AN EXTENSION OF MORITA'S p-ADIC LOG-GAMMA FUNCTION

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. . . that he not busy being born is busy dying
–Robert A. Zimmerman

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Resumen

A mediados de los años 70, Y. Morita y J. Diamond definieron dos funciones p-ádicas distintas, ambas análogas a la función $\log\Gamma(x)$ clásica. Estas funciones comparten propiedades similares, pero ninguna de ellas tiene a todo $\mathbb{C}_p$ como su dominio. Definimos un análogo p-ádico de la función $\log\Gamma(x)$, que tiene a $\mathbb{C}_p$ como dominio, y demostramos que las funciones de Morita y Diamond son casos particulares sobre ciertos subdominios.

Abstract

Two different analogues of the classical $\log \Gamma(x)$ function were defined in the mid 1970's by Y. Morita and J. Diamond. Although they share similar properties, neither of them has the whole of $\mathbb{C}_p$ as its domain. We define a p-adic analogue of $\log \Gamma(x)$, with $\mathbb{C}_p$ as its domain, and show that it specializes to Morita and Diamond's functions on certain sub-domains.

Contents

1. Introduction

In the mid 1970's, two different (but not that different) p-adic analogues of the classical $\log\Gamma(x)$ function were defined by Yasuo Morita [Mo] and Jack Diamond [Di]. We recall that $\log\Gamma(x)$ is the logarithm of the classical $\Gamma(x)$ function, and that it satisfies the difference equation

$$\log\Gamma(x+1) - \log\Gamma(x) = \log x \qquad (x > 0) \tag{1}$$

and the integral formula

$$\int_0^1 \log\left(\frac{\Gamma(x+t)}{\sqrt{2\pi}}\right) dt = x\log x - x \qquad (x > 0)\,, \tag{2}$$

discovered by Joseph Ludwig Raabe in the 1840's [Ra1, §2] [Ra2, §1] [Ni, §34]. Also, $\log\Gamma(x)$ satisfies a characterization theorem. Namely, it is the unique convex function defined on $(0,\infty)$ satisfying $\log\Gamma(1) = 0$ and the difference equation (1).

In Morita's paper [Mo], he defines a p-adic analogue of $\Gamma(x)$, that we will call Γ_{M}, which has $\mathbb{Z}_p$ as its domain and takes values in $\mathbb{Z}_p^*$.[1] For positive integers n, Γ_{M} is defined as

$$\Gamma_{\mathrm{M}}(n) := (-1)^n \prod_{\substack{1\le j<n \\ p\nmid j}} j\,.$$

Morita proves that this function is continuous over $\mathbb{N}$ (with the p-adic topology) and extends to a continuous function on $\mathbb{Z}_p$. He also shows that $\Gamma_{\mathrm{M}} : \mathbb{Z}_p \to \mathbb{Z}_p^*$ satisfies the functional equation

$$\frac{\Gamma_{\mathrm{M}}(x+1)}{\Gamma_{\mathrm{M}}(x)} = \begin{cases} -x & \text{if } x \in \mathbb{Z}_p^*\,, \\ -1 & \text{if } x \in p\mathbb{Z}_p\,. \end{cases} \tag{3}$$

Since Γ_{M} is continuous on $\mathbb{Z}_p$, it is completely characterized by its value $\Gamma_{\mathrm{M}}(1) = -1$ and by (3). The function Γ_{M} also satisfies an analogue of the well known formula for the classical Γ-function [Ni, p. 14]

$$\Gamma(1-x)\,\Gamma(x) = \frac{\pi}{\sin\pi x} \qquad (x > 0).$$

[1]For p a prime number, we let $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote, respectively, the ring of p-adic numbers, the field of fractions of $\mathbb{Z}_p$, and the completion of the algebraic closure of $\mathbb{Q}_p$. If A is a commutative ring with identity, A^* will denote the group of units of A.

Namely, if p is an odd prime[2] and $x \in \mathbb{Z}_p$, then [GK, p. 572, Lemma 2.3]

$$\Gamma_{\mathrm{M}}(1-x)\,\Gamma_{\mathrm{M}}(x) = (-1)^{\ell} \qquad (x \in \mathbb{Z}_p), \tag{4}$$

where ℓ is the unique integer with $1 \le \ell \le p$ and $\ell \equiv x \pmod{p\mathbb{Z}_p}$.

To prove analytic properties of his function, Morita actually works with the Iwasawa p-adic logarithm $\log_p$ of Γ_{M} [Ro, §V.4.5] [Sc, §45]. We will write this function $\mathrm{Log}\Gamma_{\mathrm{M}}$, i.e.,

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) := \log_p \Gamma_{\mathrm{M}}(x) \qquad (x \in \mathbb{Z}_p).$$

Taking the Iwasawa logarithm on both sides of (3), we find that $\mathrm{Log}\Gamma_{\mathrm{M}}$ satisfies the difference equation

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{M}}(x) = \begin{cases} \log_p x & \text{if } x \in \mathbb{Z}_p^*, \\ 0 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{5}$$

in analogy to (1). It is then immediate that $\mathrm{Log}\Gamma_{\mathrm{M}}$ is uniquely determined by the value $\mathrm{Log}\Gamma_{\mathrm{M}}(1) = 0$ and by the difference equation (5). Also, by taking the logarithm on both sides of (4), we find that $\mathrm{Log}\Gamma_{\mathrm{M}}$ satisfies the reflection formula

$$\mathrm{Log}\Gamma_{\mathrm{M}}(1-x) + \mathrm{Log}\Gamma_{\mathrm{M}}(x) = 0 \qquad (x \in \mathbb{Z}_p). \tag{6}$$

Morita's $\mathrm{Log}\Gamma_{\mathrm{M}}$ function can be given by the integral formula [Sc, §58]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi_{\mathbb{Z}_p^*}(x+t)\,dt \qquad (x \in \mathbb{Z}_p), \tag{7}$$

where $\chi_{\mathbb{Z}_p^*}$ denotes the characteristic function of $\mathbb{Z}_p^*$, and where the integral on the right is the Volkenborn integral.[3] Henri Cohen and Eduardo Friedman [CF, p. 370, Prop. 2.4] showed that $\mathrm{Log}\Gamma_{\mathrm{M}}$ satisfies a Raabe-type formula, similar to formula (2) for the classic $\log\Gamma(x)$ function. Namely, we have the integro-differential equation

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{M}}(x+t)\,dt = (x-1)\,\mathrm{Log}\Gamma_{\mathrm{M}}'(x) - x + \left\lceil \frac{x}{p} \right\rceil \qquad (x \in \mathbb{Z}_p), \tag{8}$$

[2]For $p=2$, equation (4) takes the form $\Gamma_{\mathrm{M}}(1-x)\,\Gamma_{\mathrm{M}}(x) = (-1)^{a_1+1}$ where $x = a_0 + a_1 2 + a_2 2^2 + \cdots$ is the expansion of $x \in \mathbb{Z}_2$ such that $a_i \in \{0,1\}$ for $i \ge 0$ [Sc, p. 111, Prop. 37.2].

[3]If $g : \mathbb{Z}_p \to \mathbb{C}_p$ we say that g is Volkenborn integrable if the limit

$$\int_{\mathbb{Z}_p} g(t)\,dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(j)$$

exists, and we will call it the Volkenborn integral of g [Ro, §V.5] [Sc, §55].

where $\left\lceil \frac{x}{p} \right\rceil$ is the p-adic limit of the usual integer ceiling function $\left\lceil \frac{x_n}{p} \right\rceil$ as $x_n \to x$ through $x_n \in \mathbb{Z}$, and where the integral on the left is again the Volkenborn integral.

Another important property of $\mathrm{Log}\Gamma_{\mathrm{M}}$ is that this function is locally analytic on $p\mathbb{Z}_p$, in the sense that it has a power series expansion around 0, convergent for all $x \in p\mathbb{Z}_p$. Namely [Sc, p. 177, Lemma 58.1], for all $x \in p\mathbb{Z}_p$ we have the identity

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \lambda_1 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\lambda_{n+1}}{n(n+1)}\,x^{n+1},$$

where

$$\lambda_1 := \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\,\log_p t\,dt\,, \quad \lambda_{n+1} := \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\,t^{-n}\,dt \qquad (n \in \mathbb{N}).$$

Actually, this power series defines an analytic function on the open unit ball

$$B(0;1^-) := \{x \in \mathbb{C}_p \mid |x|_p < 1\}$$

of $\mathbb{C}_p$ [Sc, p. 177, Lemma 58.2]. Hence, we can extend the domain of $\mathrm{Log}\Gamma_{\mathrm{M}}$ to $B(0;1^-) \cup \mathbb{Z}_p$ by defining

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) := \lambda_1 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\lambda_{n+1}}{n(n+1)}\,x^{n+1} \qquad (x \in B(0;1^-)),$$

but it can be shown that with this extended definition, $\mathrm{Log}\Gamma_{\mathrm{M}}$ is no longer the Iwasawa logarithm of a p-adic function.

Diamond [Di] defines his p-adic analogue of the classical $\log\Gamma(x)$ function, which we will write $\mathrm{Log}\Gamma_{\mathrm{D}}$, by the Volkenborn integral

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\,dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{9}$$

Diamond showed that his function is locally analytic on $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, takes values in $\mathbb{C}_p$, and that it satisfies the difference equation [Di, p. 326, Theorem 5]

$$\mathrm{Log}\Gamma_{\mathrm{D}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{D}}(x) = \log_p x \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p), \tag{10}$$

and the reflection formula [Di, p. 327, Theorem 8]

$$\mathrm{Log}\Gamma_{\mathrm{D}}(1-x) + \mathrm{Log}\Gamma_{\mathrm{D}}(x) = 0 \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{11}$$

The function $\mathrm{Log}\Gamma_{\mathrm{D}}$ also satisfies a Raabe-type formula like (2) and (8), and a characterization theorem [CF, p. 364]. Namely, $\mathrm{Log}\Gamma_{\mathrm{D}}$ satisfies

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{D}}(x+t)\,dt = (x-1)\,\mathrm{Log}\Gamma_{\mathrm{D}}'(x) - x + \frac{1}{2} \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p), \tag{12}$$

and it is the only locally analytic function $f : \mathbb{C}_p \setminus \mathbb{Z}_p \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = \log_p x \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p)$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \frac{1}{2} \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p).$$

Comparing formulas (7) and (9), we notice that $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ have very similar expressions involving a Volkenborn integral. Equations (6) and (11) shows that $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ have identical reflection formulas, and (5), (10), (8) and (12) show that these functions satisfy similar difference and integro-differential equations. Also, notice that the domains of $\mathrm{Log}\Gamma_{\mathrm{M}}$ and $\mathrm{Log}\Gamma_{\mathrm{D}}$ are disjoint and complementary in $\mathbb{C}_p$. Finally, we mention that if we would like to extend $\mathrm{Log}\Gamma_{\mathrm{D}}$ to $\mathbb{C}_p$, then the difference equation forces $\mathrm{Log}\Gamma_{\mathrm{D}}$ to be discontinuous on either the positive integers or the negative integers. Since both these sets are dense in $\mathbb{Z}_p$, $\mathrm{Log}\Gamma_{\mathrm{D}}$ cannot be extended continuously at any point of $\mathbb{Z}_p$.

The aim of this thesis will be to define a new p-adic log-gamma function, with $\mathbb{C}_p$ as its domain. We will define a new p-adic log-gamma function $\mathrm{Log}\Gamma_{\mathrm{p}} : \mathbb{C}_p \to \mathbb{C}_p$ by the Volkenborn integral

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x) := \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi_1(x+t)\,dt,$$

where χ_1 the characteristic function of the complement of the open unit ball $B(0;1^-)^{\mathrm{c}} = \{x \in \mathbb{C}_p \mid |x|_p \geq 1\}$.[4] This function will be proved to be locally analytic on $\mathbb{C}_p$, and to satisfy the difference equation (Proposition 3.2)

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x+1) - \mathrm{Log}\Gamma_{\mathrm{p}}(x) = \chi_1(x)\log_p x \qquad (x \in \mathbb{C}_p),$$

as well as the reflection formula (Proposition 3.6)

$$\mathrm{Log}\Gamma_{\mathrm{p}}(1-x) + \mathrm{Log}\Gamma_{\mathrm{p}}(x) = 0 \qquad (x \in \mathbb{C}_p).$$

[4] Actually, our definition (Definition 3.1) will be somewhat more general, but the one given above is sufficient for the case. What we now call $\mathrm{Log}\Gamma_{\mathrm{p}}(x)$, will be the special case $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1)$.

On certain sub-domains of $\mathbb{C}_p$, $\mathrm{Log}\Gamma_{\mathrm{p}}(x)$ coincides with Morita's function, on other with Diamond's. Namely

$$\begin{aligned} \mathrm{Log}\Gamma_{\mathrm{p}}(x) &= \mathrm{Log}\Gamma_{\mathrm{M}}(x) && (x \in \mathbb{Z}_p), \\ \mathrm{Log}\Gamma_{\mathrm{p}}(x) &= \mathrm{Log}\Gamma_{\mathrm{D}}(x) && (x \in \mathbb{C}_p\,,\ |x|_p > 1), \end{aligned}$$

as we will show in §4. These identities will also help us compute the local power series for $\mathrm{Log}\Gamma_{\mathrm{p}}(x)$ (Theorem 4.8).

The function $\mathrm{Log}\Gamma_{\mathrm{p}}(x)$ satisfies a Raabe-type formula and a characterization theorem similar to the one satisfied by $\mathrm{Log}\Gamma_{\mathrm{D}}(x)$. Namely (Theorem 3.3),

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{p}}(x+t)\,dt = (x-1)\,\mathrm{Log}\Gamma_{\mathrm{p}}{}'(x) - R_p(x) \qquad (x \in \mathbb{C}_p)\,,$$

where $R_p : \mathbb{C}_p \to \mathbb{C}_p$ is the function defined by the Volkenborn integral

$$R_p(x) := \int_{\mathbb{Z}_p} (x+t)\chi_1(x+t)\,dt\,,$$

which we calculate explicitly in Proposition 3.4.

Moreover, $\mathrm{Log}\Gamma_{\mathrm{p}}(x)$ is the unique locally analytic function $f : \mathbb{C}_p \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = \chi_1(x)\log_p x \qquad (x \in \mathbb{C}_p)$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - R_p(x) \qquad (x \in \mathbb{C}_p).$$

2. Some preliminary results

2.1. **Residue field and a partition of $\mathbb{C}_p$.** For $x \in \mathbb{C}_p$, $|x|_p$ will denote the p-adic absolute value of x normalized by $|p|_p = p^{-1}$. For $r \in \mathbb{R}\,,\ r \geq 0\,,$ we will write

$$B(x;r^-) = \{y \in \mathbb{C}_p \mid |x-y|_p < r\} \text{ and } B(x;r) = \{y \in \mathbb{C}_p \mid |x-y|_p \leq r\}$$

for the open and closed ball, respectively, centered at x with radius r.[5] Also, let

$$\mathcal{Z}_p = B(0;1) \text{ and } \mathfrak{M} = B(0;1^-)$$

be the ring of integers of $\mathbb{C}_p$ and the maximal ideal of $\mathcal{Z}_p$, respectively. Then the quotient $\mathcal{Z}_p/\mathfrak{M}$ is a field, called the residue field of $\mathbb{C}_p$, and it is isomorphic to $\overline{\mathbb{F}}_p$, the algebraic

[5]For a brief review of the basic properties of $|\ |_p$ and of open and closed balls see [Ro, pp. 73, 77].

closure of the finite field with p elements $\mathbb{F}_p$.[6] If $x \in \mathcal{Z}_p$ then we will write $\overline{x}$ for its natural image in $\overline{\mathbb{F}}_p$.

Now, let $a \in \mathbb{C}_p^*$. If $|x|_p \le |a|_p$, or equivalently, if $x \in B(0; |a|_p)$, then $xa^{-1} \in B(0;1) = \mathcal{Z}_p$, so it makes sense to write $\overline{xa^{-1}}$. For $0 \le \ell \le p-1$, define the sets

$$S_{a,\ell} := \{x \in B(0; |a|_p) \mid \overline{xa^{-1}} = \overline{\ell}\}, \tag{13}$$

where $\overline{\ell} \in \mathbb{F}_p \subset \overline{\mathbb{F}}_p$ as $\ell \in \mathbb{Z}$. Actually, we can define the sets $S_{a,\ell}$ for all $\ell \in \mathbb{Z}$, and it is easy to show that we have $S_{a,\ell} = S_{a,k}$ if $\ell, k \in \mathbb{Z}$ with $\ell \equiv k \pmod p$. Also, define the sets

$$S_{a,\infty} := \{x \in B(0; |a|_p) \mid \overline{xa^{-1}} \notin \mathbb{F}_p\} \text{ and } T_a := \{x \in \mathbb{C}_p \mid |x|_p > |a|_p\}. \tag{14}$$

Definition 2.1. *Let $a \in \mathbb{C}_p^*$ and write $a\mathbb{Z}_p := \{at \mid t \in \mathbb{Z}_p\}$. We will say that an arbitrary subset $D \subset \mathbb{C}_p$ is $a\mathbb{Z}_p$-invariant, if for all $x \in D$ and for all $t \in \mathbb{Z}_p$ we have $x + at \in D$.*

Trivial examples of $a\mathbb{Z}_p$-invariant subsets of $\mathbb{C}_p$ are $\mathbb{C}_p$, $a\mathbb{Z}_p$ and $\mathbb{C}_p \setminus a\mathbb{Z}_p$. Later we will need the following lemma

Lemma 2.2. *With the above notation, we have*

(i) $\displaystyle\bigcup_{\ell=0}^{p-1} S_{a,\ell} = \{x \in B(0; |a|_p) \mid \overline{xa^{-1}} \in \mathbb{F}_p\}$.

(ii) *Let* $S_a := \displaystyle\bigcup_{\ell=0}^{p-1} S_{a,\ell}$. *Then S_a is $a\mathbb{Z}_p$-invariant.*

(iii) *For $0 \le \ell \le p-1$, $S_{a,\ell} = \{x \in \mathbb{C}_p \mid |x - a\ell|_p < |a|_p\} = B(a\ell; |a|_p^-)$, so that the sets $S_{a,\ell}$ are open balls.*

(iv) *The sets T_a and $S_{a,\infty}$ are open and closed in $\mathbb{C}_p$.*

(v) *The sets T_a and $S_{a,\infty}$ are $a\mathbb{Z}_p$-invariant, hence $T_a \cup S_{a,\infty}$ is $a\mathbb{Z}_p$-invariant.*

Proof.

(i) Clear since $\mathbb{F}_p = \{\overline{0}, \dots, \overline{p-1}\}$.

(ii) Suppose $x \in S_{a,i}$ for some $0 \le i \le p-1$, and let $t \in \mathbb{Z}_p$. Then $\overline{t} = \overline{j}$ for some $0 \le j \le p-1$, and we deduce that $\overline{xa^{-1} + t} = \overline{i + j} = \overline{k}$ for some $0 \le k \le p-1$. Thus, $x + at \in S_{a,k}$, and the result follows.

(iii) The second equality is just the definition of $B(a\ell; |a|_p^-)$. For the first equality, let $x \in \mathbb{C}_p$. Then $x \in B(a\ell; |a|_p^-) \iff |x - a\ell|_p < |a|_p \iff |xa^{-1} - \ell|_p < 1 \iff \overline{xa^{-1}} = \overline{\ell} \iff x \in S_{a,\ell}$.

[6]See [Sc, p. 25, Definition 11.2] and [Sc, p. 45, Corollary 17.2].

(iv) Since the closed ball $B(0; |a|_p)$ is open and closed, its complement T_a is open and closed. Also, by (iii) we have that the set S_a is open and closed, thus $T_a \cup S_a$ is open and closed, and then its complement $S_{a,\infty}$ is open and closed.

(v) Let $t \in \mathbb{Z}_p$. Then, in particular, $|t|_p \leq 1$ and $|at|_p \leq |a|_p$. If $x \in T_a$, then $|x|_p > |a|_p \geq |at|_p$, so that $|x + at|_p = |x|_p > |a|_p$, which means that $x + at \in T_a$. Now, if $y \in S_{a,\infty}$, then $|y + at|_p \leq \max\{ |y|_p , |at|_p \} \leq |a|_p$, hence $\overline{(y + at)\, a^{-1}}$ is defined. Suppose that $\overline{(y + at)\, a^{-1}} = \overline{ya^{-1} + t} \in \mathbb{F}_p$, so that $y + at \notin S_{a,\infty}$. Since $\mathbb{F}_p$ is a subfield of $\overline{\mathbb{F}}_p$ and $\overline{t} \in \mathbb{F}_p$, then $\overline{ya^{-1}} = \overline{ya^{-1} + t - t} = \overline{ya^{-1} + t} - \overline{t} \in \mathbb{F}_p$, contradicting $y \in S_{a,\infty}$.

□

Notice that, by (i), (iii), (iv) and by its definitions, the sets $S_{a,0}, \ldots, S_{a,p-1}$, $S_{a,\infty}$ and T_a are open, mutually disjoint, and its union is $\mathbb{C}_p$. Hence, we have a partition of $\mathbb{C}_p$ into open subsets.

2.2. **Locally analytic functions.** Let D be an open or closed ball in $\mathbb{C}_p$ with center $y \in D$ and with positive radius. We will call a function $f : D \to \mathbb{C}_p$ analytic on D if f can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - y)^n \tag{15}$$

convergent on D, where $a_n \in \mathbb{C}_p$ for all $n \in \mathbb{N}_0$.[7]

Now, let A be any subset of $\mathbb{C}_p$. We will call a function $f : A \to \mathbb{C}_p$ locally analytic on A if for each $a \in A$ there is an open ball $D \subset A$ with positive radius, that contains a, such that f is analytic on D. It is easily seen that we may replace the word open for the word closed in this definition.

Now we state a result due to Diamond [Di] that will allow us to define a new p-adic $\log \Gamma$-function.

Proposition 2.3. *Let $f : \mathbb{C}_p \to \mathbb{C}_p$ be a locally analytic function on $\mathbb{C}_p$, and let $a \in \mathbb{C}_p^*$ be fixed. Then, for all $b \in \mathbb{N}$, the limit*

$$F(x) := \lim_{n \to \infty} \frac{1}{bp^n} \sum_{j=0}^{bp^n - 1} f(x + aj) \tag{16}$$

[7]We will use the convention $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

exists and is independent of b. Moreover, $F(x)$ defines a locally analytic function on $\mathbb{C}_p$, and we have the identity

$$F'(x) = \lim_{n\to\infty} \frac{1}{bp^n} \sum_{j=0}^{bp^n-1} f'(x+aj)\,. \tag{17}$$

Proof. The existence of (16) is a special case of [Di, p. 324, Corollary], and the identity (17) follows immediately from [Di, p. 325, Theorem 3].

□

We can restate Proposition 2.3 using the Volkenborn integral [Ro, §V.5] [Sc, §55].

Lemma 2.4. *Let $f : \mathbb{C}_p \to \mathbb{C}_p$ be a locally analytic function on $\mathbb{C}_p$, and let $a \in \mathbb{C}_p^*$ be fixed. Then the Volkenborn integral*

$$F(x) := \int_{\mathbb{Z}_p} f(x+at)\,dt \tag{18}$$

exists and $F(x)$ defines a locally analytic function on $\mathbb{C}_p$. Moreover, we can differentiate under the integral sign, that is

$$F'(x) = \int_{\mathbb{Z}_p} f'(x+at)\,dt\,. \tag{19}$$

Proof. This follows immediately from the definition of the Volkenborn integral, letting $b = 1$ in Proposition 2.3.

□

Remark. *The Volkenborn integral is usually defined for strictly differentiable functions [Ro, § V.1.1][Sc, §27]. Let X be any non empty subset of $\mathbb{C}_p$ with no isolated points and let $f : X \to \mathbb{C}_p$. We say that f is strictly differentiable[8] at a point $a \in X$ if*

$$\lim_{(x,y)\to(a,a)} \frac{f(x)-f(y)}{x-y} \tag{20}$$

exists, where we take the limit over $x, y \in X$ such that $x \neq y$. We say that f is strictly differentiable on X, or that $f \in C^1(X)$, if f is strictly differentiable for all $a \in X$. If $f \in C^1(\mathbb{Z}_p)$, then the Volkenborn integral

$$\int_{\mathbb{Z}_p} f(t)\,dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} f(j)$$

[8]Schikhof call such functions continuously differentiable.

of f exists [Ro, § V.5.1][Sc, §55]. All the properties of the Volkenborn integral that we will mention from [CF], [Ro] and [Sc] are proved there for strictly differentiable functions. Since any locally analytic function on an open set $X \subset \mathbb{C}_p$ is actually strictly differentiable on X [Sc, p. 91, Corollary 29.11], these properties hold for locally analytic functions on X.

Perhaps the simplest non trivial property of a function F defined by (18) is that it satisfies the difference equation [Di, p. 325, Theorem 4] [Sc, p. 168, Prop. 55.5]

$$F(x+a) - F(x) = af'(x) \qquad (x \in \mathbb{C}_p). \tag{21}$$

The proof is simple since by definition we have

$$\begin{aligned} F(x+a) - F(x) &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} \big(f(x+a(j+1)) - f(x+aj)\big) \\ &= \lim_{n\to\infty} \frac{1}{p^n} \big(f(x+ap^n) - f(x)\big) \qquad \text{[telescoping sum]} \\ &= a \lim_{n\to\infty} \frac{1}{ap^n} \big(f(x+ap^n) - f(x)\big) \qquad [a \neq 0] \\ &= af'(x)\,. \end{aligned}$$

Finally, we will use the following result [CF, p. 369, Lemma 2.2].

Lemma 2.5. *Let f and F be as in Lemma 2.4, and let $a \in \mathbb{C}_p^*$. Then we have the identity*

$$\int_{\mathbb{Z}_p} F(x+at)\,dt = F(x) + (x-a)F'(x) - \int_{\mathbb{Z}_p} (x+at)f'(x+at)\,dt\,, \tag{22}$$

valid for all $x \in \mathbb{C}_p$.

Proof. Let $w(x) := F(x+a) - F(x) = af'(x)$, by equation (21). Then w is locally analytic on $\mathbb{C}_p$, and we have the telescoping sum

$$F(x+aj) = F(x) + \sum_{k=0}^{j-1} w(x+ak) \qquad (j \in \mathbb{N}_0).$$

Hence,

$$\begin{aligned} \frac{1}{p^n} \sum_{j=0}^{p^n-1} F(x+aj) &= F(x) + \frac{1}{p^n} \sum_{j=0}^{p^n-1} \sum_{k=0}^{j-1} w(x+ak) \\ &= F(x) + \frac{1}{p^n} \sum_{k=0}^{p^n-2} (p^n-1-k) w(x+ak) \end{aligned}$$

$$= F(x) + \frac{1}{p^n}\sum_{k=0}^{p^n-1}(p^n - 1 - k)w(x+ak)$$

$$= F(x) + \sum_{k=0}^{p^n-1} w(x+ak) - \frac{1}{p^n}\sum_{k=0}^{p^n-1}(k+1)w(x+ak).$$

Since w is locally analytic, we have

$$\lim_{n\to\infty}\sum_{k=0}^{p^n-1} w(x+ak) = \Big(\lim_{n\to\infty} p^n\Big)\Big(\lim_{n\to\infty}\frac{1}{p^n}\sum_{k=0}^{p^n-1} w(x+ak)\Big) = 0.$$

Thus,

$$\begin{aligned}\int_{\mathbb{Z}_p} F(x+at)\,dt &= F(x) - \int_{\mathbb{Z}_p}(t+1)w(x+at)\,dt\\ &= F(x) - \int_{\mathbb{Z}_p}(at+a)f'(x+at)\,dt\\ &= F(x) + (x-a)\int_{\mathbb{Z}_p} f'(x+at)\,dt - \int_{\mathbb{Z}_p}(x+at)f'(x+at)\,dt\\ &= F(x) + (x-a)F'(x) - \int_{\mathbb{Z}_p}(x+at)f'(x+at)\,dt\,.\end{aligned}$$

□

3. A new p-adic log-gamma function

In this section we define a new version of the p-adic log-gamma function and prove some of its properties.

For a fixed $a \in \mathbb{C}_p^*$ let us define the function $h : \mathbb{C}_p \to \mathbb{C}_p$ by

$$h(x) = h(x|a) := \begin{cases} 0 & \text{if } |x|_p < |a|_p\,, \\ x\log_p x - x & \text{if } |x|_p \geq |a|_p\,. \end{cases} \tag{23}$$

If we call χ_a the characteristic function of the set $B(0;|a|_p^-)^c = \{x \in \mathbb{C}_p \mid |x|_p \geq |a|_p\}$, we can write

$$h(x|a) = x(\log_p x - 1)\chi_a(x). \tag{24}$$

Since the open ball $B(0;|a|_p^-)$ is also closed, its complement in $\mathbb{C}_p$ is open, so in (23) we have defined h by its restriction to disjoint open sets. Now, the null function is trivially analytic on $\mathbb{C}_p$, and so is the identity function. Also, $\log_p x$ is locally analytic on $B(0;|a|_p^-)^c$, in fact on $\mathbb{C}_p^*$ [Sc, p. 131, Prop. 45.7]. Thus the function $x\log_p x - x$ is also locally analytic on the open set $B(0;|a|_p^-)^c$. Hence, the function h is locally analytic on $\mathbb{C}_p$. Therefore, by Lemma 2.4, the following definition makes sense.

Definition 3.1. *For $a \in \mathbb{C}_p^*$ fixed, define the locally analytic[9] funtion $\mathrm{Log}\Gamma_\mathrm{p} : \mathbb{C}_p \to \mathbb{C}_p$ by the Volkenborn integral*

$$\mathrm{Log}\Gamma_\mathrm{p}(x|a) := \frac{1}{a}\int_{\mathbb{Z}_p} h(x+at|a)\,dt\,, \tag{25}$$

where χ_a is the characteristic function of the set $B(0;|a|_p^-)^c = \{x \in \mathbb{C}_p \mid |x|_p \geq |a|_p\}$, and where $h(x|a) = x(\log_p x - 1)\chi_a(x)$.

Hence, we can write

$$\mathrm{Log}\Gamma_\mathrm{p}(x|a) = \frac{1}{a}\int_{\mathbb{Z}_p} (x+at)\big(\log_p(x+at) - 1\big)\chi_a(x+at)\,dt\,.$$

The simplest property of $\mathrm{Log}\Gamma_\mathrm{p}$ is the following difference equation.[10]

Proposition 3.2. *For all $x \in \mathbb{C}_p$ we have the difference equation*

$$\mathrm{Log}\Gamma_\mathrm{p}(x+a|a) - \mathrm{Log}\Gamma_\mathrm{p}(x|a) = \chi_a(x)\log_p x\,, \tag{26}$$

where χ_a is as in Definition 3.1.

Proof. This follows from (21), noticing that

$$h'(x|a) = \chi_a(x)\log_p x\,. \tag{27}$$

□

The function $\mathrm{Log}\Gamma_\mathrm{p}$ satisfies a Raabe-type formula and a characterization theorem similar to the one satisfied by $\mathrm{Log}\Gamma_\mathrm{D}$ [CF, p. 364].

Theorem 3.3. *The function $\mathrm{Log}\Gamma_\mathrm{p}$ satisfies*

$$\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_\mathrm{p}(x+at|a)\,dt = (x-a)\,\mathrm{Log}\Gamma_\mathrm{p}'(x|a) - \frac{1}{a}\,R_p(x|a) \qquad (x \in \mathbb{C}_p)\,, \tag{28}$$

where $R_p : \mathbb{C}_p \to \mathbb{C}_p$ is defined by the Volkenborn integral

$$R_p(x|a) := \int_{\mathbb{Z}_p} (x+at)\chi_a(x+at)\,dt\,. \tag{29}$$

[9]Also by Lemma 2.4, $\mathrm{Log}\Gamma_\mathrm{p}$ is locally analytic on $\mathbb{C}_p$. We will give its local power series in §4.

[10]From now on, we will write $\mathrm{Log}\Gamma_\mathrm{p}$ for the function $\mathrm{Log}\Gamma_\mathrm{p}(\cdot\,|a)$ when $a \in \mathbb{C}_p^*$ is fixed.

Moreover, for a fixed $a \in \mathbb{C}_p^*$, $\mathrm{Log}\Gamma_\mathrm{p}$ *is the unique locally analytic function* $f : \mathbb{C}_p \to \mathbb{C}_p$ *satisfying the difference equation*

$$f(x+a) - f(x) = \chi_a(x) \log_p x \tag{30}$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+at)\, dt = (x-a) f'(x) - \frac{1}{a} R_p(x|a)\,. \tag{31}$$

Proof. First we prove formula (28). Using (22) and (27) we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_\mathrm{p}(x+at|a)\, dt &= \mathrm{Log}\Gamma_\mathrm{p}(x|a) + (x-a)\, \mathrm{Log}\Gamma_\mathrm{p}{}'(x|a) \\
&\qquad - \frac{1}{a}\int_{\mathbb{Z}_p} (x+at) \log_p(x+at) \chi_a(x+at)\, dt \\
&= \mathrm{Log}\Gamma_\mathrm{p}(x|a) + (x-a)\, \mathrm{Log}\Gamma_\mathrm{p}{}'(x|a) \\
&\qquad - \mathrm{Log}\Gamma_\mathrm{p}(x|a) - \frac{1}{a}\int_{\mathbb{Z}_p} (x+at) \chi_a(x+at)\, dt \\
&= (x-a)\, \mathrm{Log}\Gamma_\mathrm{p}{}'(x|a) - \frac{1}{a} R_p(x|a)\,.
\end{aligned}$$

To prove the uniqueness we follow [CF]. Suppose $f : \mathbb{C}_p \to \mathbb{C}_p$ is a locally analytic function satisfying (30) and (31), and let $g(x) := \mathrm{Log}\Gamma_\mathrm{p}(x|a) - f(x)$. From equations (26) and (30) it is easily seen that $g(x+a) - g(x) = 0$. Inductively, we have that $g(x+aj) = g(x)$ for all $j \in \mathbb{N}_0$, and then

$$\int_{\mathbb{Z}_p} g(x+at)\, dt = \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(x+aj) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(x) = g(x)\,.$$

On the other hand, from formulas (28) and (31), we have the equality

$$\int_{\mathbb{Z}_p} g(x+at)\, dt = (x-a)\, g'(x)\,.$$

Since $g(x+ap^n) = g(x)$ for all $n \in \mathbb{N}_0$, it is clear that $g'(x) = 0$ for all $x \in \mathbb{C}_p$. Hence

$$g(x) = \int_{\mathbb{Z}_p} g(x+at)\, dt = 0\,.$$

□

We now compute the function R_p defined by (29).

Proposition 3.4. *For $x \in \mathbb{C}_p$, we have the explicit expression*

$$R_p(x|a) = \begin{cases} x - \dfrac{x}{p} + \dfrac{a\ell}{p} - a\left\lceil \dfrac{\ell}{p} \right\rceil & \text{if } \left|xa^{-1} - \ell\right|_p < 1 \text{ for some } \ell \in \mathbb{Z}, \\ x - \dfrac{a}{2} & \text{otherwise,} \end{cases}$$

where $\lceil c \rceil$ is the usual integer ceiling function for $c \in \mathbb{Q}$.

A consequence of the proposition is that

$$\frac{1}{a} R_p(x|a) = R_p(xa^{-1}|1). \tag{32}$$

One checks that $\left|xa^{-1} - \ell\right|_p < 1$ implies that $|x|_p \leq |a|_p$, and that $\dfrac{a\ell}{p} - a\left\lceil \dfrac{\ell}{p} \right\rceil$ only depends on ℓ modulo p.

Proof. We begin the proof with the easy case, which is when $|x|_p > |a|_p$. Then $|x + at|_p = |x|_p > |a|_p$. Thus, $\chi_a(x + at) = 1$ for all $t \in \mathbb{N}_0$ and

$$\begin{aligned} R_p(x|a) &= \int_{\mathbb{Z}_p} (x + at)\chi_a(x + at)\, dt \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} (x + aj)\chi_a(x + aj) \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} x \;+\; a \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} j \\ &= x + a \lim_{n\to\infty} \frac{p^n - 1}{2} \\ &= x - \frac{a}{2}. \end{aligned}$$

Now, suppose that $|x|_p \leq |a|_p$ and write $u = xa^{-1}$, so $|u|_p \leq 1$. Then

$$\begin{aligned} R_p(x|a) &= \int_{\mathbb{Z}_p} (x + at)\chi_a(x + at)\, dt \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0 \leq j < p^n} (x + aj)\chi_a(x + aj) \\ &= \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0 \leq j < p^n \\ |x+aj|_p = |a|_p}} (x + aj) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |u+j|_p=1}} (x+aj) \\
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\le j<p^n} (x+aj) - \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |u+j|_p<1}} (x+aj) \\
&= \lim_{n\to\infty} \frac{1}{p^n} \sum_{0\le j<p^n} (x+aj) - a \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |u+j|_p<1}} (u+j) \\
&= x - \frac{a}{2} - a \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |u+j|_p<1}} (u+j)\,. \qquad (33)
\end{aligned}
$$

In the last sum above, the condition $|u+j|_p < 1$ is equivalent to $\left|xa^{-1}+j\right|_p < 1$ and hence, if such j does not exist, this sum is 0. Then, in this case, we also have

$$
R_p(x|a) = x - \frac{a}{2}\,.
$$

The remaining case is when $|x|_p \le |a|_p$ and $\left|xa^{-1}-\ell\right|_p < 1$ for some integer ℓ, which we may choose to satisfy $0 \le \ell \le p-1$. Thus $|u-\ell|_p < 1$. Then, in the sum in (33), the condition $|u+j|_p < 1$ is equivalent, by the strong triangle inequality, to $|\ell+j|_p < 1$. Since $\ell + j \in \mathbb{N}_0$, this is also equivalent to the simpler condition $p \mid (\ell+j)$. Hence we have

$$
\begin{aligned}
\sum_{\substack{0\le j<p^n \\ |u+j|_p<1}} (u+j) &= \sum_{\substack{0\le j<p^n \\ p|(\ell+j)}} (u-\ell+\ell+j) \\
&= \sum_{\substack{\ell\le i<\ell+p^n \\ p|i}} (u-\ell+i) \\
&= \sum_{\frac{\ell}{p}\le j<\frac{\ell}{p}+p^{n-1}} (u-\ell+pj) \\
&= \sum_{\lceil\frac{\ell}{p}\rceil\le i<\lceil\frac{\ell}{p}\rceil+p^{n-1}} (u-\ell+pi) \\
&= \sum_{0\le j<p^{n-1}} \left(u-\ell+pj+p\left\lceil\frac{\ell}{p}\right\rceil\right)
\end{aligned}
$$

$$= p^{n-1}(u-\ell) + p^n \left\lceil \frac{\ell}{p} \right\rceil + p \sum_{0 \le j < p^{n-1}} j\,.$$

Replacing this in (33) we obtain

$$\begin{aligned} R_p(x|a) &= x - \frac{a}{2} - a \lim_{n\to\infty} \frac{1}{p^n} \sum_{\substack{0\le j<p^n \\ |u+j|_p<1}} (u+j) \\ &= x - \frac{a}{2} - \frac{au}{p} + \frac{a\ell}{p} - a \left\lceil \frac{\ell}{p} \right\rceil - a \lim_{n\to\infty} \frac{1}{p^{n-1}} \sum_{0\le j<p^{n-1}} j \\ &= x - \frac{x}{p} + \frac{a\ell}{p} - a \left\lceil \frac{\ell}{p} \right\rceil . \end{aligned}$$

□

The next result reduces the calculation of $\mathrm{Log}\Gamma_{\mathrm{p}}(x|a)$ to the case $a = 1$.

Proposition 3.5. *For all $a \in \mathbb{C}_p^*$ and $x \in \mathbb{C}_p$, we have the identity*

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = \mathrm{Log}\Gamma_{\mathrm{p}}(xa^{-1}|1) + R_p(xa^{-1}|1)\log_p a\,. \tag{34}$$

Proof. By definition we have

$$\begin{aligned} \mathrm{Log}\Gamma_{\mathrm{p}}(x|a) &= \frac{1}{a}\int_{\mathbb{Z}_p} (x+at)\big(\log_p(x+at) - 1\big)\chi_a(x+at)\,dt \\ &= \int_{\mathbb{Z}_p} (xa^{-1}+t)\big(\log_p(xa^{-1}+t) - 1 + \log_p a\big)\chi_1(xa^{-1}+t)\,dt \\ &= \mathrm{Log}\Gamma_{\mathrm{p}}(xa^{-1}|1) + \log_p a \int_{\mathbb{Z}_p} (xa^{-1}+t)\chi_1(xa^{-1}+t)\,dt \\ &= \mathrm{Log}\Gamma_{\mathrm{p}}(xa^{-1}|1) + \frac{\log_p a}{a}\int_{\mathbb{Z}_p} (x+at)\chi_a(x+at)\,dt\,, \end{aligned}$$

and the result follows by (29) and (32).

□

Finally, we give a reflection formula for $\mathrm{Log}\Gamma_{\mathrm{p}}$.

Proposition 3.6. *For all $x \in \mathbb{C}_p$ we have the reflection formula*

$$\mathrm{Log}\Gamma_{\mathrm{p}}(a-x|a) + \mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = 0\,. \tag{35}$$

Proof. To prove this, following [CF], we will use the characterization of $\mathrm{Log}\Gamma_{\mathrm{p}}$ given in Theorem 3.3. Write $f(x) = -\mathrm{Log}\Gamma_{\mathrm{p}}(a-x|a)$. We will show that f satisfies (30) and (31).

The difference equation (30) is easy, since by (26), we have

$$\begin{aligned} f(x+a)-f(x) &= \mathrm{Log}\Gamma_{\mathrm{p}}(-x+a|a) - \mathrm{Log}\Gamma_{\mathrm{p}}(-x|a) \\ &= \chi_a(-x)\log_p(-x) \\ &= \chi_a(x)\log_p x\,, \end{aligned} \tag{36}$$

where in the last equality we used the fact that $\chi_a(x)$ and $\log_p x$ are even functions.

To show that f also satisfies the integro-differential equation (31) we will use the identity [Sc, p. 169, Prop. 55.7]

$$\int_{\mathbb{Z}_p} f(x+at)\,dt = \int_{\mathbb{Z}_p} f(x+a(-1-t))\,dt\,. \tag{37}$$

Replacing the definition of f in (37), we have

$$\int_{\mathbb{Z}_p} f(x+at)\,dt = -\int_{\mathbb{Z}_p} \mathrm{Log}\Gamma_{\mathrm{p}}(2a-x+at|a)\,dt\,.$$

Applying formula (28) to the right hand side, we obtain

$$\int_{\mathbb{Z}_p} f(x+at)\,dt = (x-a)\,\mathrm{Log}\Gamma_{\mathrm{p}}{}'(2a-x|a) + \frac{1}{a}\,R_p(2a-x|a)\,.$$

One easily sees, using the derivative of formula (26), that

$$\begin{aligned} \mathrm{Log}\Gamma_{\mathrm{p}}{}'(2a-x|a) &= \mathrm{Log}\Gamma_{\mathrm{p}}{}'(a-x|a) - \frac{\chi_a(a-x)}{x-a} \\ &= f'(x) - \frac{\chi_a(a-x)}{x-a}\,, \end{aligned}$$

and thus

$$\int_{\mathbb{Z}_p} f(x+at)\,dt = (x-a)f'(x) - \chi_a(a-x) + \frac{1}{a}\,R_p(2a-x|a)\,.$$

Hence, we need to show that for all $x \in \mathbb{C}_p$, we have

$$-\frac{1}{a}\,R_p(x|a) = -\chi_a(a-x) + \frac{1}{a}\,R_p(2a-x|a)\,. \tag{38}$$

By (32), equation (38) is equivalent to

$$-R_p(xa^{-1}|1) = -\chi_1(1-xa^{-1}) + R_p(2-xa^{-1}|1)\,. \tag{39}$$

Reordering and writing $u = 1 - xa^{-1}$ in (39), we obtain that (38) is also equivalent to showing that for all $u \in \mathbb{C}_p$, we have

$$\chi_1(u) = R_p(1-u|1) + R_p(1+u|1)\,, \tag{40}$$

thus reducing us to the case where $a = 1$.

Let $S_{1,\ell}$, $S_{1,\infty}$ and T_1 be the subsets of $\mathbb{C}_p$ defined by replacing $a = 1$ in (13) and (14). If $u \in S_{1,\infty} \cup T_1$, since by (v) in Lemma 2.2 this set is $\mathbb{Z}_p$-invariant, then $1-u$, $1+u \in S_{1,\infty} \cup T_1$. Also, since $u \in S_{1,\infty} \cup T_1$, then $\chi_1(u) = 1$. Thus, by Proposition 3.4, we obtain

$$R_p(1-u|1) + R_p(1+u|1) = (1-u-\frac{1}{2}) + (1+u-\frac{1}{2}) = 1 = \chi_1(u).$$

Now, let $u \in S_{1,\ell}$ for some integer ℓ, which we may choose to satisfy $1 \leq \ell \leq p$. Then $1-u \in S_{1,1-\ell}$ and $1+u \in S_{1,1+\ell}$, and by Proposition 3.4,

$$\begin{aligned} R_p(1-u|1) + R_p(1+u|1) &= 1-u-\frac{1-u}{p} + \frac{(1-\ell)}{p} - \left\lceil \frac{1-\ell}{p} \right\rceil \\ &\quad + 1 + u - \frac{1+u}{p} + \frac{(1+\ell)}{p} - \left\lceil \frac{1+\ell}{p} \right\rceil \\ &= 2 - \left\lceil \frac{1-\ell}{p} \right\rceil - \left\lceil \frac{1+\ell}{p} \right\rceil . \end{aligned} \tag{41}$$

Since $1 \leq \ell \leq p$, then $1-p \leq 1-\ell \leq 0$, and we have that $\left\lceil \frac{1-\ell}{p} \right\rceil = 0$. Also since $1 \leq \ell \leq p$, we have that $\chi_1(u) = 0$ if and only if $\ell = p$. By this and by (41), we obtain that (40) is equivalent to showing that

$$2 - \left\lceil \frac{1+\ell}{p} \right\rceil = \begin{cases} 1 & \text{if } 1 \leq \ell \leq p-1, \\ 0 & \text{if } \ell = p, \end{cases}$$

which is immediate. Since $\mathbb{C}_p = S_{1,\infty} \cup T_1 \cup \bigcup_{\ell=1}^{p} S_{1,\ell}$, we have covered all cases, and the proposition follows.

□

4. Relation with the functions of Morita and Diamond

We now take a look at the relation of $\mathrm{Log}\Gamma_\mathrm{p}$ with Morita's and Diamond's functions. We also compute the power series expansion for $\mathrm{Log}\Gamma_\mathrm{p}$. We use of the notation given in §2.1.

Let us start with Diamond's function,

$$\mathrm{Log}\Gamma_\mathrm{D}(x) := \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t)-1\big)\,dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p).$$

This function satisfies a characterization theorem [CF, p. 364] that states that if $D \subset \mathbb{C}_p \setminus \mathbb{Z}_p$ is any nonempty $\mathbb{Z}_p$-invariant subset, then $\mathrm{Log}\Gamma_\mathrm{D}$ is the only locally analytic function $f : D \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = \log_p x \qquad (x \in D) \tag{42}$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \frac{1}{2} \qquad (x \in D). \tag{43}$$

Actually, the original statement is for $D = \mathbb{C}_p \setminus \mathbb{Z}_p$, but it is easily seen that its proof in [CF] is valid for any nonempty $\mathbb{Z}_p$-invariant subset D. Thus, we can take $D = T_1 \cup S_{1,\infty}$ (see (14)) since, from (i) in Lemma 2.2 and from the remark after the same lemma, we have that $T_1 \cup S_{1,\infty} \subset \mathbb{C}_p \setminus \mathbb{Z}_p$. Also, from (v) in Lemma 2.2, the set $T_1 \cup S_{1,\infty}$ is $\mathbb{Z}_p$-invariant.

Similarly, Theorem 3.3 can be extended to conclude that $\mathrm{Log}\Gamma_\mathrm{p}(x|1)$ is the only locally analytic function on $T_1 \cup S_{1,\infty}$ satisfying both

$$f(x+1) - f(x) = \chi_1(x)\log_p x \qquad (x \in T_1 \cup S_{1,\infty})$$

and

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - R_p(x|1) \qquad (x \in T_1 \cup S_{1,\infty}).$$

Since $x \in T_1 \cup S_{1,\infty}$, by Proposition 3.4 and by its proof, we immediately see that

$$\chi_1(x) = 1 \qquad \text{and} \qquad R_p(x|1) = x - \frac{1}{2} \qquad (x \in T_1 \cup S_{1,\infty}).$$

We deduce that $\mathrm{Log}\Gamma_\mathrm{p}(x|1)$ is a locally analytic function on $T_1 \cup S_{1,\infty}$ satisfying both (42) and (43). By uniqueness, we deduce

Proposition 4.1. *For $x \in S_{1,\infty} \cup T_1$, where $S_{1,\infty}$ and T_1 are defined by replacing $a = 1$ in (14), we have*

$$\mathrm{Log}\Gamma_\mathrm{p}(x|1) = \mathrm{Log}\Gamma_\mathrm{D}(x) \qquad (x \in S_{1,\infty} \cup T_1). \tag{44}$$

More generally, since by Proposition 3.4, $R_p(x|a) = x - a/2$ for $x \in S_{a,\infty} \cup T_a$, replacing this value and (44) in equation (34), we obtain

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = \mathrm{Log}\Gamma_{\mathrm{D}}(xa^{-1}) + \big(\frac{x}{a} - \frac{1}{2}\big) \log_p a \qquad (x \in S_{a,\infty} \cup T_a). \tag{45}$$

Equation (45) is useful because of the following. By Lemma 2.4, $\mathrm{Log}\Gamma_{\mathrm{p}}$ is locally analytic on $\mathbb{C}_p$, and hence, it is also locally analytic on the open subset $S_{a,\infty} \cup T_a$. We will use (45) to compute the power series expansion for $\mathrm{Log}\Gamma_{\mathrm{p}}$. The power series expansion for $\mathrm{Log}\Gamma_{\mathrm{D}}$ is known [Sc, p. 183, Theorem 60.2] and runs as follows. Let $y_0 \in \mathbb{C}_p \setminus \mathbb{Z}_p$ and $\rho = \rho(y_0) := \inf_{y \in \mathbb{Z}_p} \{ |y - y_0|_p \}$. Then we have the power series expansion

$$\mathrm{Log}\Gamma_{\mathrm{D}}(y) = \mathrm{Log}\Gamma_{\mathrm{D}}(y_0) + \tau_1(y - y_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \tau_{n+1}(y - y_0)^{n+1} \tag{46}$$

valid for all $y \in B(y_0; \rho^-)$, where

$$\tau_1 := \int_{\mathbb{Z}_p} \log_p(y_0 + t)\, dt\,, \quad \tau_{n+1} := \int_{\mathbb{Z}_p} (y_0 + t)^{-n}\, dt \qquad (n \in \mathbb{N}).$$

Now, let $x_0 \in S_{a,\infty} \cup T_a$ and write $y_0 = x_0 a^{-1}$. Then $y_0 \in S_{1,\infty} \cup T_1 \subset \mathbb{C}_p \setminus \mathbb{Z}_p$, and a simple calculation shows that in this case $\rho = \left|x_0 a^{-1}\right|_p \geq 1$. Hence, if $x \in B(x_0; |x_0|_p^-)$, or equivalently, if $xa^{-1} \in B(y_0; |y_0|_p^-)$, replacing in (46) we obtain

$$\mathrm{Log}\Gamma_{\mathrm{D}}(xa^{-1}) = \mathrm{Log}\Gamma_{\mathrm{D}}(x_0 a^{-1}) + \frac{\tau_1}{a}(x - x_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, \tau_{n+1}}{n(n+1)\, a^{n+1}} (x - x_0)^{n+1} \tag{47}$$

with the τ_n defined as above with $y_0 = x_0 a^{-1}$. With the help of identity (45), we deduce the following.

Proposition 4.2. *Let $x_0 \in S_{a,\infty} \cup T_a$, where $S_{a,\infty}$ and T_a are defined in* (14). *Then we have the power series expansion*

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = \mathrm{Log}\Gamma_{\mathrm{p}}(x_0|a) + \widetilde{\tau}_1(x - x_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \widetilde{\tau}_{n+1}(x - x_0)^{n+1} \tag{48}$$

valid for all $x \in B(x_0; |x_0|_p^-)$, where

$$\widetilde{\tau}_1 := \frac{1}{a} \int_{\mathbb{Z}_p} \log_p(x_0 + at)\, dt\,, \quad \widetilde{\tau}_{n+1} := \frac{1}{a} \int_{\mathbb{Z}_p} (x_0 + at)^{-n}\, dt \qquad (n \in \mathbb{N}).$$

Proof. Using (45) and (47), and setting $y_0 = x_0 a^{-1}$, we arrive at

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = \mathrm{Log}\Gamma_{\mathrm{p}}(x_0|a) + \frac{(x-x_0)}{a}\log_p a + \frac{(x-x_0)}{a}\tau_1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,\tau_{n+1}}{n(n+1)\,a^{n+1}}(x-x_0)^{n+1}\,.$$

For $n \in \mathbb{N}$, it is easily seen that $\widetilde{\tau}_{n+1} = a^{-(n+1)}\,\tau_{n+1}$. Also, since $\int_{\mathbb{Z}_p} 1\,dt = 1$,

$$\begin{aligned}\tau_1 + \log_p a &= \int_{\mathbb{Z}_p} \log_p(x_0 a^{-1} + t)\,dt + \int_{\mathbb{Z}_p} \log_p a\,dt \\ &= \int_{\mathbb{Z}_p} \log_p(x_0 + at)\,dt \;=\; a\,\widetilde{\tau}_1\,.\end{aligned}$$

□

We now consider the relation of $\mathrm{Log}\Gamma_{\mathrm{p}}$ with Morita's function $\mathrm{Log}\Gamma_{\mathrm{M}}$. Recall that $\mathrm{Log}\Gamma_{\mathrm{M}}$ is defined by the Volkenborn integral

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \int_{\mathbb{Z}_p} (x+t)\big(\log_p(x+t) - 1\big)\chi_{\mathbb{Z}_p^*}(x+t)\,dt \qquad (x \in \mathbb{Z}_p)\,.$$

The direct relation between $\mathrm{Log}\Gamma_{\mathrm{p}}$ and $\mathrm{Log}\Gamma_{\mathrm{M}}$ is easy since $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$ is actually $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1)$ restricted to $\mathbb{Z}_p$. To see this, let us go back to the function h defined at the beginning of §3. We have

$$h(x|1) = \begin{cases} 0 & \text{if } |x|_p < 1\,, \\ x\log_p x - x & \text{if } |x|_p \geq 1\,, \end{cases}$$

and if we restrict ourselves to $x \in \mathbb{Z}_p$, then

$$\{x \in \mathbb{Z}_p \mid |x|_p < 1\} = p\mathbb{Z}_p \;\text{ and }\; \{x \in \mathbb{Z}_p \mid |x|_p \geq 1\} = \mathbb{Z}_p^*.$$

Hence we have

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|1) = \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(x+t)(x+t)\big(\log_p(x+t) - 1\big)\,dt \qquad (x \in \mathbb{Z}_p),$$

and this is exactly Morita's function $\mathrm{Log}\Gamma_{\mathrm{M}}$.

An important property of $\mathrm{Log}\Gamma_{\mathrm{M}}$ is that it has a power series expansion around 0, valid for all $x \in p\mathbb{Z}_p$. Namely [Sc, p. 177, Lemma 58.1], for all $x \in p\mathbb{Z}_p$ we have the identity

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) = \lambda_1 x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,\lambda_{n+1}}{n(n+1)}\,x^{n+1}\,, \tag{49}$$

where

$$\lambda_1 := \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t) \log_p t \, dt \,, \quad \lambda_{n+1} := \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\, t^{-n} \, dt \qquad (n \in \mathbb{N}).$$

Moreover, the right side of (49) defines an analytic function on the open unit ball $B(0;1^-) \subset \mathbb{C}_p$ [Sc, p. 177, Lemma 58.2]. Hence, we extend the domain of $\mathrm{Log}\Gamma_{\mathrm{M}}$ to $B(0;1^-) \cup \mathbb{Z}_p$ by defining[11]

$$\mathrm{Log}\Gamma_{\mathrm{M}}(x) := \lambda_1 x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, \lambda_{n+1}}{n(n+1)}\, x^{n+1} \qquad (x \in B(0;1^-)). \tag{50}$$

We notice here that if $n \in \mathbb{N}$ is odd, then the function $t \to \chi_{\mathbb{Z}_p^*}(t)\, t^{-n}$ is an odd function, and then [Ro, p. 269, Corollary][12] implies that

$$\lambda_{n+1} = \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\, t^{-n} \, dt = -\frac{1}{2} \frac{d}{dt} \chi_{\mathbb{Z}_p^*}(t)\, t^{-n} \Big|_{t=0} = 0 \,.$$

Thus, we could rewrite the sum in the right of (50) as a sum over even n, but we prefer to leave it like that because of the resemblance with the expansion (46) of the function $\mathrm{Log}\Gamma_{\mathrm{D}}$.

Now, we will see the deeper relation between $\mathrm{Log}\Gamma_{\mathrm{p}}$ and Morita's function $\mathrm{Log}\Gamma_{\mathrm{M}}$. We already computed the power series expansion for $\mathrm{Log}\Gamma_{\mathrm{p}}$ at each point of the open set $S_{a,\infty} \cup T_a$, and now we will compute it at each point of the open set $S_a := \bigcup_{\ell=0}^{p-1} S_{a,\ell}$ (see (13)), i.e., on the complement of $S_{a,\infty} \cup T_a$ in $\mathbb{C}_p$. We will show that the expansions of $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$ and $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1)$ are identical for $x \in S_{1,0} = B(0;1^-)$.[13]

First we need the following lemmas.

Lemma 4.3. *Let $x, y \in S_{a,\ell} = \{ x \in \mathbb{C}_p \mid |x - a\ell|_p < |a|_p \}$ for some $0 \le \ell \le p-1$. Then $\chi_a(x + at) = \chi_a(y + at)$ for all $t \in \mathbb{Z}_p$.*

Proof. By (iii) in Lemma 2.2 we know that $S_{a,\ell} = B(a\ell; |a|_p^-)$. Since every point in an open ball is also its center, we have that $B(a\ell; |a|_p^-) = B(x; |a|_p^-) = B(y; |a|_p^-)$. Hence, $x, y \in S_{a,\ell}$ implies $|x-y|_p < |a|_p$. Now, suppose that $t \in \mathbb{Z}_p$ and that $\chi_a(x+at) = 0$, i.e., $|x+at|_p < |a|_p$. Then $|y+at|_p = |y-x+x+at|_p \le \max\{ |y-x|_p, |x+at|_p \} < |a|_p$

[11] It can be shown that with this extended definition $\mathrm{Log}\Gamma_{\mathrm{M}}$ is no longer the Iwasawa logarithm of a p-adic function.

[12] Namely, when f is an odd function, then $\int_{\mathbb{Z}_p} f(t)\, dt = -\frac{f'(0)}{2}$.

[13] This equality is actually a simple consequence of p-adic analytic continuation, but to stay in the spirit of this exposition we will give a direct proof.

and thus, $\chi_a(y+at)=0$. By symmetry, $\chi_a(y+at)=0$ implies $\chi_a(x+at)=0$, and the desired equality follows.

$\square$

Lemma 4.4. *Let* $x,y \in S_{a,\ell} = \{x \in \mathbb{C}_p \mid |x-a\ell|_p < |a|_p\}$ *for some* $0 \le \ell \le p-1$, *and suppose* $\chi_a(y+at)=1$ *for some* $t \in \mathbb{Z}_p$. *Then*

$$\left|\frac{x+at}{y+at}-1\right|_p < 1\,.$$

Proof. Rewrite the above inequality as

$$\left|\frac{x-y}{y+at}\right|_p < 1\,.$$

But $|x-y|_p < |a|_p$ as $x,y \in S_{a,\ell}$, and $|y+at|_p \ge |a|_p$ as $\chi_a(y+at)=1$, by assumption.

$\square$

Lemma 4.5. *For all* $a \in \mathbb{C}_p^*$ *we have the special value* $\mathrm{Log}\Gamma_\mathrm{p}(0|a)=0$.

Proof. Replacing $x=0$ in equations (26) and (35), we see that $\mathrm{Log}\Gamma_\mathrm{p}(a|a)-\mathrm{Log}\Gamma_\mathrm{p}(0|a)=0$ and $\mathrm{Log}\Gamma_\mathrm{p}(a|a)+\mathrm{Log}\Gamma_\mathrm{p}(0|a)=0$. Hence, $\mathrm{Log}\Gamma_\mathrm{p}(0|a)=\mathrm{Log}\Gamma_\mathrm{p}(a|a)=0$.

$\square$

Proposition 4.6. *Let* $x_0 \in S_{a,\ell} = \{x \in \mathbb{C}_p \mid |x-a\ell|_p < |a|_p\}$ *for some* $0 \le \ell \le p-1$. *Then we have the power series expansion*

$$\mathrm{Log}\Gamma_\mathrm{p}(x|a) = \mathrm{Log}\Gamma_\mathrm{p}(x_0|a) + \widetilde{\lambda}_1(x-x_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}\, \widetilde{\lambda}_{n+1}(x-x_0)^{n+1} \tag{51}$$

valid for all $x \in B(x_0;|a|_p^-) = B(a\ell;|a|_p^-) = S_{a,\ell}$, *where*

$$\widetilde{\lambda}_1 := \frac{1}{a}\int_{\mathbb{Z}_p} \log_p(x_0+at)\chi_a(x_0+at)\,dt, \quad \widetilde{\lambda}_{n+1} := \frac{1}{a}\int_{\mathbb{Z}_p} (x_0+at)^{-n}\chi_a(x_0+at)\,dt \quad (n \in \mathbb{N}).$$

Proof. Let $x_0, x \in S_{a,\ell}$. Then, by Lemma 4.3,

$$\mathrm{Log}\Gamma_\mathrm{p}(x|a) = \frac{1}{a}\int_{\mathbb{Z}_p} (x+at)\big(\log_p(x+at)-1\big)\chi_a(x_0+at)\,dt\,.$$

Subtracting the same expression with x replaced by x_0, after some reordering we have

$$
\begin{aligned}
a\,\mathrm{Log}\Gamma_\mathrm{p}(x|a) - a\,\mathrm{Log}\Gamma_\mathrm{p}(x_0|a) &= (x-x_0)\int_{\mathbb{Z}_p} \log_p(x_0+at)\chi_a(x_0+at)\,dt \\
&\quad - (x-x_0)\int_{\mathbb{Z}_p} \chi_a(x_0+at)\,dt \\
&\quad + \int_{\mathbb{Z}_p} (x+at)\big(\log_p(x+at) - \log_p(x_0+at)\big)\chi_a(x_0+at)\,dt \\
&= a(x-x_0)\widetilde{\lambda}_1 - (x-x_0)\int_{\mathbb{Z}_p} \chi_a(x_0+at)\,dt \\
&\quad + \int_{\mathbb{Z}_p} (x+at)\log_p\big(\frac{x+at}{x_0+at}\big)\chi_a(x_0+at)\,dt\,. \qquad (52)
\end{aligned}
$$

For convenience, define

$$
(*) := a\,\mathrm{Log}\Gamma_\mathrm{p}(x|a) - a\,\mathrm{Log}\Gamma_\mathrm{p}(x_0|a) - a(x-x_0)\widetilde{\lambda}_1\,. \qquad (53)
$$

Then we have

$$
(*) = \int_{\mathbb{Z}_p} \Big(-x + x_0 + (x+at)\log_p\big(\frac{x+at}{x_0+at}\big)\Big)\chi_a(x_0+at)\,dt\,. \qquad (54)
$$

By Lemma 4.4, in (54) we can replace the logarithm by its power series [Sc, p. 131, Prop. 45.7] to obtain

$$
\begin{aligned}
(*) &= \int_{\mathbb{Z}_p} \Big(-x + x_0 + (x+at)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\big(\frac{x+at}{x_0+at} - 1\big)^n\Big)\chi_a(x_0+at)\,dt \\
&= \int_{\mathbb{Z}_p} \Big(-x + x_0 + (x+at)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\big(\frac{x-x_0}{x_0+at}\big)^n\Big)\chi_a(x_0+at)\,dt \\
&= \int_{\mathbb{Z}_p} \chi_a(x_0+at)(x-x_0)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\big(\frac{x-x_0}{x_0+at}\big)^n\,dt \qquad [\text{see } (56)]\,, \qquad (55)
\end{aligned}
$$

where in the last step we used the identity [Ro, p. 377]

$$
(\mu+\nu)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,\mu^n}{n\,\nu^n} = \mu + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,\mu^{n+1}}{n(n+1)\,\nu^n} \qquad (|\mu|_p < |\nu|_p) \qquad (56)
$$

with $\mu = x - x_0$ and $\nu = x_0 + at$.

Now, an immediate consequence of the proof of [Di, p. 324, Theorem 2], is that we can interchange the Volkenborn integral with the summation in (55), since the function inside is analytic. Thus we obtain

$$(*) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} (x - x_0)^{n+1} \int_{\mathbb{Z}_p} \chi_a(x_0 + at)(x_0 + at)^{-n}\, dt$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} (x - x_0)^{n+1}\, a\widetilde{\lambda}_{n+1}\,.$$

The proposition follows from (52) and (53).

□

The following corollary relates $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1)$ and $\mathrm{Log}\Gamma_{\mathrm{M}}(x)$. Recall that the power series (50) naturally extends the domain of $\mathrm{Log}\Gamma_{\mathrm{M}}$ from $\mathbb{Z}_p$ to $\mathbb{Z}_p \cup B(0;1^-)$.

Corollary 4.7. *For all* $x \in \mathbb{Z}_p \cup B(0;1^-)$ *we have* $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1) = \mathrm{Log}\Gamma_{\mathrm{M}}(x)\,.$

Proof. We already showed that if $x \in \mathbb{Z}_p$ then $\mathrm{Log}\Gamma_{\mathrm{p}}(x|1) = \mathrm{Log}\Gamma_{\mathrm{M}}(x)$. Now, since $0 \in B(0;1^-)$, we can choose $x_0 = 0$ in Proposition 4.6. Then, for $x \in B(0;1^-)$ we have

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|1) = \mathrm{Log}\Gamma_{\mathrm{p}}(0|1) + x \int_{\mathbb{Z}_p} \chi_1(t) \log_p t\, dt + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} \int_{\mathbb{Z}_p} \chi_1(t)\, t^{-n}\, dt\,.$$

By Lemma 4.5, $\mathrm{Log}\Gamma_{\mathrm{p}}(0|1) = 0$. Also, if $t \in \mathbb{Z}_p$, then $\chi_1(t) = \chi_{\mathbb{Z}_p^*}(t)$. Hence, replacing this in the above equation, we obtain for $x \in B(0;1^-)$

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|1) = x \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t) \log_p t\, dt + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)\, t^{-n}\, dt$$
$$= \mathrm{Log}\Gamma_{\mathrm{M}}(x) \qquad [\text{see } (50)\,].$$

□

We can summarize Propositions 4.2 and 4.6 in a single theorem.

Theorem 4.8. *Let* $x_0 \in \mathbb{C}_p$ *and* $a \in \mathbb{C}_p^*$, *and let* $\sigma = \max\{\, |x_0|_p\,, |a|_p \,\}$. *Then we have the convergent power series expansion*

$$\mathrm{Log}\Gamma_{\mathrm{p}}(x|a) = \mathrm{Log}\Gamma_{\mathrm{p}}(x_0|a) + \widetilde{\lambda}_1(x - x_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \widetilde{\lambda}_{n+1}(x - x_0)^{n+1} \qquad (57)$$

valid for all $x \in B(x_0;\sigma^-)$, *where*

$$\widetilde{\lambda}_1 := \frac{1}{a} \int_{\mathbb{Z}_p} \log_p(x_0 + at)\chi_a(x_0 + at)\, dt, \quad \widetilde{\lambda}_{n+1} := \frac{1}{a} \int_{\mathbb{Z}_p} (x_0 + at)^{-n}\chi_a(x_0 + at)\, dt \quad (n \in \mathbb{N}).$$

Proof. If $x_0 \in S_a := \bigcup_{\ell=0}^{p-1} S_{a,\ell}$, then $\sigma = |a|_p$, and the proof is immediate by comparing with Proposition 4.6. Now, if $x_0 \in S_{a,\infty} \cup T_a$, then $\sigma = |x_0|_p$. Also, we obtain that $\chi_a(x_0 + at) = 1$ for all $t \in \mathbb{Z}_p$, and the result follows by comparing coefficients in (48) and (57).

□

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