ON GROUP ACTIONS ON 1-DIMENSIONAL MANIFOLDS

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A mi Madre y Padre

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Resumen

En la tesis consideramos acciones de grupos en variedades unidimensionales.

En el primer capítulo probamos que la entropía de la acción de un grupo en el círculo, por difeomorfismos de clase C^2, es igual a la entropía de la acción restringida al conjunto de puntos no errantes.

Específicamente

Teorema A. *Si G es un subgrupo finitamente generado de $\mathrm{Diff}^2_+(\mathrm{S}^1)$, entonces para cada sistema finito de generadores Γ de G, se tiene $h_\Gamma\,(G \circlearrowleft S^1) = h_\Gamma\,(G \circlearrowleft \Omega)$, donde Ω es el conjunto de puntos no errantes.*

Teorema B. *Si G es un subgrupo finitamente generado de $\mathrm{Homeo}_+(\mathrm{S}^1)$ sin elementos subexponencialmente distorsionados entonces para cada sistema finito de generadores Γ de G, se tiene $h_\Gamma(G \circlearrowleft S^1) = h_\Gamma\,(G \circlearrowleft \Omega)$.*

En el segundo capítulo consideramos el problema de hacer actuar grupos nilpotentes en el intervalo, por difeomorfismos de clase $C^{1+\alpha}$ y abordamos la siguiente pregunta.

Dado un grupo nilpotente, finitamente generado, libre de torsión, no abeliano G encontrar el supremo $\alpha(G)$ de los valores $\alpha \geq 0$ tal que G se incrusta en $Diff_+^{1+\alpha}([0,1])$ y probamos los siguientes resultados.

Teorema C. *Para todo $n \in \mathbb{N}$ y $\alpha < 1$ existe un subgrupo nilpotente, metabeliano de $Diff_+^{1+\alpha}([0,1])$ de grado de nilpotencia n.*

Teorema D. *Para todo $n \geq 2$ y $\alpha < \frac{2}{n(n-1)}$ el grupo N_{n+1} se incrusta en $Diff_+^{1+\alpha}([0,1])$, donde N_n denota el grupo (nilpotente) de las matrices triangulares inferiores de $n \times n$ con entradas enteras y unos en la diagonal.*

Abstract

We consider group actions on one–dimensional manifolds.

In Chapter 1, we show that the topological entropy of a group action, by C^2- diffeomorphisms, on the circle is equal to the topological entropy of the action restricted to the non–wandering set. More precisely, we prove the following two results.

Theorem A. *If G is a finitely generated subgroup of $\mathrm{Diff}^2_+(\mathrm{S}^1)$, then for every finite system of generators Γ of G, we have $h_\Gamma\left(G \circlearrowright S^1\right) = h_\Gamma\left(G \circlearrowright \Omega\right)$.*

Here Ω denotes the non-wandering set.

Theorem B. *If G is a finitely generated subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$ without sub–exponentially distorted elements, then for every finite system of generators Γ of G, we have $h_\Gamma(G \circlearrowright S^1) = h_\Gamma\left(G \circlearrowright \Omega\right)$.*

In Chapter 2, we consider nilpotent group actions, by $C^{1+\alpha}$-diffeomorphisms, on the interval. We tackle the following problem.

Given a finitely generated, torsion-free, non-Abelian, nilpotent group G, find the supremum $\alpha(G)$ of the values of $\alpha \geq 0$ such that G embeds into $\mathrm{Diff}^{1+\alpha}_+([0,1])$.

We prove the following.

Theorem C. *For each $n \in \mathbb{N}$ and each $\alpha < 1$, there exists a metabelian, nilpotent subgroup of $\mathrm{Diff}^{1+\alpha}_+([0,1])$ whose nilpotence degree equals n.*

Theorem D. *For each $n \geq 2$ and each $\alpha < \frac{2}{n(n-1)}$, the group N_{n+1} embeds into $\mathrm{Diff}^{1+\alpha}_+([0,1])$.*

Here N_n denotes the (nilpotent) group of $n \times n$ lower-triangular matrices with integer entries, all of which are equal to 1 on the diagonal.

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Introduction

In recent years, the study of group actions on manifolds has attracted the interest of many people because of its connexions with classical subjects as Rigidity Theory [7] and Foliation Theory [2]. This work deals with group actions on the circle and the interval, mainly by diffeomorphisms. Two different (though related) subjects are treated: the topological entropy for actions on the circle, and the differentiability for nilpotent group actions on the interval. To describe the first of these, we need to introduce some notation.

Let $(X, dist)$ be a compact metric space and G a group of homeomorphisms of X generated by a finite family of elements $\Gamma = \{g_1, \ldots, g_n\}$. To simplify, we will always assume that Γ is symmetric, that is, $g^{-1} \in \Gamma$ for every $g \in \Gamma$. For each $n \in \mathbb{N}$ we denote by $B_\Gamma(n)$ the ball of radius n in G (w.r.t. Γ), that is, the set of elements $f \in G$ which may be written in the form $f = g_{i_m} \cdots g_{i_1}$ for some $m \leq n$ and $g_{i_j} \in \Gamma$. For $g \in G$ we let $\|f\| = \|f\|_\Gamma := min\{n \colon f \in B_\Gamma(n)\}$

As in the classical case, given $\varepsilon > 0$ and $n \in \mathbb{N}$, two points x, y in X are said to be (n, ε)-separated if there exists $g \in B_\Gamma(n)$ such that $dist(g(x), g(y)) \geq \varepsilon$. A subset $A \subset X$ is (n, ε)-separated if all $x \neq y$ in A are (n, ε)-separated. We denote by $s(n, \varepsilon)$ the maximal possible cardinality (perhaps infinite) of a (n, ε)-separated set. The topological entropy for the action at the scale ε is defined by

$$h_\Gamma(G \circlearrowleft X, \varepsilon) = \limsup_{n \uparrow \infty} \frac{\log\big(s(n, \varepsilon)\big)}{n},$$

and the topological entropy is defined by

$$h_\Gamma(G \circlearrowleft X) := \lim_{\varepsilon \downarrow 0} h_\Gamma(G \circlearrowleft X, \varepsilon).$$

Notice that, although $h_\Gamma(G \circlearrowleft X, \varepsilon)$ depends on the system of generators, the properties of having zero, positive, or infinite entropy, are independent of this choice.

The definition above was proposed in [10] as an extention of the classical topological entropy of single maps (the definition extends to pseudo-groups of homeomorphisms, and hence is suitable for applications in Foliation Theory). Indeed, for a homeomorphism f, the topological entropy of the action of $\mathbb{Z} \sim \langle f \rangle$ equals two times the (classical) topological entropy of f. Nevertheless, the functorial properties of this notion remain unclear. For example, the following fundamental question is open.

General Question. Is it true that $h_\Gamma(G \circlearrowleft X)$ is equal to $h_\Gamma(G \circlearrowleft \Omega)$?

Here $\Omega = \Omega(G \circlearrowleft X)$ denotes the *non-wandering* set of the action, that is, the set of points $x \in X$ such that for every neighborhood U of x we have $f(U) \cap U \neq \emptyset$ for some $f \neq id$ in G. This is a closed invariant set whose complement Ω^c corresponds to the *wandering set* of the action.
The following example, due to Kleptsyn, shows that the answer to the question above is negative when X is the unit sphere (nevertheless, the action that we will discuss is not an action by diffeomorphisms).

Example. Let G_k be the fundamental group of a closed surface S_k of genus at least $k \geq 2$ endowed with the canonical generating set Γ_k. Then G_k freely acts by isometries of the hyperbolic plane $\mathbb{H}^2$, so that the quotient $\mathbb{H}^2/G_k$ corresponds to S_k. Let us consider the one-point compactification of $\mathbb{H}^2$, which is isomorphic to the two-sphere S^2. The action of G_k continuosly extends to an action on S^2 so that the point at infinity p_∞ is fixed by every element. It is not hard to see that $\{p_\infty\}$ coincides with the non-wandering set Ω_k of this action. Hence, $h_{\Gamma_k}(G_k \circlearrowleft \Omega_k) = 0$. We claim, however, that the entropy for the action of G_k on S^2 is strictly positive. More precisely, let us fix a point $q \in \mathrm{S}^2 \setminus \{p_\infty\}$. Let ε_1 be the minimum among the (finitely many, positive) numbers $dist_{\mathbb{S}^2}(q, q')$, where q' projects into the same point of S_k as q and they belong to "contiguous" fundamental regions (viewed in $\mathbb{H}^2$). Let $\varepsilon_2 := dist_{\mathrm{S}^2}(q, \mathrm{S}^2 \setminus R_q)$, where R_q is the union of the fundamental region containing q and all the regions contiguous to it. Finally, let $\varepsilon := min\{\varepsilon_1, \varepsilon_2\}$. We claim that

$$h_{\Gamma_k}(G_k \circlearrowleft \mathrm{S}^2, \varepsilon) > 0.$$

To show this, for each $g \in B_{\Gamma_k}(n)$ let $q_g := g(q)$. If we are able to show that these points are (n, ε)-separated then we are done, as the number of them coincides with the cardinality of $B_{\Gamma_k}(n)$, which growths exponentially on n (*c.f.* §1.1). Now, to see that these points are (n, ε)-separated, choose any two of them, say $q_g \neq q_h$. Then, by definition of ε, the distance between the points q and $g^{-1}h(q)$ is greater than or equal to ε. Notice that $q = g^{-1}(q_g)$ and $g^{-1}h(g) = g^{-1}(q_h)$. Since g^{-1} belongs to $B_{\Gamma_k}(g)$, the claim follows.

The first chapter of this work deals with the General Question in the case where X is a one–dimensional manifold. In this context, the notion of topological entropy for group actions is quite appropriate. In fact, in this case, the topological entropy is necessarily finite (*c.f.* §1.1). Moreover, in the case of actions by diffeomorphisms, the dichotomy $h_{top} = 0$ or $h_{top} > 0$ is well understood. Indeed, according to a result originally proved by Ghys, Langevin, and Walczak, for groups of C^2 diffeomorphisms [10], and extended by Hurder to groups of C^1 diffeomorphisms (see for instance [24]), we have $h_{top} > 0$ if and only if there exists a resilient orbit for the action. This means that there exist a group element f contracting by one side to a fixed point x_0, and another element g which sends x_0 into its basin of contraction by f.

The results of the first chapter of this work (which are reproduced from [12]) give a positive answer to the General Question in the context of group actions on the circle under certain mild assumptions.

Theorem A. *If G is a finitely generated subgroup of* $\mathrm{Diff}^2_+(\mathrm{S}^1)$*, then for every finite system of generators* Γ *of* G*, we have* $h_\Gamma(G \circlearrowleft S^1) = h_\Gamma(G \circlearrowleft \Omega)$.

Our proof for Theorem A actually works in the Denjoy class C^{1+bv}, and applies to general codimension-one foliations on compact manifolds. Here bv means bounded variation. In the class C^{1+Lip}, it is quite possible that we could give an alternative proof using standard techniques from Level Theory [4, 11].

It is unclear whether Theorem A extends to actions of lower regularity. However, it still holds under certain algebraic hypotheses. In fact, (quite unexpectedly) the regularity hypothesis is used to rule out the existence of elements $f \in G$ that fix some connected component of the wandering set and which are *distorted*, that is, those elements which satisfy

$$\lim_{n\to\infty} \frac{\|f^n\|}{n} = 0.$$

Actually, for the equality between the entropies it suffices to require that no elememt in G be *sub-exponentially distorted.* In other words, it suffices to require that, for each element $f \in G$ with infinite order, there exist a non–decreasing function $q\colon \mathbb{N} \to \mathbb{N}$ (depending on f) with sub–exponential growth satisfying $q(\|f^n\|) \geq n$, for every $n \in \mathbb{N}$. This is an algebraic condition which is satisfied by many groups, as for example nilpotent or free groups. (We refer the reader to [3] for a nice discussion on distorted elements, as well as [1] for a proof that every irrational rotation is distorted inside some finitely generated group of circle diffeomorphism.) Under this hypothesis, the following result holds.

Theorem B. *If G is a finitely generated subgroup of* $\mathrm{Homeo}_+(\mathrm{S}^1)$ *without sub–exponentially distorted elements, then for every finite system of generators* Γ *of* G*, we have*

$$h_\Gamma(G \circlearrowleft S^1) = h_\Gamma(G \circlearrowleft \Omega).$$

The hypothesis of the theorem above is natural, since distorted elements and the entropy of general group actions seem to be related in an interesting manner. Indeed, though the topological entropy of a single homeomorphism f may be equal to zero, if this map appears as a sub–exponentially distorted element inside an acting group, then this map may create positive entropy for the group action.

The second part of this work (Chapter 2) concerns the differentiability of nilpotent group actions on the interval. This fits into the general study of the algebraic constraints of finitely generated subgroups of $\mathrm{Diff}([0,1])$. The origin of this study relies on classical works on centralizers of C^2-diffeomorphisms of the interval [6, 14, 21, 22]. As a sample classical result which will be relevant for us, we can mention that –using the well-known Kopell lemma on commuting diffeomorphisms– Plante and Thurston showed in [19] that nilpotent groups of C^2-diffeomorphisms of $[0,1[$ (resp. $]0,1[$) are Abelian (resp. metabelian).

As is well known, most of the rigidity properties are lost when we consider centralizers of C^1-diffeomorphisms. In relation to Plante-Thurston's theorem above, this fact is corroborated by the work of Farb and Franks. In [8], they construct an embedding of N_n into $\mathrm{Diff}^1_+([0,1])$, where N_n denotes the (nilpotent) group of lower-triangular matrices whose entries are integers which equal 1 on the diagonal (and $n \geq 3$). Since every finitely generated, torsion-free, nilpotent group embeds into N_n for some n (see [20]), one concludes that all these groups can be realized as groups of C^1-diffeomorphisms of the (closed) interval. (Since the center of a nilpotent group is nontrivial, this provides examples of centralizers of diffeomorphisms with rich dynamics.)

In recent years, the study of intermediate differentiability classes (*i.e.* between C^1 and C^2) has become particularly relevant from both the dynamical and the group-theoretical viewpoints (see [5, 13, 15]). Recall that, for $0 < \alpha < 1$, a diffeomorphism f is said to be of class $C^{1+\alpha}$ if its derivative is α-continuous. In other words, there exists a constant M such that for all x, y we have

$$|f'(x) - f'(y)| \leq M|x-y|^\alpha.$$

We denote the group of $C^{1+\alpha}$-diffeomorphisms of $[0,1]$ by $\mathrm{Diff}^{1+\alpha}_+([0,1])$.

Main Problem. Given a finitely generated, torsion-free, non-Abelian, nilpotent group G, find the supremum $\alpha(G)$ of the values of $\alpha \geq 0$ such that G embeds into $\mathrm{Diff}^{1+\alpha}_+([0,1])$.

Although there is a big hope of answering this question by using the ideas from [5, 13, 15], the value of $\alpha(G)$ remains a mystery to guess in the general case. At the beginning we suspected that it was related to the nilpotence degree of the group. However, the following result (which is the main content of the first part of Chapter 2) shows that this is not the case.

Theorem C. *For each $n \in \mathbb{N}$ and each $\alpha < 1$, there exists a metabelian, nilpotent subgroup of $\mathrm{Diff}^{1+\alpha}_+([0,1])$ whose nilpotence degree equals n.*

The proof of this theorem uses techniques introduced by Denjoy and Pixton (a brilliant exposition of these techniques appears in [23]). Nevertheless, putting these methods in practice in the present case is far from being a trivial issue. The computations are quite involved, and some of them are only sketched.

In the second part of Chapter 2, we further elaborate on these techniques. Extending the main result of [8], we show the following theorem.

Theorem D. *For each $n \geq 2$ and each $\alpha < \frac{2}{n(n-1)}$, the group N_{n+1} embeds into $\mathrm{Diff}^{1+\alpha}_+([0,1])$.*

We suspect that this theorem is sharp though no result in this direction is given. Anyway, we point out that [5, Théorème B] implies that the actions constructed for the proof cannot be topologically conjugate to actions by $C^{1+\alpha}$-diffeomorphisms for any $\alpha > 1/(n-1)$. Filling the gap between the exponents $1 + 2/n(n-1)$ and $1 + 1/(n-1)$ is at the core of the Main

Problem above. Nevertheless, we strongly believe that Theorems C and D should shed some light in the pursue of a (hopefully, prompt) solution for it.

Chapter 1

On the topological entropy for group actions on the circle

1.1 Some background

In this chapter we will consider the normalized length on the circle, and every homeomorphism will be orientation preserving.

We begin by noticing that if G is a finitely generated group of circle homeomorphisms and Γ is a finite generating system for G, then for all $n \in \mathbb{N}$ and all $\varepsilon > 0$ one has

$$s(n, \varepsilon) \leq \frac{1}{\varepsilon} \# B_\Gamma(n). \tag{1.1}$$

Indeed, let A be a (n, ε)-separated set of cardinality $s(n, \varepsilon)$. Then for every two adjacent points x, y in A there exists $f \in B_\Gamma(n)$ such that $dist(f(x), f(y)) \geq \varepsilon$. For a fixed f, the intervals $[f(x), f(y)]$ which appear have disjoint interior. Since the total length of the circle is 1, any given f can be used in this construction at most $1/\varepsilon$ times, which immediately gives (1.1).

Notice that, taking the logarithm at both sides of (1.1), dividing by n, and passing to the limits, this gives

$$h_\Gamma(G \circlearrowright S^1) \leq gr_\Gamma(G),$$

where $gr_\Gamma(G)$ denotes the *growth* of G with respect to Γ, that is,

$$gr_\Gamma(G) = \lim_{n \to \infty} \frac{\log(\# B_\Gamma(n))}{n}.$$

Some easy consequences of this fact are the following ones:

- If G has sub-exponential growth, that is, if $gr_\Gamma(G) = 0$ (in particular, if G is nilpotent, or if G is the Grigorchuk-Maki's group considered in [15]), then $h_\Gamma(G \circlearrowright S^1) = 0$ for all finite generating systems Γ.

– In the general case, if $\#\Gamma = q \geq 1$, then from the relations

$$\#B_\Gamma(n) \leq 1 + \sum_{j=1}^{n} 2q(2q-1)^{j-1} = \begin{cases} 1 + \left(\frac{q}{q-1}\right)\left((2q-1)^n - 1\right), & q \geq 2, \\ 1 + 2n, & q = 1, \end{cases}$$

one concludes that

$$h_\Gamma(G \circlearrowright S^1) \leq \log(2q-1).$$

This shows in particular that the entropy of the action of G on S^1 is finite. Notice that this may be also deduced from the probabilistic arguments of [5] (see Théorème D therein). However, these arguments only yield the weaker estimate $h_\Gamma(G \circlearrowright S^1) \leq \log(2q)$ when Γ has cardinality q.

1.2 Some preparation for the proofs

The statement of Theorems A and B are obvious when the non-wandering set of the action equals the whole circle. Hence, we will assume in what follows that Ω is a proper subset of S^1, and we will currently denote by I some of the connected components of the complement of Ω. Let $Est(I)$ denote the stabilizer of I in G.

Lemma 1.2.1. *The stabilizer $Est(I)$ is either trivial or infinite cyclic.*

Proof. The (restriction to I of the) nontrivial elements of $Est(I)|_I$ have no fixed points, for otherwise these points would be non-wandering. Thus $Est(I)|_I$ acts freely on I, and according to Hölder Theorem [9, 16], its action is semiconjugate to an action by translations. We claim that, if $Est(I)|_I$ is nontrivial, then it is infinite cyclic. Indeed, if not then the corresponding group of translations is dense. This implies that the preimage by the semiconjugacy of any point whose preimage is a single point corresponds to a non-wandering point for the action. Nevertheless, this contradicts the fact that I is contained in Ω^c.

If $Est(I)|_I$ is trivial then $f|_I$ is trivial for every $f \in Est(I)$, and hence f itself must be the identity. We then conclude that $Est(I)$ is trivial.

Analogously, $Est(I)$ is cyclic if $Est(I)|_I$ is cyclic. In this case, $Est(I)|_I$ is generated by the restriction to the interval I of the generator of $Est(I)$. □

Definition 1.2.2. A connected component I of Ω^c will be called of *type 1* if $Est(I)$ is trivial, and will be called of *type 2* if $Est(I)$ is infinite cyclic.

Notice that the type of an interval is preserved by the action, that is, for each $f \in G$ the interval $f(I)$ is of type 1 (resp. of type 2) if I is of type 1 (resp. of type 2). Moreover, given two connected components of type 1 of Ω^c, there exists at most one element in G sending the former into the latter. Indeed, if $f(I) = g(I)$ then $g^{-1}f$ is in the stabilizer of I, and hence $f = g$ if I is of type 1.

Lemma 1.2.3. Let $x_1, \ldots, x_m$ be points contained in a single type 1 connected component of Ω^c. If for some $\varepsilon > 0$ the points x_i, x_j are (ε, n)-separated for every $i \neq j$, then $m \leq 1 + \frac{1}{\varepsilon}$.

Proof. Let $I =]a, b[$ be the connected component of type 1 of Ω^c containing the points $x_1, \ldots, x_m$. After renumbering the x_i's, we may assume that $a < x_1 < x_2 < \ldots < x_m < b$. For each $1 \leq i \leq m-1$ one can choose an element $g_i \in B_\Gamma(n)$ such that $dist(g_i(x_i), g_i(x_{i+1})) \geq \varepsilon$. Now, since I is of type 1, the intervals $]g_i(x_i), g_i(x_{i+1})[$ are two by two disjoint. Therefore, the number of these intervals times the minimal length among them is less than or equal to 1. This gives $(m-1)\varepsilon \leq 1$, thus proving the lemma. □

The case of connected components I of type 2 of Ω^c is much more complicated than the one of type 1 connected components. The difficulty is related to the fact that, if the generator of the stabilizer of I is sub-exponentially distorted in G, then this would imply the existence of exponentially many (n, ε)-separated points inside I, and hence a relevant part of the entropy would be "concentrated" in I. To deal with this problem, for each connected component I of type 2 of Ω^c we denote by p_I its middle point, and then we define $\ell_I \colon G \to \mathbb{N}_0$ as follows. Let h be the generator of the stabilizer of I such that $h(x) > x$ for all x in I. For each $f \in G$ the element fhf^{-1} is the generator of the stabilizer of $f(I)$ with the analogous property. We then let $\ell_I(f) := |r|$, where r is the unique integer number such that

$$fh^r f^{-1}(p_{f(I)}) \leq f(p_I) < fh^{r+1} f^{-1}(p_{f(I)}).$$

Lemma 1.2.4. For all f, g in G one has

$$\ell_I(g \circ f) \leq \ell_{f(I)}(g) + \ell_I(f) + 1.$$

Proof. Let r be the unique integer number such that

$$(fhf^{-1})^r(p_{f(I)}) \leq f(p_I) < (fhf^{-1})^{r+1}(p_{f(I)}), \tag{1.2}$$

and let s be the unique integer number such that

$$(gfhf^{-1}g^{-1})^s(p_{gf(I)}) \leq g(p_{f(I)}) < (gfhf^{-1}g^{-1})^{s+1}(p_{gf(I)}),$$

so that

$$l_I(f) = |r|, \quad l_{f(I)}(g) = |s|.$$

We then have

$$g^{-1}(gfhf^{-1}g^{-1})^s(p_{gf(I)}) \leq p_{f(I)} < g^{-1}(gfhf^{-1}g^{-1})^{s+1}(p_{gf(I)}),$$

that is

$$(fhf^{-1})^s g^{-1}(p_{gf(I)}) \leq p_{f(I)} < (fhf^{-1})^{s+1} g^{-1}(p_{gf(I)}).$$

Therefore,

$$(fhf^{-1})^r(fhf^{-1})^s g^{-1}(p_{gf(I)}) \leq f(p_I) < (fhf^{-1})^{r+1}(fhf^{-1})^{s+1}g^{-1}(p_{gf(I)}),$$

and hence

$$(fhf^{-1})^{r+s}g^{-1}(p_{gf(I)}) \leq f(p_I) < (fhf^{-1})^{r+s+2}g^{-1}(p_{gf(I)}).$$

This easily gives

$$g(fhf^{-1})^{r+s}g^{-1}(p_{gf(I)}) \leq gf(p_I) < g(fhf^{-1})^{r+s+2}g^{-1}(p_{gf(I)}),$$

and thus

$$(gfhf^{-1}g^{-1})^{r+s}(p_{gf(I)}) \leq gf(p_I) < (gfhf^{-1}g^{-1})^{r+s+2}(p_{gf(I)}).$$

This shows that $l_I(gf)$ equals either $|r+s|$ or $|r+s+1|$, which concludes the proof. □

The following corollary is a direct consequence of the preceding lemma, but may be proved independently.

Corollary 1.2.5. For every $f \in G$ one has

$$|\ell_I(f) - \ell_{f(I)}(f^{-1})| \leq 1.$$

Proof. From (1.2) one obtains

$$h^{-(r+1)}(p_I) < f^{-1}(p_{f(I)}) \leq h^{-r}(p_I) < h^{-r+1}(p_I),$$

and hence $\ell_{f(I)}(f^{-1})$ equals either $|r|$ or $|r+1|$. Since $\ell_I(f) = |r|$, the corollary follows. □

1.3 The proof in the smooth case: Theorem A

To rule out the possibility of "concentration" of the entropy on a type 2 connected component I of Ω^c, in the C^2 case we will use classical control of distortion arguments in order to construct, starting from the function ℓ_I, a kind of quasi-morphism from G into $\mathbb{N}_0$. Slightly more generally, let $\mathcal{F}$ be any finite family of connected components of type 2 of Ω^c. We denote by $\mathcal{F}^G$ the family formed by all the intervals contained in the orbits of the intervals in $\mathcal{F}$. For each $f \in G$ we then define

$$\ell_{\mathcal{F}}(f) := \sup_{I \in \mathcal{F}^G} \ell_I(f).$$

A priori, the value of $\ell_{\mathcal{F}}$ could be infinite. We claim however that, for groups of C^2 diffeomorphisms, its value is necessarily finite for every element f.

Proposition 1.3.1. Let $\mathcal{F}$ be a finite family of type 2 connected components of Ω^c. Then, for all $f \in G$, the value of $\ell_{\mathcal{F}}(f)$ is finite.

To show this proposition, we will need to estimate the function $\ell_I(f)$ in terms of the distortion of f on the interval I.

Lemma 1.3.2. For each fixed type 2 connected component I of Ω^c and every $g \in G$, the value of $\ell_I(g)$ is bounded from above by a number $L(V)$ depending on $V = var(\log(g'|_I))$, the total variation of the logarithm of the derivative of the restriction of g to I.

Proof. Denote $]a,b[= I$ and $]\bar{a},\bar{b}[= g(I)$. If h is a generator for the stabilizer of I, then for every $f \in G$ the value of $\ell_I(f)$ corresponds (up to some constant ± 1) to the number of fundamental domains for the dynamics of fhf^{-1} on $f(I)$ between the points $p_{f(I)}$ and $f(p_I)$, which in its turn corresponds to the number of fundamental domains for the dynamics of h on I between $f^{-1}(p_{f(I)})$ and p_I. Therefore, we need to show that there exists $c < d$ in $]a,b[$ depending on V and such that $g^{-1}(p_{g(I)})$ belongs to $[c,d]$. We will show that this happens for the values

$$c = a + \frac{|I|}{2e^V} \qquad \text{and} \qquad d = b - \frac{|I|}{2e^V}.$$

We will just check that the first choice works, leaving the second one to the reader. By the Mean Value Theorem, there exists $x \in g(I)$ and $y \in [\bar{a}, p_{g(I)}]$ such that

$$(g^{-1})'(x) = \frac{|I|}{|g(I)|}$$

and

$$(g^{-1})'(y) = \frac{|g^{-1}([\bar{a}, p_{f(I)}])|}{|[\bar{a}, p_{g(I)}]|} = \frac{g^{-1}(p_{g(I)}) - a}{|g(I)|/2}.$$

By the definition of the constant V, we have $(g^{-1})'(x)/(g^{-1})'(y) \leq e^V$. This gives

$$e^V \geq \frac{|I|/|g(I)|}{2(g^{-1}(p_{g(I)}) - a)/|g(I)|} = \frac{|I|}{2(g^{-1}(p_{g(I)}) - a)},$$

thus proving that $g^{-1}(p_{g(I)}) \geq a + \frac{|I|}{2e^V}$, as we wanted to show. □

Proof of Proposition 1.3.1. Let $J =]\bar{a},\bar{b}[$ be an interval in the orbit by G of $I =]a,b[$. If $g = g_{i_n} \cdots g_{i_1}$, $g_{i_j} \in \Gamma$, is an element of minimal length sending I into J, then the intervals $I, g_{i_1}(I), g_{i_2}g_{i_1}(I), \ldots, g_{i_{n-1}} \cdots g_{i_2}g_{i_1}(I)$ are pairwise disjoint. Therefore,

$$var(\log(g'|_I)) \leq \sum_{j=0}^{n-1} var(\log(g'_{i_{j+1}}|_{g_{i_j}\cdots g_{i_1}(I)}) \leq \sum_{h\in\Gamma} var(\log(h')) =: W.$$

Moreover, denoting $V = var(\log(f'))$,

$$var(\log((fg)'|_I)) \leq var(\log(g'|_I)) + var(\log(f')) = W + V.$$

By Lemmas 1.2.4 and 1.3.2 and Corollary 1.2.5,

$$\begin{aligned}\ell_J(f) &\leq \ell_J(g^{-1}) + \ell_I(fg) + 1\\ &\leq \ell_I(g) + \ell_I(fg) + 2\\ &\leq L(W) + L(W+V) + 2.\end{aligned}$$

This shows the proposition when $\mathcal{F}$ consists of a single interval. The case of general finite $\mathcal{F}$ follows easily. □

For a given $\varepsilon > 0$ we define $\ell_\varepsilon := \ell_{\mathcal{F}_\varepsilon}$, where $\mathcal{F}_\varepsilon = \{I_1, \ldots, I_k\}$ is the family of the connected components of Ω^c having length greater than or equal to ε, with $k = k(\varepsilon)$. Notice that, by Lemma 1.2.4, for every f, g in Γ one has

$$\ell_\varepsilon(gf) \leq \ell_\varepsilon(g) + \ell_\varepsilon(f) + 1 \tag{1.3}$$

Lemma 1.3.3. There exists constants $A(\varepsilon) > 0$ and $B(\varepsilon)$ satisfying the following property: If $x_1, \ldots, x_m$ are points contained in a single connected component of type 2 of Ω^c and x_i, x_j are (ε, n)-separated for every $i \neq j$, then $m \leq A(\varepsilon)n + B(\varepsilon)$.

Proof. Denote $c_\varepsilon = \max\{\ell_\varepsilon(g) \colon g \in \Gamma\}$ (according to Proposition 1.3.1, the value of c_ε is finite). Let I be the type 2 connected component of Ω^c containing $x_1, \ldots, x_m$. We may assume that $x_1 < x_2 < \ldots < x_m$. For each $1 \leq i \leq k$ let h_i be the generator of $Est(I_i)$. Notice that $\ell_\varepsilon(h_i^r) \geq |r|$ for all $r \in \mathbb{Z}$.

If f is an element in $B_\Gamma(n)$ sending I into some I_i, then the number of points which are ε-separated by f is less than or equal to $1/\varepsilon + 1$. We claim that the number of elements in $B_\Gamma(n)$ sending I into I_i is bounded above by $4nc_\varepsilon + 4n - 1$. Indeed, if g also sends I onto I_i then $gf^{-1} \in Est(I_i)$, hence $gf^{-1} = h_i^r$ some r. Therefore, using (1.3) one obtains $|r| \leq \ell_\varepsilon(h_i^r) \leq 2nc_\varepsilon + 2n - 1$.

Since the previous arguments apply to each type 2 interval I_i, we have

$$m \leq k\Big(\frac{1}{\varepsilon} + 1\Big)(4nc_\varepsilon + 4n - 1).$$

Therefore, letting

$$A(\varepsilon) := \Big(4k + \frac{4k}{\varepsilon}\Big)(1 + c_\varepsilon) \qquad \text{and} \qquad B(\varepsilon) = -\Big(k + \frac{k}{\varepsilon}\Big),$$

this concludes the proof. □

To conclude the proof of Theorem A, we will use the following notation. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we will denote by $s(n, \varepsilon)$ the largest cardinality of a (n, ε)−separated subset of S^1. Likewise, $s_\Omega(n, \varepsilon)$ will denote the largest cardinality of a (n, ε)−separated set contained in the non-wandering set.

Proof of Theorem A. Fix $0 < \varepsilon < 1/(2L)$, where L is a common Lipschitz constant for the elements in Γ. We will show that, for some function p_ε growing linearly on n (and whose coefficients depend on ε), one has

$$s(n,\varepsilon) \leq p_\varepsilon(n) s_\Omega(n,\varepsilon) + p_\varepsilon(n). \tag{1.4}$$

Actually, any function p_ε with sub-exponential growth and verifying such an inequality suffices. Indeed, taking the logarithm in both sides, dividing by n, and passing to the limit, this implies that

$$h_\Gamma(G \circlearrowleft \mathrm{S}^1, \varepsilon) = h_\Gamma(G \circlearrowleft \Omega, \varepsilon).$$

Letting ε go to zero, this gives

$$h_\Gamma(G \circlearrowleft \mathrm{S}^1) \leq h_\Gamma(G \circlearrowleft \Omega).$$

Since the opposite inequality is obvious, this shows the desired equality between the entropies.

To show (1.4), fix a (n,ε)-separated set S containing $s(n,\varepsilon)$ points. Let n_Ω (resp. n_{Ω^c}) be the number of points in S which are in Ω (resp. in Ω^c). Obviously, $s(n,\varepsilon) = n_\Omega + n_{\Omega^c}$. Let $t = t_S$ be the number of connected components of Ω^c containing points in S, and let $l := [\frac{t}{2}]$, where $[\cdot]$ denotes the integer part function. We will show that there exists a (n,ε)-separated set T contained in Ω having cardinality l. This will obviously give $s_\Omega(n,\varepsilon) \geq l$. Using the inequalities $t \leq 2l+1$ and $n_\Omega \leq s_\Omega(n,\varepsilon)$, and by Lemmas 1.2.3 and 1.2.4, this will imply that

$$\begin{aligned} s(n,\varepsilon) &= n_\Omega + n_{\Omega^c} \\ &\leq n_\Omega + tk\big(1+\frac{1}{\varepsilon}\big)(4nc_\varepsilon + 4n - 1) \\ &\leq s_\Omega(n,\varepsilon) + (2s_\Omega(n,\varepsilon)+1)k\big(1+\frac{1}{\varepsilon}\big)(4nc_\varepsilon + 4n - 1), \end{aligned}$$

thus showing (1.4).

To show the existence of the set T with the properties above, we proceed in a constructive way. Let us number the connected components of Ω^c containing points in S in a cyclic way by $I_1, \ldots, I_t$. Now for each $1 \leq i \leq l$ choose a point $t_i \in \Omega$ between I_{2i-1} and I_{2i}, and let $T := \{t_1, \ldots, t_l\}$. We need to check that, for $i \neq j$, the points t_i and t_j are (n,ε)-separated. Now by construction, for each $i \neq j$ there exist at least two different points x, y in S contained in the interval of smallest length in S^1 joining t_i and t_j. Since S is a (n,ε)-separated set, there exist $m \leq n$ and $g_{i_1}, \ldots, g_{i_m}$ in Γ so that $dist(h(x), h(y)) \geq \varepsilon$, where $h = g_{i_m} \cdots g_{i_2} g_{i_1}$. Unfortunately, because of the topology of the circle, this does not imply that $dist(h(t_i), h(t_j)) \geq \varepsilon$. However, the proof will be finished if we show that

$$dist\big(g_{i_r} \cdots g_{i_1}(t_i), g_{i_r} \cdots g_{i_1}(t_j)\big) \geq \varepsilon \quad \text{for some} \quad 0 \leq r \leq m. \tag{1.5}$$

This claim is obvious if $dist(t_i, t_j) \geq \varepsilon$. If this is not the case then, by the definition of the constants ε and L, the length of the interval $[g_{i_1}(t_i), g_{i_1}(t_j)]$ is smaller than $1/2$, and hence it coincides with the distance between its endpoints. If this distance is at least ε, then we are done. If not, the same argument shows that the length of the interval $[g_{i_2}g_{i_1}(t_i), g_{i_2}g_{i_1}(t_j)]$ is smaller than $1/2$ and coincides with the distance between its endpoints. If this length is at least ε, then we are done. If not, we continue the procedure... Clearly, there must be some integer $r \leq m$ such that the length of the interval $[g_{i_{r-1}} \cdots g_{i_1}(t_i), g_{i_{r-1}} \cdots g_{i_1}(t_j)]$ is smaller than ε, but the one of $[g_{i_r} \cdots g_{i_1}(t_i), g_{i_r} \cdots g_{i_1}(t_j)]$ is greater than or equal to ε. As before, the length of the later interval will be forced to be smaller than $1/2$, and hence it will coincide with the distance between its endpoints. This shows (1.5) and concludes the proof of Theorem A. □

1.4 The proof in the case of nonexistence of sub-exponentially distorted elements: Theorem B

Recall that the topological entropy is invariant under topological conjugacy. Therefore, due to [5, Théorème D], in order to prove Theorem B we may assume that G is a group of bi-Lipschitz homeomorphisms. Let L be a common Lipschitz constant for the elements in Γ. Fix again $0 < \varepsilon < 1/2L$, and let $I_1, \ldots, I_k$ be the connected components of Ω^c having length greater than or equal to ε. Let h_i be a generator for the stabilizer of I_i (with $h_i = Id$ in case where I_i is of type 1). Consider the minimal non decreasing function q_ε such that, for each of the nontrivial h_i's, one has $q_\varepsilon(\|h_i^r\|) \geq r$ for all positive r. We will show that (1.4) holds for the function

$$p_\varepsilon(n) = 2k\big(1 + \frac{1}{\varepsilon}\big)(2q_\varepsilon(2n) + 1) + 1.$$

Notice that, by assumption, this function p_ε growths at most sub-exponentially on n. Hence, as in the case of Theorem A, inequality (1.4) allows to finish the proof of the equality between the entropies.

The main difficulty for showing (1.4) in this case is that Lemma 1.3.3 is no longer available. However, the following still holds.

Lemma 1.3.4. If $x_1, \ldots, x_m$ are points contained in a single type 2 connected component I of Ω^c having length at least ε, and x_i, x_j are (ε, n)-separated for every $i \neq j$, then $m \leq k\big(\frac{1}{\varepsilon} + 1\big)(2q_\varepsilon(2n) + 1)$.

Proof. Let I be the type 2 connected component of Ω^c containing $x_1, \ldots, x_m$. We may assume that $x_1 < x_2 < \ldots < x_m$. If f is an element in $B_\Gamma(n)$ sending I into some I_i, then the number of points which are ε-separated by f is less than or equal to $1/\varepsilon + 1$. We claim that the number of elements in $B_\Gamma(n)$ sending I into I_i is bounded above by $q_\varepsilon(r)$. Indeed, if g also sends I onto I_i then $gf^{-1} \in Est(I_i)$, hence $gf^{-1} = h_i^r$ some r. Therefore,

$$2n \geq \|gf^{-1}\| = \|h_i^r\|,$$

and hence

$$q_\varepsilon(2n) \geq q_\varepsilon(\|h_i^r\|) \geq |r|.$$

Since the previous arguments apply to each type 2 interval I_i, this gives

$$m \leq k\big(\frac{1}{\varepsilon} + 1\big)(2q_\varepsilon(2n) + 1),$$

thus proving the lemma. □

To show (1.4) in the present case, we proceed as in the proof of Theorem A. We fix a (n, ε)-separated set S containing $s(n, \varepsilon)$ points. We let n_Ω (resp. n_{Ω^c}) be the number of points in S which are in Ω (resp. in Ω^c), so that $s(n,\varepsilon) = n_\Omega + n_{\Omega^c}$. Let $t = t_S$ be the number of connected components of Ω^c containing points in S, and let $l := [\frac{t}{2}]$, where $[\cdot]$ denotes the integer part function. As before, one can show that there exists a (n, ε)-separated set T contained in Ω having cardinality l. This will obviously give $s_\Omega(n, \varepsilon) \geq l$. Inequalities $t \leq 2l + 1$ and $n_\Omega \leq s_\Omega(n, \varepsilon)$ still holds. Using Lemmas 1.2.3 and 1.3.4 one now obtains

$$\begin{aligned} s(n,\varepsilon) &= n_\Omega + n_{\Omega^c} \\ &\leq n_\Omega + tk\big(1 + \frac{1}{\varepsilon}\big)(2q_\varepsilon(2n) + 1) \\ &\leq s_\Omega(n,\varepsilon) + (2s_\Omega(n,\varepsilon) + 1)k\big(1 + \frac{1}{\varepsilon}\big)(2q_\varepsilon(2n) + 1). \end{aligned}$$

This concludes the proof of Theorem B.

Chapter 2

Nilpotent groups of diffeomorphisms of the interval

2.1 A reminder on Denjoy-Pixton actions

For the constructions leading to the proofs of Theorems C and D, we will use Pixton's technique [18]. The main technical tool will be the following lemma from [23].

Lemma 2.1.1. *For a certain universal constant M, there exists a family of diffeomorphisms $\varphi_{I,I'}^{J,J'} : I \to J$ where I, I', J, J' are any non-degenerate intervals such that I' (resp. J') is contiguous to I (resp. J) on the left, satisfying $\varphi_{I,I'}^{J,J'} \circ \varphi_{J,J'}^{K,K'} = \varphi_{I,I'}^{K,K'}$. Moreover, one has*

$$\frac{\left| \log\left(\frac{\partial}{\partial x}\varphi_{I,I'}^{J,J'}(u)\right) - \log\left(\frac{\partial}{\partial x}\varphi_{I,I'}^{J,J'}(v)\right)\right|}{|u-v|} \leq \frac{M}{|I|}\left|\frac{|I||J'|}{|J||I'|} - 1\right|$$

for all u, v in I provided that $1/2 \leq |I|/|J| \leq 2$. Furthermore, the derivative of $\varphi_{I,I'}^{J,J'}$ equals $|J|/|I|$ (resp. $|J'|/|I'|$) at the right endpoint (resp. left endpoint) of I.

The proof of this lemma given in [23] proceeds as follows. Let $\xi(x)(\frac{\partial}{\partial x})$ be a C^∞ vector field on $[0,1]$ such that $\xi(x) = x$ near 0, and $\xi(x) = 0$ on $[1/2, 1]$. Moreover, assume that, for all x,

$$\left|\frac{\partial \xi}{\partial x}(x)\right| \leq 1.$$

Let $\psi_t(x)$ be the solution of the differential equation

$$\frac{d\psi_t}{dt}(x) = \xi(\psi_t(x)), \quad \psi_0(x) = x.$$

Let us consider the diffeomorphism $x \mapsto b\psi_t(x/a)$ sending the interval $[0,a]$ onto the interval $[0,b]$. For any real numbers a', a, b', b such that $a' < 0 < a$ and $b' < 0 < b$, let $\phi_{b',b}^{a',a}$ be the diffeomorphism from $[0,a]$ onto $[0,b]$ defined by

$$\phi_{a',a}^{b',b}(x) := b\psi_{\log(b'a/a'b)}(x/a).$$

Its is easy to check that for all positive a, b, c and all negative a', b', c', one has

$$\phi_{b',b}^{c',c} \circ \phi_{a',a}^{b',b} = \phi_{a',a}^{c',c}$$

Moreover, as is shown in [23],

$$\log \frac{\partial \phi_{a',a}^{b',b}}{\partial x}(x) = \log \frac{b}{a} + \log \frac{\partial \psi_{\log(b'a/a'b)}}{\partial x}\left(\frac{x}{a}\right), \tag{2.1}$$

$$\left|\log \frac{\partial \psi_{\log(b'a/a'b)}}{\partial x}\right| \leq \left|\log \frac{b'a}{a'b}\right| = \left|\log \frac{b'}{a'} - \log \frac{b}{a}\right|. \tag{2.2}$$

Furthermore, letting $C > 0$ be a constant such that $|\frac{\partial^2 \xi}{\partial x^2}| \leq C$ for all x, we have

$$\left|\left(\frac{\partial}{\partial x}\right) \log \frac{\partial \phi_{a',a}^{b',b}}{\partial x}(x)\right| \leq \frac{C}{a}\left|\frac{b'a}{a'b} - 1\right|. \tag{2.3}$$

Starting with the maps $\phi_{a',a}^{b',b}$, we construct the desired family $\{\varphi_{I,I'}^{J,J'}\}$ as follows. Letting $I := [w, w+a]$, $I' := [w+a', w]$, $J := [w', w'+b]$, and $J' := [w'+b', w']$, where $a' < 0 < a$ and $b' < 0 < b$, we let

$$\varphi_{I,I'}^{J,J'} := \phi_{a',a}^{b',b}(x-w) + w'.$$

2.2 A family of metabelian subgroups of $\mathrm{Diff}_+^{1+\alpha}([0,1])$: Theorem C

In what follows, M will denote a universal constant whose explicit value is irrelevant for our purposes.

For each pair of integers (i,j), let $I_{i,j}$ be an interval of length $|I_{i,j}|$ so that the sum $\sum_{i,j} |I_{i,j}|$ is finite. Joining these intervals lexicographically, we obtain a closed interval I. Following [8, §2.3], we will deal with a particular family of nilpotent groups G_n acting on I. Each G_n has nilpotence degree $n+1$, and G_1 coincides with the Heisenberg group.

The group G_n has a presentation

$$\big\langle f, g_0, g_1, \ldots, g_n \colon [g_i, g_j] = id,\ [f, g_0] = id,\ [f, g_i] = g_{i-1} \text{ for all } i \geq 1 \big\rangle.$$

As maps, the generators send each interval $I_{i,j}$ into a certain $I_{i',j'}$ and coincide with the diffeomorphism $\varphi_{I_{i,j},I_{i,j-1}}^{I_{i',j'},I_{i',j'-1}}$ therein. The map f sends $I_{i,j}$ into $I_{i+1,j}$. The maps g_0 and g_1

send $I_{i,j}$ into $I_{i,j+1}$ and $I_{i,j+i}$, respectively. To describe the dynamics of $g_2, \ldots, g_n$, for each $0 < k \leq n$ and each $i \in \mathbb{Z}$, we let

$$r_k(i) := \frac{i(i+1)(i+2)\ldots(i+k-1)}{k!},$$

and we define $r_0(i) := 1$ for all i. (Note that $|r_k(i)| \leq |i|^k$ for $k > 0$.) Then the element g_k sends the interval $I_{i,j}$ into $I_{i,j+r_k(i)}$.

Now fix a positive number $\alpha < 1$. To carry out the preceding construction so that the resulting maps are $C^{1+\alpha}$-diffeomorphisms of I, we need to make a careful choice of the lengths $|I_{i,j}|$. We let $q > 1$ be such that the following conditions are satisfied:

($\mathrm{i_C}$) $1 < q < 2$,

($\mathrm{ii_C}$) $\alpha < 2 - q$,

($\mathrm{iii_C}$) $\alpha < \frac{q}{2q-1}$,

($\mathrm{iv_C}$) $\alpha < \frac{1}{q}$.

Note that since $\alpha < 1$ and the preceding right-side expressions go to 1 or to infinity as q tends to 1 from above, we may choose q very near to 1 in such a way that these conditions are fulfilled.

Now let $p := \frac{2q-1}{q-1}$. Clearly, we may also impose the following supplementary conditions:

($\mathrm{v_C}$) $p > nq$,

($\mathrm{vi_C}$) $\alpha \leq \frac{1}{q} - \frac{n}{p}$,

($\mathrm{vii_C}$) $\alpha < \frac{1}{q-1} - \frac{nq}{2q-1}$.

We then define

$$|I_{i,j}| := \frac{1}{|i|^p + |j|^q + 1}.$$

Since $1/p + 1/q < 1$, it follows from [13, §3] that the sum $\sum_{i,j} |I_{i,j}|$ is finite. We claim that the group G_n thus obtained is formed by $C^{1+\alpha}$-diffeomorphisms of I.

2.2.1 The map f is of class $C^{1+\alpha}$

For simplicity, we only deal with points in the intervals $I_{i,j}$ with $i \geq 0$ and $j \geq 0$ (the other cases are analogous).

First we consider x, y in the same interval $I_{i,j}$. We have

$$\frac{|\log f'(x) - \log f'(y)|}{|x-y|} \leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right|.$$

Hence

$$\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha} \leq \frac{M}{|I_{i,j}|} \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right| |I_{i,j}|^{1-\alpha} = \frac{M}{|I_{i,j}|^\alpha} \left| \frac{|I_{i,j}|}{|I_{i+1,j}|} \frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} - 1 \right|.$$

This yields

$$\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha} \leq M \left| \frac{(i+1)^p + j^q + 1}{i^p + j^q + 1} \, \frac{i^p + (j-1)^q + 1}{(i+1)^p + (j-1)^q + 1} - 1 \right| (i^p + j^q + 1)^\alpha.$$

Therefore, the value of $\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha}$ is bounded from above by

$$M \left| \frac{((i+1)^p + j^q + 1)(i^p + (j-1)^q + 1) - (i^p + j^q + 1)((i+1)^p + (j-1)^q + 1)}{(i^p + j^q + 1)^{1-\alpha}((i+1)^p + (j-1)^q + 1)} \right|$$

which equals

$$M \frac{\big((i+1)^p - i^p\big)\big(j^q - (j-1)^q\big)}{(i^p + j^q + 1)^{1-\alpha}((i+1)^p + (j-1)^q + 1)}.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M \frac{i^{p-1} j^{q-1}}{(i^p + j^q + 1)^{1-\alpha}((i+1)^p + (j-1)^q + 1)}.$$

Thus

$$\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha} \leq M \frac{i^{p-1}}{(i+1)^p} \frac{j^{q-1}}{j^{q(1-\alpha)}}.$$

Now notice that the last expression is uniformly bounded when $q - 1 \leq q(1-\alpha)$, which is satisfied by condition ($\mathrm{iv_C}$).

Next we consider $x \in I_{i,j}$ and $y \in I_{i,j'}$, with $j < j'$. The definition of f and property (2.1) yield

$$\log f'(x) = \log \frac{\partial \phi_{b',b}^{a',a}}{\partial x}(x - w) = \log \frac{b}{a} + \log \frac{\partial \psi_{\log(b'a/a'b)}}{\partial x} \left(\frac{x-w}{a} \right)$$

where $I_{i,j} = [w, w+a]$, $I_{i,j-1} = [w+a', w]$, $I_{i+1,j} = [w', w'+b]$, and $I_{i+1,j-1} = [w'+b', w']$. Analogously,

$$\log f'(y) = \log \frac{\partial \phi_{d',d}^{c',c}}{\partial y}(y-u) = \log \frac{d}{c} + \log \frac{\partial \psi_{\log(d'c/c'd)}}{\partial y} \left(\frac{y-u}{c} \right)$$

where $I_{i,j'} = [u, u+c]$, $I_{i,j'-1} = [u+c', u]$, $I_{i+1,j'} = [u', u'+d]$, and $I_{i+1,j'-1} = [u'+d', u']$. By property (2.2),

$$\begin{aligned} |\log f'(x) - \log f'(y)| &\leq \left| \log \frac{b}{a} - \log \frac{d}{c} \right| + \left| \log \frac{\partial \psi_{\log(b'a/a'b)}}{\partial x} \left(\frac{x-w}{a} \right) \right| + \left| \log \frac{\partial \psi_{\log(d'c/c'd)}}{\partial y} \left(\frac{y-u}{c} \right) \right| \\ &\leq \left| \log \frac{b}{a} - \log \frac{d}{c} \right| + \left| \log \frac{b'}{a'} - \log \frac{b}{a} \right| + \left| \log \frac{d'}{c'} - \log \frac{d}{c} \right|. \end{aligned}$$

Notice that the last expression corresponds to

$$\left| \log \left(\frac{|I_{i+1,j}|}{|I_{i,j}|} \right) - \log \left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|} \right) \right| + \left| \log \left(\frac{|I_{i+1,j-1}|}{|I_{i,j-1}|} \right) - \log \left(\frac{|I_{i+1,j}|}{|I_{i,j}|} \right) \right| + \left| \log \left(\frac{|I_{i+1,j'-1}|}{|I_{i,j'-1}|} \right) - \log \left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|} \right) \right|.$$

Since the function $j \mapsto \frac{|I_{i+1,j}|}{|I_{i,j}|}$ is non-decreasing, the preceding inequality yields

$$\begin{aligned}\big|\log f'(x) - \log f'(y)\big| &\leq 3\left|\log\left(\frac{|I_{i+1,j'}|}{|I_{i,j'}|}\right) - \log\left(\frac{|I_{i+1,j-1}|}{|I_{i,j-1}|}\right)\right| \\ &= 3\left|\log \frac{|I_{i+1,j'}|}{|I_{i,j'}|}\frac{|I_{i,j-1}|}{|I_{i+1,j-1}|}\right| \\ &= 3\left|\log \frac{(i^p + j'^q + 1)}{((i+1)^p + j'^q + 1)}\frac{((i+1)^p + (j-1)^q + 1)}{(i^p + (j-1)^q + 1)}\right|.\end{aligned}$$

Hence the value of $\big|\log f'(x) - \log f'(y)\big|$ is bounded from above by

$$M\left|\log\left(1 + \frac{(i^p + j'^q + 1)((i+1)^p + (j-1)^q + 1) - ((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}{((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}\right)\right|. \tag{2.4}$$

Since $\frac{i^p + j'^q + 1}{(i+1)^p + j'^q + 1}\frac{(i+1)^p + (j-1)^q + 1}{i^p + (j-1)^q + 1}$ is uniformly bounded from below, namely

$$\frac{i^p + j'^q + 1}{(i+1)^p + j'^q + 1}\frac{(i+1)^p + (j-1)^q + 1}{i^p + (j-1)^q + 1} \geq \frac{i^p + j'^q + 1}{2^p i^p + j'^q + 1} \geq \frac{i^p + j'^q + 1}{2^p i^p + 2^p j'^q + 2^p} = \frac{1}{2^p},$$

the expression (2.4) is bounded from above by

$$M\left|\frac{(i^p + j'^q + 1)((i+1)^p + (j-1)^q + 1) - ((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}{((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}\right|,$$

which equals

$$M\left|\frac{(j'^q - (j-1)^q)((i+1)^p - i^p)}{((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}\right|.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M\frac{i^{p-1}j'^{q-1}(j' - j + 1)}{((i+1)^p + j'^q + 1)(i^p + (j-1)^q + 1)}.$$

Therefore,

$$\big|\log f'(x) - \log f'(y)\big| \leq M\frac{i^{p-1}j'^{q-1}(j' - j)}{(i^p + j'^q)(i^p + j^q)}. \tag{2.5}$$

We will split the general case into the following four cases:

(a) $j' \leq 2j + 1$,

(b) $j'^q \leq i^p$,

(c) $j' > 2j + 1\,, j'^q > i^p\,, j^q \geq i^p$,

(d) $j' > 2j + 1\,, j'^q > i^p\,, j^q < i^p$.

In cases (a) and (b), notice that from

$$|x - y| \geq (j' - j - 1)I_{i,j'}$$

it follows that

$$|x-y|^\alpha \geq \left(\frac{j'-j-1}{i^p+j'^q+1}\right)^\alpha.$$

Hence by (2.5),

$$\frac{|\log f'(x)-\log f'(y)|}{|x-y|^\alpha} \leq M\frac{i^{p-1}j'^{q-1}(j'-j)(i^p+j'^q+1)^\alpha}{(i^p+j'^q)(i^p+j^q)(j'-j-1)^\alpha},$$

that is,

$$\frac{|\log f'(x)-\log f'(y)|}{|x-y|^\alpha} \leq M\frac{i^{p-1}j'^{q-1}(j'-j)^{1-\alpha}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}. \tag{2.6}$$

In case (a), we have $j' \leq 2j+1$, and hence the right-side of (2.6) is bounded from above by

$$M\frac{i^{p-1}j^{q-1}j^{1-\alpha}}{(i^p+j^q)^{1-\alpha}(i^p+j^q)} = M\frac{i^{p-1}j^{q-\alpha}}{(i^p+j^q)^{2-\alpha}}.$$

On the one hand, if $i \leq j^{\frac{1}{p-1}}$, then this expression is bounded by $\frac{j^{q-\alpha+1}}{j^{q(2-\alpha)}} = j^{(\alpha-1)(q-1)}$. Since $\alpha < 1$, the expression is uniformly bounded. On the other hand, if $j \leq i^{p-1}$, then we have the upper bound

$$\frac{i^{p-1+(p-1)(q-\alpha)}}{i^{p(2-\alpha)}} = i^{\alpha-1-p-q+pq}.$$

Now this expression is uniformly bounded by condition (ii_C).

In case (b), we have $j'^q \leq i^p$, and hence the right-side expression of (2.6) is bounded from above by

$$\frac{i^{p-1}i^{\frac{p}{q}(q-1)}i^{\frac{p}{q}(1-\alpha)}}{i^{p+p(1-\alpha)}} = i^{\alpha(p-\frac{p}{q})-1},$$

which is uniformly bounded by condition (iii_C).

In case (c), we have

$$|x-y| \geq \sum_{j<n<j'} |I_{i,n}| = \sum_{j<n<j'} \frac{1}{i^p+n^q+1} \geq \sum_{j<n<j'} \frac{1}{j^q+n^q+1} \geq \sum_{j<n<j'} \frac{1}{3n^q} \geq \int_{j+1}^{j'} \frac{dx}{3x^q},$$

and hence

$$|x-y| \geq M\frac{1}{(j+1)^{q-1}}\Big(1-\big(\frac{j+1}{j'}\big)^{q-1}\Big) \geq \frac{1}{(j+1)^{q-1}}\Big(1-\big(\frac{1}{2}\big)^{q-1}\Big).$$

Therefore,

$$|x-y|^\alpha \geq M\frac{1}{(j+1)^{(q-1)\alpha}}, \tag{2.7}$$

and this yields

$$\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha} \leq M \frac{i^{p-1} j'^{q-1}(j'-j) j^{(q-1)\alpha}}{(i^p + j'^q)(i^p + j^q)} = M\Big(\frac{j'^{q-1}(j'-j)}{i^p + j'^q}\Big)\Big(\frac{i^{p-1} j^{(q-1)\alpha}}{i^p + j^q}\Big).$$

In the last expression, the first factor is uniformly bounded, while the second one is bounded by

$$M \frac{j^{\frac{q}{p}(p-1)} j^{(q-1)\alpha}}{j^q}.$$

This last expression is uniformly bounded when $\frac{q}{p}(p-1) + (q-1)\alpha \leq q$, which is ensured by the condition ($\mathrm{iii_c}$).

The last case (d) is

$$j' > 2j+1, \quad j'^q > i^p, \quad j^q < i^p.$$

For the distance between x and y we now have the estimate

$$|x-y| \geq \sum_{j<n<j'} I_{i,n} = \sum_{j<n<j'} \frac{1}{i^p + n^q + 1} \geq \int_{j+1}^{j'} \frac{1}{i^p + x^q + 1} dx \geq \int_{j+1}^{j'} \frac{1}{(i^{\frac{p}{q}} + x + 1)^q} dx.$$

The last integral is essentially

$$M \frac{(i^{\frac{p}{q}} + j' + 1)^{q-1} - (i^{\frac{p}{q}} + j + 2)^{q-1}}{(i^{\frac{p}{q}} + j + 2)^{q-1}(i^{\frac{p}{q}} + j' + 1)^{q-1}},$$

and by the Mean Value Theorem, this is larger than

$$M \frac{j' - j - 1}{(i^{\frac{p}{q}} + j' + 1)^{2-q}(i^{\frac{p}{q}} + j + 2)^{q-1}(i^{\frac{p}{q}} + j' + 1)^{q-1}}.$$

Therefore,

$$|x-y|^\alpha \geq M \frac{(j'-j-1)^\alpha}{(i^{\frac{p}{q}} + j' + 1)^\alpha (i^{\frac{p}{q}} + j + 2)^{(q-1)\alpha}}. \tag{2.8}$$

This yields

$$\begin{aligned}
\frac{|\log f'(x) - \log f'(y)|}{|x-y|^\alpha} &\leq M \frac{i^{p-1} j'^{q-1}(j'-j)(i^{\frac{p}{q}} + j' + 1)^\alpha (i^{\frac{p}{q}} + j + 2)^{(q-1)\alpha}}{(i^p + j'^q)(i^p + j^q)(j'-j-1)^\alpha} \\
&\leq M \frac{i^{p-1}(i^{\frac{p}{q}} + j' + 1)^\alpha (i^{\frac{p}{q}} + j + 2)^{(q-1)\alpha}}{(i^p + j^q)(j'-j-1)^\alpha} \\
&\leq M \frac{i^{p-1}(2j'+1)^\alpha (2i^{\frac{p}{q}} + 2)^{(q-1)\alpha}}{(i^p + j^q)(\frac{j'}{2})^\alpha} \\
&\leq M \frac{i^{p-1}(i^{\frac{p}{q}} + 1)^{(q-1)\alpha}}{(i^p + j^q)},
\end{aligned}$$

and by condition ($\mathrm{iii_C}$) the last expression is uniformly bounded.

We finally consider the case where $x \in I_{i,j}$ and $y \in I_{i',j'}$ for different i, i'. In this case, we let z (resp. z') be the endpoint of the interval I_i (resp. I_i') obtained as the union of all the $I_{i,k}$'s (resp. $I'_{i',k}s$), with $k \in \mathbb{Z}$, situated between x and y. The estimates so far show that the derivative of f at z equals 1, and

$$\begin{aligned}\frac{|\log(f'(x)) - \log(f'(y))|}{|x-y|^{\alpha}|} &\leq \frac{|\log(f'(x)) - \log(f'(z))|}{|x-y|^{\alpha}} + \frac{|\log(f'(z)) - \log(f'(y))|}{|x-y|^{\alpha}} \\ &\leq \frac{|\log(f'(x)) - \log(f'(z))|}{|x-z|^{\alpha}} + \frac{|\log(f'(z)) - \log(f'(y))|}{|z-y|^{\alpha}} \\ &\leq 2M.\end{aligned}$$

This completes the proof of the $C^{1+\alpha}$ regularity of f. Similar arguments apply to its inverse f^{-1}, thus showing that f is a $C^{1+\alpha}$-diffeomorphism of I.

2.2.2 Each map g_k is of class $C^{1+\alpha}$

Again, we will only consider the case of positive i, j. First, we take x, y in the same interval $I_{i,j}$. We have

$$\frac{|\log g_k'(x) - \log g_k'(y)|}{|x-y|} \leq \frac{M}{|I_{i,j}|}\left|\frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|}\frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1\right|.$$

Hence

$$\begin{aligned}\frac{|\log g_k'(x) - \log g_k'(y)|}{|x-y|^{\alpha}} &\leq \frac{M}{|I_{i,j}|}\left|\frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|}\frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1\right| |I_{i,j}|^{1-\alpha} \\ &= M\left|\frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|}\frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|} - 1\right| |I_{i,j}|^{-\alpha} \\ &\leq M\left|\frac{i^p + (j+r_k(i))^q + 1}{i^p + j^q + 1}\frac{i^p + (j-1)^q + 1}{i^p + (j + r_k(i) - 1)^q + 1} - 1\right| (i^p + j^q + 1)^{\alpha}.\end{aligned}$$

The last expression may be rewritten as

$$M\left|\frac{(i^p + (j+r_k(i))^q + 1)(i^p + (j-1)^q + 1) - (i^p + j^q + 1)(i^p + (j + r_k(i) - 1)^q + 1)}{(i^p + j^q + 1)^{1-\alpha}(i^p + (j + r_k(i) - 1)^q + 1)}\right|.$$

By the Mean Value Theorem, the value of this expression is bounded from above by

$$M\frac{i^p j^{q-1} + i^p(j+r_k(i))^{q-1} + j^{q-1} + (j+r_k(i))^{q-1} + (j+r_k(i))^q j^{q-1} + (j+r_k(i))^{q-1}j^q}{(i^p + j^q)^{1-\alpha}(i^p + (j + r_k(i) - 1)^q)},$$

and hence by

$$M\frac{i^p(j+r_k(i))^{q-1}+(j+r_k(i))^q j^{q-1}+(j+r_k(i))^{q-1}j^q}{(i^p+j^q)^{2-\alpha}}\leq$$

$$\leq M\frac{i^p(j+i^k)^{q-1}+(j+i^k)^q j^{q-1}+(j+i^k)^{q-1}j^q}{(i^p+j^q)^{2-\alpha}}.$$

We claim that the preceding right-expression is uniformly bounded. Indeed, if $i^p \leq j^q$, then it is smaller than or equal to

$$M\frac{j^q(j+j^{\frac{qk}{p}})^{q-1}+(j+j^{\frac{qk}{p}})^q j^{q-1}+(j+j^{\frac{qk}{p}})^{q-1}j^q}{j^{q(2-\alpha)}},$$

which is uniformly bounded by the conditions $(\mathrm{iv_C})$ and $(\mathrm{v_C})$. If $j^q \leq i^p$, then it is smaller than or equal to

$$M\frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}+(i^{\frac{p}{q}}+i^k)^q i^{\frac{p}{q}(q-1)}+(i^{\frac{p}{q}}+i^k)^{q-1}i^p}{i^{p(2-\alpha)}},$$

which is again uniformly bounded by the conditions $(\mathrm{iv_C})$ and $(\mathrm{v_C})$.

Next we consider the case where $x \in I_{i,j}$ and $y \in I_{i,j'}$, with $j \leq j'$. In this case, $|\log g_k'(x) - \log g_k'(y)|$ is smaller than or equal to

$$\left|\log\frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|}-\log\frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|}\right|+\left|\log\frac{|I_{i,j+r_k(i)-1}|}{|I_{i,j-1}|}-\log\frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|}\right|+\left|\log\frac{|I_{i,j'+r_k(i)-1}|}{|I_{i,j'-1}|}-\log\frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|}\right|.$$

The estimates for the last two terms are similar to those above, and we leave the computations to the reader. The first term equals

$$\left|\log\frac{|I_{i,j}|}{|I_{i,j+r_k(i)}|}\frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|}\right|=\left|\log\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1}\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1}\right|,$$

that is,

$$\left|\log\left(1+\frac{(i^p+j'^q+1)(i^p+(j+r_k(i))^q+1)-(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}{(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}\right)\right|.$$

We claim that the expression $\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1}\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1}$ is bounded from below by a positive number. Indeed, the first factor is uniformly bounded because:

– if $j'^q \leq i^p$, then $\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \geq \frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \geq \frac{i^p+1}{i^p+(i^{\frac{p}{q}}+i^k)^q+1}$, and the last expression is uniformly bounded from below by a positive number;

– if $i^p \leq j'^q$, then $\frac{i^p+j'^q+1}{i^p+(j'+r_k(i))^q+1} \geq \frac{i^p+j'^q+1}{i^p+(j'+i^k)^q+1} \geq \frac{j'^q+1}{j'^q+(j'+j'^{\frac{qk}{p}})^q+1}$, which is uniformly bounded from below by a positive number.

The second factor is uniformly bounded as well because:
– if $i^p \le j^q$, then $0 \le j - j^{\frac{qk}{p}} \le j + r_k(i)$, thus $\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1} \ge \frac{i^p+(j-j^{\frac{qk}{p}})^q+1}{i^p+j^q+1} \ge \frac{(j-j^{\frac{qk}{p}})^q+1}{2j^q+1}$, which is uniformly bounded from below by a positive number;
– if $j^q \le i^p$, then $\frac{i^p+(j+r_k(i))^q+1}{i^p+j^q+1} \ge \frac{i^p+1}{2i^p+1}$, which is uniformly bounded from below by a positive number.

From what precedes, we deduce the estimate

$$\left|\log\frac{|I_{i,j+r_k(i)}|}{|I_{i,j}|} - \log\frac{|I_{i,j'+r_k(i)}|}{|I_{i,j'}|}\right| \le$$
$$\le M\left|\frac{(i^p+j'^q+1)(i^p+(j+r_k(i))^q+1)-(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}{(i^p+(j'+r_k(i))^q+1)(i^p+j^q+1)}\right|.$$

Using the Mean Value Theorem and the inequalities $j < j'$ and $r_k(i) \le i^k$, the right-side term above is easily seen to be smaller than or equal to

$$M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}.$$

To get an upper bound for this expression, we separately consider the cases (a), (b), (c), and (d), from the previous section.

The first case (a) is $j' \le 2j+1$. We have $|x-y| \ge \frac{j'-j-1}{i^p+j'^q+1}$, and hence

$$\begin{aligned}\frac{|\log g_k'(x)-\log g_k'(y)|}{|x-y|^\alpha} &\le M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\frac{(i^p+j'^q+1)^\alpha}{(j'-j-1)^\alpha}+M\\ &\le M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}+M\\ &\le M\frac{i^{p+k}(j+i^k)^{q-1}+2j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}+M.\end{aligned}$$

We will deal with the expressions $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}$ and $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)}$ separately. For the first we have

$$\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)} \le \frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}}.$$

Now notice that
– if $i^p \le j^q$, then $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \le \frac{j^{\frac{q}{p}(p+k)}(j+j^{\frac{qk}{p}})^{q-1}}{j^{q(2-\alpha)}}$, and this is bounded when $\frac{q}{p}(p+k)+q-1 \le q(2-\alpha)$, that is, when $\alpha \le \frac{1}{q}-\frac{k}{p}$, which is our condition ($\mathrm{vi_C}$);
– if $j^q \le i^p$, then $\frac{i^{p+k}(j+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \le \frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{i^{p(2-\alpha)}}$, and this is bounded when $p+k+\frac{p}{q}(q-1) \le p(2-\alpha)$, that is, when $\alpha \le \frac{1}{q}-\frac{k}{p}$.

For the second expression we have

$$\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)^{1-\alpha}(i^p+j^q)} \leq \frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}.$$

Again, notice that

– if $i^p \leq j^q$, then $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, and as before, this is bounded when $\alpha \leq \frac{1}{q}-\frac{k}{p}$;

– if $j^q \leq i^p$, then $\frac{j^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, and as before, this is bounded when $\alpha \leq \frac{1}{q}-\frac{k}{p}$.

The second case (b) is $j'^q \leq i^p$. The inequality $|x-y| \geq \frac{j'-j-1}{i^p+j'^q+1}$ yields

$$\begin{aligned}\frac{|\log g_k'(x)-\log g_k'(y)|}{|x-y|^\alpha} &\leq M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\frac{(i^p+j'^q+1)^\alpha}{(j'-j-1)^\alpha}+M\\ &\leq M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}+M\\ &\leq M\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}+j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k+i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}+M.\end{aligned}$$

To estimate the last expression, we will bound the following three expressions:

$$\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}}, \quad \frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}, \quad \text{and} \quad \frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}}.$$

For the first we have

$$\frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{(i^p+j^q)^{2-\alpha}} \leq \frac{i^{p+k}(i^{\frac{p}{q}}+i^k)^{q-1}}{i^{p(2-\alpha)}},$$

and the right-side member is bounded provided that $p+k+\frac{p}{q}(q-1) \leq p(2-\alpha)$, that is, $\alpha \leq \frac{1}{q}-\frac{k}{p}$, which is our condition (vi_C). For the second expression, notice that

– if $i^p \leq j^q$, then $\frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, and this is bounded when $q+\frac{qk}{p}+q-1 \leq q(2-\alpha)$, that is, $\alpha \leq \frac{1}{q}-\frac{k}{p}$;

– if $j^q \leq i^p$, then $\frac{j^q(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, and the last term is bounded because $\alpha \leq \frac{1}{q}-\frac{k}{p}$.

For the third expression, we have that

– if $i^p \leq j^q$, then $\frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{j^q(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}}{j^{q(2-\alpha)}}$, which is bounded when $\alpha \leq \frac{1}{q}-\frac{k}{p}$;

– if $j^q \leq i^p$, then $\frac{i^p(j+i^k)^{q-1}i^k}{(i^p+j^q)^{2-\alpha}} \leq \frac{i^p(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^{p(2-\alpha)}}$, which is also bounded when $\alpha \leq \frac{1}{q}-\frac{k}{p}$.

The third case (c) is $j' > 2j+1\,, j'^q > i^p\,, j^q \geq i^p$. Using (2.7) we obtain

$$\frac{|\log g_k'(x)-\log g_k'(y)|}{|x-y|^\alpha} \leq M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}+M.$$

To estimate the preceding right-side expression, we deal separately with

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}, \quad \frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}, \quad \text{and} \quad \frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha}.$$

For the first one has

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \leq \frac{i^k(j'+i^k)^{q-1}j'^{(q-1)\alpha}}{i^p+j'^q} \leq \frac{j'^{\frac{qk}{p}}(j'+j'^{\frac{qk}{p}})^{q-1}j'^{(q-1)\alpha}}{j'^q},$$

and the right-side term is bounded by condition ($\mathrm{vii_C}$). For the second expression one has

$$\frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \leq \frac{(j'+i^k)^{q-1}i^kj'^{(q-1)\alpha}}{i^p+j'^q} \leq \frac{(j'+j'^{\frac{qk}{p}})^{q-1}j'^{\frac{qk}{p}}j'^{(q-1)\alpha}}{j'^q},$$

and the right-side term is uniformly bounded by the condition ($\mathrm{vii_C}$). Finally, for the third expression one has

$$\frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}j^{(q-1)\alpha} \leq \frac{(j+i^k)^{q-1}i^kj^{(q-1)\alpha}}{i^p+j^q} \leq \frac{(j+j^{\frac{qk}{p}})^{q-1}j^{\frac{qk}{p}}j^{(q-1)\alpha}}{j^q},$$

and the right-side term is also bounded by condition ($\mathrm{vii_C}$).

The last case (d) is $j' > 2j+1$, $j'^q > i^p$, $j^q < i^p$. Inequality (2.8) shows that $\frac{|\log g_k'(x)-\log g_k'(y)|}{|x-y|^\alpha}$ is smaller than or equal to

$$M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}\frac{(i^{\frac{p}{q}}+j'+1)^\alpha(i^{\frac{p}{q}}+j+2)^{(q-1)\alpha}}{(j'-j-1)^\alpha}+M.$$

In this expression, the term $\frac{(i^{\frac{p}{q}}+j'+1)^\alpha}{(j'-j-1)^\alpha}$ is bounded by $\frac{(2j'+1)^\alpha}{(\frac{j'}{2})^\alpha}$, and hence it is uniformly bounded. Therefore,

$$\frac{|\log g_k'(x)-\log g_k'(y)|}{|x-y|^\alpha} \leq M\frac{i^{p+k}(j'+i^k)^{q-1}+j^q(j'+i^k)^{q-1}i^k+j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}.$$

To estimate the right-side expression, we will deal separately with

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}, \frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}, \text{ and } \frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha}.$$

For the first one has

$$\frac{i^{p+k}(j'+i^k)^{q-1}}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{i^k(j'+i^k)^{q-1}}{(i^p+j'^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{j'^{\frac{qk}{p}}(j'+j'^{\frac{qk}{p}})^{q-1}}{j'^q}(j'+j')^{(q-1)\alpha},$$

and, as before, we know that the right-side term is uniformly bounded by the condition ($\mathrm{vii_C}$). For the second expression one has

$$\frac{j^q(j'+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j'+i^k)^{q-1}i^k}{(i^p+j'^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j'+j'^{\frac{qk}{p}})^{q-1}j'^{\frac{qk}{p}}}{j'^q}(j'+j')^{(q-1)\alpha},$$

and the right-side term is uniformly bounded by the condition ($\mathrm{vii_C}$). Finally, for the third expression one has

$$\frac{j'^q(j+i^k)^{q-1}i^k}{(i^p+j'^q)(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(j+i^k)^{q-1}i^k}{(i^p+j^q)}(i^{\frac{p}{q}}+j)^{(q-1)\alpha} \leq \frac{(i^{\frac{p}{q}}+i^k)^{q-1}i^k}{i^p}(i^{\frac{p}{q}}+i^{\frac{p}{q}})^{(q-1)\alpha}.$$

Here the right-side term is bounded when $\frac{p}{q}(q-1)+k+\frac{p}{q}(q-1)\alpha \leq p$, which is true by the condition ($\mathrm{vii_C}$).

To conclude the proof of the regularity of g_k, notice that the case where $x \in I_{i,j}$ and $y \in I_{i',j'}$ for different i, i' can be ruled out by using the same argument as that of f.

2.3 Realizations of lower-triangular matrix groups as groups of interval diffeomorphisms: Theorem D

We now deal with the group N_n of $n \times n$ lower-triangular matrices with integer entries, all of which are equal to 1 on the diagonal. Notice that N_3 corresponds to the Heisenberg group. In general, N_n is a nilpotent group of nilpotence degree $n-1$. A nice system of generators of N_{n+1} is $\{f_1, \ldots, f_n\}$, where the only non-zero entry of f_j outside the diagonal is the $(j+1, j)$-entry and equals 1.

The group N_{n+1} acts naturally on the interval. Indeed, let $\{I_{i_1,\ldots,i_n}\colon (i_1,\ldots,i_n) \in \mathbb{Z}^n\}$ be a family of intervals so that the sum $\sum_{i_1,\ldots,i_n} |I_{i_1,\ldots,i_n}|$ is finite. Joining these intervals lexicographically, we obtain a closed interval I. Then an embedding of N_{n+1} into $\mathrm{Homeo}_+(I)$ is obtained by identifying f_j to the (unique) homeomorphism whose restriction to each interval $I = I_{i_1,\ldots,i_n}$ coincides with $\varphi_{I,I'}^{J,J'}$, where:

– for $j = 1$:

$$I' = I_{i_1,i_2,\ldots,i_{n-1},i_n-1}, \qquad J = I_{i_1+1,i_2,\ldots,i_{n-1},i_n}, \qquad J' = I_{i_1+1,i_2,\ldots,i_{n-1},i_n-1};$$

– for $2 \leq j \leq n-1$:

$$I' = I_{i_1,\ldots,i_j,\ldots,i_n-1}, \qquad J = I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n}, \qquad J' = I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n-1};$$

– for $j = n$:

$$I' = I_{i_1,\ldots,i_n-1,i_n-1}, \qquad J = I_{i_1,\ldots,i_{n-1},i_n+i_{n-1}}, \qquad J' = I_{i_1,\ldots,i_{n-1},i_n+i_{n-1}-1}.$$

This realization comes from the natural action of N_{n+1} on $\mathbb{Z}^{n+1}$, noticing that the affine hyperplane $1 \times \mathbb{Z}^n$ is invariant under this action.

In order to prove Theorem D, we fix once and for all an arbitrary positive number $\alpha < \frac{2}{n(n-1)}$. Our aim is to show that, for a good choice of the lengths $|I_{i_1,\dots,i_n}|$, the f_j's previously defined are $C^{1+\alpha}$-diffeomorphisms of the corresponding interval I. From now on, we will assume that $n \geq 3$: the case of the Heisenberg group N_3 is covered by Theorem C.

We begin by choosing number $p_n \in]1, 5/4]$, and for $1 \leq j \leq n-1$ we choose $p_j > 0$ so that the following properties are satisfied:

($\mathrm{i_D}$) $p_1 > p_2 > \ldots > p_{n-1} > p_n > 1$,

($\mathrm{ii_D}$) $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{n-1}} + \frac{1}{p_n} < 1$,

($\mathrm{iii_D}$) $\alpha \leq \frac{p_n}{(p_n-1)p_1}$,

($\mathrm{iv_D}$) $\alpha \leq \frac{p_n}{p_n-1}\left(\frac{1}{p_j} - \frac{1}{p_{j-1}}\right)$ for all $1 < j < n$,

($\mathrm{v_D}$) $\alpha \leq \frac{1}{p_n} - \frac{1}{p_{n-1}}$.

A concrete choice is $p_j := \frac{1}{j\alpha(1-1/p_n)}$. (Hence, one may take $p_n := 5/4$ and $p_j := \frac{5}{j\alpha}$ for $1 \leq j \leq n-1$.) Indeed, the first property is easy to check. For the second one, we have

$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{p_n} + \sum_{j=1}^{n-1} j\alpha\big(1 - 1/p_n\big) = \frac{1}{p_n} + \alpha\big(1 - 1/p_n\big)\frac{n(n-1)}{2} < \frac{1}{p_n} + \big(1 - 1/p_n\big) = 1,$$

where the inequality comes from the hypothesis $\alpha < \frac{2}{n(n-1)}$. For the third and fourth properties, we actually have equalities with our choice. Finally, since $n \geq 3$,

$$\alpha < \frac{2}{n(n-1)} < \frac{2}{3} \leq \frac{1/p_n}{2 - 1/p_n} \leq \frac{1/p_n}{1 + (n-1)(1 - 1/p_n)}.$$

Hence,

$$\alpha\Big[1 + (n-1)\big(1 - 1/p_n\big)\Big] \leq \frac{1}{p_n},$$

that is

$$\alpha \leq \frac{1}{p_n} - \alpha(n-1)\big(1 - 1/p_n\big) = \frac{1}{p_n} - \frac{1}{p_{n-1}},$$

which shows ($\mathrm{v_D}$).

It is worth mentioning that, for $\alpha \geq \frac{2}{n(n-1)}$, the properties above are incompatible. (This is one of the main reasons why we suspect that our construction is optimal.) Indeed, from ($\mathrm{iii_D}$) we get $\frac{1}{p_1} \geq \frac{\alpha(p_n-1)}{p_n}$. Using ($\mathrm{iv_D}$) inductively, we obtain $\frac{1}{p_j} \geq \frac{j\alpha(p_n-1)}{p_n}$ for $1 \leq j \leq n-1$. This yields

$$\sum_{j=1}^{n} \frac{1}{p_j} \geq \sum_{j=1}^{n-1} \frac{j\alpha(p_n-1)}{p_n} + \frac{1}{p_n} = \frac{\alpha(p_n-1)n(n-1)}{2p_n} + \frac{1}{p_n}.$$

If $\alpha \geq \frac{2}{n(n-1)}$, the right-side expression is greater than or equal to 1, contrary to $(\mathrm{ii})_{\mathrm{D}}$.

Now fixing any choice of the p_j's as above, we let

$$|I_{i_1,\ldots,i_n}| := \frac{1}{|i_1|^{p_1} + \cdots + |i_n|^{p_n} + 1}.$$

According to [13, §3], property $(\mathrm{ii}_{\mathrm{D}})$ implies that the sum of the lengths $|I_{i_1,\ldots,i_n}|$ is finite. We next proceed to show that the induced maps f_j are $C^{1+\alpha}$-diffeomorphisms of the corresponding interval I.

2.3.1 The map f_1 is of class $C^{1+\alpha}$

First we consider x, y in the same interval $I_{i_1,\ldots,i_n}$. We have

$$\frac{|\log(f_1'(x)) - \log(f_1'(y))|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_n}|}\left|\frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1+1,\ldots,i_n}|}\frac{|I_{i_1+1,\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1\right|.$$

Hence,

$$\frac{|\log(f_1'(x)) - \log(f_1'(y))|}{|x-y|^{\alpha}} \leq M\left|\frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1+1,\ldots,i_n}|}\frac{|I_{i_1+1,\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1\right| |I_{i_1,\ldots,i_n}|^{-\alpha}.$$

The right-side expression is bounded from above by

$$M\left|\frac{(|i_1|^{p_1}+\ldots+|i_n-1|^{p_n}+1)(|i_1+1|^{p_1}+\ldots+|i_n|^{p_n}+1)}{(|i_1+1|^{p_1}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)} - 1\right| (|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{\alpha},$$

which equals

$$M\left|\frac{(|i_n|^{p_n}-|i_n-1|^{p_n})(|i_1|^{p_1}-|i_1+1|^{p_1})}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n|^{p_n}+1)}\right|(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{\alpha}.$$

By the Mean Value Theorem, this expression is bounded from above by

$$M\frac{(|i_n|+1)^{p_n-1}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n|^{p_n}+1)^{2-\alpha}}. \tag{2.9}$$

In the case $|i_1|^{p_1} \leq |i_n|^{p_n}$, this is bounded by

$$M\frac{(|i_n|+1)^{p_n-1}(|i_n|^{\frac{p_n}{p_1}}+1)^{p_1-1}}{(|i_n|^{p_n}+1)^{2-\alpha}}.$$

This expression is uniformly bounded when $p_n - 1 + \frac{p_n}{p_1}(p_1 - 1) \leq p_n(2-\alpha)$, that is, when $\alpha \leq \frac{1}{p_1} + \frac{1}{p_n}$, which is ensured by the condition $(\mathrm{v}_{\mathrm{D}})$. In the case $|i_n|^{p_n} \leq |i_1|^{p_1}$ we have the upper bound

$$M\frac{(|i_1|^{\frac{p_1}{p_n}}+1)^{p_n-1}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{2-\alpha}}.$$

This expression is uniformly bounded when $p_1 - 1 + \frac{p_1}{p_n}(p_n - 1) \leq p_1(2-\alpha)$, that is, when $\alpha \leq \frac{1}{p_1} + \frac{1}{p_n}$, which –as we have already seen– is ensured by the condition ($\mathrm{v_D}$).

Now we consider x, y so that $x \in I_{i_1,\ldots,i_{n-1},i_n}$ and $y \in I_{i_1,\ldots,i_{n-1},i'_n}$ for some i_n, i'_n such that $i'_n - i_n \geq 2$. To simplify, we will just deal with positive i_n, i'_n, the other cases being analogous.

By property (2.2), the value of $|\log(f'_1(x)) - \log(f'_1(y))|$ is bounded from above by

$$\left|\log\frac{|I_{i_1+1,\ldots,i_n}|}{|I_{i_1,\ldots,i_n}|} - \log\frac{|I_{i_1+1,\ldots,i'_n}|}{|I_{i_1,\ldots,i'_n}|}\right| + \left|\log\frac{|I_{i_1+1,\ldots,i_n}|}{|I_{i_1,\ldots,i_n}|} - \log\frac{|I_{i_1+1,\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|}\right| + \left|\log\frac{|I_{i_1+1,\ldots,i'_n}|}{|I_{i_1,\ldots,i'_n}|} - \log\frac{|I_{i_1+1,\ldots,i'_n-1}|}{|I_{i_1,\ldots,i'_n-1}|}\right|.$$

Since $i \mapsto \frac{|I_{i_1+1,\ldots,i_{n-1},i}|}{|I_{i_1,\ldots,i_{n-1},i}|}$ is a monotonous function, this expression is smaller than or equal to

$$\begin{aligned}
3&\left|\log\left(\frac{|I_{i_1+1,\ldots,i'_n}|}{|I_{i_1,\ldots,i'_n}|}\right) - \log\left(\frac{|I_{i_1+1,\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|}\right)\right| = \\
&= \left|\log\frac{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)}\right| = \\
&= \left|\log\left(1 + \frac{(i'^{p_n}_n - (i_n-1)^{p_n})(|i_1|^{p_1} - |i_1+1|^{p_1})}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)}\right)\right|.
\end{aligned}$$

Since the expression in brackets in the right-side term equals

$$\frac{(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)},$$

it is bounded from below by a positive number. Therefore,

$$\left|\log(f'_1(x)) - \log(f'_1(y))\right| \leq M\left|\frac{(i'^{p_n}_n - i^{p_n}_n)(|i_1|^{p_1} - |i_1+1|^{p_1})}{(|i_1+1|^{p_1}+|i_2|^{p_2}+\ldots+|i_n|^{p_n}+1)(|i_1|^{p_1}+|i_2|^{p_2}+\ldots+|i'_n|^{p_n}+1)}\right|.$$

By The Mean Value Theorem, the last expression is bounded from above by

$$M\frac{i'^{p_n-1}_n(i'_n - i_n)(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+\ldots+i^{p_n}_n+1)(|i_1|^{p_1}+\ldots+i'^{p_n}_n+1)}.$$

Thus, in order to get an upper bound for $\frac{|\log(f'_1(x)) - \log(f'_1(y))|}{|x-y|^\alpha}$, we need to estimate the expression

$$\frac{i'^{p_n-1}_n(i'_n - i_n)(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+\ldots+i^{p_n}_n+1)(|i_1|^{p_1}+\ldots+i'^{p_n}_n+1)|x-y|^\alpha}. \tag{2.10}$$

We will split the general case into four ones:

(a) $i'_n \leq 2i_n + 1$,

(b) $i'^{p_n}_n \leq |i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}}$,

(c) $i'_n \geq 2i_n + 2$ and $i^{p_n}_n \geq |i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}}$,

(d) $i'_n \geq 2i_n + 2$ and $i_n^{p_n} \leq |i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}} \leq i_n'^{p_n}$.

In case (a), the estimate $|x-y| \geq (i'_n - i_n - 1)|I_{i_1,i_2,\ldots,i'_n}|$ shows that the expression (2.10) is bounded from above by

$$M \frac{i_n'^{p_n-1}(i'_n - i_n)^{1-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + i_n^{p_n} + 1)(|i_1|^{p_1} + \ldots + i_n'^{p_n} + 1)^{1-\alpha}}. \tag{2.11}$$

By the condition $i'_n \leq 2i_n + 1$, the latter expression is smaller than or equal to

$$M \frac{i_n^{p_n-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + i_n^{p_n} + 1)^{2-\alpha}}.$$

If $|i_1|^{p_1} \leq i_n^{p_n}$, then $\frac{i_n^{p_n-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+i_n^{p_n}+1)^{2-\alpha}} \leq \frac{i_n^{p_n-\alpha}(i_n^{\frac{p_n}{p_1}}+1)^{p_1-1}}{(i_n^{p_n}+1)^{2-\alpha}}$, and the last expression is uniformly bounded by condition (iii$_{\mathrm{D}}$). If $i_n^{p_n} \leq |i_1|^{p_1}$, then $\frac{i_n^{p_n-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+i_n^{p_n}+1)^{2-\alpha}} \leq \frac{|i_1|^{\frac{p_1}{p_n}(p_n-\alpha)}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{2-\alpha}}$, and this is uniformly bounded again by condition (iii$_{\mathrm{D}}$).

In case (b), the expression (2.10) is still bounded from above by (2.11), which in its turn is smaller than or equal to

$$M \frac{i_n'^{p_n-\alpha}(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}} + 1)^{2-\alpha}}.$$

Now using the condition $i_n'^{p_n} \leq |i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}}$, we see that this last expression is bounded from above by

$$M \frac{(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}} + 1)^{1-\alpha+\frac{\alpha}{p_n}}} \leq \frac{(|i_1|+1)^{p_1-1}}{(|i_1|^{p_1}+1)^{1-\alpha+\frac{\alpha}{p_n}}}.$$

Finally, the right-side expression is uniformly bounded by condition (iii$_{\mathrm{D}}$).

In case (c), we first need to estimate the value of $|x-y|$:

$$\begin{aligned}
|x-y| \geq \sum_{i_n<j<i'_n} |I_{i_1,\ldots,i_{n-1},j}| &= \sum_{i_n<j<i'_n} \frac{1}{|i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}} + j^{p_n} + 1} \geq \\
&\geq \sum_{i_n<j<i'_n} \frac{1}{i_n^{p_n} + j^{p_n} + 1} \geq \sum_{i_n<j<i'_n} \frac{1}{3j^{p_n}} \geq \int_{i_n+1}^{i'_n} \frac{1}{3x^{p_n}} dx \geq \\
&\geq \frac{M}{(i_n+1)^{p_n-1}} \left(1 - \left(\frac{i_n+1}{i'_n}\right)^{p_n-1}\right) \geq \\
&\geq \frac{M}{(i_n+1)^{p_n-1}} \left(1 - \left(\frac{1}{2}\right)^{p_n-1}\right) \geq \frac{M}{(i_n+1)^{p_n-1}},
\end{aligned}$$

where in the second inequality we used the hypothesis $i_n^{p_n} \geq |i_1|^{p_1} + \ldots + |i_{n-1}|^{p_{n-1}}$. Using this, the value of (2.10) is easily seen to be smaller than or equal to

$$M \frac{i_n'^{p_n-1}(i_n' - i_n)(|i_1|+1)^{p_1-1}(i_n+1)^{(p_n-1)\alpha}}{(|i_1|^{p_1} + \ldots + i_n^{p_n} + 1)(|i_1|^{p_1} + \ldots + i_n'^{p_n} + 1)} \leq M \frac{(|i_1|+1)^{p_1-1}(i_n+1)^{(p_n-1)\alpha}}{|i_1|^{p_1} + \ldots + i_n^{p_n} + 1}.$$

Since by hypothesis we have $i_n^{p_n} \geq |i_1|^{p_1}$, the right-side expression above is bounded from above by

$$M \frac{(i_n^{\frac{p_n}{p_1}} + 1)^{p_1-1}(i_n+1)^{(p_n-1)\alpha}}{i_n^{p_n} + 1},$$

which is uniformly bounded by the condition (iii$_{\mathrm{D}}$).

Let us finally consider the case (d). Letting

$$S := 1 + |i_1|^{p_1} + |i_2|^{p_2} \ldots |i_{n-1}|^{p_{n-1}},$$

we first observe that

$$|x - y| \geq \sum_{i_n < j < i_n'} |I_{i_1,\ldots,i_{n-1},j}| = \sum_{i_n < j < i_n'} \frac{1}{S + j^{p_n}} \geq \int_{i_n+1}^{i_n'} \frac{dx}{x^{p_n} + S} \geq \int_{i_n+1}^{i_n'} \frac{dx}{(x + S^{1/p_n})^{p_n}}.$$

The last integral equals

$$\frac{1}{(p_n-1)} \left[\frac{1}{(i_n+1+S^{1/p_n})^{p_n-1}} - \frac{1}{(i_n'+S^{1/p_n})^{p_n-1}} \right] = \frac{1}{(p_n-1)} \left[\frac{(i_n' + S^{1/p_n})^{p_n-1} - (i_n + 1 + S^{1/p_n})^{p_n-1}}{(i_n+1+S^{1/p_n})^{p_n-1}(i_n'+S^{1/p_n})^{p_n-1}} \right].$$

Using the Mean Value Theorem, we conclude that

$$|x - y| \geq \frac{i_n' - i_n - 1}{(i_n + 1 + S^{1/p_n})^{p_n-1}(i_n' + S^{1/p_n})}.$$

Using this, we conclude that (2.10) is smaller than or equal to

$$\frac{i_n'^{p_n-1}(i_n' - i_n)(|i_1|+1)^{p_1-1}(i_n + 1 + S^{1/p_n})^{\alpha(p_n-1)}(i_n' + S^{1/p_n})^{\alpha}}{(|i_1|^{p_1} + \ldots + i_n^{p_n} + 1)(|i_1|^{p_1} + \ldots + i_n'^{p_n} + 1)(i_n' - i_n - 1)^{\alpha}} \leq$$

$$\leq \frac{(|i_1|+1)^{p_1-1}(i_n + 1 + S^{1/p_n})^{\alpha(p_n-1)}(i_n' + S^{1/p_n})^{\alpha}}{(|i_1|^{p_1} + \ldots + i_n^{p_n} + 1)(i_n' - i_n - 1)^{\alpha}}. \quad (2.12)$$

By hypothesis, $1 + i_n^{p_n} \leq S$, thus $i_n \leq S^{1/p_n}$. Since by definition, $|i_1|^{p_1} \leq S$, this yields

$$\frac{(|i_1|+1)^{p_1-1}(i_n + 1 + S^{1/p_n})^{\alpha(p_n-1)}}{(|i_1|^{p_1} + \ldots + i_n^{p_n} + 1)} \leq M S^{\frac{p_1-1}{p_1} + \frac{\alpha(p_n-1)}{p_n} - 1}. \quad (2.13)$$

By hypothesis, we also have $S \leq 1 + i_n'^{p_n}$ and $i_n' \geq 2i_n + 2$, which gives

$$\frac{(i_n' + S^{1/p_n})^\alpha}{(i_n' - i_n - 1)^\alpha} \leq M. \tag{2.14}$$

Putting together (2.13) and (2.14), we conclude that the expression in (2.12) is bounded from above by

$$MS^{\frac{p_1-1}{p_1} + \frac{\alpha(p_n-1)}{p_n} - 1},$$

which is uniformly bounded by the condition (iii$_D$).

To conclude the proof of the regularity of f_1, notice that the case where $x \in I_{i_1,\ldots,i_n}$ and $y \in I_{i_1',\ldots,i_n'}$ for different $(i_1, \ldots, i_{n-1})$ and $(i_1', \ldots, i_{n-1}')$ can be ruled out by an argument similar to those used for the maps f and g_k of previous sections.

2.3.2 For $2 \leq j \leq n-1$, the map f_j is of class $C^{1+\alpha}$

We first consider x, y in the same interval $I_{i_1,\ldots,i_n}$. We have

$$\frac{|\log(f_j'(x)) - \log(f_j'(y))|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_n}|} \left| \frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n}|} \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1 \right|.$$

Hence,

$$\frac{|\log(f_j'(x)) - \log(f_j'(y))|}{|x-y|^\alpha} \leq M \left| \frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n}|} \frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1 \right| |I_{i_1,\ldots,i_n}|^{-\alpha}.$$

One readily checks that the right-side expression equals

$$M \left| \frac{(|i_n|^{p_n} - |i_n - 1|^{p_n})(|i_j|^{p_j} - |i_j + i_{j-1}|^{p_j})}{(|i_1|^{p_1} + \ldots + |i_j + i_{j-1}|^{p_j} + \ldots + |i_n - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_j|^{p_j} + \ldots + |i_n|^{p_n} + 1)^{1-\alpha}} \right|.$$

By The Mean Value Theorem, and since $p_{j-1} > p_j$, the last expression is bounded from above by

$$M \frac{(|i_n|+1)^{p_n-1}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1} + \ldots + |i_j|^{p_j} + \ldots + |i_n|^{p_n} + 1)^{2-\alpha}}. \tag{2.15}$$

To estimate this expression, let us first assume that $|i_j|^{p_j} \leq |i_{j-1}|^{p_{j-1}}$. In this case, (2.15) is bounded from above by

$$M \frac{(|i_n|+1)^{p_n-1}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}} + |i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}} + |i_n|^{p_n} + 1)^{2-\alpha}}. \tag{2.16}$$

If $|i_n|^{p_n} \leq |i_{j-1}|^{p_{j-1}}$, then this expression is smaller than

$$M \frac{(|i_{j-1}|^{\frac{p_{j-1}}{p_n}} + 1)^{p_n-1}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}} + |i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}} + 1)^{2-\alpha}}.$$

Since $p_{j-1}/p_j \geq 1$, this is uniformly bounded if

$$\frac{p_{j-1}}{p_n}(p_n-1)+\frac{p_{j-1}}{p_j}(p_j-1)+1-p_{j-1}(2-\alpha) \;\leq\; 0,$$

that is, $\alpha \leq \frac{1}{p_n}+\frac{1}{p_j}-\frac{1}{p_{j-1}}$, and this is ensured by conditions ($\mathrm{i_D}$) and ($\mathrm{v_D}$). If $|i_{j-1}|^{p_{j-1}} \leq |i_n|^{p_n}$, then the expression (2.16) is smaller than or equal to

$$M\frac{(|i_n|+1)^{p_n-1}(|i_n|^{\frac{p_n}{p_j}}+|i_n|^{\frac{p_n}{p_{j-1}}})^{p_j-1}|i_n|^{\frac{p_n}{p_{j-1}}}}{(|i_n|^{p_n}+1)^{2-\alpha}}.$$

Since $p_n/p_{j-1} \leq p_n/p_j$, this is uniformly bounded if

$$p_n-1+\frac{p_n}{p_j}(p_j-1)+\frac{p_n}{p_{j-1}}-p_n(2-\alpha) \;\leq\; 0,$$

which is again ensured by conditions $(\mathrm{i})_\mathrm{D}$ and ($\mathrm{v_D}$).

Assume now that $|i_{j-1}|^{p_{j-1}} \leq |i_j|^{p_j}$. In this case, (2.15) is bounded from above by

$$M\frac{(|i_n|+1)^{p_n-1}(|i_j|+|i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j}+|i_n|^{p_n}+1)^{2-\alpha}}.$$

Proceeding as in the previous case, one readily checks that this expression is uniformly bounded when $\alpha \leq \frac{1}{p_n}+\frac{1}{p_j}-\frac{1}{p_{j-1}}$, which is ensured by conditions $(\mathrm{i})_\mathrm{D}$ and ($\mathrm{v_D}$).

Now we consider the case where $x \in I_{i_1,i_2,\ldots,i_n}$ and $y \in I_{i_1,i_2,\ldots,i'_n}$, with $i'_n - i_n \geq 2$. (For the case where $x \in I_{i_1,\ldots,i_n}$ and $y \in I_{i'_1,\ldots,i'_n}$ for different $(i_1,\ldots,i_{n-1})$ and $(i'_1,\ldots,i'_{n-1})$, one may apply the same argument as that of f_1.) To simplify, we will only deal with positive i_n, i'_n: the other cases may be treated in a similar way.

Once again, property (2.2) implies that $|\log(f'_j(x))-\log(f'_j(y))|$ is smaller than or equal to the sum

$$\left|\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n}|}{|I_{i_1,\ldots,i_j,\ldots,i_n}|}-\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_n}|}{|I_{i_1,\ldots,i_j,\ldots,i'_n}|}\right|+$$
$$+\left|\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n}|}{|I_{i_1,\ldots,i_j,\ldots,i_n}|}-\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n-1}|}{|I_{i_1,\ldots,i_j,\ldots,i_n-1}|}\right|+$$
$$+\left|\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_n-1}|}{|I_{i_1,\ldots,i_j,\ldots,i'_n-1}|}-\log\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_n}|}{|I_{i_1,\ldots,i_j,\ldots,i'_n}|}\right|.$$

As in previous estimates of similar expressions, we have

$$|\log(f'_j(x))-\log(f'_j(y))| \leq 3\left|\log\left(\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i_n-1}|}{|I_{i_1,\ldots,i_j,\ldots,i_n-1}|}\right)-\log\left(\frac{|I_{i_1,\ldots,i_j+i_{j-1},\ldots,i'_n}|}{|I_{i_1,\ldots,i_j,\ldots,i'_n}|}\right)\right|.$$

The last expression equals

$$3\left|\log\frac{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i'_n|^{p_n}+1)}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_n|^{p_n}+1)}\right|,$$

that is,

$$3\left|\log\left(1+\frac{(|i_n-1|^{p_n}-i_n'^{p_n})(|i_j+i_{j-1}|^{p_j}-|i_j|^{p_j})}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_n|^{p_n}+1)}\right)\right|. \tag{2.17}$$

The expression into brackets in the right-side term equals

$$\frac{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i'_n|^{p_n}+1)}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_n-1|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_n|^{p_n}+1)},$$

hence it is uniformly bounded from below by a positive number. Therefore, the value of (2.17) is smaller than or equal to

$$M\left|\frac{(i_n^{p_n}-i_n'^{p_n})(|i_j+i_{j-1}|^{p_j}-|i_j|^{p_j})}{(|i_1|^{p_1}+\ldots+|i_j+i_{j-1}|^{p_j}+\ldots+|i_n|^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+|i'_n|^{p_n}+1)}\right|.$$

Using the Mean Value Theorem and the condition $p_{j-1}>p_j$, this last expression is easily seen to be bounded from above by

$$M\frac{i_n'^{p_n-1}(i'_n-i_n)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n'^{p_n}+1)}.$$

Therefore, $\frac{|\log(f'_j(x))-\log(f'_j(y))|}{|x-y|^\alpha}$ is smaller than or equal to

$$M\frac{i_n'^{p_n-1}(i'_n-i_n)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n'^{p_n}+1)|x-y|^\alpha}. \tag{2.18}$$

In order to estimate this expression, we will again consider separately the cases (a), (b), (c) and (d) of the previous section.

The case (a) is $i'_n\leq 2i_n+1$. Here the estimate $|x-y|\geq(i'_n-i_n-1)|I_{i_1,i_2,\ldots,i'_n}|$ shows that (2.18) is bounded from above by

$$M\frac{i_n'^{p_n-1}(i'_n-i_n)^{1-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n'^{p_n}+1)^{1-\alpha}}, \tag{2.19}$$

which is smaller than or equal to

$$M\frac{i_n^{p_n-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)^{2-\alpha}}. \tag{2.20}$$

There are three subcases:

– If $|i_j|^{p_j} \leq |i_{j-1}|^{p_{j-1}}$ and $i_n^{p_n} \leq |i_{j-1}|^{p_{j-1}}$, then

$$M\frac{i_n^{p_n-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)^{2-\alpha}} \leq M\frac{|i_{j-1}|^{\frac{p_{j-1}}{p_n}(p_n-\alpha)}(|i_{j-1}|^{\frac{p_{j-1}}{p_j}}+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}}+1)^{2-\alpha}}.$$

The last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_D}$).

– If $|i_j|^{p_j} \leq i_n^{p_n}$ and $|i_{j-1}|^{p_{j-1}} \leq i_n^{p_n}$, then

$$M\frac{i_n^{p_n-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)^{2-\alpha}} \leq M\frac{i_n^{p_n-\alpha}(i_n^{\frac{p_n}{p_j}}+i_n^{\frac{p_n}{p_{j-1}}})^{p_j-1}i_n^{\frac{p_n}{p_{j-1}}}}{(i_n^{p_n}+1)^{2-\alpha}},$$

and the last expression is uniformly bounded by condition ($\mathrm{iv_D}$).

– If $|i_{j-1}|^{p_{j-1}} \leq |i_j|^{p_j}$ and $i_n^{p_n} \leq |i_j|^{p_j}$, then one has

$$M\frac{i_n^{p_n-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)^{2-\alpha}} \leq M\frac{|i_j|^{\frac{p_j}{p_n}(p_n-\alpha)}(|i_j|+|i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j}+1)^{2-\alpha}},$$

and the last expression is uniformly bounded by condition ($\mathrm{iv_D}$).

In case (b), we still have the upper bound (2.19) for (2.18). Now using the condition $i_n'^{p_n} \leq |i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}$, the value of (2.19) is easily seen to be bounded from above by

$$M\frac{i_n'^{p_n-\alpha}(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}+1)^{2-\alpha}} \leq M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}+1)^{1-\alpha+\frac{\alpha}{p_n}}}.$$

To estimate the right-side expression of this inequality, we consider two subcases:

– If $|i_{j-1}|^{p_{j-1}} \leq |i_j|^{p_j}$, then

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}+1)^{1-\alpha+\frac{\alpha}{p_n}}} \leq M\frac{(|i_j|+|i_j|^{\frac{p_j}{p_{j-1}}})^{p_j-1}|i_j|^{\frac{p_j}{p_{j-1}}}}{(|i_j|^{p_j}+1)^{1-\alpha+\frac{\alpha}{p_n}}},$$

and the last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_D}$).

– If $|i_j|^{p_j} \leq |i_{j-1}|^{p_{j-1}}$, then

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}+1)^{1-\alpha+\frac{\alpha}{p_n}}} \leq M\frac{(|i_{j-1}|^{\frac{p_{j-1}}{p_j}}+|i_{j-1}|)^{p_j-1}|i_{j-1}|}{(|i_{j-1}|^{p_{j-1}}+1)^{1-\alpha+\frac{\alpha}{p_n}}},$$

and the last expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_D}$).

In case (c), we had the estimate

$$|x-y| \geq \frac{M}{(i_n+1)^{p_n-1}}, \tag{2.21}$$

which shows that (2.18) is bounded from above by

$$M\frac{i_n'^{p_n-1}(i_n'-i_n)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_n+1)^{(p_n-1)\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n'^{p_n}+1)}.$$

This is smaller than or equal to

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_n+1)^{(p_n-1)\alpha}}{|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1}. \tag{2.22}$$

Now from the condition $i_n^{p_n} \geq |i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}$ it follows that $|i_j|^{p_j} \leq i_n^{p_n}$ and $|i_{j-1}|^{p_{j-1}} \leq i_n^{p_n}$. Therefore, (2.22) is bounded from above by

$$M\frac{(i_n^{p_n/p_j}+i_n^{p_n/p_{j-1}})^{p_j-1}i_n^{p_n/p_{j-1}}(i_n+1)^{(p_n-1)\alpha}}{i_n^{p_n}+1},$$

and this expression is easily seen to be uniformly bounded by condition ($\mathrm{iv_D}$).

In case (d), we had the estimate

$$|x-y| \geq \frac{i_n'-i_n-1}{(i_n+1+S^{1/p_n})^{p_n-1}(i_n'+S^{1/p_n})}, \tag{2.23}$$

where $S := 1+|i_1|^{p_1}+|i_2|^{p_2}\ldots|i_{n-1}|^{p_{n-1}}$. Thus, (2.18) is bounded from above by

$$M\frac{i_n'^{p_n-1}(i_n'-i_n)(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_n+1+S^{1/p_n})^{\alpha(p_n-1)}(i_n'+S^{1/p_n})^{\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n'^{p_n}+1)(i_n'-i_n-1)^{\alpha}},$$

hence by

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_n+1+S^{1/p_n})^{\alpha(p_n-1)}(i_n'+S^{1/p_n})^{\alpha}}{(|i_1|^{p_1}+\ldots+|i_j|^{p_j}+\ldots+i_n^{p_n}+1)(i_n'-i_n-1)^{\alpha}}.$$

Since the condition $1+i_n^{p_n} \leq S$ yields $i_n \leq S^{1/p_n}$, this expression is smaller than or equal to

$$M\frac{(|i_j|+|i_{j-1}|)^{p_j-1}|i_{j-1}|(i_n'+S^{1/p_n})^{\alpha}}{(i_n'-i_n-1)^{\alpha}}S^{\frac{\alpha(p_n-1)}{p_n}-1}.$$

The conditions $1+i_n^{p_n} \leq S$ and $p_{j-1} \geq p_j$ also yield $|i_j| \leq S^{1/p_j}$ and $|i_{j-1}| \leq S^{1/p_{j-1}} \leq S^{1/p_j}$, thus showing that the last expression is smaller than or equal to

$$M\frac{(i_n'+S^{1/p_n})^{\alpha}}{(i_n'-i_n-1)^{\alpha}}S^{\frac{1}{p_j}(p_j-1)+\frac{1}{p_{j-1}}+\frac{\alpha(p_n-1)}{p_n}-1}.$$

Using the conditions $i_n' \geq 2i_n+2$ and $S \leq 1+i_n'^{p_n}$, this last expression is easily seen to be bounded from above by

$$MS^{\frac{1}{p_j}(p_j-1)+\frac{1}{p_{j-1}}+\frac{\alpha(p_n-1)}{p_n}-1},$$

which is uniformly bounded by the condition ($\mathrm{iv_D}$).

2.3.3 The map f_n is of class $C^{1+\alpha}$

First we consider x, y in the same interval $I_{i_1,\ldots,i_n}$. We have

$$\frac{|\log(f_n'(x)) - \log(f_n'(y))|}{|x-y|} \leq \frac{M}{|I_{i_1,\ldots,i_n}|} \left| \frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1,\ldots,i_n+i_{n-1}}|} \frac{|I_{i_1,\ldots,i_n+i_{n-1}-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1 \right|,$$

hence

$$\frac{|\log(f_n'(x)) - \log(f_n'(y))|}{|x-y|^\alpha} \leq M \left| \frac{|I_{i_1,\ldots,i_n}|}{|I_{i_1,\ldots,i_n+i_{n-1}}|} \frac{|I_{i_1,\ldots,i_n+i_{n-1}-1}|}{|I_{i_1,\ldots,i_n-1}|} - 1 \right| |I_{i_1,\ldots,i_n}|^{-\alpha}.$$

The right-side term above is smaller than or equal to

$$M \left| \frac{(|i_1|^{p_1} + \ldots + |i_n - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n + i_{n-1}|^{p_n} + 1)}{(|i_1|^{p_1} + \ldots + |i_n + i_{n-1} - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)} - 1 \right| (|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)^\alpha,$$

which equals

$$M \left| \frac{\sum_{k=1}^{n-1} |i_k|^{p_k} (|i_n + i_{n-1}|^{p_n} - |i_n|^{p_n}) + \sum_{k=1}^{n-1} |i_k|^{p_k} (|i_n - 1|^{p_n} - |i_n + i_{n-1} - 1|^{p_n}) + C}{(|i_1|^{p_1} + \ldots + |i_n + i_{n-1} - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)} \right| (S + |i_n|^{p_n})^\alpha;$$

where

$$C := |i_n - 1|^{p_n} |i_n + i_{n-1}|^{p_n} - |i_n + i_{n-1} - 1|^{p_n} |i_n|^{p_n} + |i_n - 1|^{p_n} - |i_n + i_{n-1} - 1|^{p_n} + |i_n + i_{n-1}|^{p_n} - |i_n|^{p_n}$$

and, as before, $S := 1 + |i_1|^{p_1} + |i_2|^{p_2} \ldots |i_{n-1}|^{p_{n-1}}$. By the Mean Value Theorem, the last expression is bounded from above by

$$M \frac{\sum_{k=1}^{n-1} |i_k|^{p_k} (|i_n| + |i_{n-1}|)^{p_n - 1} |i_{n-1}| + \sum_{k=1}^{n-1} |i_k|^{p_k} (|i_n| + |i_{n-1}| + 1)^{p_n - 1} |i_{n-1}| + C'}{(|i_1|^{p_1} + \ldots + |i_n + i_{n-1} - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)^{1-\alpha}},$$

where

$$\begin{aligned} C' := \; & |i_n + i_{n-1}|^{p_n} (|i_n| + 1)^{p_n - 1} + |i_n|^{p_n} (|i_n| + |i_{n-1}| + 1)^{p_n - 1} + \\ & + (|i_n| + |i_{n-1}| + 1)^{p_n - 1} |i_{n-1}| + (|i_n| + |i_{n-1}|)^{p_n - 1} |i_{n-1}|. \end{aligned}$$

To get an upper bound for this last expression, it is enough to do so for

$$\frac{|i_n + i_{n-1}|^{p_n} (|i_n| + 1)^{p_n - 1}}{(|i_1|^{p_1} + \ldots + |i_n + i_{n-1} - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)^{1-\alpha}} \tag{2.24}$$

and

$$\frac{|i_k|^{p_k} (|i_n| + |i_{n-1}|)^{p_n - 1} |i_{n-1}|}{(|i_1|^{p_1} + \ldots + |i_n + i_{n-1} - 1|^{p_n} + 1)(|i_1|^{p_1} + \ldots + |i_n|^{p_n} + 1)^{1-\alpha}}, \tag{2.25}$$

where $1 \leq k \leq n$.

Expression (2.24) may be written as

$$\frac{|i_n+i_{n-1}|^{p_n}}{(|i_1|^{p_1}+\ldots+|i_n+i_{n-1}-1|^{p_n}+1)}\,\frac{(|i_n|+1)^{p_n-1}}{(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{1-\alpha}}.$$

The first factor is uniformly bounded, whereas the second is smaller than or equal to

$$\frac{(|i_n|+1)^{p_n-1}}{(|i_n|^{p_n}+1)^{1-\alpha}}.$$

This last expression is uniformly bounded provided that $p_n-1-p_n(1-\alpha)\leq 0$, which is a consequence of condition ($\mathrm{v_D}$).

Concerning expression (2.25), notice that, since $p_{n-1}>p_n$, it is smaller than or equal to

$$\frac{|i_k|^{p_k}(|i_n|+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{2-\alpha}}\leq\frac{(|i_n|+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{1-\alpha}}.$$

On the one hand, if $|i_n|^{p_n}\leq|i_{n-1}|^{p_{n-1}}$, then

$$\frac{(|i_n|+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{1-\alpha}}\leq\frac{(|i_{n-1}|^{\frac{p_{n-1}}{p_n}}+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_{n-1}|^{p_{n-1}}+1)^{1-\alpha}},$$

and the last term is uniformly bounded when $\frac{p_{n-1}}{p_n}(p_n-1)+1\leq p_{n-1}(1-\alpha)$, which is ensured by condition ($\mathrm{v_D}$). On the other hand, if $|i_{n-1}|^{p_{n-1}}\leq|i_n|^{p_n}$, then

$$\frac{(|i_n|+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+|i_n|^{p_n}+1)^{1-\alpha}}\leq\frac{(|i_n|+|i_n|^{\frac{p_n}{p_{n-1}}})^{p_n-1}|i_n|^{\frac{p_n}{p_{n-1}}}}{(|i_n|^{p_n}+1)^{1-\alpha}},$$

which is uniformly bounded when $p_n-1+\frac{p_n}{p_{n-1}}\leq p_n(1-\alpha)$, that is when condition ($\mathrm{v_D}$) holds.

Next we consider the case where $x\in I_{i_1,i_2,\ldots,i_n}$ and $y\in I_{i_1,i_2,\ldots,i'_n}$, with $i'_n-i_n\geq 2$. (For the case where $x\in I_{i_1,\ldots,i_n}$ and $y\in I_{i'_1,\ldots,i'_n}$ for different $(i_1,\ldots,i_{n-1})$ and $(i'_1,\ldots,i'_{n-1})$, one applies the same argument as that of f_1.) Once again, we will only deal with positive i_n,i'_n. As in previous cases, $|\log(f'_n(x))-\log(f'_n(y))|$ is bounded from above by

$$3\left|\log\frac{|I_{i_1,\ldots,i_n+i_{n-1}-1}|}{|I_{i_1,\ldots,i_n-1}|}\,\frac{|I_{i_1,\ldots,i'_n}|}{|I_{i_1,\ldots,i'_n+i_{n-1}}|}\right|\leq M\left|\log\frac{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i'_n+i_{n-1}|^{p_n}+1)}{(|i_1|^{p_1}+\ldots+|i_n+i_{n-1}|^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)}\right|.$$

Notice that the right-side term may be rewritten as

$$M\left|\log\left(1+\frac{\sum_{k=1}^{n-1}|i_k|^{p_k}(|i'_n+i_{n-1}|^{p_n}-i_n'^{p_n})+\sum_{k=1}^{n-1}|i_k|^{p_k}(i_n^{p_n}-|i_n+i_{n-1}|^{p_n})+\bar{C}}{(|i_1|^{p_1}+\ldots+|i_n+i_{n-1}|^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)}\right)\right|,\tag{2.26}$$

where

$$\bar{C}:=i_n^{p_n}|i'_n+i_{n-1}|^{p_n}-|i_n+i_{n-1}|^{p_n}i_n'^{p_n}+i_n^{p_n}-i_n'^{p_n}+|i'_n+i_{n-1}|^{p_n}-|i_n+i_{n-1}|^{p_n}$$
$$=i_n^{p_n}\left[|i'_n+i_{n-1}|^{p_n}-i_n'^{p_n}\right]+i_n'^{p_n}\left[i_n^{p_n}-|i_n+i_{n-1}|^{p_n}\right]+\left[i_n^{p_n}-|i_n+i_{n-1}|^{p_n}\right]+\left[|i'_n+i_{n-1}|^{p_n}-i_n'^{p_n}\right].$$

Since the expression

$$\frac{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+|i'_n+i_{n-1}|^{p_n}+1)}{(|i_1|^{p_1}+\ldots+|i_n+i_{n-1}|^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)}$$

is uniformly bounded from below by a positive number, (2.26) is bounded from above by

$$M\left|\frac{\sum_{k=1}^{n-1}|i_k|^{p_k}(|i'_n+i_{n-1}|^{p_n}-i_n'^{p_n})+\sum_{k=1}^{n-1}|i_k|^{p_k}(i_n^{p_n}-|i_n+i_{n-1}|^{p_n})+\bar{C}}{(|i_1|^{p_1}+\ldots+|i_n+i_{n-1}|^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)}\right|.$$

By The Mean Value Theorem, and since $p_{n-1}>p_n$, this last expression is smaller than or equal to

$$M\frac{\sum_{k=1}^{n-1}|i_k|^{p_k}(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|+\sum_{k=1}^{n-1}|i_k|^{p_k}(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|+\bar{C}'}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)},$$

where $\bar{C}'$ equals

$$i_n^{p_n}(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|+i_n'^{p_n}(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|+(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|+(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|.$$

Therefore, in order to get an upper bound for the value of $\frac{|\log(f'_n(x))-\log(f'_n(y))|}{|x-y|^\alpha}$, we only need to do so with

$$\frac{|i_k|^{p_k}(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)|x-y|^\alpha},\quad \text{where } 1\leq k\leq n, \tag{2.27}$$

and

$$\frac{i_n'^{p_n}(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)|x-y|^\alpha}. \tag{2.28}$$

Expression (2.27) is easy to deal with. Indeed, since

$$|x-y|\geq(i'_n-i_n-1)|I_{i_1,i_2,\ldots,i'_n}|=\left(\frac{i'_n-i_n-1}{|i_1|^{p_1}+\ldots+i_n'^{p_n}+1}\right), \tag{2.29}$$

we have

$$\frac{|i_k|^{p_k}(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)|x-y|^\alpha}\leq$$

$$\leq\frac{|i_k|^{p_k}(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)^{1-\alpha}}\leq$$

$$\leq\frac{(i'_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)^{1-\alpha}}.$$

To estimate the right-side expression, we consider two cases. If, on the one hand, we have $i_n'^{p_n} \leq |i_{n-1}|^{p_{n-1}}$, then

$$\frac{(i_n' + |i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)^{1-\alpha}} \leq \frac{(|i_{n-1}|^{\frac{p_{n-1}}{p_n}}+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_{n-1}|^{p_{n-1}}+1)^{1-\alpha}}.$$

This is uniformly bounded when $\frac{p_{n-1}}{p_n}(p_n-1)+1 \leq p_{n-1}(1-\alpha)$, which is equivalent to condition ($\mathrm{v_D}$). On the other hand, if $|i_{n-1}|^{p_{n-1}} \leq i_n'^{p_n}$, then

$$\frac{(i_n' + |i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)^{1-\alpha}} \leq \frac{(i_n' + (i_n')^{\frac{p_n}{p_{n-1}}})^{p_n-1}(i_n')^{\frac{p_n}{p_{n-1}}}}{(i_n'^{p_n}+1)^{1-\alpha}},$$

and the right-side term is uniformly bounded provided that condition ($\mathrm{v_D}$) holds.

To obtain an upper bound for (2.28), we will consider separately the cases (a), (b), (c) and (d) of the previous two sections.

In case (a) we have $i_n \leq i_n' \leq 2i_n+1$. Hence, the upper bound already obtained for (2.27) with $k=n$ is an upper bound for (2.28).

In case (b), we have $i_n'^{p_n} \leq |i_1|^{p_1}+\ldots+|i_{n-1}|^{p_{n-1}}$. Hence, (2.28) is smaller than or equal to

$$\sum_{k=1}^{n-1} \frac{|i_k|^{p_k}(i_n'+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)|x-y|^{\alpha}},$$

and we have already seen that each term of this sum is uniformly bounded.

In case (c), we use (2.21) to obtain

$$\frac{i_n'^{p_n}(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(|i_1|^{p_1}+\ldots+i_n'^{p_n}+1)|x-y|^{\alpha}} \leq$$
$$\leq M \frac{(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)^{1-\alpha}} \frac{(i_n+1)^{(p_n-1)\alpha}}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)^{\alpha}}.$$

In the right-side expression, the second factor $\frac{(i_n+1)^{(p_n-1)\alpha}}{(i_n^{p_n}+1)^{\alpha}}$ is uniformly bounded. To show that the same holds with the first factor, one may proceed as at the end of the estimates for (2.27) just changing i_n' by i_n.

Finally, in case (d), the estimate (2.23) shows that (2.28) is smaller than or equal to

$$\frac{(i_n+|i_{n-1}|)^{p_n-1}|i_{n-1}|(i_n+1+S^{1/p_n})^{\alpha(p_n-1)}(i_n'+S^{1/p_n})^{\alpha}}{(|i_1|^{p_1}+\ldots+i_n^{p_n}+1)(i_n'-i_n-1)^{\alpha}}.$$

Since the condition $1+i_n^{p_n} \leq S$ yields $i_n \leq S^{1/p_n}$, this expression is smaller than or equal to

$$M\,\frac{(i_n + |i_{n-1}|)^{p_n-1}|i_{n-1}|(i'_n + S^{1/p_n})^{\alpha}}{(i'_n - i_n - 1)^{\alpha}}\, S^{\frac{\alpha(p_n-1)}{p_n}-1}.$$

Moreover, by the definition of S, we have $|i_{n-1}| \leq S^{1/p_{n-1}} \leq S^{1/p_n}$, which shows that the last expression is smaller than or equal to

$$M\,\frac{(i'_n + S^{1/p_n})^{\alpha}}{(i'_n - i_n - 1)^{\alpha}}\, S^{\frac{p_n-1}{p_n}+\frac{1}{p_{n-1}}+\frac{\alpha(p_n-1)}{p_n}-1}.$$

Because of the conditions $i'_n \geq 2i_n + 2$ and $S \leq 1 + i'^{p_n}_n$, this last expression is bounded from above by

$$M\,S^{\frac{p_n-1}{p_n}+\frac{1}{p_{n-1}}+\frac{\alpha(p_n-1)}{p_n}-1},$$

which is uniformly bounded by the condition $(\mathrm{v_D})$.

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