Lower bounds for the Artin conductor

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A mis hijos Sergio y Amalia.

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Resumen

Mejoramos las cotas inferiores de Odlyzko para conductores de Artin. Comenzamos adaptando los métodos de Odlyzko al lenguaje de las formulas explícitas. Esto da una primera mejora a los resultados de Odlyzko. Después introducimos una técnica que aprovecha la contribución de los primos, mejorando aun más las cotas inferiores. Estas mejoras se presentan independientemente del comportamiento de la representación en los diversos primos.

Abstract

We improve on Odlyzko's lower bounds for the Artin conductor. We begin by translating Odlyzko's methods to the language of explicit formulas. This yields an initial improvement on Odlyzko's bound. Then we introduce a technique to take advantage of the contribution of the primes to further improve the lower bounds. This improvement occurs regardless of the behavior of the representation at the various primes.

Contents

Introduction

Let K be an algebraic number field such that $K/\mathbb{Q}$ is Galois and let χ be a linear character of $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$. We denote by f_χ the Artin conductor associated to χ ([3], pp. 525). The purpose of this thesis is to improve the known lower bounds for the Artin conductor due to Odlyzko by using Weil's explicit formulas. Also we introduce a technique for taking advantage of the contribution of the prime numbers in these formulas.

Our work, like Odlyzko's, is heavily influenced by the search for lower bounds for the discriminant D_K of a number field K. Kronecker conjectured that $|D_K| > 1$. Later, Minkowski proved this and found a lower bound increasing exponentially with the degree of K by using geometry of numbers. In 1976, Odlyzko greatly improved on this bound by using analytic methods applied to the Dedekind zeta function of K. J.P. Serre showed [7] that Odlyzko's results could be obtained from Weil's explicit formulas. G. Poitou and Odlyzko developed this method in [5] and [6], obtaining good results.

Odlyzko [4] also found lower bound for the Artin conductor f_χ, applying his original methods to the Artin L-function of a character χ ([3], pp. 540). He obtained

$$f_\chi \geq (3.70)^{a_\chi}(2.38)^{b_\chi},$$

where a_χ and b_χ are integer numbers such that $a_\chi + b_\chi = \chi(1)$, $a_\chi - b_\chi = \chi(g_0)$, with $g_0 \in \mathcal{G}$ a complex conjugation. He also observed that the conductor $f_{\chi\overline{\chi}}$, where $\overline{\chi}$ is the character of the contragredient representation, divides $f_\chi^{2(\chi(1)-1)}$. Using this, for irreducible characters he obtained

$$f_\chi^{1/\chi(1)} \geq (7.75)^{\frac{(a_\chi - b_\chi)^2}{\chi(1)^2}}(4.71)^{\frac{4a_\chi b_\chi}{\chi(1)^2}} + o(1), \qquad \text{as}\, \chi(1) \to \infty.$$

Odlyzko assumed the Artin conjecture for χ and $\chi\overline{\chi}$. We also need to make the same assumption.

This thesis has been divided in two chapters. In chapter 1, following Mestre [1], we adapt Weil's explicit formulas to Artin L- functions. More precisely, consider a function $F : \mathbb{R} \to \mathbb{R}$ which satisfies $F(0) = 1$ and some conditions for insuring the convergence of series and integrals and suppose that $(s-1)^r L(s,\chi)$ is entire. Putting $\chi(1)=n$ we get

$$\begin{aligned}\frac{1}{n}\log f_\chi &= \log(\pi)+\frac{a_\chi}{n}I_F+\frac{b_\chi}{n}J_F-\frac{4r}{n}R_F\\ &+ \frac{2}{n}\sum_{\text{p prime}}\sum_{m=1}^{\infty}\frac{\log(p)}{p^{m/2}}\mathrm{Re}(\chi(p^m))F(m\log p)+\sum_\rho \phi(\rho),\end{aligned}$$

where I_F, J_F and R_F are integrals depending only on the function F, and $\chi(p^m)$ is the character χ evaluated on the m-th power of a Frobenius element associated to a prime ideal β above p, acting on the subspace of V fixed by the inertia group of β/p. Also, ρ runs over all the zeros of the L-function and the transform $\phi(s)$ is defined by

$$\phi(s)=\int_{-\infty}^{\infty}F(x)e^{(s-\frac{1}{2})x}dx.$$

In order to obtain a lower bound for f_χ, we take F positive and such that $\mathrm{Re}(\phi(s))\geq 0$ when $0<\mathrm{Re}(s)<1$. If in addition, we suppose that $\mathrm{Re}(\chi)$ is positive, we can dispose of the sum over the primes and the contribution of zeros. Nevertheless, the bounds obtained like that are not valid for all characters. To deal with this, following Odlyzko, we consider the character $\widetilde{\chi}=\chi+\chi(1)\chi_0$ which always has positive real part and verifies $f_\chi=f_{\widetilde{\chi}}$. Thus, with a suitable choice of a function F we can improve Odlyzko's bounds for any character χ of $\mathcal{G}$. In Theorem 1.3.2 we obtain

$$f_\chi \geq (4.90)^{a_\chi}(2.91)^{b_\chi}.$$

This is close to best possible as there are reducible representation with conductor $5^{a_\chi}3^{b_\chi}$, as is clear on taking direct sum of the 1-dimensional representation with conductor 5 and 3 asociated to the quadratic fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-3})$.

Another way to obtain lower bounds is to consider the character $\chi\overline{\chi}$ and to use the relationship between f_χ and $f_{\chi\overline{\chi}}$ found by Odlyzko. For irreducible characters, in Theorem 1.4.1 we obtain

$$f_\chi^{1/n}\geq 4.73(1.648)^{\frac{(a_\chi-b_\chi)^2}{n^2}}e^{-(13.34/n)^2}.$$

This bound hardly improves on Odlyko, as he obtained excellent ones for large degrees. However, for small degrees we are able to improve substantially on Odlyzko's lower bounds (see the table at the end of Chapter 1).

In chapter 2, we introduce a new idea into Odlyzko's method (rather than just clean up his method using the explicit formulas, as we do in chapter 1).

The main observation is that the first method (valid for all characters) does not yield good lower bounds only because the primes may contribute negative terms. When we pass to irreducible characters and consider $\chi\overline{\chi}$, the primes always contribute positively. However, without further information on the primes, we have to just drop these terms. Thus we consider simultaneously both inequalities and remark that we need not take the worst possible case in both methods. If the primes hurt us (that is, amount to a negative term) in the first method, then they exist and will help us in the second one. This stragegy is carried out in chapter 2. It yields substantial improvements when $a_\chi \neq 0$. In particular, we obtain in Corollaries 2.0.2 and 2.0.3,

$$f_\chi^{1/n} \geq 9.482\, e^{-10.359/n},$$

for $a_\chi = n$, and

$$f_\chi^{1/n} \geq 5.542\, e^{-21.537/n}$$

for $a_\chi = b_\chi$. This improves on the lower bounds

$$f_\chi^{1/n} \geq 7.797\, e^{-(13.34/n)^2}$$

and

$$f_\chi^{1/n} \geq 4.73\, e^{-(13.34/n)^2},$$

respectively from chapter 1. A table for small degrees giving improved lower bounds by this method is given at the end of chapter 2.

Chapter 1

Explicit Formulas and Odlyzko's Method

1.1 Summary

In this chapter we improve on Odlyzko's lower bounds for Artin conductors [4] using Weil's explicit formulas [9] as simplified by Mestre [1]. Our approach here is entirely analogous to Poitou's [6] (and Odlyzko's) use of these same formulas to improve on and simplify Odlyzko's original discriminant bounds. Following Odlyzko, we assume throughout the truth of Artin's conjecture.

We begin by explaining Mestre's explicit formula in §1.2 and then give lower bounds valid for arbitrary characters in §1.3. In §1.3.1 we give a slight improvement on these bounds using the zeroes of the Riemann zeta function. In §1.4.1 we again follow Odlyzko's methods to give lower bounds for conductors of irreducible characters. We tabulate our results and compare them with Odlyzko's in §1.4.1.

1.2 Mestre's Explicit Formulas

Let K be an algebraic number field, i.e. a finite field extension of $\mathbb{Q}$. Suppose that $K/\mathbb{Q}$ is Galois, χ is a linear character of $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$ and f_χ is its Artin conductor ([3], p. 527). Let us define the completed Artin L-function by

$$\Lambda(s,\chi) = \left(\frac{f_\chi}{\pi^{\chi(1)}}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)^{a_\chi}\Gamma\left(\frac{s+1}{2}\right)^{b_\chi}L(s,\chi), \qquad (1.2.1)$$

where $L(s,\chi)$ is the Artin L-function associated to χ with base field $\mathbb{Q}$, a_χ and b_χ are integers such that

$$a_\chi + b_\chi = \chi(1), \qquad a_\chi - b_\chi = \chi(g_0), \qquad (1.2.2)$$

with 1 the identity element of $\mathcal{G}$ and $g_0 \in \mathcal{G}$ a complex conjugation ([3], pp. 522, 540). This function verifies the functional equation ([3], p. 540)

$$\Lambda(1-s,\overline{\chi}) = W(\chi)\Lambda(s,\chi), \tag{1.2.3}$$

where $W(\chi) \in \mathbb{C}$ is such that $|W(\chi)| = 1$ and $\overline{\chi}$ is the character of the dual (or contragredient) representation of χ ([8], p. 12).

We will need Mestre's form ([1], pp. 212–213) of Weil's explicit formulas for rather general L-functions. We assume our L-functions L_i have Euler products of the type

$$L_1(s) = \prod_p \prod_{i=1}^{M'} \left(1 - \alpha_{i,p}\, p^{-s}\right)^{-1},$$

$$L_2(s) = \prod_p \prod_{i=1}^{M'} \left(1 - \beta_{i,p}\, p^{-s}\right)^{-1},$$

where p runs over the prime numbers and $\alpha_{i,p}, \beta_{i,p}$ are complex numbers such that

$$|\alpha_{i,p}|, |\beta_{i,p}| \le p^c. \tag{1.2.4}$$

For positive real numbers A, B, a_i and a'_i $(1 \le i \le M)$ such that $\sum_{i=1}^{M} a_i = \sum_{i=1}^{M} a'_i$ and complex numbers b_i and b'_i, with $\mathrm{Re}(b_i) \ge 0$ and $\mathrm{Re}(b'_i) \ge 0$, we consider meromorphic functions

$$\Lambda_1(s) = A^s L_1(s) \prod_{i=1}^{M} \Gamma(a_i s + b_i),$$

$$\Lambda_2(s) = B^s L_2(s) \prod_{i=1}^{M} \Gamma(a'_i s + b'_i),$$

verifying by assumption

$$\Lambda_1(1-s) = \omega \Lambda_2(s),$$

for some $\omega \in \mathbb{C}^*$.

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that:

$\bigstar$

$$(1.2.5)$$

($\star$) There exists $\varepsilon > 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is integrable over $\mathbb{R}$, with $c \geq 0$ satisfying (1.2.4).

($\star\star$) There exists $\varepsilon > 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is of bounded variation, the value at each point being the average of the right- and left-hand limits.

($\star\star\star$) The function $\dfrac{F(x)-F(0)}{x}$ is of bounded variation.

We define the Mellin transform of F by

$$\phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-\frac{1}{2})x}dx, \qquad (-\epsilon < \mathrm{Re}(s) < 1+\epsilon). \tag{1.2.6}$$

Define[1]

$$I(a,b) = a\int_0^{\infty} (\frac{F(ax)e^{-(\frac{a}{2}+b)x}}{1-e^{-x}} - \frac{F(0)e^{-x}}{x})dx$$

and

$$J(a,b) = a\int_0^{\infty} (\frac{F(-ax)e^{-(\frac{a}{2}+b)x}}{1-e^{-x}} - \frac{F(0)e^{-x}}{x})dx.$$

For a function F verifying the conditions ($\bigstar$), Mestre obtained the following explicit formula ([1], pp. 212–213):

$$\sum_{\rho}\phi(\rho) - \sum_{\mu}\phi(\mu) + \sum_{i=1}^{M} I(a_i,b_i) + \sum_{i=1}^{M} J(a_i',b_i') \tag{1.2.7}$$

$$= F(0)\log(AB) - \sum_{i=1}^{M'}\sum_{p \text{ prime}}\sum_{m=1}^{\infty} \big(\alpha_{i,p}^m F(m\log p) + \beta_{i,p}^m F(-m\log p)\big)\frac{\log p}{p^{m/2}},$$

where ρ and μ run respectively over all zeros and poles of Λ_1 (counted according to their multiplicity) in the vertical strip $\{s \in \mathbb{C} \mid -c \leq \mathrm{Re}(s) \leq 1+c\}$.

We will apply Mestre's formula as follows. Let χ be any character of $\mathcal{G}$. If $\rho : \mathcal{G} \longrightarrow \mathrm{GL}(V)$ with V a $\mathbb{C}$-vector space, is the representation associated

[1] There is a slight misprint in the definition of $I(a,b)$ in ([1], p. 212), where $f(ax)$ appears instead of $F(ax)$

to χ, β is any prime ideal of K over p and φ_β is a corresponding Frobenius automorphism, we can write the Artin L-function as a product of Euler factors for each prime as

$$L(s,\chi) = \prod_{p \text{ prime}} \left(\det(\mathrm{Id} - p^{-s}\rho(\varphi_\beta); V^{I_\beta})\right)^{-1},$$

where V^{I_β} is the subspace of invariants in V under the inertia group I_β ([3], p. 518). If $\lambda_{1,p}, \ldots, \lambda_{m_p,p}$ are the eigenvalues of $\rho(\varphi_\beta)$ acting on V^{I_β}, then $m_p \leq n = \chi(1)$ and

$$\det(\mathrm{Id} - p^{-s}\rho(\varphi_\beta); V^{I_\beta}) = \prod_{i=1}^{m_p} \left(1 - p^{-s}\lambda_{i,p}\right) = \prod_{i=1}^{n} \left(1 - p^{-s}\lambda_{i,p}\right),$$

where we have put $\lambda_{i,p} = 0$ if $n \geq i > m_p$. Thus,

$$L(s,\chi) = \prod_p \prod_{i=1}^{n} \left(1 - p^{-s}\lambda_{i,p}\right)^{-1}. \tag{1.2.8}$$

In Mestre's formula take

$$\begin{aligned} L_1(s) &= L(s,\chi), & L_2(s) &= L(s,\overline{\chi}), \\ \Lambda_1(s) &= \Lambda(s,\chi) \quad \text{and} & \Lambda_2(s) &= \Lambda(s,\overline{\chi}), \end{aligned} \tag{1.2.9}$$

with Λ the completed Artin L-function in (1.2.1). Note that $|\lambda_{i,p}| \leq 1$, because $\mathcal{G}$ is a finite group. Take $\alpha_{i,p} = \lambda_{i,p}$, so that $\alpha_{i,p} = \overline{\beta_{i,p}}$ and $c = 0$ in (1.2.4). As $\lambda_{i,p}^m$ is an eigenvalue of $\rho(\varphi_\beta^m)$, if we denote by $\chi(p^m)$ the character χ evaluated on φ_β^m acting on V^{I_β}, we have

$$\chi(p^m) = \sum_{i=1}^{n} \lambda_{i,p}^m, \qquad 2\mathrm{Re}\big(\chi(p^m)\big) = \sum_{i=1}^{n} \left(\alpha_{i,p}^m + \beta_{i,p}^m\right). \tag{1.2.10}$$

We also take

$$M' = M = n = \chi(1) = a_\chi + b_\chi, \qquad a_i' = a_i = 1/2 \quad \text{for } 1 \leq i \leq n,$$

$$b_i' = b_i = 0 \quad \text{for } 1 \leq i \leq a_\chi, \qquad b_i' = b_i = 1/2 \quad \text{for } a_\chi + 1 \leq i \leq n,$$

and

$$A = \left(\frac{f_\chi}{\pi^{\chi(1)}}\right)^{1/2}, \qquad B = \left(\frac{f_{\overline{\chi}}}{\pi^{\overline{\chi}(1)}}\right)^{1/2}.$$

Actually,

$$A = B, \tag{1.2.11}$$

because $f_\chi = f_{\overline{\chi}}$ and $\chi(1) = n = \overline{\chi}(1)$. Here is an analytic proof of $f_\chi = f_{\overline{\chi}}$. Take absolute values of both sides of the functional (1.2.3) for $s = \frac{1}{2} + it$ and $t \in \mathbb{R}$ such that $L(s,\chi) \neq 0$, to get

$$\left|\frac{f_\chi}{f_{\overline{\chi}}}\right|^{1/4} = \left|\frac{\overline{L(\frac{1}{2}+it,\chi)}}{L(\frac{1}{2}+it,\chi)}\right| = 1.$$

Here we used $\overline{L(s,\chi)} = L(\overline{s},\overline{\chi})$ and $|W(\chi)| = 1$. As the conductor is a positive integer, we conclude that $f_\chi = f_{\overline{\chi}}$.

A more satisfactory arithmetic proof of this same fact can be carried out as follows. If ψ is any representation of $\mathcal{G}$ let $f_\psi = \prod_{p\nmid\infty} p^{f_p(\psi)}$. If G_j is the j-th ramification group at p in the lower numbering, we have for each prime p

$$f_p(\psi) = \frac{1}{|G_0|}\sum_{j\geq 0} |G_j|\psi(1) - \psi(G_j),$$

where $\psi(G_j) = \sum_{g\in G_j} \psi(g)$ (see [3], pp. 528–530). Since $f_p(\psi)$ is a real number, we have $f_p(\psi) = \overline{f_p(\psi)} = f_p(\overline{\psi})$ for each prime p.

Let us now assume that the function F in Mestre's formula (1.2.7) verifies $F(-x) = F(x)$ and $F(0) = 1$. Thus $I(a,b) = J(a,b)$, since F is even, and

$$\sum_{i=1}^{n} I(a_i,b_i) + \sum_{j=1}^{n} J(a_i',b_i') = 2\sum_{i=1}^{n} I(a_i,b_i) = 2a_\chi I(\tfrac{1}{2},0) + 2b_\chi I(\tfrac{1}{2},\tfrac{1}{2}),$$

where

$$\begin{aligned} I(\tfrac{1}{2},0) &= \frac{1}{2}\int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx\,, \qquad (1.2.12)\\ I(\tfrac{1}{2},\tfrac{1}{2}) &= \frac{1}{2}\int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx\,. \end{aligned}$$

Finally, if $(s-1)^r L(s,\chi)$ is entire, with r being exactly the order of the pole at $s = 1$, from (1.2.7) to (1.2.12), we obtain the explicit formula

$$\begin{aligned} \log f_\chi &= \sum_\rho \phi(\rho) - r\big(\phi(0)+\phi(1)\big) + \chi(1)\log(\pi)\\ &+ a_\chi \int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx + b_\chi \int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx\\ &+ 2\sum_{p \text{ prime}}\sum_{m=1}^{\infty} \frac{\log(p)}{p^{m/2}}\mathrm{Re}\big(\chi(p^m)\big)F(m\log p), \qquad (1.2.13) \end{aligned}$$

where $\phi(s)$ is like (1.2.6) and ρ runs over all the zeros of $\Lambda(s,\chi)$ in the critical strip $0 < \mathrm{Re}(\rho) < 1$.[2]

Remark 1. We shall obtain lower bounds for conductors by controlling the signs of various terms appearing in the explicit formula. For this we will have to impose sign conditions on F and its Mellin transform. On the other hand, since we only want an inequality, we may weaken slightly some of the analytic conditions

Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that:

(♣)

(1.2.14)

(♣) There exists $\varepsilon \geq 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is integrable over $\mathbb{R}$, with $c \geq 0$ satisfying (1.2.4). If $\varepsilon = 0$ assume in addition that

$$\sum_{p \text{ prime}} \sum_{m=1}^{\infty} \log(p) \frac{F\big(m\log(p)\big)}{p^{m/2}} < \infty.$$

(♣♣) There exists $\varepsilon \geq 0$ such that $F(x)e^{(\frac{1}{2}+c+\varepsilon)x}$ is of bounded variation, the value at each point being the average of the right- and left-hand limits.

(♣♣♣) The function $\dfrac{F(x)-F(0)}{x}$ is of bounded variation.

(♣♣♣♣) F is even, $F(0) = 1$, $F(x) \geq 0$ for all $x \in \mathbb{R}$, and $\mathrm{Re}(\phi(s)) \geq 0$ for $0 < \mathrm{Re}(s) < 1$.

The purpose of the last condition is to ensure that the contributions from the zeroes ρ are all non–negative. In ♣ and ♣♣ we have weakened ★ by allowing $\varepsilon = 0$. This can be achieved by a limiting argument, replacing $F(x)$ by $F(x)\exp(-\varepsilon|x|)$ and taking $\varepsilon \to 0^+$ using Lebesgue dominated convergence (cf. Proposition 5 in [6]).

[2] $L(s,\chi)$ has neither zeroes nor poles on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$, except possibly at $s = 1$, where there may only be a pole. Its order is exactly the multiplicity of the trivial representation in χ (see [2], p. 6). This will be important when we consider $L(s,\chi\overline{\chi})$ in the last chapter.

Under (♣) we have then[3]

$$\begin{aligned}\log f_\chi &\geq \chi(1)\log(\pi) + a_\chi \int_0^\infty \Big(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx \\ &+ b_\chi \int_0^\infty \Big(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\Big)\,dx - 4r\int_0^\infty F(x)\cosh(x/2)\,dx. \\ &+ 2\sum_{p \text{ prime}}\sum_{m=1}^{\infty} \frac{\log(p)}{p^{m/2}}\mathrm{Re}\big(\chi(p^m)\big)F(m\log p)\,.\end{aligned} \tag{1.2.15}$$

As Odlyzko pointed out (cf. [6]), the conditions of nonnegativity on $F(x)$, and on $\mathrm{Re}\big(\phi(s)\big)$ on the critical strip, are equivalent to the requirement that

$$F(x) = \frac{f(x)}{\cosh(x/2)}, \tag{1.2.16}$$

where $f(x) \geq 0$ and $f(x)$ has a nonnegative Fourier transform. Indeed, note that $\phi(s) = \phi(1-s)$, because F is assumed even, and $\mathrm{Re}\big(\phi(s)\big)$ is a harmonic function on the strip $0 < \mathrm{Re}(s) < 1$. To show that $\mathrm{Re}\big(\phi(s)\big) \geq 0$ in the critical strip, the maximum principle for harmonic functions tells us that we need only check that $\mathrm{Re}\big(\phi(s)\big) \geq 0$ on the boundary. But there we have

$$\mathrm{Re}\big(\phi(1+it)\big) = \mathrm{Re}\big(\phi(it)\big) = \int_{-\infty}^{\infty} f(x)\cos(tx)dx.$$

If we assume the Riemann Hypothesis for $L(s,\chi)$ (i.e. $L(\rho,\chi) = 0$ for $0 < \mathrm{Re}(\rho) < 1$ implies $\mathrm{Re}(\rho) = \frac{1}{2}$) we have only to ensure $\mathrm{Re}\big(\phi(\frac{1}{2}+it)\big) \geq 0$ for all real t. In this case we will only need to assume that $F(x) \geq 0$ and that F has a nonnegative Fourier transform.

1.3 Bounds for arbitrary characters

A preliminary result is the following:

Theorem 1.3.1. *Suppose that χ is a character of $\mathcal{G}$ such that $Re\big(\chi(g)\big) \geq 0$ for all $g \in \mathcal{G}$ and that for some integer r, $(s-1)^r L(s,\chi)$ is entire. Then*

$$f_\chi \geq (6.5735)^{a_\chi}(3.9046)^{b_\chi}(0.1134)^r. \tag{1.3.17}$$

[3] Use

$$\phi(0) + \phi(1) = 4\int_0^\infty F(x)\cosh(x/2)\,dx.$$

Proof. Consider the family of functions (introduced by L. Tartar [6])

$$F_y(x) = \frac{f(x\sqrt{y})}{\cosh(x/2)}, \tag{1.3.18}$$

where

$$f(x) = \frac{9\big(\sin(x) - x\cos(x)\big)^2}{x^6} \tag{1.3.19}$$

and $y > 0$ is a positive parameter. F_y satisfies (♣) (see [6]). In ([6], p. 13) it is shown that f has a non-negative Fourier transform.[4] Since we have assumed $\mathrm{Re}\big(\chi(g)\big) \geq 0$ we may drop from inequality (1.2.15) the sum over the primes. Putting $a_\chi + b_\chi = \chi(1)$, $F = F_y$ and $y = 12$ in (1.2.15), yields numerically

$$\log f_\chi \geq 1.88305\, a_\chi + 1.36216\, b_\chi - 2.17656\, r,$$

and this is equivalent to (1.3.17). □

In general $\mathrm{Re}(\chi)$ is not positive, so Theorem 1.3.1 does not apply. Nevertheless, following Odlyzko we can prove

Theorem 1.3.2. *Let χ be a character of $\mathcal{G}$ such that its Artin L-function $L(s,\chi)$ is entire. Then its conductor f_χ satisfies*

$$f_\chi \geq (4.90)^{a_\chi}(2.91)^{b_\chi},$$

where a_χ (resp. b_χ) is the number of $\Gamma(\frac{s}{2})$ $\big($resp. $\Gamma(\frac{1+s}{2})\big)$ factors in the completed Artin L-function.

Odlyzko ([4], p. 382) obtained

$$f_\chi \geq (3.70)^{a_\chi}(2.38)^{b_\chi}.$$

Our bounds are nearly best possible for (possibly) reducible characters. Indeed, the quadratic field $\mathbb{Q}(\sqrt{5})$ has a character χ_5 with $\chi_5(1) = 1$, $a_\chi = 1$, $b_\chi = 0$ and $f_{\chi_5} = 5$, and $\mathbb{Q}(\sqrt{-3})$ has a character χ_3 with $\chi_3(1) = 1$, $a_\chi = 0$, $b_\chi = 1$ and $f_{\chi_3} = 3$. Thus, if $\chi := a\chi_5 + b\chi_3$ (for arbitrary non-negative integers a and b),

$$f_\chi = 5^a\, 3^b,$$

with $a = a_\chi$ and $b = b_\chi$.

[4] We note that there is an error in ([6], p. 13) concerning the normalization constant required to ensure $F_y(0) = 1$. There the 9 in (1.3.19) is wrongly replaced by $4/\pi^2$.

Proof. Consider the character

$$\widetilde{\chi} = \chi + \chi(1)\chi_0,$$

where χ_0 is the one-dimensional identity character. Since $L(s,\chi)$ is assumed entire, we see that $(s-1)^{\chi(1)}L(s,\widetilde{\chi})$ is entire. Indeed,

$$\begin{aligned} L(s,\chi+\chi(1)\chi_0) &= L(s,\chi)L(s,\chi(1)\chi_0) \\ &= L(s,\chi)L(s,\chi_0)^{\chi(1)} \\ &= L(s,\chi)\zeta(s)^{\chi(1)}, \end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. Since $|\chi(g)| \leq \chi(1)$ for all $g \in \mathcal{G}$, we have

$$\operatorname{Re}\big(\widetilde{\chi}(g)\big) = \chi(1) + \operatorname{Re}\big(\chi(g)\big) \geq 0.$$

From the properties of the conductor ([3], p. 533),

$$f_{\widetilde{\chi}} = f_{\chi+\chi(1)\chi_0} = f_\chi f_{\chi(1)\chi_0} = f_\chi. \tag{1.3.20}$$

Also (see (1.2.1)),

$$a_{\widetilde{\chi}} = a_\chi + \chi(1) = 2a_\chi + b_\chi, \qquad b_{\widetilde{\chi}} = b_\chi, \qquad \widetilde{\chi}(1) = 2\chi(1).$$

Applying Theorem 1.3.1 to the character $\widetilde{\chi}$ we obtain

$$\begin{aligned} f_\chi &\geq (6.5735)^{(2a_\chi+b_\chi)}(3.9046)^{b_\chi}(0.1134)^{\chi(1)} \\ &= (6.5735)^{(2a_\chi+b_\chi)}(3.9046)^{b_\chi}(0.1134)^{(a_\chi+b_\chi)} \\ &> (4.90)^{a_\chi}(2.91)^{b_\chi}. \end{aligned}$$

□

1.3.1 Contribution of zeros

So far, we have not considered the positive contribution from the zeros in the explicit formulas. In general, we know almost nothing about the location of zeros of $L(s,\chi)$, but in the proof of Theorem 1.3.2 we introduced the Riemann zeta function and dropped the contribution from its zeros. If we restore the contribution from the lowest zeros $\rho_0 = \frac{1}{2} \pm i14.134725142$ of the Riemann zeta function we gain $2\operatorname{Re}\phi_y(\rho_0)$, where ϕ_y is the Mellin transform of F_y. In this way we obtain, with $y = 10.35$

$$f_\chi \geq (4.947)^{a_\chi}(2.833)^{b_\chi},$$

which is slightly better than Theorem 1.3.2 if a_χ is much larger than b_χ.

Another possibility is to take $y = 13.5$ to obtain likewise,

$$f_\chi \geq (4.832)^{a_\chi}(2.95)^{b_\chi}.$$

With $y = 12$ we obtain a (minor) improvement for all a_χ and b_χ. Namely, under the hypotheses of Theorem 1.3.2,

$$f_\chi \geq (4.905)^{a_\chi}(2.913)^{b_\chi}.$$

1.4 Bounds for irreducible characters

We have seen that our results above are nearly optimal for arbitrary (i.e., possibly reducible) characters. In this section we again follow Odlyzko to obtain better lower bounds for irreducible characters. We will need the following lemma, valid for any character χ.

Lemma 1.4.1. *(Odlyzko) $f_{\chi\overline{\chi}}$ divides $f_\chi^{2(\chi(1)-1)}$.*

Proof. (Odlyzko) Since the conductor f_χ is a product of local conductors $p^{f_p(\chi)}$ ([3], p. 532), we need to prove that

$$f_p(\chi\overline{\chi}) \leq 2(\chi(1)-1)f_p(\chi). \tag{1.4.21}$$

For this, we will show that for every subgroup H of $\mathcal{G}$

$$|H|\chi(1)^2 - \chi\overline{\chi}(H) \leq 2(\chi(1)-1)\big(|H|\chi(1) - \chi(H)\big), \tag{1.4.22}$$

where $f(H) = \sum_{h\in H} f(h)$ and $|H|$ denotes the cardinality of H. We decompose

$$\chi|_H = r\phi_0 + \sum_{i\geq 1} r_i\phi_i, \tag{1.4.23}$$

where ϕ_0 is the trivial character of H, the ϕ_i are distinct, irreducible, nontrivial characters of H, and $r_i \geq 0$, $r \geq 0$. We have that

$$\chi(H) = r\sum_{h\in H}\phi_0(h) + \sum_{i\geq 1} r_i \sum_{h\in H}\phi_i(h)$$

and $\sum_{h\in H}\phi_0(h) = |H|$, and that $\sum_{h\in H}\phi_i(h) = 0$ (see [8], p. 17). Hence $\chi(H) = r|H|$. Also,

$$\begin{aligned}\chi\overline{\chi}|_H &= r^2\phi_0 + r\sum_{i\geq 1} r_i\overline{\phi_i} + r_1|\phi_1|^2 + r_1\sum_{i\neq 1} r_i\phi_1\overline{\phi}_i + r_2|\phi_2|^2 + r_2\sum_{i\neq 2} r_i\phi_2\overline{\phi}_i + \dots \\ &+ r_k^2|\phi_k|^2 + r_k\sum_{i\neq k} r_i\phi_k\overline{\phi}_i.\end{aligned}$$

Thus,

$$\chi\overline{\chi}(H) = r^2|H| + r\sum_{i\geq 1} r_i \sum_{h\in H} \overline{\phi}_i(h) + r_1^2 \sum_{h\in H} |\phi_1(h)|^2 + \ldots r_k^2 \sum_{h\in H} |\phi_k(h)|^2,$$

and so $\chi\overline{\chi}(H) = \big(r^2 + \sum_{i\geq 1} r_i^2\big)|H|$. But (1.4.22) is equivalent to

$$\chi(1)^2 - r^2 - \sum_{i\geq 1} r_i^2 \leq 2(\chi(1)-1)(\chi(1)-r),$$

and to

$$-\sum_{i\geq 1} r_i^2 \leq (\chi(1)-r)(\chi(1)-r-2).$$

From (1.4.23), $\chi(1) = r + \sum_{i\geq 1} r_i\phi_i(1)$, so the last inequality is equivalent to

$$-\sum_{i\geq 1} r_i^2 \leq \Big(\sum_{i\geq 1} r_i\phi_i(1)\Big)\Big(\sum_{i\geq 1} r_i\phi_i(1) - 2\Big). \tag{1.4.24}$$

The right side is negative only if $\sum_{i\geq 1} r_i\phi_i(1) < 2$, and this can happen only if $r_j = \phi_j(1) = 1$, for some j and $r_i = 0$ if $i \neq j$. In this case we obtain equality in (1.4.24), so (1.4.22) is true.

Returning to the proof of the lemma, let G_j be the j-th ramification group in the lower numbering ([3], p. 528) associated to a prime of K above p. Then, from (1.4.21)and (1.4.22) we obtain[5]:

$$\begin{aligned}
f_p(\chi\overline{\chi}) &= \frac{1}{|G_0|}\sum_{j\geq 0}\big(|G_j|\chi(1)^2 - \chi\overline{\chi}(G_j)\big) \qquad \text{(see [3], p. 530)}\\
&\leq \frac{1}{|G_0|}\sum_{j\geq 0} 2(\chi(1)-1)(|G_j|\chi(1) - \chi(G_j))\\
&= 2(\chi(1)-1)\frac{1}{|G_0|}\sum_{j\geq 0}(|G_j|\chi(1) - \chi(G_j))\\
&= 2(\chi(1)-1)f_p(\chi).
\end{aligned}$$

□

[5] Note that Neukirch defines $f(H)$ slightly differently from us. He divides by $|H|$. This explains an apparent difference between our formula for f_χ and his on the first line displayed below.

If we take χ an irreducible character and assume the Artin Conjecture for the (reducible) character $\chi\overline{\chi}$ of $\mathcal{G}$, then $(s-1)L(s,\chi\overline{\chi})$ is entire. In fact, the number of times that the representation $\rho_{\chi\overline{\chi}}$ associated to $\chi\overline{\chi}$ contains the trivial representation with character χ_0 is

$$< \chi\overline{\chi}, \chi_0 > = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}|\chi(g)|^2 = < \chi, \chi > = 1,$$

the last step being precisely because χ is irreducible ([8], p. 16).

Lemma 1.4.1 implies

$$f_{\chi\overline{\chi}} \leq f_{\chi}^{2(\chi(1)-1)}, \tag{1.4.25}$$

and therefore

$$f_{\chi\overline{\chi}}^{1/2\chi(1)} \leq f_{\chi}. \tag{1.4.26}$$

Now, applying (1.2.15) to the character $\chi\overline{\chi}$ with $r=1$,

$$\log f_{\chi\overline{\chi}} \geq a_{\chi\overline{\chi}}(I_F(y)+\log(\pi)) + b_{\chi\overline{\chi}}(J_F(y)+\log(\pi)) - 4R_F(y) \tag{1.4.27}$$

where

$$I_F(y) := \int_0^\infty (\frac{e^{-x/4}F_y(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x})\,dx \tag{1.4.28}$$

$$J_F(y) := \int_0^\infty (\frac{e^{-3x/4}F_y(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x})\,dx \tag{1.4.29}$$

and

$$R_F(y) := \int_0^\infty F_y(x)\cosh(x/2)\,dx. \tag{1.4.30}$$

Observe that (1.4.27) is equivalent to

$$\log f_{\chi\overline{\chi}} \geq (a_{\chi\overline{\chi}}-b_{\chi\overline{\chi}})(I_F(y)+\log(\pi))+2b_{\chi\overline{\chi}}(\frac{I_F(y)+J_F(y)+2\log(\pi)}{2})-4R_F(y). \tag{1.4.31}$$

Hence, from (1.4.26) with $n=\chi(1)$,

$$\begin{aligned}\frac{1}{n}\log f_{\chi} &\geq \frac{1}{2n^2}\log f_{\chi\overline{\chi}}\\ &\geq \frac{(a_{\chi\overline{\chi}}-b_{\chi\overline{\chi}})}{n^2}(\frac{I_F(y)+\log(\pi)}{2}) + \frac{2b_{\chi\overline{\chi}}}{n^2}(\frac{I_F(y)+J_F(y)+2\log(\pi)}{4}) - \frac{2R_F(y)}{n^2}.\end{aligned}$$

Using the definition (1.2.2) of $a_{\chi\overline{\chi}}$ and $b_{\chi\overline{\chi}}$

$$a_{\chi\overline{\chi}} - b_{\chi\overline{\chi}} = \chi\overline{\chi}(g_0) = \chi(g_0)^2 = (a_\chi - b_\chi)^2$$

and

$$b_{\chi\overline{\chi}} = \frac{\chi\overline{\chi}(1) - \chi\overline{\chi}(g_0)}{2} = \frac{\chi(1)^2 - \chi(g_0)^2}{2} = 2a_\chi b_\chi,$$

we have

$$\begin{aligned} \frac{\log f_\chi}{n} &\geq \frac{(a_\chi - b_\chi)^2}{n^2}\left(\frac{I_F(y) + \log(\pi)}{2}\right) + \frac{4a_\chi b_\chi}{n^2}\left(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}\right) \\ &- \frac{2R_F(y)}{n^2}. \end{aligned} \tag{1.4.32}$$

From here, we obtain a lower bound that is useful for large n.

Theorem 1.4.1. *Let χ be an irreducible character of degree n with conductor f_χ such that $L(s, \chi\overline{\chi})$ satisfies the Artin conjecture. Then*

$$f_\chi^{1/n} \geq 4.73(1.648)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(13.34/n)^2}. \tag{1.4.33}$$

Proof. Evaluate (1.4.32) with $y = 0.0045$ to obtain

$$f_\chi^{1/n} \geq (7.797)^{\frac{(a_\chi - b_\chi)^2}{n^2}} (4.73)^{\frac{4a_\chi b_\chi}{n^2}} e^{-(13.34/n)^2},$$

which is equivalent to (1.4.33) since $a_\chi + b_\chi = n$ and $\frac{7.797}{4.73} > 1.648$. □

If we assume the Generalized Riemann Hypothesis (see the end of Remark 1), we can improve the lower bounds.

Theorem 1.4.2. *Let χ be an irreducible character of degree n with conductor f_χ such that $L(s, \chi\overline{\chi})$ satisfies the Artin conjecture and the Riemann hypothesis. Then*

$$f_\chi^{1/n} \geq 6.59(2.163)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(13278.42/n)^2}. \tag{1.4.34}$$

Proof. Consider the even function[6] $F = F_{(y)} : \mathbb{R} \to \mathbb{R}$ which vanishes for $x > y^{-1/2}$ and which for $x \in [0, y^{-1/2}]$ is given by

$$F_{(y)}(x) = (1 - x\sqrt{y})\cos(\pi x\sqrt{y}) + \frac{\sin(\pi x\sqrt{y})}{\pi}. \tag{1.4.35}$$

Setting $y = 0.0004$ and using (1.4.32) we obtain (1.4.34). □

[6] Introduced by Odlyzko, cf. [6]. The crucial property of f_y is that it and its Fourier transform are non-negative.

Odlyzko ([4], p. 385) obtained

$$f_\chi^{1/n} \geq 4.71(1.645)^{\frac{(a_\chi - b_\chi)^2}{n^2}} + O(1/n^2), \quad \text{as } n \to \infty,$$

and, assuming the Riemann hypothesis,

$$f_\chi^{1/n} \geq 6.44(2.13)^{\frac{(a_\chi - b_\chi)^2}{n^2}} + O(1/n^2), \quad \text{as } n \to \infty.$$

Taking $y = .001$ in the above proof, we can get (still under the Riemann hypothesis for $L(s, \chi\overline{\chi})$)

$$f_\chi^{1/n} \geq 6.458(2.094)^{\frac{(a_\chi - b_\chi)^2}{n^2}} e^{-(260.81/n)^2}.$$

For large n our bounds are only marginally better than Odlyzko's. In the next subsection we shall substantially improve on his bounds for small degrees.

1.4.1 Tables for small degrees

In the previous section we used inequality (1.4.26), since we were interested only in large n. In this section we are interested in small n, so we use the stronger original inequality (1.4.25). The net effect is to replace every n^2 on the right-hand side of (1.4.32) by $n(n-1)$. From (1.4.31) we therefore obtain

$$\begin{aligned}\frac{\log f_\chi}{n} \geq\ & \frac{(a_\chi - b_\chi)^2}{n(n-1)}\left(\frac{I_F(y) + \log(\pi)}{2}\right) + \frac{4a_\chi b_\chi}{n(n-1)}\left(\frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}\right) \\ & - \frac{2R_F(y)}{n(n-1)}. \end{aligned} \qquad (1.4.36)$$

As before, we obtain bounds by evaluating (1.4.36) with Tartar's F_y as in (1.3.18) and y as given in the table below.

From (1.4.28) and (1.4.29) we find $J_F(y) < I_F(y)$. Hence, from (1.4.36) we have the lower bound, valid for any non-negative a_χ, b_χ with $a_\chi + b_\chi = n > 1$,

$$\frac{\log f_\chi}{n} \geq \frac{n}{n-1} \cdot \frac{I_F(y) + J_F(y) + 2\log(\pi)}{4} - \frac{2R_F(y)}{n(n-1)}, \quad \text{for } n \text{ even} \quad (1.4.37)$$

and

$$\begin{aligned}\frac{\log f_\chi}{n} \geq\ & \frac{n}{n-1} \cdot \frac{I_F(y) + J_F(y) + 2\log(\pi)}{4} + \frac{1}{n(n-1)} \cdot \frac{I_F(y) - J_F(y)}{4} \\ & - \frac{2R_F(y)}{n(n-1)}, \quad \text{for } n \text{ odd.} \end{aligned} \qquad (1.4.38)$$

These bounds are given in the third column of the table below for $2 \leq n \leq 20$. We also give lower bounds for the extreme cases in which $a_\chi = 0$ or $b_\chi = 0$, this time using (1.4.36). Finally, for the bounds under GRH we use Odlyzko's function (1.4.35) with y as shown.

Table 1. Lower bounds for irreducible characters.[7]

	Assuming Artin's Conjecture						Artin's Conjecture and G.R.H.			
	Any a_χ, b_χ			$a_\chi b_\chi = 0$			Any a_χ, b_χ		$a_\chi b_\chi = 0$	
n	y	$f_\chi^{1/n} \geq$	Odl	y	$f_\chi^{1/n} \geq$	Odl	y	$f_\chi^{1/n} \geq$	y	$f_\chi^{1/n} \geq$
2	4.71	3.255	2.83	2.65	5.067	4.21	0.2353	3.266	0.14	5.127
3	1.5	4.103	–	0.84	6.370	–	–	–	0.053	6.615
4	0.8	4.245	3.74	0.460	7.059	5.86	0.052	4.347	0.033	7.544
5	0.48	4.528	–	0.300	7.432	–	–	–	0.024	8.169
6	0.4	4.553	4.07	0.220	7.649	6.47	0.0269	4.785	0.019	8.619
7	0.25	4.681	–	0.169	7.782	–	–	–	0.016	8.962
8	0.2	4.684	4.22	0.140	7.867	6.47	0.0192	5.0227	0.014	9.235
9	0.19	4.748	–	0.11	7.922	–	–	–	0.013	9.460
10	0.18	4.738	4.30	0.096	7.960	6.88	0.0153	5.175	0.012	9.647
11	0.120	4.782	–	0.084	7.984	–	–	–	0.011	9.810
12	0.11	4.776	4.35	0.074	8.002	7.06	0.0129	5.283	0.01	9.952
13	0.1	4.799	–	0.065	8.013	–	–	–	0.0094	10.076
14	0.116	4.776	4.39	0.059	8.020	7.38	0.0113	5.365	0.0091	10.185
15	0.08	4.808	–	0.064	8.025	–	–	–	0.0085	10.287
16	0.09	4.798	4.47	0.049	8.027	7.57	0.0101	5.431	0.0081	10.377
17	0.067	4.812	–	0.045	8.028	–	–	–	0.0077	10.46
18	0.06	4.806	4.55	0.042	8.028	7.69	0.00925	5.484	0.0074	10.536
19	0.05	4.813	–	0.039	8.026	–	–	–	0.0071	10.606
20	0.036	4.809	4.61	0.036	8.025	7.77	0.00855	5.529	0.0069	10.671

[7] The columns labeled Odl show the lower bounds obtained by Odlyzko ([4], p. 404). Cases not covered by Odlyzko's tables have a – in the Odl column. All bounds are rounded down so that the inequality is rigorous. We assume χ is irreducible and that $L(s, \chi\overline{\chi})$ is analytic for $s \neq 1$. The last four columns on the right apply when we also assume GRH, i.e. that all zeroes ρ of $L(s, \chi\overline{\chi})$ satisfy $\mathrm{Re}(\rho) = \frac{1}{2}$. The first lower bound in each case (Columns labeled Any a_χ, b_χ) apply for any value of a_χ or b_χ with $a_\chi + b_\chi = n$. The lower bounds in the columns labeled $a_\chi b_\chi = 0$ only apply when $a_\chi = n$ or $b_\chi = n$.

We note that for large n our non-GRH bounds will *drop* toward 4.78 because the term $-\frac{2R_F(y)}{n(n-1)}$ in (1.4.37) becomes irrelevant (it approaches 0) and the decrease in the factor $\frac{n}{n-1}$ takes over.

Chapter 2

Beyond Odlyzko's Method

In the previous chapter we obtained lower bounds for the conductor f_χ of the irreducible character χ by two different methods. In the first one (where irreducibility was irrelevant) we had to compensate for the possible negativity of $\mathrm{Re}(\chi)$. In the second method the primes entered positively, but we dropped them. In this section we improve on these bounds by noting that if the first method requires primes to be compensated for, then they must make a substantial contribution to the second method. If primes do not require compensation, then the first method can be substantially improved. Thus we are able to obtain an improvement regardless of the behavior of the primes.

We shall need a lemma which will allow us to balance gains against losses in the two methods.

Lemma 2.0.2. *Let j run over a finite set of indices and let τ, δ_j and β_j be real numbers, with $\tau > 0$ and $\delta_j > 0$ for all j. If*

$$\sum_j x_j \beta_j \leq -\tau, \tag{2.0.1}$$

then

$$\sum_j x_j^2 \delta_j \geq \frac{\tau^2}{\Gamma}, \qquad \textit{where} \qquad \Gamma = \sum_j \frac{\beta_j^2}{\delta_j}. \tag{2.0.2}$$

Proof. Since the δ_j are assumed positive, there is a minimum value m of the positive quadratic form $\sum_j x_j^2 \delta_j$ as the x_j range over the region defined by $\sum_j x_j \beta_j \leq -\tau$. First we show that m can only be a assumed on the boundary. Indeed, suppose that there exist $\widetilde{x}_j$ such that $\sum_j \widetilde{x}_j \beta_j < -\tau$ and m is assumed at $\widetilde{x} = (\widetilde{x}_j)$. Then $\widetilde{x}$ is a critical point of the quadratic form. Taking partial derivatives we find $2\widetilde{x}_j \delta_j = 0$ for all j. Hence $\widetilde{x} = 0$, contradicting $\sum_j \widetilde{x}_j \beta_j < -\tau$, since τ is assumed positive.

Thus we seek the to minimize the expression (2.0.2) using the condition (2.0.1) with equality. We will use Lagrange multipliers. Note that the minimum is known to exist, and hence will be given as a critical point of the auxiliary function $F(\mathbf{x}, \lambda)$ used with Lagrange multipliers. We shall see that there is a unique critical point, and hence this yields the minimum m. Consider the function

$$F(\mathbf{x}, \lambda) = g(\mathbf{x}) - \lambda h(\mathbf{x}),$$

where

$$g(\mathbf{x}) = \sum_j x_j^2 \delta_j$$

and

$$h(\mathbf{x}) = \tau + \sum_j x_j \beta_j.$$

Now, we will find critical point for F. This is equivalent to solving the system

$$\frac{\partial F}{\partial x_j} = 0, \qquad \frac{\partial F}{\partial \lambda} = 0,$$

which is equivalent to

$$\frac{\partial F}{\partial x_j} = 2x_j\delta_j - \lambda\beta_j = 0$$
$$\frac{\partial F}{\partial \lambda} = \tau + \sum_j x_j\beta_j = 0.$$

Thus,

$$x_j = \lambda\frac{\beta_j}{2\delta_j},$$

and so

$$-\tau = \sum_j x_j\beta_j = \frac{\lambda}{2}\sum_j \frac{\beta_j^2}{\delta_j} = \frac{\lambda\Gamma}{2}.$$

Hence, $x_j = -\frac{\tau\beta_j}{\Gamma\delta_j}$. Moreover,

$$\sum_j x_j^2\delta_j = \frac{\tau^2}{\Gamma^2}\sum_j \frac{\beta_j^2}{\delta_j^2}\delta_j = \frac{\tau^2}{\Gamma^2}\sum_j \frac{\beta_j^2}{\delta_j} = \frac{\tau^2}{\Gamma}.$$

Therefore,

$$\sum_j x_j^2\delta_j \geq \frac{\tau^2}{\Gamma},$$

as claimed in the lemma. □

To describe our main inequality we need some notation. Fix non-negative integers a and b, and set $n = a+b$. For $F : \mathbb{R} \to \mathbb{R}$ an even function satisfying condition ♣ (1.2.14), set

$$I_F := \int_0^\infty \left(\frac{e^{-x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\right) dx, \tag{2.0.3}$$

$$J_F := \int_0^\infty \left(\frac{e^{-3x/4}F(x/2)}{1-e^{-x}} - \frac{e^{-x}}{x}\right) dx, \tag{2.0.4}$$

$$R_F := \int_0^\infty F(x)\cosh(x/2)\, dx, \tag{2.0.5}$$

$$G_F := \log(\pi) + \frac{a}{n}I_F + \frac{b}{n}J_F, \tag{2.0.6}$$

$$H_F := \frac{(a-b)^2}{n(n-1)}\left(\frac{I_F+\log(\pi)}{2}\right) + \frac{4ab}{n(n-1)}\left(\frac{I_F+J_F+2\log(\pi)}{4}\right) - \frac{2}{n(n-1)}R_F, \tag{2.0.7}$$

$$\alpha_{F,p,m} := \frac{F(m\log(p))\log(p)}{p^{m/2}}, \tag{2.0.8}$$

We note H_F is exactly the right-hand side of inequality (1.4.36) for $\frac{1}{n}\log f_\chi$ for irreducible characters, while G_F would also be a lower bound for $\frac{1}{n}\log f_\chi$ (cf. (1.2.15) with $r=0$) if the primes had not forced us to replace χ by $\tilde{\chi}$ to ensure $\mathrm{Re}\big(\tilde{\chi}(p^m)\big) \geq 0$. Terms like $\alpha_{F,p,m}$ had not appeared in our inequalities as we had arranged to drop all terms coming from the primes in the explicit formulas.

Theorem 2.0.3. *Let χ be an irreducible character of $\mathcal{G}$ of dimension $n \geq 2$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Suppose further that F and $\widetilde{F}$ satisfy condition ♣ (1.2.14) and $G_F > H_{\widetilde{F}}$, with F compactly supported and $\widetilde{F} > 0$ on the support of F. Then*

$$\frac{1}{n}\log f_\chi \geq H_{\widetilde{F}} + \frac{(n-1)\Gamma}{n}\cdot\left(\sqrt{1+\frac{n}{n-1}\cdot\frac{G_F - H_{\widetilde{F}}}{\Gamma}} - 1\right)^2, \tag{2.0.9}$$

where $\Gamma = \sum_{p,m} \frac{\alpha_{F,p,m}^2}{\alpha_{\widetilde{F},p,m}}$, the sum ranging over all primes p and positive integers m such that $m\log p$ is contained in the support of F.

The way to interpret the messy expression (2.0.9) is to think of $H_{\widetilde{F}}$ as the lower bound we had from the previous chapter, with the rest of the expression as the gain from the primes. In calculating (2.0.6) and (2.0.7) we take $a = a_\chi$, $b = b_\chi$ and $n = a + b$.

Proof. From the basic inequality (1.2.15) with $r = 0$, we obtain

$$\frac{1}{n}\log f_\chi \geq G_F + \frac{2}{n}\sum_{p,m}\alpha_{F,p,m}\cdot c_{p,m}, \qquad \text{where} \qquad c_{p,m} := \mathrm{Re}\big(\chi(p^m)\big). \tag{2.0.10}$$

Consider now the character $\chi\overline{\chi}$, for which we have proved in Lemma 1.4.1

$$\frac{1}{n}\log f_\chi \geq \frac{1}{2n(n-1)}\log f_{\chi\overline{\chi}} = \frac{n}{(n-1)}\frac{1}{2n^2}\log f_{\chi\overline{\chi}}.$$

We now apply (1.2.15) to $\chi\overline{\chi}$ (which corresponds to a representation of dimension n^2) the basic inequality (1.2.15) with $r = 1$ and $F = \widetilde{F}$ to obtain

$$\begin{aligned}\frac{1}{n}\log f_\chi &\geq H + \frac{1}{n(n-1)}\sum_{p,m}\alpha_{\widetilde{F},p,m}\cdot|\chi(p^m)|^2 \\ &\geq H + \frac{1}{n(n-1)}\sum_{p,m}\alpha_{\widetilde{F},p,m}\cdot c_{p,m}^2 .\end{aligned} \tag{2.0.11}$$

In the last sum over p and m we may (and do) drop all p and m for which $F(m\log(p)) = 0$.[1] Dropping these terms ensures that sums over p and m are finite, which will be required when we apply lemma 2.0.2 below. From the hypotheses in the theorem we have the strict inequality $\alpha_{\widetilde{F},p,m} > 0$ for terms p and m remaining in the sum.[2]

Let

$$T := H_{\widetilde{F}} + \frac{(n-1)\Gamma}{n}\cdot\left(\sqrt{1+\frac{n}{n-1}\cdot\frac{G_F - H_{\widetilde{F}}}{\Gamma}} - 1\right)^2. \tag{2.0.12}$$

We claim,

$$G_F > T > H_{\widetilde{F}}.$$

Indeed, the second inequality is trivial and the first one is equivalent to (on letting $\Gamma' = \frac{(n-1)\Gamma}{n}$)

$$G_F - H_{\widetilde{F}} > \Gamma'\left(\sqrt{1+\frac{G_F - H_{\widetilde{F}}}{\Gamma'}} - 1\right)^2,$$

[1] This is permissible since the last condition in ♣ ensures $\alpha_{\widetilde{F},p,m} \geq 0$.

[2] In (1.4.36) previously we had simply dropped all of the sum over the primes using $\chi\overline{\chi}(p^m) \geq 0$. We wish to exploit in the explicit formula for $\chi\overline{\chi}$ the finitely many primes appearing in the explicit formula for χ with non-zero coefficients.

which, on expanding the square, is equivalent to

$$G_F - H_{\tilde{F}} > G_F - H_{\tilde{F}} + 2\Gamma'\left(1 - \sqrt{1 + \frac{G_F - H_{\tilde{F}}}{\Gamma'}}\right),$$

which is clearly true since $G_F - H_{\tilde{F}} > 0$ by assumption. Let us write (2.0.10) as

$$\frac{1}{n}\log f_X \geq \ T + t + \frac{2}{n}\sum_{p,m} c_{p,m} \cdot \alpha_{F,p,m},$$

where

$$t := G_F - T > 0. \tag{2.0.13}$$

If we had

$$t + \frac{2}{n}\sum_{p,m} c_{p,m} \cdot \alpha_{F,p,m} \geq 0, \tag{2.0.14}$$

we would have

$$\frac{1}{n}\log f_X \geq T,$$

proving the theorem in this case. Hence, we may suppose that (2.0.14) is false, i.e.

$$\sum_{p,m} c_{p,m} \cdot \alpha_{F,p,m} < -\frac{nt}{2}.$$

As consequence of lemma 2.0.2, with j indexed by p and m as in the lemma, $x_j = c_{p,m},\ \beta_j = \alpha_{F,p,m},\ \delta_j = \alpha_{\tilde{F},p,m},\ \tau = nt/2$ we have

$$\sum_{p,m} c_{p,m}^2 \alpha_{\tilde{F},p,m} \geq \frac{t^2}{4\Gamma} n^2.$$

Therefore, in (2.0.11) using (2.0.13),

$$\frac{1}{n}\log f_X \geq H_{\tilde{F}} + \frac{t^2}{4\Gamma}\frac{n}{(n-1)} = H_{\tilde{F}} + \frac{(G_F - T)^2}{4\Gamma}\frac{n}{(n-1)} = T, \tag{2.0.15}$$

where at the end we used definition (2.0.12) and some algebraic manipulations.[3] Our last inequality proves the Theorem. □

[3] Here are the details. Let us abbreviate $\Gamma' = (n-1)\Gamma/n,\ H = H_{\tilde{F}},\ G = G_F$. Then we have from the definition of T

$$T - H = \Gamma'\left(\sqrt{1 + \Gamma'^{-1}(G - H)} - 1\right)^2. \tag{2.0.16}$$

We now apply the above theorem to obtain improved lower bounds for large degrees.[4] In this case, we can replace every occurrence of $n-1$ in (2.0.7) and (2.0.9) by n.[5] Then (2.0.9) simplifies to

$$\frac{1}{n}\log f_{\chi} \geq H_{\widetilde{F}} + \Gamma \cdot \left(\sqrt{1+\frac{G_F - H_{\widetilde{F}}}{\Gamma}} - 1\right)^2. \tag{2.0.20}$$

In (2.0.20) we will take F to be Bernardette Perrin-Riou's function, introduced in [6], p. 13,[6]

$$F(x) := \frac{fr(x\sqrt{y_G})}{\cosh(x/2)},$$

where y_G is a positive parameter to be specified later, $fr(x)$ is even, vanishes for $x > 2\pi$ and for $x \in [0, 2\pi]$ is given by

$$fr(x) = \frac{1}{3\pi}\left(2\pi - x + \frac{3\sin(x) + \pi\cos(x) - (x-\pi)\cos(x)}{2}\right). \tag{2.0.21}$$

Since now F in Theorem 2.0.3 depends on an extra parameter, we add it everywhere to the notation, writing for example $G_F(y_G)$ for G_F in (2.0.6).

For $\widetilde{F}$ in (2.0.20) we will take Tartar's function

$$\widetilde{F}(x) = \frac{f(x\sqrt{y_H})}{\cosh(x/2)}, \tag{2.0.22}$$

where

$$f(x) = \frac{9\big(\sin(x) - x\cos(x)\big)^2}{x^6} \tag{2.0.23}$$

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and the trivial equation

$$G - H = 2\Gamma'\big(\sqrt{1+\Gamma'^{-1}(G-H)} - 1\big) + \Gamma'\big(1 + \Gamma'^{-1}(G-H) + 1 - 2\sqrt{1+\Gamma'^{-1}(G-H)}\big). \tag{2.0.17}$$

Subtracting (2.0.16) from (2.0.17) we obtain

$$G - T = 2\Gamma'\big(\sqrt{1+\Gamma'^{-1}(G-H)} - 1\big), \tag{2.0.18}$$

whence

$$\frac{(G-T)^2}{4\Gamma'} = \Gamma'\big(\sqrt{1+\Gamma'^{-1}(G-H)} - 1\big)^2, \tag{2.0.19}$$

which is equivalent to (2.0.15), in view of (2.0.16).

[4] In the next section we will tabulate such bounds for small degrees.

[5] To see this, note that the $n-1$ comes from inequality (1.4.36), which becomes strictly weaker if we replace every ocurrence of $n-1$ by n.

[6] There the function is described as a convolution square, but not explicitly calculated. The formula we give in (2.0.21) is the result of carrying out the calculation of this convolution square.

and $y_H > 0$ is another positive parameter to be specified below. Since Tartar's (non-negative function) function is positive for $0 \le x \le 4.49/\sqrt{y_H}$, one finds that $\widetilde{F}$ is positive on the support of Perrin-Riou's F if $y_H < y_G/2$.

We have the following numerical corollaries of the Theorem 2.0.3.[7]

Corolary 2.0.1. *Let χ be an irreducible character of $\mathcal{G}$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then*

$$\begin{aligned}\frac{1}{n}\log f_\chi &\ge 1.378 + \frac{(a-b)^2}{n^2}0.374 - \frac{4.743}{n^2}\\ &+ 0.609\Big(\sqrt{1.1 + \frac{a}{n}1.677 + \frac{4a(n-a)}{n^2}0.374 + \frac{7.782}{n^2}} - 1\Big)^2\end{aligned}$$

Proof. Take $y_H = 0.632$ and $y_G = 2.968$, which satisfy $y_H < y_G/2$. Also we must verify $G_F(y_G) > H_{\widetilde{F}}(y_H)$ for all a, b. In fact, we can consider $G_F(y_G) - H_{\widetilde{F}}(y_H)$ as a function of a letting $b = n-a$, then with $G_F(y_G) = g(a)$ and $H_{\widetilde{F}}(y_H) = h(a)$ we have (replacing n by $n-1$)

$$\begin{aligned}g(a) - h(a) &= \log(\pi) + J_F(y_G) - \widetilde{I}_{\widetilde{F}}(y_H) + \frac{a}{n}(I_F(y_G) - J_F(y_G))\\ &+ \frac{4a(n-a)}{n^2}(\widetilde{I}_{\widetilde{F}}(y_H) - \widetilde{J}_{\widetilde{F}}(y_H)) + \frac{2R_{\widetilde{F}}(y_H)}{n^2}, \qquad (2.0.24)\end{aligned}$$

where $I_F(y)$, $J_F(y)$ and $R_F(y)$ were defined in (2.0.3), (2.0.4) and (2.0.5) respectively (but we have now put the dependence on y into the notation), and

$$\widetilde{I}_F(y) = \frac{I_F(y) + \log(\pi)}{2},$$

$$\widetilde{J}_F(y) = \frac{I_F(y) + J_F(y) + 2\log(\pi)}{4}.$$

Since $I_F(y) - J_F(y) > 0$, the expression (2.0.24) is quadratic in a with negative leading coefficient, its minimal value in any interval is attained at one of the interval extremes. One thus checks that on the interval $[0,n]$ the minimum is attained at $a = 0$. This implies

$$\Gamma\left(\sqrt{1 + \frac{g(a)-h(a)}{\Gamma}} - 1\right)^2 \ge \Gamma\left(\sqrt{1 + \frac{g(0)-h(0)}{\Gamma}} - 1\right)^2.$$

We have that $g(0) - h(0) = 0.0615$, then $G_F(y_G) > H_{\widetilde{F}}(y_H)$ for all a. After this one simply evaluates (2.0.20). ☐

[7] The case $a = 0$ is not treated below as Theorem 2.0.3 gives no significant improvement in this case.

Corolary 2.0.2. *Let χ be an irreducible character of $\mathcal{G}$ with $a = n$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n} \geq 9.482\, e^{-10.359/n}. \tag{2.0.25}$$

This improves on the lower bound $f_\chi^{1/n} \geq 7.797\, e^{-(13.34/n)^2}$ from Theorem 1.4.1 in the previous chapter.

Proof. Evaluating 2.0.20 with $y_G = 1.2$ and $y_H = 0.033$, we obtain

$$\begin{aligned}
\frac{1}{n}\log f_\chi &\geq 2.0302 - \frac{20.7526}{n^2} + 1.2925\Big(\sqrt{1.9934 + \frac{16.0556}{n^2}} - 1\Big)^2 \\
&\geq 5.8933 - 2.585\sqrt{1.9934 + \frac{16.0556}{n^2}} \\
&\geq 2.2494 - \frac{10.359}{n},
\end{aligned}$$

where in the last step we used $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$ for A and B positive. □

Corolary 2.0.3. *Let χ be an irreducible character of $\mathcal{G}$ with $a = b = n/2$ and assume the Artin conjecture for $L(s,\chi)$ and $L(s,\chi\overline{\chi})$. Then,*

$$f_\chi^{1/n} \geq 5.542\, e^{-16.859/n} \tag{2.0.26}$$

This improves on the lower bound $f_\chi^{1/n} \geq 4.73\, e^{-(13.34/n)^2}$ from the previous chapter.

Proof. We evaluate 2.0.20 with $y_G = 2.069$ and $y_H = 0.05$ and use the same procedure as in the previous corollary. □

2.1 Small degrees

In this section we use Theorem 2.0.3 to improve on the lower bounds in the previous chapter for small degrees. We use the same functions (Tartar's and Perrin-Rious's) as in the previous section, with different values of y_G and y_H, as given in the table below. Unlike in the previous section, we keep the $n-1$ in Theorem 2.0.3 as this improves lower bounds for small n.

In the following tables, we tabulate the lower bound for $2 \leq n \leq 20$. We have omitted the case $a = 0$ as the gains over the Chapter 2 are minor in this case.

Table 1 Lower bounds for irreducible characters

	$a_\chi = n$			$a_\chi = b_\chi$		
n	y_G	y_H	$f_\chi^{1/n} \geq$	y_G	y_H	$f_\chi^{1/n} \geq$
2	3.147	1.651	7.469	5.997	2.696	4.599
3	1.890	0.675	8.636	–	–	–
4	1.517	0.378	9.207	2.968	0.632	5.336
5	1.35	0.265	9.509	–	–	–
6	1.263	0.190	9.68	2.433	0.301	5.559
7	1.21	0.153	9.781	–	–	–
8	1.016	0.121	9.834	2.253	0.197	5.645
9	1.153	0.105	9.882	–	–	–
10	1.115	0.088	9.906	2.169	0.135	5.684
11	1.102	0.079	9.920	–	–	–
12	1.1	0.069	9.929	2.134	0.116	5.7
13	1.1	0.059	9.934	–	–	–
14	1.142	0.052	9.935	2.099	0.08	5.71
15	1.138	0.047	9.935	–	–	–
16	1.136	0.043	9.933	2.085	0.072	5.714
17	1.160	0.041	9.930	–	–	–
18	1.1	0.038	9.928	2.081	0.060	5.715
19	1.13	0.035	9.924	–	–	–
20	1.2	0.033	9.917	2.069	0.050	5.714

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