by virtue of Proposition 4.9 and Corollary 4.1. Hence, for $I_{21}^{(a)}$ we obtain the following estimate

$$\begin{aligned} I_{21}^{(a)} &= 3s^3 \lambda^4 \int_{Q_h} \varphi^3 (\partial_x \psi)^4 |u_h|^2 + \int_{Q_h} (s\lambda\varphi)^3 \mathcal{O}(1) |u_h|^2 - \int_{Q_h} (s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2)) |u_h|^2 \\ &\quad - \int_{Q_h^*} s \mathcal{O}_\lambda((sh)^2) |D_h u_h|^2 - s^2 \int_{\partial Q_h} \mathcal{O}_\lambda(sh) |u_h|^2. \end{aligned} \quad (6.7)$$

On the other hand, $I_{21}^{(b)}$ can be estimated as

$$I_{21}^{(b)} = \int_{\partial Q_h} -(s\lambda\varphi \partial_x \psi)^3 + s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2) t_r(A_h(|u_h|^2)) n_h, \quad (6.8)$$

since $\gamma_{21} = -(s\lambda\varphi \partial_x \psi)^3 + s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2)$, due to Proposition 4.9. Thus, combining (6.6) with (6.7) and (6.8) the Lemma follows.

6.4. Proof of Lemma 5.6

Let $I_{22} := \langle C_2 u_h, B_2 u_h \rangle_{Q_h}$. By definition of C_2 and B_2 , let us estimate the semi-discrete integral

$$I_{22} = -s \int_{Q_h} \alpha_2 \partial_x^2 \varphi A_h^2(u_h) u_h.$$

To this end, by using (2.2), I_{22} can be rewritten as

$$I_{22} = -s \int_{Q_h} \alpha_2 \partial_x^2 \varphi |u_h|^2 + \frac{-sh^2}{4} \int_{Q_h} \alpha_2 \partial_x^2 \varphi u_h D_h^2 u_h := I_{22}^{(a)} + I_{22}^{(b)}.$$

Since $sh \leq 1$, from Proposition 4.5 and Lemma 4.3 we have for $\alpha_2 := e^{s\varphi} D_h^2(e^{-s\varphi})$ the following estimate

$$\alpha_2 = (s\lambda\varphi)^2 (\partial_x \psi)^2 + s \mathcal{O}_\lambda(1) + s^2 \mathcal{O}_\lambda((sh)^2).$$

Furthermore, noting that $\partial_x^2 \varphi = \lambda^2 (\partial_x \psi)^2 \varphi + \lambda \varphi \partial_x^2 \psi$, with the previous estimate for α_2 we obtain

$$\alpha_2 \partial_x^2 \varphi = s^2 \lambda^4 \varphi^3 (\partial_x \psi)^4 + s^2 \lambda^3 \varphi^2 (\partial_x \psi)^2 \partial_x^2 \psi + s \mathcal{O}_\lambda(1) + s^2 \mathcal{O}_\lambda((sh)^2) = s^2 \mathcal{O}_\lambda(1). \quad (6.9)$$

Then, $I_{22}^{(a)}$ is estimated as

$$I_{22}^{(a)} = -s^3 \lambda^4 \int_{Q_h} \varphi^3 (\partial_x \psi)^4 |u_h|^2 - \int_{Q_h} (-s^3 \lambda^3 \varphi^2 (\partial_x \psi)^2 \partial_x^2 \psi + s^2 \mathcal{O}_\lambda(1) + s^3 \mathcal{O}_\lambda((sh)^2)) |u_h|^2. \quad (6.10)$$

Similarly, for $I_{22}^{(b)}$, a semi-discrete integration by parts yields

$$I_{22}^{(b)} = \frac{sh^2}{4} \int_{Q_h^*} D_h(\alpha_2 \partial_x^2 \varphi u_h) D_h u_h - \frac{sh^2}{4} \int_{\partial Q_h} \alpha_2 \partial_x^2 \varphi u_h t_r(D_h u_h) n_h := I_{22}^{(b_1)} - I_{22}^{(b_2)}.$$

Let us estimate $I_{22}^{(b_2)}$. Note that by using (6.9) and Young's inequality, $I_{22}^{(b_2)}$ can be bounded as

$$|I_{22}^{(b_2)}| \leq s \int_{\partial Q_h} |\mathcal{O}_\lambda((sh)^2)| |u_h|^2 n_h + s \int_{\partial Q_h} |\mathcal{O}_\lambda((sh)^2)| t_r(|D_h u_h|^2). \quad (6.11)$$

Now, let us focus on $I_{22}^{(b_1)}$. Using Lemma 2.1 we write $D_h(|u_h|^2) = 2D_h u_h A_h u_h$. Thus, $I_{22}^{(b_1)}$ can be written as

$$I_{22}^{(b_1)} = \frac{sh^2}{8} \int_{Q_h^*} D_h(\alpha_2 \partial_x^2 \varphi) D_h(|u_h|^2) + \frac{sh^2}{4} \int_{Q_h^*} A_h(\alpha_2 \partial_x^2 \varphi) |D_h u_h|^2.$$

We now use a semi-discrete integration by parts on the first integral above to obtain

$$I_{22}^{(b_1)} = -\frac{sh^2}{8} \left(\int_{Q_h} D_h^2(\alpha_2 \partial_x^2 \varphi) |u_h|^2 + \int_{\partial Q_h} |u_h|^2 t_r(D_h(\alpha_2 \partial_x^2 \varphi)) n_h \right) + \frac{sh^2}{4} \int_{Q_h^*} A_h(\alpha_2 \partial_x^2 \varphi) |D_h u_h|^2.$$

To obtain an estimate for $I_{22}^{(b_1)}$ we claim that

$$A_h(\alpha_2 \partial_x^2 \varphi) = s^2 \mathcal{O}_\lambda(1), \quad (6.12)$$

$$D_h^2(\alpha_2 \partial_x^2 \varphi) = s^2 \mathcal{O}_\lambda(1), \quad (6.13)$$

$$D_h(\alpha_2 \partial_x^2 \varphi) = s^2 \mathcal{O}_\lambda(1). \quad (6.14)$$

Indeed, to prove the estimate (6.12) we use Lemma 2.1 to write

$$A_h(\alpha_2 \partial_x^2 \varphi) = A_h(\alpha_2) A_h(\partial_x^2 \varphi) + \frac{h^4}{2} D_h(\alpha_2) D_h(\partial_x^2 \varphi).$$

Then, thanks to Lemma 4.8, we obtain

$$A_h(\alpha_2) = s^2 \mathcal{O}_\lambda(1),$$

$$D_h(\alpha_2) = s^2 \mathcal{O}_\lambda(1).$$

Moreover, using Proposition 4.1 we have

$$A_h(\partial_x^2 \varphi) = \partial_x^2 \varphi + h^2 \mathcal{O}_\lambda(1),$$

$$D_h(\partial_x^2 \varphi) = \partial_x^3 \varphi + h^2 \mathcal{O}_\lambda(1),$$

and since $\partial_x^2 \varphi = \mathcal{O}_\lambda(1)$, (6.12) follows. For the estimate (6.13), applying (2.5) it follows that

$$D_h^2(\alpha_2 \partial_x^2 \varphi) = D_h^2(\alpha_2) A_h^2(\partial_x^2 \varphi) + 2A_h D_h(\alpha_2) A_h D_h(\partial_x^2 \varphi) + D_h^2(\partial_x^2 \varphi) A_h^2(\alpha_2) \quad (6.15)$$

Similarly, by using Lemma 4.8 and Proposition 4.1 we have

$$\begin{aligned} D_h^2(\alpha_2) &= s^2 \mathcal{O}_\lambda(1), \quad A_h D_h(\alpha_2) = s^2 \mathcal{O}_\lambda(1), \quad A_h^2(\alpha_2) = s^2 \mathcal{O}_\lambda(1), \\ A_h^2(\partial_x^2 \varphi) &= \partial_x^2 \varphi + h^2 \mathcal{O}_\lambda(1), \quad A_h D_h(\partial_x^2 \varphi) = \partial_x^3 \varphi + h^2 \mathcal{O}_\lambda(1), \end{aligned}$$

These estimates and (6.15) establishes (6.13). The same methodology works for (6.14).

We thus have, from (6.12)–(6.14), the following estimate for $I_{22}^{(b_1)}$

$$I_{22}^{(b_1)} = -s \int_{Q_h} \mathcal{O}_\lambda(sh) |u_h|^2 + s \int_{Q_h^*} \mathcal{O}_{\lambda, \epsilon}((sh)^2) |D_h u_h|^2 + s \int_{\partial Q_h} \mathcal{O}_\lambda(1) |u_h|^2. \quad (6.16)$$

Therefore, combining (6.10) with (6.16) and (6.11) proves the estimate for I_{22} .

6.5. Proof of Lemma 5.7

We begin proving the first inequality (5.16) of our Lemma. Recalling that $v_h = u_h e^{-s\varphi}$, thanks to Lemma 2.1 and Young's inequality, we have

$$\begin{aligned} \|e^{s\varphi} D_h v_h\|_{L_h^2(Q_h^*)}^2 &\leq \|e^{s\varphi} D_h(u_h) A_h(e^{-s\varphi})\|_{L_h^2(Q_h^*)}^2 + \|e^{s\varphi} D_h(e^{-s\varphi}) A_h(u_h)\|_{L_h^2(Q_h^*)}^2 \\ &:= J_1 + J_2, \end{aligned} \quad (6.17)$$

Let us first estimate J_2 . Using (2.3) and a discrete integration by part respect to the average operator we obtain

$$J_2 = \int_{Q_h} A_h((e^{s\varphi} D_h(e^{-s\varphi}))^2) |u_h|^2 + \frac{h}{2} \int_{\partial Q_h} t_r((e^{s\varphi} D_h(e^{-s\varphi}))^2) |u_h|^2. \quad (6.18)$$

Then, by virtue of Proposition 4.5, J_2 can be estimated as follows

$$J_2 \leq s \int_{Q_h} |u_h|^2 + s \mathcal{O}_\lambda(sh) \int_{Q_h} |u_h|^2. \quad (6.19)$$

It remains to prove that

$$J_1 \leq \mathcal{O}_\lambda(1) \int_{Q_h} |D_h u|^2, \quad (6.20)$$

which it follows from Proposition 4.5, and the proof for (5.16) is complete.

To prove the inequality (5.17), we note that

$$D_h u_h(h/2, t) = D_h(e^{s\varphi}) A_h v_h(h/2, t) + D_h(v_h) A_h(e^{s\varphi})(h/2, t), \quad (6.21)$$

due to Lemma 2.1. Hence, Young's inequality and Proposition 4.5 yield

$$e^{-2s\varphi} |D_h u_h|^2(h/2, t) \leq C_\lambda (s^2 |A_h v_h|^2(h/2, t) + |D_h v_h|^2(h/2, t)), \quad (6.22)$$

which establishes inequality (5.17).

We proceed similarly for (5.18). From (2.2) we have

$$A_h(|u_h|^2) = |A_h u_h|^2 + \frac{h^2}{4} |D_h u_h|^2.$$

Repeated application of Lemma 2.1 and Young's inequality lead to

$$A_h(|u_h|^2) \leq C (A_h(v_h^2)|A_h e^{s\varphi}|^2 + h^4|D_h v_h|^2|D_h e^{s\varphi}|^2 + h^2|D_h v_h|^2|A_h e^{s\varphi}|^2 + h^2|D_h e^{s\varphi}|^2|A_h v|^2).$$

Then, using Proposition 4.5 we obtain

$$e^{-2s\varphi} A_h(u_h^2) \leq (\mathcal{O}_\lambda(1) + \mathcal{O}_\lambda((sh)^2))A_h(|v_h|^2) + (h^2 + h^2\mathcal{O}_\lambda((sh)^2))|D_h v_h|^2, \quad (6.23)$$

which completes the proof.

7. COMMENTS

The results presented in Section 4 are of independent interest in view of its potential applications on problems related to semi-discrete Carleman estimates. For instance, it could be used to answer the challenge proposed by C. Zheng in [17], that is, to obtain a semi-discrete global Carleman estimates for fourth-order Schrödinger equation and establish a semi-discrete counterpart of the main results presented in that paper. Even in the continuous setting, there are few papers about the stability of an inverse problem for higher-order equations, via Carleman estimates, due to tedious computation and the increased complexity. To our knowledge, there are no results about discrete or semi-discrete Carleman estimates for higher-order operators. Thus, Theorem 4.9 can be a useful tool to obtain results in that direction.

A possible extension of this paper could be to reformulate Theorem 1.2 for some families of non-uniform meshes. The Carleman estimate (1.11) is established for uniform mesh and could be adapted to some non-uniform meshes obtained as the smooth image of a uniform grid, following the methodology of [3].

Another interesting question is to consider the fully discrete case of our problem, particularly due to the term $\partial_x^2 \partial_t$, which mixes time and space. Perhaps a first attempt is just consider the time-discrete case

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{\partial_x^2 u^{n+1} - \partial_x^2 u^n}{\Delta t} = p^{n+1} \partial_x u^{n+1} + q^{n+1} u^{n+1}, \quad n = 0, 1, \dots,$$

which is a possible discretization in time.

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