ON THE GEOMETRY OF THE ACTION OF FINITE LINKED GROUPS: Isogenous Jacobian varieties via intermediate coverings.

Tesis
Entregada a la
Universidad de Chile
en cumplimiento parcial de los requisitos
para optar al grado de
Doctora en Ciencias con mención en Matemáticas

Facultad de Ciencias

Por

Estefanía Nicole Bravo Rubio

Marzo, 2022

Directora de Tesis Dra. Anita Maria Rojas Rodriguez

FACULTAD DE CIENCIAS

UNIVERSIDAD DE CHILE

INFORME DE APROBACIÓN

TESIS DE DOCTORADO

Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por la candidata.

Estefanía Nicole Bravo Rubio

Ha sido aprobada por la Comisión de Evaluación de la tesis como requisito para optar al grado de Doctora en Ciencias con mención en Matemáticas, en el examen de Defensa de Tesis rendido el día 30 de Marzo de 2022.

Directora de Tesis:

Dra. Anita Maria Rojas Rodriguez ....

Comisión de Evaluación de la Tesis

Dra. Jennifer Paulhus ....

Dr. Sebastián Reyes-Carocca ....

Dr. Robert Auffarth ....

DEDICATORIA

A mi Porrorrito, a mis Padres, a mi Mimi
y a los que hoy sólo guardo en mi corazón.

"Defiende tu derecho a pensar,
porque incluso pensar de manera errónea
es mejor que no pensar."
Hipatía.

Biografia

Nací el 9 de junio del año 1990 en la ciudad de Rancagua, VI Región de Chile, sin embargo viví toda mi infancia y adolescencia en Graneros, ubicado en la misma región. Desde que tengo memoria fui una persona curiosa y hambrienta de saber. Más aún, durante mi etapa escolar, cada vez que me preguntaban "¿Qué quieres ser cuando grande?" mi respuesta siempre fue "Profesora de Matemáticas", a pesar de que me gustaban mucho otras asignaturas y la mayoría de ellas tuve profesoras y profesores que confiaron en mis capacidades.

La enseñanza básica la cursé en el Colegio Graneros y luego me cambié al Colegio Cuisenaire de Rancagua, lugar en que cursé la enseñanza media. Pese a que el cambio fue un desafío para mí, obtuve durante toda mi escolaridad el primer lugar en calificaciones cada año. Sentía gran pasión cuando enseñaba a mis compañeros de colegio materias que les eran difíciles y mi Profesora de enseñanza media pudo percibir ese fulgor en mí, y me apoyó gestionando talleres de matemáticas en diferentes niveles de educación media del colegio que reafirmaron aún más mi pasión por las matemáticas y la pedagogía.

Cuando egresé de educación media tenía mis dudas respecto a que me gustaba más; si la pedagogía o las matemáticas. Por esa razón ingresé a la Pontificia Universidad Católica de Valparaíso a la carrera Matemáticas, en la cuál fui el primer puntaje de ingreso y también la mejor egresada. Mi elección se debió a que el primer año era en común entre ambas carreras y luego podías decidir por cuál continuar.

Luego del primer año sabía que no podía elegir entre ambas carreras y que quería seguir con ambas, sin embargo los miedos y experiencias de otros compañeros me llenaron de temores y decidí seguir por el camino de la pedagogía. Culminé el segundo año de pregrado de forma exitosa, las asignaturas de geometría abstracta me envolvieron por completo, cada vez que estudiaba sobre estructuras algebraicas sentía que mi mente se expandía. Entonces me armé de coraje, confié en mí y empecé a tomar las dos carreras al mismo tiempo, fue una senda difícil de atravesar, a veces tenía muchos ramos al mismo tiempo, siempre tenía mucho que estudiar y tareas que hacer, pero mi pasión y mi vocación siempre fueron mayor que el cansancio.

Aunque mi experiencia en mi seminario de investigación de Licenciatura no fue motivadora, tuve profesores que me incentivaron a seguir en el camino de la investigación matemática, fue entonces cuando postulé al Magíster en Ciencias Matemáticas de la Universidad de Chile. Desde que llegué a Santiago no sólo conocí un nuevo mundo académico y de investigación, también me abrí espacio dentro del mundo de la divulgación científica, la cual me ha permitido compartir el amor por la Ciencia. Luego de todas las experiencias que involucran el proceso de investigación en mi paso en el Magíster, pude contemplar que esa gran pasión que sentía enseñando matemáticas no sólo estaba en el enseñar, también venía de mi fascinación por la belleza de las matemáticas, y entonces en mi segundo año de Magister decidí cambiarme al Doctorado en Matemáticas.

Este trabajo representa el esfuerzo, coraje y el trabajo de mucho tiempo, pero por sobretodo, un amor incondicional y el punto cúlmine de uno de los capítulos que forma parte de un largo libro que aún sigo escribiendo.

Agradecimientos

Primero quisiera agradecer a mi tutora Anita Rojas por compartir sus conocimientos, por su apoyo y motivación durante todo el postgrado Ha sido una gran fuente de inspiración en muchos ámbitos; me motivó a no sólo ser una mejor matemática, también una mejor divulgadora y por sobre todo, a ser una mujer mucho más valiente. Agradezco a los correctores de este trabajo: Jennifer Paulhus, Sebastián Carocca y Robert Auffarth por su dedicación y paciencia, su ayuda fue de gran valor y sus comentarios e ideas permitieron enriquecer esta investigación.

También quisiera agradecer a Ángel Carocca y Milagros Izquierdo, quienes tuvieron la mejor disposición y cooperaron con valiosas ideas en el proceso de mi investigación. Junto a ellos, quisiera también dar gracias por la calidez con que siempre fui recibida por los miembros del grupo “Geometry at the Frontier” de la Universidad de la Frontera, sobre todo a Rubí Rodriguez, Rubén Hidalgo, Saúl Quispe, Paola Comparin, Mauricio Godoy y a los estudiantes de postgrado quienes siempre me hicieron sentir parte de su familia matemática.

Al departamento de Matemáticas le agradezco por recibirme como miembro y abrir las puertas a un nuevo mundo en la investigación, en especial a Luis Arenas, quien tuvo una paciencia infinita para enseñarme topología y responder todas mis preguntas, y a Giancarlo Lucchini quien me ha motivado a desafiarme en presentaciones matemáticas.

A los miembros del Instituto de Matemáticas de la Pontificia Universidad Católica de Valparaíso, pues cada uno de ellos fueron parte importante de mi formación en pregrado, sobre todo a los profesores Roberto Johnson y Luisa Aburto por su confianza y su absoluto apoyo.

Agradezco a la agencia de investigación ANID por el financiamiento de mi beca de Doctorado Nacional con folio 2017-21171358.

A mis padres porque a pesar de sus aprensiones inherentes por ser hija única, confiaron en mí, en los valores y principios que me han inculcado, por eso y mucho más, gracias por su amor y su apoyo incondicional. A mi amor Ignacio, le doy gracias por cada minuto desde que llegó a mi vida, por ser mi apoyo, mi compañero de trasnoches de estudio y mi mejor amigo, has sido mi piedra angular a lo largo de todo este camino. A la Mimi, por sus palabras de aliento y sus bendiciones a diario, quién me enseñó lo entretenido de hacer tareas y me dejó siempre experimentar y jugar a pesar del desorden. Por supuesto quiero agradecer a los que no están y han formado parte de lo que soy ahora, a mi Mami Tencha y a Carlitos quienes me llenan de angelitos protectores cada día. Gracias también a mis primas, primos, tías y tíos que me han apoyado, que escuchan y admiran mis locuras, y aquellos que siempre me envían amor.

Gracias a mis amigos, a los que están hoy y también a los que estuvieron, pues ellos me han enseñado mucho sobre la vida y se transformaron en familia.

Contents

Notation

symbol	means
$\mathbb{C}$	The set of complex numbers.
$\|G\|$	Order of the group G.
$[G : H]$	Index of the subgroup H in G.
$H \rtimes K$	Semidirect product of H by K ($H \trianglelefteq H \rtimes K$).
$\operatorname{Tr} M$	The Trace of a square matrix M.
$GL(n, \mathbb{C})$	The set of $n \times n$ invertible matrices with entries from $\mathbb{C}$.
S_p	Symmetric group of degree p .
$\mathbb{Q}[G]$	The rational group algebra of group G.
JX_H	Jacobian of the Riemann surface X/H (or $J(X/H)$) .
$\mathcal{M}_g$	Conformal equivalence class space of Riemann surfaces of a genus g.
$\mathcal{A}_g$	Moduli space for Abelian variety of genus g.
$GCD(x, y)$	Greatest common divisor between x and y.

Abstract

Let G be a finite group acting on a compact Riemann surface X. This action induces the so called *group algebra decomposition* of the corresponding Jacobian variety JX. Moreover, consider a subgroup $H \leq G$ of G and the intermediate quotient X/H arising from this action restricted to H. The group algebra decomposition of JX determines a decomposition of the Jacobian variety $J(X/H)$ of X/H.

In this work, we prove a condition under which two intermediate quotients, X/H and X/K for $H, K \leq G$, correspond to isogenous Jacobian varieties. The condition is that they induce the same permutation character, a concept that has been widely studied in the context of Representation Theory, where it is said that H and K are *linked subgroups in* G.

For every (odd) prime $p \geq 3$, we study a family of groups $G_p \cong (\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ having two linked subgroups which are not conjugate. We describe their elements, irreducible complex (and rational) representations, different signatures for their actions on Riemann surfaces, and the corresponding impact on the group algebra decomposition of the associated Jacobian varieties.

Resumen

Sea G un grupo finito actuando en una superficie de Riemann compacta X. Esta acción induce la llamada *descomposición según el álgebra de grupo* de la variedad Jacobiana JX correspondiente a X. Más aún, considere $H \leq G$ subgrupo de G y la superficie cuociente (intermedia) X/H determinada por la acción restringida a H. La descomposición de JX determina una descomposición de la Jacobiana de X/H, $J(X/H)$.

En este trabajo demostramos una condición bajo la cual las variedades Jacobianas de dos cubrientes intermedios, X/H y X/K para $H, K \leq G$, son isógenas. Esta condición es que H y K inducen la misma representación permutacional. Ello ha sido ampliamente estudiado en el contexto de Teoría de Representaciones, donde se dice que *H y K son subgroups ligados en G*.

Para todo primo (impar) $p \leq 3$, estudiamos una familia de grupos $G_p \cong (\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ que tienen dos subgrupos ligados no conjugados. Describimos sus elementos, caracteres irreducibles complejos (y racionales), diferentes firmas y acciones, y las consecuencias en la descomposición de las variedades Jacobianas asociadas.

Introduction

The classification of objects is essential in Mathematics. The question of whether or not two objects are equivalent appears in every theory as soon as the objects of study are defined. Frequently, there is more than one possible definition of equivalence that depends on the properties we are interested in.

In Field Theory, we find the concept of *arithmetically equivalent fields* and *isomorphic fields*. Isomorphic fields are arithmetically equivalent but the reciprocal is not true. In fact, a field K that is isomorphic to every field to which it is arithmetically equivalent is said to be *arithmetically solitary*. In 1925, Gassmann [14, 31] discovered the first non-solitary fields: He proved the existence of two fields K, K' of degree $[K : \mathbb{Q}] = [K' : \mathbb{Q}] = 180$ which are arithmetically equivalent but not isomorphic.

Now, turning to the context of Riemannian Geometry, in 1985, a work of Sunada [40] was published where he brought these ideas, that were used in the context of number fields, to Riemannian manifolds. Specifically, he sought to study the question of the existence of isospectral but not isometric Riemannian manifolds. Roughly speaking, he replaced *field extensions* by *Riemannian coverings* to use the parallel between Galois theory for covering spaces and field extensions. We consider his work as a door to the idea of using all these results living in the world of algebra to our field: Riemann surfaces and Abelian varieties.

Let us now go to the area of Riemann surfaces. Because of the Torelli Theorem, isomorphic Riemann surfaces have isomorphic polarized Jacobian varieties. One question, of the same flavor as the ones discussed above and that has interest in the context of Riemann surfaces and Abelian varieties, is the existence of non-isomorphic Riemann surfaces with isomorphic Jacobian varieties (not considering the polarization). In other words, the question deals with the existence of non-isomorphic Riemann surfaces whose Jacobian varieties are isomorphic as complex tori. This is the deep question that motivates our work.

Ciliberto and Van der Geer [11] in 1994 constructed non-isomorphic Riemann surfaces of genus 4 with isomorphic Jacobians, as non-polarized Abelian varieties.

Howe [21] in 1996 constructed non-isomorphic curves over finite fields with isomorphic Jacobians. Later [22], in 2000, he moves to complex curves and constructs n distinct plane quartics and one hyperelliptic curve all of whose Jacobians are isomorphic to one another as complex tori (i.e. as Abelian varieties without considering the polarization).

These works [11, 22] are intricately connected to the question of understanding Abelian varieties with several principal polarizations, treated for instance in Lange's work [27] from 1987. In these two papers examples are provided where two non-isomorphic polarized Jacobians which are isomorphic only as complex tori correspond to two (different) principal polarizations on the same Abelian variety.

Let us return to non-solitary fields. Perlis [31] in 1977 investigated the phenomenon of non-solitary fields more closely. In fact, he established a connection between group and field theory. He gave the definition of *Gassmann equivalence* for subgroups of a group G, relating it with Dedekind zeta functions of fields, and hence with arithmetically equivalent fields. He uses this property to construct examples of pairs of non-isomorphic and arithmetically equivalent number fields.

Two subgroups H, H' of a finite group G are called *Gassmann equivalent* if every $x \in G$ satisfies $\left|x^G \cap H\right| = \left|x^G \cap H'\right|$, where x^G denotes the conjugacy class of x in G. Notice that if H and H' are Gassmann equivalent, then $[G : H] = [G : H']$.

The relation that Perlis established between Gassmann equivalent subgroups and arithmetically equivalent fields is as follows:

Consider two number fields K, K' with common Galois extension N. Let G be the Galois group of N over $\mathbb{Q}$ and H, H' subgroups of G such that they correspond to the Galois group of K, K' over $\mathbb{Q}$ respectively. Then, K and K' are arithmetically equivalent if and only if H and H' are Gassmann equivalent as subgroups of G. In the case that H is not conjugate to H', the fields K and K' are not isomorphic.

In 1980, Feit [13], in a work where he was studying consequences of the classification of finite simple groups, found what he called "an apparently unrelated result". He proved that if two number fields K, K' are arithmetically equivalent non-isomorphic number fields of degree p (prime), then p has certain restrictions. For this, he used that two arithmetically equivalent number fields have the same Galois closure F (in some algebraic closure) and the permutation representations of the Galois group G of F on the cosets of the subgroups H, H' of G corresponding to the fields K, K', afford the same character.

As a result, a relation among Gassmann equivalence (Group Theory), arithmetically equivalent (Field Theory) and equal characters (Representation Theory) came into play.

Guralnick [18] took this idea in 1983 and showed the equivalence between Gassmann equivalence and the following property:

Two subgroups H, H' of a finite group G are Gassmann equivalent if and only if H and H' induce the same permutational character of G (which corresponds to the action of this group on the cosets).

Since conjugate subgroups have the same permutational character, Guralnick was interested in the case when H and H' are not conjugate. He constructed explicit groups having such subgroups for indices p^2 and p^3, for p prime.

In 1985, Guralnick and Wales [19] studied groups G with subgroups H, H' of index $[G : H] = [G : H'] = pq$, with p, q different primes, with the same permutational character and found conditions on the primes. Moreover, they addressed the general situation on the index of the subgroups, and they proved that for $n \leq 40$ but $n \neq 18$ there are groups containing non-conjugate subgroups H, H' of index n with the same permutational character if and only if $n = 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 23, 25, 29, 37, 38$. As far as we know, it is still unknown whether there are groups with such subgroups of index $n = 18$.

Years later, the property of H and H' being Gassmann equivalent in G was captured and defined in terms of representations. Caranti et. al. [7] in 1994 gave the following definition:

If G is a finite group, two subgroups H, H' of G are *linked in* G if and only if the character of $\mathrm{Ind}_H^G(1_H)$ is equal to the character of $\mathrm{Ind}_{H'}^G(1_{H'})$, where $\mathrm{Ind}_H^G(1_H)$ stands for the induced representation of the trivial representation of the subgroup H.

Subgroups that are conjugate are trivially linked, but there are linked subgroups that are not even isomorphic [7].

In 1997, Gavioli [15] found necessary and sufficient conditions so that given H and H' soluble finite groups, there exists a soluble finite group G such that H and H' are linked in G.

Therefore, and as shown in the preceding discussion, in the context of Group and Field Theory, as well as in Riemannian Geometry, finding nontrivial linked subgroups (that is, linked subgroups that are not conjugate) is of interest for their applications in studying questions about classification, such as finding non-isomorphic arithmetically equivalent fields, as well as non-isometric isospectral Riemannian manifolds. In this work, we propose to use the theory of linked subgroups to construct non-isomorphic Riemann surfaces with isomorphic (without polarization) Jacobian varieties. We proved the following Theorem:

Theorem (Theorem 3, Chapter 2) Let X be a compact Riemann surface with the action of a finite group G. If H and H' are linked groups in G, then the Jacobian varieties JX_H and $JX_{H'}$ corresponding to the intermediate quotients X/H and X/H' are isogenous.

This is a first step in answering the general question of finding non-isomorphic Riemann surfaces with isomorphic Jacobians, since an isogeny between Abelian varieties is a surjective homomorphism with finite kernel. Therefore, to ask for non-isomorphic Riemann surfaces with isogenous Jacobian varieties is a weaker question.

For doing this, we use the theory of group actions on Abelian varieties [28, 8, 33], and particularly on Jacobian varieties [35]. It is known that this theory has been a fruitful ground for understanding aspects of the geometry of moduli spaces of Abelian varieties. The same happens with using group actions to study the moduli space of compact Riemann surfaces [1, 10]. A further understanding of both topics is achieved when combining viewpoints [26, 24, 29, 35].

We recall that when a group G acts on an Abelian variety A, it induces a morphism

$$\rho : \mathbb{Q}[G] \to End_{\mathbb{Q}}(A)$$

from the rational group algebra $\mathbb{Q}[G]$ to the endomorphism algebra $End_{\mathbb{Q}}(A) = End(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of A. This morphism allows us to carry the decompositions of $\mathbb{Q}[G]$ to A.

The decomposition of A corresponding to the decomposition of $\mathbb{Q}[G]$ into a product of simple algebras is called *the isotypical decomposition*, and each factor is called an *isotypical factor*.

Since each simple algebra is decomposed into a product of minimal (left) ideals, there is a finer decomposition of $\mathbb{Q}[G]$ as a product of minimal ideals. The corresponding decomposition of A induced by this one is called the *group algebra decomposition*, and each factor is called a *primitive factor*.

In particular, when a group G acts on a (compact) Riemann surface S, there is an action of G on the corresponding Jacobian variety JS of S. Therefore, JS is decomposed in these two ways as a consequence. We point out that the isotypical decomposition is unique, so the isotypical factors are uniquely defined. But that is not the case of the group algebra decomposition, where there are several sets of primitive factors decomposing the Abelian variety. While the dimension of the factors will remain fixed regardless of these choices, their induced polarization and the kernel of the isogeny can change.

Notice that, the primitive factors can be simple or not, depending on A

and on the action of G. Moreover, they are not, in general, principally polarized. A secondary fact that we want to point out, although it is not related to what we are presenting here, is that the group algebra decomposition does not coincide, in general, with the Poincaré decomposition of the variety.

There are several tools [8, 28, 33, 35, 25] that can be used to study the geometry of these decompositions (dimension of the factors, induced polarizations, kernel, etc). We use here, in order to relate the knowledge that exists about linked subgroups with these decompositions, the results in [8], where this bridge between algebra and geometry is deepened by constructing idempotents in the group algebra that describe the primitive factors that decompose the Jacobian varieties corresponding to Riemann surfaces arising from taking intermediate coverings. This is, Jacobians of Riemann surfaces S/H where S is a Riemann surface with the action of a group G and $H \leq G$.

As said, this thesis is a first step towards merging the geometric context associated to decompositions of Jacobian varieties and the algebraic condition of linked subgroups that is already quite developed in the context of Group and Galois Theory. We expect that some questions in complex geometry that are currently being studied, such as the already mentioned question about several principal polarizations on Jacobian varieties or non-isomorphic Riemann surfaces with isomorphic Jacobians (as tori), can be tackled by combining the known results about Gassmann equivalence, linked subgroups, their applications to Riemannian manifolds and Field Theory, as well as group actions on varieties and the associated decompositions.

We work with a family of groups (depending on a prime $p \neq 2$) proposed in [18], and also studied in [15]. Each group G_p in the family has linked subgroups H and K, we prove that the minimal dimension of a family of Riemann surfaces whose elements have G_p action is one (see Section 2.3, Chapter 2). We study the group algebra decomposition of the corresponding Jacobian varieties for one of these families. Moreover, we study new families of larger dimension of this family by extending the signature in a suitable way, and study the consequences.

One last remark, according to Singerman's work [39], if a family of compact Riemann surfaces with the action of a (finite) group G has dimension greater than 4, then G is the full automorphism group of the general element in the family. In our case, G_p is a p-group, therefore it has no involutions. A direct consequence of this is that generically the elements in our families are not hyperelliptic curves. We point out that some of these families could contain a subfamily where the general element has more automorphisms, besides from those in G, but this subfamily has to be lower dimensional.

Our framework has two parts.

- Algebraic objects: For each prime $p \neq 2$, we present the group G_p, and the subgroups H, K linked in G_p. We use:

  $G_p = A \rtimes H$, where $A = \langle a, b \rangle \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ with a of order p^2, b of order p, and $H = \langle x, y \rangle \leqslant \operatorname{Aut} A$, where the semidirect product is given by $a^x = ab$, $b^x = ba^p$, $a^y = a^{p+1}$, and $b^y = b$.

  Since p is odd, $H \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Let $K = \langle x, ya^p \rangle$. According to [18], $1_H^G = 1_K^G$, where 1_H^G stands for permutation representation on the cosets of the subgroup H. Besides, both subgroups, H and K, are not conjugate. This is, H, K are linked in G_p for all $p \neq 2$ a prime number.

- Geometric objects: We consider a collection of families of Riemann surfaces that depend on four discrete parameters $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$ of genus

  $$(t_1 + t_2 + t_3 + t_4 + 1)p^5 - (t_3 + t_4 + 1)p^4 - p^3(t_1 + t_2 + 1) + 1$$

  which each admit the action of G_p with *extended signature*

  $$(0; (p^2)^{2t_1+1}, (p^2)^{2t_2+1}, (p)^{2t_3+1}, (p)^{2t_4+1}),$$

  and *extended generating vector*

  $$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}],$$

  where $(\alpha, \beta)^t$ means $\alpha, \beta, \overset{t}{\ldots}, \alpha, \beta$.

  These are generating vectors corresponding to the obvious extension of the generating vector

  $$\left(a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}\right),$$

  which corresponds to the signature $(0; p^2, p^2, p, p)$ determined by the tuple $(t_1, t_2, t_3, t_4) = (0, 0, 0, 0)$. This signature captures a one-dimensional family of Riemann surfaces of genus

  $$g = p^5 - p^4 - p^3 + 1$$

  admitting the action of G_p.

  Notice that 4 is the smallest length that a generating vector for G_p can have, since $\{a, x, y\}$ is a minimal generating set of the group G_p.

The thesis is divided into the following chapters:

1. Preliminaries: Here we present definitions and results related to decompositions of Abelian varieties.

2. A family of groups and linked groups: In this chapter we develop the necessary algebraic aspects of G_p, such as complex irreducible representations, rational irreducible representations, and actions on Riemann surfaces, in order to describe the consequences of its action on Riemann surfaces, and on the corresponding Jacobian varieties.

3. Decomposition of Jacobian varieties associated with linked groups in G_p: In this chapter we analyze the isotypical and group algebra decompositions of the Jacobian varieties with the action of G_p that comes from its action on the previously described Riemann surfaces.

4. Jacobians of intermediate quotients by linked subgroups: In this chapter we study decompositions of the Jacobians associated to quotients by the linked subgroups H and K in G_p.

5. Action of the group G_3 on Riemann surfaces and Jacobian varieties: In this chapter we specialize the results obtained in Chapters 3 and 4 to the case $p = 3$. Historically, this chapter was the starting point of this research.

Chapter 1

Preliminary Results

This chapter introduces the content necessary for the presentation and development of the results in this research work.

1.1 Abelian Variety

A complex torus $T = V/L$ of dimension g is the quotient of a complex vector space V of dimension g by a lattice L (a discrete subgroup of maximal rank $2g$) in V. Thus T is a compact complex manifold (of dimension g) and a commutative complex Lie group, and the natural quotient map $p : V \to T$ is holomorphic. Conversely, any connected compact complex Lie group of dimension g is a complex torus of dimension g.

We will be mainly interested in complex tori that are also projective varieties; these correspond to complex tori that possess sufficiently many meromorphic functions, the so-called abelian varieties.

A polarization (or a Riemann form) on a torus $T = V/L$ is a non-degenerate real alternating form E on V such that

$$E(\imath u, \imath v) = E(u, v),$$

for all u, v in V, and $E(L \times L) \subseteq \mathbb{Z}$; here $\imath$ denotes a complex number with $\imath^2 = -1$. A polarized abelian variety $\mathcal{A} = (T, E)$ of dimension g is a pair consisting of a complex torus $T = V/L$ of dimension g and a polarization E on T. An abelian variety is a complex torus that admits a polarization.

1.2 Decomposition of an Abelian Variety with group action

Let G be a finite group and let F be a field. A representation of G (or a representation of G over F, or an F-representation of G) is a group homomorphism $\rho : G \to GL(V)$ where V is a $F-$vector space. The degree of ρ is the dimension $\dim\,(V)$ of V. We also say that G acts linearly on V, and that V is a $G-$vector space.

A matrix representation of G over F is a group homomorphism

$$R : G \longrightarrow GL_n(F)$$

for a certain $n \in \mathbb{N}$, called the degree of R.

A representation $\rho : G \to GL(V)$ gives rise to a left action of G on V:

$$\begin{aligned} \cdot : G \times V &\to V \\ (g, v) &\mapsto g \cdot v = \rho(g) \cdot v = \rho(g)(v), \end{aligned}$$

such that for all $g \in G$, for all $x, y \in$V and for all $\lambda \in F$:

(i) $g \cdot (x + y) = g \cdot x + g \cdot y$;

(ii) $g \cdot (\lambda x) = \lambda(g \cdot x)$.

Conversely an action $\cdot : G \times V \to V$ satisfying (i) and (ii) gives rise to a representation

$$\begin{array}{rccll} \rho : & G & \longrightarrow & GL(V) & \\ & g & \longmapsto & \rho(g) : & V \to V \\ & & & & v \mapsto p(g). \end{array}$$

If V is a $G-$vector space with corresponding representation ρ, then

(a) $V' \leqslant V$ is called a $G-$invariant subspace of V if and only if $g \cdot V' := \rho(g)\,(V') \subseteq V'$ for all $g \in G$ (in fact then $\rho(g)\,(V') = V'$ since $\rho(g)$ is bijective).

(b) If there exists a $G-$invariant subspace $0 < V' < V$, then ρ is called reducible; else irreducible.

We denote by $\mathrm{Irr}_F(G)$ the set of irreducible representations of G over F. We identify the representation $\rho : G \to GL(V)$ with the underlying $G-$vector space V, let $V \in \mathrm{Irr}_{\mathbb{C}}(G)$, $K_V = \mathbb{Q}(\chi_{V(g)}, g \in G)$ is called the character field of V and L_V the field of definition of V. Then $K_V \subseteq L_V$ and

$$m_V := m_{\mathbb{Q}}(V) = [L_V : K_V]$$

is the Schur index of the complex representation V.

If $\mathrm{Gal}(L_V/\mathbb{Q}), \mathrm{Gal}(L_V/K_V)$ and $\mathrm{Gal}(K_V/\mathbb{Q})$ denote the respective Galois groups, then for $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ we define the set

$$G(V) := \{V^\sigma : \sigma \in \mathrm{Gal}(L_V/\mathbb{Q})\}\,,$$

where each representation V^σ is a conjugate of V by an element σ in $\mathrm{Gal}(L_V/\mathbb{Q})$. That is, if for $g \in G$, $V(g) := [a_{ij}]_{ij}$ is a matrix in a chosen basis of V, then $V^\sigma(g) = [\sigma(a_{ij})]_{ij}$. Let us observe that V^σ is also defined over L_V and both V and V^σ share the same character field.

Moreover, we obtain the rational irreducible representation W of G associated to V by [[12, Thm 70.15]] with the following expression

$$W \otimes_{\mathbb{Q}} L_V \simeq \bigoplus_{\sigma \in \mathrm{Gal}(L_V/\mathbb{Q})} V^\sigma := \bigoplus_{\sigma \in \mathrm{Gal}(K_V/\mathbb{Q})} (m_V V)^\sigma\,. \tag{1.1}$$

Any V is this sum it is called a complex irreducible representation asociated to W and V^σ is called a G Galois-conjugate to V, where $G \in \mathrm{Gal}(L_V/\mathbb{Q})$. In this way one finds the set of the rational irreducible representations of G up to equivalence which we denote

$$\mathrm{Irr}_{\mathbb{Q}}(G) = \{W_1, \ldots, W_r\}\,.$$

Let $\mathcal{A}$ be an abelian variety over the field $\mathbb{C}$ of dimension g with a faithful action by G. That is, there is a monomorphism from G to $\mathrm{Aut}(\mathcal{A})$. We say in this case that $\mathcal{A}$ is a G-abelian variety. This action induces a homomorphism of semisimple $\mathbb{Q}$-algebras

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}).$$

Each element $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety

$$\mathcal{A}^\alpha := \mathrm{im}(\tau\alpha) \subseteq \mathcal{A},$$

where τ is some positive integer such that $\rho(\alpha) \in \mathrm{End}(\mathcal{A})$ or equivalently $\tau\alpha \in \mathbb{Z}[G]$. This definition is independent on the chosen integer τ, up to isogeny.

Since $\mathbb{Q}[G]$ is a semisimple $\mathbb{Q}$-algebra of finite dimension, it admits a unique decomposition as a product of simple Q-algebras

$$\mathbb{Q}[G] = Q_1 \times \ldots \times Q_r\,.$$

The factors Q_i, with $i \in \{1, \dots, r\}$, are uniquely determined by central idempotents $e_i \in \mathbb{Q}[G]$ such that $1 = e_1 + e_2 + \dots + e_r$ and $e_i \in Q_i \, \forall i \in \{1, \dots, r\}$. Furthermore, these are described by [12, Thm. 33.8]

$$e_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \mathrm{tr}_{K_{V_i}/\mathbb{Q}}(\chi_i(g^{-1}))g \, , \tag{1.2}$$

where V_i is a complex irreducible representation of G associated to W_i and K_{V_i}.

The idempotent e_i defines an abelian subvariety $\mathcal{A}_{W_i} := \mathcal{A}^{e_i}$. These are called the isotypical components and they are uniquely determined by W_i. Hence there is an isogeny

$$\mu : \mathcal{A}_{W_1} \times \dots \times \mathcal{A}_{W_r} \to \mathcal{A} \, , \tag{1.3}$$

given by the addition. This is called the **isotypical decomposition** of $\mathcal{A}$.

Furthermore, the isotypical components $\mathcal{A}_{Wi}$ decompose further. There are sets of primitive idempotents $\{q_{i1}, \dots, q_{in_i}\}$ in $Q_i \subseteq \mathbb{Q}[G]$ such that

$$e_i = q_{i1} + \dots + q_{in_i} \, ,$$

where $n_i = \dfrac{\dim V_i}{m_{V_i}}$. The idempotents q_{ij} define subvarieties of A_{W_i}, which are called primitive factors defined by $B_{ij} := \mathcal{A}^{q_{ij}}$. Therefore we have the following isogenies for $i = 1, \dots, r$

$$\nu_i : B_{i1} \times \dots \times B_{in_i} \to \mathcal{A}_{W_i}. \tag{1.4}$$

From (1.3) and (1.4) we obtain the isogeny

$$\nu : \prod_{j=1}^{n_1} B_{1j} \times \dots \times \prod_{j=1}^{n_r} B_{rj} \to \mathcal{A}, \tag{1.5}$$

which is called the **group algebra decomposition** and its components are called primitive factors. It is important to point out that this descomposition is no longer unique, since it depends on the choice of q_{ij} for a fixed i.

Additionally, the abelian subvarieties B_{ij} are mutually isogenous for a fixed i and for all $j = 1, \dots, n_i$. If B_{W_i} denotes one of them, then it results in an isogeny $B_{W_i}^{n_i} \to \mathcal{A}_{W_i}$, so by replacing the factors we obtain

$$\hat{\nu} : B_{W_1}^{n_1} \times \dots \times B_{W_r}^{n_r} \to \mathcal{A}. \tag{1.6}$$

The isogeny (1.6) is the classic way of writing the group algebra decomposition, which is equivalent to (1.5) but given the objectives to be developed here, the first expression is more useful for the analysis of the isogenies.
More details may be found in for instance [25, 28, 33, 34, 35].

1.3 Jacobians with group action.

A branched covering $f : X \to Y$, between Riemann surfaces X and Y, is by definition a surjective holomorphic map (in particular, nonconstant). We say that the covering f is Galois if there exists a subgroup G of the group of automorphisms of X such that $Y = X/G := X_G$ and such that f is the canonical projection. For each g in G we denote by the same symbol g the automorphism induced by g on JX, by $< g >$ the subgroup of G generated by g, and by JX^N the set of fixed points of N in JX, for each subgroup N of G.

Let V be a complex representation of G, and let H be a subgroup of G. Then $\mathrm{Ind}_H^G(1_H)$ denote the representation of G induced by the trivial representation of H. It follows from Frobenius reciprocity Theorem (see [37], Ch. 7-Theo. 13) that

$$\dim_{\mathbb{C}} V^H = \left\langle \mathrm{Ind}_H^G(1_H), V \right\rangle_G , \tag{1.7}$$

where V^H is the subspace of V fixed under H and $\langle \cdot, \cdot \rangle_G$ denotes the usual inner product between characters of the representations of G.

Each compact Riemann surface X of genus g, has associated with it a principally polarized abelian variety JX, that is, a complex torus with a principal polarization. This variety is called the Jacobian variety of X of complex dimension g. If G acts on X, then it acts on JX and the corresponding group algebra decomposition is given by

$$JX \sim JX_G \times B_{W_2}^{n_1} \times \ldots \times B_{W_r}^{n_r} , \tag{1.8}$$

where W_i and n_i are as in the (1.6). Without loss of generality, we assume that W_1 is the trivial representation, and $JX_G = J(X/G)$ is the isotypical component associated to W_1 and corresponds to the Jacobian variety of the total quotient X_G.

For the case of a G-action on the Jacobian variety JX, we get even more information on intermediate geometric components. From [8] we get the following results

Theorem 1. *([8, Thm. 4.4]) Let W_i be a rational irreducible representation of a group G, and denote by e_i the associated central idempotent in $\mathbb{Q}[G]$. We denote by V_i a complex irreducible representations of G associated to W_i. For any subgroup H of G, let*

$$p_H = \frac{1}{|H|} \sum_{h \in H} h \tag{1.9}$$

be the central idempotent in $\mathbb{Q}[H]$ corresponding to the trivial representation of H. Then

$$f_H^i := p_H e_i = e_i p_H \tag{1.10}$$

is an element of the simple algebra $\mathbb{Q}[G]e_i$ satisfying the following conditions.

(1) ${f_H^i}^2 = f_H^i$,

(2) $hf_H^i = f_H^i = f_H^i h$ *for every* $h \in H$, *and*

(3) $f_H^i = 0$ *if and only if* $\dim V_i^H = 0$.

Furthermore, in the case $f_H^i \neq 0$, the left ideal $\mathbb{Q}[G]f_H^i$ generated by the idempotent f_H^i is a left $\mathbb{Q}[G]$-module affording the representation W_i with multiplicity given by $\dfrac{\dim V_i^H}{m_{V_i}}$.

Theorem 2. *([8, Prop. 5.2]) Given a Galois cover $X \to X_G$, consider the associated isotypical decomposition* (1.8) *of JX.*
Let H be a subgroup of G and denote by $\pi_H : X \to X_H$ the corresponding quotient map. Then the corresponding isotypical decomposition of JX_H is given as follows:

$$JX_H \sim JX_G \times B_{W_2}^{\frac{\dim V_2^H}{m_{V_2}}} \times \ldots \times B_{W_r}^{\frac{\dim V_r^H}{m_{V_r}}}. \tag{1.11}$$

Furthermore, considering p_H and f_H^i as in Theorem 1, we have

(1) If $\pi_H^*(JX_H)$ is the pull-back of JX_H by π_H, then $\mathrm{im}(p_H) = \pi_H^*(JX_H)$.

(2) If $\dim V_i^H \neq 0$ *then* $\mathrm{im}(f_H^i) = B_{W_i}^{\frac{\dim V_i^H}{m_i}}$.

The previous propositions allow us to determine the decomposition of the Jacobian varieties of intermediate coverings. Further information about the decomposition, such as the dimension of each factor B_{W_i} depends on the geometry of the action.

1.3.1 Signature of actions

Let $f : X \to Y$ be a branched covering between Riemann surfaces X and Y, a point in X is a branch point for f if f fails to be locally one-to-one in there. The image of a branch point is a branch value of f. Let B be the set of branch values of f. For $q \in B$ consider its fiber $f^{-1}(q) = \{p_1, \ldots, p_s\} \subset X$.

Then the cycle structure of f at q is the s-tuple $(n_1, \ldots, n_s)$ where n_j is the ramification index of f at p_j. That is, f is n_j-to-1 at $p_j, n_j > 1$.

Let X be compact Riemann surface with G a group of automorphisms of G and $\{p_1, \ldots, p_t\} \subset X$ be a maximal collection of non-equivalent branch points with respect to action of G (i.e. the p_j are in different G-orbits). For each $j = 1, \ldots, t$, consider the stabilizer G_j of p_j. The **signature of G on X** (see [39]) for the cover $\pi_G : X \to X_G$ is the tuple $(\gamma; m_1, \ldots, m_t)$, where γ is the genus of X_G and $m_j = |G_j|$ for each j.

On the other hand, let G_j be a (non-trivial) cyclic subgroup of G, a branch value $q \in X_G$ is called of type G_j, if G_j is the stabilizer of at least one point in the fiber of q. If there is a point $p \in X$ with non-trivial stabilizer G_p, then the points in its orbit have stabilizers running through the complete conjugacy class of G_p. Hence we will call $q \in X_G$ of type C_j if the stabilizer of the points in its fiber are the elements of the (complete) conjugacy class C_j of G_j. For the computations developed in the following sections, it is not critical to know all the conjugacy classes of cyclic subgroups of G. The type of the branch values can be given by a cyclic subgroup G_j instead of a conjugacy class. As above, let X be a compact Riemann surface and G a group of automorphisms of X. Let $\{q_1, \ldots, q_t\} \subset X_G$ be a maximal collection of branch values for the covering $\pi_G : X \to X_G$. We define the **geometric signature of G on X** (see [35]) as the tuple $(\gamma; [m_1, C_1], \ldots, [m_t, C_t])$, where γ is the genus of X_G, C_j is the type of the branch value q_j and m_j is the order of any subgroup in C_j.

Let us consider G as before, following ([5], Def. 2.2), we call a $(2\gamma + t)-$tuple

$$(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_t) \in G^{2\gamma+t}$$

of elements of G a **generating vector** of type $(\gamma; m_1, \ldots, m_t)$ if the following conditions are satisfied:

(i) G is generated by the elements $\{a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_t\}$,

(ii) $\operatorname{order}(c_j) = m_j$;

(iii) $\prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{t} c_j = 1$, where $[a_i, b_i] = \left(a_i \cdot b_i \cdot a_i^{-1} \cdot b_i^{-1}\right)$.

Broughton in [5] gives a precise modern treatment of **Riemann's Existence Theorem** which is a fundamental result which translates the problem of constructing group actions on Riemann surfaces to a problem in finite group theory. Later, in [35], this result is extended to include conditions for the geometric signature. This is, to describe the action by giving the stabilizers of the ramification points, not just the signature.

Proposition 1. *(see [35], Thm. 4.1) Given a finite group G, there is a compact Riemann surface S of genus g on which G acts with geometric signature $(\gamma; [m_1, C_1], \ldots, [m_t, C_t])$ if and only if the following three conditions are met:*

(i) (Riemann-Hurwitz)

$$g = |G|(\gamma - 1) + 1 + \frac{|G|}{2} \sum_{j=1}^{t} \left(1 - \frac{1}{m_j}\right). \tag{1.12}$$

(ii) The group G has a generating vector $(a_1, b_1, \ldots, a_\gamma, b_\gamma, c_1, \ldots, c_t)$ of type $(\gamma; m_1, \ldots, m_t)$.

(iii) The elements $c_1, \ldots, c_t$ of the generating vector are such that the subgroup generated by c_j is in the conjugacy class C_j, $j = 1, \ldots, t$.

The Riemann existence theorem is also described in [38] in terms of Fuchsian groups: A Fuchsian group Γ is a finitely generated discrete subgroup of $\mathrm{PSL}(2, R)$, the group of conformal homeomorphisms of the upper-half plane $\mathbb{H}$.
The most general presentation for Γ is

Generators:	$a_1, b_1, \ldots, a_g, b_\theta$	(Hyperbolic) .
	$x_1, x_2 \ldots, x_r$	(Elliptic).
	$p_1, \ldots, p_s$	(Parabolic).
	$h_1, \ldots, h_{\mathrm{t}}$	(Hyperbolic boundary elements).

Relations:

$$x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1.$$

We then say Γ has signature

$$(g; m_1, m_2, \ldots, m_r; s; t) \tag{1.13}$$

The integers $m_1, m_2, \ldots, m_r$ are called the periods of Γ.

If the orbit surface $\mathbb{H}/\Gamma$ is compact of genus γ, then the algebraic structure of Γ is determined by its signature; namely, the tuple $\sigma = (\gamma; m_1, ..., m_r)$ where the m_j are the branch indices in the associated universal projection $\mathbb{H} \to \mathbb{H}/\Gamma$. If $r = 0$, then it is said that Γ is a surface Fuchsian group. Define
$M(\Gamma) = 2\gamma - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right)$

Let Γ_1 be a group of automorphisms of $\mathbb{H}$. If Γ_1 is a subgroup of Γ of finite index then Γ_1 is also Fuchsian.

Let n be a period of Γ_1. Then n is the order of an elliptic element $y \in \Gamma_1$, and y will be a power of a conjugate of one the generators $x_j \in \Gamma$ of order m_j. We shall then say that n has been induced by m_j. Then, we have the following theorem:

Theorem 3. *(see [38],Thm.1) Let* Γ *have signature* (1.13). *Then* Γ *contains a subgroup* Γ_1 *of index* N *with signature*

$$(g'; n_{11}, n_{12}, \ldots, n_{1\rho_1}, \ldots, n_{r1}, n_{r2}, \ldots, n_{r p_r}; s'; t')$$

if and only if

(a) There exists a finite transitive permutation group G *on* N *points, and an epimorphism* $\theta : \Gamma \to G$ *satisfying the following conditions:*

(i) The permutation $\theta\left(x_j\right)$ *has precisely* ρ_j *cycles of lengths less than* m_j*, the lengths of these cycles being* $m_j/n_{j1}, \ldots, m_j/n_{j\rho_j}$*.*

(ii) If we denote the number of cycles in the permutation $\theta(\gamma)$ *by* $\delta(\gamma)$ *then*

$$s' = \sum_{k=1}^{s} \delta\left(p_k\right), t' = \sum_{1=1}^{t} \delta\left(h_l\right).$$

(b) $M\left(\Gamma_1\right)/M(\Gamma) = N$

Definition 1. (See [6]) We say that the finite group G acts on genus g if G is (isomorphic to) a group of automorphisms of some compact Riemann surface X of genus g. Further, we say that G acts as a full group on genus g if G is the full automorphism group of some compact Riemann surface of genus g.

Suppose G acts on genus g, and let X be a compact Riemann surface for which $G \subseteq \mathrm{Aut}(X)$. Write $G = \Gamma/\Lambda$, where Γ and Λ are Fuchsian groups such that Λ has signature $(g; -)$ and is normal in Γ. If Γ has signature $(\gamma; m_1, \ldots, m_r)$, then we say that G acts on genus g with signature $(\gamma; m_1, \ldots, m_r)$, and further, if $G = \mathrm{Aut}(X)$, then we say that G acts as a full group on genus g with signature $(\gamma; m_1, \ldots, m_r)$. Of course G may act with different signatures on the same genus g.

With regards to this, Singerman in [39] says that if the group G can be written as Γ/Λ, where the signature of Γ does not appear in Singerman's table, then, generically, G acts as a full group on the corresponding genus g. Conversely, if the signature appears in Singerman's table, it means that the action *may* extend.

TABLE 1. Non-maximal Fuchsian signatures *(Singerman's table)*

Signature $\sigma = \sigma(\Gamma)$	$\sigma' = \sigma(\Gamma')$	$\lvert\Gamma' : \Gamma\rvert$
$(2; -)$	$(0; 2, 2, 2, 2, 2, 2)$	2
$(1; t, t)$	$(0; 2, 2, 2, 2, t)$	2
$(1; t)$	$(0; 2, 2, 2, 2t)$	2
$(0; t, t, t, t), t \geq 3$	$(0; 2, 2, 2, t)$	4
$(0; t, t, u, u), t + u \geq 5$	$(0; 2, 2, t, u)$	2
$(0; t, t, t), t \geq 4$	$(0; 3, 3, t)$	3
$(0; t, t, t), t \geq 4$	$(0; 2, 3, 2t)$	6
$(0; t, t, u), t \geq 3, t + u \geq 7$	$(0; 2, t, 2u)$	2

TABLE 1. (Continuation)

Signature $\sigma = \sigma(\Gamma)$	$\sigma' = \sigma(\Gamma')$	$\lvert\Gamma' : \Gamma\rvert$
$(0; 7, 7, 7)$	$(0; 2, 3, 7)$	24
$(0; 2, 7, 7)$	$(0; 2, 3, 7)$	9
$(0; 3, 3, 7)$	$(0; 2, 3, 7)$	8
$(0; 4, 8, 8)$	$(0; 2, 3, 8)$	12
$(0; 3, 8, 8)$	$(0; 2, 3, 8)$	10
$(0; 9, 9, 9)$	$(0; 2, 3, 9)$	12
$(0; 4, 4, 5)$	$(0; 2, 4, 5)$	6
$(0; n, 4n, 4n), n \geq 2$	$(0; 2, 3, 4n)$	6
$(0; n, 2n, 2n), n \geq 3$	$(0; 2, 4, 2n)$	4
$(0; 3, n, 3n), n \geq 3$	$(0; 2, 3, 3n)$	4
$(0; 2, n, 2n), n \geq 4$	$(0; 2, 3, 2n)$	3

The following theorem of Ries [36] describes conditions and how the extension of actions works.

Theorem 4. *([36], Thm. of Section 2) Let* $\psi : \Gamma \to G$ *and* $i : K_g \to \Gamma$ *induce the inclusion of* G *as a subgroup of* Mod_g. *Suppose one of the following is true:*

1. Γ *has signature* $[2; -]$ *and there is an automorphism* α *of* G *such that, if* $d = a_1^{-1} b_1^{-1} a_2 b_2$, $\alpha(a_1) = a_1^{-1}$, $\alpha(b_1) = b_1^{-1}$, $\alpha(a_2) = d a_2^{-1} d^{-1}$, $\alpha(b_2) = d b_2^{-1} d^{-1}$.

2. Γ *has signature* $[1; k, k]$ *and there is an* $\alpha \in \mathrm{Aut}(G)$ *such that, if* $d = a_1^{-1} b_1^{-1} c_1$, $\alpha(a_1) = a_1^{-1}$, $\alpha(b_1) = b_1^{-1}$, $\alpha(c_1) = d c_2 d^{-1}$, $\alpha(c_2) = d c_1 d^{-1}$.

3. Γ *has signature* $[1;k]$ *and there is an* $\alpha \in \mathrm{Aut}(G)$ *such that* $\alpha(a_1) = a_1^{-1}, \quad \alpha(b_1) = b_1^{-1}, \quad \alpha(c_1) = a_1^{-1}b_1^{-1}c_1b_1a_1.$

4. Γ *has signature* $[0;k,l,k,l]$ *for some* k,l *with* $2 \leq k$ *and* $3 \leq l$*. If* $k \neq l$*, there is an* $\alpha \in \mathrm{Aut}(G)$ *such that*
$$\alpha(c_1) = c_3, \quad \alpha(c_2) = c_4, \quad \alpha(c_3) = c_1, \quad \alpha(c_4) = c_2.$$
If $k = l$*, then* α *exists after replacing* ψ *by* $\psi \circ \mu$ *for some* $\mu \in \mathrm{Aut}^+(\Gamma)$*.*

5. Γ *has signature* $[0;k,k,k,k]$ *for some* k *with* $3 \leq k$ *and there are* $\alpha, \beta \in \mathrm{Aut}(G)$ *such that*
$$\alpha(c_1) = c_3, \quad \alpha(c_2) = c_4, \quad \alpha(c_3) = c_1, \quad \alpha(c_4) = c_2$$
$$\beta(c_1) = c_2, \quad \beta(c_2) = c_1, \quad \beta(c_3) = c_1^{-1}c_4c_1, \quad \beta(c_4) = c_2c_3c_2^{-1}$$

6. Γ *has signature* $[0;l,l,k]$ *for some* k,l *with* $3 \leq l, 2 \leq k$ *and at least one of the inequalities is strict. If* $k \neq l$*, there is an* $\alpha \in \mathrm{Aut}(G)$ *such that*
$$\alpha(c_1) = c_2, \quad \alpha(c_2) = c_1, \quad \alpha(c_3) = c_2c_3c_2^{-1}.$$
If $k = l$*, then* α *exists after replacing* ψ *by* $\psi \circ \mu$ *for some* $\mu \in \mathrm{Aut}^+(\Gamma)$*.*

7. Γ *has signature* $[0;k,k,k]$ *for some* $k \geq 4$ *and there is a* $\beta \in \mathrm{Aut}(G)$ *such that*
$$\beta(c_1) = c_2, \quad \beta(c_2) = c_3, \quad \beta(c_3) = c_1.$$

8. Γ *has signature* $[0;k,k,k]$ *for some* $k \geq 4$ *and there are* $\alpha, \beta \in \mathrm{Aut}(G)$ *such that*
$$\alpha(c_1) = c_2, \quad \alpha(c_2) = c_1, \quad \alpha(c_3) = c_2c_3c_2^{-1}$$
$$\beta(c_1) = c_2, \quad \beta(c_2) = c_3, \quad \beta(c_3) = c_1$$

If H *is the corresponding group with presentation*

(1,2,3,4,6.) $\quad H = \langle G, a | \cdots, a^2 = 1, aga = \alpha(g),\ for\ g \in G \rangle,$

5. . $H = \langle G, a, b | \cdots, a^2 = b^2 = 1, aga = \alpha(g), bgb = \beta(g),\ for\ g \in G, abab = (c_2c_3)^{-1} \rangle,$

7. . $H = \langle G, b | \cdots, b^3 = 1, bgb^2 = \beta(g),\ for\ g \in G \rangle,$

8. $H = \langle G, a, b | \cdots, a^2 = b^3 = 1, aga = \alpha(g), bgb^2 = \beta(g),\ for\ g \in G, abab = c_1^{-1} \rangle,$

where the dots denote the relations of G, then there exist Γ_0, an inclusion $j : \Gamma \to \Gamma_0$, and an epimorphism $\psi_0 : \Gamma_0 \to H$ such that $\psi = \psi_0 \circ j$ and $(j \circ i, \psi_0)$ induces the inclusion of H as a subgroup of Mod_g with the property that $G \neq H, G \lhd H$ and $\mathcal{M}_g^{[G]} = \mathcal{M}_g^{[H]}$.

Conversely, any subgroup H of Mod_g *with the above property arises in this way.*

Notice that the converse in Ries' theorem mentions Mod_g. This means that we have to be careful when considering generating vectors. This is, to consider their equivalence classes. We write the following corollary to describe the part of Ries' theorem we need (item 4.), and translated to the context of actions captured by generating vectors.

Corollary 1. *Let G be a finite group acting on a Riemann surface X with signature $s = (0; k, l, k, l)$ and generating vector ν for s. Then, that action extends if and only if there is an $\alpha \in \mathrm{Aut}(G)$ such that*

$$\alpha(c_1) = c_3, \quad \alpha(c_2) = c_4, \quad \alpha(c_3) = c_1, \quad \alpha(c_4) = c_2$$

for some generating vector (c_1, c_2, c_3, c_4) equivalent to ν.

Therefore, if we prove that there is no such α for every generating vector equivalent to ν, then that action does not extend generically.

Finally, concerning some of the geometry involved when having group actions, we use the following results.

First, we need to say something about the dimension of the primitive factors of the group algebra decomposition (1.8). A theorem of this kind was obtained by Ksir in [26] for groups with rational irreducible representations absolutely irreducible; i.e., irreducible over the complex field, and later it was generalized in [35] to any group. We need this last one for our work here.

Proposition 2. *(see [35], Prop. 5.12) Let G be a finite group acting on a Riemann surface S with geometric signature $(\gamma; [m_1, C_1], \ldots, [m_t, C_t])$. Then the dimension of any primitive factor B_i associated to a non trivial rational irreducible representation W_i, in the G -equivariant isogeny decomposition of the corresponding Jacobian variety JS, is given by*

$$dim\ B_i = k_i \left(dim\ V_i(\gamma - 1) + \frac{1}{2} \sum_{k=1}^{t} \left(dim\ V_i - dim\ V_i^{G_k} \right) \right), \qquad (1.14)$$

where G_k is a representative of the conjugacy class C_k, $dim\, V_i$ is the dimension of a complex irreducible representation V_i associated to W_i, $K_{V_i} = \mathbb{Q}\left(\chi_{V_i}(g) : g \in G\right), \ell_i$ is the Schur index of V_i, and $k_i = \ell_i . \mid \mathrm{Gal}\left(K_{V_i} : \mathbb{Q}\right) \mid$.

We also use here the following result about intermediate coverings.

Proposition 3. *([35, Prop.3.4],[26, Eq.(10)]) Let S be a Riemann Surface with G–action of geometric signature $(\gamma; [m_1, C_1], \dots, [m_t, C_t])$. Then for each subgroup $H \leqslant G$ the genus of S/H is given by*

$$g_{S/H} = [G:H](\gamma - 1) + 1 + \frac{1}{2}\sum_{j=1}^{t}([G:H] - |H\backslash G/G_j|) \tag{1.15}$$

where $H\backslash G/G_j$ is the corresponding set of double cosets and G_k is a representative of the conjugacy class C_k.

Chapter 2

A family of groups and linked subgroups acting on Riemann surfaces and Jacobian varieties

In this chapter we develop the necessary algebraic aspects of groups with an interesting algebraic property. They are groups that have two nonconjugate subgroups inducing the same permutation character. That is, their trivial representation induces the same representation of the group. We study a family of groups with this property.

2.1 Linked Finite Subgroups

Let us consider G a finite group and H a subgroup of G. Using the notation of [18] and [7], we will denote by 1_H^G the character of the permutation representation of G on the right (or left) cosets of H. If $H \leqslant G$ and T is a right (or left) transversal for H in G, then

$$\begin{aligned} 1_H^G(x) &= |\{g \in T \mid xgH = gH\}| \\ &= |\{g \in T \mid x^g \in H\}| \end{aligned}$$

where $x^g = g^{-1}xg$. In [15] the following definition is introduced:

Definition 2. Let H, K be two subgroups of a finite group G. We will say that H and K are linked in G if and only if they induce the same permutational character. That is

$$1_H^G = 1_K^G . \tag{2.1}$$

Remark 1. Note that conjugate subgroups are trivially linked. However, there are linked subgroups which are not conjugate.

Let G be a finite group, two subgroups H, H' of G are linked in G if and only if $\mathrm{Ind}_H^G(1_H)$ is equivalent to $\mathrm{Ind}_{H'}^G(1_{H'})$, where $\mathrm{Ind}_H^G(1_H)$ stand for the induced representation of the trivial representation of the subgroup H.

Proposition 4. *Let G be a finite group and $H, K \leqslant G$. H and K are linked in G if and only if*

$$\chi_{\mathrm{Ind}_H^G(1_H)} = \chi_{\mathrm{Ind}_K^G(1_K)}, \tag{2.2}$$

where $\chi_{\mathrm{Ind}_H^G(1_H)}$ denotes the character of the induced representation $\mathrm{Ind}_H^G(1_H)$. Analogous with K.

Proof. We have the following equalities

$$\begin{aligned} 1_H^G(x) &= |\{g \in T \mid xgH = gH\}| \\ &= \left|\left\{g \in T \mid (g^{-1}xg)H = H\right\}\right| \\ &= \left|\left\{g \in T \mid (g^{-1}xg) \in H\right\}\right| \\ &= \sum_{g \in T} \dot{\mathrm{i}}_{g^{-1}xg} \\ &= \chi_{\mathrm{Ind}_H^G(1_H)} \end{aligned}$$

where $\dot{\mathrm{i}}_{g^{-1}xg} := \begin{cases} 1 & , x \in H \\ 0 & , x \notin H \end{cases}$.

Using the previous definition, we are able to conclude that H and K are linked in G. □

In this work, we understand linked subgroups to be the nontrivial case; that is, nonconjugate. We bring these ideas to the context of group actions on Riemann surfaces and establish the following theorem.

Theorem 5. *Let X be a compact Riemann surface with the action of a finite group G. If H and K are linked groups in G, then the Jacobian varieties JX_H and JX_K corresponding to the intermediate quotients X/H and X/K are isogenous.*

Proof. If H and K are linked groups in G, then by definition and Frobenius reciprocity theorem, we know that for every $\mathbb{C}-$irreducible representation of G the following statement are equivalent:

(i) $\left\langle \mathrm{Ind}_H^G(1_H), V \right\rangle_G = \left\langle \mathrm{Ind}_K^G(1_K), V \right\rangle_G$.

(ii) $\dim_{\mathbb{C}} V^H = \dim_{\mathbb{C}} V^K$.

By Theorem 2 (Section 1.2), we are in a position to conclude that the isotypical components of isotypical decomposition are equal, therefore the Jacobians JX_H and JX_K are isogenous. $\square$

Linked subgroups have been studied in the context of Group Theory. Below we recall results from [18] that will be fundamental in the analysis of this work, since they provide a family of groups whose action we shall study here.

Theorem 6. *([18, Thm.A]) Suppose $H, K \leqslant G$ and $1_H^G = 1_K^G$ with $[G : H] = p^2$. Then either $H = K^g$ for some $g \in G$ or $p^\varepsilon = (q^n - 1)/(q - 1)$ with $\varepsilon \leqslant 2, n \geqslant 3$, and q a prime power. Furthermore, for any such prime p, there is a group G with $1_H^G = 1_K^G$ for nonconjugate subgroups H and K with $[G : H] = p^2$.*

2.1.1 The Group G_p

Let p be a prime, $p > 2$. Consider G_p a group that generates a family of groups satisfying the hypotheses of Theorem 5 given in [18, example 4.1]. If $A = \langle a, b\rangle \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $H = \langle x, y\rangle \leqslant \operatorname{Aut} A$, then $G_p = A \rtimes H$ and its presentation is:

$$\left\langle a, b, x, y \mid a^{p^2} = b^p = x^p = y^p = 1, x^{-1}ax = ab, x^{-1}bx = ba^p, y^{-1}ay = a^{p+1}, y^{-1}by = b\right\rangle \tag{2.3}$$

With the previous relations and $K = \langle x, ya^p\rangle$, it is concluded that $1_H^G = 1_K^G$ and H and K are not conjugate.

The interest to us of G_p comes from the following remark. If G_p acts on a compact Riemann surface and subgroups H and K are linked groups in G_p, then the Jacobian varieties JX_H and JX_K corresponding to the intermediate quotients X/H and X/H' are isogenous. Let us study some algebraic properties of G_p.

Proposition 5. *Let p be a prime, $p \neq 2$. The group G_p satisfies the following statements:*

(i) A minimal generator set for G_p is $\{a, x, y\}$.

(ii) The element $xya^{p+1}b$ has order p^2.

(iii) $\forall k, n \in \mathbb{N}$, the following equalities hold:

(iii.1.) $x^{-n}ax^k = a^{1+\frac{pn(n-1)}{2}}b^n x^{-n+k}$.

(iii.2.) $x^{-n}bx^k = a^{np}bx^{-n+k}$.

(iii.3.) $y^{-k}x^{-n}ax^n y^k = a^{pk+1+\frac{pn(n-1)}{2}}b^n$.

(iii.4.) $y^{-k}x^{-n}bx^n y^k = a^{np}b$.

Proof. *(i)* Notice that $a^{p^2-1}x^{p-1}ax = b$. From this we are in a position to conclude that $b \in \langle x, a\rangle$, then $\langle x, y, a, b\rangle = \langle x, y, a\rangle$.

On the other hand, the Frattini subgroup $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G, it is also the set of non-generators of G. By Burnside's Basis Theorem, if G is a finite p-group, then every maximal subgroup of G is normal and hence, $G' \subseteq \Phi(G)$. For the case of G_p, we obtain

$$\Phi(G_p) = G_p' = \langle a^p, b\rangle .$$

Then $G_p/\Phi(G_p) \cong \mathbb{Z}_{p^3}$, from which we conclude that every minimal set of generators has three elements and the set $\{a, x, y\}$ is a minimal generating set of the G_p.

(ii) Consider the prime $p = 3$. Then we have the following equality with respect to the order of the element $xya^{p+1}b \in G_3$:

$$(xya^{p+1}b)^3 = (xya^{p+1}b)^2(xya^{p+1}b) = (x^2y^2a^5)(xya^{p+1}b) = x^3y^3a^6 = a^6.$$

From the above we deduce that

$$(xya^{p+1}b)^9 = ((xya^{p+1}b)^3)^3 = (a^6)^3 = 1,$$

therefore the element $xya^{p+1}b$ has order 3^2. Now, let p be a prime greater than 3. Then the order of the element xya^{p+1} in G_p satisfies the following statements:

(1) For all $n \in \mathbb{N}$, it follows that $(xya^{p+1}b)^n = x^n y^n a^{\alpha(n)}b^{\beta(n)}$, where $\alpha : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}$ are functions that depend on n and $\alpha(n), \beta(n)$ are the exponent of a and b respectively in the n-th power of the element $xya^{p+1}b$.

We shall prove by induction, we have that $(xya^{p+1}b)^2 = x^2y^2a^{\alpha(2)}b^{\beta(2)}$, where $\alpha(2) = 4p + 2$ and $\beta(2) = 3$. Then by inductive hypothesis $(xya^{p+1}b)^n = x^n y^n a^{\alpha(n)}b^{\beta(n)}$, we have that

$$\begin{aligned}
(xya^{p+1}b)^{n+1} &= (xya^{p+1}b)^n(xya^{p+1}b) \\
&= (x^n y^n a^{\alpha(n)}b^{\beta(n)})(xya^{p+1}b) \\
&= x^{n+1}y^{n+1}a^{(p+1)(\alpha(n)+p\beta(n)+1)}b^{\alpha(n)+\beta(n)+1} \\
&= x^{n+1}y^{n+1}a^{\alpha(n+1)}b^{\beta(n+1)}.
\end{aligned}$$

This proves the claim for all $n \in \mathbb{N}$ and $p > 3$ prime.

(2) The value of the exponents of a and b in $(xya^{p+1}b)^n$, which will be denoted by $\alpha(n)$ and $\beta(n)$ respectively, are defined by recurrence by the following expressions:

$$\begin{cases} \beta(1) &= 1, \\ \beta(n) &= n + \beta(n-1), \ n \geq 2 \end{cases},$$

$$\begin{cases} \alpha(n) &= 0, n = 1, 2 \\ \alpha(n) &= n(np+1) + p\gamma(n), \ n \geq 3 \end{cases},$$

where

$$\begin{cases} \gamma(n) &= 0, \ n = 1, 2 \\ \gamma(n) &= \gamma(n-1) + \beta(n-2), \ n \geq 3\,. \end{cases}$$

Based on the above, we will describe $\gamma(n)$ as a function of the sequence $\beta(\cdot)$ with the following expression:

$$\gamma(n) = \sum_{k=1}^{n-2} \beta(k), \text{ for all } n \geq 3. \tag{2.4}$$

In fact, for $n = 3$ the equation (2.4) is satisfied. For any $n \geq 3$ fixed, we study $\gamma(n+1)$:

$$\begin{aligned} \gamma(n+1) &= \gamma(n) + \beta(n-1) = \left(\sum_{k=1}^{n-2} \beta(k) \right) + \beta(n-1) \\ &= \sum_{k=1}^{n-1} \beta(k). \end{aligned}$$

An important property of the sequence $\beta(\cdot)$ is that $\beta(n) = \frac{n(n+1)}{2}$, for all $n \geq 1$. Indeed, $\beta(1)$ satisfies the condition. If we consider a fixed $n \geq 1$ such that $\beta(n) = \frac{n(n+1)}{2}$, then we have to calculate $\beta(n+1)$:

$$\beta(n+1) = \beta(n) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.$$

In addition, we have that $\beta(p-2) \equiv 1(\text{mod p})$ for all $p \geq 3$ with p prime. Indeed let be $p \geq 3$ prime, then

$$\begin{aligned} \beta(p-2) &= \tfrac{(p-2)(p-1)}{2} = \tfrac{p^2-p-2p+2}{2}, \\ \beta(p-2) &\equiv 1(\text{mod p}). \end{aligned}$$

On the other hand, the sequence $\gamma(\cdot)$ is $\gamma(p-1) \equiv -1(\text{mod p})$ for all $p \geq 3$ with p prime. We note that

$$\begin{aligned}
\gamma(p-1) &= \sum_{k=1}^{p-3} \beta(k) = \sum_{k=1}^{p-3} \frac{k(k+1)}{2} = \sum_{k=1}^{p-3} \frac{k^2}{2} + \frac{k}{2} \\
&= \frac{(p-3)(p-2)(2p-5)}{2 \cdot 6} + \frac{(p-3)(p-2)}{4} \\
&\equiv \frac{-30}{2 \cdot 6} + \frac{6}{4} \text{ (mod p)} \\
&\equiv -1 \text{ (mod p)}.
\end{aligned}$$

Suppose that $(xya^{p+1}b)^n = x^n y^n a^{\alpha(n)} b^{\beta(n)}$, then now we calculate $(xya^{p+1}b)^{n+1}$ using the identities involving x, y, a and b:

$$\begin{aligned}
(xya^{p+1}b)^{n+1} &= (x^n y^n a^{\alpha(n)} b^{\beta(n)})(xya^{p+1}b) \\
&= x^n y^n a^{\alpha(n)} b^{\beta(n)} yxa^{p+1}b \\
&= x^n y^n a^{\alpha(n)} y b^{\beta(n)} xa^{p+1}b \\
&= x^n y^n a^{\alpha(n)} yxa^{p\beta(n)} b^{\beta(n)} a^{p+1}b \\
&= x^n y^n a^{\alpha(n)} yxa^{p\beta(n)+p+1} b^{\beta(n)+1} \\
&= x^n y^n ya^{(p+1)\alpha(n)} xa^{p\beta(n)+p+1} b^{\beta(n)+1} \\
&= x^n y^{n+1} xa^{(p+1)\alpha(n)} b^{(p+1)\alpha(n)} a^{p\beta(n)+p+1} b^{\beta(n)+1} \\
&= x^{n+1} y^{n+1} a^{(p+1)\alpha(n)+p\beta(n)+p+1} b^{(p+1)\alpha(n)+\beta(n)+1}.
\end{aligned}$$

In order to get a simpler expression, we must analyze the exponents of a and b. To determine the exponent of a we must consider that a has order p^2, then

$$\begin{aligned}
&(p+1)\alpha(n)+p\beta(n)+p+1\\
=&(p+1)[n(np+1)+p\gamma(n)]+p\beta(n)+p+1\\
=&p^2n^2+pn+pn^2+n+p^2\gamma(n)+p\gamma(n)+p\beta(n)+p+1\\
=&pn^2+pn+n+p\gamma(n)+p\beta(n)+p+1\\
=&pn^2+pn+n+p\gamma(n)+p(\beta(n-1)+n)+p+1\\
=&pn^2+pn+n+p[\gamma(n)+\beta(n-1)]+pn+p+1\\
=&pn^2+pn+n+p\gamma(n+1)+pn+p+1\\
=&(n+1)[(n+1)p+1]+p\gamma(n+1)\\
=&\alpha(n+1).
\end{aligned}$$

On the other hand, to determine the exponent of b we must consider that b has order p, then

$$\begin{aligned}
(p+1)\alpha(n)+\beta(n)+1 &= (p+1)[n(np+1)+p\gamma(n)]+\beta(n)+1\\
&= p^2n^2+pn+pn^2+n+p^2\gamma(n)+p\gamma(n)+\beta(n)+1\\
&= (n+1)+\beta(n)\\
&= \beta(n+1).
\end{aligned}$$

From the above, we are in a position to conclude that $xya^{p+1}b$ does not have order p, and furthermore, that $\forall n\in\mathbb{N}, n\geq 3$:

$$(xya^{p+1}b)^n = x^ny^na^{\frac{(n-2)(n-1)(2n-3)}{12}+\frac{(n-1)(n-2)}{4}}b^{\frac{n(n+1)}{2}}, \tag{2.5}$$

therefore the element $xya^{p+1}b$ has order p^2.

(iii) Considering the relations defined in G_p we will prove the equalities using the induction method.

(iii.1) Consider the relation defined in G_p: $ax = xab$, suppose that for n the equality $ax^n = x^na^{1+\frac{pn(n-1)}{2}}b^n$ is satisfied, then we have the following equivalences:

$$\begin{aligned}
ax^{n+1} &= (ax^n)x\\
&= (x^na^{1+\frac{pn(n-1)}{2}}b^n)x\\
&= x^na^{1+\frac{pn(n-1)}{2}}xa^{pn}b^n\\
&= x^{n+1}a^{1+\frac{p(n+1)n}{2}}b^{n+1}.
\end{aligned}$$

In conclusion $x^{-n}ax^n = a^{1+\frac{pn(n-1)}{2}}b^n$ for all $n\in\mathbb{N}$, from this the requested equality is deduced.

(iii.2) Second, consider the relation $bx = xa^pb$ defined in G_p, suppose that for $n \in \mathbb{N}$ the equality $bx^n = x^n a^{pn} b$ is satisfied, then we have the following equalities:

$$\begin{aligned} bx^{n+1} &= (bx^n)x \\ &= (x^n a^{pn} b)x \\ &= x^n a^{pn} x a^p b \\ &= x^{n+1} x a^{p(n+1)} b. \end{aligned}$$

Therefore, the equality $x^{-n}bx^n = a^{pn}b$ is satisfied for all $n \in \mathbb{N}$, from this the requested equality is deduced.

(iii.3) Third, let us consider the relations defined in G_p: $by = yb$ and $ay = ya^{p+1}$. Moreover, using the equality proved in *(iii.1)* for all $n \in \mathbb{N}$, we assume that the equality $x^{-n}ax^n y^k = (a^{1+\frac{pn(n-1)}{2}} b^n) y^k = y^k a^{pk+1+\frac{pn(n-1)}{2}} b^n$ is satisfied for $k \in \mathbb{N}$, then, developing the induction on k we have the following equalities:

$$\begin{aligned} x^{-n}ax^n y^{k+1} &= (x^{-n}ax^n y^k)y \\ &= y^k a^{pk+1+\frac{pn(n-1)}{2}} b^n y \\ &= y^k a^{pk+1+\frac{pn(n-1)}{2}} y b^n \\ &= y^{k+1} (a^{pk+1+\frac{pn(n-1)}{2}})^{p+1} b^n \\ &= y^{k+1} a^{p(k+1)+1+\frac{pn(n-1)}{2}} b^n. \end{aligned}$$

The result follows.

(iii.4) Finally, let us consider the relations defined in G_p described in the previous point and using the equality proved in *(iii.2)* for all $n \in \mathbb{N}$, we assume that the equality $x^{-n}bx^n y^k = (a^{np}b)y^k = y^k a^{np} b$ is satisfied for $k \in \mathbb{N}$. Then, developing the induction on k we have the following equalities:

$$\begin{aligned} x^{-n}bx^n y^{k+1} &= (x^{-n}bx^n y^k)y \\ &= (y^k a^{np} b)y \\ &= y^k a^{np} y b \\ &= y^{k+1} a^{np(p+1)} b \\ &= y^{k+1} a^{np} b. \end{aligned}$$

This concludes the equality of point *(iii)* of the proposition.

$\square$

2.2 Algebraic description of G_p.

In this section we develop the necessary algebraic aspects of G_p to describe the consequences of its action on Riemann surfaces, and on the corresponding Jacobian varieties. This is conjugacy classes and complex and rational representations.

2.2.1 Complex Irreducible representations of G_p.

G_p is a semi-direct product of H and A (details given in the previous section). We observe that A is a normal subgroup of G, therefore, we use the method of Little Groups of Winger and Mackey ([37, Prop. 25, ch.8]), to find its irreducible representations.

Let $\mathbb{X}_A$ be the set of all complex irreducible representations of the group A. Since it is abelian, all its representations are of degree 1. If $w = e^{\frac{2\pi i}{p^2}}$, for each pair $(t, s) \in \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, the 1-representations are defined by:

$$\begin{array}{rrl} \chi_{(t,s)}: & A \to & \mathbb{C}^* \\ & a^\mu \mapsto & w^{t\mu} \\ & b^\gamma \mapsto & w^{ps\gamma} \end{array}$$

with $0 \leq \mu \leq p^2 - 1$ and $0 \leq \gamma \leq p - 1$.

The group $H = \langle x, y \rangle$ acts on $\mathbb{X}_A^H$ as follows

$$\begin{array}{rcl} \cdot : H & \to & \mathbb{X}_A^H \\ h & \mapsto & h \cdot \chi_{(t,s)}(g) = \chi_{(t,s)}(h^{-1}gh). \end{array} \tag{2.6}$$

To describe the set of orbits determined by the action of H on $\mathbb{X}_A^H$, consider the following proposition:

Proposition 6. *Let x and y be the generating elements of $H \leq G_p$ and $t, s, n \in \mathbb{N}$. If $\chi_{(t,s)}$ is a representation of A with $(t, s) \in \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then for $0 \leq n \leq p - 1$ the action satisfies the following statements*

$$\begin{array}{rcl} x^n \cdot \chi_{(t,s)} & = & \chi_{((1+\zeta)t+nsp\ ,\ nt+s)}, \text{ where } \zeta = \displaystyle\sum_{j=1}^{n-1} jp, \\ y^n \cdot \chi_{(t,s)} & = & \chi_{((np+1)t\ ,\ s)}. \end{array}$$

Proof. For all $g \in A$ there exist $k_1 \in \{0, 1, \ldots, p^2 - 1\}$ and $k_2 \in \{0, 1, \ldots, p - 1\}$ such that $g = a^{k_1}b^{k_2}$, then

$$\begin{aligned} x\cdot\chi_{(t,s)}(a^{k_1}) &= \chi_{(t,s)}(x^{-1}a^{k_1}x)\\ &= \chi_{(t,s)}((x^{-1}ax)^{k_1})\\ &= \chi_{(t,s)}((ab)^{k_1})\\ &= \chi_{(t,s)}(a^{k_1})\chi_{(t,s)}(b^{k_1})\\ &= w^{k_1(t+ps)}. \end{aligned}$$

$$\begin{aligned} x\cdot\chi_{(t,s)}(b^{k_2}) &= \chi_{(t,s)}(x^{-1}b^{k_2}x)\\ &= \chi_{(t,s)}((x^{-1}bx)^{k_2})\\ &= \chi_{(t,s)}((ba^{p})^{k_2})\\ &= \chi_{(t,s)}(b^{k_2})\chi_{(t,s)}(a^{pk_2})\\ &= w^{pk_2(t+s)}. \end{aligned}$$

We deduce the following:

$$\begin{aligned} x\cdot\chi_{(t,s)}(a^{k_1}b^{k_2}) &= \chi_{(t,s)}(x^{-1}a^{k_1}b^{k_2}x)\\ &= \chi_{(t,s)}\left((x^{-1}a^{k_1}x)(x^{-1}b^{k_2}x)\right)\\ &= \chi_{(t,s)}(x^{-1}a^{k_1}x)\chi_{(t,s)}(x^{-1}b^{k_2}x)\\ &= \left(x\cdot\chi_{(t,s)}(a^{k_1})\right)\cdot\left(x\cdot\chi_{(t,s)}(b^{k_2})\right)\\ &= w^{k_1(t+ps)}w^{pk_2(t+s)}\\ &= (w^{k_1})^{(t+ps)}(w^{k_2})^{p(t+s)}\\ &= \chi_{(t+ps,t+s)}(a^{k_1}b^{k_2}). \end{aligned}$$

Analogously for $y\in H$ we have:

$$\begin{aligned} y\cdot\chi_{(t,s)}(a^{k_1}) &= \chi_{(t,s)}(y^{-1}a^{k_1}y)\\ &= \chi_{(t,s)}((y^{-1}ay)^{k_1})\\ &= \chi_{(t,s)}(a^{(p+1)k_1})\\ &= w^{k_1(p+1)t}. \end{aligned}$$

$$\begin{aligned} y\cdot\chi_{(t,s)}(b^{k_2}) &= \chi_{(t,s)}(y^{-1}b^{k_2}y)\\ &= \chi_{(t,s)}((y^{-1}by)^{k_2})\\ &= \chi_{(t,s)}(b^{k_2})\\ &= w^{k_2ps}. \end{aligned}$$

We have that

$$\begin{aligned} y\cdot\chi_{(t,s)}(a^{k_1}b^{k_2}) &= \chi_{(t,s)}(y^{-1}a^{k_1}b^{k_2}y)\\ &= \chi_{(t,s)}\left((y^{-1}a^{k_1}y)(y^{-1}b^{k_2}y)\right)\\ &= \chi_{(t,s)}(y^{-1}a^{k_1}y)\chi_{(t,s)}(y^{-1}b^{k_2}y)\\ &= \left(y\cdot\chi_{(t,s)}(a^{k_1})\right)\cdot\left(y\cdot\chi_{(t,s)}(b^{k_2})\right)\\ &= (w^{k_1})^{(p+1)t}(w^{k_2})^{ps}\\ &= \chi_{((p+1)t,s)}(a^{k_1}b^{k_2}). \end{aligned}$$

Since the elements are arbitrary, we claim that

$$\begin{aligned} x \cdot \chi_{(t,s)} &= \chi_{(t+ps,t+s)}, \\ y \cdot \chi_{(t,s)} &= \chi_{((p+1)t,s)}. \end{aligned}$$

By proceeding inductively, if we assume that the proposition is true for $n-1$ then:

$$\begin{aligned} x^n \cdot \chi_{(t,s)} &= x \cdot \left(x^{n-1} \cdot \chi_{(t,s)}\right) \\ &= x \cdot \left(\chi_{(\,[1+\sum_{j=1}^{n-2} jp]t+(n-1)sp\,,\,(n-1)t+s)}\right) \\ &= \chi_{(\,[1+\sum_{j=1}^{n-2} jp]t+(n-1)sp+p(n-1)t+ps\,,\,t+(n-1)t+s)} \\ &= \chi_{\left(\,[1+\sum_{j=1}^{n-1} jp]t+nsp\,,\,nt+s\right)}. \end{aligned}$$

$$\begin{aligned} y^n \cdot \chi_{(t,s)} &= y \cdot \left(y^{n-1} \cdot \chi_{(t,s)}\right) \\ &= y \cdot \left(\chi_{([(n-1)p+1]t\,,\,s)}\right) \\ &= \chi_{((n+1)[(n-1)p+1]t\,,\,s)} \\ &= \chi_{((np+1)t\,,\,s)}. \end{aligned}$$

□

The action described in Proposition 6 allows us to identify how many orbits we have and how many elements they have. We use it in the following proposition.

Proposition 7. *Let us consider $H \leq G_p$ as before. The action of H on $\mathbb{X}_A$ determines $3p-2$ orbits of which there are:*

(i) p orbits with one element each.

(ii) $p-1$ orbits with p elements.

(iii) $p-1$ orbits with p^2 elements.

Proof. By group theory, we know that $\#\mathrm{Orb}_H(\chi_{(t,s)}) \in \{1, p, p^2, p^3\}$ for $(t,s) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.
If $t = pk$ with $k \in \{0, \ldots, p-1\}$, then we identify two cases:
1) Let $s = 0$, using the previous proposition $\mathrm{Orb}_H(\chi_{(pk,0)}) = \{\chi_{(pk,0)}\}$, then $\#\mathrm{Orb}_H(\chi_{(pk,0)}) = 1$, therefore there are p orbits of one element.
2) Let $s \neq 0$. By using the previous proposition

$$\mathrm{Orb}_H(\chi_{(pk,s)}) = \{\chi_{(p(k+ns),s)} : n \in \mathbb{Z} \ \wedge \ 0 \leq n \leq p-1\}.$$

This implies that $\#\mathrm{Orb}_H(\chi_{(pk,s)}) = p$, given that $s \in \{1, \ldots, p-1\}$, then there are $p-1$ orbits of p elements.

On the other hand, $\underline{\text{if } GCD(t,p) = 1}$, then for each fixed s and $n \in \{0, \ldots, p-1\}$ we have $y \cdot \chi_{(t,s)} = \chi_{((np+1)t,s)}$. Thus

$$\chi_{((np+1)t,s)} \in \mathrm{Orb}_H(\chi_{(t,s)}).$$

Moreover, let us observe the following

$$x \cdot \chi_{((np+1)t,s)} = \chi_{((np+1)t+ps,t+s)},$$

under the assumption that $GCD(t,p) = 1$. Then, this element is different from those defined by p. Therefore, for this case $\#\mathrm{Orb}_H(\chi_{(t,s)}) > p$ and the action is not transitive, so it is concluded that $\#\mathrm{Orb}_H(\chi_{(t,s)}) = p^2$.

Since the elements of the orbit are determined by $s \in \{1, \ldots, p-1\}$ and also by considering the total number of 1-representations and the number of elements that are already known from the previous orbits: $p^3-(p+p(p-1)) = p^2(p-1)$, it follows that there are $p-1$ different classes of orbits with p^2 elements. $\square$

Once the number of orbits and their number of elements are know, we look for the representatives of each of them. For that, the following proposition is presented.

Proposition 8. *The representatives of the orbits determined by the action of H on $\mathbb{X}_A$ according to their number of elements as in Proposition 7, are*

(i) $\overline{\chi_{(pk,0)}}$ *for* $0 \leq k \leq p-1$ *are representatives of the orbits with 1 element.*

(ii) $\overline{\chi_{(0,s)}}$ *for* $1 \leq s \leq p-1$ *are representatives of the orbits with p elements.*

(iii) $\overline{\chi_{(t,0)}}$ *for* $1 \leq t \leq p-1$ *are representatives of the orbits with p^2 elements.*

Proof. By the previous proposition, it is enough to prove that these elements represent different orbits. By considering the results in Proposition 6, we say the following:

Statement *(i)* is trivial, since $\mathrm{Orb}_H(\chi_{(pk,0)}) = \{\chi_{(pk,0)}\}$.

For statement *(ii)*, we see that

$$\begin{aligned} \mathrm{Orb}_H(\chi_{(0,s)}) &= \{x^n y^m \cdot \chi_{(0,s)} : n, m \in \mathbb{Z} \ \wedge \ 0 \leq n, m \leq p-1\} \\ &= \{\chi_{(nps,s)} : n \in \mathbb{Z} \ \wedge \ 0 \leq n \leq p-1\}. \end{aligned} \tag{2.7}$$

Therefore, for $\hat{s} \in \{1, \ldots, p-1\}$, $\chi_{(0,\hat{s})} \in \mathrm{Orb}_H(\chi_{(0,s)})$ if and only if $s = \hat{s}$.

Finally, for statement *(iii)*, we have

$$\begin{aligned} \mathrm{Orb}_H(\chi_{(t,0)}) &= \{x^n y^m \cdot \chi_{(0,s)} : n,m \in \mathbb{Z} \ \wedge \ 0 \le n,m \le p-1\} \\ &= \{\chi_{((1+p+\ldots+(n-1)p)t+mpt\ ,\ nt)} : n,m \in \mathbb{Z} \ \wedge \ 0 \le n,m \le p-1\}. \end{aligned} \tag{2.8}$$

Then, for $\hat{t} \in \{1,\ldots,p-1\}$, $\chi_{(\hat{t},0)} \in \mathrm{Orb}_H(\chi_{(t,0)})$ if and only if there exist n,m such that

$$np \equiv 0(\operatorname{mod} p^2) \quad \wedge \quad (1+p+\ldots+(n-1)p)t + mpt \equiv \hat{t}(\operatorname{mod} p).$$

Then $n=0$ and this implies that $t(1+mp) \equiv \hat{t}(\operatorname{mod} p^2)$, therefore $t \equiv \hat{t}(\operatorname{mod} p)$, i.e. $t = \hat{t}$. □

Remark 2. We describe the set $\mathbb{X}_A^H$ by

$$\{\mathrm{Orb}_H(\chi_{(pk,0)})\}_{0\le k\le p-1}\dot{\cup}\{\mathrm{Orb}_H(\chi_{(0,s)})\}_{1\le s\le p-1}\dot{\cup}\{\mathrm{Orb}_H(\chi_{(t,0)})\}_{1\le t\le p-1}$$

The set I indexing the representations we chose as representatives is given as the disjoint union of the set I_1, I_p, I_{p^2} where

$$\begin{aligned} I_1 &:= \{(pk,0) \in \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} : 0 \le k \le p-1\}, \\ I_p &:= \{(0,s) \in \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} : 1 \le s \le p-1\}, \\ I_{p^2} &:= \{(t,0) \in \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} : 1 \le t \le p-1\}. \end{aligned}$$

Continuing with the Little Groups method, for each $(i,j) \in I$ we define the following subgroup of G_p

$$H_{(i,j)} = \{h \in H : h \cdot \overline{\chi_{(i,j)}} = \overline{\chi_{(i,j)}}\} := Stab(\overline{\chi_{(i,j)}}).$$

Then, using equations (2.7) and (2.8), we obtain the following stabilizers $H_{(i,j)}$

(i) $\forall (i,j) \in I_1,\ H_{(i,j)} = \langle x,y \rangle = H \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$,

(ii) $\forall (i,j) \in I_p,\ H_{(i,j)} = \langle y \rangle \simeq \mathbb{Z}/p\mathbb{Z}$,

(iii) $\forall (i,j) \in I_{p^2},\ H_{(i,j)} = \{id\}$.

Then, define $G_{(i,j)} = A \rtimes H_{(i,j)}$, and the representation $\overline{\chi}_{(i,j)}$ extends to $G_{(i,j)}$ as

$$\widehat{\overline{\chi}}_{(i,j)}(ah) = \overline{\chi}_{(i,j)}(a), \text{ for } a \in A \text{ and } h \in H_{(i,j)}.$$

According to the Little Groups method, we take every ρ irreducible representation of $H_{(i,j)}$, and compose it with the projection $\pi : G_{(i,j)} \to H_{(i,j)}$

obtaining in this way a representation $\widetilde{\rho}$ of $G_{(i,j)}$. Finally, we take tensor product $\widehat{\widetilde{\chi}}_{(i,j)} \otimes \widetilde{\rho}$ to provide an induced representation of $G_{(i,j)}$. The corresponding induced representation of G_p is irreducible, and in this way all the complex irreducible representations of G_p are obtained. We describe the irreducible $\mathbb{C}-$representations of $H_{(i,j)}$ with $(i,j) \in I$ as follows. Notice that we use a convenient notation.

(i) If $(i,j) \in I_1$, then the representations of $H_{(pk,0)}$ are determined for each $\alpha, \beta \in \{0, \ldots, p-1\}$ such that

$$\begin{aligned} \rho^{(\alpha,\beta)}_{H_{(pk,0)}} : H_{(pk,0)} &\to \mathbb{C}^* \\ x &\mapsto w^{p\alpha} \\ y &\mapsto w^{p\beta}\,. \end{aligned}$$

(ii) If $(i,j) \in I_p$, then the representations of $H_{(0,s)}$ are determined for each $\beta \in \{0, \ldots, p-1\}$ such that

$$\begin{aligned} \rho^{(0,\beta)}_{H_{(0,s)}} : H_{(0,s)} &\to \mathbb{C}^* \\ y &\mapsto w^{p\beta}. \end{aligned}$$

(iii) If $(i,j) \in I_{p^2}$, then the representations of $H_{(t,0)}$ are determined for the trivial representation, i.e. $\rho^{(0,0)}_{H_{(t,0)}} = Id_{H_{(t,0)}}$.

As said if $\pi : G_{(i,j)} \to H_{(i,j)}$ is the canonical projection for $(i,j) \in I$, then $\forall \alpha, \beta \in \{0, \ldots, p-1\}$ the irreducible representation of $G_{(i,j)}$ is defined by the function $\widetilde{\rho}^{(\alpha,\beta)}_{(i,j)} := \rho^{(\alpha,\beta)}_{H_{(i,j)}} \circ \pi$. Note that this representation is of degree 1 as well.

Finally, if we take the tensor product $\widehat{\widetilde{\chi}}_{(i,j)} \otimes \widetilde{\rho}^{(\alpha,\beta)}_{(i,j)} = \widehat{\widetilde{\chi}}_{(i,j)} \cdot \widetilde{\rho}^{(\alpha,\beta)}_{(i,j)}$, we obtain all the complex irreducible representations of G_p which correspond to the following induced representations:

$$\mathrm{Ind}^{G_p}_{G_i}(\widehat{\widetilde{\chi}}_{(i,j)} \otimes \widetilde{\rho}^{(\alpha,\beta)}_{(i,j)}). \tag{2.9}$$

In the following section, we determine each complex irreducible representation of G_p, we present them organized by its degree.

2.2.1.1 Complex irreducible representations of G_p of degree 1.

Observe that for all $(pk,0) \in I_1$ and for all $\alpha, \beta \in \{0, \ldots, p-1\}$, we have that $[G_p : G_{(pk,0)}] = 1$. Then,

$$\deg\left(\mathrm{Ind}^{G_p}_{G_{(pk,0)}}(\widehat{\widetilde{\chi}}_{(pk,0)} \otimes \widetilde{\rho}^{(\alpha,\beta)}_{(pk,0)})\right) = 1.$$

Therefore, the following equalities are satisfied:

$$\operatorname{Ind}_{G_{(pk,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(pk,0)}\otimes\widetilde{\rho}_{(pk,0)}^{(\alpha,\beta)}\right)=\widehat{\widehat{\chi}}_{(pk,0)}\otimes\widetilde{\rho}_{(pk,0)}^{(\alpha,\beta)}=\widehat{\widehat{\chi}}_{(pk,0)}\cdot\widetilde{\rho}_{(pk,0)}^{(\alpha,\beta)}.$$

Therefore, we are in a position to conclude that G_p has p^3 irreducible representations of degree 1, which are:

$$\begin{array}{rrl}\operatorname{Ind}_{G_{(pk,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(pk,0)}\otimes\widetilde{\rho}_{(pk,0)}^{(\alpha,\beta)}\right): & G_p & \to \mathbb{C}^*\\ & a & \mapsto w^{pk}\\ & b & \mapsto 1\\ & x & \mapsto w^{p\alpha}\\ & y & \mapsto w^{p\beta}.\end{array}$$

Notation 1. From now on, for all $k,\alpha,\beta\in\{0,1,\ldots,p-1\}$, we will denote these representations as:

$$V_{(k,\alpha,\beta)}^{(1)}:=\operatorname{Ind}_{G_{(pk,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(pk,0)}\otimes\widetilde{\rho}_{(pk,0)}^{(\alpha,\beta)}\right). \tag{2.10}$$

2.2.1.2 Complex irreducible representations of G_p of degree p

First, let us note that for all $(0,s)$ in I_p, we have that $[G_p:G_{(0,s)}]=p$. Then,

$$\deg\left(\operatorname{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)\right)=p,$$

for every $\beta\in\{0,\ldots,p-1\}$, Then G_p has $p(p-1)$ complex irreducible representations of degree p.

Since $G_p/G_{(0,s)}=G_p/(A\rtimes\langle y\rangle)\simeq\langle x\rangle$, then $\{1,x,x^2,\ldots,x^{p-1}\}$ is a (left) transversal of $G_{(0,s)}$ in G_p. Moreover, using $\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}=\widehat{\widehat{\chi}}_{(0,s)}\cdot\widetilde{\rho}_{(0,s)}^{(0,\beta)}$, we get

$$\begin{array}{rcl}\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}: & A\rtimes\langle y\rangle & \to \mathbb{C}^*\\ & a & \mapsto 1\\ & b & \mapsto w^{ps}\\ & y & \mapsto w^{p\beta}.\end{array}$$

Using Proposition 5, item (iii) (Section 2.1.1) and the properties of group representations, we describe these representations for each generator of G_p.

(i) Let us denote by $[a_{ij}]_{0\leq i,j\leq p-1}$ the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(0,s)}}^{G}(\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)})(a)$ in the chosen basis of left cosets of $G_{(0,s)}$ on G_p. Using Proposition 5, we have:

$$\begin{aligned} a_{ii} &= \widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}(x^{-i}ax^{i}) \\ &= \widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}(a^{1+\frac{pi(i-1)}{2}}b^{i}) \\ &= w^{psi}. \end{aligned}$$

Additionally, we have

$$x^{-i}ax^{j} = a^{1+\frac{pi(i-1)}{2}}b^{i}x^{-i+j}.$$

Therefore, $x^{-i}ax^{j}\in G_{(0,s)}$ if and only if $i=j$. From this statement it follows that only the elements of the diagonal of the matrices are non-zero. Using this information we obtain the representation induced for $a\in G_p$:

$$\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(a) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & w^{ps} & 0 & \dots & 0 \\ 0 & 0 & w^{2ps} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w^{(p-1)ps} \end{pmatrix} \in M_p(\mathbb{C}).$$

Remark 3. We observe that for all $k\in\{0,\dots,p^2-1\}$ the character of $\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)$ evaluated in a^k is given by the sum $\sum_{j=0}^{p-1}(w^{kps})^{j}$. We are in a position to conclude that for $k\neq p$, the sum is 0. On the other hand, if $k=p$, the sum is p.

(ii) Let us denote by $[b_{ij}]_{0\leq i,j\leq p-1}$ to the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(b)$ in the chosen basis of left cosets of $G_{(0,s)}$ on G_p. Using Proposition 5, we have:

$$\begin{aligned} b_{ii} &= \widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}(x^{-i}bx^{i}) \\ &= \widehat{\widehat{\chi}}_{(0,s)}\otimes\widetilde{\rho}_{(0,s)}^{(0,\beta)}(a^{ip}b) \\ &= w^{ps}. \end{aligned}$$

Additionally, we have the equality

$$x^{-i}bx^{j} = a^{ip}bx^{-i+j},$$

therefore $x^{-i}bx^j \in G_{(0,s)}$ if and only if $i = j$. From this statement it follows that only the elements of the diagonal on the matrix are non-zero. Using this information we obtain the representation of $b \in G_p$:

$$\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(b) = \begin{pmatrix} w^{ps} & 0 & 0 & \dots & 0 \\ 0 & w^{ps} & 0 & \dots & 0 \\ \vdots & 0 & w^{ps} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & w^{ps} \end{pmatrix} \in M_p(\mathbb{C}).$$

Remark 4. We observe that for all $k \in \{0, \dots, p-1\}$ the character of $\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)$ evaluated in b^k is $\sum_{j=0}^{p-1} w^{psk} = pw^{kps}$.

(iii) Let us denote by $[y_{ij}]_{0\leq i,j\leq p-1}$ the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(0,s)}}^{G}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(y)$ in the chosen basis of left cosets of $G_{(0,s)}$ on G_p. Using Proposition 5, we have:

$$\begin{aligned} y_{ii} &= \widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}(x^{-i}yx^i) \\ &= \widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}(y) \\ &= w^{p\beta}. \end{aligned}$$

Additionally, since H is an abelian group, we have the equality $x^{-i}yx^j = yx^{-i+j}$. Therefore, $x^{-i}yx^j \in G_{(0,s)}$ if and only if $i = j$.

As before, it follows that only the elements of the diagonal of the matrix are non-zero. Using this, we get for $y \in G_p$:

$$\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(y) = \begin{pmatrix} w^{p\beta} & 0 & 0 & \dots & 0 \\ 0 & w^{p\beta} & 0 & \dots & 0 \\ \vdots & 0 & w^{p\beta} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & w^{p\beta} \end{pmatrix} \in M_p(\mathbb{C}).$$

Remark 5. We observe that for all $k \in \{0, \dots, p-1\}$ the character of $\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)$ evaluated in y^k is $\sum_{j=0}^{p-1} w^{p\beta k} = pw^{p\beta k}$.

(iv) Let us denote by $[x_{ij}]_{0\leq i,j\leq p-1} \in M_p(\mathbb{C})$ the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(x)$ in the chosen basis of left cosets of $G_{(0,s)}$ on G_p. Using Proposition 5, we have $x^{-i}xx^j = x^{-i+j+1}$. Then the elements of the matrix for $i,j \in \mathbb{Z}$ and $0 \leq i,j \leq p-1$ are defined by:

$$x_{ij} = \begin{cases} 1 & ,if \;\; i-j \equiv 1(\operatorname{mod} p), \\ 0 & ,if \;\; i-j \not\equiv 1(\operatorname{mod} p). \end{cases}$$

If we define for $n,m \in \mathbb{N}$, the element $O_{m\times n}$ as the null matrix in $M_{m\times n}(\mathbb{Z})$, and Id_n as the identity matrix in $M_{n\times n}(\mathbb{Z})$, then, interpreting the above information, we get the description of the representation of G_p on x which is:

$$\operatorname{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widetilde{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(x) = \begin{pmatrix} O_{\frac{p-1}{2}\times\frac{p+1}{2}} & Id_{\frac{p-1}{2}} \\ Id_{\frac{p+1}{2}} & O_{\frac{p+1}{2}\times\frac{p-1}{2}} \end{pmatrix} \in M_p(\mathbb{C}).$$

Remark 6. The matrix $\operatorname{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widetilde{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)(x)$ is a permutation matrix corresponding to the permutation $\sigma_p = (a_1\ a_2\ \ldots\ a_p) \in S_p$, where the indices of a_i for all $i \in \{1,2,\ldots,p\}$ are given by:

$$a_i = \begin{cases} b_{\frac{i+1}{2}}, & \text{if } i \text{ is odd}, \\ c_{\frac{i}{2}}, & \text{if } i \text{ is even}, \end{cases} \tag{2.11}$$

with the sequences $\{b_j\}$ and $\{c_j\}$ described by the following expressions:

$$\begin{aligned} b_j &= j, \\ c_j &= \frac{p+3}{2} + (j-1), \end{aligned}$$

for $1 \leq j \leq p$.

Note that this is a cyclic permutation of length p, therefore, for every $k \in \{0,\ldots,p-1\}$ the character of $\operatorname{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widetilde{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right)$ evaluated in x^k is 0. The fact that the permutation is described by a cycle implies that it has order p (length of the cycle) and that powers smaller than its order do not fix elements.

From now on, for all $s \in \{1,\ldots,p-1\}$, for all $\beta \in \{0,1,\ldots,p-1\}$, we will denote these representations by:

$$V_{(s,\beta)}^{(p)} := \operatorname{Ind}_{G_{(0,s)}}^{G_p}\left(\widehat{\widetilde{\chi}}_{(0,s)} \otimes \widetilde{\rho}_{(0,s)}^{(0,\beta)}\right). \tag{2.12}$$

2.2.1.3 Complex irreducible representations of G_p of degree p^2

Finally, we will consider $(t,0)$ in I_{p^2}. We have that $[G_p : G_{(t,0)}] = p^2$, then,

$$\deg\left(\mathrm{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)\right) = p^2.$$

Additionally, we have the following equality:

$$\mathrm{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right) = \mathrm{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(t,0)} \otimes Id_{G_{(t,0)}}\right),$$

this implies that G_p has $p-1$ complex irreducible representations of degree p^2.

Given that $G_p/G_{(t,0)} = G_p/(A \rtimes \{1\}) \simeq H$, then $\{x^i y^j : 0 \leq i, j \leq p-1\}$ is a set of left cosets representatives $G_{(t,0)}$ in G_p. Moreover, we have the expression $\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)} = \widehat{\widehat{\chi}}_{(t,0)} \cdot 1_{G_{(t,0)}}$, therefore

$$\begin{array}{rrcl} \widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)} : & A \rtimes \{1\} & \longrightarrow & \mathbb{C}^* \\ & a & \mapsto & w^t \\ & b & \mapsto & 1. \end{array}$$

Considering the above result and using Proposition 5, item (iii) (Section 2.1.1), we describe the value of the representations for each generator of G_p by means of the following expressions:

(i) Let us denote by $[a_{ij}]_{0\leq i,j\leq p^2-1} \in M_{p^2}(\mathbb{C})$ the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(t,0)}}^{G_p}(\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)})(a)$ in the chosen basis of left cosets of $G_{(t,0)}$ on G_p. Using Proposition 5, we have

$$y^{-j}x^{-i}ax^iy^j = a^{pj+1+\frac{pi(i-1)}{2}}b^i.$$

This implies the equality of the following sets:

$$\{a_{ii} = a^{p\left(\frac{i(i+1)}{2}\right)+1}b^i : 0 \leq i \leq p^2-1\} = \{a^{pk+1}b^j : 0 \leq j, k \leq p-1\},$$

and the elements of the matrix are non-zero on the diagonal.
If $\sigma_{(i,j)}^{(column)}$ denotes the permutation of columns i and j in a matrix, and if $\sigma_{(i,j)}^{(row)}$ denotes the permutation of rows i and j of a matrix, then

$$\mathrm{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(a) = \sigma_{(1,p^2)}^{(column)}\sigma_{(1,p^2)}^{(row)}\left(\mathrm{diag}\left[A_t^{(0)}, \ldots, A_t^{(p-1)}\right]\right),$$

where $\forall k \in \{0, \dots, p-1\}$, $A_t^{(k)}$ is defined by

$$\begin{aligned} A_t^{(k)} &= \begin{pmatrix} w^{t(pk+1)} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & {\scriptstyle (p-times)} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & w^{t(pk+1)} \end{pmatrix} \in M_p(\mathbb{C}), \\ &= \operatorname{diag}\left[\underbrace{w^{t(pk+1)}, w^{t(pk+1)}, \dots, w^{t(pk+1)}}_{p-times}\right]. \end{aligned}$$

Remark 7. We calculate the character of the previous representation evaluated in a by:

$$\begin{aligned} \chi_{\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)}(a) &= \sum_{k=0}^{p-1} p w^{(pk+1)t}, \\ &= p w^t \sum_{k=0}^{p-1} (w^{pt})^k, \\ &= -p w^t. \end{aligned}$$

(ii) Let us denote by $[b_{ij}]_{0 \le i,j \le p^2-1} \in M_{p^2}(\mathbb{C})$ the coefficients of the matrix corresponding to $\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(b)$, in the chosen basis of left cosets of $G_{(t,0)}$ on G_p. Then, $\forall i,j \in \{0, \dots, p-1\}$, we know the expression $y^{-j}x^{-i}bx^iy^j = a^{ip}b$, this implies that

$$\{b_{ii} = a^{ip}b : 0 \le i \le p^2-1\} = \{a^{jp}b : 0 \le j \le p-1\}$$

and the elements of the matrix are non-zero on the diagonal. Therefore, the matrix is described as follows:

$$\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(b) = \begin{pmatrix} B & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & {\scriptstyle p-times} & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & B \end{pmatrix} = \operatorname{diag}\left[\underbrace{B, B, \dots, B}_{p-times}\right],$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & w^t & 0 & \dots & \dots & 0 \\ \vdots & 0 & w^{pt} & \ddots & & \vdots \\ \vdots & & \ddots & w^{2pt} & \ddots & \vdots \\ 0 & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & w^{p(p-1))t} \end{pmatrix} = \operatorname{diag}\left[1, w^t, w^{pt}, w^{2pt}, \dots, w^{p(p-1)t}\right].$$

Remark 8. We calculate the character of the above representation evaluated at b as:

$$\begin{aligned} \chi_{\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)}(b) &= p\sum_{k=0}^{p-1}(w^{tp})^k, \\ &= -p. \end{aligned}$$

(iii) Let us denote by $[x_{ij}]_{0\leq i,j\leq p^2-1}$ the coefficients of the matrix corresponding to $\operatorname{Ind}_{G_{(t,0)}}^{G_p}(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)})(x)$ in the chosen basis of left cosets of $G_{(t,0)}$ on G_p. Since $H \leqslant G_p$ is abelian and $x \in H$, then using Proposition 5 we have $y^{-j_1}x^{-i_1}xx^{i_2}y^{j_2} = x^{-i_1+i_2+1}y^{-j_1+j_2}$. This implies that the elements $x_{i,j}$ are nonzero if and only if $i_1 - i_2 \equiv 1(\operatorname{mod} p)$ and $j_1 \equiv j_2(\operatorname{mod} p)$, therefore

$$\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(x) = \operatorname{diag}\left[\underbrace{X, X, \dots, X}_{p-times}\right],$$

where the element X is described by

$$X = \begin{pmatrix} O_{\frac{p+1}{2}\times\frac{p-1}{2}} & Id_{\frac{p-1}{2}} \\ Id_{\frac{p+1}{2}} & O_{\frac{p-1}{2}\times\frac{p+1}{2}} \end{pmatrix} \in M_p(\mathbb{Z}).$$

Remark 9. We observe that the character of the previous representation in x is 0. Additionally, from the previous subsection, we know that the matrix X is a permutation matrix, therefore, $\operatorname{Ind}_{G_{(t,0)}}^{G_p}\left(\widehat{\overline{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(x)$ is also a permutation matrix and represented by ζ_{p^2} as a product of p disjoint permutations of order p described by the following expression:

$$\zeta_{p^2} = \prod_{k=0}^{p-1}\sigma_{p_k}, \text{ where } \sigma_{p_k} = (a_{1+pk} \;\; a_{2+pk} \;\; \dots \;\; a_{p+pk}),$$

and the sequence $\{a_i\}_{1\leq i\leq p}$ is described as shown in Remark 6 (see Section 2.2.1.2). From this we are in a position to conclude that this permutation matrix has order p and powers less than this number do not fix any element.

(iv) Let us denote by $[y_{ij}]_{0\leq i,j\leq p^2-1} \in M_{p^2}(\mathbb{C})$ the coefficients of the matrix corresponding to $\mathrm{Ind}_{G_{(t,0)}}^{G_p}(\widehat{\chi}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)})(y)$ in the chosen basis of left cosets of $G_{(t,0)}$ on G_p.

Since $H \leqslant G_p$ is abelian and $y \in H$, then $\forall i,j \in \{0,\ldots,p-1\}$, using Proposition 5,we have $y^{-j_1}x^{-i_1}yx^{i_2}y^{j_2} = x^{-i_1i_2+1}y^{-j_1+j_2+1}$. This implies that the elements of the matrix $y_{i,j}$ are nonzero if and only if $i_1 \equiv i_2 (\mathrm{mod}\, p)$ and $j_1 - j_2 \equiv 1 (\mathrm{mod}\, p)$, therefore:

$$Ind_{G_{(t,0)}}^{G_p}\left(\widehat{\chi}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)}\right)(y) = \begin{pmatrix} O_p & O_p & \ldots & \ldots & O_p & I_p \\ I_p & O_p & \ldots & \ldots & \ldots & O_p \\ O_p & I_p & O_p & \ldots & \ldots & O_p \\ O_p & O_p & I_p & O_p & \ldots & O_p \\ \vdots & \vdots & \ldots & \ddots & & \vdots \\ O_p & \ldots & \ldots & \ldots & I_p & O_p \end{pmatrix},$$

$$= \begin{pmatrix} O_{p\times p(p-1)} & Id_p \\ Id_{p(p-1)} & O_{p(p-1)\times p} \end{pmatrix} \in M_{p^2}(\mathbb{Z}).$$

Remark 10. Analogously to the previous case, we describe the representation by a permutational matrix given by the permutation ε_{p^2}, where

$$\varepsilon_{p^2} = \prod_{k=0}^{p-1} \epsilon_{pk} \quad \text{where } \epsilon_{pk} = (k, d_1 + pk, d_2 + pk, \ldots, d_{p-1} + pk),$$

and the sequence $\{d_j\}$ is described by the expression: $d_j = (p-j)p + 1$, with $1 \leq j \leq p-1$.

From the above, we deduce that the character of the above representation for y is 0, since the permutation described is composed of disjoint cycles of order p, and therefore, the permutation matrix of this representation has order p and any lower power does not fix any element. Moreover, the composition of the permutations ζ and ε satisfies these characteristics, since it is also composed of disjoint cycles of order p.

From now on, for all $t \in \{1, \ldots, p-1\}$, we denote these group representations by:

$$V_{(t)}^{(p^2)} := \mathrm{Ind}_{G_{(t,0)}}^{G_p} \left(\widehat{\widehat{\chi}}_{(t,0)} \otimes \widetilde{\rho}_{(t,0)}^{(0,0)} \right).$$

In summary, the complex irreducible representations of G_p are collected in the following theorem.

Theorem 7. *Let the notation be as in the previous results. The group G_p has a total of $p^3 + p^2 - 1$ complex irreducible representations, which are as follows:*

(i) For k, α and β integers such that $0 \leq k, \alpha, \beta \leq p-1$; $V_{(k,\alpha,\beta)}^{(1)}$ are the representations of degree 1.

(ii) For s, β integers such that $1 \leq s \leq p-1$ and $0 \leq \beta \leq p-1$; $V_{(s,\beta)}^{(p)}$ are the representations of degree p.

(iii) For t integer such that $1 \leq t \leq p-1$; $V_{(t)}^{(p^2)}$ are the representations of degree p^2.

2.2.2 Rational irreducible representations of G_p .

Using the method of (1.1) (see [12, Thm 70.15]), one obtains the irreducible rational representations of G_p by a direct sum, which depends on the Schur index of the irreducible complex representations and their Galois conjugates.

First, we study the Schur index of each complex representations of G_p.

Proposition 9. *Let V be a complex irreducible representation of the group G_p, then its Schur index m_V is 1.*

Proof. We know that $|G_p| = p^5$, then the proof follows directly from the fact that the Schur index m_V associated to an irreducible representation V, satisfies the equality $m_V = 1$ unless $p = 2$ and $\sqrt{-1} \notin F$ in which case $m_V \leq 2$. See Roquette corollary [23, Coroll. 10.14] of *Goldschmidt-Isaacs* Theorem [23, Thm. 10.12]

□

We obtain the rational irreducible representations by adding, in a direct sum, the Galois conjugate [12, Thm. 70.15 and Ex. 70.2]. We collect that in the following theorem.

Theorem 8. *Let the notation be as in Section 2.2.1. Then the rational irreducible representations of G_p are follows:*

(i) There is one rational irreducible representation of degree 1, this is defined by $W_{(0)} := V_{(0,0,0)}$ and it is the trivial representation.

(ii) There are p^2+p+1 rational irreducible representations of degree $p-1$, which are:

$$\begin{cases} W_{(0,0,1)} & := \displaystyle\bigoplus_{\sigma\in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(1)}_{(0,0,1)}\right)^{\sigma}, \\ W_{(0,1,k)}, & := \displaystyle\bigoplus_{\sigma\in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(1)}_{(0,1,k)}\right)^{\sigma}, \quad \text{for } k\in\mathbb{Z} \text{ and } 0\le k\le p-1, \\ W_{(1,m,n)} & := \displaystyle\bigoplus_{\sigma\in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(1)}_{(1,m,n)}\right)^{\sigma}, \quad \text{for } m,n\in\mathbb{Z} \text{ and } 0\le m,n\le p-1. \end{cases}$$

(iii) There are p rational irreducible representations of degree $p(p-1)$, which are

$$W_{(1,u)} := \bigoplus_{\sigma\in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(p)}_{(1,u)}\right)^{\sigma}, \text{ for } u\in\mathbb{Z} \text{ and } 0\le u\le p-1,$$

(iv) There is one rational irreducible representation of degree $p^2(p-1)$, which is

$$W_{(1)} := \bigoplus_{\sigma\in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(p^2)}_{(1)}\right)^{\sigma}.$$

Proof. We recall the following notations: let K_V be the field associated to the representation V obtained by adjoining to $\mathbb{Q}$ the values of the character χ_V and m_V the Schur index of the complex representation V. From Proposition 9, we have $m_V = 1$ for all complex irreducible representations of G_p. Thus every field of definition L_V correspond to the character field K_V. We have the following statements for the characters:

(i) First, the representation $V^{(1)}_{(0,0,0)}$ is the trivial representation and we denote

$$V^{(1)}_{(0,0,0)} =: W_{(0)}.$$

(ii) Second, for $0 \leq k, \alpha, \beta \leq p-1$ and $(k, \alpha, \beta) \neq 0$, the representations $V^{(1)}_{(k,\alpha,\beta)}$ have as character fields $K_{V^{(1)}_{(k,\alpha,\beta)}} = \mathbb{Q}(w^p)$, then

$$\mathrm{Gal}\left(K_{V^{(1)}_{(k,\alpha,\beta)}} \Big/ \mathbb{Q}\right) = \left\{\sigma_i \in \mathrm{Aut}\left(\mathbb{Q}(w^p)/\mathbb{Q}\right) : \sigma_i(w^p) = w^{pi}, 1 \leq i \leq p-1\right\} \simeq (\mathbb{Z}/p\mathbb{Z})^{\times}.$$

This implies

$$\left\{\left(V^{(1)}_{(k,\alpha,\beta)}\right)^{\sigma_i}\right\}_{\sigma_i \in \mathrm{Aut}(\mathbb{Q}(w^p)/\mathbb{Q})} = \left\{V^{(1)}_{(ik,i\alpha,i\beta)} : 1 \leq i \leq p-1\right\}.$$

Hence, there are p^2+p+1 rational irreducible representations of degree $p-1$. If we condition the first element as null or not null, and then the same with the second, we describe the representations by the following expressions:

$$\begin{cases} W_{(0,1,k)} & := \displaystyle\bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(1 \cdot V^{(1)}_{(0,1,k)}\right)^{\sigma_i}, \quad \text{for } 0 \leq k \leq p-1. \\ W_{(0,0,1)} & := \displaystyle\bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(1 \cdot V^{(1)}_{(0,0,1)}\right)^{\sigma_i}. \\ W_{(1,m,n)} & := \displaystyle\bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(1 \cdot V^{(1)}_{(1,m,n)}\right)^{\sigma_i}, \quad \text{for } 0 \leq m, n \leq p^2. \end{cases}$$

(iii) Third, for the case of complex irreducible representations of degree p; $V^{(p)}_{(s,\beta)}$ with $1 \leq s \leq p-1$ and $0 \leq \beta \leq p-1$, we have the fields $K_{V^{(1)}_{(s,\beta)}} = \mathbb{Q}(w^p)$, then

$$\mathrm{Gal}\left(K_{V^{(p)}_{(s,\beta)}} \Big/ \mathbb{Q}\right) = \left\{\sigma_i \in \mathrm{Aut}(\mathbb{Q}(w^p)/\mathbb{Q}) : \sigma_i(w^p) = w^{pi}, 1 \leq i \leq p-1\right\} \simeq (\mathbb{Z}/p\mathbb{Z})^{\times}.$$

From the above we are in a position to conclude that

$$\left\{\left(V^{(p)}_{(s,\beta)}\right)^{\sigma_i}\right\}_{\sigma_i \in \mathrm{Aut}(\mathbb{Q}(w^p)/\mathbb{Q})} = \left\{V^{(p)}_{(is,i\beta)} : 1 \leq i \leq p-1\right\},$$

which implies that there are p rational irreducible representations of degree $p(p-1)$. These are described by:

$$W_{(1,u)} := \bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(1 \cdot V^{(p)}_{(1,u)}\right)^{\sigma_i} = \bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V^{(p)}_{(1,u)}\right)^{\sigma_i}, \text{ for } 0 \leq u \leq p-1.$$

(iv) Finally, for the case of complex irreducible representations of degree p^2; $V_{(t)}^{(p^2)}$ with $1 \leq t \leq p-1$, we have fields $K_{V_t^{(p^2)}} = \mathbb{Q}(w^p)$, then

$$\mathrm{Gal}\left(K_{V_{(t)}^{(p^2)}} \Big/ \mathbb{Q}\right) = \left\{\sigma_i \in \mathrm{Aut}(\mathbb{Q}(w^p)/\mathbb{Q}) : \sigma_i(w^p) = w^{pi}, 1 \leq i \leq p-1\right\} \simeq (\mathbb{Z}/p\mathbb{Z})^{\times}.$$

This implies that

$$\left\{\left(V_{(t)}^{(p^2)}\right)^{\sigma_i}\right\}_{\sigma_i \in \mathrm{Aut}(\mathbb{Q}(w^p)/\mathbb{Q})} = \left\{V_{(it)}^{(p^2)} : 1 \leq i \leq p-1\right\}.$$

Therefore, there is 1 rational irreducible representation of degree $p^2(p-1)$. This can be described by:

$$W_{(1)} := \bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(1 \cdot V_{(1)}^{(p^2)}\right)^{\sigma_i} = \bigoplus_{\sigma_i \in \mathrm{Gal}(\mathbb{Q}(w^p)/\mathbb{Q})} \left(V_{(1)}^{(p^2)}\right)^{\sigma_i}.$$

This concludes the description of the rational irreducible representations of the group G_p. □

2.3 Action of G_p on Riemann surfaces

We recall the known fact that the dimension of a family of Riemann surfaces with the action of a finite group G with signature $(\gamma; a_1, \ldots, a_t)$ is

$$3\gamma - 3 + t.$$

According to Proposition 6, the group G_p does not act with signature $(0; a, b, c)$, since this happens if and only if the group can be generated by two elements, which is not the case for G_p.

Since the minimal set of generators of G_p has 3 elements, the minimal number of branch values for an action with total quotient of genus 0 is 4. Such actions correspond to one dimensional families. Another way of getting a one dimensional family is acting with one branch value and total quotient of genus 1.

Considering that G_p has elements of orders $1, p$ and p^2, only, the list of possible signatures corresponding to one dimensional families is in the following table.

We study whether there is a generating vector for each signature corresponding to one dimensional families, in order to prove which actions actually exist.

We collect the information obtained computationally (using Sage) for $p = 3$ in the same table, and give the answer for the general case in Remark 11.

Case.	Signature	Conclusion for $p = 3$
(i)	$(1; p)$	It exists
(ii)	$(1; p^2)$	Does not exist.
(iii)	$(0; p, p, p, p)$	Does not exist.
(iv)	$(0; p, p, p, p^2)$	Does not exist.
(v)	$(0; p, p, p^2, p^2)$	It exists and it is the starting point for the families we study here
(vi)	$(0; p, p^2, p^2, p^2)$	No result with [3], computed by hand (see below)
(vii)	$(0; p^2, p^2, p^2, p^2)$	No result with [3], computed by hand (see below)

Remark 11. Although we could not find computationally (using the algorithms in [3]) generating vectors for cases (vi) and (vii) for $p = 3$, we know from the analysis of the general situation that they do exist. Besides, using the algorithm in [30], code available in https://github.com/jenpaulhus/breuer-modified, Prof. Paulhus told us that she found several for $p = 3$ for both cases.

The general situation, meaning $p \geq 3$ a prime, is as follows:

- Case (i) exists, take the generating vector $(x, a; b^{-1})$, but we prefer to focus our analysis on cases where the total quotient is of genus 0.

- Case (ii) does not exist since the order of the commutator subgroup G_p' is p^2 and it is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. So it has no elements of order p^2.

- Cases (iii) and (iv) do not exist. Since all elements of order p are in the subgroup $R = \langle a^p, b, x, y\rangle$, then if g is not in R, $g = a^k b^r x^n y^m$ with $k, r, n, m \in \mathbb{N}_0$, $GCD(k, p) = 1$, $k > 1$, has order p^2. Therefore, signatures are not realizable in G_p, since G_p cannot be generated only by elements of order p and three elements of R do not generate an element of order p^2.

- Cases (v), (vi) and (vii) do exist.

  We work out case (v) in detail, since the techniques involved in the other two ((vi) and (vii)) are similar. We leave these cases, together with signature $(1; p)$ for future research.

  For (vi) and (vii), it is enough to consider the elements a^{p-2}, ax, ay of order p^2. It follows that $a^{p-2}(ax)(ay)$ has order p and $\langle a^{p-2}, ax, ay\rangle = G_p$, then the generating vector $[(a^{p-2}(ax)(ay))^{-1}, a^{p-2}, ax, ay]$ realizes the signature $(0; p, p^2, p^2, p^2)$.

On the other hand, the element $a(ax)(ay) \in G_p$ has order p^2, we can consider a^2, so $a^2(ax)(ay)$ has order p^2 because $GCD(p,2) = 1$. Moreover, $\langle a, ax, ay \rangle = G_p$, then it is sufficient to consider the generating vector $[(a(ax)(ay))^{-1}, a, ax, ay]$ that realizes the signature $(0; p^2, p^2, p^2, p^2)$.

For (v) see Theorem 9.

Theorem 9. *There exists a Riemann surface X of genus $g = p^5 - p^4 - p^3 + 1$, on which G_p acts with signature $s = (0; p^2, p^2, p, p)$ and G_p has a generating s-vector described by $[a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$.*

Proof. By Riemann's Existence Theorem, since the Riemann-Hurwitz (1.12) equation is satisfied, we need to prove the existence of a generating vector to be able to conclude that the Riemann surface X exists. According to the structure of G_p the following information about the orders composing the generating vector is known:

(i) $|a^{-1}| = |a| = p^2$,

(ii) $|xya^{p+1}b| = p^2$ (proved in Proposition 5),

(iii) $|x^{-1}| = |x| = |y^{-1}| = |y| = p$.

Therefore, $[a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$ is an $(0; p^2, p^2, p, p)$-generating vector. □

Proposition 10. *Let X be a Riemann Surface with G–action of geometric signature $(0; [p^2, \langle a^{-1}\rangle], [p^2, \langle xya^{p+1}b\rangle], [p, \langle y^{-1}\rangle], [p, \langle x^{-1}\rangle])$. Then for H and K subgroups of G_p defined in Section 2.1.1, the genus of X/H and X/K is $p^3 - \frac{3}{2}p^2 - p + \frac{3}{2}$.*

Proof. We define the subgroups $G_1 := \langle a^{-1}\rangle$, $G_2 := \langle xya^{p+1}b\rangle$, $G_3 := \langle y^{-1}\rangle$, $G_4 := \langle x^{-1}\rangle$. Then the orders of the set corresponding to the double cosets are

- $|H \setminus G / G_1| = \left|\left\{H1G_1, HbG_1, Hb^2G_1, \ldots, Hb^{p-1}G_1\right\}\right| = p.$

- $|H \setminus G / G_2| = \left|\left\{H1G_2, HbG_2, Hb^2G_2, \ldots, Hb^{p-1}G_2\right\}\right| = p.$

- $$|H \setminus G / G_3| = \left|\left\{Ha^ib^jG_3 : 0 \le i, j \le p-1\right\} \cup \left\{Ha^{ip}b^jG_3 : 0 \le j \le p-1,\ 1 \le i \le p-1\right\}\right|, \\ = 2p^2 - p.$$

- $$|H \setminus G / G_4| = \left|\left\{H1G_4, HaG_4, Ha^2G_4, Ha^{p^2-1}G_4, HbG_4, Hb^2G_4, \ldots, Hb^{p-1}G_4\right\}\right| \\ = p^2 + p - 1.$$

Finally, using [35, Prop.3.4], we are in a position to conclude that

$$g_{X/H} = p^3 - \frac{3}{2}p^2 - p + \frac{3}{2}.$$

We proceed analogously to conclude for K. $\square$

Remark 12. We point out that using the algorithms of [3], we computed that for $p = 3$ there are exactly 16 non-equivalent actions for the signature $(0; p, p, p^2, p^2)$ (see Section 2.4). It then follows that there are 16 equisymmetric strata, see [24, 5] for definitions, in the singular locus of the moduli space $\mathcal{M}_{136}$. This work starts with one of these actions and explores its geometry, describing in this way one of these strata and the corresponding family in $\mathcal{A}_{136}$. Moreover, we study the geometry of this action for any prime p, describing in this way one stratum in $\mathcal{M}_{p^5-p^4-p^3+1}$ and the corresponding family in $\mathcal{A}_{p^5-p^4-p^3+1}$. The same work can be done for the other generating vectors, for $p = 3$ as well as in the general case.

Finally, we expand the signature and this generating vector in a natural way (see Section 3.2), and study the geometry of this new action. As before, the same study could be done for every other generating vector.

Expanding the signature is an idea that interests us because it allows us to extend the analysis of the decomposition of Jacobian varieties of curves in our original one-dimensional family to Jacobians corresponding to curves of higher genera and in higher dimensional families.

We say that *we extend a signature*, if we add pairs of branch points repeating the orders in the original signature. Analogously, we describe an extended version of a generating vector by adding appropriate elements of the group (see sections 3.2 for details).

In the case of the signature and generating vector presented in Theorem 9, we consider a collection of families of Riemann surfaces with four discrete parameters $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$ which admit the action of G_p with extended signature

$$(0; (p^2)^{2t_1+1}, (p^2)^{2t_2+1}, (p)^{2t_3+1}, (p)^{2t_4+1}),$$

and extended generating vector

$$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}],$$

where $(\alpha, \beta)^t$ means $\alpha, \beta, \overset{t-times}{\ldots}, \alpha, \beta$.

2.4 The signature $(0; p^2, p^2, p, p)$ in actions of G_p on Riemann surfaces for $p = 3$.

The existence of the equisymmetric stratification of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, consists of the fact that each stratum is formed by the points in the moduli space corresponding to equisymmetric surfaces.

Two closed Riemann surfaces X and Y of genus g are said to be equisymmetric if their automorphism groups determine finite conjugate subgroups in the mapping class group of genus g; i.e., the actions of their automorphism groups are topologically equivalent. The branch locus of the moduli space consists of the strata corresponding to surfaces of genus $g > 2$ that admit nontrivial automorphisms.

The actions σ_1, σ_2 of G on X are topologically equivalent if there exists $\Phi \in \mathrm{Aut}(G)$ and h in the group of orientation-preserving homeomorphisms $\mathrm{Homeo}^+(X)$ of X, such that the following diagram is commutative for all $g \in G$.

$$\begin{tikzcd} X \arrow[r, "\sigma_1(\Phi(g))"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ X \arrow[r, "\sigma_2(g)"'] & X \end{tikzcd}$$

In other words, $\sigma_2(g) = h\,\sigma_1(\Phi(g))\,h^{-1}$, for all $g \in G$.

If $h \in \mathrm{Aut}(X)$, then we will say that σ_1, σ_2 are analytically (or conformally) equivalent.

Remark 13. Actions that are analytically equivalent are topologically equivalent, but not conversely. We will not address this question in this work, see [17, 9] for details.

Using the algorithms found in [3], which run on Sagemath, we calculate that for $p = 3$ there are exactly 16 topologically non-equivalent actions for the signature $(0; p^2, p^2, p, p)$, therefore there are 16 equisymmetric strata. The description of the commands is displayed below:

```
F. <a, b, x, y> = FreeGroup(4)
G = F/[a^9, b^3, x^3, y^3, a * b * a^ - 1 * b^ - 1, x * y * x^ - 1 * y^ - 1,
a^ - 1 * x^ - 1 * a * x * b^ - 1, b^ - 1 * x^ - 1 * b * x * a^ - 3, y^ - 1 * a * y * a^ - 4,
y^ - 1 * b * y * b^ - 1]
Gs = G.as_permutation_group()
A = G.as_permutation_group().0
B = G.as_permutation_group().1
X = G.as_permutation_group().2
Y = G.as_permutation_group().3
gen = find_generator_representatives(Gs, [9, 9, 3, 3])
L2 = [[ as_word (k, Gs, [A, B, X, Y]) for k in x] for x in gen ]
```

We get the description of the 16 non-equivalent actions for the signature $(0; p, p, p^2, p^2)$ with $p = 3$:

$[a, a^8x, y, x^2y^2]$
$[a, a^8x, y^2, yx^2]$
$[a, a^8x, by, b^2x^2y^2]$
$[a, a^8x, by^2, b^2x^2y]$
$[a, a^8x, yb^2, bx^2y^2]$
$[a, a^8x, y^2b^2, ybx^2]$
$[a, a^8x, yx, xy^2]$
$[a, a^8y, x, x^2y^2]$
$[a, a^8y^2, x, yx^2]$
$[ax, a^8x, y, x^2b^2x^2y^2]$
$[ax, a^8x, y^2.ybxb]$
$[ax, a^8x, by, y^2xb]$
$[ax, a^8x, by^2, yxb]$
$[ax, a^8x, yx^2, b^2x^2y^2]$
$[ax, a^8x^2y, x, b^2x^2y^2]$
$[ax, a^8x^2y^2, x, b^2x^2y]$

The generating vector of Theorem 9, which is the one studied in this work, is equivalent to the second one of these 16 non-equivalent generating vectors, listed above.

On Extendability of the $[a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$−action on compact Riemann surfaces.

Among the 16 generating vectors already described for G_3, there are some of them that correspond to actions of G_3 as the full group of automorphisms of the general element in the family, and others for which the action extends (see [6]). That is, there exists a super-group $\tilde{G}$ such that $G_3 < \tilde{G} < \mathrm{Aut}(X)$ and the action of $\tilde{G}$ restricts to the original action of G. The question we address here is to determine whether our action extends or not.

It follows from Singerman's table (see [39]) that for the signature $(0; p^2, p^2, p, p)$, if the action extends it does to a group $\tilde{G}$ such that $[\tilde{G} : G_p] = 2$. Moreover, the signature of the action of $\tilde{G}$ on X has signature $(0; 2, 2, p, p^2)$. Moreover, in [36] a condition is given under which the action extends, as well as a description of $\tilde{G}$.

Proposition 11. *Let X be a compact Riemann surface. The group G_3 acts on X with signature $(0; 9, 9, 3, 3)$ and generating vector $[a^{-1}, xya^4b, y^{-1}, x^{-1}]$, and G_3 corresponds to the full automorphisms group of the general element; that is, the action does not extend.*

Proof. The existence of this action was proved in Theorem 9. In order to prove that it does not extend, fix $\nu := [a^{-1}, xya^4b, y^{-1}, x^{-1}]$. By Ries Theorem statement 4 (see Corollary 1), there exists a group $\tilde{G}$ extending the action of G_3 determined by ν if and only if there exists $\alpha \in \mathrm{Aut}(G_3)$ commuting with the elements of the same order for some generating vector equivalent to ν.

Using Magma, Prof. Paulhus wrote a code which confirmed that there is no such α for every generating vector equivalent to ν.

□

Unfortunately, to prove that our action does not extend for every p, is beyond our possibilities since it would require describing all generating vectors equivalent to our ν. We leave this as an open question for future work.

Chapter 3

Decomposition of Jacobian varieties with G_p action.

In this chapter we describe the group algebra decomposition (GAD) of Jacobian varieties with the action of G_p induced by the action determined by the generating vector described in Section 2.3. In particular, we compute the dimensions of the primitive factors.

3.1 Group algebra descomposition

In Chapter 1 we recalled what the group algebra decomposition of an abelian variety with a group action is, see equation (1.8). Since we have obtained the irreducible reresentations of G_p, we are in conditions to describe how the group algebra decomposition of an abelian variety with the action of G_p is in general. Then, we specialize the action to the one we are interested in (see Section 2.3).

Theorem 10. *Let X be a Riemann surface with the action of G_p. Then, the group algebra decomposition of the corresponding Jacobian variety decomposes generically as follows*

$$JX \sim JX_{G_p} \times \prod_{i=1}^{p^2+p+1} B_i^1 \times \prod_{i=1}^{p} C_i^p \times D^{p^2}, \tag{3.1}$$

where we use the following notation to classify each primitive factor:

(i) B_i is a primitive factor associated to the rational irreducible representations of degree $p-1$ for all $1 \leq i \leq p^2+p+1$.

(ii) C_i *is a primitive factor associated to the rational irreducible representations of degree* $p(p-1)$ *for all* $1 \leq i \leq p$.

(iii) D *is the primitive factor associated to the unique rational irreducible representation of degree* $p^2(p-1)$.

Proof. It is a direct application of the equation (1.5) to the group G_p according to the irreducible complex and rational representations described in Theorem 7 and Theorem 8, respectively (see Sections 2.2.1 and 2.2.2 for more details).

Finally, the exponent of each one of these factors, as indicated in equation (1.6), corresponds to the degree of the complex irreducible representation associated to the rational irreducible representation corresponding to each isotypical factor divided by the Schur index, which in each case, as shown in Proposition 9, is 1.

□

Since we have a concrete action of G_p we are interested in understanding it with signature $(0; p^2, p^2, p, p)$ and its extended versions parameterized by $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$ (see Section 2.3). We compute the dimensions of the primitive factors using Proposition 2 [33], recalled in Chapter 1, in the decomposition of the group algebra of the corresponding Jacobian variety, as in the equation (1.8). In particular, many of them vanish for this action of G_p.

Theorem 11. *Let* p *be a prime number* $p > 2$. *Consider the group* G_p *acting on a Riemann surface* X *with geometric signature*

$$(0; [p^2, \overline{\langle a^{-1}\rangle}], [p^2, \overline{\langle xya^{p+1}b\rangle}], [p, \overline{\langle y^{-1}\rangle}], [p, \overline{\langle x^{-1}\rangle}]).$$

Then, the associated variety JX *is described by the following decomposition:*

$$JX \sim \prod_{i=1}^{4p-8} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,p-1)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(2p-3)}{2})} \times C^p_{(1,\frac{(p-1)(p-3)}{2})} \times D^{p^2}_{(1,p(p-1)^2)}, \tag{3.2}$$

where the pair (i, j) *as a subscript has to be read as follows:* i *refers to the positional value in the product, and* j *is the dimension of the primitive factor indexed by* i.

Proof. To compute the dimensions of each factor we use Proposition 2. First, we need to compute the dimension of each V^{G_k} for each

$$G_k \in \{\langle a^{-1}\rangle, \langle xya^{p+1}b\rangle, \langle y^{-1}\rangle, \langle x^{-1}\rangle\}$$

and V complex irreducible representation. We use the notation of Theorem 7, denote $Fix_{G_k}V$ by V^{G_k}, Res^G_H the restriction representation of the subgroup H of G, and w the p^2-th root of the unit $e^{\frac{2\pi i}{p^2}}$. Then, $\sum_{j=1}^{p}(w^p)^j = 0$ and for each complex irreducible representation, we have the following statements:

1) For the representation $V_{(0,0,1)}$, we calculate:

$$\begin{aligned}(i)\ \dim Fix_{\langle a^{-1}\rangle}V^{(1)}_{(0,0,1)} &= \left\langle 1_{\langle a^{-1}\rangle}, Res^G_{\langle a^{-1}\rangle}V^{(1)}_{(0,0,1)}\right\rangle_{\langle a^{-1}\rangle}\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle a^{-1}\rangle}(a^{-j})\cdot Res^G_{\langle a^{-1}\rangle}V^{(1)}_{(0,0,1)}(a^j)\right)\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}1\\ &= 1.\end{aligned}$$

$$\begin{aligned}(ii)\ \dim Fix_{\langle xya^{p+1}b\rangle}V^{(1)}_{(0,0,1)} &= \left\langle 1_{\langle xya^{p+1}b\rangle}, Res^G_{\langle xya^{p+1}b\rangle}V^{(1)}_{(0,0,1)}\right\rangle_{\langle xya^{p+1}b\rangle}\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle xya^{p+1}b\rangle}((xya^{p+1}b)^j)\cdot Res^G_{\langle xya^{p+1}b\rangle}V^{(1)}_{(0,0,1)}((xya^{p+1}b)^{-j})\right)\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}V^{(1)}_{(0,0,1)}\left((xya^{p+1}b)^j\right)\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(V^{(1)}_{(0,0,1)}(xya^{p+1}b)\right)^j\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left((V^{(1)}_{(0,0,1)}(x))^j(V^{(1)}_{(0,0,1)}(y))^j(V^{(1)}_{(0,0,1)}(a^{p+1}))^j(V^{(1)}_{(0,0,1)}(b))^j\right)\\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}(w^p)^j\\ &= 0.\end{aligned}$$

$$
\begin{aligned}
(iii)\ \dim Fix_{\langle x^{-1}\rangle} V^{(1)}_{(0,0,1)} &= \left\langle 1_{\langle x^{-1}\rangle}, Res^G_{\langle x^{-1}\rangle} V^{(1)}_{(0,0,1)} \right\rangle_{\langle x^{-1}\rangle} \\
&= \frac{1}{p}\sum_{j=1}^{p} \left(1_{\langle x^{-1}\rangle}((x^{-1})^j) \cdot Res^G_{\langle x^{-1}\rangle} V^{(1)}_{(0,0,1)} \left((x^{-1})^{-j}\right)\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} V^{(1)}_{(0,0,1)} \left(x^j\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} 1 \\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ \dim Fix_{\langle y^{-1}\rangle} V^{(1)}_{(0,0,1)} &= \left\langle 1_{\langle y^{-1}\rangle}, Res^G_{\langle y^{-1}\rangle} V^{(1)}_{(0,0,1)} \right\rangle_{\langle y^{-1}\rangle} \\
&= \frac{1}{p}\sum_{j=1}^{p} \left(1_{\langle y^{-1}\rangle}((y^{-1})^j) \cdot Res^G_{\langle y^{-1}\rangle} V^{(1)}_{(0,0,1)} \left((y^{-1})^{-j}\right)\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} V^{(1)}_{(0,0,1)} \left(y^j\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} (w^p)^j \\
&= 0.
\end{aligned}
$$

Then, from $(i), (ii), (iii), (iv)$, and the calculation of the Galois group in the proof of Th.8 and Prop. 9 (see Section 2.2.2), we are in a position to conclude

$$\dim B^1_{W_{(0,0,1)}} = 0$$

2) For the representation $V^{(1)}_{(0,1,k)}$, with $0 \leq k \leq p-1$, we have the following values:

$$
\begin{aligned}
(i)\ \dim Fix_{\langle a^{-1}\rangle} V^{(1)}_{(0,1,k)} &= \left\langle 1_{\langle a^{-1}\rangle}, Res^G_{\langle a^{-1}\rangle} V^{(1)}_{(0,1,k)} \right\rangle_{\langle a^{-1}\rangle} \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2} \left(1_{\langle a^{-1}\rangle}(a^{-j}) \cdot Res^G_{\langle a^{-1}\rangle} V^{(1)}_{(0,1,k)}(a^j)\right) \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2} 1 \\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ \dim Fix_{\langle xya^{p+1}b\rangle} V^{(1)}_{(0,1,k)} &= \left\langle 1_{\langle xya^{p+1}b\rangle}, Res^{G}_{\langle xya^{p+1}b\rangle} V^{(1)}_{(0,1,k)} \right\rangle_{\langle xya^{p+1}b\rangle} \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle xya^{p+1}b\rangle}((xya^{p+1}b)^j)\cdot Res^{G}_{\langle xya^{p+1}b\rangle} V^{(1)}_{(0,1,k)}((xya^{p+1}b)^{-j})\right) \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2} V^{(1)}_{(0,1,k)}\left((xya^{p+1}b)^j\right) \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(V^{(1)}_{(0,1,k)}(xya^{p+1}b)\right)^j \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left((V^{(1)}_{(0,1,k)}(x))^j\cdot(V^{(1)}_{(0,1,k)}(y))^j\cdot(V^{(1)}_{(0,1,k)}(a^{p+1}))^j\right) \\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(w^{p(1+k)}\right)^j \\
&= \begin{cases} 1 \quad , & k = p-1, \\ 0 \quad , & 0 \le k \le p-2. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ \dim Fix_{\langle x^{-1}\rangle} V^{(1)}_{(0,1,k)} &= \left\langle 1_{\langle x^{-1}\rangle}, Res^{G}_{\langle x^{-1}\rangle} V^{(1)}_{(0,1,k)} \right\rangle_{\langle x^{-1}\rangle} \\
&= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle x^{-1}\rangle}((x^{-1})^j)\cdot Res^{G}_{\langle x^{-1}\rangle} V^{(1)}_{(0,1,k)}\left((x^{-1})^{-j}\right)\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} V^{(1)}_{(0,1,k)}\left(x^j\right) \\
&= \frac{1}{p}\sum_{j=1}^{p}(w^p)^j \\
&= 0.
\end{aligned}
$$

$$(iv)\ \dim Fix_{\langle y^{-1}\rangle} V^{(1)}_{(0,1,k)} = \left\langle 1_{\langle y^{-1}\rangle}, Res^{G}_{\langle y^{-1}\rangle} V^{(1)}_{(0,1,k)} \right\rangle_{\langle y^{-1}\rangle}$$
$$= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle y^{-1}\rangle}((y^{-1})^{j}) \cdot Res^{G}_{\langle y^{-1}\rangle} V^{(1)}_{(0,1,k)}\left((y^{-1})^{-j}\right)\right)$$
$$= \frac{1}{p}\sum_{j=1}^{p} V^{(1)}_{(0,1,k)}\left(y^{j}\right)$$
$$= \frac{1}{p}\sum_{j=1}^{p}\left(w^{pk}\right)^{j}$$
$$= \begin{cases} 1 & , \text{ if } \quad k = 0, \\ 0 & , \text{ if } \quad 1 \leq k \leq p-1. \end{cases}$$

Then, from $(i), (ii), (iii), (iv)$, and the calculation of the Galois group in the proof of Th.8 and Prop. 9 (see Section 2.2.2), we conclude

$$\dim B^{1}_{W_{(0,1,k)}} = \begin{cases} 0 & , \text{if} \quad k = 0, p-1, \\ \frac{p-1}{2} & , \text{if} \quad 1 \leq k \leq p-2. \end{cases}$$

3) Consider the representation $V_{(1,m,n)}$, then we calculate:

$$(i)\ \dim Fix_{\langle a^{-1}\rangle} V^{(1)}_{(1,m,n)} = \left\langle 1_{\langle a^{-1}\rangle}, Res^{G}_{\langle a^{-1}\rangle} V^{(1)}_{(1,m,n)} \right\rangle_{\langle a^{-1}\rangle}$$
$$= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle a^{-1}\rangle}(a^{-j}) \cdot Res^{G}_{\langle a^{-1}\rangle} V^{(1)}_{(1,m,n)}(a^{j})\right)$$
$$= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(w^{p}\right)^{j}$$
$$= 0.$$

$$
\begin{aligned}
(ii)\ \dim Fix_{\langle xya^{p+1}b\rangle}V^{(1)}_{((1,m,n)} &= \left\langle 1_{\langle xya^{p+1}b\rangle}, Res^G_{\langle xya^{p+1}b\rangle}V^{(1)}_{(1,m,n)}\right\rangle_{\langle xya^{p+1}b\rangle}\\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle xya^{p+1}b\rangle}((xya^{p+1}b)^j)\cdot Res^G_{\langle xya^{p+1}b\rangle}V^{(1)}_{(1,m,n)}((xya^{p+1}b)^{-j})\right)\\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}V^{(1)}_{(1,m,n)}\left((xya^{p+1}b)^j\right)\\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(V^{(1)}_{(1,m,n)}(xya^{p+1}b)\right)^j\\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left((V^{(1)}_{(1,m,n)}(x))^j\cdot(V^{(1)}_{(1,m,n)}(y))^j\cdot(V^{(1)}_{(1,m,n)}(a^{p+1}))^j\right)\\
&= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(w^{p(m+n+1)}\right)^j\\
&= \begin{cases}1 & ,\text{if}\quad n=p-m-1,\\ 0 & ,\text{if}\quad n\neq p-m-1.\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ \dim Fix_{\langle x^{-1}\rangle}V^{(1)}_{((1,m,n)} &= \left\langle 1_{\langle x^{-1}\rangle}, Res^G_{\langle x^{-1}\rangle}V^{(1)}_{((1,m,n)}\right\rangle_{\langle x^{-1}\rangle}\\
&= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle x^{-1}\rangle}((x^{-1})^j)\cdot Res^G_{\langle x^{-1}\rangle}V^{(1)}_{((1,m,n)}\left((x^{-1})^{-j}\right)\right)\\
&= \frac{1}{p}\sum_{j=1}^{p}V^{(1)}_{((1,m,n)}\left(x^j\right)\\
&= \frac{1}{p}\sum_{j=1}^{p}(w^{pm})^j\\
&= \begin{cases}1 & ,\text{if}\qquad m=0,\\ 0 & ,\text{if}\quad 1\leq m\leq p-1.\end{cases}
\end{aligned}
$$

$$(iv)\ \dim\ Fix_{\langle y^{-1}\rangle} V^{(1)}_{((1,m,n)} = \left\langle 1_{\langle y^{-1}\rangle}, Res^{G}_{\langle y^{-1}\rangle} V^{(1)}_{((1,m,n)} \right\rangle_{\langle y^{-1}\rangle}$$

$$= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle y^{-1}\rangle}((y^{-1})^{j}) \cdot Res^{G}_{\langle y^{-1}\rangle} V^{(1)}_{((1,m,n)}\left((y^{-1})^{-j}\right)\right)$$

$$= \frac{1}{p}\sum_{j=1}^{p} V^{(1)}_{((1,m,n)}\left(y^{j}\right)$$

$$= \frac{1}{p}\sum_{j=1}^{p} (w^{pn})^{j}$$

$$= \begin{cases} 1 & ,\text{if} \quad n = 0, \\ 0 & ,\text{if} \quad 1 \le n \le p-1. \end{cases}$$

Then, from $(i), (ii), (iii), (iv)$, and the calculation of the Galois group in the proof of Th.8 and Prop. 9 (see Section 2.2.2), we conclude

$$\dim B^{1}_{W_{(1,m,n)}} = \begin{cases} 0 & ,\text{if} \quad (m,n) \in \{(0,0),(0,p-1),(p-1,0)\}, \\ \frac{p-1}{2} & ,\text{if} \quad (m,n) \in \{(i,0),(0,i),(i,p-i-1) : 1 \le i \le p-2\}, \\ p-1 & ,\text{if} \quad (m,n) \in \{(i,j) : 1 \le i,j \le p-1 \ \wedge\ j \ne p-i-1\}. \end{cases}$$

4) Consider the representation $V^{(p)}_{(1,u)}$, $u \in \mathbb{N}_0$, $0 \le u \le p-1$, using the information in Remarks 3, 4 and 5 (see Section 2.2.1.2), we obtain the following equalities:

$$\chi_{V^{(p)}_{(1,u)}}(a^{j}) = \sum_{i=1}^{p}\left(w^{pj}\right)^{i} = \begin{cases} 0 & ,\text{if} \quad GCD(j,p) = 1, \\ p & ,\text{if} \quad GCD(j,p) \ne 1. \end{cases}$$

$$\chi_{V^{(p)}_{(1,u)}}(x^{j}) = \begin{cases} p & ,\text{if} \quad GCD(j,p) \ne 1, \\ 0 & ,\text{if} \quad GCD(j,p) = 1. \end{cases}$$

$$\chi_{V^{(p)}_{(1,u)}}(y^{j}) = \begin{cases} p & ,\text{if} \quad u = 0, \\ 0 & ,\text{if} \quad u \ne 0. \end{cases}$$

For the element $xya^{p+1}b$ we need to analyze the character of the representations. Let us look at the following equality:

$$\chi_{V^{(p)}_{(1,u)}}((xya^{p+1}b)^{j}) = \mathrm{Tr}\left(V^{(p)}_{(1,u)}((xya^{p+1}b)^{j})\right).$$

As a consequence of the fact that the matrices $V^{(p)}_{(1,u)}(y)$, $V^{(p)}_{(1,u)}(a)$, $V^{(p)}_{(1,u)}(y)$ are of diagonal type, we know that they commute. Furthermore, $V^{(p)}_{(1,u)}(y)$ has an associated permutation matrix defined in Observation 6 as σ_p (see Section 2.2.1.2). We see that

$$\operatorname{Tr}\left(V^{(p)}_{(1,u)}((xya^{p+1}b)^j)\right) = \operatorname{Tr}\left(\left(\sigma_p V^{(p)}_{(1,u)}(ya^{p+1}b)\right)^j\right),$$

where $V^{(p)}_{(1,u)}(ya^{p+1}b) = \operatorname{diag}(d_1, d_2, \ldots, d_p)$ with

$$\begin{aligned} d_i &= w^p \cdot w^{pu} \cdot w^{p(i-1)}, \\ &= w^{p(u+i)}, \qquad\qquad \text{for } 1 \leq i \leq p. \end{aligned}$$

From this, we are able to conclude that the character we are searching for is given by the expression:

$$\chi_{V^{(p)}_{(1,u)}}((xya^{p+1}b)^j) = \begin{cases} p & \text{, if } \; GCD(j,p) \neq 1, \\ 0 & \text{, if } \; GCD(j,p) = 1. \end{cases}$$

Now, with this information, we calculate the dimention of the fixed subespaces:

$$\begin{aligned} (i)\ \ \dim Fix_{\langle a^{-1}\rangle} V^{(p)}_{(1,u)} &= \left\langle 1_{\langle a^{-1}\rangle}, Res^G_{\langle a^{-1}\rangle} V^{(p)}_{(1,u)} \right\rangle_{\langle a^{-1}\rangle} \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle a^{-1}\rangle}(a^{-j}) \cdot Res^G_{\langle a^{-1}\rangle} V^{(p)}_{(1,u)}(a^j)\right) \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(\chi_{V^{(p)}_{(1,u)}}(a^j)\right) \\ &= \frac{1}{p^2}(p + \overset{p}{\ldots} + p) \\ &= 1. \end{aligned}$$

$$\begin{aligned} (ii)\ \ \dim Fix_{\langle xya^{p+1}b\rangle} V^{(p)}_{(1,u)} &= \left\langle 1_{\langle xya^{p+1}b\rangle}, Res^G_{\langle xya^{p+1}b\rangle} V^{(p)}_{(1,u)} \right\rangle_{\langle xya^{p+1}b\rangle} \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle xya^{p+1}b\rangle}((xya^{p+1}b)^j) \cdot Res^G_{\langle xya^{p+1}b\rangle} V^{(p)}_{(1,u)}((xya^{p+1}b)^{-j})\right) \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\chi_{V^{(p)}_{(1,u)}}\left((xya^{p+1}b)^j\right) \\ &= \frac{1}{p^2}(p + \overset{p}{\ldots} + p) \\ &= 1. \end{aligned}$$

$$(iii)\ \dim Fix_{\langle x^{-1}\rangle} V^{(p)}_{(1,u)} = \left\langle 1_{\langle x^{-1}\rangle}, Res^{G}_{\langle x^{-1}\rangle} V^{(p)}_{(1,u)} \right\rangle_{\langle x^{-1}\rangle}$$
$$= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle x^{-1}\rangle}((x^{-1})^{j}) \cdot Res^{G}_{\langle x^{-1}\rangle} V^{(p)}_{(1,u)}\left((x^{-1})^{-j}\right)\right)$$
$$= \frac{1}{p}\sum_{j=1}^{p}\chi_{V^{(p)}_{(1,u)}}\left(x^{j}\right)$$
$$= \frac{1}{p}\left(p + 0 + \overset{p-1}{\ldots} + 0\right)$$
$$= 1.$$

$$(iv)\ \dim Fix_{\langle y^{-1}\rangle} V^{(p)}_{(1,u)} = \left\langle 1_{\langle y^{-1}\rangle}, Res^{G}_{\langle y^{-1}\rangle} V^{(p)}_{(1,u)} \right\rangle_{\langle y^{-1}\rangle}$$
$$= \frac{1}{p}\sum_{j=1}^{p}\left(1_{\langle y^{-1}\rangle}((y^{-1})^{j}) \cdot Res^{G}_{\langle y^{-1}\rangle} V^{(p)}_{(1,u)}\left((y^{-1})^{-j}\right)\right)$$
$$= \frac{1}{p}\sum_{j=1}^{p}\chi_{V^{(p)}_{(1,u)}}\left(y^{j}\right)$$
$$= \begin{cases} \frac{1}{p}\left(p + 0 + \overset{p-1}{\ldots} + 0\right) = 1 & \text{, if } u = 0, \\ \frac{1}{p}(0 + \overset{p}{\ldots} + 0) = 0 & \text{, if } u \neq 0. \end{cases}$$

Then, from $(i), (ii), (iii), (iv)$, and the calculation of the Galois group in the proof of Th.8 and Prop. 9 (see Section 2.2.2), we conclude

$$\dim C^{p}_{W_{(1,u)}} = \begin{cases} \dfrac{(p-1)(p-3)}{2} & , \qquad u = 0, \\[2ex] \dfrac{(p-1)(2p-3)}{2} & , \quad 1 \leq u \leq p-1. \end{cases}$$

5) Consider the representation $V^{(p^2)}_{(1)}$, using Remarks 7, 8 (see Section 2.2.1.3) and the information of the permutations $V^{(p^2)}_{(1)}(x) := \zeta_{p^2}$ and $V^{(p^2)}_{(1)}(y) := \varepsilon_{p^2}$ in Remarks 9 and 10 (see Section 2.2.1.3), we get the

following results:

$$\chi_{V_{(1)}^{(p^2)}}(a^j) = \sum_{j=1}^{p^2} w^{pj+1} = \sum_{j=1}^{p} pw^{pj+1} = pw\sum_{j=1}^{p} w^{pj} = 0.$$

$$\chi_{V_{(1)}^{(p^2)}}(x^j) = \mathrm{Tr}\left((\zeta_{p^2})^j\right) = \begin{cases} p^2 & , \quad GCD(p,j) \neq 1, \\ 0 & , \quad GCD(p,j) = 1. \end{cases}$$

$$\chi_{V_{(1)}^{(p^2)}}(y^j) = \mathrm{Tr}\left((\varepsilon_{p^2})^j\right) = \begin{cases} p^2 & , \quad GCD(p,j) \neq 1, \\ 0 & , \quad GCD(p,j) = 1. \end{cases}$$

$$\chi_{V_{(1)}^{(p^2)}}((xya^{p+1}b)^j) = \mathrm{Tr}\left(((\zeta_{p^2}\varepsilon_{p^2})D_{ab})^j\right) = \begin{cases} p\displaystyle\sum_{j=1}^{p} w^{pi} = 0 & , \quad GCD(p,j) \neq 1, \\ \qquad 0 & , \quad GCD(p,j) = 1. \end{cases}$$

Where $D_{ab} := \mathrm{diag}[(a_{1\,1}b_{1\,1}), (a_{2\,2}b_{2\,2}), \ldots, (a_{p^2\,p^2}b_{p^2\,p^2})]$ with $V_1^{(p^2)}(a) := [a_{ij}]_{0\leq i,j\leq p^2-1}$ and $V_1^{(p^2)}(b) := [b_{ij}]_{0\leq i,j\leq p^2-1}$.

With the above information, we determine the dimension of the fixed subspaces as follows:

$$\begin{aligned} (i)\ \dim Fix_{\langle a^{-1}\rangle}V_{(1)}^{(p^2)} &= \left\langle 1_{\langle a^{-1}\rangle}, Res^G_{\langle a^{-1}\rangle}V_{(1)}^{(p^2)}\right\rangle_{\langle a^{-1}\rangle} \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle a^{-1}\rangle}(a^{-j}) \cdot Res^G_{\langle a^{-1}\rangle}V_{(1)}^{(p^2)}(a^j)\right) \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(\chi_{V_{(1)}^{(p^2)}}(a^j)\right) \\ &= 0. \end{aligned}$$

$$\begin{aligned} (ii)\ \dim Fix_{\langle xya^{p+1}b\rangle}V_{(1)}^{(p^2)} &= \left\langle 1_{\langle xya^{p+1}b\rangle}, Res^G_{\langle xya^{p+1}b\rangle}V_{(1)}^{(p^2)}\right\rangle_{\langle xya^{p+1}b\rangle} \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\left(1_{\langle b\rangle}((xya^{p+1}b)^j) \cdot Res^G_{\langle xya^{p+1}b\rangle}V_{(1)}^{(p^2)}((xya^{p+1}b)^{-j})\right) \\ &= \frac{1}{p^2}\sum_{j=1}^{p^2}\chi_{V_{(1)}^{(p^2)}}\left((xya^{p+1}b)^j\right) \\ &= 0. \end{aligned}$$

$$
\begin{aligned}
(iii)\ \dim Fix_{\langle x^{-1}\rangle} V_{(1)}^{(p^2)} &= \left\langle 1_{\langle x^{-1}\rangle}, Res^G_{\langle x^{-1}\rangle} V_{(1)}^{(p^2)} \right\rangle_{\langle x^{-1}\rangle} \\
&= \frac{1}{p}\sum_{j=1}^{p} \left(1_{\langle x^{-1}\rangle}((x^{-1})^j) \cdot Res^G_{\langle x^{-1}\rangle} V_{(1)}^{(p^2)}\left((x^{-1})^{-j}\right)\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} \chi_{V_{(1)}^{(p^2)}}\left(x^j\right) \\
&= \frac{1}{p}\left(p^2 + 0 + \overset{p-1}{\cdots} + 0\right) \\
&= p.
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ \dim Fix_{\langle y^{-1}\rangle} V_{(1)}^{(p^2)} &= \left\langle 1_{\langle y^{-1}\rangle}, Res^G_{\langle y^{-1}\rangle} V_{(1)}^{(p^2)} \right\rangle_{\langle y^{-1}\rangle} \\
&= \frac{1}{p}\sum_{j=1}^{p} \left(1_{\langle y^{-1}\rangle}((y^{-1})^j) \cdot Res^G_{\langle y^{-1}\rangle} V_{(1)}^{(p^2)}\left((y^{-1})^{-j}\right)\right) \\
&= \frac{1}{p}\sum_{j=1}^{p} \chi_{V_{(1)}^{(p^2)}}\left(y^j\right) \\
&= p^2 + 0 + \overset{p-1}{\cdots} + 0 \\
&= p.
\end{aligned}
$$

Then, from $(i), (ii), (iii)$,(iv), and the calculation of the Galois group in the proof of Th.8 and Prop. 9 (see Section 2.2.2), we conclude that

$$\dim D^{p^2}_{W_{(1)}} = p(p-1)^2 .$$

The above information is summarized in the following points:

- The union of the sets $\left\{B^1_{W_{(0,1,k)}} : 1 \le k \le p-2\right\}$ and $\left\{B^1_{W_{(1,m,n)}} : (m,n) \in \{(i,0),(0,i),(i,p-i-1)\} \wedge 1 \le i \le p-2\right\}$ corresponds to the factors associated to the rational irreducible representations of degree $p-1$ having dimension $\frac{p-1}{2}$. Observe that there are $4(p-2)$ of these factors. Then from now on, we denote these primitive components by $B^1_{\left(i,\frac{p-1}{2}\right)}$ with $1 \le i \le 4(p-2)$. This is the first factor presented in the decomposition (3.2).
- The set $\left\{B^1_{W_{(1,m,n)}} : (m,n) \in \{(i,j) : 1 \le i,j \le p-1\ \wedge\ j \ne p-i-1\}\right\}$ corresponds to the factors associated to the rational irreducible

representations of degree $p-1$ having dimension $p-1$. Observe that there are $(p-1)^2-(p-2)=p^2-3p+3$ of these factors, so from now on we denote these primitive components by $B^1_{(i,p-1)}$ with $1 \leq i \leq p^2-3p+3$. This is the second factor presented in decomposition (3.2).

- The set $\left\{C^p_{W_{(1,u)}} : 1 \leq u \leq p-1\right\}$ corresponds to the factors associated to the rational irreducible representations of degree $p(p-1)$ having dimension $\dfrac{(p-1)(2p-3)}{2}$. Observe that there are $p-1$ of these factors, so from now on, we denote these primitive components by $C^p_{\left(i,\frac{(p-1)(2p-3)}{2}\right)}$. This is the third factor presented in decomposition (3.2).
- The factor $C^p_{W_{(1,0)}}$ corresponds to the rational irreducible representation of degree $p(p-1)$ having dimension $\frac{(p-1)(p-3)}{2}$. So from now on we denote this primitive component by $C^p_{\left(1,\frac{(p-1)(p-3)}{2}\right)}$. This is the fourth factor presented in decomposition 3.2.
- The factor $D^{p^2}_{W_{(1)}}$ corresponds to the rational irreducible representation of degree $p^2(p-1)$ having dimension $p(p-1)^2$. So from now on, we denote this primitive component by $D^{p^2}_{(p(p-1)^2)}$. This is the last factor presented in decomposition 3.2.

Using the redefinition of the factors indicated in the previous points, we are in a position to conclude that the *GAD* of JX is described by the following expression:

$$JX \sim \prod_{i=1}^{4p-8} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,p-1)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(2p-3)}{2})} \times C^p_{(1,\frac{(p-1)(p-3)}{2})} \times D^{p^2}_{(p(p-1)^2)}.$$

This concludes the proof.

□

3.2 Decomposition of the Jacobian variety with G_p acting with extended signature

In this section we study how the *GAD* of a Jacobian variety is with the action of G_p on the corresponding Riemann Surface with an extended version of the signature.

Definition 3. Let $c_1, c_2, \ldots, c_r$ be elements of a group G. Let us note that if $[c_1, c_2, \ldots, c_r]$ is a $(0; m_1, m_2, \ldots, m_r)$-generating vector then we can construct an extended version of this vector by adding elements of the group and preserving the characteristics that a generating vector possesses.

For example, given that $|c_i^{-1}| = |c_i^{-1}| = m_i$ for some $i \in \{1, 2, \ldots, r\}$, then $[c_1, c_2, \ldots, c_i, c_i^{-1}, c_i, \ldots, c_r]$ is a $(0; m_1, m_2, \ldots, m_i, m_i, m_i, \ldots, m_r)$-generating vector and corresponds to an extension of the vector described above. We will call these elements the extended signature and the extended generating vector respectively.

For the purpose of describing extended signatures and vectors, we will use the following notation:

Notation 2. We will write

$$[c_1, c_2, \ldots, (c_i, c_i^{-1})^t, c_i, \ldots, c_r]$$

with $t \in \mathbb{N}$ to refer to the $(0; m_1, m_2, \ldots, \underbrace{m_i, m_i, \ldots, m_i}_{2t+1}, \ldots, m_r)$-generating vector $[c_1, c_2, \ldots, \underbrace{c_i, c_i^{-1}, \ldots, c_i, c_i^{-1}}_{t-times\ c_i, c_i^{-1}}, c_i, \ldots, c_r]$.
In addition, we will use the notation $(0; m_1, m_2, \ldots, \{m_i\}^{2t+1}, \ldots, m_r)$ for the signature $(0; m_1, m_2, \ldots, \underbrace{m_i, m_i, \ldots, m_i}_{2t+1}, \ldots, m_r)$.

The concept of extended signature interests us because it allows us to expand the analysis of the decomposition of a Jacobian variety from our original one dimensional family, to a higher dimensional one. This is convenient since the signature $(0; p^2, p^2, p, p)$ we are considering for the action of G on X, appears in Singerman's work [39] as one of the signatures for which the general member of the family may have a bigger group acting on it. This means that G_p could actually be a subgroup of the (full) automorphisms group $\mathrm{Aut}(X)$ of X, see for instance [10, 24]. If this is the case for the generating vector we are considering, it could happen that H and K were conjugate in $\mathrm{Aut}(X)$, and so trivially linked there. This situation is not interesting for us in the bigger picture of getting non-isomorphic Riemann surfaces, namely X/H and X/K, with isogenous Jacobians (see Section 4.1). There are tools to study whether or not our generating vector of length 4 admits an extension of the actual group acting on every member of the family, see for instance [3], but we decided to leave this question for a future work and instead increase the length of the signature by using these discrete parameters $t_1, \ldots, t_4$ (see Section 2.3). This ensures us that the group does not grow (see [39]).

From the above, if we consider for G_p the extended signature

$$\left(0; \{p^2\}^{2t_1+1}, \{p^2\}^{2t_2+1}, \{p\}^{2t_3+1}, \{p\}^{2t_4+1}\right),$$

and the associated generating vector

$$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}],$$

we obtain the results presented below.

Theorem 12. *Let $p > 2$ prime and $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$. The group G_p acts on a Riemann surface X of genus*

$$(t_1 + t_2 + t_3 + t_4 + 1)p^5 - (t_3 + t_4 + 1)p^4 - p^3(t_1 + t_2 + 1) + 1$$

with signature $(0; \{p^2\}^{2t_1+1}, \{p^2\}^{2t_2+1}, \{p\}^{2t_3+1}, \{p\}^{2t_4+1})$ and generating vector

$$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}].$$

Under this action, the GAD of the Jacobian variety JX is given by the following expression

$$\begin{aligned}
JX \sim & B^1_{(1,(p-1)(t_1+t_2))} \times B^1_{(1,(p-1)(t_1+t_3))} \times B^1_{(1,(p-1)(t_1+t_4))} \times B^1_{(1,(p-1)(t_2+t_3))} \times B^1_{(1,(p-1)(t_2+t_4))} \\
& \times B^1_{(1,(p-1)(t_3+t_4))} \times \prod_{i=1}^{p-2} B^1_{(i,\frac{(p-1)(2(t_1+t_2+t_3)+1)}{2})} \times \prod_{i=1}^{p-2} B^1_{(i,\frac{(p-1)(2(t_1+t_3+t_4)+1)}{2})} \\
& \times \prod_{i=1}^{p-2} B^1_{(i,\frac{(p-1)(2(t_1+t_2+t_4)+1)}{2})} \times \prod_{i=1}^{p-2} B^1_{(i,\frac{(p-1)(2(t_1+t_2+t_3)+1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,(t_1+t_2+t_3+t_4+1)(p-1))} \\
& \times C^p_{(1,\frac{1}{2}(p-1)(p-3)+(p-1)^2(t_1+t_2+t_4))} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{1}{2}(p-1)(2p-3)+(p-1)^2(t_1+t_2+t_4)+(p-1)pt_3)} \\
& \times D^{p^2}_{(1,p(p-1)^2(1+t_3+t_4)+p^2(p-1)(t_1+t_2))},
\end{aligned}$$

where the pair (i, j) as a subscript has to be read as follows: i refers to the positional value in the product, and j is the dimension of the primitive factor indexed by i.

Proof. First, according to the structure of G_p studied in Proposition 5, the elements of the generating vector satisfy the indicated orders. Moreover, by Riemann's Existence Theorem, we are in a position to conclude that there exists a Riemann surface X on which G_p acts with the indicated signature. By the Riemann-Hurwitz equation (1.12), the genus is the one indicated.

On the other hand, by the previous section we have that the given dimension of the fixed subspaces depends on exactly the same generators in the extended generating vector. Then, it is now sufficient to calculate the dimension of each of the factors described in (3.1), which is done using Proposition 2 (see Chapter 1).

For the primitive factor varieties associated to the rational irreducible representations of degree p, we calculate the following:

$$\dim B^1_{W_{(0,0,1)}} = (p-1)(t_2+t_3),$$

$$\dim B^1_{W_{(0,1,k)}} = \begin{cases} (p-1)(t_2+t_4) & \text{, if} \quad k=0, \\ \dfrac{(p-1)(1+2t_2+2t_3+2t_4)}{2} & \text{, if} \quad 1 \leq k \leq p-2, \\ (p-1)(t_3+t_4) & \text{, if} \quad k=p-1. \end{cases}$$

$$\dim B^1_{W_{(1,m,n)}} = \begin{cases} (p-1)(t_1+t_2) & \text{, if} \quad (m,n)=(0,0), \\ (p-1)(t_1+t_3) & \text{, if} \quad (m,n)=(0,p-1), \\ (p-1)(t_1+t_4) & \text{, if} \quad (m,n)=(p-1,0), \\ \dfrac{(p-1)(1+2t_1+2t_2+2t_4)}{2} & \text{, if} \quad (m,n)=(i,0) \wedge 1 \leq i \leq p-2, \\ \dfrac{(p-1)(1+2t_1+2t_2+2t_3)}{2} & \text{, if} \quad (m,n)=(0,i) \wedge 1 \leq i \leq p-2, \\ \dfrac{(p-1)(1+2t_1+2t_3+2t_4)}{2} & \text{, if} \quad (m,n)=(i,p-i-1) \wedge 1 \leq i \leq p-2, \\ (p-1)(1+t_1+t_2+t_3+t_4) & \text{, if} \quad 1 \leq m,n \leq p-1 \wedge n \neq p-m-1. \end{cases}$$

Observe that there are $(p-1)^2-(p-2)=p^2-3p+3$ of these last factors where it is satisfied that $1 \leq m,n \leq p-1$ and $n \neq p-m-1$. This implies that there are p^2-3p+3 primitive factors of dimension $(p-1)(1+t_1+t_2+t_3+t_4)$.

On the other hand, with respect to the primitive factor associated to the rational irreducible representations of degree $p(p-1)$ we have the following:

$$\dim C^p_{W_{(1,0)}} = \frac{(p-1)(p-3)}{2} + (p-1)^2(t_1+t_2+t_4),$$

$$\dim C^p_{W_{(1,u)}} = \frac{(p-1)(2p-3)}{2} + (p-1)^2(t_1+t_2+t_4) + p(p-1)t_3,$$

if $1 \leq u \leq p-1$.

Finally, with respect to the primitive factor associated to the rational irreducible representation of degree $p^2(p-1)$ we have the following:

$$\dim D^p_{W_{(1)}} = p(p-1)^2(1+t_3+t_4) + p^2(p-1)(t_1+t_2).$$

Then, knowing the dimensions of the primitive factors, we redefine the factors with the subscript (i,j), where i is the counter and j is the dimension of the factor. From this we obtain the decomposition described in the statement of the theorem.

Since we have the discrete parameters $(t_1,t_2,t_3,t_4) \in \mathbb{N}_0^4$ determining several families of Riemann surfaces with G_p-action, hence several families of Jacobians together with their *GAD*, it is natural to ask what the geometric situation is when the discrete parametres are on the axes, so to speak. We collect these results in the following corollaries.

$\square$

Corollary 2. *Let $p > 2$ be a prime, and $t_1 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface of genus $(t_1+1)p^5 - p^4 - (t_1+1)p^3 + 1$ with signature $(0; \{p^2\}^{2t_1+1}, p^2, p^2, p, p)$ and generating vector*

$$[(a^{-1}, a)^{t_1}, a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}].$$

Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface X is given by the following expression

$$JX \sim \prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,t_1(p-1))} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{(2t_1+1)(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,(t_1+1)(p-1))}$$
$$\times \prod_{i=1}^{p-1} C^p_{(i,\frac{1}{2}(p-1)(2p-3)+t_1(p-1)^2)} \times C^p_{(1,\frac{1}{2}(p-1)(p-3)+t_1(p-1)^2)} \times D^{p^2}_{(1,p(p-1)^2+t_1p^2(p-1))}.$$

Example 1. *For p a prime $p > 2$ and $t_1 = 1$, the group G_p acts on a Riemann surface X of genus $2p^5 - p^4 - 2p^3 + 1$ with signature $(0; p^2, p^2, p^2, p^2, p, p)$ and generating vector $[a^{-1}, a, a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety JX is given by the following expression:*

$$\prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,p-1)} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{3(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,2p-2)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(4p-5)}{2})} \times C^p_{(1,\frac{(p-1)(3p-5)}{2})} \times D^{p^2}_{(1,p(2p-1)(p-1))}.$$

Corollary 3. *Let p be a prime $p > 2$ and $t_2 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X of genus $(t_2+1)p^5 - p^4 - (t_2+1)p^3 + 1$ with signature $(0; p^2, \{p^2\}^{2t_2+1}, p, p)$ and generating vector*

$$[a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, y^{-1}, x^{-1}].$$

With these conditions, the Jacobian variety of the associated surface is given by the following expression:

$$JX \sim \prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,t_2(p-1))} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{(2t_2+1)(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,(t_2+1)(p-1))} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{1}{2}(p-1)(2p-3)+t_2(p-1)^2)} \times C^p_{(1,\frac{1}{2}(p-1)(p-3)+t_2(p-1)^2)} \times D^{p^2}_{(1,p(p-1)^2+t_2p^2(p-1))}.$$

Example 2. *For p a prime $p > 2$ and $t_2 = 1$, the group G_p acts on a Riemann surface X of genus $2p^5 - p^4 - 2p^3 + 1$ with signature $(0; p^2, p^2, p^2, p^2, p, p)$ and generating vector $[a^{-1}, xya^{p+1}b, (xya^{p+1}b)^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety JX is given by the following expression:*

$$\prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,p-1)} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{3(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,2p-2)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(4p-5)}{2})} \times C^p_{(1,\frac{(p-1)(3p-5)}{2})} \times D^{p^2}_{(1,p(2p-1)(p-1))}.$$

Corollary 4. *Let p be a prime $p > 2$ and $t_3 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X of genus $(t_3+1)p^5 - (t_3+1)p^4 - p^3 + 1$ with signature $(0; p^2, p^2, \{p\}^{2t_3+1}, p)$ and generating vector*

$$[a^{-1}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, x^{-1}].$$

Under these conditions, the GAD of the Jacobian variety corresponding to the Riemann surface is given by the following expression:

$$JX \sim \prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,t_3(p-1))} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{(2t_3+1)(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,(t_3+1)(p-1))} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{1}{2}(p-1)(2p-3)+(p-1)pt_3)} \times C^p_{(1,\frac{1}{2}(p-1)(p-3))} \times D^{p^2}_{(1,(t_3+1)p(p-1)^2)}.$$

Example 3. *For p prime $p > 2$ and $t_3 = 1$, the group G_p acts on a Riemann surface X of genus $2p^5 - p^4 - 2p^3 + 1$ with signature $(0; p^2, p^2, p, p, p, p)$ and generating vector $[a^{-1}, xya^{p+1}b, y^{-1}, y, y^{-1}, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety JX is given by the following expression:*

$$\prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,p-1)} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{3(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,2p-2)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(4p-3)}{2})} \times C^p_{(1,\frac{(p-1)(p-3)}{2})} \times D^{p^2}_{(1,2p(p-1)^2)}.$$

Corollary 5. *Let $p > 2$ be a prime and $t_4 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface of genus $(t_4+1)p^5 - (t_4+1)p^4 - p^3 + 1$ with signature $(0; p^2, p^2, p, \{p\}^{2t_4+1})$ and generating vector $[a^{-1}, xya^{p+1}b, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}]$. Under these conditions, the GAD of Jacobian variety corresponding to the Riemann surface X is given by the following expression:*

$$JX \sim \prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,t_4(p-1))} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{(2t_4+1)(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,(t_4+1)(p-1))} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{1}{2}(p-1)(2p-3)+(p-1)^2t_4)} \times C^p_{(1,\frac{1}{2}(p-1)(p-3)+(p-1)^2t_4)} \times D^{p^2}_{(1,(t_4+1)p(p-1)^2)}.$$

Example 4. *For $p > 2$ be a prime and $t_4 = 1$. The group G_p acts on a Riemann surface X of genus $2p^5 - p^4 - 2p^3 + 1$ with signature $(0; p^2, p^2, p, p, p, p)$ and generating vector $[a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}, x, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety JX is given by the following expression:*

$$\prod_{i=1}^{p-2} B^1_{(i,\frac{p-1}{2})} \times \prod_{i=1}^{3} B^1_{(i,p-1)} \times \prod_{i=1}^{3p-6} B^1_{(i,\frac{3(p-1)}{2})} \times \prod_{i=1}^{p^2-3p+3} B^1_{(i,2p-2)} \times \prod_{i=1}^{p-1} C^p_{(i,\frac{(p-1)(4p-5)}{2})} \times C^p_{(1,\frac{(p-1)(3p-5)}{2})} \times D^{p^2}_{(1,2p(p-1)^2)}.$$

In fact, Theorem 12 presented above allows us to describe the decomposition of JX with any extension of the geometric signature

$$(0; [p^2, \overline{\langle a^{-1}\rangle}], [p^2, \overline{\langle xya^{p+1}b\rangle}], [p, \overline{\langle y^{-1}\rangle}], [p, \overline{\langle x^{-1}\rangle}]).$$

and any prime p greater than 2. We compute the case $p = 3$ in Chapter 5.

Remark 14. We have done all the heavy computations for describing G_p (e.g. Proposition 5) and some associated objects; such as, representations (e.g. Theorems 7 and 8), signatures and generating vectors for its actions on Riemann surfaces (e.g. Theorem 9, Remark 11), decompositions of the corresponding Jacobian varieties (e.g. Theorems 10, 11 and 12), etc. So now we can pursue in the description the action of its linked subgroups on Riemann surfaces and their impact on the corresponding Jacobian varieties. We make some progress on this subject in Chapter 4 where we find the induced decomposition for the Jacobian variety of the intermediate quotients by the linked subgroups H and K of G_p. Nevertheless, we leave some open questions for future work.

The main question that we leave open is to find out whether the intermediate quotients X/H and X/K are isomorphic or not for the actions we study here. If this is the case, we would have isogenous Jacobian varieties from non-isomorphic Riemann surfaces, which is our long range goal.

In this direction, we know from Proposition 11 that G_p is the full group of automorphisms of the general element in the one dimensional family from Theorem 9 for $p = 3$. Then, for this family, there is no larger group acting (generically) in such a way that H and K are conjugate in it. Hence the intermediate quotients X/H and X/K are not isomorphic in an obvious way. The same happens with the general element in the families arising from extending the signature (see Section 3.2).

Other open questions we leave are:

- Identify (if possible) the Jacobian varieties associated to the linked subgroups H or K of G_p as factors (up to isogeny) decomposing JX.

- Describe primitive factors as Jacobians or Prym varieties of intermediate coverings.

- Describe (degree, kernel, etc) the isogeny between the Jacobian $J(X/H)$ and $J(X/K)$. We have some progress on this in Theorem 14.

Chapter 4

Jacobians of intermediate quotiens by linked subgroups and primitive idempotents

4.1 Intermediate Jacobians by H and K

For this chapter we assume $p > 2$ prime and $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$.
From the previous chapter we know that the intermediate Jacobians associated to the linked subgroups H and K of G_p are isogenous. If we consider the action determined by the generating vector from section 2.3, then we obtain the following theorems about JX_K and JX_H.

Theorem 13. *Let H, K be the linked subgroup given in G_p. Consider the action of G_p on a Riemann surface X with signature $(0; \{p^2\}^{2t_1+1}, \{p^2\}^{2t_2+1}, \{p\}^{2t_3+1}, \{p\}^{2t_4+1})$ and generating vector*

$$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}],$$

as in Theorem 12. Then, the GAD of Jacobians JX_K and JX_H associated to the intermediate coverings generated by taking quotients by H and K is given by the following expression:

$$B^1_{(1,(p-1)(t_1+t_2))} \times C^1_{(1,\frac{1}{2}(p-1)(p-3)+(p-1)^2(t_1+t_2+t_4))} \times D^1_{(1,(t_1+t_2)p^2(p-1)+(1+t_3+t_4)p(p-1)^2)},$$

and the genus of the Riemann surfaces X/H (and X/K) is

$$p^3(t_1+t_2+t_3+t_4+1) - p^2\left(2t_3+t_4-\frac{3}{2}\right) - p(t_1+t_2-t_3+t_4+1) + t_4 + \frac{3}{2}.$$

Proof. Considering Theorem 2 (See Chapter 1) and the information about GAD of the JX in Theorem 12. Then, the primitive factors appearing in the decomposition of JX_H are determined by calculating $\dim V^H$ for each complex representation V , which coincides with $\dim V^H$ as shown above.

Calculating the fixed spaces by H and K, it is concluded that the only ones with non zero dimension are described below:

$$\begin{aligned}
\dim\,(V^1_{(1,0,0)})^H &= \dim\,(V^1_{(1,0,0)})^K = 1,\\
\dim\,(V^p_{(1,0)})^H &= \dim\,(V^p_{(1,0)})^K = 1,\\
\dim\,(V^{p^2}_{(1)})^H &= \dim\,(V^{p^2}_{(1)})^K = 1.
\end{aligned}$$

Consequently, the only three factors of the decomposition are determined by the rational representations associated with the complex representations $V^1_{(1,0,0)}$, $V^{p^2}_{(1,0)}$, and $V^{p^2}_{(1)}$. Using the above notation, we are able to conclude the decomposition of JX_K and JX_H is in the statement of the Theorem 13, that is, the primitive factors in the Jacobian decomposition are those associated with these representations, which are:

$$B_{(1,(p-1)(t_1+t_2))},\ C_{(1,\frac{1}{2}(p-1)(p-3)+(p-1)^2(t_1+t_2+t_4))}\,,\ D_{(1,(t_1+t_2)p^2(p-1)+(1+t_3+t_4)p(p-1)^2)}\,.$$

On the other hand, to calculate the genus of the surface generated by the intermediate covering determined by the subgroups H and K we apply the expression given in the Proposition 3. If we define

$$\begin{aligned}
G_1 &:= \left\langle a^{-1}\right\rangle,\\
G_2 &:= \left\langle xya^{p+1}b\right\rangle,\\
G_3 &:= \left\langle y^{-1}\right\rangle,\\
G_4 &:= \left\langle x^{-1}\right\rangle,
\end{aligned}$$

then we have the following equalities:

$$|H\setminus G\,/G_1| = |\left\{H1G_1, HbG_1, Hb^2G_1, \ldots, Hb^{p-1}G_1\right\}| = p.$$

$$|H\setminus G\,/G_2| = |\left\{H1G_2, HbG_2, Hb^2G_2, \ldots, Hb^{p-1}G_2\right\}| = p.$$

$$\begin{aligned}
|H\setminus G\,/G_3| &= |\left\{Ha^ib^jG_3 : 0\le i,j\le p-1\right\}\cup\left\{Ha^{ip}b^jG_3 : 0\le j\le p-1,\ 1\le i\le p-1\right\}|,\\
&= 2p^2-p.
\end{aligned}$$

$$\begin{aligned}
|H\setminus G\,/G_4| &= |\left\{H1G_4, HaG_4, Ha^2G_4, Ha^{p^2-1}G_4, HbG_4, Hb^2G_4, \ldots, Hb^{p-1}G_4\right\}|,\\
&= p^2+p-1.
\end{aligned}$$

Replacing what is obtained in expression (1.15), we are in a position to conclude that the genus of X/H, and X/K, is

$$p^3(t_1+t_2+t_3+t_4+1)-p^2\left(2t_3+t_4-\frac{3}{2}\right)-p(t_1+t_2-t_3+t_4+1)+t_4+\frac{3}{2}\,.$$

$\square$

4.2 Image of primitive idempotents

As said in Theorem 1 (see Chapter 1), for any action of G_p, JX_H and JX_K are isogenous to a subvariety of JX defined as the image of certain idempotent, we study this in what follows, for the action we are considering.

Theorem 14. *Let H, K, G_p be as before. Consider X a Riemann surface with the action of G_p with signature*

$$\left(0; \{p^2\}^{2t_1+1}, \{p^2\}^{2t_2+1}, \{p\}^{2t_3+1}, \{p\}^{2t_4+1}\right)$$

and generating vector

$$[(a^{-1},a)^{t_1}, a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1},y)^{t_3}, y^{-1}, (x^{-1},x)^{t_4}, x^{-1}].$$

Then, the Jacobian variety JX_H of X/H is isogenous to the subvariety of JX defined as the image of the idempotent

$$f_H := p_H(e_{B_{W_{(1,0,0)}}} + e_{C_{V_{(1,0)}}} + e_{D_{(1)}}) \tag{4.1}$$

and the Jacobian variety JX_K is isogenous to the subvariety of JX defined as the image of the idempotent

$$f_K := p_K(e_{B_{W_{(1,0,0)}}} + e_{C_{V_{(1,0)}}} + e_{D_{(1)}}), \tag{4.2}$$

where the idempotents are described by

$$e_{B_{W_{(1,0,0)}}} = \frac{1}{p^5}\left(\sum_{(i,j,k,l)\in J_1}(p-1)x^iy^ja^{pk}b^l + \sum_{(i,j,k,l)\in J_2}(-1)x^iy^ja^kb^l\right),$$
$$J_1 = \left\{(i,j,k,l) \in (\mathbb{Z}_0^+)^4 : 0 \leq i,j,k,l \leq p-1\right\},$$
$$J_2 = \left\{(i,j,k,l) \in (\mathbb{Z}_0^+)^4 : 0 \leq i,j,l \leq p-1 \wedge 1 \leq k < p^2 \wedge GCD(k,p) = 1\right\}.$$

$$e_{C_{V_{(1,0)}}} = \frac{1}{p^4}\left(\sum_{0\leq k,i\leq p-1}(p(p-1))y^ka^{pi} + \sum_{(h,i,j)\in I}(-1)y^ha^{pi}b^j\right),$$
$$I = \left\{(h,i,j) \in (\mathbb{Z}_0^+)^3 : 0 \leq h,i,j \leq p-1,\ (h,i,j) \neq (0,0,0)\right\}.$$

$$e_{D_{(1)}} = \frac{1}{p^3}\left(p^2(p-1)1_{G_p} + \sum_{i=1}^{p-1}(-p^2)a^{pi}\right).$$

Proof. The proof follows from the proof of Theorem 13 (see Section 4.1), and a direct application of Theorem 1 (see Section 1.2). □

Remark 15. Note that we can establish an isomorphism between the subgroups H and K by means of the relation $y \to ya^{p+1}$ which induces a morphism on $\mathbb{Q}[G]$.

Corollary 6. *Let $p > 2$ be a prime and $t_1 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X with signature $(0; \{p^2\}^{2t_1+1}, p^2, p^2, p, p)$ and generating vector*

$$[(a^{-1},a)^{t_1}, a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}],$$

as in Corollary 2. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression.

$$JX_H \sim JX_K \sim B^1_{(1,t_1(p-1))} \times C^1_{(1,\frac{1}{2}(p-1)(p-3)+t_1(p-1)^2)} \times D^1_{(1,p(p-1)^2+t_1p^2(p-1))},$$

and the genus of intermediate surfaces X/H and X/K is

$$p^3(t_1+1) - \frac{3}{2}p^2 - p(2t_1+1) + t_1 + \frac{3}{2}.$$

Example 5. *Consider $p > 2$ a prime and $t_2 = 1$ in the Corollary 6. Then, the decomposition of the Jacobian varieties of the Riemann surface arising* *from taking quotients by the subgroups H and K, is given by the following expression*

$$JX_H \sim JX_K \sim B^1_{(1,p-1)} \times C^1_{(1,\frac{(p-1)(3p-5)}{2})} \times D^1_{(1,p(2p-1)(p-1))},$$

and the genus of intermediate surfaces X/H and X/K is $2p^3 - \frac{3}{2}p^2 - 2p + \frac{3}{2}$

Corollary 7. *Let $p > 2$ be a prime and $t_2 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X with signature $(0; p^2, \{p^2\}^{2t_2+1}, p, p)$ and generating vector*

$$[a^{-1}, (xya^{p+1}b, (xya^{p+1}b)^{-1})^{t_2}, xya^{p+1}b, y^{-1}, x^{-1}]$$

as in corollary 3. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression

$$JX_H \sim JX_K \sim B^1_{(1,t_2(p-1))} \times C^1_{(1,\frac{1}{2}(p-1)(p-3)+t_2(p-1)^2)} \times D^1_{(1,p(p-1)^2+t_2p^2(p-1))},$$

and the genus of intermediate surfaces X/H and X/K is

$$p^3(t_2+1) - \frac{3}{2}p^2 - p(2t_2+1) + t_2 + \frac{3}{2}.$$

Example 6. *Consider $p > 2$ a prime and $t_2 = 1$ in the previous corollary. Then the intermediate Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H and K have genus $2p^3 - \frac{3}{2}p^2 - 2p + \frac{3}{2}$ and the decomposition of JX_H isogenous to JX_K is given by*

$$JX_H \sim JX_K \sim B^1_{(1,p-1)} \times C^1_{(1,\frac{(p-1)(3p-5)}{2})} \times D^1_{(1,p(2p-1)(p-1))},$$

Corollary 8. *Let $p > 2$ be a prime and $t_3 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X with signature $(0; p^2, p^2, \{p\}^{2t_3+1}, p)$ and generating vector*

$$[a^{-1}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, x^{-1}]$$

as in corollary 4. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression

$$JX_H \sim JX_K \sim C^1_{(1,\frac{1}{2}(p-1)(p-3))} \times D^1_{(1,(t_3+1)p(p-1)^2)},$$

and the genus of intermediate surfaces X/H and X/K is

$$p^3(t_3+1) - p^2(2t_3 + \frac{3}{2}) + (t_3-1)p + \frac{3}{2}.$$

Example 7. *Consider $p > 2$ a prime and $t_3 = 1$ in the previous corollary. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression.*

$$JX_H \sim JX_K \sim C^1_{(1,\frac{(p-1)(p-3)}{2})} \times D^1_{(1,2p(p-1)^2)},$$

and the genus of intermediate surfaces X/H and X/K is $2p^3 - \frac{7}{2}p^2 + \frac{3}{2}$.

Corollary 9. *Let $p > 2$ be a prime and $t_4 \in \mathbb{N}_0$. The group G_p acts on a Riemann surface X with signature with signature $(0; p^2, p^2, p, \{p\}^{2t_4+1})$ and generating vector $[a^{-1}, xya^{p+1}b, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}]$ as in corollary 5. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression*

$$JX_H \sim C^1_{(1,\frac{1}{2}(p-1)(p-3)+(p-1)^2t_4)} \times D^1_{(1,(t_4+1)p(p-1)^2)},$$

and the genus of intermediate surfaces X/H and X/K is

$$p^3(t_4+1) - p^2\left(t_4 - \frac{3}{2}\right) - p(t_4+1) + t_4 + \frac{3}{2}.$$

Example 8. *Consider $p > 2$ a prime and $t_4 = 1$ in the previous corollary. Then, the decomposition of the Jacobian varieties of the Riemann surface arising from taking quotients by the subgroups H and K, is given by the following expression.*

$$JX_H \sim C^1_{(1,\frac{(p-1)(3p-5)}{2})} \times D^1_{(1,2p(p-1)^2)},$$

and the genus of intermediate surfaces X/H and X/K is $2p^3 - \frac{5}{2}p^2 - 2p + \frac{5}{2}$.

This concludes the decomposition analysis of the Jacobian of the quotient Riemann surfaces arising from taking quotient by the linked groups H, K of G_p. From the above, we observe that the only case where the decomposition of JX_H, or JX_K, has only two factors (at most), is when t_3 is not null. This also means that the idempotent decomposition will have only two factors.

Regarding the variables $t_i \in \mathbb{N}_0$ with $i = 1, 2, 3, 4$, we can conclude the following:

Corollary 10. *Let t_1, t_2, t_3, t_4 be elements in $\mathbb{N}_0$. If we denote as $JX_H^{(t_1,t_2,t_3,t_4)}$ the decomposition given in Theorem 12 for JX_H and JX_K, then*

$$JX_H^{(t_1,r-t_1,t_3,t_4)} \sim JX_H^{(r-t_2,t_2,t_3,t_4)}, \tag{4.3}$$

for all $r \in \mathbb{N}$ such that $r > t_1, t_2$.

Chapter 5

Action of the group G_3 on Riemann surfaces and Jacobian varieties

In this chapter we show the results obtained in Chapter 3 and 4 for case $p = 3$; this is, the group

$$G_3 = \left\langle \; a, b, x, y \; : \; \begin{array}{l} a^9 = b^3 = x^3 = y^3 = 1, \\ ab = ba, xy = yx, \\ x^{-1}ax = ab, x^{-1}bx = ba^3, y^{-1}ay = a^4, y^{-1}by = b \end{array} \right\rangle .$$

5.1 Decomposition of Jacobian varieties induced by G_3 group action

If we use Theorems 12 and 13 for the prime 3 we obtain the following results:

Proposition 12. *Let $(t_1, t_2, t_3, t_4) \in \mathbb{N}_0^4$. The group G_3 acts on a Riemann surface of genus*

$$3^5(t_1 + t_2 + t_3 + t_4 + 1) - 3^4(t_3 + t_4 + 1) - 3^3(t_1 + t_2 + 1) + 1$$

with signature $(0; \{9\}^{2t_1+1}, \{9\}^{2t_2+1}, \{3\}^{2t_3+1}, \{3\}^{2t_4+1})$ *and generating vector*

$$[(a^{-1}, a)^{t_1}, a^{-1}, (xya^4b, (xya^4b)^{-1})^{t_2}, xya^{p+1}b, (y^{-1}, y)^{t_3}, y^{-1}, (x^{-1}, x)^{t_4}, x^{-1}].$$

Under these conditions, the GAD of the Jacobian variety of the corresponding

Riemann surface is given by the following expression

$$\begin{aligned} JX \sim & B^1_{(1,2(t_1+t_2))} \times B^1_{(1,2(t_1+t_3))} \times B^1_{(1,2(t_1+t_4))} \times B^1_{(1,2(t_2+t_3))} \times B^1_{(1,2(t_2+t_4))} \times B^1_{(1,2(t_3+t_4))} \\ & \times B^1_{(1,2(t_1+t_2+t_3)+1)} \times B^1_{(1,2(t_1+t_3+t_4)+1} \times B^1_{(1,2(t_1+t_2+t_4)+1)} \times B^1_{(1,2(t_1+t_2+t_3)+1)} \\ & \times \prod_{i=1}^{3} B^1_{(i,2(t_1+t_2+t_3+t_4+1))} \times C^3_{(1,4(t_1+t_2+t_4))} \times \prod_{i=1}^{2} C^3_{(i,3+4(t_1+t_2+t_4)+6t_3)} \\ & \times D^{p^2}_{(1,18(t_1+t_2)+12(1+t_3+t_4))}. \end{aligned}$$

Besides, the decomposition of the Jacobian JX_K and JX_H corresponding to the intermediate quotients by the linked groups H and K is given by the following expression:

$$B^1_{(1,2(t_1+t_2))} \times C^1_{(1,4(t_1+t_2+t_4))} \times D^1_{(1,18(t_1+t_2)+12(1+t_3+t_4))},$$

and the genus of intermediate surfaces X/H and X/K is

$$24t_1 + 24t_2 + 12t_3 + 16t_4 + 39.$$

Considering particular cases of this theorem, the following examples are concluded.

Example 9. *The group G_3 acts on a Riemann surface of genus* 136 *with signature* $(0; 9, 9, 3, 3)$ *and generating vector* $[a^{-1}, xya^{p+1}b, y^{-1}, x^{-1}]$.
Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface is given by the following expression

$$JX \sim \prod_{i=1}^{4} B^1_{(i,1)} \times \prod_{i=1}^{3} B^1_{(i,2)} \times \prod_{i=1}^{2} C^3_{(i,3)} \times D^9_{(1,12)},$$

Moreover, the decomposition of the intermediate Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H and K is given by the following expression.

$$JX_H \sim JX_K \sim D^1_{(1,12)},$$

and the genus of intermediate surfaces X/H and X/K is 12.

Example 10. *The group G_3 acts on a Riemann surface of genus 352 with signature* $(0; 9, 9, 9, 9, 3, 3)$ *and generating vector* $[a^{-1}, a, a^{-1}, xya^4b, y^{-1}, x^{-1}]$.
Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface is given by the following expression

$$JX \sim B^1_{(1,1)} \times \prod_{i=1}^{3} B^1_{(i,2)} \times \prod_{i=1}^{3} B^1_{(i,3)} \times \prod_{i=1}^{3} B^1_{(i,4)} \times \prod_{i=1}^{2} C^3_{(i,7)} \times C^3_{(i,4)} \times D^9_{(1,30)}.$$

Moreover, the decomposition of the Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H and K is given by the following expression.

$$JX_H \sim JX_K \sim B^1_{(1,2)} \times C^1_{(1,4)} \times D^1_{(1,30)}\,,$$

and the genus of intermediate surfaces X/H and X/K is 36.

Example 11. *The group G_3 acts on a Riemann surface of genus 352 with signature $(0; 9, 9, 9, 9, 3, 3)$ and generating vector $[a^{-1}, xya^4b, (xya^4b)^{-1}, xya^4b, y^{-1}, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface is given by the following expression*

$$JX \sim B^1_{(1,1)} \times \prod_{i=1}^{3} B^1_{(i,2)} \times \prod_{i=1}^{3} B^1_{(i,3)} \times \prod_{i=1}^{3} B^1_{(i,4)} \times \prod_{i=1}^{2} C^3_{(i,7)} \times C^3_{(i,4)} \times D^9_{(1,30)}.$$

Moreover, the decomposition of the Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H and K is given by the following expression.

$$JX_H \sim JX_K \sim B^1_{(1,2)} \times C^1_{(1,4)} \times D^1_{30}\,,$$

and the genus of intermediate surfaces X/H and X/K is 36.

Example 12. *The group G_3 acts on a Riemann surface of genus 298 with signature $(0; 9, 9, 3, 3, 3, 3)$ and generating vector $[a^{-1}, y^{-1}, x^{-1}, x, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface is given by the following expression*

$$JX \sim B^1_{(1,1)} \times \prod_{i=1}^{3} B^1_{(i,2)} \times \prod_{i=1}^{3} B^1_{(i,3)} \times \prod_{i=1}^{3} B^1_{(i,4)} \times \prod_{i=1}^{2} C^3_{(i,7)} \times C^3_{(1,4)} \times D^9_{(1,24)}.$$

Moreover, the decomposition of the Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H and K is given by the following expression.

$$JX_H \sim JX_K \sim D^9_{24},$$

and the genus of intermediate surfaces X/H and X/K is 24.

Example 13. *The group G_3 acts on a Riemann surface of genus 298 with signature $(0; 9, 9, 3, 3, 3, 3)$ and generating vector $[a^{-1}, y^{-1}, x^{-1}, x, x^{-1}]$. Under these conditions, the GAD of the Jacobian variety of the corresponding Riemann surface is given by the following expression*

$$JX \sim B^1_{(1,1)} \times \prod_{i=1}^{3} B^1_{(i,2)} \times \prod_{i=1}^{3} B^1_{(i,3)} \times \prod_{i=1}^{3} B^1_{(i,4)} \times \prod_{i=1}^{2} C^3_{(i,7)} \times C^3_{(1,4)} \times D^9_{(1,24)}.$$

Moreover, the decomposition of the Jacobian varieties of the Riemann surfaces arising as quotients by the subgroups H *and* K *is given by the following expression.*

$$JX_H \sim JX_K \sim C^3_{(1,4)} \times D^9_{(1,24)}\,,$$

and the genus of intermediate surfaces X/H *and* X/K *is 28.*

Let us note that in the examples there are coincident intermediate Jacobian decompositions, this happens because the conditions of Theorem 12 are satisfied.

Bibliography

[1] G. Bartolini, M. Izquierdo. *On the connectedness of the branch locus of the moduli space of Riemann surfaces of low genus.* Proceedings of the American Mathematical Society 140 Number 1 (2012), 35–45.

[2] A. Behn, R.E. Rodriguez and A.M. Rojas, *Adapted hyperbolic and symplectic representations for group actions on Riemann surfaces,* Journal of Pure and Applied Algebra 217 (2013), 409–426.

[3] A.Behn, A.M. Rojas, M.Tello-Carrera. *A SAGE package for n-gonal equisymmetric stratification in $\mathcal{M}_g$.* Exp. Math. (2020).

[4] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 302, Springer-Verlag, Berlin (2004).

[5] Broughton, S.A., *Classifying finite group actions on surfaces of low genus.* Journal of Pure and Applied Algebra 69 (1990), 233–270.

[6] E. Bujalance, F. J. Cirre and M. Conder, *On Extendability of Group Actions on Compact Riemann Surfaces.* , Transactions of the American Mathematical Society, Vol. 355, no. 4 (2002), 1537–1557.

[7] A. Caranti, N. Gavioli, S. Mattarei, *Subgroups of finite p-groups inducing the same permutation character*, Comm. Algebra 22 (1994), 877–895

[8] A. Carocca, R. Rodriguez, *Jacobians with group actions and rational idempotents*, Journal of Algebra 306 (2006), 322–343.

[9] M. Carvacho, Nonequivalent families of group actions on Riemann surfaces. *Journal of Pure and Applied Algebra* 217, no. 12 (2013), 2345–2355.

[10] A.F. Costa, M. Izquierdo. *Equisymmetric strata of the singular locus of the moduli space of Riemann surfaces of genus 4.* Geometry of Riemann surfaces, London Math. Soc. Lecture Note Ser 368 (2013), 120–138.

[11] C. Ciliberto and G. van der Geer, *Non-isomorphic curves of genus four with isomorphic (non-polarized) jacobians.* Contemporary Mathematics, Vol. 162, (1994).

[12] C.W. Curtis, I. Reiner. *Representation Theory of Finite Groups and Associative Algebras*, Wiley, New York, (1988).

[13] W. Feit, *Some consequences of the classification of finite simple groups*, Proceedings, Symposia in Pure Mathematics 37 (1980).

[14] F. Gassmann, *Bemerkungen zu der vorstehenden Arbeit von Hurwitz*, Math. Z. 25 (1926), 124–143.

[15] N. Gavioli. *Subgroups of finite soluble groups inducing the same permutation character.* Transactions of the American Mathematica Society 349, No.7 (1997), 2969–2980.

[16] V. Gónzalez-Aguilera and R.Rodríguez. *Fermat's quartic curve, Klein's curve and the Tetrahedron*, Contemporary Math. 201 (1997), 43–62.

[17] G. González-Diez, R. Hidalgo, Conformal versus topological conjugacy of automorphism on compact Riemann surfaces Bull. London Math. Soc. 29 (1997), 280–284.

[18] R. M. Guralnick, *Subgroups inducing the same permutation representation*, J. Algebra 81 (l983), 312–319.

[19] R. M. Guralnick and D. B. Wales, *Subgroups inducing the same permutation representation, II*, J. Algebra 96 (1985)) 94–113.

[20] R. Hidalgo y R. Rodríguez, *Introducción a las Variedades Abelianas y Grupos Kleinianos.* Monografías Matemáticas UTFSM (2005).

[21] E. W. Howe, *Constructing distinct curves with isomorphic Jacobians in characteristic zero*, Journal of number theory 56 (1996), 381–390.

[22] E. W. Howe, *Plane Quartics with Jacobians isomorphic to a Hyperelliptic Jacobian*, Proceedings of the American Mathematical Society, Volume 129, Number 6 (2000), 1647–1657.

[23] I.M. Isaacs, *Character theory of finite groups.* Dover Publications (1994).

[24] M. Izquierdo, L. Jiménez, A.M. Rojas. *Decomposition of Jacobian varieties of curves with dihedral actions via equisymmetric stratification.* Revista Matemática Iberoamericana 35, Issue 4 (2019), 1259–1279.

[25] L. Jiménez. *On the Group Algebra Descomposition of a Jacobian Variety.* Doctoral Thesis. Universidad de Chile, (2014).

[26] A. Ksir, *Dimensions of Prym Varieties.* Int. J. Math. Math. Sci. 26 ,No. 2 (2001), 107–116.

[27] H. Lange, *Abelian varieties with several principal polarizations.* Duke Mathematical Journal Vol. 55, Issue 3 (1987), 617–628.

[28] H. Lange, S. Recillas: *Abelian varieties with group action*, J. reine angew. math. 575 (2004), 135–155.

[29] H. Lange and A. M. Rojas. *Polarizations of isotypical components of Jacobians with group action*, Archiv der Mathematik 98 (2012), 513–526.

[30] J. Paulhus, *Branching data for curves up to genus 48*, arXiv:1512.07657v1 (2015), 1–9.

[31] R. Perlis, *On the equation* $\zeta_K(s) = \zeta_{K'}(s)$, J. Number Theory 9 (1977), 342–360.

[32] H. Poincaré, *Sur les fonctions abéliennes*, Amer. J. Math. 8 (1886), 289–342.

[33] R. Rodríguez, *Abelian varieties and group actions.* Contemporary Mathematics, Volume 629, (2014).

[34] A. M. Rojas, *Introducción a las Superficies de Riemann y Curvas algebraicas.* Course Notes (2009).

[35] A. M. Rojas, *Group Actions on Jacobian Varieties.* Revista Matemática Iberoamericana, 23, No. 2 (2007), 397–420.

[36] J. F. X. Ries, *Subvarieties of Moduli Space Determined by Finite Groups Acting on Surfaces.* Transactions of the American Mathematical Society, Vol. 335, No. 1 (1993), 385-406.

[37] J.-P. Serre, *Linear Representations of Finite Groups.* Graduate Texts in Math. 42, Springer-Verlag (1977).

[38] D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. London. Math. Soc, 2 (1970), 319-323.

[39] D. Singerman, *Finitely maximal Fuchsian groups*, J. London Math. Soc. (2) 6 (1972), 29–38.

[40] T. Sunada, *Riemannian coverings and isospectral manifolds.* Ann. Math. 121 (1985), 169–186.