ECUACIONES CUASILINEALES CON PESOS

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Facultad de Ciencias
por

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FACULTAD DE CIENCIAS

UNIVERSIDAD DE CHILE

INFORME DE APROBACION

TESIS DE DOCTORADO

Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por el candidato.

LEONELO PATRICIO ITURRIAGA PASTENE

Ha sido aprobada por la Comisión de Evaluación de la tesis como requisito para optar al grado de Doctor en Ciencias con mención en Matemáticas, en el exámen de Defensa de Tesis rendido el día 9 de Noviembre de 2005.

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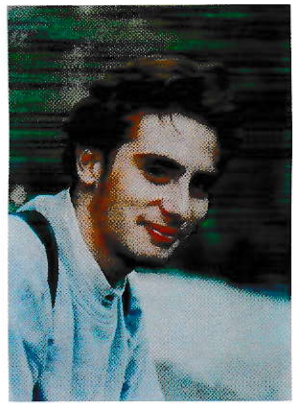
A todos los que me ayudaron a salir adelante



Es muy difícil escribir tu biografía sin caer en el egocentrismo, por que si bien es una forma de auto-evaluarse, también es muy importante destacar y comunicar de cierta forma las derrotas y logros vividos. No es fácil quedar al desnudo de lo que muchas veces guardas para ti mismo. Pero en esta oportunidad el orgullo de lo que soy llevan a reflejar fácilmente mi vida en corto bajo estas letras.

Mi vida comienza un 20 de Marzo de 1978, en Santiago. Criándome bajo el refugio de 2 grandes mujeres, mi madre Ana María y mi Yeya (mi abuela) Yolanda. Rodeado de mucho amor tuve una infancia entretenida, con muchos amigos de mi edad con quienes compartí y exploré lo que este mundo nos estaba entregando, haciendo muchas travesuras como todo niño. A pesar de la ausencia de figura paterna, siento que mi vida transcurre y se desarrolla a ojos de la sociedad de forma normal.

Mi adolescencia fue un poco mas desordenada, amante del equipo de fútbol "Colo Colo", me sentía todo un garrero y me rodee de amistades poco aceptables de los que puedo asegurar que si seguía su camino, hoy no estaría escribiendo estas líneas. "Malas Juntas" esa es una forma de interpretar lo que viví en mi adolescencia. Pero la verdad, no todo fue malo porque pude rescatar de esa etapa o vivencia que no quería eso para mi, sentía que yo podía ser mas que eso si me lo proponía, fue el click en mi vida para descubrir que el futuro tal vez me tenia preparado algo mejor.

Bueno el destino efectivamente me juega una mala pasada y mi vida cambia bruscamente con la muerte de mi Yeya. Siempre dijo que yo era su razón de ser y de una u otra forma ella también lo fue para mi... "Sólo tú sabes como marcaste mi vida". Luego de ese quiebre y como todo en la vida debí continuar. Terminé mi enseñanza media en 1995 y sentía que mis aptitudes estaban en el área matemática.

Por eso decidí estudiar licenciatura en matemáticas en la Universidad de Santiago de Chile, destacándome desde los inicios como uno de los mejores alumnos de la facultad de ciencias de dicha escuela. Entre los años 1998 y 2001 participe en diversos proyectos de investigación y docencia, destacando mi participación en la elaboración del libro de cálculo de la Universidad de Santiago.

En el año 2002 dicté mi primera cátedra en la Universidad de Santiago y en ese mismo período fui aceptado en el programa de doctorado en ciencias con mención en matemáticas de la Universidad de Chile. Para entonces ya sentía que mi vida iba abriéndose paso velozmente y empecé a valorar enormemente mis logros y la compañía de mi familia y amigos que han sido pilar fundamental del cumplimiento de mis metas.

El 2003, obtuve la beca CONICYT, la que me permitió dedicarme completamente a mis estudios de doctorado.

Tal vez se pueda pensar que las oportunidades son solo para algunos, pero yo me atrevo a decir que son para quien las sepa valorar y aprovechar. Tengo un futuro y estoy dispuesto a aprovecharlo, tener una familia y darles lo mejor de mi.

Leo.

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Resumen

En este trabajo se estudiarán ecuaciones cuasilineales asociadas al p-Laplaciano con pesos singulares en el origen. Este estudio generará resultados de existencia y multiplicidad de soluciones. Para esto consideraremos no linealidades continuas con crecimiento subcrítico. Las técnicas principales usadas son el Teorema del Paso de la Montaña, el principio variacional de Ekeland y la inclusión compacta en espacios de Sobolev con pesos, donde la desigualdad de Caffarelli–Kohn–Nirenberg es fundamental. Además, imponiendo una dependencia de cierto parámetro en la no linealidad, se estudiará el comportamiento asintótico de las soluciones, tanto cuando el parámetro tiende a cero, como cuando tiende al infinito. Por último usando técnicas de truncación, regularidad e iteración monótona, se probará la existencia de un problema no variacional, donde la no linealidad depende del gradiente.

Abstract

We study quasilinear equations with singular weights at the origin associated to the $p-$Laplacian. The study generates results about existence and multiplicity of solutions. We consider continuous nonlinearitites with subcritical growth. The principal techniques used are the Mountain Pass Theorem, Ekeland's variational principle, and compact embeddings in Sobolev spaces with weights for which the Caffarelli–Kohn–Nirenberg inequality plays a fundamental role. Moreover, under some parameter dependence of the nonlinearity, we study the asymptotic behavior of the solutions when the parameter tends to zero or infinity. Finally, using truncation and regularity techniques as well as the method of monotone iteration, we show the existence of solutions of a non–variational problem where the nonlinearity depends on the gradient.

Contents

Introduction

We consider weak solutions of the quasilinear elliptic problem with singular weights

$$\begin{cases} -div(|x|^{-ap}A(|\nabla u|)\nabla u) = |x|^{-(a+1)p+c}f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary, $1 < p < N$, $0 \in \Omega$, $-\infty < a < \frac{N-p}{p}$, $c > 0$, and the functions A and f satisfy certain conditions.

In the case $A(t) = 1$, observe that equations such as (0.1) are studied as models for several physical phenomena related to an equilibrium of anisotropic media which may somewhere be perfect insulators or perfect conductors (see [14, p. 79]), and they may, in particular, describe standing waves of an anisotropic Schrödinger equation (see [6],[35],[36],[42]). Also, if $A(t) = 1$, the Problem (0.1) is of certain interest in the framework of optimization and $G-$convergence. (See for example [20] and the references therein.)

Degenerate elliptic problems with weights have been intensively studied starting with the pioneering work of M. K. V. Murthy and G. Stampacchia [29]. (See for example [11],[18],[30],[37].) In the special case where $A(t) = t^{p-2}$ for some $p > 1$, the differential operator becomes a weighted $p-$Laplacian.

In the radial case, that is, when $u = u(|x|)$, ordinary differential equations methods apply, and many results about existence, non–existence and asymptotic behavior of solutions are available (see for example [12],[13]). For problems with weights other than powers of $|x|$, see [3],[21],[22].

In the non–radial case, some progress has also been made in the case $A(t) = t^{p-2}$, with $1 < p < N$, in recent years. In 2001, the equation (0.1) for $p = 2$, that is, the weighted Laplacian case, where the nonlinearity is a power of u on $\mathbb{R}^N$, is studied by F. Catrina and Z. Wang in [9]. They obtain existence of solutions within a prescribed symmetry group. In 2003, also in the weighted Laplacian case, the fact that the solutions of Problem (0.1) are Hölder continuous in bounded domains Ω with smooth boundary, provided that the nonlinearity has subcritical growth is proved by V. Felli and M. Schneider in [19]. That same year, existence of solutions, in the sense of entropy, of Problem (0.1) with $1 < p < N$, where the nonlinearity satisfies various structural assumptions, is studied by A. Abdellaoui and I. Peral in [1]. Blow–up fenomena of the solutions are also discussed.

Problem (0.1) is also studied under various aspects in the weighted p–Laplacian case by B. Xuan. In 2003, using the Mountain Pass Theorem and linking arguments, the existence of a solution of Problem (0.1) for a special type of nonlinearity f is proved in [44]. Subsequently, existence and multiplicity of solutions for an asymptotically linear f is shown in [45]. In 2004, the eigenvalue problem associated to Problem (0.1), with $1 < p < N$ and $a \geq 0$, is studied in [44]. In particular, he shows that the first eigenvalue is simple and that the first eigenfunctions do not change sign.

Finally, we mention that existence results for equations associated to a weighted p–Laplacian operator with general singular weights have been recently studied by M.–F. Bidaut–Veron and M. García–Huidobro in [3].

In this work we study Problem (0.1) in various situations. We will assume that the functions A and f satisfy the following conditions:

(H1) $A \in C(\mathbb{R}^+, \mathbb{R})$ and $A(t)t$ is locally integrable on $\mathbb{R}_0^+$.

(H2) There exist constants $b_1, b_2 > 0$ satisfying

$$b_1 \leq \liminf_{t \to +\infty} t^{2-p} A(t) \leq \limsup_{t \to +\infty} t^{2-p} A(t) \leq b_2 .$$

(H3) The mapping $t \mapsto tA(t)$ is strictly increasing and $\lim_{t \to 0^+} A(t)t = 0$.

(f1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$.

(f2) There exist non–negative constants a_1 and a_2, and $q \in (1, r)$, where $r = \min\{Np/(N-p), p(N-(a+1)p+c)/(N-p(a+1))\}$, such that

$$|f(x,t)| \leq a_1 + a_2 |t|^{q-1}.$$

Typical examples of the function A of the differential operator in equation (0.1) are the following:

(i) $A(t) = b_1 t^{p-2} + c_1 t + d t^{q-2}$, where $1 < q < p$, $b_1 > 0$ and $c_1, d \geq 0$.

(ii) $A(t) = (1+t^q)^{\frac{p}{q}-1} t^{q-2}$, where $1 < q < p$.

(iii) $A(t) = (1+|t|)^{-1/2}/ln(1+|t|^{1-q})$, where $1 < q < p - 1/2$.

(iv) $A(t) = \left(1 + \frac{1}{(1+|t|^p)^p}\right) t^{p-2}$, where $p \geq 2$.

Our aim is to obtain solutions of Problem (0.1) as critical points of the energy functional

$$I(u) = \frac{1}{p}\int_\Omega |x|^{-ap} S(|\nabla u|^p) - \int_\Omega |x|^{-(a+1)p+c} F(x,u)$$

where $S(t) = p\int_0^{t^{(1/p)}} A(s)s\, ds$ and $F(x,t) = \int_0^t f(x,s)ds$.

Note first that well known techniques, like the Mountain Pass Theorem, ensure both existence and multiplicity of positive nodal solutions of problems of type (0.1) in the unweighted case, that is, when $a = 0$. (See for example [2],[33], [24],[23], [16], [25], [40], [41].) If $a \neq 0$, then we cannot apply classical methods directly. In this case, the analysis of existence, multiplicity and regularity of the solutions becomes a delicate matter due to the degenerate character of the differential equation.

The following integral inequality due to Caffarelli, Kohn and Nirenberg plays a central role in our variational approach to equation (0.1).

$$\Big(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q\, dx\Big)^{p/q} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\, dx \tag{0.2}$$

where

$$\begin{aligned} -\infty < a < \frac{N-p}{p}, &\quad \text{for } a \leq b \leq a+1, \\ q = p^*(a,b) = \frac{Np}{N-dp}, &\quad \text{for } d = 1+a-b. \end{aligned} \tag{0.3}$$

(See [8].)

Let us now introduce some weighted function spaces we will work with. We let $W_0^{1,p}(\Omega, |x|^{-ap})$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \Big(\int_\Omega |x|^{-ap}|\nabla u|^p\, dx\Big)^{1/p}.$$

It follows from the boundedness of Ω and a standard approximation argument that, for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$, inequality (0.2) holds, in the sense that, for $1 \le r \le \dfrac{Np}{N-p}$ and $\alpha \le (1+a)r + N(1-\frac{r}{p})$, we have

$$\Big(\int_\Omega |x|^{-\alpha}|u|^r\,dx\Big)^{p/r} \le C\int_\Omega |x|^{-ap}|\nabla u|^p\,dx \tag{0.4}$$

or in other words, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted L^r–space endowed with the norm

$$\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega,|x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha}|u|^r\,dx\Big)^{1/r}.$$

The Thesis is organized as follows:

In Chapter 1, we study the differentiability of our functional I. Furthermore, imposing an Ambrosetti–Rabinowitz type condition on the nonlinearity f, we show that our functional has the Palais–Smale property.

Chapter 2 consists of four sections. In Section 2.1, we show the regularity of solutions. In Section 2.2, we assume that the parameters a, c, r and q satisfy one of the following three conditions:

(i) $a > 0$, $c > p(N-p(a+1))/(N-p)$ and

$$\frac{p(a+1)-c}{a} \le q < p^* = r.$$

(ii) $a \le 0$, $c \ge p(N-(a+1)p)/(N-p)$ and

$$1 < q < p^* = r.$$

(iii) $a < 0$, $0 < c < p(N-(a+1)p)/(N-p)$ and

$$1 < q < \frac{p(N-(a+1)p+c)}{N-(a+1)p} = r.$$

These conditions, together with some restrictions on the nonlinearity f, ensure the existence of an unbounded sequence of solutions of Problem (0.1). Section 2.3 is devoted to the parameter dependent problem

$$\begin{cases} -div(|x|^{-ap}A(|\nabla u|)\nabla u) = \lambda |x|^{-(a+1)p+c} f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We show the existence of positive solutions under certain conditions on the functions A and f, and we study their multiplicity and behavior as λ tends to 0. Section 2.4 treats the non–variational problem

$$\begin{cases} -div(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $0 \leq a < (N-2)/2$ and $c > 1$. In other words, when $A = 1$ and $p = 2$ in Problem (0.1). Since the nonlinearity f depends on ∇u, we cannot deal Problem (0.1) directly with variational methods. Our approach is based on an idea of De Figueiredo–Girardi–Matzeu (see [16], and compare [25]) for an equation involving the Laplacian. The idea consists in analyzing a family of associated elliptic equations without dependence on the gradient. More precisely, given $w \in W_0^{1,2}(\Omega, |x|^{-2a})$, we consider the problem

$$\begin{cases} -\text{div}\,(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x,u,\nabla w) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where w is a given Lipschitz continuous function. We then show that the problem above has a solution u which is again Lipschitz continuous. Combining truncation techniques, the Mountain Pass Theorem and monotone iteration, we obtain the existence of a non–trivial solution of the original problem.

In Chapter 3, we study non–existence of solutions using a variant of Pohozaev's identity due to P. Pucci and J. Serrin. (See [32].)

The Thesis concludes with an Appendix that contains the proofs of some technical results used in Chapter 1.

Chapter 1

Preliminaries

1.1 Sobolev space with weights

We first establish some notation.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^1 boundary. If $\alpha \in \mathbb{R}$ and $l \geq 1$, we let $L^l(\Omega, |x|^{-\alpha})$ denote the weighted L^l−space endowed with the norm

$$\|u\|_{l,\alpha} := \|u\|_{L^l(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^l dx\right)^{1/l}.$$

In particular, in the space $L^2(\Omega, |x|^{-2\alpha})$ we will also work with the weighted scalar product

$$\langle f, g\rangle_{L^2(\Omega,|x|^{-2\alpha})} = \int_\Omega |x|^{-2\alpha} f(x)g(x)\, dx.$$

If $1 \leq p \leq N$ and $-\infty < a < (N-p)/p$, then $W_0^{1,p}(\Omega, |x|^{-ap})$ will denote the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$, defined by

$$\|u\| = \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}.$$

Next, we next point out a Caffarelli–Kohn–Nirenberg type inequality without proof. (See [8] for a proof.)

Assume that $1 \leq l \leq p^* := Np/(N-p)$ and that $\alpha \leq (1+a)l + N(1 - (l/p))$. Then, for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$, the following inequality holds:

$$\left(\int_\Omega |x|^{-\alpha}|u|^l dx\right)^{p/l} \leq C\int_\Omega |x|^{-ap}|\nabla u|^p dx\,. \tag{1.1}$$

In other words, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is continuous.

The following compactness theorem is due to B. Xuan (compare [44], [45]). For the sake of completeness, we will give its proof in the Appendix.

Theorem 1.1 (Compact embedding theorem). *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, where $1 < p < N$, $-\infty < a < (N-p)/p$, $1 \leq l < Np/(N-p)$ and $\alpha < (1+a)l + N(1 - (l/p))$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is compact.*

Finally, we study the eigenvalue problem

$$\begin{cases} Lu := -div(|x|^{-2a}\nabla u) = \lambda |x|^{-2a} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

together with the (its) associated bilinear form
$B : W_0^{1,2}(\Omega, |x|^{-2a}) \times W_0^{1,2}(\Omega, |x|^{-2a}) \longrightarrow \mathbb{R}$ given by

$$B[u, v] = \int_\Omega |x|^{-2a} \nabla u \nabla v.$$

Theorem 1.2. *(i) The operator L has a real discrete spectrum. Repeating each eigenvalue according to its (finite) multiplicity, we have*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

and $\lambda_k \to \infty$ as $k \to \infty$.

(ii) For $k \in \mathbb{N}$, there exists an orthonormal basis $\{\varphi_k\}_{k \geq 1}$ of $L^2(\Omega, |x|^{-2a})$ so that

$$\begin{cases} L\varphi_k = \lambda_k \varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $\varphi_k \in W_0^{1,2}(\Omega, |x|^{-2a})$ is an eigenfunction corresponding to the eigenvalue λ_k.

The proofs of Theorems 1.1 and 1.2 will be given in the Appendix.

1.2 Differentiable functionals

In this section we recall definitions and notation from differentiability, and we prove two preliminary results.

Let X be a Banach space, and let X' denote its dual. If $f \in X'$ and $u \in X$, we let $\langle f, u \rangle$ denote the value of f in u.

Definition 1.1. *Let $\varphi : U \to \mathbb{R}$ be a functional where U is an open subset of a Banach space X. We will say that the functional φ has a* **Gateaux derivative** *$f \in X'$ at $u \in U$ if, for every $h \in X$, we have*

$$\lim_{t\to 0} \frac{1}{t}[\varphi(u+th) - \varphi(u) - \langle f, th\rangle] = 0.$$

The Gateaux derivative at u is denoted by $\varphi'(u)$.

We will say that the functional φ has a **Fréchet derivative** *$f \in X'$ at $u \in U$ if*

$$\lim_{h\to 0} \frac{1}{\|h\|}[\varphi(u+h) - \varphi(u) - \langle f, h\rangle] = 0.$$

The functional φ belongs to $C^1(U,\mathbb{R})$ if the Fréchet derivative of φ exists and is continuous on U.

Remark 1.1.

- *The Gateaux derivative is given by*

$$\langle \varphi'(u), h\rangle := \lim_{t\to 0} \frac{1}{t}[\varphi(u+th) - \varphi(u)].$$

- *Any Fréchet derivative is a Gateaux derivative.*

It follows easily from the mean value theorem that:

Proposition 1.1. *If φ has a continuous Gateaux derivative on U, then $\varphi \in C^1(U,\mathbb{R})$.*

We next consider the functional $\psi : W_0^{1,p}(\Omega, |x|^{-ap}) \to \mathbb{R}$ given by

$$\psi(u) = \int_\Omega |x|^{-(a+1)p+c} F(x,u)\, dx$$

where $F(x,t) = \int_0^t f(x,s)\, ds$, the domain Ω is bounded in $\mathbb{R}^N$, for $N \geq 3$, $-\infty < a < (N-p)/p$, $1 < p < N$ and $c > 0$.

Proposition 1.2. *Suppose that f satisfies assumptions (f1) and (f2). Then the functional ψ is of class $C^1(W_0^{1,p}(\Omega, |x|^{-ap}), \mathbb{R})$, and its derivative in $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ is given by*

$$\langle \psi'(u), h\rangle = \int_\Omega |x|^{-(a+1)p+c} f(x,u) h\, dx$$

for any $h \in W_0^{1,p}(\Omega, |x|^{-ap})$.

Proof. **Existence of the Gateaux derivative.** Let $u, h \in W_0^{1,p}(\Omega, |x|^{-ap})$. Given $x \in \Omega$ and $0 < |t| < 1$, according to the mean value theorem, there exists $\lambda \in (0,1)$ so that

$$\begin{aligned}\frac{|F(x,u(x)+th(x)) - F(x,u(x))|}{|t|} &= |f(x,u(x)+\lambda th(x))h(x)| \\ &\leq C(1+(|u(x)|+|h(x)|)^{q-1})|h(x)| \\ &\leq C(1+2^{q-1}(|u(x)|^{q-1}+|h(x)|^{q-1}))|h(x)|.\end{aligned}$$

Hölder's inequality then implies

$$(1+2^{q-1}(|u(x)|^{q-1}+|h(x)|^{q-1}))|h(x)| \in L^1(\Omega, |x|^{-(a+1)p+c}).$$

It follows from Lebesgue's dominated convergence theorem that ψ is Gateaux differentiable and that

$$\langle \psi'(u), h\rangle = \int_\Omega |x|^{-(a+1)p+c} f(x,u)h\, dx\,.$$

Continuity of the Gateaux derivative. Assume that $u_n \to u$ in $W_0^{1,p}(\Omega, |x|^{-ap})$. According to the compact embedding theorem, $u_n \to u$ in $L^q(\Omega, |x|^{-(a+1)p+c})$. Then, for a subsequence again denoted by u_n, we have that $u_n \to u$ a. e. and, for some $g \in L^q(\Omega, |x|^{-(a+1)p+c})$, we have

$$|u(x)|, |u_n(x)| \leq g(x).$$

Therefore,

$$|f(x,u_n) - f(x,u)|^{q'} \leq 2^{q'}C^{q'}(1+|g|^{q/q'})^{q'} \in L^1(\Omega, |x|^{-(a+1)p+c}).$$

According to Lebesgue's dominated convergence theorem, $f(x,u_n) \to f(x,u)$ in $L^{q'}(\Omega, |x|^{-(a+1)p+c})$ where $q' = q/(q-1)$. By Hölder's inequality, we have

$$\begin{aligned}|\langle \psi'(u_n) - \psi'(u), h\rangle| &\leq c\|f(x,u_n) - f(x,u)\|_{L^{q'}(\Omega,|x|^{-(a+1)p+c})} \cdot \|h\|_{L^q(\Omega,|x|^{-(a+1)p+c})} \\ &\leq C\|f(x,u_n) - f(x,u)\|_{L^{q'}(\Omega,|x|^{-(a+1)p+c})} \cdot \|h\|,\end{aligned}$$

and hence

$$\|\psi'(u_n) - \psi'(u)\| \leq C\|f(x,u_n) - f(x,u)\|_{L^{q'}(\Omega,|x|^{-(a+1)p+c})} \xrightarrow[n\to\infty]{} 0.$$

□

1.3 The $(S)_+$ Condition

Let $J : W_0^{1,p}(\Omega, |x|^{-ap}) \to \mathbb{R}$ be the functional defined by

$$J(u) = \frac{1}{p}\int_\Omega |x|^{-ap} S(|\nabla u|^p) \tag{1.4}$$

where $S(t) = p\int_0^{t^{(1/p)}} A(v)v\,dv$. Under the conditions $(H2)$ and $(H3)$ it follows that there exist positive constants α_1, α_2, β_1 and β_2 such that for every $t > 0$ we have that

$$A(t)t \leq \alpha_1 + \alpha_2 t^{p-1} \text{ and} \tag{1.5}$$

$$S(t) \geq \beta_1 t - \beta_2. \tag{1.6}$$

Note that if conditions (H1) through (H3) are satisfied, then J is a C^1–functional.

Now we slightly generalize a result of F. Browder [4],[5] in the theory of mappings of class $(S)_+$ of elliptic operators in generalized divergence form.

Lemma 1.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = S(|t|^p)$. Suppose that h is strictly convex and that S satisfies inequalities (1.5) and (1.6). Then J' belongs to the class $(S)_+$. In other words, for all sequences $\{u_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ such that*

$$\begin{cases} u_n \rightharpoonup u \\ \limsup_{n\to+\infty}\langle J'(u_n), u_n - u\rangle \leq 0 \end{cases} \tag{1.7}$$

we have $u_n \to u$.

Proof. Our proof follows [40]. Using (H3), it is not difficult to verify that $\nabla u_n(x) \to \nabla u$ a. e.. We then observe that

$$\langle J'(u_n), u_n - u\rangle = \int_\Omega |x|^{-ap} A(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u).$$

Setting $\gamma_n(x) := |x|^{-ap}\nabla u_n(x)\cdot(\nabla u_n(x) - \nabla u(x))$, we have

$$\langle J'(u_n), u_n - u\rangle = \int_\Omega A(|\nabla u_n|)\gamma_n(x)\chi_{Q_n}(x) + \int_\Omega A(|\nabla u_n|)\gamma_n(x)\chi_{Q_n^c}(x),$$

where $Q_n = \{x \in \Omega \,|\, |\nabla u_n(x)| \leq M\}$ and M is a positive constant. Our aim is to show that

$$\int_\Omega A(|\nabla u_n|)\gamma_n(x)\chi_{Q_n}(x) \xrightarrow[n\to+\infty]{} 0. \tag{1.8}$$

Defining

$$\eta_n(x) = A(|\nabla u_n|)\gamma_n(x)\chi_{Q_n}(x),$$

we have

$$\begin{aligned} &\eta_n(x) \to 0 \qquad \text{in } \Omega, \text{ and} \\ &|\eta_n(x)| \leq A(M)M(M + |\nabla u(x)|). \end{aligned}$$

Hence, using Lebesgue's dominated convergence theorem, we obtain (1.8). Thus, for any $M > 0$, we have

$$\limsup_{n\to+\infty} \langle J'(u_n), u_n - u\rangle = \limsup_{n\to+\infty} \int_\Omega A(|\nabla u_n|)\gamma_n(x)\chi_{Q_n^c}(x). \tag{1.9}$$

Setting

$$\begin{aligned} \Gamma_n^+ &= \{x \in \Omega \,|\, \gamma_n(x) \geq 0\} \text{ and} \\ \Gamma_n^- &= \{x \in \Omega \,|\, \gamma_n(x) < 0\}, \end{aligned}$$

we have by (H3) and (1.6),

$$A(t)t^2 \geq \frac{S(t^p)}{p} \geq \frac{\beta_1}{p}t^p - \frac{\beta_2}{p}.$$

Hence, for M sufficiently large, it follows that

$$\begin{aligned} \int_\Omega A(|\nabla u_n|)\gamma_n\chi_{Q_n^c} &= \int_\Omega A(|\nabla u_n|)\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^+} + \int_\Omega A(|\nabla u_n|)\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^-} \\ &\geq \left(\frac{\beta_1}{p} - \delta_1\right)\int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^+} + \\ &\quad + (\alpha_2 + \delta_2)\int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^-} \end{aligned}$$

where $0 < \delta_1 < \beta_1/p$ and $\delta_2 > 0$. Thus

$$\begin{aligned} \int_\Omega A(|\nabla u_n|)\gamma_n\chi_{Q_n^c} \geq &\left(\frac{\beta_1}{p} - \delta_1\right)\int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c} \\ &+ \left(\alpha_2 + \delta_2 - \frac{\beta_1}{p} + \delta_1\right)\int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^-}. \end{aligned} \tag{1.10}$$

Next we claim

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c}\chi_{\Gamma_n^-} = 0. \tag{1.11}$$

Defining

$$\Upsilon_n = |\nabla u_n|^{p-2}\gamma_n \cdot \chi_{Q_n^c} \cdot \chi_{\Gamma_n^-},$$

it is not difficult to verify that $|\nabla u_n(x)| \le |\nabla u(x)|$ if $x \in \Gamma_n^-$. Hence

$$|\Upsilon_n(x)| \le 2|x|^{-ap}|\nabla u(x)|^p, \qquad \text{for any } x \in \Omega.$$

However, $\Upsilon_n(x) \xrightarrow[n\to+\infty]{} 0$ a. e. in Ω. Thus (1.11) follows by Lebesgue dominated convergence theorem. Using (1.9) and (1.10), we obtain

$$\left(\frac{\beta_1}{p} - \delta_1\right) \limsup_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n^c} \le \limsup_{n\to+\infty} \langle J'(u_n), u_n - u\rangle \le 0.$$

Consequently, since

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\gamma_n\chi_{Q_n} = 0,$$

we have

$$\limsup_{n\to+\infty} \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u) \le 0.$$

This means that

$$\limsup_{n\to+\infty} \langle J_p'(u_n), u_n - u\rangle \le 0$$

where $J_p(u) = \int_\Omega |x|^{-ap}|\nabla u|^p$. Using the monotonicity property of the p–Laplacian, it is then not difficult to verify that J_p' belongs to the class $(S)_+$.

□

1.4 The (PS) Condition

This section discusses the issue of variational integrals of the type

$$I(u) := J(u) - \int_\Omega |x|^{-(a+1)p+c}F(x,u).$$

Definition 1.2. *Let E be a Banach space. Given $C \in \mathbb{R}$, we will say that $I \in C^1(E,\mathbb{R})$ satisfies the* **$(\mathrm{PS})_\mathrm{C}$ condition** *if any sequence $\{u_n\} \subset E$ such that $I(u_n) \to C$ and $I'(u_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence. If $I \in C^1(E,\mathbb{R})$ satisfies the $(PS)_C$ condition for every $C \in \mathbb{R}$, we will say that I satisfies the (PS)* **condition**.

Proposition 1.3. *Suppose that f satisfies assumptions (f1) and (f2), and that $F(x,t) = \int_0^t f(x,s)ds$. Suppose furthermore that J satisfies the hypotheses of Lemma 1.1. Then I satisfies the (PS) condition if every sequence $\{u_n\}$ in $W_0^{1,p}(\Omega, |x|^{-ap})$ such that*

$$|I(u_n)| \leq C \quad \text{and} \quad I'(u_n) \to 0, \tag{1.12}$$

where C is a constant, is bounded.

Proof. Let $C \in \mathbb{R}$, and let $\{u_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ be a sequence such that

$$\begin{cases} I(u_n) \to C & \text{and} \\ I'(u_n) \to 0. \end{cases} \tag{1.13}$$

It suffices to prove that $\{u_n\}$ contains a subsequence which converges in the norm of $W_0^{1,p}(\Omega, |x|^{-ap})$. Since $\{u_n\}$ is bounded, there is a subsequence $\{u_{n_j}\}$ converging weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ to some u.

On the other hand, the second assertion of (1.13) means that, for all $v \in W_0^{1,p}(\Omega, |x|^{-ap})$, we have

$$\left| \int_\Omega |x|^{-ap} A(|\nabla u_{n_j}|) \nabla u_{n_j} \nabla v - \int_\Omega |x|^{-(a+1)p+c} f(x, u_{n_j}) v \right| \leq \varepsilon_{n_j} \|v\|_{W_0^{1,p}(\Omega, |x|^{-ap})},$$

where $\varepsilon_{n_j} \to 0$. Choosing $v = u_{n_j} - u$ and taking limits over subsequences, we obtain

$$\int_\Omega |x|^{-ap} A(|\nabla u_{n_j}|) \nabla u_{n_j} (\nabla u_{n_j} - \nabla u) \to 0,$$

or in other words

$$\lim_{j \to +\infty} \langle J'(u_{n_j}), u_{n_j} - u \rangle = 0.$$

This means, according to Lemma 1.1, that $u_{n_j} \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$. □

In order to guarantee that the functional I satisfy the (PS) condition, we assume on the nonlinearity f the following further Ambrosetti–Rabinowitz type condition:

(f3) There exist $\theta \in \left(\frac{1}{r}, \frac{1}{p}\right)$ and $t_0 \geq 0$ so that, for $|t| \geq t_0$, we have

$$\theta t f(x,t) \geq F(x,t) > 0$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Proposition 1.4. *Suppose that the functions A and f satisfy, respectively, conditions (H1), (H2), (H3) and assumptions (f1), (f2), (f3). Suppose also that the numbers b_1 and b_2 of condition $(H2)$ satisfy $b_2\theta < b_1/p$.*
Then the functional I satisfies the (PS) condition.

Proof. Let $\{u_n\}$ be a sequence satisfying (1.12). According to Proposition 1.3 it suffices to verify that $\{u_n\}$ is bounded.

It follows from (1.12) that

$$\frac{1}{p}\int_\Omega |x|^{-ap} S(|\nabla u_n|^p) - \theta \int_\Omega |x|^{-ap} A(|\nabla u_n|)|\nabla u_n|^2 \\ + \int_\Omega |x|^{-(a+1)p+c}(\theta f(x,u_n)u_n - F(x,u_n)) \le C + \varepsilon_n \theta \|u_n\|_{W_0^{1,p}(\Omega,|x|^{-ap})},$$

where $\varepsilon_n \to 0$.

Using (H2) and the hypothesis that $b_2\theta < b_1/p$, we find constants $\xi_1, \xi_2 > 0$, $c_0 > 0$ and $t_0 \ge 0$ such that

$$\begin{cases} S(t) \ge \xi_1 b_1 t - c_0, & \text{for all } t \in \mathbb{R}^+, \\ t^{\frac{2-p}{p}} A(t^{1/p}) \le \xi_2 b_2, & \text{for all } t \ge t_0, \text{ and} \\ \xi_2 b_2 \theta < \dfrac{\xi_1 b_1}{p}. \end{cases}$$

Hence, it follows from relation (f3) that there exists a constant c_1 such that

$$\left(\frac{\xi_1 b_1}{p} - \xi_2 b_2 \theta\right) \int_\Omega |x|^{-ap} |\nabla u_n|^p \le c_1 + \varepsilon_n \theta \|u_n\|_{W_0^{1,p}(\Omega,|x|^{-ap})}$$

which means that $\{u_n\}$ is bounded. □

Chapter 2

Existence Results

2.1 A regularity result

In this section we henceforth assume the slightly stronger condition $(H2')$ below instead of condition $(H2)$.

$(H2')$ There exist positive constants b_1 and b_2 satisfying $b_1 \le A(t)t^{2-p} \le b_2$, for all $t \ge 0$.

Thus we propose to prove both boundedness and Hölder regularity of the solutions of our Problem (0.1). We will make use of the following two lemmata. The first, Lemma 2.1, is proved in the special case of the p–Laplacian operator with right hand side $f + m(1 + |v|^{p-2}v)$ where $f, m \in L^{N/p}(D)$ (see [26]), although the proof carries over without difficulties to the general case. The second lemma, Lemma 2.2 in the case of the p–Laplacian operator.

Lemma 2.1. *Let D be a bounded domain in $\mathbb{R}^N$, with $1 < p < N$. Let $l : D \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function so that, for some function $\varphi \in L^{N/p}(D)$, we have*

$$|l(y,t)| \le \varphi(y)(1 + |t|^{p-1}), \quad \textit{for all } (y,t) \in D \times \mathbb{R}, \tag{2.1}$$

let $a_i : D \times \mathbb{R}^N \to \mathbb{R}$ be such that $a_i = a_i(y,\xi)$ is measurable in y and continuous in ξ, with $i = 1, \dots, N$ and such that, for all $(y,\xi) \in D \times \mathbb{R}^N$ and some positive numbers k_1 and k_2, the following two inequalities hold.

$$\xi_i a_i(y,\xi) \ \ge \ k_1|\xi|^p \quad \textit{and} \tag{2.2}$$

$$\sqrt{\sum_{i=1}^{N} a_i^2(y,\xi)} \ \le \ k_2|\xi|^{p-1}, \quad \textit{for all } (y,\xi) \in D \times \mathbb{R}^N. \tag{2.3}$$

Furthermore, let $v \in W_0^{1,p}(D)$ satisfy

$$-\frac{\partial}{\partial y_i} a_i(y, \nabla v) = l(y, v) \quad in \ D. \tag{2.4}$$

Then $v \in L^r(D)$, for every $r \geq 1$.

Lemma 2.2. *Let D and a_i, for $i = 1, \dots, N$, be as in Lemma 2.1. Let $v \in W_0^{1,p}(D)$ satisfy*

$$-\frac{\partial}{\partial y_i} a_i(y, \nabla v) = f(y) \quad in \ D \tag{2.5}$$

where $f \in L^\rho(D)$ for some $\rho > N/p$. Then $v \in L^\infty(D) \cap C^{0,\alpha}_{loc}(D)$, for some $\alpha \in (0, 1]$. Moreover, if $\partial D \in C^{0,1}$, then $u \in C^{0,\alpha}(\overline{D})$.

The main result of this section is the following.

Theorem 2.1. *Let $c > 0$, $\beta := p(a + 1) - c$, and*

$$p - 1 < q < \min\left\{\frac{Np}{N-p} - 1\,; p - 1 + \frac{c}{N - p(a+1)}\right\}. \tag{2.6}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function so that, for all $(x, t) \in \Omega \times \mathbb{R}$ and some $c_1 > 0$, we have

$$|g(x,t)| \leq c_1(1 + |t|^q). \tag{2.7}$$

Let $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ satisfy weakly

$$-div(|x|^{-ap} A(|\nabla u|)\nabla u) = |x|^{-\beta} g(x, u) \ \ in \ \Omega. \tag{2.8}$$

Then $u \in L^\infty(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$, for some $\alpha \in (0, 1]$. Moreover, if $\partial\Omega \in C^{0,1}$, then $u \in C^{0,\alpha}(\overline{\Omega})$.

Proof. We introduce new coordinates given by

$$x = |y|^{k-1} y\,, \ \text{ for } x \in \mathbb{R}^N, \quad \text{where } k = \frac{N-p}{N - p(a+1)}.$$

We set $D := \{y : x \in \Omega\}$, $v(y) := u(x)$, $h(y, t) := g(x, t)$, $(x \in \Omega, t \in \mathbb{R})$. Then it is not difficult to see that $v \in W_0^{1,p}(D)$ and that v satisfies weakly

$$-\frac{\partial}{\partial y_i} a_i(y, \nabla v) = k^p |y|^{-\gamma} h(y, v) \ \text{ in } \ D \tag{2.9}$$

where $\gamma = p - c(N-p)/(N-p(a+1))$, and where, for $y \in D$ and $\xi \in \mathbb{R}^N$, we have

$$\begin{aligned} a_i(y,\xi) &= A\left(\frac{|y|^{(1-k)}}{k}B(y,\xi)\right)\cdot\left(\frac{|y|^{(1-k)}}{k}B(y,\xi)\right)^{2-p} \times \\ &\quad \frac{1}{k^{p-1}}B(y,\xi)^{p-2}\left(k^2\xi_i + (1-k^2)\frac{y_i y_j \xi_j}{|y|^2}\right) \end{aligned}$$

and

$$B(y,\xi) := \left\{k^2|\xi|^2 + (1-k^2)\frac{(y_j\xi_j)^2}{|y|^2}\right\}^{1/2}.$$

Note that $k > 0$ and that, for $i = 1, \dots, N$, the functions a_i, satisfy the conditions (2.2) and (2.3), with $k_1 = b_1 \cdot \min\{1,k\}$ and $k_2 = b_2 \cdot \max\{1,k\}$. We will write

$$d(y) := \frac{|y|^{-\gamma}h(y,v(y))}{1+|v(y)|^{p-1}}, \quad \text{for } y \in D.$$

Now it follows from the Sobolev embedding theorem that $v \in L^{Np/(N-p)}(\Omega)$. Since

$$q < p - 1 + \frac{c}{N-p(a+1)},$$

Hölder's inequality yields

$$\begin{aligned} \int_D |d|^{\frac{N}{p}} &\leq c_1 \int_D |y|^{-\frac{\gamma N}{p}}\left(1+|v|^{\frac{N(q-p+1)}{p}}\right) \\ &\leq c_1\left(\int_D |y|^{-\frac{\gamma Np}{p^2-(N-p)(q-p+1)}}\right)^{\frac{p^2-(N-p)(q-p+1)}{p^2}}\left(c_2 + \left(\int_D |v|^{\frac{Np}{N-p}}\right)^{\frac{(N-p)(q-p+1)}{p^2}}\right) \\ &< +\infty \end{aligned}$$

where c_2 is some positive constant. That is, v satisfies the conditions of Lemma 2.1. Hence $v \in L^r(D)$, for every $r \geq 1$, that is, $|y|^{-\gamma}g(y,v(y)) \in L^\rho(D)$, for some $\rho > N/p$. The theorem now follows from Lemma 2.2. □

Let us mention that it seems rather difficult to prove $C^\alpha-$regularity of solutions under the weaker condition $(H2)$. On the other hand, if our differential operator satisfies assumption $(H2')$, Theorem 2.1 shows that the solutions of Problem (0.1) which we obtain in subsequent sections are locally Hölder continuous.

2.2 Problem 1

In this section we study the elliptic problem with singular weights

$$\begin{cases} -div(|x|^{-ap}A(|\nabla u|)\nabla u) = |x|^{-(a+1)p+c}f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary, $2 \le p < N$, $0 \in \Omega$, $-\infty < a < \frac{N-p}{p}$,

$$r = \min\left\{ \frac{Np}{N-p}, \frac{p(N-(a+1)p+c)}{N-p(a+1)} \right\},$$

and $c > 0$. Note that results similar to ours, but without weights, have been obtained in [41].

Main Theorem

Let $k \in \mathbb{N}$. Denote by H_k the finite dimensional space spanned by the first k eigenfunctions, which correspond to the eigenvalues $\lambda_1, \ldots, \lambda_k$, of the singular elliptic equation

$$\begin{cases} -div(|x|^{-2a}\nabla u) = \lambda |x|^{-2a} u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that since $p \ge 2$, we conclude that $W_0^{1,p}(\Omega, |x|^{-ap})$ is a subset of $W_0^{1,2}(\Omega, |x|^{-2a})$. Let W_k be a subspace of $W_0^{1,p}(\Omega, |x|^{-ap})$ such that

$$W_0^{1,p}(\Omega, |x|^{-ap}) = H_k \oplus W_k\,.$$

It follows from the definitions of H_k and W_k that

$$\|u\|^2_{W_0^{1,2}(\Omega,|x|^{-2a})} \ge \lambda_{k+1} \|u\|^2_{L^2(\Omega,|x|^{-2a})}\,, \quad \text{for all} \quad u \in W_k$$

and

$$\lambda_k \|u\|^2_{L^2(\Omega,|x|^{-2a})} \ge \|u\|^2_{W_0^{1,2}(\Omega,|x|^{-2a})}\,, \quad \text{for all} \quad u \in H_k\,.$$

In what follows, we will assume that the parameters a, c, r and q satisfy one of the following three conditions:

(i) $a > 0$, $c > p(N-p(a+1))/(N-p)$ and

$$\frac{p(a+1)-c}{a} \le q < p^* = r.$$

(ii) $a \leq 0\,,\ \ c \geq p(N-(a+1)p)/(N-p)$ and

$$1 < q < p^* = r.$$

(iii) $a < 0\,,\ \ 0 < c < p(N-(a+1)p)/(N-p)$ and

$$1 < q < \frac{p(N-(a+1)p+c)}{N-(a+1)p} = r.$$

The proof of our main result will depend on the following three lemmata.

Lemma 2.3. *For each finite dimensional subspace* $E \subset W_0^{1,p}(\Omega, |x|^{-ap})$, *there exists a number* $R = R(E)$ *so that* $I \leq 0$ *on* $E \setminus B_R$, *where* $B_R = \{v \in W_0^{1,p}(\Omega, |x|^{-ap}) \mid \|v\| < R\}$.

Proof. According to assumption $(f3)$, there exist $k_1 > 0$ and $k_2 \in \mathbb{R}$ so that, for all $t \in \mathbb{R}$, we have

$$F(t) \geq k_1|t|^{1/\theta} + k_2.$$

By condition $(H2)$, there exist constants $\eta_1, \eta_2 > 0$ and $C > 0$ such that

$$-C + \eta_1 t \leq S(t) \leq \eta_2 t + C, \qquad \text{for all } t \geq 0. \tag{2.11}$$

Consequently, for some $\widetilde{C} > 0$, for every $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ we have

$$\begin{aligned} I(u) &= \int_\Omega \left\{ |x|^{-ap}\frac{S(|\nabla u|^p)}{p} - |x|^{-(a+1)p+c}F(u) \right\} \\ &\leq \frac{\eta_2}{p}\int_\Omega |x|^{-ap}|\nabla u|^p + \widetilde{C} - K_1\int_\Omega |x|^{-(a+1)p+c}|u|^{1/\theta}. \end{aligned}$$

Since $p < 1/\theta$ and $\|\cdot\|_{W_0^{1,p}(\Omega,|x|^{-ap})}$, $\|\cdot\|_{L^{(1/\theta)}(\Omega,|x|^{-(a+1)p+c})}$ are equivalent norms in E, we conclude that there exists an $R > 0$ so that

$$I \leq 0 \text{ on } E \setminus B_R.$$

□

Lemma 2.4. *For every* $u \in W_0^{1,p}(\Omega, |x|^{-ap})$, *we have*

$$\|u\|_{L^q(\Omega,|x|^{-(a+1)p+c})} \leq C\|u\|^{\alpha}_{L^2(\Omega,|x|^{-2a})}\|u\|^{1-\alpha}_{W_0^{1,p}(\Omega,|x|^{-ap})}$$

where $\alpha \in (0,1)$.

Proof. Without loss of generality we may assume that $2 < q < r \leq p^*$. Since $q < r \leq p^*$, we conclude that

$$\|u\|_{L^q(\Omega,|x|^{-(a+1)p+c})} \leq \|u\|^{\alpha}_{L^2(\Omega,|x|^{-2a})} \|u\|^{1-\alpha}_{L^{p^*}(\Omega,|x|^{-\beta})} \tag{2.12}$$

by Hölder's inequality, where

$$\beta = \frac{p^*}{q}\left(\frac{(a+1)p - c - q\alpha a}{1-\alpha}\right)$$

and

$$\alpha = \frac{2(Np - (N-p)q)}{q(Np - 2(N-p))}\,.$$

The result now follows from conditions (i) through (iii) and inequalities (2.12) and (1.1). □

Lemma 2.5. *There exist constants $\rho, \beta > 0$ and $k \in \mathbb{N}$ such that $I\big|_{\partial B_\rho \cap W_k} \geq \beta$.*

Proof. According to assumption $(f2)$ and (2.11), we have

$$I(u) \geq \frac{\xi_1}{p}\int_\Omega |x|^{-ap}|\nabla u|^p - C_2\int_\Omega |x|^{-(a+1)p+c}|u|^q - C_3\,.$$

Let $k \in \mathbb{N}$. It follows from Lemma 2.4 and the definition of W_k that, for all $u \in W_k \cap W_0^{1,p}(\Omega, |x|^{-ap})$, we have

$$\|u\|_{L^q(\Omega,|x|^{-(a+1)p+c})} \leq \frac{C_4}{\lambda_{k+1}^{\alpha/2}}\|u\|_{W_0^{1,p}(\Omega,|x|^{-ap})}$$

where C_4 is some positive constant. If $u \in \partial B_\rho$ it follows that

$$I(u) \geq \rho^p\left(\frac{\xi_1}{p} - \frac{C_5}{\lambda_{k+1}^{\alpha q/2}}\rho^{q-p}\right) - C_6$$

where C_5 and C_6 are positive constants. Without loss of generality, we may suppose $q > p$.

Choosing $\rho = \rho_k$ so that

$$\frac{\xi_1}{p} - \frac{C_5}{\lambda_{k+1}^{\alpha q/2}}\rho_k^{q-p} = \frac{\xi_1}{q}\,,$$

we find

$$I(u) \geq \frac{\xi_1}{q}\rho_k^p - C_6\,.$$

Observe that $\rho_k \to +\infty$ as $k \to +\infty$. Hence, there is a $\beta > 0$ so that, for k sufficiently large, we have

$$\frac{\xi_1}{q}\rho_k^p - C_6 \geq \beta\,, \qquad \text{for all } u \in \partial B_\rho \cap W_k\,.$$

□

We are now in a position to prove our main result.

Theorem 2.2. *Suppose that functions A and f satisfy, respectively, conditions (H1), (H2), (H3) and assumptions (f1), (f2), (f3). Assume that one further condition is satisfied:*

(f4) The function F is even.

Suppose also that $p \geq 2$ *and that the numbers* b_1 *and* b_2 *of condition (H2) satisfy* $b_2\theta < b_1/p$. *Finally, suppose that the parameters* a, c, r *and* q *satisfy one of the conditions (i) through (iii).*

Then Problem (2.10) possesses an unbounded sequence of weak solutions.

Remark. Conditions $(H1)$ through $(H3)$ are satisfied for the $p-$Laplacian, that is, $A(t) = |t|^{p-2}$ and $a = 0$. A model case for assumptions $(f1)$ through $(f3)$ is given by $f(t) = a_1(1 + |t|^{q-1})$.

Proof of Theorem 2.2. According to Proposition 1.4, the functional I satisfies the (PS) condition. By Lemmata 2.3 and 2.5, we can apply Theorem 9.12 of [34]. Therefore, I possesses an unbounded sequence of critical values $c_k = I(u_k)$, where u_k is a weak solution of (2.10).
We claim that $\{u_k\}$ is an unbounded sequence of $W_0^{1,p}(\Omega, |x|^{-ap})$. In fact, since $I'(u_k)u_k = 0$, we have

$$\int_\Omega |x|^{-ap} A(|\nabla u_k|)|\nabla u_k|^2 = \int_\Omega |x|^{-(a+1)p+c} f(u_k)u_k\,. \tag{2.13}$$

Now $c_k = I(u_k)$ implies

$$\frac{1}{p}\int_\Omega |x|^{-ap} S(|\nabla u_k|^p) - \int_\Omega |x|^{-(a+1)p+c} F(u_k) = c_k \to +\infty. \tag{2.14}$$

Multiplying by $1/p$ in (2.13) and subtracting from (2.14) yields

$$\begin{aligned} c_k =& \frac{1}{p}\int_\Omega |x|^{-ap}(S(|\nabla u_k|^p) - A(|\nabla u_k|)|\nabla u_k|^2) \\ &+ \int_\Omega |x|^{-(a+1)p+c}\left(\frac{f(u_k)u_k}{p} - F(u_k)\right). \end{aligned} \tag{2.15}$$

According to condition (H3), we have

$$A(t)t^2 \geq \frac{S(t^p)}{p}, \quad \text{for all} \quad t > 0. \tag{2.16}$$

Combining (2.15) with (2.16), we obtain

$$\left(1 - \frac{1}{p}\right) \int_\Omega |x|^{-ap} A(|\nabla u_k|)|\nabla u_k|^2 + \int_\Omega |x|^{-(a+1)p+c} \left(\frac{f(u_k)u_k}{p} - F(u_k)\right) \geq c_k \tag{2.17}$$

which implies that $\{u_k\}$ is unbounded. □

2.3 Problem 2

In this section, we consider the elliptic problem with singular weights

$$(P)_\lambda \qquad \begin{cases} -div(|x|^{-ap} A(|\nabla u|)\nabla u) = \lambda |x|^{-p(a+1)+c} f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, is an open bounded domain with C^1 boundary, $0 \in \Omega$, $1 < p < N$, $-\infty < a < \dfrac{N-p}{p}$, and $c > 0$.

We consider the functional associated with the Problem $(P)_\lambda$

$$I_\lambda(u) = \frac{1}{p} \int_\Omega S(|\nabla u|^p) - \lambda \int_\Omega |x|^{-(a+1)p+c} F(x,u)$$

where

$$S(t) = p \int_0^{t^{\frac{1}{p}}} A(s)sds \qquad \text{and} \qquad F(x,t) = \int_0^t f(x,s)ds.$$

Observe that if conditions $(H1)$, $(H2)$, $(H3)$, $(f1)$ and $(f2)$ are satisfied, then I_λ is a C^1–functional on the space $W_0^{1,p}(\Omega, |x|^{-ap})$. Furthermore, the same conditions imply the following three properties:

(I) The function $t \to S(|t|^p)$ is strictly convex.

(II) There are positive constants c_1, and c_2 satisfying

$$S(t) \leq c_1 t + c_2, \quad \text{for } t \geq 0.$$

(III) There are positive constants b_0 and b_1 such that

$$|F(x,t)| \leq b_0|t|^l + b_1\,, \quad \text{for all} \quad x \in \Omega$$

where $l < r = \min\left\{\frac{Np}{N-p}, \frac{p(N-(a+1)p+c)}{N-p(a+1)}\right\}$.

We will further assume the following condition

(H4) There is a positive constant c_0 such that

$$c_0 t \leq S(t)\,, \text{ for } t \geq 0.$$

We mention that (H4) is, for instance, satisfied if

$$A(|y|)y \sim |y|^{q-2}y \quad \text{near } 0$$

for some $1 < q \leq p$.

In addition, in the superlinear case, we will assume one further Ambrosetti–Rabinowitz type condition:
There exist $\theta \in (1/r, 1/p)$ and $t_0 > 0$ so that, for all $x \in \Omega$, we have

$$(AR)_p \qquad \theta f(x,t)t > F(x,t) > 0\,, \quad \text{for } 0 < t_0 < |t|\,.$$

Under conditions $(H1)$, $(H2)$, $(H3)$, $(f1)$ and $(f2)$, and $(AR)_p$, the functional I_λ satisfies the Palais–Smale condition. (See Section 1.4.)

We mention that equations similar to Problem $(P)_\lambda$, though without weights, have been studied by H. Prado and P. Ubilla. (See [31].)

The proofs of our main results depend essentially on the Mountain Pass Theorem due to Rabinowitz [34] and Ekeland's variational principle [15], which have often been used to obtain existence results. Note that we will proceed similarly to [31].

Main Theorems

Theorem 2.3 (The superlinear case). *Assume conditions $(H1)$ through $(H4)$, $(f1)$, $(f2)$ and $(AR)_p$. Then there exists a positive constant λ^* such that, for any $0 < \lambda < \lambda^*$, there is a non-trivial solution u_λ of Problem $(P)_\lambda$ in $W_0^{1,p}(\Omega, |x|^{-ap})$. Moreover,*

$$\lim_{\lambda \to 0} \|u_\lambda\| = \infty.$$

The proof of the above theorem is obtained by an application of the following two lemmata.

Lemma 2.6. *There are numbers* $\alpha_\lambda, \rho_\lambda > 0$ *satisfying* $\lim_{\lambda\to 0^+} \alpha_\lambda = +\infty$ *and* $I_\lambda(u) > \alpha_\lambda$, *for* $\|u\| = \rho_\lambda$.

Proof. It follows from property (III) and condition (H4) that, for each $u \in W_0^{1,p}(\Omega, |x|^{-ap})$, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{c_0}{p}\int_\Omega |x|^{-ap}|\nabla u|^p - \lambda b_0 \int_\Omega |x|^{-(a+1)p+c}|u|^l - \lambda\widetilde{b_1} \\ &\geq \frac{c_0}{p}\|u\|^p - \lambda\widetilde{b_0}\|u\|^l - \lambda\widetilde{b_1} \end{aligned} \tag{2.18}$$

where $\widetilde{b_0}$ and $\widetilde{b_1}$ are positive constants. Let

$$\|u\| = \lambda^{-\beta} \text{ where } 0 < \beta < \frac{1}{l-p} \cdot$$

Define $\rho_\lambda = \lambda^{-\beta}$. Then

$$I_\lambda(u) \geq \frac{c_0}{p}\lambda^{-\beta p} - \widetilde{b_0}\lambda^{1-l\beta} - \lambda\widetilde{b_1}\,.$$

Note that $p < l$ by condition $(AR)_p$. Defining $\alpha_\lambda = \frac{c_0}{p}\lambda^{-p\beta} - \widetilde{b_0}\lambda^{1-l\beta} - \lambda\widetilde{b_1}$ and $\rho_\lambda = \lambda^{-\beta}$, the Lemma now follows. □

Lemma 2.7. *Let* $v \neq 0$ *in* $W_0^{1,p}(\Omega, |x|^{-ap})$. *Then*

$$\lim_{t\to+\infty} I_\lambda(tv) = -\infty\,.$$

Proof. According to property (II) and condition $(AR)_p$, we have

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{p}\int_\Omega |x|^{-ap}S(t^p|\nabla v|^p) - \int_\Omega |x|^{-(a+1)p+c}F(x,tv) \\ &\leq \frac{c_1t^p}{p}\int_\Omega |x|^{-ap}|\nabla v|^p + \widetilde{c_2} - k_0t^{1/\theta}\int_\Omega |x|^{-(a+1)p+c}|v|^{1/\theta} + k_1 \\ &\leq \frac{c_1t^p}{p}\|v\|^p + \widehat{c_2} - \widetilde{k_0}t^{1/\theta}\|v\|^{1/\theta} \end{aligned}$$

where c_1, $\widehat{c_2}$, $\widetilde{k_0}$ and k_1 are positive constants. The required limit now follows from the fact that $p < \frac{1}{\theta}\cdot$ □

Proof of Theorem 2.3 . The conditions $(H1)$ through $(H4)$, $(f1)$, $(f2)$ and $(AR)_p$ imply that I_λ satisfies the (PS) condition. Hence the preceding lemmata allow us to apply the Mountain Pass Theorem. Thus there exists a non trivial critical point u_λ for I_λ such that

$$I_\lambda(u_\lambda) = c_\lambda \geq \alpha_\lambda. \tag{2.19}$$

Moreover, from properties (II) and (III) we conclude that

$$I(u_\lambda) \leq \frac{c_1}{p}\|u_\lambda\|^p + \widehat{c_2}\,.$$

In view of (2.19) and Lemma 2.6 we then deduce that

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = +\infty.$$

□

Lemma 2.8. *Let X be a Banach space, and let $I : X \to \mathbb{R}$ be a lower semicontinuous functional which is Gateux differentiable. Suppose that I is bounded below on the set $\overline{B(0,\delta)}$ and $\inf\{I(u) \mid u \in \overline{B(0,\delta)}\} < 0$. Suppose further that $I(u) \geq 0$ when $\|u\| = \delta$. Then, for each $0 < \varepsilon < -\inf\{I(u) \mid u \in \overline{B(0,\delta)}\}$, there exists a u_ε such that $\|u_\varepsilon\| < \delta$ and*

(i) $I(u_\varepsilon) \leq \inf\{I(u) \mid u \in \overline{B(0,\delta)}\} + \varepsilon$

(ii) $\|I'(u_\varepsilon)\| \leq \varepsilon$.

Proof. We apply Ekeland's variational principle to the function I restricted to $\overline{B(0,\delta)}$. Hence, for each $\varepsilon > 0$, there exists a point $u_\varepsilon \in \overline{B(0,\delta)}$ so that

$$I(u_\varepsilon) - I(u) \leq \varepsilon\|u - u_\varepsilon\| \tag{2.20}$$

for every $u \in \overline{B(0,\delta)}$ such that $u \neq u_\varepsilon$. Now if $\|u_\varepsilon\| = \delta$, then $I(u_\varepsilon) \geq 0$ and assertion (i) is satisfied. Thus

$$0 \leq I(u_\varepsilon) \leq \inf\{I(u) \mid u \in \overline{B(0,\delta)}\} + \varepsilon.$$

Concerning (ii), since $c = \inf\{I(u) \mid u \in \overline{B(0,\delta)}\}$ so whenever $0 < \varepsilon < -c$ we arrive at a contradiction. Thus $\|u_\varepsilon\| < \delta$. Moreover from inequality (2.20) we obtain assertion (ii). □

Theorem 2.4. *Suppose that $f(x,t) \geq 0$, for all $x \in \Omega$ and $t \geq 0$. Suppose further that, for some $r_0 > 0$, the following two conditions hold:*

(a) $\lim_{t\to 0} \dfrac{F(x,t\sigma)}{F(x,t)} = \sigma^{r_0}$, *for every* $0 < \sigma < 1$ *and all* $x \in \Omega$,

(b) $\lim_{t\to 0} \dfrac{F(x,t)}{S(t^p)} = +\infty$, *for all* $x \in \Omega$.

Then, under conditions $(H1)$ *through* $(H4)$, $(f1)$ *and* $(f2)$, *there exists a positive constant* λ^* *so that, for every* $0 < \lambda < \lambda^*$, *there is a solution* u_λ *of Problem* $(P)_\lambda$ *in* $W_0^{1,p}(\Omega, |x|^{-ap})$. *Moreover,*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Proof. Let $\rho_\lambda = \lambda^\alpha$ for $\alpha > 0$. By inequality of (2.18), for every $u \in B(0,\rho_\lambda)$, we have

$$I_\lambda(u) \geq C_0\lambda^{\alpha p} - k_0\lambda^{1+\alpha l} - \lambda\widetilde{b_1}.$$

Choosing $0 < \alpha < \frac{1}{p}$ and taking λ^* sufficiently small, for $0 < \lambda < \lambda^*$ we find

$$I_\lambda(u) \geq 0$$

whenever $\|u\| = \rho_\lambda = \lambda^\alpha$.

Moreover, it follows from inequality (2.18) that I_λ is bounded below on the set $B[0,\rho_\lambda] = \{u \in W_0^{1,p}(\Omega,|x|^{-ap}) \mid \|u\| \leq \rho_\lambda\}$.

Let $\phi_t = tv$, for $t \geq 0$, where $v \in C_0^\infty(\Omega)$ is such that $0 < v \leq 1$ and $0 \leq |\nabla v| \leq 1$. Then

$$\begin{aligned} I_\lambda(\phi_t) &= \frac{1}{p}\int_\Omega |x|^{-ap} S(t^p|\nabla v|^p) - \lambda \int_\Omega |x|^{-(a+1)p+c} F(x,tv) \\ &= S(t^p)\left[\frac{1}{p}\int_\Omega |x|^{-ap}\frac{S(t^p|\nabla v|^p)}{S(t^p)} - \frac{\lambda}{S(t^p)}\int_\Omega |x|^{-(a+1)p+c} F(x,t)\frac{F(x,tv)}{F(x,t)}\right]. \end{aligned}$$

Since

$$\frac{S(t^p|\nabla v|^p)}{S(t^p)} \leq 1, \qquad \text{for } t > 0,$$

and since

$$\frac{F(x,tv)}{F(x,t)}, \leq 1 \qquad \text{for } t > 0, x \in \Omega,$$

and in view of conditions (a) and (b), the dominated convergence theorem yields the existence of $\delta > 0$ such that

$$I_\lambda(tv) < 0 \qquad \text{whenever } 0 < t < \delta.$$

Thus $c_\lambda = \inf\{I_\lambda(u) \mid u \in \overline{B(0,\delta)}\}$ is negative, and the assumptions of the preceding lemma are verified. Since I_λ satisfies the (PS) condition, there exists a non–trivial minimizer u_λ in the interior of $B(0,\rho_\lambda)$, or in other words a non–trivial weak solution of Problem $(P)_\lambda$. Moreover $\|u_\lambda\| < \lambda^\alpha$, for $0 < \lambda < \lambda^*$. Hence if $\alpha > 0$, then u_λ tends to zero as λ tends to 0^+, which completes the proof. □

Corollary 2.1. *Under the hypotheses of Theorems 2.3 and 2.4, there exist at least two solutions u_λ and v_λ of Problem $(P)_\lambda$ so that*

$$\lim_{\lambda\to 0}\|u_\lambda\| = +\infty \qquad \text{and} \qquad \lim_{\lambda\to 0^+}\|v_\lambda\| = 0.$$

Theorem 2.5 (The sublinear case). *Assume conditions $(H1)$ through $(H4)$, $(f1)$ and $(f2)$. Suppose that $a \geq 0$ and that $l < p$. Suppose further that, for all $x \in \Omega$ and all $0 < \sigma < 1$, the following three conditions hold:*

(i) $\displaystyle\lim_{t\to 0}\frac{F(x,t\sigma)}{F(x,t)} = \sigma^{r_0}$, *where* $r_0 < p$.

(ii) $\displaystyle\liminf_{t\to 0}\frac{F(x,t)}{S(t^p)} > 0.$

(iii) There exist positive constants c_1, c_2 and $r_1 > 0$ satisfying

$$F(x,t) \geq c_1 t^{r_1} - c_2.$$

Then there is a $\lambda^* > 0$ such that, for every $\lambda > \lambda^*$, there exists a solution u_λ of the Problem $(P)_\lambda$ in $W_0^{1,p}(\Omega,|x|^{-ap})$. Moreover, $\lim_{\lambda\to\infty}\|u_\lambda\| = +\infty$.

Lemma 2.9. *Under the hypotheses of Theorem 2.5, for all $\lambda > 0$, the functional I_λ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS)–sequence. According to Proposition 1.3, it suffices to verify that $\{u_n\}$ is bounded. Let λ_1 be the first eigenvalue of the singular quasilinear elliptic equation

$$\begin{cases} -div(|x|^{ap}|\nabla u|^{p-2}\nabla u) = \lambda|x|^{-(a+1)p+c}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

(See [43].)
Since $r < p$, we have

$$\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^p} = 0.$$

Then, for $0 < \varepsilon < \dfrac{c_0\lambda_1}{p\lambda}$, there exists $c_\varepsilon > 0$ so that

$$|F(x,t)| \leq \varepsilon|t|^p + c_\varepsilon\,, \quad \text{for all } t \in \mathbb{R}.$$

Hence if

$$|I_\lambda(u_n)| \le C \quad \text{and} \quad I'_\lambda(u_n) \to 0$$

it follows that

$$C \ge I_\lambda(u_n) \ge \frac{c_0}{p}\|u_n\|^p - \lambda\varepsilon \int_\Omega |x|^{-(a+1)p+c}|u_n|^p - \lambda\widetilde{c_\varepsilon}$$

where $\widetilde{c_\varepsilon}$ is a positive constant. Therefore,

$$\begin{aligned} I_\lambda(u_n) &\ge \frac{c_0}{p}\|u_n\|^p - \frac{\lambda\varepsilon}{\lambda_1}\|u_n\|^p - \lambda\widetilde{c_\varepsilon} \\ &= \left(\frac{c_0\lambda_1 - p\lambda\varepsilon}{p\lambda_1}\right)\|u_n\|^p - \lambda\widetilde{c_\varepsilon}\,, \end{aligned}$$

which means that $\{u_n\}$ is bounded. □

Proof of Theorem 2.5. Since the inequality

$$I_\lambda(u) \ge \frac{c_0}{p}\|u\|^p - \lambda\widetilde{b_0}\|u\|^l - \lambda\widetilde{b_1}$$

holds for some positive constants $\widetilde{b_0}$, $\widetilde{b_1}$ and $l < p$, we conclude that $c_\lambda = \inf\{I_\lambda(u) \mid u \in W_0^{1,p}(\Omega, |x|^{-ap})\} > -\infty$, for every λ, and that there is a minimizer u_λ. According to (ii), there exist $\mu\,, \delta > 0$ so that

$$\frac{F(x,t)}{S(t^p)} \ge \mu \quad \text{whenever } 0 < t < \delta.$$

Choosing $\phi_t = tv$, for t sufficiently small, we find

$$I_\lambda(\phi_t) \le S(t^p)\left(\frac{1}{p}\int_\Omega |x|^{-ap}\frac{S(t^p|\nabla v|^p)}{S(t^p)} - \mu\lambda\int_\Omega |x|^{-(a+1)p+c}\frac{F(x,tv)}{F(x,t)}\right).$$

Thus there exists $\lambda^* > 0$ so that, for $\lambda > \lambda^*$, there is a t_λ for which

$$I_\lambda(t_\lambda v) < 0.$$

Hence $-\infty < c_\lambda < 0$. Hence u_λ is a non–trivial minimizer.

According to condition (iii), we have

$$I_\lambda(u) \le \frac{c_0}{p}\int_\Omega |x|^{-ap}|\nabla u|^p - \lambda c_1\int_\Omega |x|^{-(a+1)p+c}|u|^{r_1}.$$

Fix some $v \in W_0^{1,p}(\Omega, |x|^{-ap})$, and let $u = tv$. Then

$$c_\lambda \le \inf_{t\ge 0}\{Bt^p - \lambda Ct^{r_1}\}$$

where

$$B = \frac{c_0}{p}\int_\Omega |x|^{-ap}|\nabla v|^p dx \qquad \text{and} \qquad C = c_1\int_\Omega |x|^{-(a+1)p+c}|v|^{r_1}.$$

Hence

$$c_\lambda \le B^{-\frac{r_1}{p-r_1}} C^{\frac{p}{p-r_1}} \left[\left(\frac{r_1}{p}\right)^{\frac{p}{p-r_1}} - \left(\frac{r_1}{p}\right)^{\frac{r_1}{p-r_1}}\right] \lambda^{\frac{p}{p-r_1}} .$$

Since $p > r_1$ and $\lambda > 0$, it follows that the right hand side of the inequality is less than zero. Thus

$$c_\lambda \le -k\lambda^{\frac{p}{p-r_1}}, \quad \text{for } k > 0.$$

On the other hand, according to the properties (II), (III) and $(H4)$, we have

$$\begin{aligned} c_\lambda &= I_\lambda(u_\lambda) \\ &= \frac{1}{p}\int_\Omega |x|^{-ap} S(|\nabla u_\lambda|^p) - \lambda \int_\Omega |x|^{-(a+1)p+c} F(x, u_\lambda) \\ &\ge \frac{c_0}{p}\int_\Omega |x|^{-ap}|\nabla u_\lambda|^p - \lambda\left(\widetilde{b_0}\int_\Omega |x|^{-(a+1)p+c}|u_\lambda|^l + \widetilde{b_1}\right). \end{aligned}$$

Therefore,

$$-k\lambda^{\frac{p}{p-r_1}} \ge c_\lambda \ge \frac{c_0}{p}\|u_\lambda\|^p - \lambda(C\|u_\lambda\|^l + \widetilde{b_1})$$

where C is a positive constant. Now, if $\|u_\lambda\|$ is bounded for all $\lambda > 0$, then there is a subsequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and the sequence $\|u_{\lambda_n}\|$ converges as $\lambda_n \to \infty$. Then, by the preceding inequality, dividing by λ_n in it yields

$$-k\lambda_n^{\frac{r_1}{p-r_1}} \ge \frac{c_0}{p\lambda_n}\|u_{\lambda_n}\|^p - (C\|u_{\lambda_n}\|^l + \widetilde{b_1}),$$

passing to the limit as $\lambda_n \to \infty$ we arrive at a contraction. □

Our next result gives necessary conditions which ensure the existence of non–negative solutions of Problem $(P)_\lambda$.

Theorem 2.6. *Let the function f satisfy $(f4)$. Suppose also that $f(x,t) = 0$ for $t \le 0$ and all $x \in \Omega$, and that $f(x,t) > 0$ for $t > 0$ and all $x \in \Omega$. Then if u is a weak solution of Problem $(P)_\lambda$, u is non–negative.*

Proof. Let u be a weak solution of Problem $(P)_\lambda$. Then, for every $\phi \in W_0^{1,p}(\Omega, |x|^{-ap})$, we have

$$\int_\Omega |x|^{-ap} A(|\nabla u|) \nabla u \nabla \phi = \lambda \int_\Omega |x|^{-(a+1)p+c} f(x,u)\phi.$$

Let $u = u^+ + u^-$ and take $\phi = u^-$. Then

$$-\int_\Omega |x|^{-ap} A(|\nabla u^-|)|\nabla u^-|^2 = \lambda \int_\Omega |x|^{-(a+1)p+c} f(x,-u^-)u^- = 0.$$

Hence $|\nabla u^-| = 0$. Therefore $u \geq 0$ a.e. in Ω. □

Corollary 2.2. *Assume that f is non-negative and that it satisfies the hypotheses of Theorems 2.3 and 2.4. Then there exist at least two nonnegative solutions u_λ and v_λ of $(P)_\lambda$ such that*

$$\lim_{\lambda \to 0} \|u_\lambda\| = +\infty \qquad \text{and} \qquad \lim_{\lambda \to 0+} \|v_\lambda\| = 0.$$

Corollary 2.3. *Let f be such that $f(x,t) \geq 0$, for all $t > 0$ and all x. Then under the hypotheses of Theorem 2.5, there is a $\lambda^* > 0$ such that, for $\lambda > \lambda^*$, there exists a non-negative solution u_λ and*

$$\lim_{\lambda \to +\infty} \|u_\lambda\| = \infty.$$

2.4 Problem 3

We consider the problem

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.21}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary such that $0 \in \Omega$, $0 \leq a < \frac{N-2}{2}$ and $c > 1$. Since the nonlinearity f depends on ∇u, we cannot deal Problem (2.21) directly with variational methods. Our approach is based on an idea of De Figueiredo–Girardi–Matzeu for an equation involving the Laplacian. The idea consists in analyzing a family of associated elliptic equations without dependence on the gradient. Combining truncation techniques, the Mountain Pass Theorem and monotone iteration, we obtain the existence of a non–trivial solution. Note that, in applying the preceding techniques, a proof of Lipschitz and higher regularity of the solutions which occur in the iteration is required. (See Subsection 2.4.1.) More precisely, given $w \in W_0^{1,2}(\Omega, |x|^{-2a})$, we consider the problem

$$\begin{cases} -\text{div}\,(|x|^{-2a}\nabla u) = |x|^{-2(a+1)+c} f(x,u,\nabla w) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.22}$$

We assign the following hypotheses on the nonlinearity f:

(f_0) $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable, and $f(x,\cdot,\cdot)$ is locally bounded and Lipschitz continuous on $\mathbb{R} \times \mathbb{R}^N$, uniformly in x.

(f_1) $\displaystyle\lim_{t\to 0} \frac{f(x,t,\xi)}{t} = 0$ uniformly for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$.

(f_2) $|f(x,t,\xi)| \le a_1(1+|t|^p)(1+|\xi|^r)$, for all $(x,t,\xi) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N$, for some constants $a_1 > 0$, $1 < p < \min\left\{\dfrac{N+2}{N-2}, \dfrac{N-2(a+1)+2c}{N-2(a+1)}\right\}$ and $r \in (0,1)$.

(f_3) $0 < \theta F(x,t,\xi) \le t f(x,t,\xi)$ for all $x \in \overline{\Omega}$, $|t| \ge t_0$, $\xi \in \mathbb{R}^N$, for some constants $\theta > 2$ and $t_0 > 0$, where $F(x,t,\xi) = \int_0^t f(x,s,\xi)ds$.

We note that (f_3) implies that there exist constants $a_2, a_3 > 0$ such that

$$F(x,t,\xi) \ge a_2|t|^\theta - a_3, \quad \text{for all } x \in \overline{\Omega},\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N. \tag{2.23}$$

The hypotheses above allow us to apply Ambrosetti and Rabinowitz's Mountain Pass Theorem to equation (2.22). (Compare [2].) The solvability of Problem (2.21) is then ensured if the function f satisfies two local Lipschitz conditions given in hypothesis (f_4) below.

(f_4)
- $|f(x,t',\xi) - f(x,t'',\xi)| \le L_1|t'-t''|$, for all $x \in \overline{\Omega}$ and all $t', t'' \in [0,\rho_1]$, and $|\xi| \le \rho_2$, and
- $|f(x,t,\xi') - f(x,t,\xi'')| \le L_2|\xi'-\xi''|$, for all $x \in \overline{\Omega}$ and all $t \in [0,\rho_1]$, and $|\xi'|, |\xi''| \le \rho_2$

where ρ_1 and ρ_2 depend on $p, N, \theta, a_1, a_2, a_3$ of hypotheses (f_2) and (f_3).

The case $l = 2$ of inequality (1.1) will require special attention in our analysis: Consider the weighted Rayleigh quotient

$$Q_{a,c}(v) := \frac{\int_\Omega |x|^{-2a}|\nabla v|^2\,dx}{\int_\Omega |x|^{-2(a+1)+c}v^2\,dx}, \quad \text{for } v \in W_0^{1,2}(\Omega, |x|^{-2a}), \text{ with } v \ne 0,$$

where $a \in (-\infty, (N-2)/2)$ and $c \geq 0$. Set

$$S(\Omega, a, c) := \inf\{Q_{a,c}(v) : v \in W_0^{1,2}(\Omega, |x|^{-2a}),\ v \neq 0\}. \tag{2.24}$$

Then, if $c > 0$ it follows that $S(\Omega, a, c)$ is equal to the first eigenvalue of the problem

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) = \lambda |x|^{-2(a+1)+c} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.25}$$

and that it is attained for any first eigenfunction of Problem (2.25). (See [43].)
Now if $c = 0$ it follows that $S(\mathbb{R}^N, a, 0) = ((N-2-2a)/2)^2$, but the infimum in (2.24) is not attained (see [9]). Therefore, it is not hard to see that $S(\Omega, a, 0) = ((N-2-2a)/2)^2$ as well and that $S(\Omega, a, 0)$ is not attained.

We are now in a position to formulate our main result, which will be proved in Subsection 2.4.2.

Theorem 2.7. *Let Ω be a C^1–domain, $0 < a < (N-2)/2)$ and $c \geq 1$. Suppose f satisfies (f_0) trough (f_4). Then Problem (2.21) has both a positive and a negative solutions in $W_0^{1,2}(\Omega, |x|^{-2a})$, provided that*

$$\frac{L_1}{S(\Omega, a, c)} + \frac{L_2}{\sqrt{S(\Omega, a, 2(c-1))}} < 1. \tag{2.26}$$

2.4.1 A high regularity result

In this subsection, we obtain regularity properties for the solutions of Problem (2.22). We show boundedness and smoothness for these solutions. We consider the problem

$$\begin{aligned} &u \in W_0^{1,2}(\Omega, |x|^{-2a}), \\ &-\mathrm{div}(|x|^{-2a}\nabla u) = |x|^{-2a-2+c} f(x) \quad \text{in } \Omega, \end{aligned} \tag{2.27}$$

where $c \geq 1$ and $f \in L^\infty(\Omega)$. Note first that if $c > 0$ then a result of [19], Theorem 1.1, tells us that u is bounded and that $u \in C^{0,\alpha}(\Omega')$ for some $\alpha \in (0,1)$ and every $\Omega' \subset\subset \Omega$. Our proof is based on a blow–up argument used by Gidas and Spruck in [23]. Further, it requires the following Liouville type result.

Theorem 2.8. *Let $a \in (-\infty, (N-2)/2)$ and*

$$m_1 = -\frac{N-2}{2} + a + \sqrt{\left(\frac{N-2}{2} - a\right)^2 + N - 1}\,. \tag{2.28}$$

Then, if $u \in W^{1,2}_{loc}(\mathbb{R}^N, |x|^{-2a})$ *satisfies*

$$-div(|x|^{-2a}\nabla u) = 0 \quad and \tag{2.29}$$

$$|u(x)| \le C(1 + |x|^{m_1-\varepsilon}) \quad on\ \mathbb{R}^N \tag{2.30}$$

for some $C > 0$ *and* $\varepsilon \in (0, m_1)$, *u is constant on* $\mathbb{R}^N$.

Proof. We let (r, θ) denote $N-$dimensional polar coordinates, ($r = |x|$, $\theta \in \mathcal{S}^{N-1}$). Let $\{v_n\}$ be the sequence of orthonormal eigenfunctions for the Laplace–Beltrami operator on $\mathcal{S}^{N-1}$, or in other words

$$-\Delta_\theta v_k = \lambda_k v_k \quad \text{on } \mathcal{S}^{N-1}, \quad \text{for } k = 0, 1, 2, \ldots, \tag{2.31}$$

$$\int_{\mathcal{S}^{N-1}} v_i v_j d\theta = \delta_{ij}, \quad \text{for } i, j = 0, 1, 2, \ldots, \quad \text{and} \tag{2.32}$$

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots. \tag{2.33}$$

Note that $\lambda_0 = 0$, $v_0 = \text{const.} \neq 0$, $\lambda_1 = \ldots = \lambda_N = N - 1$, $v_k = Cx_k/|x|$, $k = 1, \ldots N$, for some $C > 0$, and the eigenvalues $\lambda (= \lambda_n)$ can be calculated from the relation

$$\lambda = n^2 + n(N-2), \qquad \text{for } n = 0, 1, 2, \ldots.$$

Let $R > 0$, and let $b_k(R)$, for $k = 0, 1, \ldots$, be (unique!) numbers so that

$$u(R, \theta) \;=\; \sum_{k=0}^{+\infty} b_k(R) v_k(\theta), \qquad \text{for all } \theta \in \mathcal{S}^{N-1}. \tag{2.34}$$

Then u has the representation

$$u(r, \theta) = \sum_{k=0}^{+\infty} b_k(R) r^{m_k} v_k(\theta), \qquad \text{for all } r \in [0, R], \text{ and all } \theta \in \mathcal{S}^{N-1} \tag{2.35}$$

where $m_k = -\frac{N-2}{2} + a + \sqrt{\left(\frac{N-2}{2} - a\right)^2 + \lambda_k}$. (See [1, Theorem 4.4, proof].) Since $R > 0$ is arbitrary, for some numbers $c_k \in \mathbb{R}$, with $k = 0, 1, 2, \ldots$, we conclude that

$$b_k(R) = c_k R^{m_k} \tag{2.36}$$

by (2.34) and (2.35). According to Parseval's identity on ∂B_R and assumption (2.30), for some $C > 0$, we find

$$C(1 + R^{2m_1 - 2\varepsilon}) \ge \int_{\mathcal{S}^{N-1}} u^2(R, \theta)\, d\theta = \sum_{k=0}^{+\infty} c_k^2 R^{2m_k}, \qquad \text{for all } R > 0. \tag{2.37}$$

Taking the limit as $R \to +\infty$, we obtain $c_k = 0$, for $k \ge 1$. Hence u is constant on $\mathbb{R}^N$. □

Lemma 2.10. *Let $a \in (-\infty, (N-2)/2)$, $c > 0$, $f \in L^\infty(\Omega)$, and let u be a solution of (2.27). Then, for every $\delta > 0$ satisfying $\delta \leq c$ and $\delta < m_1$, there exists a number $c_1 > 0$, which depends only on δ, c, a, N and Ω, so that*

$$|u(x) - u(0)| \leq c_1 M |x|^\delta, \qquad \text{for all } x \in \Omega \tag{2.38}$$

where $M := \|f\|_{L^\infty(\Omega)}$.

Proof. First assume that $M = 1$. Suppose that (2.38) is wrong. Then there is a $\delta > 0$, with $\delta \leq c$ and $\delta < m_1$, and a sequence $\{x_n\} \subset \Omega \setminus \{0\}$, with $x_n \to 0$, such that

$$\lim_{n\to\infty} |u(x_n) - u(0)||x_n|^{-\delta} = +\infty. \tag{2.39}$$

Define rotations ρ_n of the coordinate system about the origin such that $\rho_n x_n = (\varepsilon_n, 0, \ldots, 0) =: y_n$, $(\varepsilon_n > 0)$, and let $\Omega_n := \rho_n \Omega$, $f_n(x) := f(\rho_n x)$, $u_n(x) := u(\rho_n x)$, $n = 1, 2, \ldots$. We may assume without loss of generality that $\{\varepsilon_n\}$ is decreasing and that

$$|u_n(x) - u_n(0)||x|^{-\delta} \leq |u_n(y_n) - u_n(0)|\varepsilon_n^{-\delta} \quad \text{for all } x \in \Omega_n \text{ such that } |x| \geq \varepsilon_n. \tag{2.40}$$

Set $D_n := \{(1/\varepsilon_n)x : x \in \Omega_n\}$, $g_n(x) := f_n(\varepsilon_n x)$, and

$$v_n(x) := \frac{u_n(\varepsilon_n x) - u_n(0)}{u_n(y_n) - u_n(0)},$$

we would find $v_n(0) = 0$, and $v_n(e) = 1$, where e is the unit vector $(1, 0, \ldots, 0)$,

$$|v_n(x)| \leq |x|^\delta \ \text{ in } \ D_n \setminus B_1, \tag{2.41}$$

$v_n \in W_0^{1,2}(D_n, |x|^{-2a})$, and

$$-\text{div}(|x|^{-2a}\nabla v_n) = \frac{|x|^{-2a-2+c} g_n(x) \varepsilon_n^\delta}{u_n(\varepsilon_n e) - u_n(0)} =: h_n(x) \quad \text{in} \quad D_n. \tag{2.42}$$

According to (2.39), we would have

$$\lim_{n\to\infty} \frac{\varepsilon_n^\delta}{u_n(\varepsilon_n e) - u_n(0)} = 0$$

so that

$$\lim_{n\to\infty} h_n(x) = 0 \quad \text{uniformly in any compact subset of } \mathbb{R}^N. \tag{2.43}$$

Furthermore, using elliptic estimates separately in B_1 and in $D_n \setminus B_1$, we would find that the v_n's are uniformly bounded and that $v_n \in C^{0,\alpha}(D')$, for some $\alpha \in (0,1)$ and every $D' \subset\subset D_n$. (See [19].) Hence in view of relations (2.41)–(2.43), there would be a subsequence $\{v_{n'}\}$ and a function $v \in W^{1,2}_{loc}(\mathbb{R}^N, |x|^{-2a}) \cap C^{0,\alpha}(\mathbb{R}^N)$ such that

$$v'_n \longrightarrow v \text{ in } W^{1,2}(B_R, |x|^{-2a}) \text{ and in } C^{0,\alpha}(B_R), \text{ for all } R > 0, \quad (2.44)$$

$$\operatorname{div}(|x|^{-2a}\nabla v) = 0 \text{ on } \mathbb{R}^N, \quad (2.45)$$

$$|v(x)| \leq |x|^{\delta} \text{ for } |x| \geq 1, \text{ and} \quad (2.46)$$

$$v(0) = 0, \ v(e) = 1. \quad (2.47)$$

By the preceding Theorem 2.8, conditions (2.45) and (2.46) would imply that v must be constant, contrary to (2.47).
In the general case, the result follows from the above analysis replacing u by $M^{-1}u$. □

Remark. Let $R_0 > 0$ be such that $B_{R_0} \subset \Omega$, and let $u_1(x) = |x|^c$ and $u_2(x) = x_1|x|^{m_1-1}$ in B_{R_0}. Note that, for some $k \in \mathbb{R}$, we have

$$-\operatorname{div}(|x|^{-2a}\nabla u_1) = k|x|^{-2a-2+c} \text{ and } -\operatorname{div}(|x|^{-2a}\nabla u_2) = 0 \text{ in } B_{R_0}.$$

Clearly we may extend u_1 and u_2 to functions in $C^2(\overline{\Omega} \setminus \{0\})$ with compact support in Ω such that u_i is a solution of Problem (2.27) with right–hand side $|x|^{-2a-2+c_i}f_i(x)$, where $f_i \in L^\infty(\Omega)$, with $i = 1, 2$, and $c_1 = c$, $c_2 = 2$. These examples show that estimate (2.38) with $\delta > m_1$ or with $\delta > c$ does not in general hold.

We next prove the main results of this section.

Theorem 2.9. *Let $0 < a < (N-2)/2$, $c > 1$ and $f \in L^\infty(\Omega)$. Let u be a solution of Problem (2.27). Then $u \in C^{1,\beta}(\Omega)$, for every $\Omega' \subset\subset \Omega$ and every $\beta \in (0,1)$, with $\beta \leq c-1$ and $\beta < m_1$. Moreover, for every such β and Ω', there is a constant c_2 depending only on c, β, a, and Ω', such that*

$$\|u\|_{C^{1,\beta}(\Omega')} \leq c_2 M \quad (2.48)$$

where $M := \|f\|_{L^\infty(\Omega)}$. Finally, if Ω is a $C^{1,\beta}$–domain, then $u \in C^{1,\beta}(\overline{\Omega})$ and (2.48) holds with Ω' replaced by $\overline{\Omega}$.

Proof. As in the proof of the last lemma, we may assume that $M = 1$. First observe that standard regularity theory tells us that

$$u \in C^{1,\alpha}(\Omega' \setminus \overline{B_\varepsilon}) \quad \text{for every } \Omega' \subset\subset (\Omega \backslash \{0\}) \text{ and for all } \alpha \in (0,1). \quad (2.49)$$

(see e.g. [24].) Moreover, if Ω is a $C^{1,\beta}$−domain, then we have

$$u \in C^{1,\beta}(\overline{\Omega} \setminus B_\varepsilon), \qquad \text{for all } \varepsilon > 0. \tag{2.50}$$

Let $\delta \in (1, c]$, with $\delta < m_1$, $\varepsilon_0 > 0$ such that $B_{4\varepsilon_0} \subset \Omega$, and $\varepsilon \in (0, \varepsilon_0)$. Setting $u_\varepsilon(x) := \varepsilon^{-\delta}(u(\varepsilon x) - u(0))$, $f_\varepsilon(x) := f(\varepsilon x)$, and $\Omega_\varepsilon = \{(1/\varepsilon)x : x \in \Omega\}$, we have that $u_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon, |x|^{-2a})$, and

$$-\operatorname{div}(|x|^{-2a}\nabla u_\varepsilon) = |x|^{-2a-2+c} f_\varepsilon(x)\varepsilon^{c-\delta} \quad \text{in } \Omega_\varepsilon. \tag{2.51}$$

According to Lemma 2.10, the u_ε's are uniformly bounded. Hence, using elliptic estimates in $B_4 \setminus \overline{B_{1/2}}$, we obtain from (2.51) that for every $\alpha \in (0, 1)$ there is a constant $c_2(\alpha)$, independent of ε, such that

$$|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \leq c_2(\alpha)|x - y|^\alpha \text{ in } B_2 \setminus \overline{B_1},$$

which implies

$$|\nabla u(x) - \nabla u(y)| \leq c_2(\alpha)|x - y|^\alpha \varepsilon^{\delta-1-\alpha} \text{ in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}.$$

Choosing $\alpha \leq \delta - 1$, we find

$$|\nabla u(x) - \nabla u(y)| \leq c_2(\alpha)|x - y|^\alpha \text{ in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}. \tag{2.52}$$

By Lemma 2.10 and (2.49), we have that $u \in C^1_{\text{loc}}(\Omega)$ and that $\nabla u(0) = 0$. Together with (2.49) and (2.50) this proves (2.48). □

A slight modification of the above proof in the case $c = 1$ leads to the following

Theorem 2.10. *Let $0 < a < (N-2)/2$, $c = 1$, $f \in L^\infty(\Omega)$, and let u a solution of (2.27). Then $u \in C^{0,1}(\Omega')$, for every $\Omega' \subset\subset \Omega$. Moreover, there is a constant d_2 depending only on a and Ω' such that*

$$\|\nabla u\|_{L^\infty(\Omega')} \leq d_2 M \tag{2.53}$$

where M is as in Theorem 2.9. Finally, if Ω is a C^1−domain then $u \in C^{0,1}(\overline{\Omega})$ and inequality (2.53) holds, with Ω' replaced by Ω.

Proof : We proceed similarly as in the preceding proof. Note first that u satisfies (2.38) with $\delta = 1$, and that (2.49). Moreover if Ω is a C^1−domain, it follows that

$$u \in C^{0,1}(\overline{\Omega} \setminus B_\varepsilon), \qquad \text{for all } \varepsilon > 0. \tag{2.54}$$

Choosing ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$, we set $u_\varepsilon(x) := (u(\varepsilon x) - u(0))/\varepsilon$. Then we have that

$$-\text{div}(|x|^{-2a}\nabla u_\varepsilon) = |x|^{-2a-1} f_\varepsilon(x) \quad \text{in } \Omega_\varepsilon. \tag{2.55}$$

Using elliptic estimates in $B_4 \setminus \overline{B_{1/2}}$, it follows from (2.55) that there is a constant c_3 independent of ε such that

$$|\nabla u_\varepsilon(x)| \le c_3 \text{ in } B_2 \setminus \overline{B_1}.$$

This implies

$$|\nabla u(x)| \le c_3 \text{ in } B_{2\varepsilon} \setminus \overline{B_\varepsilon}.$$

Assertion (2.53) results from the continuity of u, (2.54) and inequality (2.38), with $\delta = 1$. □

2.4.2 Truncation Argument

In this subsection we henceforth that Ω is a C^1−domain. In order to obtain a solution of (2.22), we first consider a truncated problem. Fix some number $R > 0$, and let

$$\begin{aligned} f_R(x,t,\xi) &= f(x,t,\xi\varphi_R(\xi)) \quad \text{and} \\ F_R(x,t,\xi) &= \int_0^t f_R(x,\tau,\xi)d\tau\,, \qquad \text{for all } (x,t,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \end{aligned}$$

where $\varphi_R \in C^1(\mathbb{R}^N)$ and satisfies the following conditions

$$\begin{cases} |\varphi_R(\xi)| \le 1\,, & \text{for all } \xi \in \mathbb{R}^N, \\ \varphi_R(\xi) = 1\,, & \text{for all } |\xi| \le R, \\ \varphi_R(\xi) = 0\,, & \text{for all } |\xi| \ge R+1. \end{cases} \tag{2.56}$$

Furthermore, for any fixed $w \in W_0^{1,2}(\Omega, |x|^{-2a})$ we define a functional $I_w^R : W_0^{1,2}(\Omega, |x|^{-2a}) \to \mathbb{R}$ by

$$I_w^R(v) = \frac{1}{2}\int_\Omega |x|^{-2a}|\nabla v|^2 - \int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w).$$

The critical points u_w^R of I_w^R are weak solutions of the semilinear elliptic problem

$$\begin{cases} -\text{div}(|x|^{-2a}\nabla u_w^R) = |x|^{-2(a+1)+c} f_R(x, u_w^R, \nabla w) & \text{in } \Omega\,, \\ u_w^R = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to show that the functional I_w^R has a Mountain Pass type structure for any $w \in W_0^{1,2}(\Omega, |x|^{-2a})$. Indeed, one can state the following two lemmata.

Lemma 2.11. *For every $R > 0$ there exist positive numbers $\rho < 1$ and α, such that*

$$I_w^R(v) \geq \alpha \qquad \text{for all } w \in W_0^{1,2}(\Omega, |x|^{-2a}) \text{ and } \text{for all } v \in W_0^{1,2}(\Omega, |x|^{-2a}) \text{ satisfying } \|v\| = \rho. \tag{2.57}$$

Lemma 2.12. *There exists some $\overline{v} \in W_0^{1,2}(\Omega, |x|^{-2a})$, with $\overline{v} \geq 0$, $\|\overline{v}\| > 1$, such that*

$$I_w^R(\overline{v}) < 0, \qquad \text{for all } R > 0 \text{ and all } w \in W_0^{1,2}(\Omega, |x|^{-2a}). \tag{2.58}$$

Proof of Lemma 2.3. It follows from (f_1) and (f_2) that there is a positive constant k_ε, independent of R, such that

$$|F_R(x,t,\xi)| \leq \frac{\varepsilon t^2}{2} + k_\varepsilon (R+2)^r |t|^{p+1}.$$

In view of (1.1) we have that

$$\begin{aligned} \int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w) &\leq \frac{\varepsilon}{2} \int_\Omega |x|^{-2(a+1)+c} v^2 \\ &\quad + k_\varepsilon (R+2)^r \int_\Omega |x|^{-2(a+1)+c} |v|^{p+1} \\ &\leq C\left(\frac{\varepsilon}{2} + k_\varepsilon (R+2)^r \|v\|^{p-1}\right)\|v\|^2, \end{aligned} \tag{2.59}$$

for some constant $C > 0$. Now choosing

$$\|v\| < \left(\frac{\varepsilon}{2k_\varepsilon (R+2)^r}\right)^{\frac{1}{p-1}}$$

in the above inequality, one gets

$$\int_\Omega |x|^{-2(a+1)+c} F_R(x, v, \nabla w) \leq C\varepsilon \|v\|^2,$$

so that (2.57) easily follows by taking $\varepsilon < (2C)^{-1}$,
$\rho < \min\left\{1; (4k_\varepsilon (R+2)^r C)^{-1/(p-1)}\right\}$ and $\alpha = (\frac{1}{2} - C\varepsilon)\rho^2$. □

Proof of Lemma 2.5. We fix some function $v_0 \in W_0^{1,2}(\Omega, |x|^{-2a})$, with $v_0 \geq 0$, $v_0 \neq 0$. By (2.23) one gets for any $t > 0$

$$I_w^R(tv_0) \leq \frac{t^2}{2} \int_\Omega |x|^{-2a} |\nabla v_0|^2 - a_2 \int_\Omega |x|^{-2(a+1)+c} t^\theta |v|^\theta + \widetilde{a_3},$$

where $\widetilde{a_3} = a_3 \int_\Omega |x|^{-2(a+1)+c}$. Then we choose $\overline{v} = \overline{t} v_0$ with $\overline{t}$ sufficiently large such that $\|\overline{v}\| > 1$ and $I_w^R(\overline{v}) < 0$ for all $R > 0$. □

Proposition 2.1. *Let* $(f_0)-(f_3)$ *be satisfied and let* $w \in W_0^{1,2}(\Omega, |x|^{-2a})$ *and* $\overline{v}$ *given by Lemma 2.5. Then for every* $R > 0$ *there exists some* $v = v(w, R)$ *such that*

$$\begin{aligned} D(I_w^R)(v) &= 0 \qquad \text{and} \\ I_w^R(v) &= \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_w^R, \end{aligned} \tag{2.60}$$

where

$$\Gamma = \{\gamma \in C^0([0,1]; W_0^{1,2}(\Omega, |x|^{-2a})) : \gamma(0) = 0, \gamma(1) = \overline{v}\}. \tag{2.61}$$

Proof. We have that $I_w^R(0) = 0$. Furthermore, the functional I_w^R satisfies the (PS) condition in view of $(f_0)-(f_3)$. Then the existence of an element v such that (2.60) and (2.61) hold is an immediate consequence of the Lemmata 2.3, 2.5 and of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz (see [2]). □

Next we will obtain a positive and a negative solution of (2.21). To this end we fix an arbitrary element $u_0 \in W_0^{1,2}(\Omega, |x|^{-2a})$ and $R > 0$, and we consider the following iterative scheme:

Given $n \in \mathbb{N}$, fix an element $v = u_n^R$ satisfying (2.60) and (2.61) with $w = u_{n-1}^R$.

(2.62)

Note that the elements u_n^R above are not unique in general. Now we obtain a uniform estimate from above for the $W_0^{1,2}(\Omega, |x|^{-2a})$–norms of u_n^R. This will finally allow us to get rid of the dependence on R, and to pass to the following iteration scheme:

$$(P)_n \qquad \begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u_n) = |x|^{-2(a+1)+c} f(x, u_n, \nabla u_{n-1}) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.13. *There exists a positive constant* c_1 *such that*

$$\|u_n^R\| \leq c_1 \tag{2.63}$$

for every $n \in \mathbb{N}$ *and* $R > 0$.

Proof. Using the definition of u_n^R and choosing the path in Γ given by the line segment joining 0 and $\overline{v}$, one gets from (2.23)

$$I_{u_{n-1}^R}^R(u_n^R) \leq \sup_{t\geq 0} \left\{ \frac{t^2}{2} \int_\Omega |x|^{-2a} |\nabla \overline{v}|^2 - a_2 t^\theta \int_\Omega |x|^{-(a+1)p+c} |\overline{v}|^\theta + \widetilde{a_3} \right\},$$

where $\widetilde{a_3}$ is defined in the proof of Lemma 2.5. Since $\theta > 2$, the function

$$\mathbb{R}_+ \ni t \mapsto \frac{t^2}{2}\int_\Omega |x|^{-2a}|\nabla\overline{v}|^2 - a_2 t^\theta \int_\Omega |x|^{-(a+1)p+c}|\overline{v}|^\theta + \widetilde{a_3}$$

attains a positive maximum. Hence

$$I^R_{u^R_{n-1}}(u^R_n) \le const \qquad \text{for all } n \in \mathbb{N} \text{ and for all } R > 0. \tag{2.64}$$

Now (2.64), (f_3), the fact that $|\varphi_R| \le 1$, and the criticality of u^R_n for $I^R_{u^R_{n-1}}$ imply

$$\begin{aligned}\frac{1}{2}\|u^R_n\|^2 &\le const + \frac{1}{\theta}\int_\Omega |x|^{-(a+1)p+c} f_R(x, u^R_n, \nabla u^R_{n-1})u^R_n \\ &= const + \frac{1}{\theta}\|u^R_n\|^2,\end{aligned}$$

and (2.63) follows in view of $\theta > 2$. □

Using the results of section 2 we now obtain uniform estimates for the C^0–norms of $\{u^R_n\}$ and $\{\nabla u^R_n\}$, by assuming additionally that

$$u^R_0 \in C^{0,1}(\overline{\Omega}) \text{ for every } R > 0. \tag{2.65}$$

Lemma 2.14. *Assume (2.65). Then, for every $n \in \mathbb{N}$ and $R > 0$, $u^R_n \in C^{0,1}(\overline{\Omega})$.*

Proof. We have that u^R_1 is the weak solution of

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla u^R_1) = f_R(x, u^R_1, \nabla u^R_0) & \text{in } \Omega, \\ u^R_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since

$$|f_R(x, u^R_1, \nabla u^R_0)| \le M(1 + |u^R_1|^p)(2 + R)^r,$$

that is, $\|f_R(x, u^R_1, \nabla u^R_0)\|_{L^\infty(\Omega)} \le \widetilde{M}(2 + R)^r$, we may apply Theorem 2.1. Hence $u^R_1 \in C^{0,\alpha}(\overline{\Omega})$. In view of Theorem 2.4 this means that u^R_1 is Lipschitz continuous on Ω, for any $R > 0$. Our result now follows by induction. □

Lemma 2.15. *Assume (2.65). Then there exist $\mu_0 > 0$ and $\mu_1 > 0$, such that*

$$\|u^R_n\|_{L^\infty(\Omega)} \le k_0 = \mu_0(R + 2)^r, \tag{2.66}$$

$$\|\nabla u^R_n\|_{L^\infty(\Omega)} \le k_1 = \mu_1(R + 2)^r \qquad \textit{for all } R > 0 \textit{ and for all } n \in \mathbb{N}. \tag{2.67}$$

Proof. Recall that any Lipschitz function is a.e. differentiable with bounded gradient. Then, arguing as in Lemma 2.14, the condition (f_2) and the definition of f_R yield the estimates (2.66)–(2.67). □

Lemma 2.16. *Assume (2.65). Then there exists some $\overline{R} > 0$, such that*

$$\|u_n^{\overline{R}}\|_{L^\infty(\Omega)} \le k_0 = \mu_0(\overline{R}+2)^r \le \overline{R}, \tag{2.68}$$

$$\|\nabla u_n^{\overline{R}}\|_{L^\infty(\Omega)} \le k_1 = \mu_1(\overline{R}+2)^r \le \overline{R}. \tag{2.69}$$

Proof. (2.68) and (2.69) are an obvious consequence of (2.66)–(2.67) and the fact that $r \in (0,1)$. □

Lemma 2.17. *Assume (2.65). Then $u_n := u_n^{\overline{R}}$ is a solution of $(P)_n$ and the following estimates hold, for any $n \in \mathbb{N}$,*

$$\|u_n\| \le c_1, \tag{2.70}$$

$$\|u_n\|_{L^\infty(\Omega)} \le k_0 = \mu_0(\overline{R}+2)^r \tag{2.71}$$

$$\|\nabla u_n\|_{L^\infty(\Omega)} \le k_1 = \mu_1(\overline{R}+2)^r. \tag{2.72}$$

Proof. The fact that u_n solves $(P)_n$, is a consequence of the definition of f_R and the assumptions (2.56) and (2.68) with $R = \overline{R}$. Moreover, (2.63), (2.68)–(2.69), respectively, imply (2.70)–(2.72) with $R = \overline{R}$. □

The function u_n given in Lemma 2.17 is a *nontrivial* solution of $(P)_n$. More precisely, there holds

Lemma 2.18. *For any $n \in \mathbb{N}$, there exists a positive constant c_2 such that*

$$\|u_n\| \ge c_2. \tag{2.73}$$

Proof. For any $v \in W_0^{1,2}(\Omega, |x|^{-2a})$ we have that

$$\int_\Omega |x|^{-2a}\nabla u_n \nabla v = \int_\Omega |x|^{-(a+1)p+c} f(x, u_n, \nabla u_{n-1}).$$

Setting $v = u_n$ in the relation above we obtain that

$$\int_\Omega |x|^{-2a}|\nabla u_n|^2 = \int_\Omega |x|^{-(a+1)p+c} f(x, u_n, u_{n-1})u_n.$$

Hence (f_1) and (f_2) imply that for any $\delta > 0$ there exists a number $c(\delta) > 0$ such that

$$\begin{aligned}\int_\Omega |x|^{-2a}|\nabla u_n|^2 &\le \delta \int_\Omega |x|^{-(a+1)p+c}|u_n|^2 + c(\delta)\int_\Omega |x|^{-(a+1)p+c}|u_n|^{p+1} \\ &\le C(\delta\|u_n\|^2 + c(\delta)\|u_n\|^{p+1}),\end{aligned}$$

for any $n \in \mathbb{N}$ and for some constant $C > 0$. Now (2.73) follows, by choosing $\delta C < 1$. □

Lemma 2.19. *Let*

$$\overline{k_0} := \min\{k_0 > 0 : \text{(2.71) holds}\}$$
$$\overline{k_1} := \min\{k_1 > 0 : \text{(2.72) holds}\},$$

and choose $\rho_1 = \overline{k_0}$ *and* $\rho_2 = \overline{k_1}$ *in* (f_4). *Then the sequence* $\{u_n\}$ *converges strongly in* $W_0^{1,2}(\Omega, |x|^{-2a})$.

Proof. By the criticality of u_{n+1} and u_n one has for every $n \in \mathbb{N}$,

$$\int_\Omega |x|^{-2a} \nabla u_{n+1}(\nabla(u_{n+1} - u_n)) = \int_\Omega |x|^{-2(a+1)+c} f(x, u_{n+1}, \nabla u_n)(u_{n+1} - u_n), \tag{2.74}$$

$$\int_\Omega |x|^{-2a} \nabla u_n(\nabla(u_{n+1} - u_n)) = \int_\Omega |x|^{-2(a+1)+c} f(x, u_n, \nabla u_{n-1})(u_{n+1} - u_n). \tag{2.75}$$

Subtracting (2.75) from (2.74), we obtain that

$$\begin{aligned} \|u_{n+1} - u_n\|^2 = \int_\Omega |x|^{-2(a+1)+c} \Big\{ &[f(x, u_{n+1}, \nabla u_n) - f(x, u_n, \nabla u_n)](u_{n+1} - u_n) \\ &+ [f(x, u_n, \nabla u_n) - f(x, u_n, \nabla u_{n-1})](u_{n+1} - u_n) \Big\}. \end{aligned}$$

Using hypothesis (f_4), this leads to the following estimate,

$$\begin{aligned} \|u_{n+1} - u_n\|^2 \le L_1 &\int_\Omega |x|^{-2(a+1)+c} |u_{n+1} - u_n|^2 \\ &+ L_2 \int_\Omega |x|^{-2(a+1)+c} |\nabla(u_n - u_{n-1})| |u_{n+1} - u_n|. \end{aligned} \tag{2.76}$$

Using Cauchy–Schwarz and singular Poincaré inequalities, and since $c \ge 1$, we have from (2.76),

$$\begin{aligned} \|u_{n+1} - u_n\|^2 \le\; &L_1 S(\Omega, a, c)^{-1} \|u_{n+1} - u_n\|^2 \\ &+ L_2 S(\Omega, a, 2(c-1))^{-1/2} \|u_{n+1} - u_n\| \, \|u_n - u_{n-1}\|. \end{aligned}$$

This means that

$$\|u_{n+1} - u_n\| \le \frac{L_2 S(\Omega, a, 2(c-1))^{-1/2}}{1 - L_1 S(\Omega, a, c)^{-1}} \|u_n - u_{n-1}\| =: k \|u_n - u_{n-1}\|.$$

By our assumptions, we have $k < 1$. Hence the sequence $\{u_n\}$ converges in $W_0^{1,2}(\Omega, |x|^{-2a})$ to some function $u \in W_0^{1,2}(\Omega, |x|^{-2a})$. Furthermore, since $\|u_n\| \ge c_2$ by Lemma 2.18, it follows that $u \ne 0$. In this way we obtain a nontrivial solution of (2.21). ☐

Lemma 2.20. *Problem $(P)_n$ has a positive solution u_n^+ and a negative solution u_n^-. Moreover, the sequences $\{u_n^+\}$ and $\{u_n^-\}$ converge strongly in $W_0^{1,2}(\Omega, |x|^{-2a})$.*

Proof. We consider only the case of the positive solution. The argument leading to a negative solution is analogous. We replace the function $f(x,t,\xi)$ in (2.21) by the function

$$f^+(x,t,\xi) = \begin{cases} 0 & \text{if } f(x,t,\xi) < 0 \\ f(x,t,\xi) & \text{if } f(x,t,\xi) \geq 0. \end{cases}$$

Of course, f^+ satisfies (f_3) only for $t \geq 0$. But this is of no importance if we choose $v_0 > 0$ in the proof of Lemma 2.12. Indeed, proceeding analogously as before, we obtain a solution of the problem

$$\begin{cases} -\text{div}\,(|x|^{-2a}\nabla u_n) = f^+(x, u_n^+, \nabla u_{n-1}^+) & \text{in } \Omega, \\ u_n^+ = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the differential equation by the negative part of u_n and integrating by parts, we conclude that u_n is positive, that is $u_n^+ = u_n$. □

Proof of Theorem 2.7. The proof is a direct consequence of the Lemmata 2.19 and 2.20. □

Chapter 3

Pohozaev's Identity and a Non–existence Result

In this chapter we study non–existence of solutions of the Problem (0.1). We first recall a Pohozaev type identity due to P. Pucci and J. Serrin. (See [32].)

Lemma 3.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the Euler–Lagrange equation*

$$\begin{cases} div\{\mathcal{F}_{\vec{p}}(x,u,\nabla u)\} = \mathcal{F}_u(x,u,\nabla u) & in\ \Omega\,, \\ u = 0 & on\ \partial\Omega \end{cases}$$

where $\vec{p} = (p, \ldots, p_n) = \nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $\mathcal{F}_u = \partial \mathcal{F}/\partial u$. Let Λ be a scalar and h a vector–valued function of class $C^1(\Omega) \cap C(\overline{\Omega})$. Then the following equality holds

$$\begin{aligned} \oint_{\partial\Omega} \left[\mathcal{F}(x,0,\nabla u) - \frac{\partial u}{\partial x_i}\mathcal{F}_{p_i}(x,0,\nabla u)\right](h\cdot\nu)dS & \\ = \int_\Omega \Bigg\{ \mathcal{F}(x,u,\nabla u)div(h) + h_i \mathcal{F}_{x_i}(x,u,\nabla u) & \\ - \left[\frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial x_i} + u\frac{\partial \Lambda}{\partial x_i}\right]\mathcal{F}_{p_i}(x,u,\nabla u) & \\ - \Lambda\left[\frac{\partial u}{\partial x_i}\mathcal{F}_{p_i}(x,u,\nabla u) + u\mathcal{F}_u(x,u,\nabla u)\right]\Bigg\}dx & \end{aligned} \tag{3.1}$$

where repeated indices i and j are understood to be summed from 1 to n.

We consider the problem

$$\begin{cases} -div(|x|^{-ap}A(|\nabla u|)\nabla u) = g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where g satisfies $g(x,0) = 0$. Suppose $\mathcal{F}(x,u,\nabla u) = \frac{1}{p}|x|^{-ap}S(|\nabla u|^p) - G(x,u)$, where $G(x,u) = \int_0^u g(x,t)\,dt$ and $S(t) = p\int_0^{t^{1/p}} vA(v)\,dv$. Choosing $h(x) = x$, $\Lambda = constant$, equality (3.1) then becomes

$$\oint_{\partial\Omega}\left[\frac{1}{p}|x|^{-ap}S(|\nabla u|^p) - |x|^{-ap}A(|\nabla u|)|\nabla u|^2\right] x\cdot\nu(x)d\sigma =$$
$$\int_\Omega\left\{\left(\frac{N}{p} - a\right)|x|^{-ap}S(|\nabla u|^p) - NG(x,u) - x\cdot G_x(x,u) - ug(x,u)\right\}dx. \tag{3.3}$$

Observe that, according to conditions (H1) and (H2), and the strong maximum principle, the left hand side of the equality (3.3) is negative. Hence we have the following.

Theorem 3.1. *Let Ω be a smooth domain which is star–shaped with respect to the origin. Suppose*

$$\int_\Omega\left\{\left(\frac{N}{p} - a\right)|x|^{-ap}S(|\nabla u|^p) - NG(x,u) - x\cdot G_x(x,u) - ug(x,u)\right\}dx \ge 0. \tag{3.4}$$

Then there is no solution $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ of Problem (3.2).

Remark: Let $A(t) = \gamma t^{p-2} + \delta t^{q-2}$, where $\gamma > 0$, $\delta \ge 0$ and $1 < q < p$ and $g(x,u) = |x|^{-(a+1)p+c}u^{l-1}$ for some l and $u > 0$ in Ω. Then inequality (3.4) reads

$$\left[\frac{N}{p} - a - \frac{N}{l} - \frac{c-(a+1)p}{l}\right]\int_\Omega |x|^{-(a+1)p+c}u^l \ge 0$$

that is,

$$l \ge \frac{p(N-(a+1)p+c)}{N-(a+1)p}.$$

On the other hand, the condition (f2) which was needed for our existence Theorem 2.2 implies that $l < p(N-(a+1)p+c)/(N-(a+1)p)$. It is not hard to see that, in this case, equality (3.3) cannot hold. In particular, if $a = 0$ and $c = p$, then we obtain the well known non–existence result for the p–Laplacian. (Compare [32].)

Proof of Theorem 3.1. The deduction above is formal. In fact, the solution of Problem (0.1) may not be of class $C^2(\Omega)\cap C^1(\overline{\Omega})$. We need approximation arguments as of [26] and [10]. Let $\{g_\varepsilon\}$ be a sequence of $C^2(\overline{\Omega}\setminus\{0\})$ functions converging to $g(\cdot,u)$ as ε tends to 0^+, and let u_ε be a solution of the equation

$$\begin{cases} -\operatorname{div}(|x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})\nabla u_\varepsilon) = g_\varepsilon & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Then, by standard regularity results of [39], the solution u_ε belongs to $C^3(\overline{\Omega}\setminus\{0\})$ and converges to u in $C^{1,\alpha}(\overline{\Omega}\setminus\{0\})$, for some $\alpha\in(0,1)$. Multiplying equation (3.5) by $(\Lambda u_\varepsilon - x\cdot\nabla u_\varepsilon)$, where Λ is a constant, and integrating over $\Omega_\delta := \Omega\setminus\overline{B_\delta(0)}$, where $0<\delta<\operatorname{dist}(0,\partial\Omega)$, yields

$$\begin{aligned} -\int_{\Omega_\delta} \operatorname{div}(|x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})\nabla u_\varepsilon)(\Lambda u_\varepsilon - x\cdot\nabla u_\varepsilon)dx \\ = \int_{\Omega_\delta} g_\varepsilon(\Lambda u_\varepsilon - x\cdot\nabla u_\varepsilon)dx\,. \end{aligned} \tag{3.6}$$

Integrating by parts the left hand side of (3.6) over Ω_δ, we obtain

$$\begin{aligned} LHS = &-\int_{\partial\Omega_\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})(\Lambda u_\varepsilon - x\cdot\nabla u_\varepsilon)(\nabla u_\varepsilon\cdot\nu)d\sigma \\ &+\int_{\Omega_\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})\nabla u_\varepsilon\cdot\nabla(\Lambda u_\varepsilon - x\cdot\nabla u_\varepsilon)dx \\ = &-\Lambda\int_{|x|=\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})u_\varepsilon(\nabla u_\varepsilon\cdot\nu)d\sigma \\ &+\int_{\partial\Omega} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2(x\cdot\nu)d\sigma \\ &+\int_{|x|=\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2(x\cdot\nu)d\sigma \\ &+\Lambda\int_{\Omega_\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2dx \\ &-\int_{\Omega_\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})\nabla u_\varepsilon\cdot\nabla(\nabla u_\varepsilon\cdot x)dx\,. \end{aligned}$$

Since $\nabla u_\varepsilon\cdot\nabla(x\cdot\nabla u_\varepsilon) = |\nabla u_\varepsilon|^2 + \frac{1}{2}(x\cdot\nabla(|\nabla u_\varepsilon|^2))$, we conclude

$$\begin{aligned} \int_{\Omega_\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2dx \\ = \int_{\Omega_\delta} g_\varepsilon u_\varepsilon + \int_{|x|=\delta} |x|^{-ap}A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})u_\varepsilon(\nabla u_\varepsilon\cdot\nu)d\sigma \end{aligned} \tag{3.7}$$

by (3.5).

On the other hand,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |x|^{-ap} A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})(x\cdot\nabla(|\nabla u_\varepsilon|^2))dx \\
&\quad= \frac{1}{2}\int_{\Omega_\delta} |x|^{-ap} x\cdot\nabla\left(2\int_0^{\sqrt{\varepsilon+|\nabla u_\varepsilon|^2}} A(t)t\,dt\right)dx \\
&\quad= \int_{\Omega_\delta} |x|^{-ap} x\cdot\nabla\left(\frac{1}{p}S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})\right)dx \\
&\quad= \frac{1}{p}\Bigg[\int_{\partial\Omega} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \\
&\qquad+\int_{|x|=\delta} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \qquad\qquad (3.8)\\
&\qquad-\int_{\Omega_\delta} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})\cdot\nabla(x|x|^{-ap})dx\Bigg] \\
&\quad= \frac{1}{p}\Bigg[\int_{\partial\Omega} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \\
&\qquad+\int_{|x|=\delta} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \\
&\qquad-(N-ap)\int_{\Omega_\delta} |x|^{-ap} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})dx\Bigg].
\end{aligned}$$

Substituting (3.7) and (3.8) into LHS we obtain

$$\begin{aligned}
LHS &= \int_{\partial\Omega} |x|^{-ap} A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2(x\cdot\nu)d\sigma \\
&\quad+\int_{|x|=\delta} |x|^{-ap} A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})|\nabla u_\varepsilon|^2(x\cdot\nu)d\sigma + (\Lambda-1)\int_{\Omega_\delta} g_\varepsilon u_\varepsilon \\
&\quad-\int_{|x|=\delta} |x|^{-ap} A(\sqrt{\varepsilon+|\nabla u_\varepsilon|^2})u_\varepsilon(\nabla u_\varepsilon\cdot\nu)d\sigma \\
&\quad-\frac{1}{p}\int_{\partial\Omega} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \\
&\quad-\frac{1}{p}\int_{|x|=\delta} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})(x\cdot\nu)d\sigma \\
&\quad+\frac{(N-ap)}{p}\int_{\Omega_\delta} |x|^{-ap} S((\varepsilon+|\nabla u_\varepsilon|^2)^{p/2})dx.
\end{aligned}$$

Also, the right hand side of (3.6) is

$$RHS = \Lambda \int_{\Omega_\delta} g_\varepsilon u_\varepsilon - \int_{\Omega_\delta} g_\varepsilon x \cdot \nabla u_\varepsilon dx.$$

Now, letting $\varepsilon \to 0^+$ we have

$$\begin{aligned} LHS = & \int_{\partial\Omega} |x|^{-ap} A(|\nabla u|)|\nabla u|^2 (x\cdot\nu) d\sigma \\ & + \int_{|x|=\delta} |x|^{-ap} A(|\nabla u|)|\nabla u|^2 (x\cdot\nu) d\sigma + (\Lambda - 1)\int_{\Omega_\delta} gu\, dx \\ & - \int_{|x|=\delta} |x|^{-ap} A(|\nabla u|) u (\nabla u \cdot \nu) d\sigma - \frac{1}{p}\int_{\partial\Omega} S(|\nabla u|^p)(x\cdot\nu) d\sigma \\ & - \frac{1}{p}\int_{|x|=\delta} S(|\nabla u|^p)(x\cdot\nu) d\sigma + \frac{(N-ap)}{p}\int_{\Omega_\delta} |x|^{-ap} S(|\nabla u|^p) dx \end{aligned}$$

and

$$\begin{aligned} RHS &= \Lambda \int_{\Omega_\delta} gu\, dx - \int_{\Omega} g(x\cdot\nabla u)\, dx \\ &= \Lambda \int_{\Omega_\delta} gudx - \int_{\partial\Omega_\delta} G(x,u)(x\cdot\nu)d\sigma + \int_{\Omega_\delta}(x\cdot G_x(x,u))dx \\ & \quad + N\int_{\Omega_\delta} G(x,u)dx. \end{aligned}$$

Thus

$$\begin{aligned} & \int_{\partial\Omega} |x|^{-ap} A(|\nabla u|)|\nabla u|^2 (x\cdot\nu) d\sigma \\ & \quad + \int_{|x|=\delta} |x|^{-ap} A(|\nabla u|)|\nabla u|^2 (x\cdot\nu) d\sigma + (\Lambda - 1)\int_{\Omega_\delta} gu\, dx \\ & \quad - \int_{|x|=\delta} |x|^{-ap} A(|\nabla u|) u (\nabla u \cdot \nu) d\sigma \\ & \quad - \frac{1}{p}\int_{\partial\Omega} S(|\nabla u|^p)(x\cdot\nu) d\sigma \\ & \quad - \frac{1}{p}\int_{|x|=\delta} S(|\nabla u|^p)(x\cdot\nu) d\sigma \\ & \quad + \frac{(N-ap)}{p}\int_{\Omega_\delta} |x|^{-ap} S(|\nabla u|^p) dx \\ & = \Lambda \int_{\Omega_\delta} gudx - \int_{\partial\Omega_\delta} G(x,u)(x\cdot\nu)d\sigma \\ & \quad + \int_{\Omega_\delta}(x\cdot G_x(x,u))dx + N\int_{\Omega_\delta} G(x,u)dx. \end{aligned} \tag{3.9}$$

Next, we need to get rid of the boundary integral along $|x| = \delta$ of (3.9). Let u be a solution of (3.2). From the Caffarelli–Kohn–Nirenberg inequality (1.1) and Theorem 1.1, we conclude that

$$\int_\Omega |x|^{-ap}|\nabla u|^p dx, \quad \int_\Omega |x|^{-(a+1)p+c}|u|^q dx, \text{ and } \sum_{i=1}^N \int_\Omega |x|^{-ap} A(|\nabla u|) u_{x_i} u dx$$

are finite. Therefore, by mean–value theorem there exists a sequence $\{\delta_m\}$, $\delta_m \to 0^+$, as $m \to +\infty$, such that the following integrals tends to 0,

$$\int_{|x|=\delta_m} G(x,u)(x\cdot\nu)d\sigma, \int_{|x|=\delta_m} |x|^{-ap}A(|\nabla u|)u(\nabla u\cdot\nu)d\sigma$$
$$\text{and} \int_{|x|=\delta_m} |x|^{-ap}S(|\nabla u|^p)(x\cdot\nu)d\sigma,$$

as $m \to +\infty$. Thus, letting $m \to +\infty$, we obtain (3.3). □

Appendix A

Some proofs

Proof of Theorem 1.1. We follow [43]. The continuity of the embedding is a direct consequence of the Caffarelli–Kohn–Nirenberg inequality [8]. To prove the compactness, let $\{u_m\}$ be a bounded sequence in $W_0^{1,p}(\Omega, |x|^{-ap})$. For any $\rho > 0$, with $B_\rho(0) \subset \Omega$, there hold $\{u_m\} \subset W^{1,p}(\Omega \setminus B_\rho(0))$. Then the classical Rellich–Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \setminus B_\rho(0))$. Taking a diagonal sequence we may assume, without loss of generality, that for any $\rho > 0$, the sequence $\{u_m\}$ converges in $L^r(\Omega \setminus B_\rho(0))$.

On the other hand, for any $1 \le r < \frac{Np}{N-p}$, there exists $b \in (a, a+1]$ such that $r < q = p^*(a,b) = \frac{Np}{N-dp}$, $d = 1 + a - b \in [0,1)$. It follows from the Caffarelli–Kohn–Nirenberg inequality (see [8]) that $\{u_m\}$ is also bounded in $L^q(\Omega, |x|^{-bq})$. By the Hölder inequality, for any $\delta > 0$, we have

$$\begin{aligned}
&\int_{|x|<\delta} |x|^{-\alpha} |u_m - u_j|^r dx \\
&\le \left(\int_{|x|<\delta} |x|^{-(\alpha - br)\frac{q}{q-r}} dx \right)^{1-\frac{r}{q}} \left(\int_\Omega |x|^{-br} |u_m - u_j|^r dx \right)^{r/q} \\
&\le C \left(\int_0^\delta r^{N-1-(\alpha-br)\frac{q}{q-r}} dr \right)^{1-\frac{r}{q}} \\
&= C\delta^{N-(\alpha-br)\frac{q}{q-r}}
\end{aligned}$$

where $C > 0$ is a constant independent of m. Since $\alpha < (1+a)r + N(1-\frac{r}{p})$, we have $N - (\alpha - br)\frac{q}{q-r} > 0$. Therefore, for a given $\varepsilon > 0$, we may fix $\delta > 0$ so that

$$\int_{|x|<\delta} |x|^{-\alpha} |u_m - u_j|^r dx \le \frac{\varepsilon}{2}, \quad \text{for all} \quad m, j \in \mathbb{N}.$$

Then we may choose $n \in \mathbb{N}$ so that

$$\int_{\Omega\setminus B_\delta(0)} |x|^{-\alpha}|u_m - u_j|^r dx| \leq C_\alpha \int_{\Omega\setminus B_\delta(0)} |u_m - u_j|^r dx \leq \frac{\varepsilon}{2}, \quad \text{for all} \quad m, j \geq n$$

where $C_\alpha = \delta^{-\alpha}$ if $\alpha \geq 0$ and $C_\alpha = (\mathrm{diam}(\Omega))^{-\alpha}$ if $\alpha < 0$. Thus

$$\int_\Omega |x|^{-\alpha}|u_m - u_j|^r dx \leq \varepsilon, \quad \text{for all} \quad m, j \geq n,$$

or in other words $\{u_m\}$ is a Cauchy sequence in $L^q(\Omega, |x|^{-bq})$. □

Proof of Theorem 1.2.

1. For the bilinear form B, we have that

   $$|B[u,v]| \leq \|u\|_{W_0^{1,2}(\Omega,|x|^{-2a})}\|v\|_{W_0^{1,2}(\Omega,|x|^{-2a})}$$

   and

   $$B[u,u] = \|u\|^2_{W_0^{1,2}(\Omega,|x|^{-2a})}$$

   for all $u, v \in W_0^{1,2}(\Omega, |x|^{-2a})$.

   Thus B satisfies the hypothesis of Lax–Milgram Theorem. (See [17].) Now let $f : W_0^{1,2}(\Omega, |x|^{-2a}) \to \mathbb{R}$ be a bounded linear functional on $W_0^{1,2}(\Omega, |x|^{-2a})$. Then there exists a unique element $u \in W_0^{1,2}(\Omega, |x|^{-2a})$ such that

   $$B[u, v] = f(v)$$

   for all $v \in W_0^{1,2}(\Omega, |x|^{-2a})$.

2. Let $g \in L^2(\Omega, |x|^{-2a})$. Then there exists a unique $u \in W_0^{1,2}(\Omega, |x|^{-2a})$ such that

   $$B[u,v] = \langle g, v\rangle_{L^2(\Omega,|x|^{-2a})}, \qquad \text{for all } v \in W_0^{1,2}(\Omega, |x|^{-2a}).$$

   Thus, we can define $T = L^{-1} : L^2(\Omega, |x|^{-2a}) \to W_0^{1,2}(\Omega, |x|^{-2a})$, given by Tf as the unique element of $W_0^{1,2}(\Omega, |x|^{-2a})$ such that

   $$B[Tf, v] = \langle f, v\rangle_{L^2(\Omega,|x|^{-2a})}$$

3. We claim that $T : L^2(\Omega, |x|^{-2a}) \to L^2(\Omega, |x|^{-2a})$ is a bounded, linear and compact operator. We have that

   $$\begin{aligned}\|u\|^2 = B[u,u] = \langle f, u\rangle &\leq \|f\|_{L^2(\Omega,|x|^{-2a})}\|u\|_{L^2(\Omega,|x|^{-2a})}\\ &\leq C\|f\|_{L^2(\Omega,|x|^{-2a})}\|u\|_{W_0^{1,2}(\Omega,|x|^{-2a})},\end{aligned}$$

so that

$$\|Tf\|_{W_0^{1,2}(\Omega,|x|^{-2a})} \leq C\|f\|_{L^2(\Omega,|x|^{-2a})}.$$

Since the embedding $W_0^{1,2}(\Omega,|x|^{-2a}) \hookrightarrow L^2(\Omega,|x|^{-2a})$ is compact, we deduce that T is a compact operator.

4. We claim that T is symmetric. To see this, let $f,g \in L^2(\Omega,|x|^{-2a})$. The equation $Tf = u$ means that $u \in W_0^{1,2}(\Omega,|x|^{-2a})$ is the weak solution of

$$\begin{cases} Lu = |x|^{-2a}f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Analogously, $Tg = v$ means that $v \in W_0^{1,2}(\Omega,|x|^{-2a})$ solves

$$\begin{cases} Lv = |x|^{-2a}g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus

$$\langle Tf, g\rangle_{L^2(\Omega,|x|^{-2a})} = \langle f, Tg\rangle_{L^2(\Omega,|x|^{-2a})}$$

for all $f,g \in L^2(\Omega,|x|^{-2a})$. Therefore, T is symmetric.

5. Observe that

$$\langle Tf, f\rangle_{L^2(\Omega,|x|^{-2a})} = \langle u, f\rangle_{L^2(\Omega,|x|^{-2a})} = B[u,u] \geq 0$$

for all $f \in L^2(\Omega,|x|^{-2a})$. The theory of compact, symmetric operators then implies that all the eigenvalues of T are real, positive, and there are corresponding eigenfunctions which constitute an orthonormal basis of $L^2(\Omega,|x|^{-2a})$. Observe also that for, $\eta \neq 0$, we have that $Tw = \eta w$ if and only if

$$\eta \int_\Omega L(w)v = B[\eta w, v] = \int_\Omega |x|^{-2a}wv,$$

or in other words

$$\begin{cases} Lw = \frac{1}{\eta}|x|^{-2a}w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

in the weak sense.

□

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