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... a mi familia y a ti amor.

... La desorientación y reorientación que acompaña a la iniciación en cualquier misterio es la experiencia más maravillosa que vivirse pueda.

HENRY MILLER

"la esencia de la matemática radica precisamente en su libertad "

GEORG CANTOR

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Resumen

En esta tesis introducimos el concepto de *estructuras covariantes* $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ formadas por una C^*-algebra $\mathscr{A}$ separable, una acción torcida medible (a, α) de un grupo localmente compacto segundo-contable G, otra acción torcida medible $(\tilde{\mathrm{a}}, \tilde{\alpha})$ de otro grupo localmente compacto segundo-contable $\widetilde{\mathsf{G}}$ y una función estrictamente continua $\kappa : \mathsf{G} \times \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$ que conecta (a, α) y $(\tilde{\mathrm{a}}, \tilde{\alpha})$. Nociones naturales de morfismos covariantes y representaciones son consideradas en general y conducen a la construcción de una especie de producto cruzado torcido. Varias C^*-algebras emergen de un proceso de construcción de estructuras covariantes. Estas construcciones pueden ser iteradas indefinidamente. Mostramos que algunas de las C^*-algebras que aparecen en las iteraciones son isomorfas. Las construcciones son no conmutativas, pero vienen motivadas del caso Abeliano de la dualidad de Takai que es eventualmente generalizada.

Abstract

We introduce *covariant structures* $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ formed of a separable C^*-algebra $\mathscr{A}$, a measurable twisted action (a, α) of the second-countable locally compact group G, a measurable twisted action $(\tilde{\mathrm{a}}, \tilde{\alpha})$ of another second-countable locally compact group $\widetilde{\mathsf{G}}$ and a strictly continuous function $\kappa : \mathsf{G} \times \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$ suitably connected with (a, α) and $(\tilde{\mathrm{a}}, \tilde{\alpha})$. Natural notions of covariant morphisms and representations are considered, leading to a sort of twisted crossed product construction. Various C^*-algebras emerge by a procedure that can be iterated indefinitely and that also yields new pair of twisted actions. Some of these C^*-algebras are shown to be isomorphic. The constructions are non-commutative, but are motivated by Abelian Takai duality that they eventually generalize.

Contents

Introduction

Pontryagin duality establishes an isomorphism between each locally compact abelian group and its bidual group, and studies the properties of this connection. This duality result has many consequences and applications to representation theory of locally compact abelian groups and Harmonic Analysis.

A natural idea is try to generalize this duality result for non-abelian groups. In non-abelian setting the C^*-algebras play an essential role in the theory of representation of locally compact groups. There is an important development of the links between C^*-algebras and the theory of strongly unitary representations of groups [5]. Moreover, the C^*-algebras form a basic tool in the study of the representations of very extensive classes of involutive Banach algebras.

In operator algebras Pontryagin duality can be used for characterization of ceratin algebras associated to "dual actions" of abelian groups based in the construction of C^*-crossed products; this result is called Takai duality [25, 19]. On the other hand, C^*-algebras and the crossed products construction have been important tools in many fields with applications including spectral theory, pseudo-differential calculus and quantization.

Crossed products are important not only for applications, but are also the source of interesting examples of operators algebras. For example, Takesaki studied the classification of von Neumann algebras of type III [26] based in duality results involving W^*-crossed products.

This thesis can be summarized as a research about the Takai duality theorem [4]. We will be interested principally to extend this result of duality in the setting of twisted crossed products associated to twisted actions of a pair of locally compact groups weakly connected to each other. Possible future applications have motivated us to develop the duality without the theory of coactions (but since coactions are not involved, we do not obtain non-commutative versions of Takai duality [11, 17, 20]). Hopefully we are going to develop and apply this elsewhere.

In this thesis we are using the following framework:

Let $\mathscr{A}$ be a separable C^*-algebra with automorphism group $\mathsf{Aut}(\mathscr{A})$, multiplier algebra $\mathcal{M}(\mathscr{A})$ and unitary group $\mathcal{UM}(\mathscr{A})$ and let $\mathsf{G}, \tilde{\mathsf{G}}$ be two second contable locally compact groups, with units e and ε and left Haar measures dx and $d\xi$ respectively. Let also (a, α) be a measurable twisted action of G on $\mathscr{A}$ and $(\tilde{\mathrm{a}}, \tilde{\alpha})$ a measurable twisted action of $\tilde{\mathsf{G}}$ on $\mathscr{A}$. Motivated by duality issues, we are going to investigate this pair of twisted actions in the presence of a "coupling function" $\kappa : \mathsf{G} \times \tilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$, supposed strictly continuous.

The simple motivating example is given by the setting involved in the well-known (abelian) Takai duality result [25, 26, 19, 28]. In this case G is supposed to be commutative, $\tilde{\mathsf{G}} \equiv \widehat{\mathsf{G}}$ is its Pontryagin dual and $\kappa(x,\xi) := \xi(x)$ is obtained by applying the character ξ to the element x. The theory starts with a single action a of the group G (let us assume it untwisted), used to construct [6, 7, 16, 28] the crossed product $\mathscr{B} := \mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}$. On this new C^*-algebra there is a canonical action $\widehat{\mathrm{b}}^0$ of the dual group given on elements f of the dense *-subalgebra $L^1(\mathsf{G};\mathscr{A})$ by

$$\left[\widehat{\mathrm{b}}^0_\xi(f)\right](x) := f(x)\overline{\xi(x)} = f(x)\overline{\kappa(x,\xi)}\,, \quad \forall x \in \mathsf{G}\,,\, \xi \in \widehat{\mathsf{G}}\,.$$

Takai's duality result states that the second crossed product $(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}) \rtimes_{\widehat{\mathrm{b}}^0} \widehat{\mathsf{G}}$ is isomorphic to the tensor product $\mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]$ between the initial C^*-algebra $\mathscr{A}$ and the C^*-algebra of compact operators on the Hilbert space $L^2(\mathsf{G})$; this isomorphism is equivariant with respect to the canonial bi-dual action on $(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}) \rtimes_{\widehat{\mathrm{b}}^0} \widehat{\mathsf{G}}$ and a natural diagonal action on $\mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]$.

On the other hand, this dual action is not enough if one wants to fully connect the C^*-algebra $\mathscr{B}$ with the initial C^*-dynamical system $(\mathscr{A}, \mathrm{a}, \mathsf{G})$. There is also a natural strictly continuous group morphism $\lambda : \mathsf{G} \to \mathcal{UM}(\mathscr{B})$ (basically $\lambda_x = \delta_x \otimes 1$ in a suitable picture of the multiplier algebra of $\mathscr{B}$) and the covariance relation

$$\widehat{\mathrm{b}}^0_\xi(\lambda_x) = \kappa(x,\xi)\lambda_x$$

holds for each $x \in \mathsf{G}$ and $\xi \in \widehat{\mathsf{G}}$. The couple $(\widehat{\mathrm{b}}^0, \lambda)$ plays an important role [13, 16] in Landstad's characterizations of the C^*-algebras that are isomorphic to a crossed product with group G. But λ can also be seen as defining an action

$$\mathrm{b} := \mathsf{ad}_\lambda : \mathsf{G} \to \mathsf{Aut}(\mathscr{B})\,, \quad \mathrm{b}_x(f) = \mathsf{ad}_{\lambda_x}(f) \equiv \lambda_x \diamond f \diamond \lambda_x^\diamond\,,$$

where $\diamond$ denotes the composition law and $^\diamond$ the involution in the (multiplier algebra of the) crossed product. Finally $\mathscr{B}$ comes equipped with the two actions b of the group G and $\widehat{\mathrm{b}}^0$ of the group $\widehat{\mathsf{G}}$. If the initial action a is twisted by a 2-cocycle α, then λ will no longer be a group morphism and b will also aquire a 2-cocycle

$$\beta : \mathsf{G} \times \mathsf{G} \to \mathcal{UM}(\mathscr{B})\,, \quad \beta(x,y) := \lambda_x \diamond \lambda_y \diamond \lambda_{xy}^\diamond\,.$$

In addition, if initially there is also a twisted action $(\widehat{\mathrm{a}}, \widehat{\alpha})$ of the dual group $\widehat{\mathsf{G}}$ on $\mathscr{A}$, this can be converted in a modification of $\widehat{\mathrm{b}}^0$ into

$$[\widehat{\mathrm{b}}_\xi(f)](x) := \widehat{\mathrm{a}}_\xi[f(x)]\overline{\kappa(x,\xi)}$$

and this formula also requires a 2-cocycle $\widehat{\beta}(\cdot,\cdot) := 1 \otimes \widehat{\alpha}(\cdot,\cdot)$ on $\widehat{\mathsf{G}}$.

The conclusion is that, for the Pontryagin couple $(\mathsf{G}, \widehat{\mathsf{G}})$, a pair of twisted actions $\big((\mathrm{a}, \alpha, \mathsf{G}), (\widehat{\mathrm{a}}, \widehat{\alpha}, \widehat{\mathsf{G}})\big)$ on $\mathscr{A}$ generates a pair of twisted actions $\big((\mathrm{b}, \beta, \mathsf{G}), (\widehat{\mathrm{b}}, \widehat{\beta}, \widehat{\mathsf{G}})\big)$ on the twisted crossed product [3, 14, 15] $\mathscr{B} := \mathscr{A} \rtimes_{\mathsf{a}}^{\alpha} \mathsf{G}$. A different but similar pair of twisted actions $\big((\mathsf{c}, \gamma, \mathsf{G}), (\widehat{\mathsf{c}}, \widehat{\gamma}, \widehat{\mathsf{G}})\big)$ arises in the same way on the other twisted crossed product $\mathscr{C} := \mathscr{A} \rtimes_{\widehat{\mathsf{a}}}^{\widehat{\alpha}} \widehat{\mathsf{G}}$. Thus two new C^*-algebras are available: $(\mathscr{A} \rtimes_{\mathsf{a}}^{\alpha} \mathsf{G}) \rtimes_{\widehat{\mathsf{b}}}^{\widehat{\beta}} \widehat{\mathsf{G}}$ and $(\mathscr{A} \rtimes_{\widehat{\mathsf{a}}}^{\widehat{\alpha}} \widehat{\mathsf{G}}) \rtimes_{\mathsf{c}}^{\gamma} \mathsf{G}$. A very particular case of the results of our section 2.6 says that they are isomorphic in a canonical very explicit way, and this implies easily an extension of Takai's result that is recovered for $\widehat{\mathsf{a}} = \mathsf{id}$, $\alpha = 1$ and $\widehat{\alpha} = 1$.

Actually the two iterated twisted crossed products indicated above are isomorphic realizations of a new kind of object, the crossed product associated to a so-called "C^*-covariant systems". Its representations are generated by suitably defined covariant representations of this C^*-covariant system. Other realizations of this new type of crossed product are given by defining suitable twisted actions of the product group $\mathsf{G} \times \widehat{\mathsf{G}}$. The entire formalism can be seen as a far-reaching extension of the theory of Canonical Commutation Relations in Quantum Mechanics.

Chapter 1

Preliminaries

1.1 C^*-algebras and dynamical systems

We are going to recall some facts about C^*-algebras and covariant systems. Throughout, $\mathscr{A}$ will denote a C^*-algebra, $\mathbb{B}(\mathcal{H})$ will denote the C^*-algebra of all bounded operators on the Hilbert space $\mathcal{H}$ and $\mathbb{K}(\mathcal{H})$ denotes the closed ideal of compact operators on $\mathcal{H}$.

We denote by $\mathcal{M}(\mathscr{A})$ the multiplier algebra of $\mathscr{A}$. This algebra consists in the set of all double centralizers, *i.e.* pairs (M, M') of maps from $\mathscr{A}$ into $\mathscr{A}$ such that $xM(y) = M'(x)y$. We denote a couple (M, M') just by m and set $M(x) = mx$. It is known that this algebra is unital, containing $\mathscr{A}$ as an essential ideal, and when $\mathscr{A}$ is unital one has $\mathcal{M}(\mathscr{A}) = \mathscr{A}$ [16].

A homomorphism $\varphi : \mathscr{A} \to \mathcal{M}(\mathscr{A}')$ is *non-degenerate* if the set $\varphi(\mathscr{A})\mathscr{A}'$ is dense in $\mathscr{A}'$. A morphism φ between two C^*-algebras $\mathscr{A}$ and $\mathscr{A}'$ is a non-degenerate homomorphism from $\mathscr{A}$ into $\mathcal{M}(\mathscr{A}')$. All morphisms between C^*-algebras have an extension to corresponding multiplier algebras [27].

Definition 1.1.1. *A* ***non-degenerate representation*** *of $\mathscr{A}$ is a non-degenerate morphism $\pi : \mathscr{A} \to \mathcal{M}(\mathbb{K}(\mathcal{H}_\pi)) = \mathbb{B}(\mathcal{H}_\pi)$. We denote it $(\pi, \mathcal{H}_\pi)$ (or just by π). Two representations π and π' are* ***unitarily equivalent*** *(or just equivalent) if there exists a unitary isomorphism $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ such that $\pi(a) = U^*\pi'(a)U$ for all $a \in \mathscr{A}$.*

Remark 1.1.2. A homomorphism $\pi : \mathscr{A} \to \mathbb{B}(\mathcal{H}_\pi)$ is called representation. Note that if there exists a vector $\psi \in \mathcal{H}_\pi$ such that $\pi(\mathscr{A})\psi = \{0\}$, a projection in the one dimensional space generated by ψ (who is compact operator in $\mathcal{H}_\pi$) is orthogonal to the set $\pi(\mathscr{A})\mathbb{K}(\mathcal{H}_\pi)$. One can show that a representation $(\pi, \mathcal{H}_\pi)$ is non-degenerated if and only if for every $\psi \in \mathcal{H}$ there exists an $a \in \mathscr{A}$ such that $\pi(a)\psi \neq 0$ [27].

It is known that any C^*-algebra can be represented faithfully in some Hilbert space $\mathcal{H}$ [16, 1, 9, 5]. Let ϕ be a linear continuous functional; we say ϕ is a ***state*** if has norm 1 and $\phi(a) \geq 0$ for all a positive (*i.e.* ϕ is positive). We denote by $S_{\mathscr{A}}$ the set of all states of $\mathscr{A}$. The GNS construction shows that for every positive functional ϕ, there exists a representation $(\pi_\phi, \mathcal{H}_{\pi_\phi})$ and a vector $\psi \in \mathcal{H}_{\pi_\phi}$ such that

$$\phi(a) = \langle \pi_\phi(a)\psi, \psi \rangle_{\mathcal{H}_{\pi_\phi}}.$$

Definition 1.1.3. *The **universal representation** of $\mathscr{A}$ is the representation $\bigoplus_{\phi\in S_{\mathscr{A}}} \pi_\phi$ on the Hilbert space $\bigoplus_{\phi\in S_{\mathscr{A}}} \mathcal{H}_\phi$.*

Definition 1.1.4. *A representation $(\pi, \mathcal{H}_\pi)$ of a C^*-algebra $\mathscr{A}$ is called **irreducible** if a closed subspace of $\mathcal{H}_\pi$ which is stable under $\pi(\mathscr{A})$ is either $\mathcal{H}$ or $\{0\}$.*

The functionals in $S_{\mathscr{A}}$ for which the GNS construction gives an irreducible representation are just the extreme points of the set $S_{\mathscr{A}}$. These functionals are called ***pure states*** [18].

Suppose that $\mathscr{A}$ is non-degenerately represented in $\mathbb{B}(\mathcal{H})$; then the multiplier algebra coincides with the set $\{b \in \mathbb{B}(\mathcal{H}) |\, ba, ab \in \mathscr{A}\,, \forall a \in \mathcal{A}\}$ [16, 5].

The ***strict topology*** on $\mathbb{B}(\mathcal{H})$ is the weakest topology making the maps $b \mapsto ba$ and $b \mapsto ab$ norm continuous ($b \in \mathbb{B}(\mathcal{H})$, $a \in \mathscr{A}$); *i.e.* the locally convex topology generated by the semi norms $b \mapsto \|ba\|$, $b \mapsto \|ab\|$. The multiplier algebra of a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ coincides with the strict completion of $\mathscr{A}$ [16, 27].

Definition 1.1.5. *Let X be a second countably locally compact set with a Borel measure μ. We say that a function $f : X \to \mathscr{A}$ is **strictly Borel measurable** (or just strictly measurable) if the maps $x \mapsto f(x)a$ and $x \mapsto af(x)$ are measurable for all $a \in \mathscr{A}$.*

Since the multipliers can be considered bounded linear maps defined on $\mathscr{A}$, we can to perform manipulations like

$$m\left(\int f(x)\, d\mu(x)\right) = \int mf(x)\, d\mu(x)\,; \quad f \in L^1(X, \mathscr{A}),\, m \in \mathcal{M}(\mathscr{A}).$$

Definition 1.1.6. *We denote by $L^1(X, \mathcal{M}(\mathscr{A}))$ the set of strictly measurable maps $f : X \to \mathcal{M}(\mathscr{A})$ such that there is a constant C_f such that*

$$\int_X \|f(x)a\|\, d\mu(x) \leq C_f\|a\|\,, \quad \int_X \|af(x)\|\, d\mu(x) \leq C_f\|a\|\,.$$

We define for $f \in L^1(X, \mathcal{M}(\mathscr{A}))$ the multiplier

$$\left(\int_X f\, d\mu\right) a = \int_X f(x)a\, d\mu(x)\,, \quad a\left(\int_X f\, d\mu\right) = \int_X af(x)\, d\mu(x).$$

We have $\|\int_X f d\mu\| \leq C_f$.

In [28, Sect. 1.5] is defined the previous integral when X is a locally compact group whereas [14] is defined general. Here we shall use this, for example, for a map $\lambda : X \to \mathcal{UM}(\mathscr{A})$; we can extend to a linear map from $L^1(X, \mathcal{M}(\mathscr{A}))$ to $\mathcal{M}(\mathscr{A})$ via the formula

$$\Lambda(f) = \int_X f(x)\lambda(x)\, d\mu(x).$$

Note that $\|\Lambda(f)\| \leq \|\int_X f d\mu\|$.

Now we can refer to the unitary group of $\mathscr{A}$ as

$$\mathcal{UM}(\mathscr{A}) := \{m \in \mathcal{M}(\mathscr{A}) |\, m^*m = mm^* = 1\}.$$

We will consider the restriction of strict topology on $\mathcal{UM}(\mathscr{A})$. We denote by $\mathsf{Aut}(\mathscr{A})$ the group of automorphisms of $\mathscr{A}$. We consider $\mathsf{Aut}(\mathscr{A})$ with the topology of the pointwise norm convergence. When $\mathscr{A}$ is separable it is known that these sets are Polish groups (see [21] page 4).

Definition 1.1.7. *Let* G *be a locally compact group. we say that a map* $\mathrm{a} : \mathsf{G} \to \mathsf{Aut}(\mathscr{A})$ *is strongly Borel (respectively strongly continuous) if for each* $a \in \mathscr{A}$ *the map* $\mathsf{G} \ni x \to \mathrm{a}_x(a)$ *is Borel measurable (respectively continuous).*

Definition 1.1.8. A twisted action *of the locally compact group* G *on the* C^**-algebra* $\mathscr{A}$ *is a pair* (a, α) *composed of mappings* $\mathrm{a} : \mathsf{G} \to \mathsf{Aut}(\mathscr{A})$ *and* $\alpha : \mathsf{G} \times \mathsf{G} \to \mathcal{UM}(\mathscr{A})$ *such that*

$$\mathrm{a}_{\mathrm{e}} = \mathrm{id}_{\mathscr{A}}\,, \qquad \mathrm{a}_x \circ \mathrm{a}_y = \mathsf{ad}_{\alpha(x,y)} \circ \mathrm{a}_{xy}\,, \qquad \forall x, y \in \mathsf{G}\,,$$

$$\alpha(x, \mathrm{e}) = 1 = \alpha(\mathrm{e}, x)\,, \qquad \forall x \in \mathsf{G}\,,$$

$$\alpha(x, y)\, \alpha(xy, z) = \mathrm{a}_x[\alpha(y, z)]\, \alpha(x, yz)\,, \qquad \forall x, y, z \in \mathsf{G}\,.$$

If a *is strongly measurable and* α *is strictly measurable we speak of* a measurable twisted action. *If* a *is strongly continuous and* α *is strictly continuous we speak of* continuous twisted actions.

To a measurable twisted action (a, α) of the group G on the C^*-algebra $\mathscr{A}$ one associates [3, 14] the Banach *-algebra $L^1_{\mathrm{a},\alpha}(\mathsf{G}; \mathscr{A}) \equiv L^1(\mathsf{G}; \mathscr{A})$ (cf. [28, App. B]) and its enveloping C^*-algebra, the twisted crossed product $\mathscr{A} \rtimes^{\alpha}_{\mathrm{a}} \mathsf{G}$. The norm on $L^1(\mathsf{G}; \mathscr{A})$ is $\| f \|_1 := \int_{\mathsf{G}} dx \, \| f(x) \|_{\mathscr{A}}$. The composition laws are

$$(f \diamond g)(x) := \int_{\mathsf{G}} dy\, f(y)\, \mathrm{a}_y\big[g(y^{-1}x)\big]\, \alpha(y, y^{-1}x)\,,$$

$$f^{\diamond}(x) := \Delta_{\mathsf{G}}(x)^{-1} \alpha(x, x^{-1})^* \mathrm{a}_x[f(x^{-1})^*]\,.$$

Here the map Δ_{G} denotes the modular function of the group. We recall that the non-degenerate representations of $\mathscr{A} \rtimes^{\alpha}_{\mathrm{a}} \mathsf{G}$ are in one-to one correspondence with covariant representations of the twisted C^*-dynamical system $(\mathscr{A}, \mathrm{a}, \alpha)$ [14, 3]. These are triples $(\mathcal{H}, \pi, U)$ where $\mathcal{H}$ is a Hilbert space, $\pi : \mathscr{A} \to \mathbb{B}(\mathcal{H})$ a non-degenerate representation of $\mathscr{A}$ by bounded operators in $\mathcal{H}$ and $U : \mathsf{G} \to \mathbb{U}(\mathcal{H})$ a strongly measurable map whose values are unitary operators in $\mathcal{H}$, satisfying

$$U_x U_y = \pi[\alpha(x, y)] U_{xy}\,, \qquad \forall x, y \in \mathsf{G}\,,$$

$$U_x \pi(A) U_x^* = \pi[\mathrm{a}_x(A)]\,, \qquad \forall x \in \mathsf{G}\,, A \in \mathscr{A}\,.$$

The representation $\pi \rtimes U$ corresponding to $(\mathcal{H}, \pi, U)$ (its integrated form) acts on $f \in L^1(\mathsf{G};\mathscr{A})$ as

$$(\pi \rtimes U)f := \int_{\mathsf{G}} dx\, \pi[f(x)]\, U_x\,.$$

We also recall that *a covariant morphism* of $(\mathscr{A}, \mathrm{a}, \alpha)$ [15, Sect. 1] is composed of a C^*-algebra $\mathscr{B}$, a non-degenerate morphism $r : \mathscr{A} \to \mathcal{M}(\mathscr{B})$ and a strictly measurable map $u : \mathsf{G} \to \mathcal{UM}(\mathscr{B})$ satisfying for $x, y \in \mathsf{G}$ and $A \in \mathscr{A}$ the relations

$$u_x r(A) u_x^* = r[\mathrm{a}_x(A)]\,, \qquad u_x u_y = r[\alpha(x,y)]u_{xy}\,.$$

Remark 1.1.9. Defining the twisted crossed product as the enveloping C^*-algebra of the L^1 Banach algebra will be convenient in the setting of this thesis. Occasionally we are going to use the fact that this enveloping algebra has universal properties (cf. [14, Sect. 2] and [15, Sect. 1]), which can be used as alternative definitions.

There exists a canonical covariant representation for each twisted C^*-dynamical system $(\mathscr{A}, \mathsf{G}, \alpha, \omega)$. Given a representation $\pi : \mathscr{A} \to \mathbb{B}(\mathcal{H})$, we define $\mathrm{Ind}_{\mathscr{A}}^{\pi}$ as the integrated form of the covariant representation (r, U) on $L^2(\mathsf{G}) \otimes \mathcal{H} \cong L^2(\mathsf{G}, \mathcal{H})$ where

$$(r(a)\mathfrak{h})(x) = \pi(\alpha_{x^{-1}}(a))\mathfrak{h}(x)\,, \qquad (U_z\mathfrak{h})(x) = \pi(\omega(x^{-1}, z))\mathfrak{h}(z^{-1}x)\,,$$

where $\mathfrak{h} \in L^2(\mathsf{G}, \mathscr{A})$. Then

$$\begin{aligned}
[(\mathrm{Ind}_{\mathscr{A}}^{\pi}(f))\mathfrak{h}](x) &\equiv [r \rtimes U(f)\mathfrak{h}](x) \\
&= \int_{\mathsf{G}} \pi[\alpha_{x^{-1}}(f(z))]\, \pi(\omega(x^{-1}, z))\mathfrak{h}(z^{-1}x)\, dz \\
&= \int_{\mathsf{G}} \pi\big[\alpha_{x^{-1}}(f(xz^{-1}))\big]\, \pi(\omega(x^{-1}, xz^{-1}))\mathfrak{h}(z)\, dz\,,
\end{aligned} \tag{1.1.1}$$

for $f \in L^1(\mathsf{G}, \mathscr{A})$.

Definition 1.1.10. *The (twisted)* ***reduced crossed product****, denoted by* $\mathscr{A} \rtimes_{\alpha,r}^{\omega} \mathsf{G}$*, is the* C^**-algebra* $\mathrm{Ind}_{\mathscr{A}}^{\pi}(\mathscr{A} \rtimes_{\alpha}^{\omega} \mathsf{G}) \subset \mathbb{B}(L^2(\mathsf{G}, \mathscr{A}))$ *for a non-degenerate faithful representation* π.

The definition above does not depend of the initial representation π [16]. It is known that the reduced algebra is isomorphic to the (full) crossed product if and only if the group G is amenable [16]. There are important examples of amenable groups like abelian and compact, and there exist many others.

1.2 Tensor products

The theory of tensor products of operator algebras is fraught with a surprising number of technical complications. Here we will make a brief recall of a few facts about this theory [18, App. B].

The algebraic tensor product shall be denoted by $\odot$, while the tensor product completed in a norm β shall be denoted by $\otimes_\beta$. Let us to remark that the bilinear maps on the Cartesian product of vector spaces (algebras,...etc.) are canonically identified with linear maps on the corresponding tensor product: If $\Phi : X \times Y \to Z$, the corresponding (unique) linear map $\phi : X \odot Y \to Z$ is given on a elementary tensors by $\phi(x \otimes y) = \Phi(x, y)$.

First at all, we need to construct the tensor product of Hilbert spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex Hilbert spaces. There is a unique pre-Hilbert structure on $\mathcal{H}_1 \odot \mathcal{H}_2$ such that

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{\mathcal{H}_1 \odot \mathcal{H}_2} = \langle x_1, y_1 \rangle_{\mathcal{H}_1} \langle x_2, y_2 \rangle_{\mathcal{H}_2}$$

for all $x_1,\ y_1 \in \mathcal{H}_1$ and $x_2,\ y_2 \in \mathcal{H}_2$. This leads to a metric on $\mathcal{H}_1 \odot \mathcal{H}_2$. The completion of this metric space is a Hilbert space and we denote it by $\mathcal{H}_1 \otimes \mathcal{H}_2$ (with a slight abuse of notation).

Consider $L_1 \in \mathbb{B}(\mathcal{H}_1)$ and $L_2 \in \mathbb{B}(\mathcal{H}_2)$. The algebraic tensor product of L_1 and L_2 is continuous in $(\mathcal{H}_1 \odot \mathcal{H}_2; \langle \cdot, \cdot \rangle_{\mathcal{H}_1 \odot \mathcal{H}_2})$; then $L_1 \odot L_2$ extends to a linear continuous map on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and shall be denoted by $L_1 \otimes L_2$. One can show that

$$(L_1 \otimes L_2)(S_1 \otimes S_2) = L_1 S_1 \otimes L_2 S_2$$

and

$$(L_1 \otimes L_2)^* = L_1^* \otimes L_2^*.$$

for $L_1,\ S_1 \in \mathbb{B}(\mathcal{H}_1)$ and $L_2,\ S_2 \in \mathbb{B}(\mathcal{H}_2)$.

Example 1.2.1.

- Let (X, μ) and (Y, ν) be two σ-finite measure spaces. It can be shown that $L^2(X, \mu) \otimes L^2(Y, \nu) \cong L^2(X \times Y, \mu \times \nu)$ [10, Th. 7.15].
- Let $\mathcal{H}$ be a Hilbert space and $\bar{\mathcal{H}}$ the complex conjugated space. The tensorial product $\mathcal{H} \otimes \bar{\mathcal{H}}$ can be identified with the Hilbert-Smith class of operators on $\mathcal{H}$ [10].

Tensor products of C^*-algebras will play a basic role in the theory of crossed products and duality theorems.

Let $\mathscr{A}_1$, $\mathscr{A}_2$ be C^*-algebras. A C^*-cross norm for $\mathscr{A}_1 \odot \mathscr{A}_2$ is a norm $\|\cdot\|_\beta$ which satisfies $\|a_1 \otimes a_2\|_\beta = \|a_1\|_{\mathscr{A}_1} \|a_2\|_{\mathscr{A}_2}$ for all elementary tensors $a_1 \otimes a_2 \in \mathscr{A}_1 \odot \mathscr{A}_2$ and the completion $\mathscr{A}_1 \otimes_\beta \mathscr{A}_2 := \overline{\mathscr{A}_1 \odot \mathscr{A}_2}^{\|\cdot\|_\beta}$ is a C^*-algebra. $\mathscr{A}_1 \otimes_\beta \mathscr{A}_2$ is called the β-***tensor product*** of $\mathscr{A}_1$ and $\mathscr{A}_2$. We denote by $i^\beta_{\mathscr{A}_1} : \mathscr{A}_1 \to \mathcal{M}(\mathscr{A}_1 \otimes_\beta \mathscr{A}_2)$ and $i^\beta_{\mathscr{A}_2} : \mathscr{A}_2 \to \mathcal{M}(\mathscr{A}_1 \otimes_\beta \mathscr{A}_2$ the canonical maps $i^\beta_{\mathscr{A}_1}(a_1 \otimes a_2) = a_1 \otimes 1$ and $i^\beta_{\mathscr{A}_2}(a_1 \otimes a_2) = 1 \otimes a_2$.

There exist many C^*-cross norm on $\mathscr{A}_1 \odot \mathscr{A}_2$, But there is a maximal one and minimal one.

Definition 1.2.2. *We define the **maximal** C^***-cross norm** on $\mathscr{A}_1 \odot \mathscr{A}_2$ by*

$$\left\| \sum_{i=1}^n a_1 \otimes a_2 \right\|_{\max} = \sup \left\{ \left\| \sum_{i=1}^n \varphi(a_1) \otimes \psi(a_2) \right\| \right\},$$

where the supremum is taken over all non-degenerate commuting homomorphism φ, ψ of $\mathscr{A}_1$ and $\mathscr{A}_2$, i.e. $\varphi : \mathscr{A}_1 \to \mathcal{M}(\mathcal{D})$ *and* $\psi : \mathscr{A}_2 \to \mathcal{M}(\mathcal{D})$ *where* $\varphi(a_1)\psi(a_2) = \psi(a_2)\varphi(a_1)$ *for all* $a_1 \in \mathscr{A}_1$ *and* $a_2 \in \mathscr{A}_2$.

The tensor product $\mathscr{A}_1 \otimes_{\max} \mathscr{A}_2$ satisfies the following universal property [18]: If $\varphi : \mathscr{A}_1 \to \mathcal{M}(\mathcal{D})$ and $\psi : \mathscr{A}_2 \to \mathcal{M}(\mathcal{D})$ are non-degenerate commuting homomorphisms, there exists a non-degenerated homomorphism $\varphi \otimes \psi : \mathscr{A}_1 \otimes_{\max} \mathscr{A}_2 \to \mathcal{M}(\mathcal{D})$ satisfying $(\varphi \otimes \psi) \circ i^{\max}_{\mathscr{A}_1} = \varphi$ and $(\varphi \otimes \psi) \circ i^{\max}_{\mathscr{A}_2} = \psi$ [18]. In particular if $\|\cdot\|_\beta$ is another C^*-cross norm, then $i^{\max}_{\mathscr{A}_1} \otimes i^{\max}_{\mathscr{A}_2} : \mathscr{A}_1 \otimes_{\max} \mathscr{A}_2 \to \mathscr{A}_1 \otimes_\beta \mathscr{A}_2$ is a surjection (since it is the identity on $\mathscr{A}_1 \odot \mathscr{A}_2$), and hence $\|\cdot\|_\beta$ is dominated by $\|\cdot\|_{\max}$.

On the other hand, let $\pi : \mathscr{A}_1 \to \mathbb{B}(\mathcal{H}_1)$ and $\rho : \mathscr{A}_2 \to \mathbb{B}(\mathcal{H}_2)$ be two representations on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively; then there exists a representation $\pi \otimes \rho : \mathscr{A}_1 \odot \mathscr{A}_2 \to \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfying $(\pi \otimes \rho)(a_1 \otimes a_2) = \pi(a_1) \otimes \rho(a_2)$, and $\pi \otimes \rho$ is faithful on $\mathscr{A}_1 \odot \mathscr{A}_2$ if π and ρ are faithful.

Definition 1.2.3. *We define the* ***minimal C^*-cross norm*** *on* $\mathscr{A}_1 \odot \mathscr{A}_2$ *by*

$$\left\| \sum_{i=1}^{n} a_1 \otimes a_2 \right\|_{\min} = \sup \left\{ \| \sum_{i=1}^{n} \pi(a_1) \otimes \rho(a_2) \| \right\},$$

where the supremum is taken over all representation π and ρ of A_1 and A_2 respectively.

We have seen that $\|\cdot\|_{\min} \leq \|\cdot\|_{\max}$; a nontrivial fact is that if $\|\cdot\|_\beta$ is a C^*-norm then $\|\cdot\|_{\min} \leq \|\cdot\|_\beta \leq \|\cdot\|_{\max}$ [18]. Note that it follows from the minimality of $\|\cdot\|_{\min}$ that, whenever $\pi : \mathscr{A}_1 \to \mathbb{B}(\mathcal{H}_1)$ and $\rho : \mathscr{A}_2 \to \mathbb{B}(\mathcal{H}_2)$ are faithful representations of $\mathscr{A}_1$ and A_2 respectively, then $\pi \otimes \rho : \mathscr{A}_1 \otimes_{\min} \mathscr{A}_2 \to \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a faithful representation of $\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2$. Note also that $\pi \otimes \rho$ is non-degenerate on $\mathscr{A}_1 \odot \mathscr{A}_2$ if only if π and ρ are non-degenerated. This tensor product satisfies the following property:

Proposition 1.2.4. *If* $\varphi : \mathscr{A}_1 \to \mathcal{M}(\mathcal{D}_1)$ *and* $\psi : \mathscr{A}_2 \to \mathcal{M}(\mathcal{D}_2)$ *are homomorphisms then there is a homomorphism* $\varphi \otimes \psi : \mathscr{A}_1 \otimes_{\min} \mathscr{A}_2 \to \mathcal{M}(\mathcal{D}_1 \otimes_{\min} \mathcal{D}_2)$ *satisfying* $(\varphi \otimes \psi)(a_1 \otimes a_2) = \varphi(a_1) \otimes \psi(a_2)$. *If φ and ψ are non-degenerated (respectively faithful), then so is $\varphi \otimes \psi$.*

Proof. Representing $\mathcal{D}_1$ and $\mathcal{D}_2$ faithfully on a Hilbert spaces turns φ and ψ into two representations, and the results follows from the properties of the minimal tensor product mentioned above. □

Remark 1.2.5. Recall that a C^*-algebra $\mathscr{A}_1$ is called ***nuclear*** if $\|\cdot\|_{\max} = \|\cdot\|_{\min}$ on $\mathscr{A}_1 \odot \mathscr{A}_2$ for any C^*-algebra $\mathscr{A}_2$, *i.e.* all C^*-cross norms coincide on $\mathscr{A}_1 \odot \mathscr{A}_2$. Examples of nuclear C^*-algebras are the commutative C^*-algebras and the algebra $\mathbb{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$, but there are many other examples.

Remark 1.2.6. In this thesis we shall simply denote by $\mathscr{A}_1 \otimes \mathscr{A}_2$ (with a slight abuse of notation) the minimal tensor product of C^*-algebras.

1.3 Duality

There is an important characterization of crossed products due to Landstad [13]. Let $\mathscr{B}$ be a C^*-algebra and G an abelian locally compact group.

Definition 1.3.1. *A triple $(\mathscr{B}, \hat{\mathrm{a}}, \lambda)$ is said to be a G-product if*

(i) *there is an (continuous) action $\hat{\mathrm{a}}$ of $\hat{\mathsf{G}}$ (the Pontryagin dual) on $\mathscr{B}$;*

(ii) *there is an unitary representation λ of G into $\mathcal{UM}(\mathscr{B})$;*

(iii) *the equality*

$$\hat{\mathrm{a}}_\xi(\lambda_x) = \langle \xi, x\rangle \lambda_x \tag{1.3.1}$$

holds for all $x \in \mathsf{G}$ and $\xi \in \hat{\mathsf{G}}$.

This structure is studied for example in [13, 16].

Definition 1.3.2. *Let $(\mathscr{B}, \hat{\mathrm{a}}, \lambda)$ be a G-product with G abelian. We say that an element $A \in \mathcal{M}(\mathscr{B})$ satisfies the Landstad's conditions if*

1. *$\hat{\mathrm{a}}_\xi(A) = A$ for all $\xi \in \hat{\mathsf{G}}$;*
2. *$A \int_{\mathsf{G}} f(x)\lambda_x \, dx \in \mathscr{B}$ for all $f \in L^1(\mathsf{G})$;*
3. *the map $x \mapsto \lambda_x A \lambda_x^*$ is norm-continuous.*

The important result is the following:

Proposition 1.3.3. *Landstad Theorem [16][Th. 7.8.8] The triple $(\mathscr{B}, \alpha, \lambda)$ is a G-product if and only if there exists a C^*-dynamical system $(\mathscr{A}, \mathsf{G}, \mathrm{a})$ (non-twisted) such that $\mathscr{B} \cong \mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}$. The C^*-dynamical system is unique (up to a covariant isomorphism) and the algebra $\mathscr{A}$ coincides with the elements which satisfy Landstad's conditions, whereas $\mathrm{a}_x = \mathsf{ad}_{\lambda_x}$.*

Let us to check that for any $(\mathscr{A}, \mathsf{G}, \mathrm{a})$ untwisted C^*-dynamical system with G abelian, the crossed product carries a natural structure of G-product. Take $\xi \in \hat{\mathsf{G}}$, consider the map defined in the dense *-subalgebra $L^1(\mathsf{G}, \mathscr{A})$ by

$$L^1(\mathsf{G}, \mathscr{A}) \ni f \mapsto \hat{\mathrm{a}}_\xi(f) := \xi^* f \in L^1(\mathsf{G}, \mathscr{A}),$$

where $\xi^* f(x) = \overline{\langle \xi, x\rangle} f(x)$. This map has an extension to an automorphism of the algebra $\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}$, and one can show that the map $\xi \to \hat{\mathrm{a}}_\xi$ defines a continuous action of $\hat{\mathsf{G}}$ over the crossed product.

One defines the representation of G in $\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}$ by

$$L^1(\mathsf{G}, \mathscr{A}) \ni f \mapsto \lambda_x(f)(y) := f(x^{-1}y).$$

A straightforward computations shows that

$$\hat{\mathrm{a}}_\xi(\lambda_x) = \langle \xi, x\rangle \lambda_x, \qquad \hat{\mathrm{a}}(A)_\xi = A,$$

for $A \in \mathscr{A} \subset \mathcal{M}(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G})$. This proves that the triple $(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}, \hat{\mathrm{a}}, \lambda)$ is a G-product. Moreover the triple $(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}, \hat{\mathsf{G}}, \hat{\mathrm{a}})$ is an untwisted C^*-dynamical system.

Definition 1.3.4. *The action* $\hat{\mathrm{a}}$ *of* $\hat{\mathsf{G}}$ *on* $\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}$ *constructed above is called "the dual action" of* a.

It is interesting to see what happens if we take the crossed product by the dual action.

In the formulation of the next theorem we shall use the right regular representation ρ of G on $L^2(\mathsf{G})$. It is given by

$$(\rho_x \psi)(y) := \Delta_{\mathsf{G}}(y)^{1/2} \psi(yx)$$

for all $x,\ y \in \mathsf{G}$ and $\psi \in L^2(\mathsf{G})$. Note that for G abelian we have $\Delta \equiv 1$.

Theorem 1.3.5. *Takai [28][Th. 7.1] Suppose that* $(\mathscr{A}, \mathsf{G}, \mathrm{a})$ *is a* C^**-dynamical system where* G *is abelian. There exists an isomorphism from the iterated crossed product* $\big(\mathscr{A} \rtimes_{\mathrm{a}} \mathsf{G}\big) \rtimes_{\hat{\mathrm{a}}} \hat{\mathsf{G}}$ *onto* $\mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]$. *The dual action* $\hat{\hat{\mathrm{a}}}$ *to* $\hat{\mathrm{a}}$ *in* $\mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]$ *correspond to* $\mathrm{a} \otimes \mathrm{ad}_\rho$ *where* ρ *is the right regular representation of* G, i.e. *for any* $A \in \mathscr{A}$ *and* $T \in \mathbb{K}[L^2(\mathsf{G})]$ *we have*

$$\hat{\hat{\mathrm{a}}}_x(A \otimes T) = \mathrm{a}_x(A) \otimes \rho_x T \rho_x^*\,,$$

for all $x \in \mathsf{G}$.

In the following chapter we are going to give a proof of a much more general result.

Chapter 2

C^*-algebraic covariant structures

This chapter contains the important results and a generalization of Takai duality theorem in a new framework.

The first section recalls some basic facts about twisted crossed products and their unitary multipliers.

In the second section we introduce *covariant structures* $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ formed of a separable C^*-algebra $\mathscr{A}$, a measurable twisted action (a, α) of the second-countable locally compact group G, a measurable twisted action $(\tilde{\mathrm{a}}, \tilde{\alpha})$ of the second-countable locally compact group $\tilde{\mathsf{G}}$ and a strictly continuous function $\kappa : \mathsf{G} \times \tilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$. We insist on the fact that $\mathscr{A}, \mathsf{G}, \tilde{\mathsf{G}}$ can be non-commutative and the two groups G and $\tilde{\mathsf{G}}$ are very weakly connected. At the begining we worked under rather strong assumptions: κ was supposed to be a bi-character, the two "actions" a and ã were supposed to commute and each cocycle was taken to have values in the fixed-point algebra associated to the other action. Then we succeeded to isolate a much more general compatibility assumption connecting the five objects $\kappa, \mathrm{a}, \alpha, \tilde{\mathrm{a}}, \tilde{\alpha}$, that is quite meaningful and allows all the subsequent developments.

In section 2.3, this compatibility assumption is used to associate to the given covariant structure $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ two (exterior equivalent) twisted actions $(\overrightarrow{\mathrm{a}}, \overrightarrow{\alpha})$ and $(\overleftarrow{\mathrm{a}}, \overleftarrow{\alpha})$ of the product group $\mathsf{G} \times \tilde{\mathsf{G}}$ on $\mathscr{A}$.

In section 2.4 we define the (twisted crossed) *bi-product* of a covariant structure $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ by an universal property involving *covariant morphisms*; these are triples (r, u, v) such that (r, u) is a covariant morphism of the twisted C^*-dynamical system $(\mathscr{A}, \mathrm{a}, \alpha, \mathsf{G})$, (r, v) is a covariant morphism of the twisted C^*-dynamical system $(\mathscr{A}, \tilde{\mathrm{a}}, \tilde{\alpha}, \tilde{\mathsf{G}})$ and the commutation between u_x and v_ξ is ruled by the coupling function κ. Since such covariant morphisms are rigidly related to usual covariant morphisms of the twisted action $(\overrightarrow{\mathrm{a}}, \overrightarrow{\alpha})$, existence of bi-products follows easily from the theory of twisted crossed products; one can see $\mathscr{A} \rtimes^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} (\mathsf{G} \times \tilde{\mathsf{G}})$ as one of its possible realizations.

The remaining part of the chapter is dedicated to other realizations, involving iterated twisted crossed products; this will make the connection with the first half of the Introduction.

In section 2.5, associated to a covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$, we introduce the first generation covariant structures

$$\left\{(\mathscr{A}\rtimes_{\mathrm{a}}^{\alpha}\mathsf{G},\mathrm{k}),(\mathrm{b},\beta),\left(\tilde{\mathrm{b}},\tilde{\beta}\right)\right\} \quad \text{and} \quad \left\{(\mathscr{A}\rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\tilde{\mathsf{G}},\tilde{\mathrm{k}}),(\mathrm{c},\gamma),(\tilde{\mathrm{c}},\tilde{\gamma})\right\}$$

and then the second generation twisted crossed products $(\mathscr{A}\rtimes_{\mathrm{a}}^{\alpha}\mathsf{G})\rtimes_{\tilde{\mathrm{b}}}^{\tilde{\beta}}\tilde{\mathsf{G}}$ and $(\mathscr{A}\rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\tilde{\mathsf{G}})\rtimes_{\mathrm{c}}^{\gamma}\mathsf{G}$. Checking the axioms relies heavily on the compatibility assumption between κ, a, α, $\tilde{\mathrm{a}}$, $\tilde{\alpha}$.

The main result is contained in section 2.6. It is shown that the following isomorphisms hold

$$\mathscr{A}\rtimes_{\vec{\mathrm{a}}}^{\vec{\alpha}}(\mathsf{G}\times\tilde{\mathsf{G}})\cong\mathscr{A}\rtimes_{\overleftarrow{\mathrm{a}}}^{\overleftarrow{\alpha}}(\mathsf{G}\times\tilde{\mathsf{G}})\cong(\mathscr{A}\rtimes_{\mathrm{a}}^{\alpha}\mathsf{G})\rtimes_{\tilde{\mathrm{b}}}^{\tilde{\beta}}\tilde{\mathsf{G}}\cong(\mathscr{A}\rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\tilde{\mathsf{G}})\rtimes_{\mathrm{c}}^{\gamma}\mathsf{G}. \qquad (2.0.1)$$

This is obtained both by studying the covariant representations of all the structures involved and (for explicitness) by comparing the concrete form of the composition laws. All the four algebras above can be regarded as realizations of the bi-product attached to the covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$. The isomorphisms in (2.0.1) even hold in the category of covariant structures.

Some examples are presented in section 2.8. In particular, it is shown how a twisted version of the Abelian duality result can be deduced from the last isomorphism in (2.0.1).

2.1 Multipliers on crossed products

Some considerations about unitary multipliers of twisted crossed products will be needed. It is true [2, Prop. 4.19] that all the unitary multipliers of $L^1_{\mathrm{a},\alpha}(\mathsf{G};\mathscr{A})$ have the form $\delta_z\otimes m$, where δ_z is the Dirac measure in $z\in\mathsf{G}$ and $m\in\mathcal{UM}(\mathscr{A})$. One can find in [2] many other results about the interpretation of multiplier-valued regular measures on G with bounded variation as (left or bi-sided) multipliers on $L^1_{\mathrm{a},\alpha}(\mathsf{G};\mathscr{A})$. Since we only need simple facts, and since the connection between the multipliers of a Banach *-algebra and the multipliers of its enveloping C^*-algebra can be murky even in simple situations [12], we are going to give an independent treatment.

If $z\in\mathsf{G}$ and m is a multiplier of $\mathscr{A}$ the meaning of $\delta_z\otimes m$ as a measure with values in $\mathcal{M}(\mathscr{A})$ is obvious. To it we associate the operators $(\delta_z\otimes m)_l,(\delta_z\otimes m)_r : L^1(\mathsf{G};\mathscr{A})\rightarrow L^1(\mathsf{G};\mathscr{A})$ given by

$$[(\delta_z\otimes m)_l g](x)\equiv[(\delta_z\otimes m)\diamond g](x):=m\,\mathrm{a}_z\big[g(z^{-1}x)\big]\,\alpha(z,z^{-1}x)\,, \qquad (2.1.1)$$

$$[(\delta_z\otimes m)_r f](x)\equiv[f\diamond(\delta_z\otimes m)](x):=f(xz^{-1})\,\mathrm{a}_{xz^{-1}}(m)\,\alpha(xz^{-1},z)\,. \qquad (2.1.2)$$

One checks easily that $\{(\delta_z\otimes m)_l,(\delta_z\otimes m)_r\}$ is a double centralizer of the Banach *-algebra $L^1_{\mathrm{a},\alpha}(\mathsf{G};\mathscr{A})$, i.e.

$$f\diamond[(\delta_z\otimes m)_l g]=[(\delta_z\otimes m)_r f]\diamond g\,,\quad \forall\, f,g\in L^1(\mathsf{G};\mathscr{A})\,. \qquad (2.1.3)$$

The particular case $z=\mathrm{e}$ is worth mentioning:

$$[(\delta_{\mathrm{e}}\otimes m)\diamond f\diamond(\delta_{\mathrm{e}}\otimes n)](x)=mf(x)\,\mathrm{a}_x(n)\,. \qquad (2.1.4)$$

From now on we assume that m is a unitary multiplier of $\mathscr{A}$. To show that $\delta_z \otimes m$ extends to a multiplier of the full twisted crossed product, one has to examine its behavior under the integrated form $\Pi := \pi \rtimes U$ of an arbitrary covariant representations $(\pi, U, \mathcal{H})$. One has

$$\begin{aligned}
\Pi\left[(\delta_z \otimes m)_l g\right] &= \int_{\mathsf{G}} dx\, \pi\left\{m\, \mathrm{a}_z[g(z^{-1}x)]\alpha(z, z^{-1}x)\right\} U_x \\
&= \pi(m)\, U_z \int_{\mathsf{G}} dx\, \pi[g(z^{-1}x)]\, U_z^*\, \pi\left[\alpha(z, z^{-1}x)\right] U_x \\
&= \pi(m)\, U_z \int_{\mathsf{G}} dy\, \pi[g(y)]\, U_z^*\, \pi[\alpha(z, y)]\, U_{zy} \\
&= \pi(m)\, U_z \int_{\mathsf{G}} dy\, \pi[g(y)]\, U_y = \pi(m)\, U_z\, \pi(g).
\end{aligned}$$

Then, since U_z and $\pi(m)$ are unitary operators, one gets

$$\|\, \Pi\left[(\delta_z \otimes m)_l g\right] \|_{\mathbb{B}(\mathcal{H})} = \|\, \Pi(g)\, \|_{\mathbb{B}(\mathcal{H})}$$

so $(\delta_z \otimes m)_l$ extends to an isometry of the enveloping C^*-algebra $\mathscr{A} \rtimes_{\mathrm{a}}^{\alpha} \mathsf{G}$. A similar statement holds for $(\delta_z \otimes m)_r$, based on the identity $\Pi\left[(\delta_z \otimes m)_r f\right] = \Pi(f)\, \pi(m) U_z$. Then, by continuity and density, the two extensions form a double centralizer of $\mathscr{A} \rtimes_{\mathrm{a}}^{\alpha} \mathsf{G}$.

A shorter way to express the two computations above is to write $(\pi \rtimes U)(\delta_z \otimes m) = \pi(m) U_z$. One can deduce from this (or from many other arguments) the algebra of these unitary multipliers:

$$(\delta_y \otimes n) \diamond (\delta_z \otimes m) = \delta_{yz} \otimes \left[n \mathrm{a}_y(m) \alpha(y, z)\right], \tag{2.1.5}$$

$$(\delta_z \otimes m)^{\diamond} = \delta_{z^{-1}} \otimes \left[\alpha(z^{-1}, z)^* \mathrm{a}_{z^{-1}}(m^*)\right]. \tag{2.1.6}$$

Later on we are going to need the particular case

$$(\delta_{\mathsf{e}} \otimes m)^{\diamond} = \delta_{\mathsf{e}} \otimes m^*. \tag{2.1.7}$$

We close this section with two remarks that will be useful later.

Remark 2.1.1. Let $\mathsf{G}, \widetilde{\mathsf{G}}$ be two locally compact groups and (c, γ) a twisted action of $\mathsf{G} \times \widetilde{\mathsf{G}}$ on the C^*-algebra $\mathscr{A}$. Define $c^{\dagger}$ and $\gamma^{\dagger}$ respectively by the formulas $c^{\dagger}_{(\xi, x)} := c_{(x, \xi)}$ and $\gamma^{\dagger}\big((\xi, x), (\eta, y)\big) := \gamma\big((x, \xi), (y, \eta)\big)$. Then $(c^{\dagger}, \gamma^{\dagger})$ is a twisted action of the group $\widetilde{\mathsf{G}} \times \mathsf{G}$ on $\mathscr{A}$. The twisted crossed products $\mathscr{A} \rtimes_{c}^{\gamma} (\mathsf{G} \times \widetilde{\mathsf{G}})$ and $\mathscr{A} \rtimes_{c^{\dagger}}^{\gamma^{\dagger}} (\widetilde{\mathsf{G}} \times \mathsf{G})$ are isomorphic and at the level of L^1-elements the isomorphism is just composing with the flip $(x, \xi) \to (\xi, x)$.

Remark 2.1.2. We say that the two twisted actions (b, β) and (b', β') are *exterior equivalent* [14] if there exists a strictly measurable map (a normalized 1-cochain) $q : \mathsf{G} \to \mathcal{UM}(\mathscr{A})$ such that $q(\mathsf{e}) = 1$ and

$$\mathrm{b}'_x = \mathsf{ad}_{q_x} \circ \mathrm{b}_x, \quad \forall x \in \mathsf{G},$$

$$\beta'(x,y) = q_x \mathrm{b}_x(q_y)\beta(x,y)q_{xy}^*\,, \qquad \forall x,y \in \mathsf{G}\,.$$

In such a situation we are going to write $(\mathrm{b},\beta) \overset{q}{\sim} (\mathrm{b}',\beta')$. It is easy to see that $\sim$ is an equivalence relation.

Let us suppose that $(\mathrm{b},\beta) \overset{q}{\sim} (\mathrm{b}',\beta')$. Then [14, Lemma 3.3] the twisted crossed products $\mathscr{A} \rtimes_{\mathrm{b}}^{\beta} \mathsf{G}$ and $\mathscr{A} \rtimes_{\mathrm{b}'}^{\beta'} \mathsf{G}$ are canonically isomorphic. At the level of $L^1(\mathsf{G};\mathscr{A})$ the isomorphism acts as $[\iota_q(f)](x) := f(x)q_x^*$.

2.2 Covariant structures

Two second countable locally compact group are given: G with elements x,y,z, unit e and Haar measure dx and $\widetilde{\mathsf{G}}$ which has elements ξ,η,ζ, unit ε and Haar measure $d\xi$. The next definition is provisory; the really useful concept is that of Definition 2.2.4.

Definition 2.2.1. *A semi-covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ is given by a separable C^*-algebra $\mathscr{A}$ endowed with two measurable twisted actions (a,α) of G and $(\tilde{\mathrm{a}},\tilde{\alpha})$ of $\widetilde{\mathsf{G}}$ respectively, and with a strictly continuous map*

$$\mathsf{G}\times\widetilde{\mathsf{G}} \ni (x,\xi) \mapsto \kappa(x,\xi) \in \mathcal{UM}(\mathscr{A})$$

satisfying the normalization conditions

$$\kappa(\mathsf{e},\xi) = 1 = \kappa(x,\varepsilon)\,, \qquad \forall x \in \mathsf{G}\,,\ \xi \in \widetilde{\mathsf{G}}\,.$$

When extra regularity properties (as continuity) of the twisted actions will be present, this will usually be specified. One could call κ *the coupling function.*

Definition 2.2.2. *Given a semi-covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$, a covariant morphism is a quadruplet $(\mathscr{B},r,u,v)$ where*

1. *$(\mathscr{B},r,u)$ is a covariant morphism of the twisted C^*-dynamical system $(\mathscr{A},\mathrm{a},\alpha)$ with group G,*

2. *$(\mathscr{B},r,v)$ is a covariant morphism of the twisted C^*-dynamical system $(\mathscr{A},\tilde{\mathrm{a}},\tilde{\alpha})$ with group $\widetilde{\mathsf{G}}$,*

3. *the commutation relation $u_x v_\xi = r[\kappa(x,\xi)]v_\xi u_x$ holds for every $(x,\xi) \in \mathsf{G}\times\widetilde{\mathsf{G}}$.*

If $\mathscr{B} = \mathbb{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (thus $\mathcal{M}(\mathscr{B}) = \mathbb{B}(\mathcal{H})$) we speak of a covariant representation *and we use notations as $(\mathcal{H},\pi,U,V)$.*

Let us investigate under which assumptions convenient covariant morphisms exists. For a hypothetical one $(\mathscr{B},r,u,v)$ with *faithful* r and for $A \in \mathscr{A}$, $x \in \mathsf{G}$, $\xi \in \widetilde{\mathsf{G}}$ one has

$$(v_\xi u_x)r(A)(v_\xi u_x)^* = v_\xi r[\mathrm{a}_x(A)]v_\xi^* = r\{\tilde{\mathrm{a}}_\xi[\mathrm{a}_x(A)]\}$$

but also

$$\begin{aligned} v_\xi u_x r(A)(v_\xi u_x)^* &= r[\kappa(x,\xi)^*]\, u_x v_\xi\, r(A)\, v_\xi^* u_x^*\, r[\kappa(x,\xi)] \\ &= r[\kappa(x,\xi)^*]\, u_x r[\tilde{\mathrm{a}}_\xi(A)] u_x^*\, r[\kappa(x,\xi)] \\ &= r\{\kappa(x,\xi)^* \mathrm{a}_x[\tilde{\mathrm{a}}_\xi(A)]\kappa(x,\xi)\}\,. \end{aligned}$$

it follows that for all x, ξ one must have

$$\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi = \mathsf{ad}_{\kappa(x,\xi)} \circ \tilde{\mathrm{a}}_\xi \circ \mathrm{a}_x\,, \tag{2.2.1}$$

so $\mathsf{ad}_{\kappa(\cdot,\cdot)}$ measures the non-commutativity of the actions. If κ is center-valued the actions do commute.

Now, for arbitrary $x, y \in \mathsf{G}$, $\xi, \eta \in \widetilde{\mathsf{G}}$ let us compute $v_\xi u_x v_\eta u_y$ in two ways. First

$$\begin{aligned} v_\xi u_x v_\eta u_y &= v_\xi r[\kappa(x,\eta)] v_\eta u_x u_y \\ &= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\}\, v_\xi v_\eta u_x u_y \\ &= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\}\, r[\tilde\alpha(\xi,\eta)] v_{\xi\eta}\, r[\alpha(x,y)] u_{xy} \\ &= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\, \tilde\alpha(\xi,\eta)\, \tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]\} v_{\xi\eta} u_{xy}\,. \end{aligned}$$

But on the other hand

$$\begin{aligned} v_\xi u_x v_\eta u_y &= r[\kappa(x,\xi)^*] u_x v_\xi\, r[\kappa(y,\eta)^*]\, u_y v_\eta \\ &= r[\kappa(x,\xi)^*] u_x r\{\tilde{\mathrm{a}}_\xi[\kappa(y,\eta)^*]\} v_\xi u_y v_\eta \\ &= r[\kappa(x,\xi)^*] r\{(\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\} u_x r[\kappa(y,\xi)^*] u_y v_\xi v_\eta \\ &= r[\kappa(x,\xi)^*] r\{(\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\} r\{\mathrm{a}_x[\kappa(y,\xi)^*]\} u_x u_y v_\xi v_\eta \\ &= r\{\kappa(x,\xi)^* (\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\, \mathrm{a}_x[\kappa(y,\xi)^*]\}\, r[\alpha(x,y)] u_{xy} r[\tilde\alpha(\xi,\eta)] v_{\xi\eta} \\ &= r\{\kappa(x,\xi)^* (\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\, \mathrm{a}_x[\kappa(y,\xi)^*]\alpha(x,y)\}\, r\{\mathrm{a}_{xy}[\tilde\alpha(\xi,\eta)]\} u_{xy} v_{\xi\eta} \\ &= r\{\kappa(x,\xi)^* (\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\, \mathrm{a}_x[\kappa(y,\xi)^*]\alpha(x,y)\, \mathrm{a}_{xy}[\tilde\alpha(\xi,\eta)]\kappa(xy,\xi\eta)\} v_{\xi\eta} u_{xy}\,. \end{aligned}$$

The conclusion, valid for every x, y, ξ, η is

$$\begin{aligned} &\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\, \tilde\alpha(\xi,\eta)\, \tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)] \\ &\quad = \kappa(x,\xi)^* (\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\, \mathrm{a}_x[\kappa(y,\xi)^*]\alpha(x,y)\, \mathrm{a}_{xy}[\tilde\alpha(\xi,\eta)]\kappa(xy,\xi\eta)\,. \end{aligned} \tag{2.2.2}$$

The cohomological interpretation of (2.2.2) will be seen in Remark 2.3.2. This relation is sometimes hard to use, so we will reduce to it to a pair of simpler ones (also having a cohomological meaning). By taking $y = \mathrm{e}$ one gets

$$\mathrm{a}_x[\tilde\alpha(\xi,\eta)] = \kappa(x,\xi)\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\, \tilde\alpha(\xi,\eta)\kappa(x,\xi\eta)^* \tag{2.2.3}$$

and by taking $\eta = \varepsilon$ one gets

$$\tilde{\mathrm{a}}_\xi[\alpha(x,y)] = \kappa(x,\xi)^* \mathrm{a}_x[\kappa(y,\xi)^*]\, \alpha(x,y)\kappa(xy,\xi)\,. \tag{2.2.4}$$

Lemma 2.2.3. *Assume that* (a, α) *is a twisted action of* G *and* $(\tilde{\mathrm{a}}, \tilde\alpha)$ *is a twisted action of* $\widetilde{\mathsf{G}}$*, satisfying (2.2.1) for every* x, ξ*. Then (2.2.2) holds for every* x, y, ξ, η *if and only if (2.2.3) and (2.2.4) hold for every* x, y, ξ, η*.*

Proof. We only need to deduce (2.2.2) from (2.2.3) and (2.2.4). One transforms the r.h.s.

$$\begin{aligned}
&\kappa(x,\xi)^*(\mathrm{a}_x\circ\tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\,\mathrm{a}_x[\kappa(y,\xi)^*]\,\alpha(x,y)\,\mathrm{a}_{xy}[\tilde\alpha(\xi,\eta)]\,\kappa(xy,\xi\eta)\\
&\overset{(2.2.3)}{=}\kappa(x,\xi)^*(\mathrm{a}_x\circ\tilde{\mathrm{a}}_\xi)[\kappa(y,\eta)^*]\,\mathrm{a}_x[\kappa(y,\xi)^*]\,\alpha(x,y)\,\kappa(xy,\xi)\,\tilde{\mathrm{a}}_\xi[\kappa(xy,\eta)]\,\tilde\alpha(\xi,\eta)\\
&\overset{(2.2.1)}{=}(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\kappa(y,\eta)^*]\,\kappa(x,\xi)^*\,\mathrm{a}_x[\kappa(y,\xi)^*]\,\alpha(x,y)\,\kappa(xy,\xi)\,\tilde{\mathrm{a}}_\xi[\kappa(xy,\eta)]\,\tilde\alpha(\xi,\eta)\\
&\overset{(2.2.4)}{=}(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\kappa(y,\eta)^*]\,\tilde{\mathrm{a}}_\xi[\alpha(x,y)]\,\tilde{\mathrm{a}}_\xi[\kappa(xy,\eta)]\,\tilde\alpha(\xi,\eta)\\
&=\tilde{\mathrm{a}}_\xi\{\mathrm{a}_x[\kappa(y,\eta)^*]\alpha(x,y)\kappa(xy,\eta)\}\,\tilde\alpha(\xi,\eta)\\
&\overset{(2.2.4)}{=}\tilde{\mathrm{a}}_\xi\{\kappa(x,\eta)\,\tilde{\mathrm{a}}_\eta[\alpha(x,y)]\}\,\tilde\alpha(\xi,\eta)\\
&=\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\,(\tilde{\mathrm{a}}_\xi\circ\tilde{\mathrm{a}}_\eta)[\alpha(x,y)]\,\tilde\alpha(\xi,\eta)\\
&=\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\,\tilde\alpha(\xi,\eta)\,\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]
\end{aligned}$$

and we are done. □

Now we have at least one motivation for our main notion; see also Remarks 2.3.3 and 2.6.3 and the constructions of the next sections.

Definition 2.2.4. *A covariant structure is a semi-covariant structure* $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde\alpha)\}$ *for which relations (2.2.1), (2.2.3) and (2.2.4) are satisfied for all elements* $x,y\in\mathsf{G}$, $\xi,\eta\in\widetilde{\mathsf{G}}$.

Example 2.2.5. Suppose that for every x,ξ the multiplier $\kappa(x,\xi)$ is central and a fixed point for both a and ã (this happens if $\kappa(x,\xi)\in\mathbb{T}$ for instance). Also assume that it is "bilinear" (multiplicative in the second variable and anti-multiplicative in the first). Then (2.2.1), (2.2.3) and (2.2.4) simplify a lot: the two actions commute and the cocycles of each twisted action are fixed points of the other action. *A sub-particular case is one of the motivations of all our constructions:* G is an Abelian locally compact group, $\widetilde{\mathsf{G}}:=\widehat{\mathsf{G}}$ is its Pontryagin dual and $\kappa(x,\xi):=\xi(x)$ is obtained by applying the character ξ to the element x.

Example 2.2.6. Obviously a twisted action of G (or of $\widetilde{\mathsf{G}}$) can be completed by trivial objects to get a covariant structure. One might call $\{(\mathscr{A},1),(\mathrm{id},1),(\tilde{\mathrm{a}},\tilde\alpha)\}$ *a* G*-trivial covariant structure* and $\{(\mathscr{A},1),(\mathrm{a},\alpha),(\mathrm{id},1)\}$ might be called *a* $\widetilde{\mathsf{G}}$*-trivial covariant structure*. Similar examples with some non-trivial κ are also available.

Example 2.2.7. We outline now an example that will play an important role below. Let $(\tilde{\mathrm{a}},\tilde\alpha)$ be a measurable twisted action of $\widetilde{\mathsf{G}}$ on the C^*-algebra $\mathscr{A}$ and let ρ be a 1-cochain on G with values in $\mathcal{UM}(\mathscr{A})$, i.e. a map $\rho:\mathsf{G}\to\mathcal{UM}(\mathscr{A})$ satisfying $\rho_{\mathsf{e}}=1$. The family $\{(\mathscr{A},\kappa),(\rho),(\tilde{\mathrm{a}},\tilde\alpha)\}$ will be called *a* G*-particular covariant structure* if for $x\in\mathsf{G}$ and $\xi\in\widetilde{\mathsf{G}}$ one has *the covariance condition*

$$\tilde{\mathrm{a}}_\xi(\rho_x)=\kappa(x,\xi)^*\rho_x\,. \tag{2.2.5}$$

If G is commutative, $\widetilde{\mathsf{G}}$ is its dual, $\kappa(x,\xi):=\xi(x)$, $\tilde\alpha=1$ (so ã is a true action) and ρ is a group morphism, $(\mathscr{A},\rho,\tilde{\mathrm{a}})$ is traditionally called G-*product*; then the condition (2.2.5) plays an important role in Landstad duality theory [13, 16].

Lemma 2.2.8. *A measurable (resp. continuous)* G*-particular covariant structure can be turned into a measurable (resp. continuous) covariant structure.*

Proof. If $\{\mathscr{A}, (\rho), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ is a particular covariant structure, let us set

$$\mathrm{a}_x := \mathsf{ad}_{\rho_x} \quad \text{and} \quad \alpha(x,y) := \rho_x \rho_y \rho_{xy}^* .$$

Clearly (a, α) is a twisted action of G on $\mathscr{A}$. It is easy to check that it is measurable if ρ is strictly measurable and continuous if ρ is strictly continuous.

To check (2.2.1), for $x \in \mathsf{G}$, $\xi \in \widetilde{\mathsf{G}}$ one computes

$$\tilde{\mathrm{a}}_\xi \circ \mathsf{ad}_{\rho_x} = \mathsf{ad}_{\tilde{\mathrm{a}}_\xi(\rho_x)} \circ \tilde{\mathrm{a}}_\xi = \mathsf{ad}_{\kappa(x,\xi)^* \rho_x} \circ \tilde{\mathrm{a}}_\xi = \mathsf{ad}_{\kappa(x,\xi)^*} \circ \mathsf{ad}_{\rho_x} \circ \tilde{\mathrm{a}}_\xi .$$

We now verify (2.2.4):

$$\begin{aligned}\kappa(x,\xi)^* \mathrm{a}_x[\kappa(y,\xi)^*]\, \alpha(x,y) \kappa(xy,\xi) &= \kappa(x,\xi)^* \rho_x \kappa(y,\xi)^* \rho_x^*\, \rho_x \rho_y \rho_{xy}^*\, \kappa(xy,\xi) \\ &= \kappa(x,\xi)^* \rho_x\, \kappa(y,\xi)^* \rho_y\, \rho_{xy}^*\, \kappa(xy,\xi) \\ &= \tilde{\mathrm{a}}_\xi(\rho_x)\, \tilde{\mathrm{a}}_\xi(\rho_y)\, \tilde{\mathrm{a}}_\xi(\rho_{xy})^* = \tilde{\mathrm{a}}_\xi[\alpha(x,y)] .\end{aligned}$$

The relation (2.2.3) reads now

$$\rho_x \tilde{\alpha}(\xi,\eta) \rho_x^* = \kappa(x,\xi) \tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\, \tilde{\alpha}(\xi,\eta) \kappa(x,\xi\eta)^* . \tag{2.2.6}$$

Rewriting (2.2.5) in the form $\kappa(x,\xi)^* = \tilde{\mathrm{a}}_\xi(\rho_x)\rho_x^*$, the r.h.s of (2.2.6) can be transformed

$$\begin{aligned}\kappa(x,\xi)\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\, \tilde{\alpha}(\xi,\eta)\kappa(x,\xi\eta)^* &= \rho_x \tilde{\mathrm{a}}_\xi(\rho_x^*)\, \tilde{\mathrm{a}}_\xi[\rho_x \tilde{\mathrm{a}}_\eta(\rho_x^*)]\, \tilde{\alpha}(\xi,\eta)\, \tilde{\mathrm{a}}_{\xi\eta}(\rho_x)\rho_x^* \\ &= \rho_x\, \tilde{\mathrm{a}}_\xi[\tilde{\mathrm{a}}_\eta(\rho_x^*)]\, \tilde{\alpha}(\xi,\eta)\, \tilde{\mathrm{a}}_{\xi\eta}(\rho_x)\rho_x^* \\ &= \rho_x\, \tilde{\alpha}(\xi,\eta)\, \tilde{\mathrm{a}}_{\xi\eta}(\rho_x^*)\, \tilde{\mathrm{a}}_{\xi\eta}(\rho_x)\rho_x^* \\ &= \rho_x \tilde{\alpha}(\xi,\eta)\rho_x^* .\end{aligned}$$

□

Example 2.2.9. By analogy, one defines $\widetilde{\mathsf{G}}$*-particular (measurable) covariant structures* $\{(\mathscr{A},\kappa), (\mathrm{a},\alpha), (\tilde{\mathrm{a}},\tilde{\alpha})\}$ where, by definition, the twisted action (a,α) is arbitrary, but one has $\tilde{\mathrm{a}}_\xi := \mathsf{ad}_{\tilde{\rho}_\xi}$ and $\tilde{\alpha}(\xi,\eta) := \tilde{\rho}_\xi \tilde{\rho}_\eta \tilde{\rho}_{\xi\eta}^*$ for some measurable 1-cochain $\tilde{\rho} : \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$ satisfying $\mathrm{a}_x(\tilde{\rho}_\xi) = \kappa(x,\xi)\tilde{\rho}_\xi$ for all x, ξ.

Example 2.2.10. We close this section giving an example of covariant representation of a given covariant structure $\{(\mathscr{A},\kappa), (\mathrm{a},\alpha), (\tilde{\mathrm{a}},\tilde{\alpha})\}$. Let $\varpi : \mathscr{A} \to \mathbb{B}(\mathcal{H})$ be a faithful representation in a separable Hilbert space $\mathcal{H}$. We can inflate ϖ in a natural way to a representation of $\mathscr{A}$ in the Hilbert space

$$\mathscr{H} := L^2(\mathsf{G} \times \widetilde{\mathsf{G}}; \mathcal{H}) \cong L^2(\mathsf{G} \times \widetilde{\mathsf{G}}) \otimes \mathcal{H}$$

by setting

$$[\pi(A)\Omega](x,\xi) := \varpi\big[(\tilde{\mathrm{a}}_\xi \circ \mathrm{a}_x)(A)\big]\Omega(x,\xi) . \tag{2.2.7}$$

One also defines

$$(U_z\Omega)(x,\xi) := \Delta_{\mathsf{G}}(z)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\alpha(x,z)]\}\Omega(xz,\xi), \tag{2.2.8}$$

$$(V_\zeta\Omega)(x,\xi) := \Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\kappa(x,\zeta)]\tilde{\alpha}(\xi,\zeta)\}\Omega(x,\xi\zeta). \tag{2.2.9}$$

It is quite straightforward to show that $(\mathscr{H},\pi,U,V)$ is indeed a covariant representation; we say that *it is induced by* ϖ. Let us only indicate the most difficult of the relevant computations:

$$\begin{aligned}
&(U_zV_\zeta\Omega)(x,\xi) = \Delta_{\mathsf{G}}(z)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\alpha(x,z)]\}(V_\zeta\Omega)(xz,\zeta)\\
&= \Delta_{\mathsf{G}}(z)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\alpha(x,z)]\}\Delta_{\widetilde{\mathsf{G}}}(z)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\kappa(xz,\zeta)]\tilde{\alpha}(\xi,\zeta)\}\Omega(xz,\xi\zeta)\\
&= \Delta_{\mathsf{G}}(z)^{1/2}\Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\alpha(x,z)\kappa(xz,\zeta)]\}\varpi[\tilde{\alpha}(\xi,\zeta)]\Omega(xz,\xi\zeta)\\
&\overset{(2.2.4)}{=} \Delta_{\mathsf{G}}(z)^{1/2}\Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi\big[\mathrm{a}_x\big(\kappa(z,\zeta)\big)\kappa(x,\zeta)\,\tilde{\mathrm{a}}_\zeta\big(\alpha(x,z)\big)\big]\}\varpi[\tilde{\alpha}(\xi,\zeta)]\Omega(xz,\xi\zeta)\\
&= \Delta_{\mathsf{G}}(z)^{1/2}\Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\kappa(z,\zeta)]\}\varpi\{\tilde{\mathrm{a}}_\xi[\kappa(x,\zeta)](\tilde{\mathrm{a}}_\xi\circ\tilde{\mathrm{a}}_\zeta)[\alpha(x,z)]\tilde{\alpha}(\xi,\zeta)\}\\
&\times\Omega(xz,\xi\zeta)\\
&= \varpi\{(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\kappa(z,\zeta)]\}\Delta_{\mathsf{G}}(z)^{1/2}\Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\kappa(x,\zeta)]\tilde{\alpha}(\xi,\zeta)\tilde{\mathrm{a}}_{\xi\zeta}[\alpha(x,z)]\}\\
&\times\Omega(xz,\xi\zeta)\\
&= \varpi\{(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\kappa(z,\zeta)]\}\Delta_{\widetilde{\mathsf{G}}}(\zeta)^{1/2}\varpi\{\tilde{\mathrm{a}}_\xi[\kappa(x,\zeta)]\tilde{\alpha}(\xi,\zeta)\}\Delta_{\mathsf{G}}(z)^{1/2}\varpi\{\tilde{\mathrm{a}}_{\xi\zeta}[\alpha(x,z)]\}\\
&\times\Omega(xz,\xi\zeta)\\
&= \pi[\kappa(z,\zeta)](V_\zeta U_z\Omega)(x,\xi).
\end{aligned}$$

2.3 The twisted action attached to a covariant structure

Let us set for $x,y\in\mathsf{G}$ and $\xi,\eta\in\widetilde{\mathsf{G}}$

$$\overrightarrow{\mathrm{a}}_{(x,\xi)} := \tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x, \tag{2.3.1}$$

$$\overrightarrow{\alpha}\big((x,\xi),(y,\eta)\big) := \tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\,\tilde{\alpha}(\xi,\eta)\,\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]. \tag{2.3.2}$$

Proposition 2.3.1. *$(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ is a measurable twisted action of $\mathsf{G}\times\widetilde{\mathsf{G}}$ on $\mathscr{A}$. If the two twisted actions (a,α) and $(\tilde{\mathrm{a}},\tilde{\alpha})$ are continuous, then $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ is continuous.*

Proof. Using the assumptions and relations as $\Psi\circ\mathsf{ad}_B = \mathsf{ad}_{\Psi(B)}\circ\Psi$ and $\mathsf{ad}_A\circ\mathsf{ad}_B = \mathsf{ad}_{AB}$ one computes

$$\begin{aligned}
\overrightarrow{\mathrm{a}}_{(x,\xi)}\circ\overrightarrow{\mathrm{a}}_{(y,\eta)} &= \tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x\circ\tilde{\mathrm{a}}_\eta\circ\mathrm{a}_y\\
&= \tilde{\mathrm{a}}_\xi\circ\mathsf{ad}_{\kappa(x,\eta)}\circ\tilde{\mathrm{a}}_\eta\circ\mathrm{a}_x\circ\mathrm{a}_y\\
&= \mathsf{ad}_{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]}\circ\tilde{\mathrm{a}}_\xi\circ\tilde{\mathrm{a}}_\eta\circ\mathrm{a}_x\circ\mathrm{a}_y\\
&= \mathsf{ad}_{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]}\circ\mathsf{ad}_{\tilde{\alpha}(\xi,\eta)}\circ\tilde{\mathrm{a}}_{\xi\eta}\circ\mathsf{ad}_{\alpha(x,y)}\circ\mathrm{a}_{xy}\\
&= \mathsf{ad}_{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]}\circ\mathsf{ad}_{\tilde{\alpha}(\xi,\eta)}\circ\mathsf{ad}_{\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]}\circ\tilde{\mathrm{a}}_{\xi\eta}\circ\mathrm{a}_{xy}\\
&= \mathsf{ad}_{\overrightarrow{\alpha}\left((x,\xi),(y,\eta)\right)}\circ\overrightarrow{\mathrm{a}}_{(xy,\xi\eta)}.
\end{aligned}$$

One computes with a huge pacience

$$
\begin{aligned}
&\vec{\alpha}\big((x,\xi),(y,\eta)\big)\,\vec{\alpha}\big((xy,\xi\eta),(z,\zeta)\big)\\
&= \tilde{a}_\xi[\kappa(x,\eta)]\,\tilde{\alpha}(\xi,\eta)\,\tilde{a}_{\xi\eta}[\alpha(x,y)]\,\tilde{a}_{\xi\eta}[\kappa(xy,\zeta)]\,\tilde{\alpha}(\xi\eta,\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(xy,z)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)\tilde{a}_\eta[\alpha(x,y)\kappa(xy,\zeta)]\big\}\tilde{\alpha}(\xi,\eta)\,\tilde{\alpha}(\xi\eta,\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(xy,z)]\\
&\overset{(2.2.4)}{=} \tilde{a}_\xi\big\{\kappa(x,\eta)\tilde{a}_\eta[a_x(\kappa(y,\zeta))\kappa(x,\zeta)\tilde{a}_\zeta(\alpha(x,y))]\big\}\,\tilde{\alpha}(\xi,\eta)\,\tilde{\alpha}(\xi\eta,\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(xy,z)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)\tilde{a}_\eta[a_x(\kappa(y,\zeta))\kappa(x,\zeta)]\big\}\,(\tilde{a}_\xi\circ\tilde{a}_\eta\circ\tilde{a}_\zeta)[\alpha(x,y)]\\
&\quad\times\tilde{\alpha}(\xi,\eta)\,\tilde{\alpha}(\xi\eta,\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(xy,z)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)\tilde{a}_\eta[a_x(\kappa(y,\zeta))\kappa(x,\zeta)]\big\}\,\tilde{\alpha}(\xi,\eta)\,\tilde{\alpha}(\xi\eta,\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,y)]\,\tilde{a}_{\xi\eta\zeta}[\alpha(xy,z)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)\tilde{a}_\eta[a_x(\kappa(y,\zeta))\kappa(x,\zeta)]\big\}\,\tilde{a}_\xi[\tilde{\alpha}(\eta,\zeta)]\\
&\quad\times\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}\big\{a_x[\alpha(y,z)]\alpha(x,yz)]\big\}\,.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\vec{a}_{(x,\xi)}\big[\vec{\alpha}\big((y,\eta),(z,\zeta)\big)\big]\,\vec{\alpha}\big((x,\xi),(yz,\eta\zeta)\big)\\
&= (\tilde{a}_\xi\circ a_x)\{\tilde{a}_\eta[\kappa(y,\zeta)]\tilde{\alpha}(\eta,\zeta)\tilde{a}_{\eta\zeta}[\alpha(y,z)]\}\,\tilde{a}_\xi[\kappa(x,\eta\zeta)]\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&= \tilde{a}_\xi\big[a_x\{\tilde{a}_\eta[\kappa(y,\zeta)]\tilde{\alpha}(\eta,\zeta)\,\tilde{a}_{\eta\zeta}[\alpha(y,z)]\}\kappa(x,\eta\zeta)\big]\,\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&= \tilde{a}_\xi\big\{(a_x\circ\tilde{a}_\eta)[\kappa(y,\zeta)]\,a_x[\tilde{\alpha}(\eta,\zeta)]\,(a_x\circ\tilde{a}_{\eta\zeta})[\alpha(y,z)]\\
&\quad\times\kappa(x,\eta\zeta)\big\}\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&\overset{(2.2.1)}{=} \tilde{a}_\xi\big\{\kappa(x,\eta)(\tilde{a}_\eta\circ a_x)[\kappa(y,\zeta)]\kappa(x,\eta)^*\,a_x[\tilde{\alpha}(\eta,\zeta)]\\
&\quad\times\kappa(x,\eta\zeta)(\tilde{a}_{\eta\zeta}\circ a_x)[\alpha(y,z)]\big\}\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)(\tilde{a}_\eta\circ a_x)[\kappa(y,\zeta)]\kappa(x,\eta)^*\,a_x[\tilde{\alpha}(\eta,\zeta)]\\
&\quad\times\kappa(x,\eta\zeta)\big\}(\tilde{a}_\xi\circ\tilde{a}_{\eta\zeta})\{a_x[\alpha(y,z)]\}\tilde{\alpha}(\xi,\eta\zeta)\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)(\tilde{a}_\eta\circ a_x)[\kappa(y,\zeta)]\kappa(x,\eta)^*\,a_x[\tilde{\alpha}(\eta,\zeta)]\\
&\quad\times\kappa(x,\eta\zeta)\big\}\tilde{\alpha}(\xi,\eta\zeta)\tilde{a}_{\xi\eta\zeta}\{a_x[\alpha(y,z)]\}\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&\overset{(2.2.3)}{=} \tilde{a}_\xi\big\{\kappa(x,\eta)(\tilde{a}_\eta\circ a_x)[\kappa(y,\zeta)]\tilde{a}_\eta[\kappa(x,\zeta)]\tilde{\alpha}(\eta,\zeta)\big\}\\
&\quad\times\tilde{\alpha}(\xi,\eta\zeta)\tilde{a}_{\xi\eta\zeta}\{a_x[\alpha(y,z)]\}\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\\
&= \tilde{a}_\xi\big\{\kappa(x,\eta)(\tilde{a}_\eta\circ a_x)[\kappa(y,\zeta)]\tilde{a}_\eta[\kappa(x,\zeta)]\big\}\tilde{a}_\xi[\tilde{\alpha}(\eta,\zeta)]\\
&\quad\times\tilde{\alpha}(\xi,\eta\zeta)\tilde{a}_{\xi\eta\zeta}\{a_x[\alpha(y,z)]\}\,\tilde{a}_{\xi\eta\zeta}[\alpha(x,yz)]\,,
\end{aligned}
$$

the two expressions coincide and thus the 2-cocycle condition is verified. The normalization of $\vec{\alpha}$ is obvious.

The continuity and the measurability are easy. □

Remark 2.3.2. Relation (2.2.2) can be rephrased, also using (2.2.1)

$$
\begin{aligned}
\kappa(x,\xi)\,(\tilde{a}_\xi\circ a_x)[\kappa(y,\eta)]\,\{\tilde{a}_\xi[\kappa(x,\eta)]\,\tilde{\alpha}(\xi,\eta)\,\tilde{a}_{\xi\eta}[\alpha(x,y)]\}&\\
= a_x[\kappa(y,\xi)^*]\,\alpha(x,y)\,a_{xy}[\tilde{\alpha}(\xi,\eta)]\,.&
\end{aligned}
\tag{2.3.3}
$$

The right hand side of (2.3.3) defines a 2-cocycle $\overleftarrow{\alpha}$ on the group $\mathsf{G}\times\widetilde{\mathsf{G}}$ with respect to $\overleftarrow{\mathrm{a}}_{(x,\xi)} := a_x \circ \tilde{\mathrm{a}}_\xi$ and (2.2.2) can be rewritten

$$\kappa(x,\xi)\,\overrightarrow{\mathrm{a}}_{(x,\xi)}[\kappa(y,\eta)]\overrightarrow{\alpha}\big((x,\xi),(y,\eta)\big)\kappa(xy,\xi\eta)^* = \overleftarrow{\alpha}\big((x,\xi),(y,\eta)\big)\,. \qquad (2.3.4)$$

Relations (2.2.1) and (2.3.4) tell that the twisted actions $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ and $(\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$ are exterior equivalent (Remark 2.1.2 and [14]) through the 1-cochain κ. Rephrasings in terms of the group $\mathsf{H}' := \widetilde{\mathsf{G}}\times\mathsf{G}$, based on Remark 2.1.1, are left to the reader.

Remark 2.3.3. Now that we have introduced all the notations, it may be useful for the reader to recall the definition of a *covariant structure* $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$: It is defined by a twisted action (a,α) of the group G, a twisted action $(\tilde{\mathrm{a}},\tilde{\alpha})$ of the group $\widetilde{\mathsf{G}}$ and a normalized strictly continuous map $\kappa:\mathsf{G}\times\widetilde{\mathsf{G}}\to\mathcal{UM}(\mathscr{A})$ such that for all $X,Y\in\mathsf{G}\times\widetilde{\mathsf{G}}$

$$\overleftarrow{\mathrm{a}}_X = \mathsf{ad}_{\kappa(X)}\circ\overrightarrow{\mathrm{a}}_X \quad \text{and} \quad \kappa(X)\,\overrightarrow{\mathrm{a}}_X[\kappa(Y)]\overrightarrow{\alpha}(X,Y)\kappa(XY)^* = \overleftarrow{\alpha}(X,Y)\,.$$

Using a notation of Remark 2.1.2, this can be written $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha}) \overset{\kappa}{\sim} (\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$.

Proposition 2.3.4. *There are one-to-one correspondences between:*

1. *Covariant morphisms $(\mathscr{B},r,u,v)$ (cf. Def. 2.2.2) of the given covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$.*
2. *Covariant morphisms $(\mathscr{B},r,w)$ of the twisted C^*-dynamical system $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ with group $\mathsf{H}:=\mathsf{G}\times\widetilde{\mathsf{G}}$.*
3. *Covariant morphisms $(\mathscr{B},r,w')$ of the twisted C^*-dynamical system $(\mathscr{A},\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$ with group $\mathsf{H}:=\mathsf{G}\times\widetilde{\mathsf{G}}$.*

Proof. If $(\mathscr{B},r,u,v)$ is given, one defines

$$w:\mathsf{G}\times\widetilde{\mathsf{G}}\to\mathcal{UM}(\mathscr{B})\,,\quad w(x,\xi):=v_\xi u_x = r[\kappa(x,\xi)^*]u_x v_\xi\,. \qquad (2.3.5)$$

We show that $(\mathscr{B},r,w)$ is a covariant morphism of $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$. If $(x,\xi),(y,\eta)\in\mathsf{G}\times\widetilde{\mathsf{G}}$ one has

$$\begin{aligned}
w(x,\xi)w(y,\eta) &= v_\xi u_x v_\eta u_y\\
&= v_\xi r[\kappa(x,\eta)]v_\eta u_x u_y\\
&= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\}\,v_\xi v_\eta u_x u_y\\
&= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\}\,r[\tilde{\alpha}(\xi,\eta)]v_{\xi\eta}\,r[\alpha(x,y)]u_{xy}\\
&= r\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\}\,r[\tilde{\alpha}(\xi,\eta)]\,r\{\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]\}\,v_{\xi\eta}u_{xy}\\
&= r\big\{\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\,\tilde{\alpha}(\xi,\eta)\,\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]\big\}\,w(xy,\xi\eta)\\
&= r\big[\overrightarrow{\alpha}\big((x,y),(\xi,\eta)\big)\big]\,w\big((x,\xi)(y,\eta)\big)\,.
\end{aligned}$$

On the other hand, for $(x,\xi)\in\mathsf{G}\times\widetilde{\mathsf{G}}$ and $A\in\mathscr{A}$ one gets

$$\begin{aligned}
w(x,\xi)r(A)w(x,\xi)^* &= v_\xi u_x r(A)u_x^* v_\xi^*\\
&= v_\xi r[\mathrm{a}_x(A)]v_\xi^*\\
&= r\big\{\tilde{\mathrm{a}}_\xi\big[\mathrm{a}_x(A)\big]\big\}\\
&= r\big[\overrightarrow{\mathrm{a}}_{(x,\xi)}(A)\big]\,.
\end{aligned}$$

Now assume that $(\mathscr{B}, r, w)$ is a covariant representation of the twisted C^*-dynamical system $(\mathscr{A}, \vec{\mathrm{a}}, \vec{\alpha})$. Defining $u : \mathsf{G} \to \mathcal{UM}(\mathscr{B})$ and $v : \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{B})$ by

$$u_x := w(x, \varepsilon)\,, \qquad v_\xi := w(\mathsf{e}, \xi) \tag{2.3.6}$$

one gets a quadruple $(\mathscr{B}, r, u, v)$ satisfying the conditions specified at 1. We leave the easy verifications to the reader. Among others one uses the relations

$$\vec{\alpha}\big((x, \varepsilon), (y, \varepsilon)\big) = \alpha(x, y)\,, \qquad \vec{\alpha}\big((\mathsf{e}, \xi), (\mathsf{e}, \eta)\big) = \tilde{\alpha}(\xi, \eta)\,, \tag{2.3.7}$$

$$\vec{\alpha}\big((x, \varepsilon), (\mathsf{e}, \eta)\big) = \kappa(x, \eta)\,, \qquad \vec{\alpha}\big((\mathsf{e}, \xi), (y, \varepsilon)\big) = 1\,. \tag{2.3.8}$$

So we made explicit the correspondence between 1 and 2. The correspondence between 1 and 3 is analogous; just put

$$w'(x, \xi) := u_x v_\xi \quad \text{for } (x, \xi) \in \mathsf{G} \times \widetilde{\mathsf{G}}\,.$$

□

Remark 2.3.5. The last identity in (2.3.8) shows that $(\vec{\mathrm{a}}, \vec{\alpha})$ is not the most general twisted action of the product group $\mathsf{G} \times \widetilde{\mathsf{G}}$ in $\mathscr{A}$. Having in view the developments of the next section, it is natural to ask if any twisted action $(\vec{\mathrm{b}}, \vec{\beta})$ of the product group $\mathsf{G} \times \widetilde{\mathsf{G}}$ is at least exterior equivalent to some twisted action of the form $(\vec{\mathrm{a}}, \vec{\alpha})$. The answer is not known to us.

2.4 The bi-product of a covariant structure

Definition 2.4.1. *Let $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ be a given covariant structure. A (twisted crossed) bi-product is a universal covariant morphism $(\mathscr{C}, \iota_{\mathscr{A}}, \iota_{\mathsf{G}}, \iota_{\widetilde{\mathsf{G}}})$. Universality means that if $(\mathscr{B}, r, u, v)$ is another covariant morphism, there exists a unique non-degenerate morphism $s : \mathscr{C} \to \mathcal{M}(\mathscr{B})$ such that*

$$u = s \circ \iota_{\mathsf{G}}\,, \qquad v = s \circ \iota_{\widetilde{\mathsf{G}}}\,, \qquad r = s \circ \iota_{\mathscr{A}}\,. \tag{2.4.1}$$

Rather often we will call *bi-product* only the C^*-algebra $\mathscr{C}$, especially when the mappings $(\iota_{\mathscr{A}}, \iota_{\mathsf{G}}, \iota_{\widetilde{\mathsf{G}}})$ are obvious or not relevant. It could be denoted generically by $\mathscr{C} \equiv \mathscr{A}_{(\mathrm{a}, \tilde{\mathrm{a}})}^{(\alpha, \tilde{\alpha})}$, but it also depends on κ; its existence and uniqueness (up to isomorphisms) will be proved now.

Proposition 2.4.2. *Every covariant structure possesses a (twisted crossed) bi-product, that is unique up to a canonical isomorphism.*

Proof. By an easy abstract argument, if a bi-product exists, it is unique up to a canonical isomorphism. The meaning of this and the proof are the standard ones.

To prove existence, we rely on Proposition 2.3.4 and on the universality of the usual twisted crossed products. If $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ is a covariant structure, we construct as above the twisted C^*-dynamical system $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ with group $\mathsf{G}\times\widetilde{\mathsf{G}}$. Let $\left(\mathscr{C},\iota_{\mathscr{A}},\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}}\right)$ be a corresponding twisted crosed product. Recalling (2.3.6) we set

$$\iota_{\mathsf{G}}:\mathsf{G}\to\mathcal{UM}(\mathscr{C}),\qquad \iota_{\mathsf{G}}(x):=\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}}(x,\varepsilon),\tag{2.4.2}$$

$$\iota_{\widetilde{\mathsf{G}}}:\widetilde{\mathsf{G}}\to\mathcal{UM}(\mathscr{C}),\qquad \iota_{\widetilde{\mathsf{G}}}(\xi):=\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}}(\mathsf{e},\xi).\tag{2.4.3}$$

From Proposition 2.3.4 we already know that $(\mathscr{C},\iota_{\mathscr{A}},\iota_{\mathsf{G}},\iota_{\widetilde{\mathsf{G}}})$ is a covariant morphism; one must show its universality. So let $(\mathscr{B},r,u,v)$ be another covariant morphism and let us define w as in (2.3.5). Since $(\mathscr{B},r,w)$ is a covariant morphism of $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$, there exists a unique C^*-algebraic morphism $s:\mathscr{C}\to\mathcal{M}(\mathscr{B})$ such that

$$w=s\circ\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}},\qquad r=s\circ\iota_{\mathscr{A}}.\tag{2.4.4}$$

Then we have

$$(s\circ\iota_{\mathsf{G}})(x)=s[\iota_{\mathsf{G}}(x)]=s\left[\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}}(x,\varepsilon)\right]=w(x,\varepsilon)=u(x)$$

and

$$(s\circ\iota_{\widetilde{\mathsf{G}}})(\xi)=s\left[\iota_{\widetilde{\mathsf{G}}}(\xi)\right]=s\left[\iota_{\mathsf{G}\times\widetilde{\mathsf{G}}}(\mathsf{e},\xi)\right]=w(\mathsf{e},\xi)=v(\xi)$$

and we are done. □

Relying on the twisted actions $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ and $(\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$ we get new C^*-algebras

$$\mathscr{A}_{\overrightarrow{\mathrm{a}}}^{\overrightarrow{\alpha}}:=\mathscr{A}\rtimes_{\overrightarrow{\mathrm{a}}}^{\overrightarrow{\alpha}}(\mathsf{G}\times\widetilde{\mathsf{G}})\quad\text{with laws }(\overrightarrow{\#},\overrightarrow{^{\#}})$$

and

$$\mathscr{A}_{\overleftarrow{\mathrm{a}}}^{\overleftarrow{\alpha}}:=\mathscr{A}\rtimes_{\overleftarrow{\mathrm{a}}}^{\overleftarrow{\alpha}}(\mathsf{G}\times\widetilde{\mathsf{G}})\quad\text{with laws }(\overleftarrow{\#},\overleftarrow{^{\#}}).$$

They can be viewed as concrete realizations of the bi-product C^*-algebra $\mathscr{A}_{(\mathrm{a},\tilde{\mathrm{a}})}^{(\alpha,\tilde{\alpha})}$. Of course they are isomorpic, being defined by exterior equivalent twisted actions, cf. Remarks 2.3.2 and 2.1.2. It will be convenient to regard them as the enveloping C^*-algebras of the corresponding L^1 Banach *-algebras (but the abstract universal approach could also be adopted). At the L^1-level the isomorphism is given by $\overrightarrow{F}\to\overrightarrow{F}\kappa^*$. For further use, we record here the composition laws on $\mathscr{A}_{\overrightarrow{\mathrm{a}}}^{\overrightarrow{\alpha}}$

$$\begin{aligned}\left(\overrightarrow{F}\,\overrightarrow{\#}\,\overrightarrow{G}\right)(x,\xi)=&\int_{\mathsf{G}}\int_{\widetilde{\mathsf{G}}}dy\,d\eta\,\overrightarrow{F}(y,\eta)(\tilde{\mathrm{a}}_\eta\circ\mathrm{a}_y)\left[\overrightarrow{G}(y^{-1}x,\eta^{-1}\xi)\right]\\&\times\tilde{\mathrm{a}}_\eta[\kappa(y,\eta^{-1}\xi)]\tilde{\alpha}(\eta,\eta^{-1}\xi)\tilde{\mathrm{a}}_\xi[\alpha(y,y^{-1}x)],\end{aligned}\tag{2.4.5}$$

$$\begin{aligned}(\overrightarrow{F}^{\overrightarrow{\#}})(x,\xi)=&\,\Delta_{\mathsf{G}}(x^{-1})\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\alpha(x,x^{-1})^*\tilde{\alpha}(\xi,\xi^{-1})^*\\&\times\tilde{\mathrm{a}}_\xi[\kappa(x,\xi^{-1})^*]\,(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)\left[\overrightarrow{F}(x^{-1},\xi^{-1})^*\right]\end{aligned}\tag{2.4.6}$$

and on $\mathscr{A}_{\overleftarrow{\mathrm{a}}}^{\overleftarrow{\alpha}}$

$$\left(\overleftarrow{F} \mathbin{\overleftarrow{\#}} \overleftarrow{G}\right)(x,\xi) = \int_{\mathsf{G}}\int_{\widetilde{\mathsf{G}}} dy d\eta \overleftarrow{F}(y,\eta)(\mathrm{a}_y \circ \tilde{\mathrm{a}}_\eta)\left[\overleftarrow{G}(y^{-1}x,\eta^{-1}\xi)\right] \times \mathrm{a}_y[\kappa(y^{-1}x,\eta)^*]\alpha(y,y^{-1}x)\mathrm{a}_x[\tilde{\alpha}(\eta,\eta^{-1}\xi)], \tag{2.4.7}$$

$$(\overleftarrow{F}^{\overleftarrow{\#}})(x,\xi) = \Delta_{\mathsf{G}}(x^{-1})\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\tilde{\alpha}(\xi,\xi^{-1})^*\alpha(x,x^{-1})^* \times \mathrm{a}_x[\kappa(x^{-1},\xi)]\,(\mathrm{a}_x \circ \tilde{\mathrm{a}}_\xi)\left[\overleftarrow{F}(x^{-1},\xi^{-1})^*\right]. \tag{2.4.8}$$

By using Remark 2.1.1, one generates other two twisted actions of the group $\widetilde{\mathsf{G}} \times \mathsf{G}$ in $\mathscr{A}$ as well as other two twisted crossed product C^*-algebras isomorphic to the previous ones. They can also be seen as concrete realizations of the bi-product $\mathscr{A}_{(\mathrm{a},\alpha)}^{(\tilde{\mathrm{a}},\tilde{\alpha})}$.

The next Corollary is now obvious. Similar statements hold at the level of (covariant) morphisms.

Corollary 2.4.3. *There are one-to-one correspondences between:*

1. *Covariant representations $(\mathcal{H},\pi,U,V)$ of the covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$.*
2. *Covariant representations $(\mathcal{H},\pi,W)$ of the twisted C^*-dynamical system $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ with group $\mathsf{G}\times\widetilde{\mathsf{G}}$.*
3. *Covariant representations $(\mathcal{H},\pi,W')$ of the twisted C^*-dynamical system $(\mathscr{A},\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$ with group $\mathsf{G}\times\widetilde{\mathsf{G}}$.*
4. *Non-degenerate representations of the bi-product $\mathscr{A}_{(\mathrm{a},\tilde{\mathrm{a}})}^{(\alpha,\tilde{\alpha})}$.*
5. *Non-degenerate representations of the C^*-algebra $\mathscr{A}_{\overrightarrow{\mathrm{a}}}^{\overrightarrow{\alpha}}$.*
6. *Non-degenerate representations of the C^*-algebra $\mathscr{A}_{\overleftarrow{\mathrm{a}}}^{\overleftarrow{\alpha}}$.*

Example 2.4.4. In Example 2.2.10, given a representation ϖ of the C^*-algebra $\mathscr{A}$ in the Hilbert space $\mathcal{H}$, we constructed the corresponding induced covariant representation (π,U,V) of the covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ in the Hilbert space $\mathscr{H} = L^2(\mathsf{G}\times\widetilde{\mathsf{G}};\mathcal{H})$. Applying to it the construction given in the proof of Proposition 2.3.4, one gets exactly the induced covariant representation [14, Def. 3.10] $(\mathscr{H},\pi,W)$ of the twisted C^*-dynamical system $(\mathscr{A},\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ with group $\mathsf{G}\times\widetilde{\mathsf{G}}$ attached to the initial ϖ.

2.5 First and second generation twisted crossed products

Let $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ be a given covariant structure. To associate to it another (particular) covariant structure $\left\{(\mathscr{A}_{\mathrm{a}}^{\alpha},\mathsf{k}),(\mathrm{b},\beta),(\tilde{\mathrm{b}},\tilde{\beta})\right\}$, we first set $\mathscr{A}_{\mathrm{a}}^{\alpha} := \mathscr{A}\rtimes_{\mathrm{a}}^{\alpha}\mathsf{G}$ with algebraic laws $(\diamond,^{\diamond})$. Also set

$$\mathsf{k}:\mathsf{G}\times\widetilde{\mathsf{G}}\to\mathcal{UM}(\mathscr{A}_{\mathrm{a}}^{\alpha}), \quad \mathsf{k}(x,\xi) := \delta_{\mathsf{e}}\otimes\kappa(x,\xi). \tag{2.5.1}$$

From (2.1.1) and (2.1.2) and from $\| \cdot \|_{\mathscr{A}_{\rm a}^\alpha} \leq \| \cdot \|_1$ it follows easily that k is strictly continuous.

For each $\xi \in \widetilde{\mathsf{G}}$ we define $\tilde{\rm b}_\xi : L^1(\mathsf{G};\mathscr{A}) \to L^1(\mathsf{G};\mathscr{A})$ by

$$\left[\tilde{\rm b}_\xi(f)\right](y) := \tilde{\rm a}_\xi[f(y)]\kappa(y,\xi)^*, \tag{2.5.2}$$

while for $\xi, \eta \in \widetilde{\mathsf{G}}$, based on the preparations made in section 2.1, we set

$$\tilde{\beta}(\xi,\eta) := \delta_{\mathsf{e}} \otimes \tilde{\alpha}(\xi,\eta) \in \mathcal{UM}(\mathscr{A}_{\rm a}^\alpha). \tag{2.5.3}$$

Proposition 2.5.1. *The pair $(\tilde{\rm b}, \tilde{\beta})$ defines a measurable twisted action of $\widetilde{\mathsf{G}}$ on $\mathscr{A}_{\rm a}^\alpha$. If $(\tilde{\rm a}, \tilde{\alpha})$ is continuous, then $(\tilde{\rm b}, \tilde{\beta})$ is also continuous.*

Proof. 1. We need to prove that $\tilde{\rm b}_\xi$ is an automorphism of $\mathscr{A} \rtimes_{\rm a}^\alpha \mathsf{G}$. We only show that $\tilde{\rm b}_\xi : L^1(\mathsf{G};\mathscr{A}) \to L^1(\mathsf{G};\mathscr{A})$ is a *-isomorphism for the twisted crossed product structure; then the extension to the full twisted crossed product is automatic. Clearly $\tilde{\rm b}_\xi$ is well-defined and invertible and one has $\tilde{\rm b}_\varepsilon = {\rm id}$.

For the product, using the definitions, (2.2.1) and (2.2.4) one gets

$$\begin{aligned}
[\tilde{\rm b}_\xi(f) \diamond \tilde{\rm b}_\xi(g)](x) &= \int_{\mathsf{G}} dy \left[\tilde{\rm b}_\xi(f)\right](y)\, {\rm a}_y\Big\{\left[\tilde{\rm b}_\xi(g)\right](y^{-1}x)\Big\}\, \alpha(y,y^{-1}x) \\
&= \int_{\mathsf{G}} dy\, \tilde{\rm a}_\xi[f(y)]\, \kappa(y,\xi)^*\, ({\rm a}_y \circ \tilde{\rm a}_\xi)[g(y^{-1}x)]\, {\rm a}_y[\kappa(y^{-1}x,\xi)^*]\alpha(y,y^{-1}x) \\
&= \int_{\mathsf{G}} dy\, \tilde{\rm a}_\xi[f(y)]\, (\tilde{\rm a}_\xi \circ {\rm a}_y)[g(y^{-1}x)]\, \kappa(y,\xi)^*\, {\rm a}_y[\kappa(y^{-1}x,\xi)^*]\alpha(y,y^{-1}x) \\
&= \int_{\mathsf{G}} dy\, \tilde{\rm a}_\xi[f(y)]\, \tilde{\rm a}_\xi\{{\rm a}_y\, [g(y^{-1}x)]\}\, \tilde{\rm a}_\xi[\alpha(y,y^{-1}x)]\kappa(x,\xi)^* \\
&= \tilde{\rm a}_\xi\left(\int_{\mathsf{G}} dy\, f(y)\, {\rm a}_y\big[g(y^{-1}x)\big]\, \alpha(y,y^{-1}x)\right) \kappa(x,\xi)^* = \left[\tilde{\rm b}_\xi(f \diamond_{\rm a}^\alpha g)\right](x).
\end{aligned}$$

For the involution, by (2.2.1) and (2.2.3):

$$\begin{aligned}
\left[\tilde{\rm b}_\xi(f)\right]^\diamond(x) &= \Delta_{\mathsf{G}}(x^{-1})\, \alpha(x,x^{-1})^*\, {\rm a}_x\left[\tilde{\rm b}_\xi(f)(x^{-1})\right]^* \\
&= \Delta_{\mathsf{G}}(x^{-1})\, \alpha(x,x^{-1})^*\, {\rm a}_x\big\{\tilde{\rm a}_\xi\big[f(x^{-1})\big]\, \kappa(x^{-1},\xi)^*\big\}^* \\
&= \Delta_{\mathsf{G}}(x^{-1})\, \alpha(x,x^{-1})^*\, {\rm a}_x[\kappa(x^{-1},\xi)]\, {\rm a}_x\big\{\tilde{\rm a}_\xi\big[f(x^{-1})\big]\big\}^* \\
&= \Delta_{\mathsf{G}}(x^{-1})\, \alpha(x,x^{-1})^* {\rm a}_x[\kappa(x^{-1},\xi)]\kappa(x,\xi)\, \tilde{\rm a}_\xi\big\{{\rm a}_x\big[f(x^{-1})\big]\big\}^* \kappa(x,\xi)^* \\
&= \Delta_{\mathsf{G}}(x^{-1})\, \tilde{\rm a}_\xi[\alpha(x,x^{-1})^*]\, \tilde{\rm a}_\xi\big\{{\rm a}_x\big[f(x^{-1})^*\big]\big\}\, \kappa(x,\xi)^* \\
&= \tilde{\rm a}_\xi\big\{\Delta_{\mathsf{G}}(x^{-1})\alpha(x,x^{-1})^*{\rm a}_x\big[f(x^{-1})^*\big]\big\}\, \kappa(x,\xi)^* \\
&= \tilde{\rm a}_\xi[f^\diamond(x)]\, \kappa(x,\xi)^* = \left[\tilde{\rm b}_\xi(f^\diamond)\right](x).
\end{aligned}$$

2. For $\xi, \eta \in \mathsf{G}$ we show that $\tilde{\rm b}_\xi \circ \tilde{\rm b}_\eta = {\rm ad}^\diamond_{\tilde{\beta}(\xi,\eta)} \circ \tilde{\rm b}_{\xi\eta}$. One computes for $x \in \mathsf{G}$ and

$f \in L^1(\mathsf{G};\mathscr{A})$

$$\begin{aligned}\Big[\Big(\tilde{\mathrm{b}}_\xi \circ \tilde{\mathrm{b}}_\eta\Big)(f)\Big](x) &= \tilde{\mathrm{a}}_\xi\Big[\tilde{\mathrm{b}}_\eta(f)(x)\Big]\,\kappa(x,\xi)^* \\ &= (\tilde{\mathrm{a}}_\xi \circ \tilde{\mathrm{a}}_\eta)[f(x)]\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)^*]\kappa(x,\xi)^* \\ &= \tilde{\alpha}(\xi,\eta)\,\tilde{\mathrm{a}}_{\xi\eta}[f(x)]\,\tilde{\alpha}(\xi,\eta)^*\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)^*]\kappa(x,\xi)^* \\ &= \tilde{\alpha}(\xi,\eta)\,\tilde{\mathrm{a}}_{\xi\eta}[f(x)]\,\kappa(x,\xi\eta)^*\mathrm{a}_x[\tilde{\alpha}(\xi,\eta)^*] \\ &= \tilde{\alpha}(\xi,\eta)\,\Big[\tilde{\mathrm{b}}_{\xi\eta}(f)\Big](x)\,\mathrm{a}_x[\tilde{\alpha}(\xi,\eta)^*] \\ &= \Big(\tilde{\beta}(\xi,\eta)\diamond\Big[\tilde{\mathrm{b}}_{\xi\eta}(f)\Big]\diamond\tilde{\beta}(\xi,\eta)^\diamond\Big)(x)\,.\end{aligned}$$

We used (2.2.3); to justify the last equality use (2.1.4), (2.1.7).

3. Now we show that $\tilde{\beta}$ is a 2-cocycle with respect to $\tilde{\mathrm{b}}$. The normalization is clear. To check the 2-cocycle identity, from the definition of $\tilde{\beta}$, (2.1.5) and the fact (following from (2.1.1) and (2.1.2)) that $\tilde{\mathrm{b}}_\xi(\delta_{\mathrm{e}}\otimes m) = \delta_{\mathrm{e}}\otimes\tilde{\mathrm{a}}_\xi(m)$ one gets

$$\begin{aligned}\tilde{\beta}(\xi,\eta)\diamond\tilde{\beta}(\xi\eta,\zeta) &= [\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi,\eta)]\diamond[\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi\eta,\zeta)] \\ &= \delta_{\mathrm{e}}\otimes[\tilde{\alpha}(\xi,\eta)\tilde{\alpha}(\xi\eta,\zeta)] \\ &= \delta_{\mathrm{e}}\otimes[\tilde{\mathrm{a}}_\xi(\tilde{\alpha}(\eta,\zeta))\,\tilde{\alpha}(\xi,\eta\zeta)] \\ &= \{\delta_{\mathrm{e}}\otimes\tilde{\mathrm{a}}_\xi[\tilde{\alpha}(\eta,\zeta)]\}\diamond[\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi,\eta\zeta)] \\ &= \tilde{\mathrm{b}}_\xi\,[\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\eta,\zeta)]\diamond[\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi,\eta\zeta)] \\ &= \tilde{\mathrm{b}}_\xi\Big[\tilde{\beta}(\eta,\zeta)\Big]\diamond\tilde{\beta}(\xi,\eta\zeta)\,.\end{aligned}$$

4. Assuming now that $(\tilde{\mathrm{a}},\tilde{\alpha})$ is continuous, we are going to show that $(\tilde{\mathrm{b}},\tilde{\beta})$ is continuous. We indicate the rather straightforward arguments, because changes of norms are involved.

To show that $\tilde{\mathrm{b}}$ is strongly continuous, we estimate for $f = \varphi\otimes A$ in the dense algebraic tensor product $L^1(\mathsf{G})\odot\mathscr{A}$

$$\begin{aligned}\|\,\tilde{\mathrm{b}}_\eta(f)-\tilde{\mathrm{b}}_\xi(f)\,\|_{\mathscr{A}^\alpha_{\mathrm{a}}} &\le \|\,\tilde{\mathrm{b}}_\eta(f)-\tilde{\mathrm{b}}_\xi(f)\,\|_1 \\ &\le \int_{\mathsf{G}} dx\,|\varphi(x)|\,\big\|\tilde{\mathrm{a}}_\eta(A)\kappa(x,\eta)^*-\tilde{\mathrm{a}}_\xi(A)\kappa(x,\xi)^*\,\big\|_{\mathscr{A}}\,.\end{aligned}$$

By the Dominated Convergence Theorem, the integrability of φ and the bound

$$\big\|\tilde{\mathrm{a}}_\eta(A)\kappa(x,\eta)^*-\tilde{\mathrm{a}}_\xi(A)\kappa(x,\xi)^*\big\|_{\mathscr{A}} \le 2\,\|\,A\,\|_{\mathscr{A}}\,,$$

it is enough to prove that for $x\in\mathsf{G}$ the integrant converges to zero when $\eta\to\xi$, which is trivial since $\tilde{\mathrm{a}}$ is strongly continuous and $\kappa(x,\cdot)$ is strictly continuous.

Then, using (2.1.1)

$$\begin{aligned}\big\|\tilde{\beta}(\xi',\eta')\diamond f-\tilde{\beta}(\xi,\eta)\diamond f\big\|_{\mathscr{A}^\alpha_{\mathrm{a}}} &\le \big\|\,[\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi',\eta')]\diamond f-\delta_{\mathrm{e}}\otimes\tilde{\alpha}(\xi,\eta)]\diamond f\,\big\|_1 \\ &\le \int_{\mathsf{G}} dx\,|\varphi(x)|\,\|\,\tilde{\alpha}(\xi',\eta')A-\tilde{\alpha}(\xi,\eta)A\,\|_{\mathscr{A}}\,.\end{aligned}$$

Once again it follows that this converges to zero if $(\xi', \eta') \to (\xi, \eta)$, using the Dominated Convergence Theorem, the integrability of φ and the fact that $\tilde{\alpha}$ is strictly continuous. Multiplying with f to the left is treated similarly.

5. By using the definition of strong or strict measurability, one is lead to show that a map h defined from a Hausdorff, second countable locally compact space X endowed with a Radon measure μ to a separable Banach space $\mathscr{B}$ is measurable. The next criterion [28, App. B] reduces this to an easier continuity issue:

A function $h : \mathsf{X} \to \mathscr{B}$ is measurable if and only if for any compact set $K \subset \mathsf{X}$ and any $\epsilon > 0$, there exists a subset $K' \subset K$ such that $\mu(K \setminus K') \leq \epsilon$ and the restriction $h|_{K'}$ is continuous.

Now our measurable case follows rather easily from this and from the previous point 4. To illustrate the case of the action $\tilde{\mathrm{b}}$, we start once again with vectors of the form $f = \varphi \otimes A$, where $\varphi \in L^1(\mathsf{G})$ and $A \in \mathscr{A}$. Pick a compact set $K \subset \widetilde{\mathsf{G}}$ and a strictly positive number ϵ; for some subset K' of K for which the Haar measure of $K \setminus K'$ is smaller than ϵ, the restrictions to K' of the maps $\xi \to \tilde{\mathrm{a}}_\xi(A)$ and $\xi \to \kappa(x, \xi)$ are continuous for all $x \in \mathsf{G}$. By the argument above, the restriction to K' of the map $\xi \mapsto \tilde{\mathrm{b}}_\xi(\varphi \otimes A)$ is continuous. This and linearity show that the map $\xi \mapsto \tilde{\mathrm{b}}_\xi(f)$ is measurable for any vector f belonging to the dense subset $L^1(\mathsf{G}) \odot \mathscr{A}$ of $\mathscr{A}^\alpha_{\mathrm{a}}$. Passing to an arbitrary vector is easy by density, applying a $\delta/3$ trick and the criterion again. The strict measurability of $\tilde{\beta}$ is treated similarly. □

We define now the twisted action of G on the twisted crossed product. First, for $x \in \mathsf{G}$, let us set

$$\lambda_x := \delta_x \otimes 1 \in \mathcal{UM}(\mathscr{A}^\alpha_{\mathrm{a}}).$$

Deducing strict continuity or measurability from similar properties of the twisted action (a, α) is straightforward, if one takes (2.1.1) and (2.1.2) into consideration.

A computation relying on (2.1.1) leads to the covariance condition

$$\tilde{\mathrm{b}}_\xi(\lambda_x) = [\delta_{\mathrm{e}} \otimes \kappa(x, \xi)^*] \diamond \lambda_x = \mathsf{k}(x, \xi)^\diamond \diamond \lambda_x, \qquad \forall x \in \mathsf{G}, \xi \in \widetilde{\mathsf{G}}.$$

Along the lines of Example 2.2.7, define $\mathrm{b} : \mathsf{G} \to \mathsf{Aut}(\mathscr{A}^\alpha_{\mathrm{a}})$ by

$$\mathrm{b}_x(f) := \mathsf{ad}^\diamond_{\lambda_x}(f) = \lambda_x \diamond f \diamond \lambda^\diamond_x$$

and $\beta : \mathsf{G} \times \mathsf{G} \to \mathcal{UM}(\mathscr{A}^\alpha_{\mathrm{a}})$ by

$$\beta(x, y) := \lambda_x \diamond \lambda_y \diamond \lambda^\diamond_{xy} = \delta_{\mathrm{e}} \otimes \alpha(x, y).$$

All the calculations above conclude by

Theorem 2.5.2. *If $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ is a given measurable (resp. continuous) covariant structure, then $\{(\mathscr{A} \rtimes^\alpha_{\mathrm{a}} \mathsf{G}, \mathsf{k}), (\mathrm{b}, \beta), (\tilde{\mathrm{b}}, \tilde{\beta})\}$ is a measurable (resp. continuous) G-particular covariant structure.*

Starting with the same covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$, one can also construct a $\widetilde{\mathsf{G}}$-particular covariant structure $\left\{\left(\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}},\tilde{\mathrm{k}}\right),(\mathrm{c},\gamma),(\tilde{\mathrm{c}},\tilde{\gamma})\right\}$. We set $\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}:=\mathscr{A}\rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\widetilde{\mathsf{G}}$, with generic elements f,g and algebraic laws $(\tilde{\diamond},^{\tilde{\diamond}})$. The new coupling function is

$$\tilde{\mathrm{k}}:\mathsf{G}\times\widetilde{\mathsf{G}}\to\mathcal{UM}\left(\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\right),\quad \tilde{\mathrm{k}}(x,\xi):=\delta_\varepsilon\otimes\kappa(x,\xi)^*.$$

The two twisted actions are defined similarly as above, by changing suitably the roles of the groups G and $\widetilde{\mathsf{G}}$. Explicitly one has (here 1 is the unit of $\mathcal{M}(\mathscr{A})$ and $\mathsf{f}\in L^1(\widetilde{\mathsf{G}};\mathscr{A})$):

$$[\mathrm{c}_x(\mathsf{f})](\zeta)=\mathrm{a}_x[\mathsf{f}(\zeta)]\,\kappa(x,\zeta)\,, \tag{2.5.4}$$

$$\tilde{\mathrm{c}}_\xi(\mathsf{f})=(\delta_\xi\otimes 1)\,\tilde{\diamond}\,\mathsf{f}\,\tilde{\diamond}\,(\delta_\xi\otimes 1)^{\tilde{\diamond}},$$

$$\gamma(x,y)=\delta_\varepsilon\otimes\alpha(x,y)\,, \tag{2.5.5}$$

$$\tilde{\gamma}(\xi,\eta)=(\delta_\xi\otimes 1)\,\tilde{\diamond}\,(\delta_\eta\otimes 1)\,\tilde{\diamond}\,(\delta_{\xi\eta}\otimes 1)^{\tilde{\diamond}}=\delta_\varepsilon\otimes\tilde{\alpha}(\xi,\eta)\,.$$

Similarly as above one proves

Theorem 2.5.3. *If $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ is a given measurable (resp. continuous) covariant structure, then $\left\{\left(\mathscr{A}\rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\widetilde{\mathsf{G}},\tilde{\mathrm{k}}\right),(\mathrm{c},\gamma),(\tilde{\mathrm{c}},\tilde{\gamma})\right\}$ is a measurable (resp. continuous) $\widetilde{\mathsf{G}}$-particular covariant structure.*

All the 2-cocycles of the first generation are just tensor amplifications of those of the zero generation. At the level of actions, this is no longer true. But it does hold on certain *-subalgebras, as shown by the next result.

Lemma 2.5.4. *For every $x\in\mathsf{G}$, $\xi\in\widetilde{\mathsf{G}}$ and $m\in\mathcal{M}(\mathscr{A})$ we have*

$$\mathrm{b}_x(\delta_{\mathsf{e}}\otimes m)=\delta_{\mathsf{e}}\otimes\mathrm{a}_x(m)\,, \tag{2.5.6}$$

$$\tilde{\mathrm{b}}_\xi(\delta_{\mathsf{e}}\otimes m)=\delta_{\mathsf{e}}\otimes\tilde{\mathrm{a}}_\xi(m)\,, \tag{2.5.7}$$

$$\mathrm{c}_x(\delta_\varepsilon\otimes m)=\delta_\varepsilon\otimes\mathrm{a}_x(m)\,, \tag{2.5.8}$$

$$\tilde{\mathrm{c}}_\xi(\delta_\varepsilon\otimes m)=\delta_\varepsilon\otimes\tilde{\mathrm{a}}_\xi(m)\,. \tag{2.5.9}$$

Proof. One has by (2.1.5) and (2.1.6)

$$\begin{aligned}\mathrm{b}_x(\delta_{\mathsf{e}}\otimes m)&=(\delta_x\otimes 1)\diamond(\delta_{\mathsf{e}}\otimes m)\diamond[\delta_{x^{-1}}\otimes\alpha(x^{-1},x)^*]\\&=[\delta_x\otimes\mathrm{a}_x(m)]\diamond[\delta_{x^{-1}}\otimes\alpha(x^{-1},x)^*]\\&=\delta_{\mathsf{e}}\otimes\left\{\mathrm{a}_x(m)\mathrm{a}_x\left[\alpha(x^{-1},x)^*\right]\alpha(x,x^{-1})\right\}\\&=\delta_{\mathsf{e}}\otimes\mathrm{a}_x(m)\,,\end{aligned}$$

where the 2-cocycle property of α has been used for the last equality. To prove (2.5.7) one must show for $g\in L^1(\mathsf{G};\mathscr{A})$

$$\tilde{\mathrm{b}}_\xi[(\delta_{\mathsf{e}}\otimes m)\diamond g]=[\delta_{\mathsf{e}}\otimes\tilde{\mathrm{a}}_\xi(m)]\diamond\tilde{\mathrm{b}}_\xi(g)$$

and

$$\tilde{\mathrm{b}}_\xi[g\diamond(\delta_{\mathsf{e}}\otimes m)]=\tilde{\mathrm{b}}_\xi(g)\diamond[\delta_{\mathsf{e}}\otimes\tilde{\mathrm{a}}_\xi(m)]\,.$$

This follows straightforwardly from (2.1.1), (2.1.2) and the definition of $\tilde{\mathrm{b}}_\xi$. Proving (2.5.8) and (2.5.9) is similar. □

Starting from the covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ and applying the twisted crossed product construction, we obtained new (particular) measurable covariant structures $\left\{(\mathscr{A}_{\mathrm{a}}^{\alpha},\mathsf{k}),(\mathrm{b},\beta),(\tilde{\mathrm{b}},\tilde{\beta})\right\}$ and $\left\{(\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}},\tilde{\mathsf{k}}),(\mathrm{c},\gamma),(\tilde{\mathrm{c}},\tilde{\gamma})\right\}$. With all these objects one can construct (at least) two "second generation" C^*-algebras (they will be compared in the next section). First, one has

$$\mathscr{A}_{\mathrm{a},\tilde{\mathrm{b}}}^{\alpha,\tilde{\beta}} \equiv \left(\mathscr{A}_{\mathrm{a}}^{\alpha}\right)_{\tilde{\mathrm{b}}}^{\tilde{\beta}} := (\mathscr{A} \rtimes_{\mathrm{a}}^{\alpha} \mathsf{G}) \rtimes_{\tilde{\mathrm{b}}}^{\tilde{\beta}} \widetilde{\mathsf{G}},$$

with elements F, G and algebraic structure $(\square,^{\square})$. The second one is

$$\mathscr{A}_{\tilde{\mathrm{a}},\mathrm{c}}^{\tilde{\alpha},\gamma} \equiv \left(\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}}\right)_{\mathrm{c}}^{\gamma} := (\mathscr{A} \rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}} \widetilde{\mathsf{G}}) \rtimes_{\mathrm{c}}^{\gamma} \mathsf{G},$$

with composition laws $(\tilde{\square},^{\tilde{\square}})$ and elements F, G. We recall that they also depend on the coupling function κ.

Remark 2.5.5. There are other two (less interesting) second generation C^*-algebras

$$\mathscr{A}_{\mathrm{a},\mathrm{b}}^{\alpha,\beta} \equiv (\mathscr{A}_{\mathrm{a}}^{\alpha})_{\mathrm{b}}^{\beta} := (\mathscr{A} \rtimes_{\mathrm{a}}^{\alpha} \mathsf{G}) \rtimes_{\mathrm{b}}^{\beta} \mathsf{G} \quad \text{and} \quad \mathscr{A}_{\tilde{\mathrm{a}},\tilde{\mathrm{c}}}^{\tilde{\alpha},\tilde{\gamma}} \equiv (\mathscr{A}_{\tilde{\mathrm{a}}}^{\tilde{\alpha}})_{\tilde{\mathrm{c}}}^{\tilde{\gamma}} := (\mathscr{A} \rtimes_{\tilde{\mathrm{a}}}^{\tilde{\alpha}} \widetilde{\mathsf{G}}) \rtimes_{\tilde{\mathrm{c}}}^{\tilde{\gamma}} \widetilde{\mathsf{G}}.$$

2.6 They are isomorphic

The purpose now is to show that the second generation twisted crossed products $\mathscr{A}_{\mathrm{a},\tilde{\mathrm{b}}}^{\alpha,\tilde{\beta}}$ and $\mathscr{A}_{\tilde{\mathrm{a}},\mathrm{c}}^{\tilde{\alpha},\gamma}$ are isomorphic and constitute realizations of the bi-product associated to a given covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$.

Theorem 2.6.1. *There are one-to-one correspondences between:*

1. *Covariant morphisms of the covariant structure* $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$.
2. *Non-degenerate morphisms of the* C^**-algebra* $\mathscr{A}_{\mathrm{a},\tilde{\mathrm{b}}}^{\alpha,\tilde{\beta}}$.
3. *Non-degenerate morphisms of the* C^**-algebra* $\mathscr{A}_{\tilde{\mathrm{a}},\mathrm{c}}^{\tilde{\alpha},\gamma}$.

Proof. If $(\mathscr{B},r,u,v)$ is given as in Definition 2.2.2, we are going to construct covariant morphisms

$$r_{u,v} : \mathscr{A}_{\mathrm{a},\tilde{\mathrm{b}}}^{\alpha,\tilde{\beta}} \to \mathcal{M}(\mathscr{B}) \quad \text{and} \quad r_{v,u} : \mathscr{A}_{\tilde{\mathrm{a}},\mathrm{c}}^{\tilde{\alpha},\gamma} \to \mathcal{M}(\mathscr{B}).$$

Using $(\mathscr{B},r,u)$ we first construct the integrated form $r_u := r \rtimes u : \mathscr{A}_{\mathrm{a}}^{\alpha} \to \mathcal{M}(\mathscr{B})$. Let us check that $(\mathscr{B},r_u,v)$ is a covariant morphism of $(\mathscr{A}_{\mathrm{a}}^{\alpha},\tilde{\mathrm{b}},\tilde{\beta})$. First, for $f \in L^1(\mathsf{G};\mathscr{A})$ and $\xi \in \widetilde{\mathsf{G}}$ one has

$$\begin{aligned}
v_\xi r_u(f) v_\xi^* &= \int_{\mathsf{G}} dx\, v_\xi r[f(x)] v_\xi^*\, v_\xi u_x v_\xi^* \\
&= \int_{\mathsf{G}} dx\, r[\tilde{\mathrm{a}}_\xi(f(x))]\, r[\kappa(x,\xi)^*]\, u_x \\
&= \int_{\mathsf{G}} dx\, r\left[(\tilde{\mathrm{b}}_\xi f)(x)\right] u_x = r_u\left[\tilde{\mathrm{b}}_\xi(f)\right].
\end{aligned}$$

Then, since $(\mathscr{B}, r, v)$ is a covariant representation of $(\mathscr{A}, \tilde{\mathrm{a}}, \tilde{\alpha})$, for $\xi, \eta \in \widetilde{\mathsf{G}}$ we have $v_\xi v_\eta v_{\xi\eta}^* = r[\tilde{\alpha}(\xi, \eta)]$. Therefore it is enough to prove that $r_u\left[\tilde{\beta}(\xi, \eta)\right] = r[\tilde{\alpha}(\xi, \eta)]$. For $g \in L^1(\mathsf{G}; \mathscr{A})$ one computes using (2.1.1)

$$r_u\left[\tilde{\beta}(\xi, \eta) \diamond g\right] = \int_{\mathsf{G}} dx\, r\{[(\delta_{\mathrm{e}} \otimes \tilde{\alpha}(\xi, \eta)) \diamond g](x)\} u_x$$
$$= \int_{\mathsf{G}} dx\, r\{\tilde{\alpha}(\xi, \eta) g(x)\}\, u_x = r[\tilde{\alpha}(\xi, \eta)]\, r_u(g)\,.$$

Similarly one gets $r_u\left[g \diamond \tilde{\beta}(\xi, \eta)\right] = r_u(g)\, r[\tilde{\alpha}(\xi, \eta)]$ and this is exactly what we needed to show. Thus the (double) integrated form $r_{u,v} := r_u \rtimes v = (r \rtimes u) \rtimes v$ is a non-degenerate morphism of $\mathscr{A}_{\mathrm{a}, \tilde{\mathrm{b}}}^{\alpha, \tilde{\beta}}$. Analogously, $r_{v,u} := r_v \rtimes u = (r \rtimes v) \rtimes u$ will be a nondegenerate morphism of $\mathscr{A}_{\tilde{\mathrm{a}}, \mathrm{c}}^{\tilde{\alpha}, \gamma}$.

Now we show that every non-degenerate morphism $\mathcal{R}$ of $\mathscr{A}_{\mathrm{a}, \tilde{\mathrm{b}}}^{\alpha, \tilde{\beta}}$ in some C^*-algebra $\mathscr{B}$ has the form $\mathcal{R} = (r \rtimes u) \rtimes v$ with $(\mathscr{B}, r, u, v)$ as required. The reasoning for non-degenerate morphisms $\mathcal{S}$ of $\mathscr{A}_{\tilde{\mathrm{a}}, \mathrm{c}}^{\tilde{\alpha}, \gamma}$ would be similar.

The general theory, applied to the C^*-dynamical system $(\mathscr{A}_{\mathrm{a}}^{\alpha}, \tilde{\mathrm{b}}, \tilde{\beta})$, tells us that $\mathcal{R} = R \rtimes v$ for some covariant morphism $(\mathscr{B}, R, v)$. In its turn, R must have the form $r \rtimes u$ for a covariant morphism $(\mathscr{B}, r, u)$ of $(\mathscr{A}, \mathrm{a}, \alpha)$. Let us show that $(\mathscr{B}, r, v)$ is a covariant morphism of $(\mathscr{A}, \tilde{\mathrm{a}}, \tilde{\alpha})$. We already know that $v_\xi v_\eta = R\left[\tilde{\beta}(\xi, \eta)\right] v_{\xi\eta}$. So, to prove that $v_\xi v_\eta = r[\tilde{\alpha}(\xi, \eta)]\, v_{\xi\eta}$ one needs to check that $R\left[\tilde{\beta}(\xi, \eta)\right] = r[\tilde{\alpha}(\xi, \eta)]$. But this has been done above.

On the other hand, by Lemma 2.5.4, one has $\tilde{\mathrm{b}}_\xi(\delta_{\mathrm{e}} \otimes A) = \delta_{\mathrm{e}} \otimes \tilde{\mathrm{a}}_\xi(A)$ for every $\xi \in \widetilde{\mathsf{G}}$ and $A \in \mathscr{A}$. Thus one has

$$v_\xi r(A) v_\xi^* = v_\xi R(\delta_{\mathrm{e}} \otimes A) v_\xi^* = R\left[\tilde{\mathrm{b}}_\xi(\delta_{\mathrm{e}} \otimes A)\right] = R[\delta_{\mathrm{e}} \otimes \tilde{\mathrm{a}}_\xi(A)] = r[\tilde{\mathrm{a}}_\xi(A)]\,.$$

Finally we show the right commutation relations between the unitary multipliers u_x and v_ξ. The game is to deduce this only from the fact that $(\mathscr{B}, R, v)$ and $(\mathscr{B}, r, u)$ are covariant morphisms.

Note first that elements of the form $\varphi \otimes \psi \otimes A$, with $A \in \mathscr{A}$, $\varphi \in L^1(\mathsf{G})$ and $\psi \in L^1(\widetilde{\mathsf{G}})$ (thus belonging to the algebraic tensor product $L^1(\mathsf{G}) \odot L^1(\widetilde{\mathsf{G}}) \odot \mathscr{A}$) are total in $\mathscr{A}_{\mathrm{a}, \tilde{\mathrm{b}}}^{\alpha, \tilde{\beta}}$. Since $\mathcal{R} = R \rtimes v = (r \rtimes u) \rtimes v$, it is easy to check that $\mathcal{R}(\varphi \otimes \psi \otimes A) = r(A) u[\varphi] v[\psi]$, where we used the notations $u[\varphi] := \int_{\mathsf{G}} dx\, \varphi(x) u_x$ and $v[\psi] := \int_{\widetilde{\mathsf{G}}} d\xi\, \psi(\xi) v_\xi$. Thus, $\mathcal{R}$ being nondegenerate, it is enough to show for all the ingredients the identity

$$v_\xi u_x r(A) u[\varphi] v[\psi] = r[\kappa(x, \xi)^*] u_x v_\xi\, r(A) u[\varphi] v[\psi]\,.$$

Below, we are going to use the notation $g_x(\cdot) := \varphi(x^{-1} \cdot) \mathrm{a}_x(A) \alpha(x, x^{-1} \cdot) \in L^1(\mathsf{G}; \mathscr{A})$. Using properties of the two covariant representations and axioms of the covariant structure, and recalling that $R = r \rtimes u$, we compute

$$
\begin{aligned}
v_\xi u_x\, r(A)u[\varphi]v[\psi] &= v_\xi r[\mathrm{a}_x(A)]u_x \int_{\mathsf{G}} dz\, \varphi(z)u_z\, v[\psi] \\
&= v_\xi r[\mathrm{a}_x(A)] \int_{\mathsf{G}} dy\, \varphi(x^{-1}y) r[\alpha(x,x^{-1}y)]u_y v[\psi] \\
&= v_\xi \int_{\mathsf{G}} dy\, r\big\{\varphi(x^{-1}y)\mathrm{a}_x(A)\alpha(x,x^{-1}y)\big\}u_y v[\psi] \\
&= v_\xi R(g_x)v[\psi] = R\left[\tilde{\mathrm{b}}_\xi(g_x)\right] v_\xi v[\psi] \\
&= \int_{\mathsf{G}} dy\, r\big\{\varphi(x^{-1}y)\,\tilde{\mathrm{a}}_\xi\left[\mathrm{a}_x(A)\alpha(x,x^{-1}y)\right]\kappa(y,\xi)^*\big\}\, u_y v_\xi v[\psi] \\
&= \int_{\mathsf{G}} dy\, \varphi(x^{-1}y)\, r\big\{\tilde{\mathrm{a}}_\xi[\mathrm{a}_x(A)]\,\tilde{\mathrm{a}}_\xi\left[\alpha(x,x^{-1}y)\right]\kappa(y,\xi)^*\big\}\, u_y v_\xi v[\psi] \\
&\overset{(2.2.4)}{=} \int_{\mathsf{G}} dy\, \varphi(x^{-1}y)\, r\big\{\tilde{\mathrm{a}}_\xi[\mathrm{a}_x(A)]\,\kappa(x,\xi)^*\mathrm{a}_x[\kappa(x^{-1}y,\xi)^*]\alpha(x,x^{-1}y)\big\}\, u_y v_\xi v[\psi] \\
&\overset{(2.2.1)}{=} r[\kappa(x,\xi)^*]\, r\{\mathrm{a}_x[\tilde{\mathrm{a}}_\xi(A)]\} \int_{\mathsf{G}} dz\, \varphi(z)\, r\big\{\mathrm{a}_x[\kappa(z,\xi)^*]\alpha(x,z)\big\}u_{xz}v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, r\{\mathrm{a}_x[\tilde{\mathrm{a}}_\xi(A)]\} \int_{\mathsf{G}} dz\, \varphi(z)\, r\big\{\mathrm{a}_x[\kappa(z,\xi)^*]\big\}u_x\, u_z v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, r\{\mathrm{a}_x[\tilde{\mathrm{a}}_\xi(A)]\}\, u_x \int_{\mathsf{G}} dz\, \varphi(z)\, r[\kappa(z,\xi)^*]u_z v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x\, r[\tilde{\mathrm{a}}_\xi(A)] \int_{\mathsf{G}} dz\, \varphi(z)\, r[\kappa(z,\xi)^*]u_z v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x \int_{\mathsf{G}} dz\, r\big\{\varphi(z)\tilde{\mathrm{a}}_\xi(A)\kappa(z,\xi)^*\big\}u_z v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x \int_{\mathsf{G}} dz\, r\big\{\big[\tilde{\mathrm{b}}_\xi(\varphi\otimes A)\big](z)\big\}u_z v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x R\big[\tilde{\mathrm{b}}_\xi(\varphi\otimes A)\big]v_\xi v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x v_\xi R(\varphi\otimes A)v[\psi] \\
&= r[\kappa(x,\xi)^*]\, u_x v_\xi r(A)u[\varphi]v[\psi]\,,
\end{aligned}
$$

so we are done. □

Then follows straightforwardly

Corollary 2.6.2. *Both C^*-algebras $\mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}}$ and $\mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}}$ are bi-products of the covariant structure $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$. In particular, one has isomorphic C^*-algebras*

$$
\mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} \cong \mathscr{A}^{\overleftarrow{\tilde{\alpha}}}_{\overleftarrow{\tilde{\mathrm{a}}}} \cong \mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}} \cong \mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}}\,.
$$

Even if Corollary 2.6.2 can be proved directly, it is interesting and useful to have explicit forms of the isomorphisms. Actually one has a commuting diagram of isomorphisms

$$\begin{array}{ccc} \mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}} & \xrightarrow{\Upsilon} & \mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}} \\ \Phi \downarrow & & \downarrow \Psi \\ \mathscr{A}^{\overleftarrow{\alpha}}_{\overleftarrow{\mathrm{a}}} & \xrightarrow[\Gamma]{} & \mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} \end{array}$$

We have already specified Γ before

$$\left[\Gamma(\overleftarrow{F})\right](x,\xi) := \overleftarrow{F}(x,\xi)\,\kappa(x,\xi)\,,$$

as a consequence of exterior equivalence of the twisted actions $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ and $(\overleftarrow{\mathrm{a}},\overleftarrow{\alpha})$. The actions of the other three on the L^1-Banach algebras are simply

$$[\Upsilon(F)(x)](\xi) := [F(\xi)](x)\,\kappa(x,\xi)\,,$$

$$[\Phi(F)](x,\xi) := [F(\xi)](x)\,, \quad [\Psi(\mathsf{F})](x,\xi) := [\mathsf{F}(x)](\xi)\,,$$

and the diagram is already seen to commute. To convince the reader, we are going to exhibit the multiplications and the involutions of the iterated crossed products, at the level of L^1-elements. In $\mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}}$ one has

$$\begin{aligned}
&[(F\Box G)(\xi)](x) = \left\{\int_{\tilde{\mathsf{G}}} d\eta\, F(\eta)\diamond \tilde{\mathrm{b}}_\eta\big[G(\eta^{-1}\xi)\big]\diamond\tilde{\beta}(\eta,\eta^{-1}\xi)\right\}(x)\\
&= \int_{\tilde{\mathsf{G}}} d\eta \left\{F(\eta)\diamond\tilde{\mathrm{b}}_\eta\big[G(\eta^{-1}\xi)\big]\diamond[\delta_e\otimes\tilde{\alpha}(\eta,\eta^{-1}\xi)]\right\}(x)\\
&= \int_{\tilde{\mathsf{G}}} d\eta\int_{\mathsf{G}} dy\,[F(\eta)](y)\,\mathrm{a}_y\Big[\Big(\tilde{\mathrm{b}}_\eta[G(\eta^{-1}\xi)]\diamond[\delta_e\otimes\tilde{\alpha}(\eta,\eta^{-1}\xi)]\Big)(y^{-1}x)\Big]\,\alpha(y,y^{-1}x)\\
&\overset{(2.1.2)}{=} \int_{\tilde{\mathsf{G}}} d\eta\int_{\mathsf{G}} dy\,[F(\eta)](y)\,\mathrm{a}_y\Big(\tilde{\mathrm{b}}_\eta[G(\eta^{-1}\xi)](y^{-1}x)\,\mathrm{a}_{y^{-1}x}[\tilde{\alpha}(\eta,\eta^{-1}\xi)]\Big)\\
&\quad\times\alpha(y,y^{-1}x)\\
&= \int_{\tilde{\mathsf{G}}} d\eta\int_{\mathsf{G}} dy\,[F(\eta)](y)\,\mathrm{a}_y\big(\tilde{\mathrm{a}}_\eta[G(\eta^{-1}\xi)(y^{-1}x)]\kappa(y^{-1}x,\eta)^*]\,\mathrm{a}_{y^{-1}x}[\tilde{\alpha}(\eta,\eta^{-1}\xi)]\big)\\
&\quad\times\alpha(y,y^{-1}x)\\
&= \int_{\tilde{\mathsf{G}}} d\eta\int_{\mathsf{G}} dy\,[F(\eta)](y)\,(\mathrm{a}_y\circ\tilde{\mathrm{a}}_\eta)[G(\eta^{-1}\xi)(y^{-1}x)]\,\mathrm{a}_y[\kappa(y^{-1}x,\eta)^*]\,(\mathrm{a}_y\circ\mathrm{a}_{y^{-1}x})\\
&\quad\times[\tilde{\alpha}(\eta,\eta^{-1}\xi)]\big)\,\alpha(y,y^{-1}x)\\
&= \int_{\tilde{\mathsf{G}}} d\eta\int_{\mathsf{G}} dy\,[F(\eta)](y)\,(\mathrm{a}_y\circ\tilde{\mathrm{a}}_\eta)[G(\eta^{-1}\xi)(y^{-1}x)]\,\mathrm{a}_y[\kappa(y^{-1}x,\eta)^*]\\
&\quad\times\alpha(y,y^{-1}x)\,\mathrm{a}_x[\tilde{\alpha}(\eta,\eta^{-1}\xi)]\,,
\end{aligned}$$

which should be compared with (2.4.7) and

$$
\begin{aligned}
\left[F^{\square}(\xi)\right](x) &= \Big\{\Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \tilde{\beta}(\xi,\xi^{-1})^{\circ} \diamond \tilde{\mathrm{b}}_{\xi}\big[F(\xi^{-1})^{\circ}\big]\Big\}(x) \\
&= \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \Big\{\big[\delta_{\mathsf{e}} \otimes \tilde{\alpha}(\xi,\xi^{-1})^{*}\big] \diamond \tilde{\mathrm{b}}_{\xi}\big[F(\xi^{-1})^{\circ}\big]\Big\}(x) \\
&= \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \tilde{\mathrm{b}}_{\xi}\big[F(\xi^{-1})^{\circ}\big](x) \\
&= \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \tilde{\mathrm{a}}_{\xi}\big[F(\xi^{-1})^{\circ}(x)\big]\kappa(x,\xi)^{*} \\
&= \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \tilde{\mathrm{a}}_{\xi}\Big\{\Delta_{\mathsf{G}}(x^{-1})\alpha(x,x^{-1})^{*}\mathrm{a}_{x}\big[F(\xi^{-1})(x^{-1})\big]^{*}\Big\} \\
&\quad \times \kappa(x,\xi)^{*} \\
&= \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \Delta_{\mathsf{G}}(x^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \tilde{\mathrm{a}}_{\xi}\big[\alpha(x,x^{-1})^{*}\big]\, (\tilde{\mathrm{a}}_{\xi} \circ \mathrm{a}_{x}) \\
&\quad \times \big[F(\xi^{-1})(x^{-1})^{*}\big]\kappa(x,\xi)^{*} \\
&\overset{(2.2.1)}{=} \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \Delta_{\mathsf{G}}(x^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \tilde{\mathrm{a}}_{\xi}\big[\alpha(x,x^{-1})^{*}\big]\, \kappa(x,\xi)^{*}\, (\mathrm{a}_{x} \circ \tilde{\mathrm{a}}_{\xi}) \\
&\quad \times \big[F(\xi^{-1})(x^{-1})^{*}\big] \\
&\overset{(2.2.4)}{=} \Delta_{\tilde{\mathsf{G}}}(\xi^{-1})\, \Delta_{\mathsf{G}}(x^{-1})\, \tilde{\alpha}(\xi,\xi^{-1})^{*}\, \alpha(x,x^{-1})^{*}\, \mathrm{a}_{x}[\kappa(x^{-1},\xi)]\, (\mathrm{a}_{x} \circ \tilde{\mathrm{a}}_{\xi}) \\
&\quad \times \big[F(\xi^{-1})(x^{-1})^{*}\big]
\end{aligned}
$$

which should be compared with (2.4.8). In $\mathscr{A}^{\tilde{\alpha},\tilde{\gamma}}_{\tilde{\mathrm{a}},\gamma}$ one has

$$
\begin{aligned}
[(\mathsf{F}\tilde{\square}\mathsf{G})(x)](\xi) &= \Big\{\int_{\mathsf{G}} dy\; \mathsf{F}(y)\, \tilde{\diamond}\, \mathrm{c}_{y}\big[\mathsf{G}(y^{-1}x)\big]\, \tilde{\diamond}\, \gamma(y,y^{-1}x)\Big\}(\xi) \\
&= \int_{\mathsf{G}} dy\; \big\{\mathsf{F}(y)\, \tilde{\diamond}\, \mathrm{c}_{y}\big[\mathsf{G}(y^{-1}x)\big]\, \tilde{\diamond}\, [\delta_{\varepsilon} \otimes \alpha(y,y^{-1}x)\big\}(\xi) \\
&= \int_{\mathsf{G}} dy \int_{\tilde{\mathsf{G}}} d\eta\, [\mathsf{F}(y)](\eta)\, \tilde{\mathrm{a}}_{\eta}\big[\big(\mathrm{c}_{y}[\mathsf{G}(y^{-1}x)]\, \tilde{\diamond}\, [\delta_{\varepsilon} \otimes \alpha(y,y^{-1}x)]\big)(\eta^{-1}\xi)\big] \\
&\quad \times \tilde{\alpha}(\eta,\eta^{-1}\xi) \\
&\overset{(2.1.2)}{=} \int_{\mathsf{G}} dy \int_{\tilde{\mathsf{G}}} d\eta\, [\mathsf{F}(y)](\eta)\, \tilde{\mathrm{a}}_{\eta}\big(\mathrm{c}_{y}[\mathsf{G}(y^{-1}x)](\eta^{-1}\xi)\, \tilde{\mathrm{a}}_{\eta^{-1}\xi}[\alpha(y,y^{-1}x)]\big) \\
&\quad \times \tilde{\alpha}(\eta,\eta^{-1}\xi) \\
&= \int_{\mathsf{G}} dy \int_{\tilde{\mathsf{G}}} d\eta\, [\mathsf{F}(y)](\eta)\, \tilde{\mathrm{a}}_{\eta}\big(\mathrm{a}_{y}[\mathsf{G}(y^{-1}x)(\eta^{-1}\xi)]\kappa(y,\eta^{-1}\xi)] \\
&\quad \times \tilde{\mathrm{a}}_{\eta^{-1}\xi}[\alpha(y,y^{-1}x)]\big)\tilde{\alpha}(\eta,\eta^{-1}\xi) \\
&= \int_{\mathsf{G}} dy \int_{\tilde{\mathsf{G}}} d\eta\, [\mathsf{F}(y)](\eta)\, (\tilde{\mathrm{a}}_{\eta} \circ \mathrm{a}_{y})[\mathsf{G}(y^{-1}x)(\eta^{-1}\xi)]\, \tilde{\mathrm{a}}_{\eta}[\kappa(y,\eta^{-1}\xi)] \\
&\quad \times (\tilde{\mathrm{a}}_{\eta} \circ \tilde{\mathrm{a}}_{\eta^{-1}\xi})[\alpha(y,y^{-1}x)]\big)\tilde{\alpha}(\eta,\eta^{-1}\xi) \\
&= \int_{\mathsf{G}} dy \int_{\tilde{\mathsf{G}}} d\eta\, [\mathsf{F}(y)](\eta)\, (\tilde{\mathrm{a}}_{\eta} \circ \mathrm{a}_{y})[\mathsf{G}(y^{-1}x)(\eta^{-1}\xi)]\, \tilde{\mathrm{a}}_{\eta}[\kappa(y,\eta^{-1}\xi)] \\
&\quad \times \tilde{\alpha}(\eta,\eta^{-1}\xi)\, \tilde{\mathrm{a}}_{\xi}[\alpha(y,y^{-1}x)]
\end{aligned}
$$

which should be compared with (2.4.5), and

$$
\begin{aligned}
\left[\mathsf{F}^{\tilde{\Box}}(x)\right](\xi) &= \left\{\Delta_{\mathsf{G}}(x^{-1})\,\gamma(x,x^{-1})^{\tilde{\diamond}}\,\tilde{\diamond}\,\mathrm{c}_x\left[\mathsf{F}(x^{-1})^{\tilde{\diamond}}\right]\right\}(\xi) \\
&= \Delta_{\mathsf{G}}(x^{-1})\left\{[\delta_\varepsilon\otimes\alpha(x,x^{-1})^*]\,\tilde{\diamond}\,\mathrm{c}_x\left[\mathsf{F}(x^{-1})^{\tilde{\diamond}}\right]\right\}(\xi) \\
&= \Delta_{\mathsf{G}}(x^{-1})\,\alpha(x,x^{-1})^*\,\mathrm{c}_x\left[\mathsf{F}(x^{-1})^{\tilde{\diamond}}\right](\xi) \\
&= \Delta_{\mathsf{G}}(x^{-1})\,\alpha(x,x^{-1})^*\,\mathrm{a}_x\left[\mathsf{F}(x^{-1})^{\tilde{\diamond}}(\xi)\right]\kappa(x,\xi) \\
&= \Delta_{\mathsf{G}}(x^{-1})\,\alpha(x,x^{-1})^*\,\mathrm{a}_x\{\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\tilde{\alpha}(\xi,\xi^{-1})^*\tilde{\mathrm{a}}_\xi[\mathsf{F}(x^{-1})(\xi^{-1})^*]\} \\
&\times\kappa(x,\xi)
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta_{\mathsf{G}}(x^{-1})\,\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\,\alpha(x,x^{-1})^*\,\mathrm{a}_x\left[\tilde{\alpha}(\xi,\xi^{-1})^*\right] \\
&\times(\mathrm{a}_x\circ\tilde{\mathrm{a}}_\xi)\left[\mathsf{F}(x^{-1})(\xi^{-1})^*\right]\kappa(x,\xi) \\
&\overset{(2.2.1)}{=}\Delta_{\mathsf{G}}(x^{-1})\,\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\,\alpha(x,x^{-1})^*\mathrm{a}_x\left[\tilde{\alpha}(\xi,\xi^{-1})^*\right] \\
&\times\kappa(x,\xi)\,(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)[\mathsf{F}(x^{-1})(\xi^{-1})^*] \\
&\overset{(2.2.3)}{=}\Delta_{\mathsf{G}}(x^{-1})\,\Delta_{\widetilde{\mathsf{G}}}(\xi^{-1})\,\alpha(x,x^{-1})^*\tilde{\alpha}(\xi,\xi^{-1})^*\,\tilde{\mathrm{a}}_\xi\left[\kappa(x,\xi^{-1})^*\right] \\
&\times(\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x)\left[\mathsf{F}(x^{-1})(\xi^{-1})^*\right]
\end{aligned}
$$

which should be compared with (2.4.6).

Remark 2.6.3. If one tries to show directly that Υ is multiplicative, after a short computation using (2.2.1), he will realize that this is equivalent to the identity (2.2.2).

Remark 2.6.4. Naturally, by the same mechanism, the second generation C^*-algebras can also be inflated to new covariant structures
$\left\{(\mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}},\mathrm{k}^2),(\mathrm{b}^2,\beta^2),(\tilde{\mathrm{b}}^2,\tilde{\beta}^2)\right\}$ and $\left\{(\mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}},\tilde{\mathrm{k}}^2),(\mathrm{c}^2,\gamma^2),(\tilde{\mathrm{c}}^2,\tilde{\gamma}^2)\right\}$.

Then the isomorphism Υ can be upgraded to an isomorphism in a category of covariant structures, that can be easily defined. Similarly, the twisted crossed products $\mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}}$ and $\mathscr{A}^{\overleftarrow{\alpha}}_{\overleftarrow{\mathrm{a}}}$ with product group $\mathsf{G}\times\widetilde{\mathsf{G}}$ also have their natural covariant structures and the isomorphisms Γ, Φ and Ψ have their interpretation in this category. Since many formulas should be written down and also having in view a subsequent work, we shall not pursue all these here.

2.7 On the stabilization trick

Definition 2.7.1. *Two covariant systems* $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ *and* $\{(\mathscr{A},\kappa'),(\mathrm{a}',\alpha'),(\tilde{\mathrm{a}}',\tilde{\alpha}')\}$ *are said to be* covariant exterior equivalent *if there exists two exterior equivalences* $(\mathrm{a},\alpha)\overset{q}{\sim}(\mathrm{a}',\alpha')$ *and* $(\tilde{\mathrm{a}},\tilde{\alpha})\overset{\tilde{q}}{\sim}(\tilde{\mathrm{a}}',\tilde{\alpha}')$ *such that*

$$\kappa'(x,\eta)=q_x\mathrm{a}_x(\tilde{q}_\eta)\kappa(x,\eta)\tilde{\mathrm{a}}_\eta(q_x^*)\tilde{q}_\eta^*, \tag{2.7.1}$$

for all $(x,\eta)\in\mathsf{G}\times\widetilde{\mathsf{G}}$.

It is easy to check that this defines an equivalence relation on the set of covariant structures defined with $\mathscr{A}$.

Remark 2.7.2. *Note that if* $\tilde{\mathrm{a}} \equiv \tilde{\mathrm{a}}' \equiv \mathsf{id}$, *as a dynamical systems, the equation 2.7.1 is simplified to* $\kappa'(x,\eta) = q_x\kappa(x,\eta)q_x^*$. *If additionally we assume that* κ *is central, as in a* G*-products, this forces to* $\kappa \equiv \kappa'$.

We have the next result.

Proposition 2.7.3. *If two covariant systems* $\{(\mathscr{A},\kappa),(\mathrm{a},\alpha),(\tilde{\mathrm{a}},\tilde{\alpha})\}$ *and* $\{(\mathscr{A},\kappa'),(\mathrm{a}',\alpha'),(\tilde{\mathrm{a}}',\tilde{\alpha}')\}$ *are covariant exterior equivalent then the bi-product structures associated to each one are isomorphic.*

Proof. Since, the corresponding bi-products have realizations as crossed products of $\mathscr{A}$ with the twisted actions $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ and $(\overrightarrow{\mathrm{a}}',\overrightarrow{\alpha}')$ respectively, it is suffice to show that there exists an exterior equivalence between $(\overrightarrow{\mathrm{a}},\overrightarrow{\alpha})$ and $(\overrightarrow{\mathrm{a}}',\overrightarrow{\alpha}')$. Then we need to find a strictly measurable map $p:\mathsf{G}\times\widetilde{\mathsf{G}}\to\mathcal{UM}(\mathscr{A})$ that satisfies the conditions in Remark 2.1.2. Since we have the equivalences $(\mathrm{a},\alpha)\overset{q}{\sim}(\mathrm{a}',\alpha')$ and $(\tilde{\mathrm{a}},\tilde{\alpha})\overset{\tilde{q}}{\sim}(\tilde{\mathrm{a}}',\tilde{\alpha}')$, we define

$$p(x,\xi) := \tilde{q}_\xi\tilde{\mathrm{a}}_\xi(q_x).$$

Then we compute

$$\begin{aligned}
\overrightarrow{\mathrm{a}}'_{(x,\xi)} &= \tilde{\mathrm{a}}'_\xi\circ\mathrm{a}'_x\\
&= \tilde{\mathrm{a}}'_\xi\circ\mathsf{ad}_{q_x}\circ\mathrm{a}_x\\
&= \mathsf{ad}_{\tilde{q}_\xi}\circ\tilde{\mathrm{a}}_\xi\circ\mathsf{ad}_{q_x}\circ\mathrm{a}_x\\
&= \mathsf{ad}_{\tilde{q}_\xi\tilde{\mathrm{a}}_\xi(q_x)}\circ\tilde{\mathrm{a}}_\xi\circ\mathrm{a}_x\\
&= \mathsf{ad}_{p(x,\xi)}\overrightarrow{\mathrm{a}},
\end{aligned}$$

for all $(x,\xi)\in\mathsf{G}\times\widetilde{\mathsf{G}}$. On other hand, also we need to compute

$$\begin{aligned}
\overrightarrow{\alpha}'((x,y),(\xi,\eta)) &= \tilde{\mathrm{a}}'_\xi[\kappa(x,\eta)']\tilde{\alpha}'(\xi,\eta)\tilde{\mathrm{a}}'_{\xi\eta}[\alpha(x,y)]\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)']\tilde{q}_\xi^*\tilde{q}_\xi\tilde{\mathrm{a}}_\xi(\tilde{q}_\eta)\tilde{\alpha}(\xi,\eta)\tilde{q}_{\xi\eta}^*\tilde{q}_{\xi\eta}\tilde{\mathrm{a}}_{\xi\eta}[q_x\mathrm{a}_x(q_y)\alpha(x,y)q_{xy}^*]\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)'\tilde{q}_\eta]\tilde{\alpha}(\xi,\eta)\tilde{\mathrm{a}}_{\xi\eta}[q_x\mathrm{a}_x(q_y)\alpha(x,y)q_{xy}^*]\tilde{q}_{\xi\eta}^*\\
&\overset{(2.7.1)}{=} \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x]\tilde{\mathrm{a}}_\xi[a_x(\tilde{q}_\eta)\kappa(x,\eta)\tilde{\mathrm{a}}_\eta(q_x^*)]\tilde{\alpha}(\xi,\eta)\tilde{\mathrm{a}}_{\xi\eta}[q_x\mathrm{a}_x(q_y)\alpha(x,y)q_{xy}^*]\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x]\tilde{\mathrm{a}}_\xi[a_x(\tilde{q}_\eta)\kappa(x,\eta)\tilde{\mathrm{a}}_\eta(q_x^*)\tilde{\mathrm{a}}_\eta[q_x\mathrm{a}_x(q_y)\alpha(x,y)q_{xy}^*]]\tilde{\alpha}(\xi,\eta)\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x]\tilde{\mathrm{a}}_\xi[a_x(\tilde{q}_\eta)\kappa(x,\eta)\tilde{\mathrm{a}}_\eta[\mathrm{a}_x(q_y)]\tilde{\mathrm{a}}_\eta[\alpha(x,y)q_{xy}^*]]\tilde{\alpha}(\xi,\eta)\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x]\tilde{\mathrm{a}}_\xi[a_x(\tilde{q}_\eta\tilde{\mathrm{a}}_\eta(q_y))\kappa(x,\eta)\tilde{\mathrm{a}}_\eta[\alpha(x,y)q_{xy}^*]]\tilde{\alpha}(\xi,\eta)\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x](\tilde{\mathrm{a}}_\xi\circ a_x)[\tilde{q}_\eta\tilde{\mathrm{a}}_\eta(q_y)]\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)](\tilde{\mathrm{a}}_\xi\circ\tilde{\mathrm{a}}_\eta)[\alpha(x,y)q_{xy}^*]\tilde{\alpha}(\xi,\eta)\tilde{q}_{\xi\eta}^*\\
&= \tilde{q}_\xi\tilde{\mathrm{a}}_\xi[q_x](\tilde{\mathrm{a}}_\xi\circ a_x)[\tilde{q}_\eta\tilde{\mathrm{a}}_\eta(q_y)]\tilde{\mathrm{a}}_\xi[\kappa(x,\eta)]\tilde{\alpha}(\xi,\eta)\tilde{\mathrm{a}}_{\xi\eta}[\alpha(x,y)]\tilde{\mathrm{a}}_{\xi\eta}(q_{xy}^*)\tilde{q}_{\xi\eta}^*\\
&= p(x,\xi)\overrightarrow{\mathrm{a}}_{(x,\xi)}[p(y,\eta)]\overrightarrow{\alpha}((x,y),(\xi,\eta))p(xy,\xi\eta)^*.
\end{aligned}$$

for all (x,ξ), $(y,\eta)\in\mathsf{G}\times\widetilde{\mathsf{G}}$. This finish the proof. □

Note that we proved that there exists an exterior equivalence between the twisted action attached to the covariant structures. We can get the converse one. Suppose that we have $(\overrightarrow{\mathrm{a}}, \overrightarrow{\alpha}) \overset{p}{\sim} (\overrightarrow{\mathrm{a}}', \overrightarrow{\alpha}')$, for the covariant structures $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ and $\{(\mathscr{A}, \kappa'), (\mathrm{a}', \alpha'), (\tilde{\mathrm{a}}', \tilde{\alpha}')\}$. We set $q_x = p(x, \varepsilon)$ and $\tilde{q}_\xi = p(\mathrm{e}, \xi)$; then one can show that $(\mathrm{a}, \alpha) \overset{q}{\sim} (\mathrm{a}', \alpha')$ and $(\tilde{\mathrm{a}}, \tilde{\alpha}) \overset{\tilde{q}}{\sim} (\tilde{\mathrm{a}}', \tilde{\alpha}')$. The equality in 2.7.1 follows from the equation

$$\overrightarrow{\alpha}'((x,y),(\xi,\eta)) = p(x,\xi)\overrightarrow{\mathrm{a}}_{(x,\xi)}[p(y,\eta)]\overrightarrow{\alpha}((x,y),(\xi,\eta))\,p(xy,\xi\eta)^*,$$

putting $y = \mathrm{e}$ and $\xi = \varepsilon$. Therefore we have proved:

Theorem 2.7.4. *Two covariant systems* $\{(\mathscr{A}, \kappa), (\mathrm{a}, \alpha), (\tilde{\mathrm{a}}, \tilde{\alpha})\}$ *and* $\{(\mathscr{A}, \kappa'), (\mathrm{a}', \alpha'), (\tilde{\mathrm{a}}', \tilde{\alpha}')\}$ *are covariant exterior equivalent if and only if the attached twisted actions* $(\overrightarrow{\mathrm{a}}, \overrightarrow{\alpha})$ *and* $(\overrightarrow{\mathrm{a}}', \overrightarrow{\alpha}')$ *are exterior equivalent.*

There is an important result about twisted crossed products related with exterior equivalences. In [14] is proved that given a twisted dynamical system $(\mathscr{A}, \mathsf{G}, \mathrm{a}, \alpha)$, there is an untwisted action $(\mathrm{a}', 1)$ on $\mathscr{A} \otimes \mathcal{K}[L^2(G)]$ which is exterior equivalent to the twisted action $(\mathrm{a} \otimes \mathsf{id}, \alpha \otimes 1)$.

Applying this and Theorem 2.7.4 to the realization as crossed product of a bi-product structure we obtain:

Corollary 2.7.5. *There exists* $\kappa' : \mathsf{G} \times \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A} \otimes \mathcal{K}[L^2(\mathsf{G} \times \widetilde{\mathsf{G}})])$ *and actions* a', $\tilde{\mathrm{a}}'$ *of* G *and* $\widetilde{\mathsf{G}}$ *respectively such that there exist a covariant exterior equivalence between*

$$\left\{(\mathscr{A} \otimes \mathcal{K}[L^2(\mathsf{G} \times \widetilde{\mathsf{G}})], \kappa \otimes 1), (\mathrm{a} \otimes \mathsf{id}, \alpha \otimes 1), (\tilde{\mathrm{a}} \otimes \mathsf{id}, \tilde{\alpha} \otimes 1)\right\}$$

and

$$\left\{(\mathscr{A} \otimes \mathcal{K}[L^2(\mathsf{G} \times \widetilde{\mathsf{G}})], \kappa'), (\mathrm{a}', 1), (\tilde{\mathrm{a}}', 1)\right\}.$$

2.8 Takai duality and other examples

Example 2.8.1. We have seen that one realization of the bi-product $\mathscr{A}^{(\alpha,\tilde{\alpha})}_{(\mathrm{a},\tilde{\mathrm{a}})}$ is the twisted crossed product $\mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} := \mathscr{A} \rtimes^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} (\mathsf{G} \times \widetilde{\mathsf{G}})$. Applying to this well-known results [23], it follows that the bi-product is commutative if and only if $\mathscr{A}, \mathsf{G}, \widetilde{\mathsf{G}}$ are commutative, a and $\tilde{\mathrm{a}}$ are trivial and $\overrightarrow{\mathrm{a}}$ is (essentially) symmetric. But $\overrightarrow{\mathrm{a}}$ is symmetric if and only if α and $\tilde{\alpha}$ are symmetric and $\kappa = 1$.

Example 2.8.2. If $\kappa = 1$ the two actions a and $\tilde{\mathrm{a}}$ commute, the elements $\tilde{\alpha}(\xi,\eta)$ are fixed points of a, the elements $\alpha(x,y)$ are fixed points of $\tilde{\mathrm{a}}$, one has $\alpha(x,y)\tilde{\alpha}(\xi,\eta) = \tilde{\alpha}(\xi,\eta)\alpha(x,y)$ and the twisted actions $(\overrightarrow{\mathrm{a}}, \overrightarrow{\alpha})$ and $(\overleftarrow{\mathrm{a}}, \overleftarrow{\alpha})$ coincide. The isomorphism between $\mathscr{A}^{\alpha,\beta}_{\mathrm{a},\tilde{\mathrm{b}}}$ and $\mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}}$ is basically a flip of the variables. The twisted actions $(\tilde{\mathrm{b}}, \tilde{\beta})$ and (c, γ) are non-trivial only in the $\mathscr{A}$-part of the twisted crossed products.

Example 2.8.3. If the initial two actions are not twisted, i.e. $\alpha = 1$ and $\tilde{\alpha} = 1$, then κ must verify for all x, y, ξ, η

$$\kappa(x, \xi\eta) = \kappa(x, \xi)\tilde{\mathrm{a}}_\xi[\kappa(x, \eta)] \quad \text{and} \quad \kappa(xy, \xi)^* = \kappa(x, \xi)^* \mathrm{a}_x[\kappa(y, \xi)^*]. \tag{2.8.1}$$

This means that $\kappa(x, \cdot) : \widetilde{\mathsf{G}} \to \mathcal{UM}(\mathscr{A})$ and $\kappa(\cdot, \xi)^* : \mathsf{G} \to \mathcal{UM}(\mathscr{A})$ are crossed morphisms. One has

$$\overrightarrow{\alpha}\big((x, \xi), (y, \eta)\big) = \tilde{\mathrm{a}}_\xi[\kappa(x, \eta)], \quad \overleftarrow{\alpha}\big((x, \xi), (y, \eta)\big) = \mathrm{a}_x[\kappa(y, \xi)^*]. \tag{2.8.2}$$

The $\mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}}$-realization of the bi-product $\mathscr{A}^{(\alpha,\tilde{\alpha})}_{(\mathrm{a},\tilde{\mathrm{a}})}$ is still twisted and can be very complicated. The iterated crossed products $\mathscr{A}^{\alpha,\tilde{\beta}}_{\mathrm{a},\tilde{\mathrm{b}}} \equiv \mathscr{A}_{\mathrm{a},\tilde{\mathrm{b}}}$ and $\mathscr{A}^{\tilde{\alpha},\gamma}_{\tilde{\mathrm{a}},\mathrm{c}} \equiv \mathscr{A}_{\tilde{\mathrm{a}},\mathrm{c}}$ are only constructed with untwisted actions, but the actions $\tilde{\mathrm{b}}, \mathrm{c}$, besides the initial $\tilde{\mathrm{a}}, \mathrm{a}$ also contain the coupling function κ.

Example 2.8.4. Even when both twisted actions are trivial, the bi-product remembers the C^*-algebra $\mathscr{A}$ and the "coupling" between the groups G and $\widetilde{\mathsf{G}}$.

For $\{(\mathscr{A}, \kappa), (\mathsf{id}, 1), (\mathsf{id}, 1)\}$ one gets $\overrightarrow{\mathrm{a}} = \mathsf{id}$ but

$$\overrightarrow{\alpha}\big((x, \xi), (y, \eta)\big) = \kappa(x, \eta) \tag{2.8.3}$$

is still non-trivial. Relations (2.2.3) and (2.2.4) become in this case (respectively)

$$\kappa(x, \xi\eta) = \kappa(x, \xi)\kappa(x, \eta) \quad \text{and} \quad \kappa(xy, \xi) = \kappa(y, \xi)\kappa(x, \xi).$$

For Abelian $\mathscr{A}$, twisted crossed products $\mathscr{A} \rtimes^{\overrightarrow{\alpha}}_{\mathsf{id}} \mathsf{H}$ with trivial action $\overrightarrow{\mathrm{a}}$ (but with general 2-cocycle $\overrightarrow{\alpha}$) have been studied in depth in [23, 24, 8]. It is worth mentioning that our $\overrightarrow{\alpha}$ is symmetric only if $\kappa = 1$. The second generation iterated twisted crossed products have the form

$$(\mathscr{A} \rtimes_{\mathsf{id}} \mathsf{G}) \rtimes_{\tilde{\mathrm{b}}^\bullet} \widetilde{\mathsf{G}} \cong [\mathscr{A} \otimes C^*(\mathsf{G})] \rtimes_{\tilde{\mathrm{b}}^\bullet} \widetilde{\mathsf{G}} \quad \text{and} \quad (\mathscr{A} \rtimes_{\mathsf{id}} \widetilde{\mathsf{G}}) \rtimes_{\mathrm{c}^\bullet} \mathsf{G} \cong [\mathscr{A} \otimes C^*(\widetilde{\mathsf{G}})] \rtimes_{\mathrm{c}^\bullet} \mathsf{G},$$

where essentially $[\tilde{\mathrm{b}}^\bullet_\xi(f)](x) := f(x)\kappa(x, \xi)^*$ and $[\mathrm{c}^\bullet_x(\mathsf{f})](\xi) := \mathsf{f}(\xi)\kappa(x, \xi)$.

If κ is $\mathbb{T}$-valued, $\overrightarrow{\alpha}$ is a bi-character. It is easy to see that we get

$$\mathscr{A}^{(\mathsf{id},1)}_{(\mathsf{id},1)} \equiv \mathscr{A}^{\overrightarrow{\alpha}}_{\mathsf{id}} \cong \mathscr{A} \otimes C^*_\kappa(\mathsf{G} \times \widetilde{\mathsf{G}}). \tag{2.8.4}$$

We denoted by $C^*_\kappa(\mathsf{G} \times \widetilde{\mathsf{G}})$ the twisted group algebra of $\mathsf{H} := \mathsf{G} \times \widetilde{\mathsf{G}}$ corresponding to the 2-cocycle $\mathsf{H} \times \mathsf{H} \to \mathbb{T}$ given by (2.8.3). More generally, we can consider the covariant structure $\{(\mathscr{A}, \kappa), (\mathsf{id}, \alpha), (\mathsf{id}, \tilde{\alpha})\}$, where α and $\tilde{\alpha}$ are multipliers (they take values in $\mathbb{T}$). If κ is also $\mathbb{T}$-valued, then

$$\mathscr{A}^{(\alpha,\tilde{\alpha})}_{(\mathsf{id},\mathsf{id})} \cong \mathscr{A} \otimes C^*_{\overrightarrow{\alpha}}(\mathsf{G} \times \widetilde{\mathsf{G}}). \tag{2.8.5}$$

Example 2.8.5. We shall describe now briefly how *a twisted version of Takai's duality result for Abelian groups* follows from our isomorphism $\mathscr{A}^{\alpha,\tilde\beta}_{\mathrm{a},\tilde{\mathrm{b}}} \cong \mathscr{A}^{\tilde\alpha,\gamma}_{\tilde{\mathrm{a}},\mathrm{c}}$, which is written with full notations

$$(\mathscr{A} \rtimes^{\alpha}_{\mathrm{a}} \mathsf{G}) \rtimes^{\tilde\beta}_{\tilde{\mathrm{b}}} \widetilde{\mathsf{G}} \cong (\mathscr{A} \rtimes^{\tilde\alpha}_{\tilde{\mathrm{a}}} \widetilde{\mathsf{G}}) \rtimes^{\gamma}_{\mathrm{c}} \mathsf{G}\,. \tag{2.8.6}$$

Let us suppose that the group G is commutative (in additive notations) and $\widetilde{\mathsf{G}} := \widehat{\mathsf{G}}$ is its Pontryagin dual. As coupling function we choose the natural duality $\kappa(x,\xi) \equiv \kappa^0(x,\xi) := \xi(x)$. Also assume that the initial twisted action of $\widehat{\mathsf{G}}$ is trivial: $(\tilde{\mathrm{a}}, \tilde\alpha) = (\mathrm{id}, 1)$; then the 2-cocycle $\tilde\beta$ is trivial and the action $\tilde{\mathrm{b}}$ reduces to the standard dual action given by $\big[\hat{\mathrm{b}}^0_\xi(f)\big](x) := \overline{\xi(x)}f(x)$. The purpose is to express the double twisted crossed product $(\mathscr{A} \rtimes^{\alpha}_{\mathrm{a}} \mathsf{G}) \rtimes_{\hat{\mathrm{b}}^0} \widehat{\mathsf{G}}$ in a simple familiar form, using the r.h.s. of (2.8.6).

There are well-known canonical isomorphisms $\mathscr{A} \rtimes^{1}_{\mathrm{id}} \widehat{\mathsf{G}} \cong \mathscr{A} \otimes C^*(\widehat{\mathsf{G}}) \cong \mathscr{A} \otimes C_0(\mathsf{G})$, the second one being given by a partial Fourier transform $\mathrm{id}_{\mathscr{A}} \otimes \mathscr{F}$, where $\mathscr{F} : C^*(\widehat{\mathsf{G}}) \to C_0(\mathsf{G})$ is the extension of the usual Fourier transform $\mathcal{F} : L^1(\widehat{\mathsf{G}}) \to C_0(\mathsf{G})$. The twisted action (c,γ) given by (2.5.4) and (2.5.5) is carried to $(\mathrm{a} \otimes \mathrm{t}, \alpha \otimes 1)$, where $[\mathrm{t}_x(\varphi)](y) := \varphi(y+x)$ is the action of G on $C_0(\mathsf{G})$ by translations. If one finds an isomorphism

$$[\mathscr{A} \otimes C_0(\mathsf{G})] \rtimes^{\alpha\otimes 1}_{\mathrm{a}\otimes\mathrm{t}} \mathsf{G} \cong \mathscr{A} \otimes [C_0(\mathsf{G}) \rtimes_{\mathrm{t}} \mathsf{G}]\,, \tag{2.8.7}$$

then using the standard isomorphism between $C_0(\mathsf{G}) \rtimes_{\mathrm{t}} \mathsf{G}$ and the C^*-algebra $\mathbb{K}[L^2(\mathsf{G})]$ of all compact operators in the Hilbert space $L^2(\mathsf{G})$ one finally gets the desired result

$$(\mathscr{A} \rtimes^{\alpha}_{\mathrm{a}} \mathsf{G}) \rtimes_{\hat{\mathrm{b}}^0} \widehat{\mathsf{G}} \cong \mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]\,. \tag{2.8.8}$$

Using some notational abuse, the isomorphism (2.8.7) is given by

$$[\Theta(F)](z,x) := \mathrm{a}_x[F(z,x)]\alpha(x,z)\,.$$

We refer to [28, Sect. 7.1] for a more careful discussion of the case $\alpha = 1$.

The conclusion is that in this case the bi-product associated to the covariant structure $\{(\mathscr{A},\kappa^0), (\mathrm{a},\alpha), (\mathrm{id},1)\}$ is stable equivalent to the initial C^*-algebra $\mathscr{A}$. Recalling the realizations $\mathscr{A}^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}}$ and $\mathscr{A}^{\overleftarrow{\alpha}}_{\overleftarrow{\mathrm{a}}}$ of this bi-product, we get more isomorphisms that could be of some interest. In the present given situation, for example, one has

$$\overrightarrow{\mathrm{a}}_{(x,\xi)} = \mathrm{a}_x\,, \quad \overrightarrow{\alpha}\big((x,\xi),(y,\eta)\big) = \eta(x)\alpha(x,y)\,.$$

For this twisted action one gets $\mathscr{A} \rtimes^{\overrightarrow{\alpha}}_{\overrightarrow{\mathrm{a}}} (\mathsf{G} \times \widehat{\mathsf{G}}) \cong \mathscr{A} \otimes \mathbb{K}[L^2(\mathsf{G})]$.

All the isomorphisms we described above are shadows of isomorphisms of covariant systems, as indicated in Remark 2.6.4.

Appendix A

Measure and integration

We need some basic facts about the measurability of functions with values in non-separable Banach spaces. In this section X will denote a second countable locally compact space and $\mathscr{A}$ a complex Banach space. The principal aim is to define the measurability of $f : X \to \mathscr{A}$ ($\mathscr{A}$-valued function). When the Banach space is separable the theory is easier. Here we will exhibit the basic definitions and the principal goal is to characterize the measurability in terms of a certain "local continuity property" of f (Proposition A.9).

The inspiration comes from the Riesz Representation Theorem [22, Th. 2.14]: For every functional $I : C_c(X) \to \mathbb{C}$, where $C_c(X)$ denotes the spaces of complex-valued function with compact support, there exists a unique σ-algebra $B(X)$ containing the Borel sets and a unique measure μ, such that

$$I(f) = \int_X f\, d\mu\,, \quad \text{for all } f \in C_c(X),$$

and this measure satisfies the following additional properties:

(a) for $E \in B(X)$, $\mu(E) = \inf\{\mu(O) \mid E \subset O, \text{ open }\}$;

(b) the relation $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact }\}$ holds when E is an open set and for $E \in B(X)$ with $\mu(E) < \infty$;

(c) μ is ***complete***

A measure defined in the σ-algebra $B(X)$ is called ***Radon measure*** if it satisfies (a), (b) and (c).

Such a measure takes finite values in the compacts sets and is ***saturated***, *i.e.*

$$E \in B(X) \iff E \cap K \in B(X) \text{ for all } K \text{ compact.}$$

Remark A.1. It is known that any second countable locally compact set is σ-compact. Since a Radon measure is finite for each compact set, any Radon measure on X is σ-finite.

Let dx be a Radon measure on X and $f : X \to \mathscr{A}$ a function. Our definition of measurability of f must satisfy certain properties, for example, f will be a limit of a sequence of simple $\mathscr{A}$-valued functions. So we define the following:

Definition A.2. *A function $\chi : X \to \mathscr{A}$ is called simple if it takes finitely many values $a_1, \ldots, a_n \in \mathscr{A}$, and $\{x \in X \mid f(x) = a_i\} \in B(X)$ and $\mu(\{x \in X \mid f(x) = a_i\}) < \infty$ if $a_i \neq 0$.*

Definition A.3. *We say that a function $f : X \to \mathscr{A}$ is a* **measurable** *function if it is a pointwise limit of a sequence of simple $\mathscr{A}$-valued functions.*

Remark A.4. Note that if f is the limit of a sequence $\{g_n\}$ of simple functions, then the range of f is contained in the closure of the union of the images of the functions g_n, and this space is separable in $\mathscr{A}$.

In [9, Sec. 5.5] it is shown that this definition is equivalent to the following:

Definition A.5. *A function $f : X \to \mathscr{A}$ is* **w-measurable** *if*

1. *$\varphi \circ f$ is measurable for all $\varphi \in \mathscr{A}^*$;*
2. *for each $E \subset B(X)$ there is a closed separable linear subspace $A \subset \mathscr{A}$ such that $f(x) \in A$ for μ-a.e. $x \in E$.*

Another definition of measurability is given in [28, App. B.1] and is the following:

Definition A.6. *A function $f : X \to \mathscr{A}$ is* **weakly measurable** *if*

1. *$\varphi \circ f$ is measurable for all $\varphi \in \mathscr{A}^*$;*
2. *for each compact set $K \subset X$ there is a closed separable linear subspace $A \subset \mathscr{A}$, such that $f(x) \in A$ for μ-a.e. $x \in K$.*

The previous definition is inspired from the notion of saturated measure mentioned above.

In our case, the measure is σ-finite, thus we have:

Lemma A.7. *Definitions A.5 and A.6 are equivalent.*

Proof. It is clear that the w-measurability implies weakly measurability.

On the other hand, let f be a weak measurable $\mathscr{A}$-valued map. Take a measurable set E. We need to show that there exists a linear separable subspace $A \subset \mathscr{A}$ and a null-space N such that, $f(x) \in A$ for $x \in E \setminus N$. We use the fact that the space X is σ-compact (Remark A.1). We can find a sequence of compact sets $\{K_n\}$ such that

$$X = \bigcup_{n=1}^{\infty} K_n.$$

We construct a sequence $K'_1, K'_2, \ldots$ of compact sets and a sequence of linear spaces $A^1_1, A^1_2, \ldots$ as follows:

There exists $K'_1 \subset E \cap K_1$ compact set such that $\mu((E \cap K_1) \setminus K'_1) < 1$. Since f is weakly measurable, there exists a separable linear space $A^1_1 \subset \mathscr{A}$ such that $f(x) \in A^1_1$ μ-*a.e.* $x \in K'_1$. Since $(E \cap K_1) \setminus K'_1$ has finite measure, there exists $K'_2 \subset (E \cap K_1) \setminus K'_1$ such that

$$\mu(((E \cap K_1) \setminus K'_1) \setminus K'_2) < 1/2$$

and a separable space A_2^1 such that $f(x) \in A_2^1$ μ-a.e. $x \in K_2$. Continuing, we get a sequence of disjoint compact sets such that $\mu\left((E \cap K_1) \setminus \bigcup_{i=1}^{N} K_i'\right) \leq 1/N$ and a separable space A_N^1 (the linear span of $\bigcup_{i=1}^{N} A_i^1$) such that $f(x) \in A_N^1$ μ-almost $x \in \bigcup_{i=1}^{N} K_i$.

Then there exists a null set M_1 such that $E \cap K_1 = \bigcup_{i=1}^{N} K_i' \cup M_1$. Thus, we take the closure of the linear span of the set $A_1' := \bigcup_{i=1}^{\infty} A_i^1$; this is a separable space, and $f(x) \in A_1'$ for all $x \in (E \cap K_1') \setminus M_1$.

We can continue the argument for each set $E \cap K_n$ with $n = 1, 2, \ldots$; therefore we consider the closure of the linear span $A := \bigcup_{n=1}^{\infty} A_n$ and this set is a separable space. Since $E = \bigcup_{n=1}^{\infty}(E \cap K_n)$, we get $f(x) \in A$ for all $x \in \bigcup_{n=1}^{\infty}(E \cap K_n) \setminus \bigcup_{n=1}^{\infty} M_n$, but $\bigcup_{n=1}^{\infty} M_n$ is a null set, and this finishes the proof. □

Our definition of measurability coincides with the definitions in [28, App. B1]. Taking into account Lemma A.7, the Definition A.5 is equivalent to the following definition [28, Lem. B7]:

Definition A.8. *A function $f : X \to \mathscr{A}$ is **strongly measurable** if*

1. *$f^{-1}(A)$ is Borel for all $A \subset \mathscr{A}$ open set;*
2. *for each K compact set there is a closed separable linear subspace $A \subset \mathscr{A}$, such that $f(x) \in A$ for μ-a.e. $x \in K$.*

Now, our principal propose will be to give a powerful result implying measurability and this characterization play an essential role in Chapter 2. The important Lemma is the following:

Lemma A.9. *Let $f : X \to \mathscr{A}$ be a map and assume that for each compact set $K \subset X$ and $\epsilon > 0$, there exists a compact subset $K' \subset K$ such that $\mu(K \setminus K') < \epsilon$ and the restriction $f|_{K'}$ of f to K' is continuous. Then f is measurable.*

Proof. We are going to use Definition A.8. To prove the first condition, it is sufficient to show that $f^{-1}(A)$ is measurable for all $A \subset \mathscr{A}$ closed. We use the fact that μ is saturated; then we need to prove that $f^{-1}(A) \cap K$ is measurable for any compact set K.

First we construct a sequence $K_1, K_2, \ldots$ of compact sets as follows: There exists K_1 compact set such that $\mu(K \setminus K_1) < 1$ and $f|_{K_1}$ is continuous. Since $K \setminus K_1$ has finite measure, using the fact that μ is regular, there exists $K_2' \subset K \setminus K_1$ such that $\mu((K \setminus K_1) \setminus K_2') < 1/2$. Then there exists $K_2 \subset K_2'$ such that, $\mu((K \setminus K_1) \setminus K_2) < 1/2$. Continuing, we get a sequence of disjoint compact sets such that $\mu\left(K \setminus \bigcup_{i=1}^{n} K_i\right) < 1/n$.

Therefore, we can write $K = N \cup \bigcup_{i=1}^{\infty} K_i$, where N is a null set and $f|_{K_i}$ is continuous. Then

$$f^{-1}(A) \cap K = f^{-1}(A) \cap \left(N \cup \bigcup_{i=1}^{\infty} f^{-1}(A) \cap K_i\right).$$

Since $f|_{K_i}$ is continuous $f^{-1}(A) \cap K_i$ is closed and $f^{-1}(A) \cap N$ is null. This prove the first statement.

Take K a compact set. Using the above reasoning, we can write $K = N \cup \bigcup_{i=1}^{\infty} K_i$, and $f(K_i)$ is compact, thus it is separable. Then the space generated by $\bigcup_{i=1}^{\infty} f(K_i)$ is separable, and this is suffices because N is null. □

Remark A.10. The converse to Lemma A.9 is also true [28, Prop. B.20]. With this, all the definitions of measurability given in this section are equivalent (because we work with second countable locally compact spaces). We use the Definition A.3 for measurability, but we will keep in mind the equivalent definitions and Lemma A.9.

Now we can construct the ***Bochner integral***. For a simple function $f = \sum a_i \chi_{E_i}$, we define the integral of f respect to the measure μ as $\sum \mu(E_i) a_i$. A measurable map f is ***Bochner integrable*** if the map $x \to \|f(x)\|$ is integrable in the usual sense [9, Sec, 2.5]. In this case we can define the Bochner integral $\int_X f d\mu$ choosing a sequence of simple functions $\{f_n\}$ with $f_n \to f$ pointwise and $\int_X \|f_n - f\| d\mu \to 0$ and setting

$$\int_X f\, d\mu := \lim_{n \to \infty} \int_X f_n d\mu .$$

The bounded linear operators pull through Bochner integral, so we can do freely manipulations like

$$\phi\left(\int_X f d\mu\right) = \int_X \phi \circ f d\mu, \quad \phi \in \mathscr{A}^*.$$

The space of all Bochner integrable functions from X into $\mathscr{A}$ is denoted by $L^1(X, \mathscr{A})$. This is a Banach space with the norm $\|f\|_1 := \int_X \|f\| d\mu$. This space is the completion with respect to the greatest cross-norm of the algebraic tensor product of the Banach space $\mathscr{A}$ with the space of integrable complex-valued functions with respect μ [9].

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