ON THE GROUP ALGEBRA DECOMPOSITION OF A JACOBIAN VARIETY

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INFORME DE APROBACIÓN

TESIS DE DOCTORADO

Se informa a la Escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por la candidata

Leslie Alejandra Jiménez Palma

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Dr. Angel Carocca ....

Dr. Eduardo Friedman ....

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¿Para qué sirve la utopía?

La utopía está en el horizonte.
Camino dos pasos, ella se aleja dos pasos
y el horizonte se corre diez pasos más allá.
¿Entonces para que sirve la utopía?
Para eso, sirve para caminar.

Eduardo Galeano

Biografía

Nací el 17 de Enero de 1983 en el Hospital San José, comuna de Recoleta, Santiago. Viví en la comuna de Independencia hasta mi último año de colegio, el emblemático Liceo Carmela Carvajal. Ya en la Universidad viví en distintos lugares de la comuna de Maipú desde donde me dirigía a la siempre mítica Facultad de Ciencias de la Universidad de Chile. Siempre quise estudiar en la Chile y desde que pisé por primera vez esta Facultad, supe que quería estudiar aquí. Esto sucedió mientras estaba en cuarto medio, un día donde un grupo de inquietas estudiantes decidió investigar en terreno y en tenida de uniforme de qué se trataban las carreras de Matemática y Física, yo en ese tiempo estaba encantada con la Física y con ser profesora.

Despúes de terminar la Licenciatura en Matemática, hice el Magister. Luego, supe que el camino al menos inmediato era hacer un Doctorado. Yo creo que finalmente esto de estudiar de por vida siempre fue lo mío. Desde chica me lo pasaba entre libros y cuadernos. Muchas veces mi mamá me obligaba a salir porque según ella era demasiado mi fanatismo. Cuando entré a la U seguí igual, de hecho hay varias anécdotas relacionadas con mi mateismo. Pero algo pasó en los años siguientes, descubrí que además de estudiar, también me gustaba salir con mis amigos, ir a fiestas, etc. Así que fui a varias bien seguido. Fue un tiempo donde lo pasé muy bien, pero siempre me preocupé de terminar mi Licenciatura en los 4 años. Desde que inicié el postgrado he tenido que mantener el equilibrio entre mi ser mateo y mi ser más fiestero. Como reflexión, creo que ese fanatismo u obsesión que tuve en alguna época me ayudó a sobrellevar y sobreponerme a adversidades y por lo mismo a generar confianza en mi misma y lograr las metas que me había propuesto.

Por otro lado, la música siempre ha sido muy importante para mi, estuve durante toda la media en el coro del Colegio. Los que me conocen saben lo mucho que me gusta cantar, también bailar. En el colegio toqué flauta y siempre soñé con tocar piano. Poco a poco. Por lo pronto, aprender a tocar guitarra es la meta. Otra de las cosas que interioricé en estos años es que nunca es tarde para hacer lo que uno ha soñado. Eso fue lo que me pasó el año pasado, decidí aprender algo que siempre quise, que es patinar en hielo, ahora tengo mis propios patines y las piruetas van cada vez mejor. El yoga también me ayuda en eso.

Gracias a la pasantía que realicé en Suecia, apareció el gusto por los idiomas. Antes no lo tenía tanto, era más un deber, pero después de estar rodeada de gente que habla 4 o más, me di cuenta que quiero hablar bien inglés y por qué no, algún otro. En lo que viene, quiero volver a Suecia a hacer un postdoc y seguir alimentando mi gusto por las Matemáticas. Estoy ansiosa por saber lo que esas hermosas tierras nórdicas me tienen preparado.

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Sin lugar a dudas, quisiera comenzar agradeciendo a mi tutora, la Doctora Anita Rojas, por todo lo que me ha entregado durante estos años. Ha sido un gran apoyo tanto en lo académico como en lo afectivo. Siempre muy dispuesta a darme parte de su tiempo para discutir sobre Matemática, generosa con sus conocimientos y preocupada de mis avances. En el ámbito afectivo, no puedo dejar de agradecerle toda su empatía y transparencia, sus consejos como investigadora y como mujer. Le agradezco por empujarme a ir más allá siempre, por motivarme a ir al extranjero a aprender y por creer en mi.

Mi gratitud va también hacia el Grupo de Geometría Compleja de Chile, en especial a la profesora Rubí Rodríguez y el profesor Angel Carocca, por sus preguntas, sugerencias y sobre todo por el tiempo invertido en leer con detalle y hacer útiles correcciones a este trabajo. Quiero agradecer a todo el grupo en general porque me sentí muy acogida y escuchada. Creo que siempre las preguntas que recibí fueron para ayudarme a mejorar mi trabajo. Les doy las gracias en particular a los profesores Rubén Hidalgo y Víctor González, con los que tuve mayor cercanía y oportunidad de conversar.

No puedo dejar de agradecer a la profesora Milagros Izquierdo (LIU, Linköping) por recibirme en Linköping, trabajar conmigo y por toda la preocupación que tuvo hacia mi mientras estuve allá tan lejos de Chile. Le agradezco enormemente por ayudarme con el inglés del primer artículo de mi carrera. Aprovecho esta oportunidad para agradecer a Linköping University por recibirme tres meses en sus dependencias y hacerme una más en su intitución. A los estudiantes de doctorado que tuve la oprtunidad de conocer durante mi estadía les agradezco su amable acogida. Sobre todo a mi querida amiga Anna Orlof, gracias por integrarme en cada actividad; ayudarme a mejorar mi inglés y por el apoyo entregado.

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Agradezco a CONICYT por brindarme el apoyo económico durante estos años. Al proyecto MECESUP PUC 0711 por financiar mis estadías en la Universidad Federal Fluminense y Universidad Nacional de Córdoba. A BECAS CHILE por otorgarme el financiamiento para realizar una estadía de 3 meses en Linköping University.

Ya más en el ámbito afectivo, agradezco de manera especial al Departamento de Matemáticas de la Universiad de Chile. A profesores, funcionarios y compañeros, gracias por hacer del departamento un lugar al que dan ganas de ir a estudiar; en el que se puede conversar de matemáticas con cualquier profe o compañero en el pasillo. A los profesores que me hicieron clases, gracias por su dedicación e interés en los estudiantes. Quiero agradecer al profesor Nicolás Yus por apoyarme, creer en mi desde el inicio y aconsejarme en cada paso. Al profesor Eduardo Friedman por su preocupación, consejos y darme confianza en muchas ocasiones, por ejemplo para hacer la ayudanía de álgebra I en postgrado, fue una muy buena experiencia. Le agradezco además por dedicar tiempo a leer y corregir con detalle este trabajo. A la profesora Alicia Labra, por seguir atenta mis avances y felicitarme cada vez que me fue bien.

Les agradezco a Cecilia y Santiago por su amabilidad y disposición a ayudar. A Santiago la buena onda y las buenas charlas sobre fútbol. A Cecilia por su apoyo, cariño y todas las conversaciones. A mis compañeros (as) Juan Carlos, Patricio, Karen, Natalia, Fernando V, Henry y Fernando H por su buena onda y por los excelentes momentos (los de estudio y los de no-estudio). Por supuesto a las damas Katherine, Camila y especialmente a Pamela por los buenos momentos compartidos sobre todo este último tiempo, espero se vengan más.

Quiero agradecer a mi familia por confiar en mi, por hacerme barra cada vez que pueden y por apoyar todas mis decisiones. A mi madre por crearme disciplina y hábitos de estudio desde pequeña y a mi hermano Lee por celebrarme cada logro. De manera especial le agradezco a mi padre y mi hermano Cristóbal. A mi padre por haberme enseñado el valor de la responsabilidad con el ejemplo, por cada día de pequeña estar conmigo a pesar de no vivir juntos. En fin, por estar ahí cada vez que lo necesito. A Cristóbal por ser mi amigo más que mi hermano, por su apoyo incondicional y por todas las experiencias vividas. Hasta el infinito y más allá...

De manera más que especial agradezco a todos mis queridos amigos. Sin duda, sin su compañía, soporte y cariño, lograr esta meta llamada doctorado hubiera sido aún más difícil, más aún en esta última etapa. Definitivamente, son una firme red de contención, gracias totales! Agradezco a Tamara, Melanie, Jaime y Paulina por su buena amistad. Tamarilla, gracias por ser durante 5 años la mejor housemate. A mis amigos geómetras Mariela y Saúl por empujarme a creer en mi, por sus consejos desde la experiencia y su preocupación permanente. A los de la *vieja ola*: Nicolás, Amalia, Claudia, Romina y Verónica ¿Qué les puedo decir? A Nicolás y Amalia, les agradezco sus conversaciones y preocupación sobre todo cuando estuve fuera. Gracias Nico por los salud vía whatsApp para mi cumple, y gracias Amalia por las interminables skypeadas en momentos claves. A Claudita, por su preocupación permanente y por seguir haciéndome parte de los momentos importantes de su vida. A Verito y Romi, por su amistad más que incondicional sin juicios ni reproches, por escucharme atentas cada vez que lo necesito, por ponerse en mis zapatos y desde ahí aconsejar. A los *queridísimos de la nueva*: William, Vania y Sandy por traer nuevos y buenos aires a mi vida, por su hermosa y sana amistad. A William por darme ánimo siempre y hacer cada día agradable para trabajar y compartir en la oficina. A Vania por escucharme y decirme las cosas que hago bien y no tanto siempre de manera directa, eso me ayuda mucho. Vania y William gracias por siempre apañar hasta en las fiestas. A Sandyta por ser más que una amiga, una hermana (ya somos familia amarilla), por tirarme para arriba siempre, creer en mi y por correr a ayudarme cada vez que lo necesito, muchas gracias!

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Contents

LIST OF SYMBOLS

$\|G\|$; $\|x\|$	: Size of the group G; order of the element $x \in G$.
$H \leq G$; $H < G$	: H is a subgroup of G; H is a proper subgroup of G.
$N \trianglelefteq G$; $N \triangleleft G$	: N is a normal subgroup of G; N is a proper normal subgroup of G.
G/N	: quotient group of G by the normal subgroup N.
$\|G : H\|$	: index of H in G.
$G_1 \times G_2$	: direct product of G_1 by G_2.
$G_1 \rtimes G_2$	: semidirect product of G_1 by G_2.
$< x >$	: subgroup generated by x.
$\mathbf{C}_G(H)$; $\mathbf{N}_G(H)$	: centralizer of H in G; normalizer of H in G.
$\mathbb{Z}_n$; D_n	: cyclic additive group of size n ; dihedral group of size $2n$.
S_n ; A_n	: symmetric group of degree n ; alternating group of degree n.
Im ; $\Im Z$	: the image of a map ; the imaginary part of a matrix Z.

$\mathcal{W}_i; V_i$	:	irreducible rational representation of G
	;	$\mathbb{C}$-irreducible representation of G associated to W_i.
m_i	:	the Schur index of V_i.
$\mathbb{Q}[G]$	:	group algebra of the group G over the rational numbers.
$M(n,R)$; $GL(n,K)$	:	the set of the square matrices of size n over the ring R
	;	the group of invertible square matrices of size n over the field K.
I_g	:	the identity matrix of size g.
X ; X_G	:	Compact Riemann surface of genus g
	;	curve X/G given by the orbits of the action of G on X.
JX ; J_G	:	Jacobian variety of the curve X of dimension g
	;	Jacobian variety of the curve X_G.
$\mathrm{End}(JX)$; $\mathrm{End}_{\mathbb{Q}}(JX)$	:	endomorphisms of JX ; $\mathrm{End}(JX) \otimes \mathbb{Q}$.
$H_1(X,\mathbb{Z})$; $H_1(X,\mathbb{Q})$	:	the first group of homology of X over $\mathbb{Z}$; $H_1(X,\mathbb{Z}) \otimes \mathbb{Q}$.
$H^{1,0}(X,\mathbb{C})$; $H^{1,0}(X,\mathbb{C})^*$	:	the $\mathbb{C}-$vector space of holomorphic forms on X
	;	the dual space of $H^{1,0}(X,\mathbb{C})$.
$Sp_{2g}(\mathbb{Z})$; $\mathcal{H}_g$	:	the symplectic space of matrices of size $2g$
	;	the Siegel upper half space of $2g$ square matrices.
$\mathcal{A}_g$	:	the set of principally polarized abelian variety of dimension g up to isomorphism.

Resumen

Dada X una superficie de Riemann compacta con acción de un grupo finito G, el álgebra de grupo $\mathbb{Q}[G]$ induce una descomposición isógena de su variedad Jacobiana JX, conocida como la descomposición según el álgebra de grupo de JX. En este trabajo desarrollamos un método que permite construir concretamente una descomposición de este tipo. Eso permite estudiar la geometría de la descomposición. Por ejemplo, permite construir diferentes descomposiciones de tal forma de lograr aquella que corresponda a una isogénea del menor núcleo posible, de entre las construidas con nuestro método. Aplicamos este método a familias de curvas trigonales hasta género 10.

Abstract

Given a compact Riemann surface X with an action of a finite group G, the group algebra $\mathbb{Q}[G]$ provides an isogenous decomposition of its Jacobian variety JX, known as the group algebra decomposition of JX. We obtain a method to concretely build a decomposition of this kind. Our method allows us to study the geometry of the decomposition. For instance, we build several decompositions in order to determine which one has kernel of smallest order. We apply this method to families of trigonal curves up to genus 10.

Introduction

The action of a finite group G on a compact Riemann surface X of genus $g \geq 2$ induces a natural homomorphism $\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JX)$ from the rational group algebra $\mathbb{Q}[G]$ into the rational endomorphism algebra of JX. The factorization of $\mathbb{Q}[G]$ into a product of simple algebras yields a decomposition of JX into abelian subvarieties **[21]** **[18]**, up to isogeny.

This decomposition, and in general that of Jacobians with group action, has been extensively studied from different points of view **[12]** **[21]** **[19]** **[25]** **[26]** **[20]** **[3]** **[7]** **[28]** **[31]** **[18]** **[1]**.

In **[26]** Paulhus studies the decomposition of Jacobians of hyperelliptic curves into elliptic factors. She develops a nice geometrical description for these Jacobians up to genus 10. Her motivation came from **[12]**, where completely decomposable Jacobians of any dimension g were sought. They gave examples up to $g = 1297$, but leaving gaps in between. In **[31]** an explicit formula to calculate the dimension of the factors in the decomposition of JX is given. The polarizations of certain subvarieties in the decomposition of JX are studied in **[19]** and **[20]**. In general, the kernel of the decomposition of JX has not been studied. The only references treating kernels we know are: **[29]** for Jacobians with action of the symmetric group of order 3, **[24]** where the author studies families of curves whose Jacobians are isomorphic to a product of elliptic curves and **[8]** where the authors study dihedral actions on Jacobians, but the tools used to compute kernels are different from the method developed here.

This thesis is structured as follows:

- Chapter 1 is a summary of known results concerning Jacobian varieties with group action. We introduce the group algebra decomposition of a Jacobian variety **[21]** **[7]** and a useful method **[19]** for describing the lattice of each factor of such a decomposition. This method uses a symplectic representation of the action.

- Chapter 2 is the heart of this thesis as it describes and proves our main result. We present a method to concretely build an isogeny which is a group algebra decomposition.

Given a compact Riemann surface X with the action of a group G, the general theory presented in section 1.1 provides us the existence of a group algebra decomposition for the corresponding Jacobian variety JX. The results presented in section 1.4 allow us to compute the dimensions of the factors.

The method that we present in this thesis consists in finding a set of primitive idempotents $f_{i1}, \dots, f_{in_i}$ to describe subvarieties that will be factors of a product variety $B_\times$ in JX, which under certain conditions for G and the action will be isogenus to JX. We divide our method in four steps.

(1) Identification of the factors using Jacobians of intermediate coverings.
(2) Definition of distinguished subvarieties of JX.
(3) Construction of a product variety $B_\times$ of JX.
(4) Conditions for $\nu_\times : B_\times \to JX$ to be an isogeny.

These concrete constructions will allow us to easily compute the order of the kernel of the isogeny $\nu_\times$ as the determinant of the coordinate matrix L of the lattices of the factors in $B_\times$. We may choose an *optimal set* of those idempotents in the sense of getting $\nu_\times$ having a kernel of smallest possible order.

A striking fact that our method reveals is that the chosen idempotents vary among equivalent ones. That is, the corresponding subvarieties defined by them are isomorphic, nevertheless we obtain different isogenies $\nu_\times$ that may have kernels with different order. We conjecture that the explanation of this phenomenon is that isomorphic subvarieties of JX can have different intersection with the other ones inside JX (see Remark 2.2.5).

- Chapter 3 is an application of our method to trigonal curves. We start with a short introduction to these curves and their history. We then study the group algebra decomposition of the Jacobian of a trigonal curve. We obtain the isogeny $\nu_\times$ having kernel with smallest order. We also compute the induced polarization on the factors. In some cases, we compute the Riemann Matrix of the Jacobian.

- Appendix A is a summary of definitions, known results and examples concerning compact Riemann surfaces and Jacobian varieties with group action.

- Appendix B contains parts of the proof of Theorem 2.1.8, which were skipped in Chapter 3.
- Appendix C contains a small help with commands we used in MAGMA to perform some of the computations.

CHAPTER 1

Background

We introduce here some of the background we need to present and develop our results. To help the reader, we include more basic results in Appendix A.

1.1. Group algebra decomposition ν of JX

Let us consider a Jacobian JX with the action of a group G, coming from its (faithful) action on the corresponding Riemann surface X. It is known [4, Section 13.6] that this situation induces a morphism $\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JX)$ in a natural way.

For any element $\alpha \in \mathbb{Q}[G]$ we may define an abelian subvariety

$$B_\alpha := \operatorname{Im} \rho(m\alpha) \subset JX,$$

where m is some positive integer such that $\rho(m\alpha) \in \mathrm{End}_{\mathbb{Z}}(JX)$. This abelian subvariety does not depend on m.

$\mathbb{Q}[G]$ is a semisimple algebra, so it decomposes as $\mathbb{Q}[G] = Q_0 \times \cdots \times Q_r$. The simple algebras Q_i are in bijective correspondence with the rational irreducible representations of G. That is, for any rational irreducible representation $\mathcal{W}_i$ of G there is a uniquely determined central idempotent

$$e_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \mathrm{Trace}_{K_i/\mathbb{Q}}(\chi_{V_i(g^{-1})}) g,$$

where V_i is any complex irreducible representation associated to $\mathcal{W}_i$, and $K_i = \mathbb{Q}(\chi_{V_i}(g) : g \in G)$.

The idempotent e_i defines an abelian subvariety namely $A_i = B_{e_i}$. These varieties, called isotypical components, are uniquely determined by the representation $\mathcal{W}_i$, and the addition map

$$\mu : A_0 \times \cdots \times A_r \to JX \tag{1}$$

is an isogeny. This is called *the isotypical decomposition for* JX [**21**].

Moreover, the decomposition of every $Q_i = L_1 \times \cdots \times L_{n_i}$ into a product of minimal left ideals (all isomorphic) gives a further decomposition of the Jacobian. This is called the group algebra decomposition, because using the group action you cannot decompose it further. Therefore there are idempotents (not uniquely determined) $f_{i1}, \ldots, f_{in_i} \in Q_i$ such that $e_i = f_{i1} + \cdots + f_{in_i}$, where

$$n_i = \frac{\dim V_i}{m_i},$$

and $m_i = m_{V_i}$ is the Schur index for the representation V_i (see [7]).

The idempotents f_{ij} define sub-varieties of A_i, namely

$$B_{ij} := B_{f_{ij}}.$$

It is known that the factor in the isotypical decomposition of JX associated to the trivial representation of G is isogenous to $J_G = J(X/G)$. This factor will be denoted by A_0. Therefore we have the isogenies

$$\nu_i : B_{i1} \times \dots B_{in_i} \to A_i, \text{ for } i = 1..r,$$

and

$$\nu : J_G \times \Pi_1^{n_1} B_{1j} \times \cdots \times \Pi_1^{n_r} B_{rj} \to JX. \tag{2}$$

REMARK 1.1.1. The decomposition (2) is called *the group algebra decomposition of* JX, we use this name to refer to the isogeny ν as well. Using that all the minimal left ideals decomposing Q_i are isomorphic, which implies that the sub-varieties defined by the idempotents f_{ij} are isogenous, we may write the group algebra decomposition as

$$\tilde{\nu} : J_G \times B_{11}^{n_1} \times \cdots \times B_{r1}^{n_r} \to JX, \tag{3}$$

which is the classical way of writing it. The problem with this is that one of our goals is to minimize the order of the kernel of ν, and there are examples (for instance [**14**]) where you may even obtain isomorphisms by changing the components in the same isogeny class.

Therefore, we will stay with the decomposition (2) because it will allow us to reduce the kernel of the isogeny.

1.2. Rational idempotents

We look for geometric information about the components appearing in the decomposition of JX using the action of G. From [7] we get the following results.

DEFINITION 1.2.1. For any subgroup H of G, define

$$p_H = \frac{1}{|H|} \sum_{h \in H} h,$$

the central idempotent in $\mathbb{Q}[H]$ corresponding to the trivial representation of H. Also, we define $f_H^i = p_H e_i$, an idempotent element in $\mathbb{Q}[G]e_i$.

REMARK 1.2.2. It follows that f_H^i satisfies [**7**, Theorem 4.4],

- $h f_H^i = f_H^i = f_H^i h$ for every $h \in H$, and

- $f_H^i = 0$ if and only if $\dim V_i^H = 0$ if only if $\langle \rho_H, \mathcal{W}_i \rangle = 0$.

The following proposition gives us a decomposition of the Jacobian of an intermediate covering of $\pi_G : X \to X_G$ [proposition 5.2, Section 5.1, [7]].

PROPOSITION 1.2.3. *Let X be a Riemann surface with the action of a group G. Consider the associated group algebra decomposition $\tilde{\nu}$ of JX given by equation (3).*

Let H be a subgroup of G and denote by $\pi_H : X \to X_H$ the corresponding quotient map, and by JX_H the Jacobian variety of X_H. The corresponding group algebra decomposition of JX_H is

$$JX_H \sim J_G \times B_{11}^{\frac{\dim V_1^H}{m_1}} \times \cdots \times B_{r1}^{\frac{\dim V_r^H}{m_r}}, \tag{4}$$

where V_j^H is the subspace of V_j fixed by H, and m_j the Schur index.

Furthermore, setting p_H and f_H^i as in Definition 1.2.1, we have

$$\operatorname{Im}(p_H) = \pi_H^*(JX_H), \tag{5}$$

where $\pi_H^*(JX_H)$ is the pull-back of JX_H by π_H.

If $\dim V_i^H \neq 0$ *then*

$$\operatorname{Im}(f_H^i) = B_{i1}^{\frac{\dim V_i^H}{m_i}}.$$

1.3. Method: describing the factors via a symplectic representation of the action

We are interested in describing the factors of the group algebra decomposition of a Jacobian variety with group action given in the previous section.

The method we follow is to describe the lattice of such factors. We apply [**19**, Section 2], but extended to any symmetric idempotent $\alpha = \sum_{g \in G} a_g g$.

Let $\Lambda = H_1(X, \mathbb{Z})$ denote the lattice of JX and $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Let

- $\rho_s : \mathbb{Q}[G] \to \operatorname{End}(\Lambda_{\mathbb{Q}})$ denote the morphism induced by the (symplectic) rational representation of G in Λ,
- $\rho : \mathbb{Q}[G] \to \operatorname{End}_{\mathbb{Q}} JX$ is the homomorphism given by the action of the group G on X,
- ρ_r is the rational representation of $\operatorname{End}_{\mathbb{Q}}(JX)$ which completes figure 1.3 below. Thus ρ_r induces a rational representation of $\operatorname{Hom}_{\mathbb{Q}}(\Pi_{ij} B_{ij}, JX)$ given by

$$\nu \to \nu_\Lambda : \oplus_{ij} \Lambda_{ij} \to \Lambda,$$

where ν_Λ is the restriction of ν to the lattice $\oplus_{ij}\Lambda_{ij}$ of the product of the B_{ij} in ν.

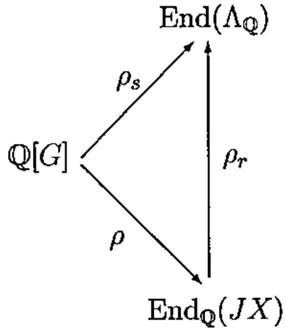

Figure 1.3

Then $\rho_s(\alpha) \in \mathrm{End}\,\Lambda_{\mathbb{Q}}$ is given by

$$\rho_s(\alpha) = \sum_{g \in G} a_g \rho_s(g).$$

Therefore we have the following facts, analogous to [**19**, section 2].

PROPOSITION 1.3.1. *Let $\alpha \in \mathbb{Q}[G]$. The sublattice of Λ defining $B_\alpha = V_\alpha/\Lambda_\alpha$ is given by*

$$\Lambda_\alpha := \rho_s(\alpha)(\Lambda_{\mathbb{Q}}) \cap \Lambda,$$

where the intersection is taken in $\Lambda_{\mathbb{Q}}$ and $\rho_s(\alpha)$ is the image of α by ρ_s. In this case, the $\mathbb{C}-$vector space V_α is generated by $\rho_s(\alpha)(\Lambda_{\mathbb{Q}})$.

The previous construction is clearer in its matrix form, once bases are chosen. From here to the end of this work, we use this form for determining the lattice of the factors in ν.

REMARK 1.3.2. Suppose we have $\Gamma = \{\alpha_1, \ldots, \alpha_{2g}\}$ a (symplectic) basis of the lattice Λ of the Jacobian JX (assumed of dimension g). Consider ρ_s in its matrix form (with respect to to this basis).

For any $g \in G$, we have that $\rho_s(g)$ is a square matrix of size $2g$. Hence, for any $\alpha \in \mathbb{Q}[G]$, we have associated a rational $2g \times 2g$-matrix

$$M = (m_{ij}).$$

The j-th column of M corresponds to the element $\rho_s(g)(\alpha_j) = \sum_{i=1}^{2g} m_{ij}\alpha_i$ and the lattice Λ_α of B_α corresponds to

$$\Lambda_\alpha := (\langle M \rangle_{\mathbb{Z}} \otimes \mathbb{Q}) \cap \Lambda,$$

where $\langle M \rangle_{\mathbb{Z}}$ denotes the lattice over $\mathbb{Z}$ generated by the columns of M.

In other words, the lattice Λ_α is obtained by considering the $\mathbb{R}$-linearly independent columns of M and intersecting it with Λ. Our next step is to look for a basis of Λ_α, which will be in terms of the elements of Γ. By computing its coordinates in the basis Γ, we get a $2g \times 2\dim B_\alpha$ coordinate matrix of the lattice Λ_α.

Moreover, V_α is the complex vector space generated by the column vectors of M.

EXAMPLE 1.3.3 (Decomposition of the Jacobian of a trigonal curve of genus 3 with an action of S_3). Considering what was presented in section 1.1, the Jacobian of a trigonal curve of genus 3 with an action of S_3, with total quotient of genus 0, decomposes as

$$JX \sim A_1 \times A_2 \sim B_1 \times B_2^2,$$

where the second isogeny corresponds to $\tilde{\nu}$.

As shown in Examples A.5.1, A.7.6 and A.8.2 below, we have that e_1 and e_2 in the symplectic basis

$$\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$$

of $H_1(X, \mathbb{Z})$ have matrix

$$[e_1]_{\mathcal{B}} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$[e_2]_{\mathcal{B}} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{pmatrix}.$$

The images of $[e_1]_{\mathcal{B}}$ and $[e_2]_{\mathcal{B}}$, i.e. the space generated by their columns, are $M_{e_1} = \langle \sigma_1, \tau_1 \rangle$ and $M_{e_2} = \langle \epsilon_1, \epsilon_2, \delta_1, \delta_2 \rangle$, where

$$\sigma_1 = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3}, \; \tau_1 = \frac{\beta_1 + \beta_2 + \beta_3}{3},$$

$$\epsilon_1 = \frac{2\alpha_1}{3} - \frac{\alpha_2}{3} - \frac{\alpha_3}{3}, \; \epsilon_2 = \frac{-\alpha_1}{3} + \frac{2\alpha_2}{3} - \frac{\alpha_3}{3},$$
$$\delta_1 = \frac{2\beta_1}{3} - \frac{\beta_2}{3} - \frac{\beta_3}{3}, \; \delta_2 = \frac{-\beta_1}{3} + \frac{2\beta_2}{3} - \frac{\beta_3}{3}.$$

To obtain the lattices L_1, L_2 of A_1, A_2, we must take the intersection of M with the lattice generated by $\mathcal{B}$ in each case. Therefore a basis for the lattice of A_1 is

$$\mathcal{B}_1 = \{\alpha_1 + \alpha_2 + \alpha_3, \beta_1 + \beta_2 + \beta_3\},$$

and a basis for the lattice of A_2 is

$$\mathcal{B}_2 = \{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \beta_1 - \beta_2, \beta_2 - \beta_3\},$$

because

$$\epsilon_1 - \epsilon_2 = \alpha_1 - \alpha_2; \quad \epsilon_1 + 2\epsilon_2 = \alpha_2 - \alpha_3,$$
$$\delta_1 - \delta_2 = \beta_1 - \beta_2; \quad \delta_1 + 2\delta_2 = \beta_2 - \beta_3.$$

The intersection product matrix for this basis of the lattice of A_1 is

$$E_1 = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}.$$

Hence, the type of polarization of A_1 is (3).

The intersection product matrix for A_2 is given by

$$E_2 = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix}.$$

Using the Frobenius algorithm, after a change of basis, we find that E_2 is similar to

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

Therefore, the polarization of A_2 is of type $(1, 3)$.

On the other hand, we know that $A_2 \sim B_2^2$. To describe B_2 we can use proposition 1.2.3 with $H =< b >$. Then

$$p_H = \frac{1}{2}(1 + b),$$

so

$$[p_H]_{\mathcal{B}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence a basis for B_2 is given by $\{\alpha_1 - \alpha_2, \beta_1 - \beta_2\}$. With respect to this basis, the polarization will be of type (2).

1.4. The dimension of the factors of ν

An important fact about the group algebra decomposition of a Jacobian variety is that the varieties B_j appearing in the Theorem 1.1, and particularly in the decomposition ν, may be of dimension zero for some particular actions of a group G. In fact, we know that the first one is given by $B_0 = J(X/G)$. If G acts on X so that $g(X/G) = 0$, then the dimension of B_0 is zero.

In [**31**] a formula was given for computing these dimensions in terms of the action of the group on the corresponding Riemann surface. We include it here because we use it in several examples.

THEOREM 1.4.1 (Dimension of the B_j's). *Let G be a finite group acting on a compact Riemann surface X with geometric signature given by $(\gamma; [m_1, C_1], ..., [m_r, C_r])$. Then the dimension of any subvariety B_i associated to a non trivial rational irreducible representation $\mathcal{W}_i$ in the $G-$equivariant isotypical decomposition of the corresponding Jacobian variety JX, is given by*

$$\dim B_i = k_i\Big(\dim V_i(\gamma - 1) + \frac{1}{2}\sum_{k=1}^{r}(\dim V_i - \dim \mathrm{Fix}_{G_k} V_i)\Big),$$

where G_k is a representative of the conjugacy class C_k, $\dim V_i$ is the dimension of a complex irreducible representation V_i associated to $\mathcal{W}_i$, $K_i = K_{V_i}$, m_i is the Schur index of V_i and $k_i = m_i[K_i : \mathbb{Q}]$.

We continue with example 1.3.3.

EXAMPLE 1.4.2. We compute the dimension of the factors in the group algebra decomposition of a Jacobian variety of a trigonal curve of genus 3 with an action of the group S_3 and signature $(0; 3, 2, 2, 2, 2)$.

We know that $JX \sim A_1 \times A_2 \sim B_1 \times B_2^2$. All rational irreducible representations of S_3 are completely irreducible [**34**], hence $k_i = m_i = 1$ for $i \in \{1, 2\}$. Therefore, the formula for the dimensions of the subvarieties B_i in this case is

$$\dim B_i = -\dim V_i + \frac{1}{2}\sum_{k=1}^{5}(\dim V_i - \dim \mathrm{Fix}_{G_k} V_i).$$

Hence, we need to calculate $\dim \mathrm{Fix}_{G_k} V_i$ for $i \in \{1, 2\}$ and for $k \in \{1, ..., 5\}$. Recall that the dimension of $\mathrm{Fix}_{G_k} V_i$ is equal to $< \mathrm{Ind}_{1_{G_k}}^{G}, V_i >$.

We had chosen

$$(0; [3, < a >], [2, < b >], [2, < ab >], [2, < ab >], [2, < ab >])$$

as a generating vector for the action of S_3 on X. Then $G_1 =< a >$, $G_2 =< b >$, $G_3 =< ab >$, $G_4 =< ab >$ and $G_5 =< ab >$.

It is not difficult to see, using the formula for the scalar product (see Definition A.7.11 below), that

(1) if $i = 1$ then $\dim Fix_{G_1} V_1 = 1$ and $\dim Fix_{G_k} V_1 = 0$ for all $k \in \{2, ..., 5\}$.
(2) if $i = 2$ then $\dim Fix_{G_1} V_2 = 2$ and $\dim Fix_{G_k} V_2 = 1$ for all $k \in \{2, ..., 5\}$.

In the following table we summarize this. The second and the third column indicate the dimension of $\mathrm{Fix}_{G_k} V_i$ for $i = 1$ and $i = 2$, respectively.

	V_1	V_2
G_1	1	2
G_2	0	1
G_3	0	1
G_4	0	1
G_5	0	1

Considering Example A.7.6 and the previous table we have:

$$\dim B_1 = -1 + \frac{1}{2}\sum_{k=1}^{5}(1 - \dim \mathrm{Fix}_{G_k} V_1) = 1,$$

and

$$\dim B_2 = -2 + \frac{1}{2}\sum_{k=1}^{5}(2 - \dim \mathrm{Fix}_{G_k} V_2) = 1.$$

Thus any trigonal curve of genus 3 with an action of the group S_3 and signature $(0; 3, 2, 2, 2, 2)$ is decomposable (via isogeny) into a product of elliptic curves.

The former kind of decomposition, i.e., when all the factors in this decomposition are elliptic curves, is a problem has been extensively studied ([12], [24], [25]). In this case, we say that JX is *completely decomposable.* Several examples of this situation are given in the Chapter 3.

CHAPTER 2

Construction of a group algebra decomposition.

In this chapter we develop the core of our thesis. We present a method to concretely build an isogeny ν as in (2).

Given a compact Riemann surface X with the action of a group G, the general theory presented in section 1.1 gives us the existence of a group algebra decomposition for the corresponding Jacobian variety JX. The results presented in section 1.4 allow us to compute the dimensions of the factors. Nevertheless to describe further geometrical properties such as induced polarization, period matrix, etc., we need an explicit description of them. The method presented in section 1.3 gives us a tool to solve some of these questions, under certain hypotheses for G.

We present here a method to find a set of primitive idempotents $f_{i1}, \ldots, f_{in_i}$ to describe the factors in this decomposition, in order to extract properties of the decomposition. This concrete construction will allow us to easily compute the order of the kernel of the isogeny ν. Hence we may choose an *optimal set* of those idempotents in the sense of getting the smallest possible kernel. We show this last application in the next chapter.

2.1. Method. A group algebra decomposition $\nu_\times$

We consider total quotient for the action of G of genus 0, for simplicity because it is known that the factor in ν corresponding to the trivial representation is the image of p_G, hence it is isogenous to $J(X/G)$.

In the following, we consider $\mathcal{W}_i$, e_i, n_i and V_i as defined in section 1.1. As before, V^H will denote the subspace of V fixed under H.

Data: Let X be a Riemann surface of genus $g \geq 2$ with the action of a group G with total quotient of genus 0. Assume that the symplectic representation ρ_s for this action is known.

1. STEP ONE: Identification of factors using Jacobians of intermediate coverings.

The following lemma gives us conditions under which a factor in the group algebra decomposition can be described as image of a concrete idempotent, in particular when it corresponds to a Jacobian of an intermediate quotient.

LEMMA 2.1.1. *Let X be a Riemann surface with an action of a finite group G such that the genus of X_G is equal to zero. Consider ν the group algebra decomposition of JX as in* (2).

(i) *If $H \leq G$ is such that $\dim_{\mathbb{C}} V_i^H = m_i$, where m_i is the Schur index of the representation V_i, then for some $j \in \{1, ..., n_i\}$ we have that*

$$\operatorname{Im}(f_H^i) = B_{ij},$$

where f_H^i is as in Definition 1.2.1.

In addition,

(ii) *if $\dim_{\mathbb{C}} V_l^H = 0$ for all l, $l \neq i$, such that $\dim_{\mathbb{C}} A_l \neq 0$ in the isotypical decomposition of JX in Equation (1), then*

$$J(X/H) \sim \operatorname{Im}(p_H) = B_{ij}.$$

PROOF. From Proposition 1.2.3 and the fact that $\dim_{\mathbb{C}} V_i^H \neq 0$, we get $\operatorname{Im}(f_H^i) = \Pi_i^{k_i} B_{ij}$. Due to $\dim_{\mathbb{C}} V_i^H = m_i$ (by hypothesis), we obtain that $k_i = 1$. Hence $\operatorname{Im}(f_H^i) := B_{ij}$ for some $j \in \{1, ..., n_i\}$.

If, in addition, $\dim_{\mathbb{C}} V_l^H = 0$ (equivalently $\dim_{\mathbb{Q}} \mathcal{W}_l^H = 0$) for all $l \neq i$ such that $\dim_{\mathbb{C}} A_l \neq 0$, then $f_H^l = 0$ for all of them. Due to $p_H = \sum_{l\in\{1,...,r\}} f_H^l$ we obtain that $p_H = f_H^i$. Moreover, by equation (4) we have $J(X/H) \sim \operatorname{Im}(p_H) = B_{ij}$. □

REMARK 2.1.2. Observe that if H satisfies Lemma 2.1.1 for some $i \in \{1, ..., r\}$, then all its conjugates satisfy it for the same i. To see this, consider $H' = H^g = gHg^{-1}$. It is clear that for all $s \in G$, we have

$$\begin{aligned}
\dim_{\mathbb{C}} V_i^H &= < \operatorname{Ind}_{1_H^G}, V_i > \\
&= \frac{1}{|G|}\sum_{t\in G} \chi_{\operatorname{Ind}_{1_H^G}}(t)\chi_{V_i}(t^{-1}) \\
&= \frac{1}{|G|}\sum_{t\in G}\Big(\sum_{s^{-1}ts\in H} \chi_{1_H}(s^{-1}ts)\Big)\chi_{V_i}(t^{-1}) \\
&= \frac{1}{|G|}\sum_{t\in G} |H|\chi_{V_i}(t^{-1}) \\
&= \frac{1}{|G|}\sum_{t\in G} |H'|\chi_{V_i}(t^{-1}) \\
&= \frac{1}{|G|}\sum_{t\in G}\Big(\sum_{s^{-1}ts\in H'} \chi_{1_{H'}}(s^{-1}ts)\Big)\chi_{V_i}(t^{-1}) \\
&= \dim_{\mathbb{C}} V_i^{H'}.
\end{aligned}$$

Our purpose is to use this result *conversely* to actually produce an isogeny ν.

STEP TWO: Definition of certain subvarieties of JX.

DEFINITION 2.1.3. Let H be a subgroup of G satisfying condition (i) of Lemma 2.1.1 for some $i \in \{1, ..., r\}$, define B_H as the image of $f_H = p_H e_i$ (see Definition 1.2.1).

Note that depending on the geometry of the action, B_H can be trivial.
From Proposition 1.3.1, we get that its lattice corresponds to

$$\Lambda_H := (\langle \rho_s(f_H) \rangle_{\mathbb{Z}} \otimes \mathbb{Q}) \cap H_1(X, \mathbb{Z}). \tag{6}$$

Using the procedure described in Remark 1.3.2, we obtain the coordinate matrix corresponding to a basis of the lattice of B_H. We use sometimes the same symbol Λ_H to denote this matrix. An example of this coordinate matrix is given in 2.1.11.

COROLLARY 2.1.4. *Let H be a subgroup of G satisfying both conditions of Lemma 2.1.1. Then $f_H = p_H e_i = p_H$, and B_H is isogenous to the Jacobian of X/H.*

PROOF. By Lemma 2.1.1, if H satisfies both conditions then the Jacobian of X/H is isogenous to one of the factors in a group algebra decomposition for JX. This is equivalent to the equality of their corresponding idempotents. □

3. STEP THREE: Construction of a product subvariety $B_\times$ of JX.

If we have *enough* subgroups from STEP TWO, we may construct the product of all the subvarieties defined by those subgroups. This will be a subvariety of JX, its lattice is described in the following definition.

DEFINITION 2.1.5. Let $r+1$ be the number of rational irreducible representations of G. Suppose for all $i \in \{1, \ldots, r\}$ and all $j = \{1, \ldots, n_i\}$ there is a subgroup H_{ij} satisfying condition (i) of Lemma 2.1.1. For each i, j take one H_{ij}, and let

$$S = \{H_{ij} : i \in \{1, \ldots, r\}, j \in \{1, \ldots, n_i\}\}$$

be the set of these subgroups, where we do not consider subgroups H_{ij} such that $\mathrm{Im}(f_{H_{ij}}) = 0$. We define $L_S \in M_{2g}(\mathbb{Z})$ to be the coordinate matrix given by the vertical join of the coordinate matrices of the lattices $\Lambda_{H_{ij}}$ (see equation 6) for $H_{ij} \in S$.

We recall here that $i = 0$ corresponds to the trivial representation whose factor is not considered here, $n_i = \dim V_i/m_i$, where m_i is the Schur index of a complex irreducible representation V_i associated to the rational irreducible representation corresponding to the factor $B_{ij}^{n_i}$ (from (2)).

The lattice $\Lambda_\times$ defined by the matrix L_S, corresponds to the sublattice $\oplus_{ij} \Lambda_{H_{ij}}$ of $\Lambda = H_1(X, \mathbb{Z})$. It is the lattice of the following subvariety of JX

$$B_\times := \Pi_{i,j} B_{H_{ij}},$$

where $B_{H_{ij}}$ is as in Definition 2.1.3.

DEFINITION 2.1.6. With the above notation define a sum map $\nu_\times : B_\times \to JX$

$$\nu_\times(b_{11}, \ldots, b_{rn_r}) = \sum_{i,j} b_{ij} \in JX.$$

STEP FOUR: Condition for $\nu_\times$ to be an isogeny.

DEFINITION 2.1.7. Let $S = \{H_{ij} : i = 1..r, j = 1..n_i\}$ be a set of subgroups of G as in Definition 2.1.5. We say that S is an *effective set for* G if the determinant of the corresponding matrix L_S is different from 0.

THEOREM 2.1.8. *Let X be a Riemann surface of genus $g \geq 2$ with an action of a group G with total quotient of genus 0. Let $S = \{H_{ij}\}_{ij}$ be an effective set for G, then the map $\nu_\times$ defined in 2.1.6 is an isogeny with kernel of order* $|\det(L_S)|$.

PROOF. As before, denote by Λ the lattice of JX. The map $\nu_\times$ induces a homomorphism of $\mathbb{Z}$–modules $\nu_\Lambda : \Lambda_\times \to \Lambda$. If the rank of $\Lambda_\times$ is $2g$, then ν_Λ is a monomorphism of lattices in $\mathbb{C}^g$. Moreover, all the sublattices $\Lambda_{H_{ij}}$ decomposing $\Lambda_\times$ correspond to subvarieties $B_{H_{ij}}$. Therefore the dimension of $B_\times$ is g.

It remains to show either $\nu_\times$ is surjective or its kernel is of finite order. For any isogeny $f : A_1 \to A_2$ between two abelian varieties, it is known [[4], section 1.2] that

$$|\mathrm{Ker}(f)| = \det \rho_r(f), \tag{7}$$

where $\rho_r(f)$ is the rational representation of the isogeny f. It is known that if the kernel is finite, then its cardinality equals the index $[\Lambda : \Lambda_\times]$. As the matrix L_S is non singular, we have that $\Lambda_\times$ is a lattice in $\mathbb{C}^g$, hence this index is finite. After columns operations, the matrix L_S is the matrix of the rational representation $\nu_\times$, therefore its kernel has order the absolute value of its determinant.

□

REMARK 2.1.9. (1) The isogeny from Theorem 2.1.8 corresponds to a group algebra decomposition isogeny ν as in (2).

(2) If the total quotient is of genus greater than 0, then we just need to include the lattice of the image of p_G, which corresponds to the fixed (by G) sublattice of $H_1(X, \mathbb{Z})$.

(3) The condition for a group G of having an effective set is not too restrictive, we will exhibit in the next chapter several groups satisfying it.

(4) We point out that the isogeny $\nu_\times$ depends on the choice of the subgroups H_{ij} for the set S. Therefore, its kernel may change if we change these subgroups. Our purpose is to move along different effective sets in order to achieve the smallest possible kernel (in the sense of its order).

In the spirit of moving along different effective sets to minimize the order of the kernel, we have the following proposition.

PROPOSITION 2.1.10. *If* $H_2 = gH_1g^{-1}$ *for some element* $g \in G$, *then*

(i) $p_{H_2} = gp_{H_1}g^{-1}$,
(ii) $\mathrm{Im}(p_{H_2})$ *is isomorphic to* $\mathrm{Im}(p_{H_1})$,
(iii) *the exponents of* $\mathrm{Im}(p_{H_2})$ *and* $\mathrm{Im}(p_{H_1})$ *are equal, and both equal* $|H_1| = |H_2|$.

PROOF. Claims (i) and (ii) are clear. Claim (iii) is obtained from the definition of the exponent of a abelian variety [4, Section 5.3]. In fact, it is obtained from the fact that the least non negative integer m such that mp_H is an element of $\mathbb{Z}[G]$ is equal to $|H|$. Let f be $|H|p_H$, then $f^2 = |H|f$. It is known that f is primitive and symmetric (see [**20**, Prop. 2.2]), hence by Criterion [4, 5.3.4] f is the norm endomorphism of $Y = \mathrm{Im}\, p_H$. Finally by [4, Cor. 5.3.3], $|H|$ is the exponent of Y. □

We include here a known and simple example where we apply our method to find a concrete group algebra decomposition of a Jacobian with the action of $G = GL(2,3)$. We calculate an effective set S for G such that the kernel of $\nu_\times$ has the smallest possible order.

EXAMPLE 2.1.11. Let X_2 be a curve of genus two admitting an action of $GL(2,3) =< a, b : a^8 = b^3 = (ab)^2 = ba^{-3}ba^{-3} = 1 >$ (see Table 2 in Chapter 3). This curve is known as Bolza's curve with equation

$$y^2 = x(x^4 - 1).$$

A generating vector for this action is $(a, b, (ab))$ of type $(0; 8, 3, 2)$. Using equation (2) we obtain that the Jacobian variety JX associated to X is completely decomposable i.e. isogenous to a product of elliptic curves. In fact, the group algebra decomposition isogeny is $\nu : B_{41} \times B_{42} \sim JX$, where the product $B_{41} \times B_{42}$ is invariant by the irreducible rational representation V_4 of degree 2 (see Table 1).

To apply our method to find an explicit decomposition $\nu_\times$, we first look for two subgroups H_1 and H_2 of $GL(2,3)$ satisfying conditions of Lemma 2.1.1. To find these subgroups, we use the symplectic representation of the action obtained from the method given in [**3**]. This allows us to write every element of $GL(2,3)$ as a square symplectic matrix of size 4.

We determine next the complex irreducible representation decomposition of the induced representation $\mathrm{Ind}_H^G 1_H$ in $GL(2,3)$ by the trivial representation in H for each $H \leq G$ (See Table 2). We know that this decomposition of $\mathrm{Ind}_H^G 1_H$ into $\mathbb{C}-$irreducible

Class of elements	1	2	3	4	5	6	7	8
Size of the classes	1	1	12	8	6	8	6	6
Order of the elements	1	2	2	3	4	6	8	8
V_1	1	1	1	1	1	1	1	1
V_2	1	1	-1	1	1	1	-1	-1
V_3	2	2	0	-1	2	-1	0	0
V_4	2	-2	0	-1	0	1	$i\sqrt{2}$	$-i\sqrt{2}$
V_5	2	-2	0	-1	0	1	$-i\sqrt{2}$	$i\sqrt{2}$
V_6	3	3	1	0	-1	0	-1	-1
V_7	3	3	-1	0	-1	0	1	1
V_8	4	-4	0	1	0	-1	0	0

TABLE 1. Character Table of $GL(2,3)$.

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
Identity element	1	1	2	2	2	3	3	4
order 2, length 1	1	1	2	0	0	3	3	0
order 2, length 12	1	0	1	1	1	2	1	2
order 3, length 4	1	1	0	0	0	1	1	2
order 4, length 3	1	1	2	0	0	1	1	0
order 4, length 6	1	0	1	0	0	2	1	0
order 6, length 4	1	1	0	0	0	1	1	0
order 6, length 4	1	0	0	0	0	1	0	1
order 6, length 4	1	0	0	0	0	1	0	1
order 8, length 1	1	1	2	0	0	0	0	0
order 8, length 3	1	0	1	0	0	0	1	0
order 8, length 3	1	0	1	0	0	1	0	0
order 12, length 4	1	0	0	0	0	1	0	0
order 16, length 3	1	0	1	0	0	0	0	0
order 24, length 1	1	1	0	0	0	0	0	0
order 48, length 1	1	0	0	0	0	0	0	0

TABLE 2. Decomposition of the induced representation by the trivial representation on each class of conjugation of subgroups of $GL(2,3)$.

representations is invariant under conjugation (see Remark 2.1.2). Table 2 shows the multiplicity of each complex irreducible representation in the induced representation $\mathrm{Ind}_H^G 1_H$, for all $H \leq G$ up to conjugacy.

We observe that the only conjugacy class of subgroups of $GL(2,3)$ whose elements satisfy the conditions in Lemma 2.1.1 consists of the class of $H =< ab >$. For each subgroup H in this class

$$< \mathrm{Ind}_H^G 1_H, V_4 >= 1.$$

By Lemma 2.1.1, each factor in the decomposition of JX is defined by $B_{4j} = \mathrm{Im}(p_{H_j}) = V_{H_j}/\Lambda_{H_j}$, where H_j is some subgroup in the class and $j \in \{1, 2\}$. We obtain $\nu_\times : B_{41} \times B_{42} \to JX$, and its kernel depends on the choice of H_j in this class.

To give an explicit description of the subgroups of $GL(2,3)$ giving the smallest order for the kernel of $\nu_\times$, we consider the above presentation of $GL(2,3)$ and the same generating vector (a, b, ab). Set $H =< ab >$, consider the following subgroups of order 2 in $GL(2,3)$

$$\begin{aligned} H_1 &= H^b = < ba^7b >, \\ H_2 &= H = < ab >, \end{aligned}$$

and write $B_{41} = \mathrm{Im}(p_{H_1})$ and $B_{42} = \mathrm{Im}(p_{H_2})$. Since

$$\rho_s(p_{H_1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \rho_s(p_{H_2}) = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

and the coordinate matrices of their lattices are

$$\Lambda_{H_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \ \Lambda_{H_2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

we find that the matrix coordinate of the lattice of the product $B_{41} \times B_{42}$ is given by

$$L_{\{H_1,H_2\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

In this case, choosing H_1 and H_2 as before $\nu_\times : B_{41} \times B_{42} \to JX$ is an isomorphism. Hence $|\mathrm{Ker}(\nu)| = 1$.

Note that we do not claim that the subgroups yielding an isomorphism are unique. In fact, there exist other subgroups in the same class such that the $|\mathrm{Ker}(\nu_\times)| = 1$.

This example shows a particular but important case of Theorem 2.1.8, as in the following corollary.

COROLLARY 2.1.12. *Under the hypothesis in Theorem 2.1.8, the isogeny $\nu_\times$ is an isomorphism if and only if* $\det(L_{\{H_{ij}\}}) = \pm 1$.

2.2. Induced polarization on the factors of $\nu_\times$.

The coordinate matrix $\Lambda_{H_{ij}}$ of $B_{H_{ij}}$ is of size $2\dim_{\mathbb{C}} B_{H_{ij}} \times 2g$. This defines the canonical embedding $i : B_{H_{ij}} \hookrightarrow JX$. We denote by E_{JX} the canonical polarization of JX of size $2g \times 2g$ and by $\Lambda^t_{H_{ij}}$, the transpose matrix of $\Lambda_{H_{ij}}$.

$$E_{JX} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

The next theorem provides the polarizations induced by E_{JX} in the factors of the isogeny $\nu_\times$ of Theorem (2.1.8).

THEOREM 2.2.1. *Let X be a Riemann surface of genus $g \geq 2$ with an action of a group G such that the genus of X/G is zero. If $S = \{H_{i1}, ..., H_{in_i}\}$ is an effective set for ν, then the induced polarization E_{ij} of the factor $B_{H_{ij}}$ in $\nu_\times$ is given by*

$$E_{ij} = \Lambda_{H_{ij}} E_{JX} \Lambda^t_{H_{ij}}. \tag{8}$$

PROOF. The proof follows the same steps as [**19**, Prop. 2.1]. By hypothesis (and Theorem 2.1.8), we know $B_{H_{ij}} = \mathrm{Im}(f_{H_{ij}})$ are enough to decompose the Jacobian $JX = \mathbb{C}^g/\Lambda$ of X. From equation 6 we have the coordinates of a basis for the lattice $\Lambda_{H_{ij}}$ with respect to the symplectic basis Γ of Λ. Then we have a basis of the lattice of each factor $B_{H_{ij}}$ given by $\{\gamma_{ij,1}, ..., \gamma_{ij,t_{ij}}, \delta_{ij,1}, ..., \delta_{ij,t_{ij}}\}$, where $t_{ij} = \dim_{\mathbb{C}} B_{H_{ij}}$.

The matrix $E_{ij} =< \gamma_{ij}, \delta_{ij} >$ given by the intersection product of those curves defines the polarization in $B_{H_{ij}}$ induced by the canonical polarization of JX. This is given by $i^*(E_{JX})$, which corresponds to the product $\Lambda_{H_{ij}} E_{JX} \Lambda^t_{H_{ij}}$. □

REMARK 2.2.2. Note that the type of the polarization E_{ij} is given by the elementary divisors of the matrix (8).

Continuing with Example 2.1.11, we calculate the polarization for every one of the factors here.

EXAMPLE 2.2.3. Recall the situation presented on example 2.1.11. Respects to the chosen basis for the lattice of the product $B_{41} \times B_{42}$, we obtain the induced polarization of each one of these factors, which are given by

$$E_{41} = E_{42} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

REMARK 2.2.4. Note that as the genus of a curve increases the possibilities that, using this method, its Jacobian can be shown as a completely decomposable variety decreases. In fact, if X be a curve of genus g which admits a G-action, and

$\tilde{\nu} : B_1^{n_1} \times ... \times B_r^{n_r} \sim JX$, is a group algebra decomposition with $g > \sum_j n_j$. Then the group algebra decomposition for JX will have factors of dimension greater than 1.

REMARK 2.2.5. Finally we point out that the principle of our method is to move along isomorphic varieties B_H, choosing different subgroups H (even in the same conjugacy class) to construct the effective set S and the corresponding isogenies $\nu_\times$, which may have kernels of different orders. This shows that the geometry of the varieties B_H as subvarieties of JX makes a difference. In fact the examples suggest that this reflects the way the subvarieties intersect each other.

CHAPTER 3

Application to $p-$gonal curves

3.1. Introduction

A compact Riemann surface X, admitting a cyclic group of automorphisms C_p of order a prime number p and such that X/C_p has genus 0, is called a cyclic $p-$gonal curve. If $p = 3$ we say X that is a trigonal curve, or a $3-$gonal curve.

The study of trigonal curves is a classical subject [28] [1], mainly concerned with properties of their Jacobians. In [36] there is a classification of all non-normal $p-$gonal curves. In [35] we find a method to compute a defining equation as a plane curve, in the normal cyclic $p-$gonal case. Bartolini, Costa and Izquierdo [2] get the full group of automorphisms of a p-gonal curve for all prime numbers p. Moreover, [10] treats the case of non-normal p-gonal curves. In particular, they describe the Hurwitz space of pairs given by a p-gonal curve and a $p-$gonal group.

In [12], among other questions, Ekedhal and Serre asked for the existence of completely decomposable Jacobians of any dimension g. That is they studied the existence of Jacobians isogenous to a product of elliptic curves. In [26] Paulhus studies the decomposition of Jacobians of hyperelliptic curves as a product of elliptic factors. She develops a nice geometrical description for these Jacobians up to genus 10. Hence, the natural next step is to study $p-$gonal curves. We focus our attention on trigonal curves, to which we apply the method developed in the previous chapter.

3.2. Known facts about $p-$gonal curves and their automorphisms

If X is a $p-$gonal curve with an action of $C_p \leq \mathrm{Aut}(X)$, the following facts are known [36][35][2][10].

- The subgroup $C_p \subseteq \mathrm{Aut}(X)$, called a $p-$gonal group for X, is not necessarily unique.
- If C_p and C_p' are two $p-$gonal groups for X, then they are conjugate in $\mathrm{Aut}(X)$.
- The group $K = N_{\mathrm{Aut}(X)}(C_p)/C_p$ is isomorphic to a finite group of automorphisms of the Riemann sphere $\Sigma = X/C_p$. It is called the reduced group or sphere group of X. Here $N_{\mathrm{Aut}(X)}(C_p)$ denotes the normalizer of C_p in $\mathrm{Aut}(X)$.

Group	Branching data
C_n	(n, n)
D_n	$(2, 2, n)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

TABLE 1. Automorphisms of the Riemann sphere and branching data.

The last fact implies that $N_{\text{Aut}(X)}(C_p)$ fits in the short exact sequence

$$1 \to C_p \to N_{\text{Aut}(X)}(C_p) \to K \to 1,$$

for all p–gonal group C_p for X. Therefore, K is one of the groups in Table 1. The branching data is a vector whose length is the number of branch points of the map $\pi_K : \Sigma \to \Sigma/K$ and whose entries are the orders of the branch points (see Appendix A for these definitions).

If X admits only one p–gonal group, it is called a normal p–gonal curve. A consequence of [**1**] is the following.

THEOREM 3.2.1. *(Wootton, [**36**, Thm 5.1]) If X is a cyclic p–gonal curve and $g > (p-1)^2$, then X is normal p–gonal.*

We consider a trigonal curve X, i.e., X has a subgroup of automorphisms isomorphic to C_3, which acts with total quotient X/C_3 of genus 0. An immediate consequence of Theorem 3.2.1 is the following.

COROLLARY 3.2.2. *If X is a trigonal surface of genus $g \geq 5$, then X is normal 3–gonal.*

Corollary 3.2.2 allows us to find a list of automorphism groups of trigonal curves (see also [**2**]). This list is obtained by an easy combination of [[**36**], Table 7], [**5**, Table 1] and [**22**], plus the computation of the reduced groups which are not in the original tables. We group the results in Tables 2 and 3. Some remarks about them are:

- Table 2, corresponds to non-normal trigonal curves. CD denotes the central diagonal subgroup of $SL(2,3)$ of order 2.
- Table 3, corresponds to normal trigonal curves with reduced group A_4, S_4 or A_5 [**5**]. We restrict to these reduced groups in the normal case mainly because the results in [**26**] suggest that these families may have completely decomposable Jacobians (at least for some dimensions of JX). In fact, [**8**] shows that curves admitting a dihedral action tend to have no elliptic factors in the group algebra decomposition of their Jacobian varieties.
- From Table 3 we obtain that any trigonal curve with reduced group S_4, A_4 or A_5 has even genus, and that there do not exist trigonal curves of genus 8 with reduced group A_4, S_4 or A_5.

Red. group	Automorphism group	Genus	Signature
D_2	$GL(2,3)$	2	(0;2,3,8)
C_4	$SL(2,3)$/CD	3	(0;2,3,12)
D_3	$D_3 \times D_3$	4	(0;2,2,2,3)
D_6	$(C_3 \times C_3) \rtimes D_4$	4	(0;2,4,6)

TABLE 2. Automorphism group for non-normal trigonal curves.

K	Automorphism group	Genus	Signature
A_4	$C_3 \times A_4$	12s-2, $s > 0$	(0;2,3,3,3^s)
A_4	$C_3 \times A_4$	12s+4	(0;6,3,3,3^s)
A_4	$(C_2 \times C_2) \rtimes C_9$	12s+6	(0;2,9,9,3^s)
A_4	$(C_2 \times C_2) \rtimes C_9$	12s+12	(0;6,9,9,3^s)
S_4	$C_3 \times S_4$	24s-2, $s > 0$	(0;2,3,4,3^s)
S_4	$C_3 \times S_4$	24s+4	(0;2,3,12,3^s)
S_4	$C_3 \times S_4$	24s+16	(0;6,3,12,3^s)
S_4	$C_3 \times S_4$	24s+10	(0;6,3,4,3^s)
S_4	$C_3 \rtimes S_4$	24s-2, $s > 0$	(0;6,3,12,3^s)
S_4	$((C_2 \times C_2) \rtimes C_9) \rtimes C_2$	24s+6	(0;2,9,4,3^s)
A_5	$C_3 \times A_5$	60s-2, $s > 0$	(0;2,3,5,3^s)
A_5	$C_3 \times A_5$	60s+10	(0;2,3,15,3^s)
A_5	$C_3 \times A_5$	60s+40	(0;6,3,15,3^s)
A_5	$C_3 \times A_5$	60s+28	(0;6,3,5,3^s)

TABLE 3. Automorphism group for normal trigonal curves with reduced group A_4, S_4 and A_5.

REMARK 3.2.3. • **Dimension** 3 (see Section B.2). The genus-3 surface in Table 2 corresponds to the curve with plane model $y^4 = x^3 - 1$ [**22**, Table 2] with an action of $SL(2,3)$/CD, where CD denotes the central diagonal subgroup of order 2. We use the presentation

$$SL(2,3)/\mathrm{CD} =< a, b : a^{12} = b^3 = (ab)^2 = a^{11}b^{-1}aba^{-1}ba^{-7} = 1 > .$$

The generating vector for the action is (a, b, ab), of type $(0; 12, 3, 2)$.

• **Dimension** 4 (see Section B.3). The genus-4 surfaces in Table 2 correspond to the cases studied in [**1**] and [**9**]. The surfaces admit four actions of C_3. Two of them are conjugate, with quotients of genus 0, and two are non-conjugate with quotients of genus 2. Moreover, the one dimensional locus in $\mathcal{A}_4$ corresponding to surfaces of genus 4 with automorphism group $D_3 \times D_3$ consists of curves with equation $ax^3y^3-(x^3+y^3)+a=0$, where $a \notin \{0,\pm1,\infty\}$. This family contains the surface with action of $(C_3\times C_3)\rtimes D_4$ [**22**, Table 4]. The generating vector for the action is (a,b,c,abc) of type $(0;3,2,2,2)$, where we consider $D_3\times D_3$ as

$$D_3\times D_3 = \langle a,b,c|a^3=b^2=c^2=(abc)^2=$$
$$=a^2ca^{-1}bca^{-1}b^{-1}a^{-2}=a^2ca^{-1}bab^{-1}ac^{-1}a^{-1}=1\rangle.$$

- **Dimension** 4: $C_3\times A_4$ and $C_3\times S_4$ act in genus 4. The group $C_3\times A_4$ is contained in $C_3\times S_4$, we study the possible actions of both in this genus. Using [**6**] and [**3**], we note that they act on the surface with planar model $y^3=x(x^4-1)$ (see [**27**] for a reference). This result completes [**1**], where only this last case was left without description, and coincides with [**22**] and [**9**]. The action of $C_3\times A_4$ (see Table 3) extends to the action of the group

$$C_3\times S_4 = \langle a,b|a^{12}=b^3=(ab)^2=a^{11}ba^{-1}bab^{-1}ab^{-1}a^{-6}=1\rangle,$$

  with generating vector (a,b,ab) of type $(0;12,3,2)$ [**9**].
- **Dimension** 6 (see section B.4.1). The groups $((C_2\times C_2)\rtimes C_9)\rtimes C_2$ and $(C_2\times C_2)\rtimes C_9$ act in genus 6. The group $(C_2\times C_2)\rtimes C_9$ is contained in $((C_2\times C_2)\rtimes C_9)\rtimes C_2$ and we study their possible actions in this genus using [**6**] and [**3**]. Both groups act on the trigonal curve [**22**] with planar model $y^3=x^8+14x^4+1$ [**32**]. The group

$$((C_2\times C_2)\rtimes C_9)\rtimes C_2 =< a,b|a^2=b^9=(ab)^4=ab^3ab^3=1>$$

  acts on the curve with generating vector $(a,b,(ab)^{-1})$ of type $(0;2,9,4)$.

We describe in detail the Jacobians of these curves in Appendix B.

3.3. Application of our results to 3–gonal curves

In this Section we apply our method (see Chapter 2) to trigonal curves. E_j denotes an elliptic curve and |Kernel| denotes the smallest possible order for the kernel of $\nu_\times$, the isogeny defined in 2.1.6.

THEOREM 3.3.1. *Let JX be the Jacobian variety of a trigonal curve X, of one of the types detailed below. Then JX is completely decomposable, and a geometrical description of the decomposition is in the following tables.*

- *If JX is the Jacobian variety of a non-normal trigonal curve, then we have the following results:*

Red. group	Automorphism group	Genus of JX	Decomposition of $\nu_\times$	|Kernel|	Induced polarization
D_2	$GL(2,3)$	2	$E_1 \times E_2$	1	(2)
C_4	$SL(2,3)/CD$	3	$E_1 \times E_2 \times E_3$	4	(2)
D_3	$D_3 \times D_3$	4	$E_1 \times \cdots \times E_4$	9	(2), (6), (2), (6)
D_6	$(C_3 \times C_3) \rtimes D_4$	4	$E_1 \times \cdots \times E_4$	9	(2)

- If JX is the Jacobian variety of a trigonal curve with reduced group A_4, S_4 or A_5, then we have the following results:

Red. group	Automorphism group	Genus of JX	Decomposition of $\nu_\times$	|Kernel|	Induced polarization
A_4	$C_3 \times A_4$	4	$E_1 \times E_2 \times E_3 \times E_4$	64	(4), (3), (3), (3)
S_4	$C_3 \times S_4$	4	$E_1 \times E_2 \times E_3 \times E_4$	16	(4)
S_4	$((C_2 \times C_2) \rtimes C_9) \rtimes C_2$	6	$E_1 \times \cdots \times E_6$	64	(4)

PROOF. We will show here the techniques applied to the group $C_3 \times A_4$, the rest of the cases are proved following the same procedure. They are included in Appendix B.

We use our method, presented in Chapter 2, all the computations were made on the software MAGMA [**6**]. The subgroups we use to obtain the effective set giving the decomposition are considering satisfying both conditions of Lemma 2.1.1. Hence, the factors that will define $B_\times$ in the isogeny $\nu_\times$ will correspond to Jacobian varieties of intermediate coverings.

From Table 3, we know that the groups $C_3 \times A_4$ and $C_3 \times S_4$ act in genus 4. It is known [**22**] that for genus 4 there is only one curve with the action of $C_3 \times S_4$, hence also the action of $C_3 \times A_4$. A planar model for this curve is [**27**]

$$X = \{(x, y) \in \mathbb{C}^2 : y^3 = x^5 - x\}.$$

The Jacobian variety JX of a curve X with action of $G = C_3 \times A_4$ and signature given by $(0; 6, 3, 3)$ is completely decomposable. This can be shown using, for instance, Theorem 1.4.1. In fact the group algebra decomposition for JX has the form

$$\tilde{\nu} : B_1 \times B_2^3 \sim JX,$$

where B_1, B_2 are elliptic curves, and G acts on each factor with the rational irreducible representations of degrees 1 and 3 respectively; V_4 and V_{11} in the character table of $C_3 \times A_4$ are shown in Table 4.

Following our method, to construct an isogeny $\nu_\times$ we look for four subgroups H_1, H_2, H_3, H_4 of G satisfying the conditions of Lemma 2.1.1 and Theorem 2.1.8. We determine, for each $H \leq C_3 \times A_4$, the decomposition into complex irreducible representations of the induced representation $\mathrm{Ind}_H^G 1_H$ of $C_3 \times A_4$, where 1_H is the trivial

Class	1	2	3	4	5	6	7	8	9	10	11	12
Size	1	3	1	1	4	4	4	4	4	4	3	3
Order	1	2	3	3	3	3	3	3	3	3	6	6
V_1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	1	1	1	1	-1-J	J	-1-J	J	-1-J	J	1	1
V_3	1	1	J	-1-J	-1-J	J	1	1	J	-1-J	J	-1-J
V_4	1	1	1	1	J	-1-J	J	-1-J	J	-1-J	1	1
V_5	1	1	-1-J	J	-1-J	J	J	-1-J	1	1	-1-J	J
V_6	1	1	J	-1-J	1	1	J	-1-J	-1-J	J	J	-1-J
V_7	1	1	-1-J	J	J	-1-J	1	1	-1-J	J	-1-J	J
V_8	1	1	-1-J	J	1	1	-1-J	J	J-1	-J	-1-J	J
V_9	1	1	J	-1- J	J	-1-j	-1-J	J	1	1	-1-J	J
V_{10}	3	-1	3	3	0	0	0	0	0	0	-1	-1
V_{11}	3	-1	-3-3J	3J	0	0	0	0	0	0	1+J	-J
V_{12}	3	-1	3J	-3-3J	0	0	0	0	0	0	-J	1+ J

TABLE 4. Character table of $C_3 \times A_4$, where $J = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}
Identity element	1	1	1	1	1	1	1	1	1	3	3	3
order 2, length 3	1	1	1	1	1	1	1	1	1	1	1	1
order 3, length 1*	1	0	1	0	0	0	0	0	1	3	0	0
order 3, length 4	1	1	0	0	0	0	1	0	0	1	1	1
order 3, length 4	1	0	0	0	1	1	0	0	0	1	1	1
order 3, length 4	1	0	0	1	0	0	0	1	0	1	1	1
order 4, length 1*	1	1	1	1	1	1	1	1	1	0	0	0
order 6, length 3	1	0	1	0	0	0	0	0	1	1	0	0
order 9, length 4	1	0	0	0	0	0	0	0	0	1	0	0
order 12, length 1	1	0	0	0	1	1	0	0	0	0	0	0
order 12, length 1	1	0	1	0	0	0	0	0	1	0	0	0
order 12, length 1	1	1	0	0	0	0	1	0	0	0	0	0
order 12, length 1	1	0	0	1	0	0	0	1	0	0	0	0
order 36, length 1	1	0	0	0	0	0	0	0	0	0	0	0

TABLE 5. Decomposition of $\mathrm{Ind}_H^G 1_H$ into $\mathbb{C}$–irreducible representations.

representation of H. This corresponds to the dimension of V^H (Frobenius reciprocity). We include this information in Table 5.

We get from Table 5 that the following subgroups satisfy both conditions of Lemma 2.1.1, and hence the hypotheses of Theorem 2.1.8, are

- the class of subgroups of order 4 and length 1 for V_4,
  and
- the class of subgroups of order 3 and length 1 for V_{11}.

Both are marked in Table 5 with a star *.

We now look inside these classes to find the subgroups H_j corresponding to the smallest order for the kernel of $\nu_\times$. To describe them, consider the presentation for G

$$G = \langle a, b | a^6 = b^3 = (ab)^3 = ba^2b^{-1}a^4 = 1\rangle.$$

An effective set S for G is $\{H_1, H_2, H_3, H_4\}$, where

$$H_1 = < ba^{-1}b^{-}1a, a^3 >,$$
$$H_2 = < ba^2 >,$$
$$H_3 = < ba^{-1} >,$$
$$H_4 = < a^{-1}b > .$$

Define $B_{H_i} = \mathrm{Im}(p_{H_i})$, for $i = 1, ..., 4$.

The matrix coordinate for the product $B_\times = B_{H_1} \times B_{H_2} \times B_{H_3} \times B_{H_4}$ is

$$L_{\{H_1,\dots,H_4\}} = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & -1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The determinant of L is 64. Therefore the isogeny

$$\nu_\times : B_{H_1} \times B_{H_2} \times B_{H_3} \times B_{H_4} \to JX$$

has kernel of order 64. Finally, using Theorem 2.2.1, we obtain that the polarization types of B_1, B_2, B_3 and B_4 are (4), (3), (3) and (3), respectively. □

REMARK 3.3.2. Combining Theorem 3.3.1, Remark 3.2.3, and the classification of [22], we may describe part of the loci of Jacobians of trigonal curves for dimension $g \leq 6$.

g	Dimension of the family	Locus description
2	0	one curve with action of GL(2, 3)
3	0	one curve with action of SL(2, 3)/CD
4	1	1-dimensional with action of $D_3 \times D_3$
		and one curve with action of $C_3 \times S_4$
6	0	one curve with action of $((C_2 \times C_2) \rtimes C_9) \rtimes C_2$

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APPENDIX A

Basics.

We include here some of the fundamental concept needed to understand our problem. we give several reference where the reader may find more details. We study a known example of the action of the group S_3 on a Riemann surface of genus 3 with computations about the decomposition of the Jacobian of this family of Riemann surfaces. This appendix is a complement to Chapter 1.

A.1. Riemann surfaces

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A Riemann surface is a topological space, Hausdorff and connected, with a complex structure given by an atlas of analytic charts. In simple words, it locally looks like an open disc in the complex plane.

DEFINITION A.1.1 (Complex chart). A complex chart on a topological space X is a homeomorphism $\phi : U \to V$, where

- $U \subset X$ is an open set in X, and
- $V \subset \mathbb{C}$ is an open set in the complex plane.

EXAMPLE A.1.2. Let $X = \mathbb{R}^2$ and let U be any open subset. Define $\phi(x, y) = x + iy$ from U to $\mathbb{C}$. This is a complex chart on $\mathbb{R}^2$.

DEFINITION A.1.3 (Complex atlas). A complex atlas $\mathcal{A}$ on X is a collection $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise *compatible* complex charts such that $X = \bigcup_\alpha U_\alpha$ (countable union).

Compatible means that when $U_i \cap U_j$ is not empty, the transition function

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a biholomorphic function.

DEFINITION A.1.4 (Riemann surface). A Riemann surface is a connected Hausdorff topological space X together with a complex atlas on it.

It is not hard to see that every Riemann surface is an orientable differential real two-manifold, due to Cauchy-Riemann equations for the transition functions, hence appealing to the classification of compact orientable two manifolds, every compact Riemann surface is diffeomorphic to a g—holed torus, for some unique $g \geq 0$. This g is called the genus of X [**23**, Prop. I.1.23].

REMARK A.1.5. To formally define a complex structure we need to be more careful. In fact there is an equivalence relation between atlases: Two atlases on X are equivalent if their union is also an atlas. There is a partial order relation in the set of equivalent atlases, given by inclusion. By Zorn's lemma, every atlas is contained in a unique (equivalent) maximal atlas. Moreover, two atlases are equivalent if and only if they are contained in the same maximal atlas. This (last) one is called the complex structure of X.

We have the following examples.

EXAMPLE A.1.6 (The complex plane). Here we have only one chart in the atlas.

- $U = \mathbb{C}$ with its natural topology (induced by the Euclidean metric). ϕ given as in the previous example: $\phi : (x, y) \to x + iy$

EXAMPLE A.1.7 (A complex torus of dimension 1). Fix α_1 and β_1 two complex numbers which are linearly independent over $\mathbb{R}$.

Consider $L = \{m_1\alpha_1 + n_1\beta_1 : m_1, n_1 \in \mathbb{Z}\}$ a lattice in $\mathbb{C}$, this is a discrete additive subgroup of rank 2 in $\mathbb{C}$ ($L \cong \mathbb{Z}^2$).

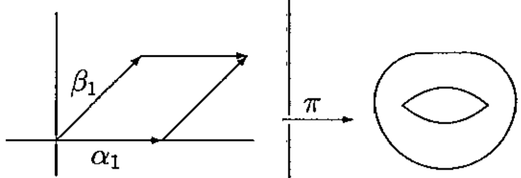

Figure A.1.7

We define $T = \mathbb{C}/L$, a complex torus of dimension 1.

To define an atlas on T we choose $\epsilon > 0$ such that $|\alpha| > 2\epsilon$ for every $\alpha \in L$. Then, for every $z_0 \in \mathbb{C}$

$$D_{z_0} = D(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

is an open set containing no equivalent points. Moreover,

$$\pi|_{D_{z_0}} : D(z_0) \to \pi(D_{z_0})$$

is a continuous, open, bijective map. Hence for each $z \in T$ choose $z_0 \in \pi^{-1}(z_0)$, the homeomorphism $\phi_z : \pi(D_{z_0}) \to D_{z_0}$ given by the restriction $\pi|_{D_{z_0}}^{-1}$, is a complex chart at z.

A.2. Group action on a compact Riemann surface

We will define the group of automorphisms $\mathrm{Aut}(X)$ of a Riemann surface X as the analytical automorphism group of X.

Definition A.2.1 (Holomorphic map). Let X, Y be Riemann surfaces. A mapping $F : X \to Y$ is holomorphic at $p \in X$ if and only if there exist charts $\phi : U_1 \to V_1$ on X ($p \in U_1$) and $\psi : U_2 \to V_2$ on Y ($F(p) \in U_2$), such that $\psi \circ F \circ \phi^{-1}$ is holomorphic at $\phi(p)$. We say that F is holomorphic on an open set $W \subseteq X$ if and only if F is holomorphic at each point of W.

Therefore,

$$\mathrm{Aut}(X) := \{F : X \to X : F \text{ is a bijective (bi)holomorphic map}\}$$

Remark A.2.2. When X is a compact Riemann surface of genus $g(X) \geq 2$, the group $\mathrm{Aut}(X)$ is finite. In fact there is a bound, due to Hurwitz:

$$|\mathrm{Aut}(X)| \leq 84(g(X) - 1).$$

Definition A.2.3 (Group action on X). We say that a finite group G acts on X if G can be mapped to $\mathrm{Aut}(X)$. This is, if there is a (group) homomorphism $\rho : G \to \mathrm{Aut}(X)$. We say that the action is effective if ρ is injective. We simplify notation letting $g.p := \rho(g)(p)$, for any $g \in G$ and $p \in X$.

The orbit of a point $p \in X$ is the set $G.p = \{g.p : g \in G\}$. The stabilizer of a point $p \in X$ is the subgroup $G_p = \{g \in G : g.p = p\}$. It is often called the isotropy subgroup of p.

We are interested in branched coverings, to study the Riemann surface X/G.

Definition A.2.4 (Branched covering). A *branched covering* $f : U \to V$, between Riemann surfaces U and V, is a surjective holomorphic map.

- A point $p \in U$ is a branch point for f if f fails of being locally one-to-one in p. The set of branch points is a discrete subset of U.

- The image of a branch point is a branch value of f. If U and V are compact Riemann surfaces, then the set B_f of branch values is finite. We observe that $f : U \setminus f^{-1}(B_f) \to V \setminus B_f$ is a holomorphic covering of finite degree.

The quotient X/G is the set of orbits. It is a Riemann surface with complex atlas induced by the holomorphic branched covering $\pi : X \to X/G$ (for details see [**23**]). The degree of π is $|G|$ and the multiplicity of π at p is $\text{mult}_p(\pi) = |G_p|$ for all $p \in X$.

We will need the Riemann-Hurwitz formula [Corollary III.3.7,[**23**]].

PROPOSITION A.2.5 (Riemann-Hurwitz). *Let G be a finite group acting (holomorphically and effectively) on a compact Riemann surface X, with quotient map $\pi : X \to X/G = Y$. Suppose that there are r branch values $y_1, ..., y_r \in Y$ with π having multiplicity m_i at the $|G|/m_i$ points above y_i. Then*

$$g(X) = |G|(g(X/G) - 1) + 1 + \frac{|G|}{2} \sum_{i=1}^{r} (1 - \frac{1}{m_i})$$

A branched covering $\pi_G : X \to X/G$ may be partially characterized by its signature which is defined as follows.

DEFINITION A.2.6 (Signature of G on X). We define the tuple $(g(X/G); m_1, ..., m_r)$ to be *the signature (or branching data) of G on X* where r, m_i are given as in Corollary A.2.5.

We introduce now a generalization of this signature which was given in [**31**]. It is a natural fusion between the definitions of *signature* in Definition A.2.6, and the *type of a branch value* (see [**31**]).

DEFINITION A.2.7. (Geometric signature of G on X) Let X be a compact Riemann surface and G be a group acting on X. Let $p_1, ..., p_r \in X$ be a maximal collection of non-equivalent branch points; i.e., points in different $G-$orbits and one by each orbit. For each $j = 1, ..., r$, we consider its stabilizer $G_j = G_{p_j}$. We define *the geometric signature of G on X* as the tuple

$$(\gamma; [m_1, C_1], ..., [m_r, C_r])$$

where γ is the genus of X/G, m_j is the order of the stabilizer subgroup G_j and C_i is the conjugacy class represented by G_i.

DEFINITION A.2.8. (Generating vector of type $(\gamma; m_1, ..., m_r)$) A $2\gamma + r$ tuple $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$ of elements of G is called a *generating vector of type* $(\gamma; m_1, ..., m_r)$ if the following are satisfied:

i) G is generated by the elements $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$;

ii) $\text{order}(c_i) = m_i$; and

iii) $\prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} c_j = 1$, where $[a_i, b_i]$ denotes the commutator of a_i and b_i.

The following theorem [**31**] assures not only the existence of a Riemann surface with the action of a given group, but also gives control on the behavior of the intermediate quotients and on the dimensions of a subvarieties appearing in the decomposition of its Jacobian. This is a subtle difference with the usual Riemann existence Theorem, due to its consideration of the geometric signature instead of the usual one (see [31] for more details).

THEOREM A.2.9 (Existence). *Given a finite group G, there is a compact Riemann surface S of genus g on which G acts with geometric signature $(\gamma; [m_1, C_1], \dots, [m_r, C_r])$ if and only if the following three conditions hold.*

i) *The Riemann-Hurwitz formula given in Corollary A.2.5.*
ii) *The group G has a generating vector*

$$(a_1, b_1, ..., a_\gamma, b_\gamma, c_1, ..., c_r)$$

of type $(\gamma; m_1, ..., m_r)$.
iii) *The elements $c_1, ..., c_r$ of the generating vector are such that the subgroup generated by c_j is in the conjugacy class C_j for $j \in \{1, ..., r\}$.*

EXAMPLE A.2.10 (action of S_3 in genus 3). S_3 has a presentation

$$S_3 =< a, b : a^3, b^2, abab > .$$

A planar model for a compact Riemann surface X of genus $g(X) = 3$ with S_3–action is

$$\{(x, y) \in \mathbb{C}^2 : \beta y^2 x^2 + y^3 + x^3 + \alpha yx + 1 = 0\},$$

where $\alpha, \beta \in \mathbb{C}^*$. Note it is a two parameter family.

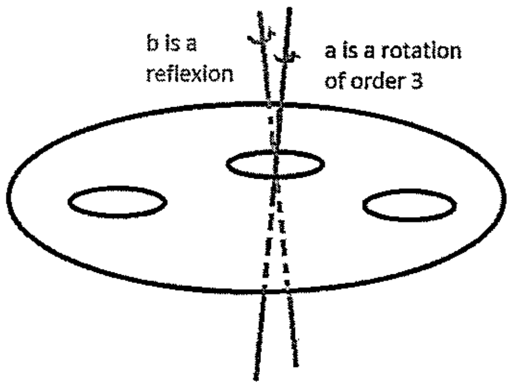

Figure A.2.10

The signature of the action of S_3 on X is $(0; 3, 2, 2, 2, 2)$. This happens due to Theorem A.2.9, because the Riemann-Hurwitz formula is satisfied, the vector

$$(a, b, ab, ab, ab)$$

is a generating vector for S_3 of type $(0; 3, 2, 2, 2, 2)$. We can take the geometric signature

$$(0; [3, \overline{\langle a\rangle}], [2, \overline{\langle b\rangle}], [2, \overline{\langle ab\rangle}], [2, \overline{\langle ab\rangle}], [2, \overline{\langle ab\rangle}]),$$

where we use the overline to denote conjugacy class of the subgroup. Observe that the branch values corresponding to branch points with stabilizer of order 2 are all of the same type (there is only one conjugacy class of subgroups of order 2). We use different representatives to see the relation with the generating vector.

Using the Riemann-Hurwitz formula we can find the signature of the action of every subgroup $H \leq S_3$. We have only 3 classes of subgroups in S_3 which are represented by $\{e, <a>, <b>\}$. We obtain that

- $<a>$ acts on X with signature $(1;3,3)$,
- $<b>$ acts on X with signature $(1;2,2,2,2)$.

In this case the calculation is simple because S_3 is a group of small order. In general, this is possible due to the formulas given in [**31**].

We quote here a proposition containing a formula for the genus of the intermediate quotients.

PROPOSITION A.2.11 (Genus of X/H). *Let X be a Riemann Surface with an action of the group G and geometric signature $(\gamma; [m_1, C_1], \dots, [m_r, C_r])$. Then for each subgroup $H \leq G$ the genus of X/H is given by*

$$g(X/H) = [G:H](\gamma - 1) + 1 + \frac{1}{2}\sum_{j=1}^{r}\sum_{l \in \Omega_{G_j}} \frac{|N_G(G_j)|}{|H|}\left(1 - \frac{|G_j^{l^{-1}} \cup H|}{G_j}\right)$$

where G_j is a representative for the conjugacy class C_j, and Ω_{G_j} is a left transversal of the normalizer $N_G(G_j)$ of G_j in G.

A.3. Complex tori and abelian varieties

It is possible to generalize the definition of complex tori of dimension 1 to higher dimensions.

DEFINITION A.3.1 (Complex tori of dimension g). Fix $2g$ vectors

$$\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$$

in $\mathbb{C}^g$ which are linearly independent over $\mathbb{R}$. Consider

$$L = \{m_1\alpha_1 + ... + m_g\alpha_g + ...n_1\beta_1 + ...n_g\beta_g : m_i, n_i \in \mathbb{Z}\}$$

as a lattice in $\mathbb{C}^g$. This is a discrete additive subgroup of rank $2g$ in $\mathbb{C}^g$, so $L \cong \mathbb{Z}^{2g}$, as $\mathbb{Z}-$modules. The quotient $T_g = \mathbb{C}^g/L$ is called a complex torus of dimension g.

T_g is a compact, connected $g-$dimensional complex manifold.

In order to describe T_g we choose a $\mathbb{C}-$basis $\{e_1, \ldots, e_g\}$ for $\mathbb{C}^g$ and write the elements $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$ in terms of $e_1, ..., e_g$.

$$\alpha_i = \sum_{j=1}^{g} \pi_{ji} e_j$$

$$\beta_i = \sum_{j=1}^{g} \widetilde{\pi}_{ji} e_j$$

The matrix

$$\Pi = ((\pi_{ji})\ (\widetilde{\pi}_{ji}))$$

in $M(g \times 2g, \mathbb{C})$ is called a period matrix for T_g. The period matrix determines completely T_g, but certainly depends on the choice of the bases of $\mathbb{C}^g$ and L (for more details see [4]).

There are two special types of holomorphic maps between complex tori: homomorphisms and translations. A homomorphism between complex tori is a holomorphic function which preserves the group structure. The translation by an element $x_0 \in T$ is defined to be the holomorphic map $t_{x_0} : x_0 \to x + x_0$.

PROPOSITION A.3.2 ([4], Chapter 1, Section 2). *Let $T = \mathbb{C}^g/L$ and $T' = \mathbb{C}^{g'}/L'$ be two complex tori of dimensions g and g' respectively. Let $h : T \to T'$ be a holomorphic map between them.*

(a) *There is a unique homomorphism $f : T \to T'$ such that $h = t_{h_0} f$, i.e. $h(x) = f(x) + h(0)$ for all $x \in T$.*
(b) *There is a unique $\mathbb{C}-$linear map $F : \mathbb{C}^g \to \mathbb{C}^{g'}$ with $F(L) \subset L'$ inducing the homomorphism f.*

REMARK A.3.3. Under addition the set of homomorphisms of T into T' forms an abelian group denoted by $\mathrm{Hom}(T, T')$. Proposition A.3.2 gives an injective homomorphism of abelian groups

$$\rho_a : \mathrm{Hom}(T, T') \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^g, \mathbb{C}^{g'}), \quad f \to F,$$

called the *analytic representation of* $\mathrm{Hom}(T, T')$. The restriction F_L of F to the lattice L is $\mathbb{Z}-$linear. F_L determines F and f. Thus we get an injective homomorphism

$$\rho_r : \mathrm{Hom}(T, T') \to \mathrm{Hom}_{\mathbb{Z}}(L, L'), \quad f \to F_L,$$

called the *rational representation of* $\mathrm{Hom}(T, T')$. We denote the extension of ρ_a and ρ_r to $\mathrm{Hom}_{\mathbb{Q}}(T, T') := \mathrm{Hom}(T, T') \otimes_{\mathbb{Z}} \mathbb{Q}$ by the same letters. In particular, if $T = T'$, ρ_a and ρ_r are representations of the ring $\mathrm{End}(T)$, or $\mathrm{End}_{\mathbb{Q}}(T) := \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $f : T \to T'$ be a homomorphism. With respect to the chosen bases the representation $\rho_a(f)$ (respectively $\rho_r(f)$) is given by the matrix $A \in M_{g' \times g, \mathbb{C}}$ (respectively $R \in M(2g' \times 2g, \mathbb{Z})$). In terms of matrices the condition $\rho_a(f)(L) \subset L'$ means

(9) $$A\Pi = \Pi' R,$$

where $\Pi \in M(g\times 2g, \mathbb{C})$ and $\Pi' \in M(g'\times 2g', \mathbb{C})$ are period matrices for T and T' with respect to the chosen bases.

Conversely, any pair of matrices $A \in M(g'\times g, \mathbb{C})$ and $R \in M(2g'\times 2g, \mathbb{Z})$ satisfying the above equality defines a homomorphism $T \to T'$.

There is a special type of homomorphisms among complex tori which we will use throughout this thesis.

DEFINITION A.3.4 (isogeny). An isogeny between two complex tori $T \to T'$ is a surjective homomorphism with finite kernel. This implies that T and T' have the same dimension as complex tori.

An important class of complex tori is given by *abelian varieties*, which are complex tori admitting a polarization. An important property of these varieties is that they are projective (see [4]).

DEFINITION A.3.5 (Polarization). A polarization of a complex torus $T = \mathbb{C}^g/L$ is a positive definite Hermitian form H on $\mathbb{C}^g$ which is non degenerate and satisfies $\Im(H)(L\times L) \subseteq \mathbb{Z}$, where $\Im$ denotes the imaginary part of H.

DEFINITION A.3.6 (Abelian variety). An abelian variety is a complex torus $T = \mathbb{C}^g/L$ admitting a polarization.

PROPOSITION A.3.7. *There exist a 1-1 correspondence between the set of Hermitian forms H on $\mathbb{C}^g$ and the set of real alternating forms E on $\mathbb{C}^g$ satisfying $E(iv, iw) = E(v, w)$ for all $v, w \in \mathbb{C}^g$. The bijection is*

$$E(v, w) = \Im(H(v, w)) \text{ and } H(v, w) = E(iv, w) + iE(v, w)$$

for all $v, w \in \mathbb{C}^g$.

In general, an abelian variety can admit many polarizations. When we fix one of them, we say that the pair (T, H) or (T, E) is a polarized abelian variety.

Due to the elementary divisor theorem it is possible to choose a base β of L with respect to which E is given by the matrix

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \operatorname{diag}(d_1, \ldots, d_g)$ with integers d_l satisfying $d_l | d_{l+1}$ for all $l \in \{1, ..., g-1\}$. The *elementary divisors* $d_1, ..., d_g$ are uniquely determined by E and L.

The vector $(d_1, ..., d_g)$ is called the *type of* E and the basis β a *symplectic basis* of L for E.

Definition A.3.8 (Principally polarized abelian variety). If (T, E) is polarized abelian variety such that the type of E is $(1, ..., 1)$ then we say that (T, E) is a principally polarized abelian variety.

A.4. Jacobian varieties.

A special example of principally polarized abelian variety is the case of the *Jacobian varieties* which admit a canonical principal polarization. Their construction is as follows (see [4, 11.1]).

Given a compact Riemann surface X of genus g, consider the $g-$dimensional $\mathbb{C}-$vector space $H^{1,0}(X, \mathbb{C})$ of holomorphic 1-forms on $\mathbb{C}$, and the first homology group $H_1(X, \mathbb{Z})$, a free abelian group of rank $2g$. For convenience we use the same letter for (topological) 1-cycles on X and their corresponding classes in $H_1(X, \mathbb{C})$. By Stokes' theorem any element $\gamma \in H_1(X, \mathbb{Z})$ yields in a canonical way a linear form on the vector space $H^{1,0}(X, \mathbb{C})$, which we also denote by γ:

$$\gamma : H^{1,0}(X, \mathbb{C}) \to \mathbb{C};\ \omega \to \int_\gamma \omega.$$

Remark A.4.1. The canonical map

$$\varphi : H_1(X, \mathbb{C}) \to H^{1,0}(X, \mathbb{C})^* = \mathrm{Hom}(H^{1,0}(X, \mathbb{C}), \mathbb{C})$$

given above is injective.

Since $\varphi(H_1(X, \mathbb{Z}))$ is a lattice in $H^{1,0}(X, \mathbb{C})^*$, we have

Definition A.4.2 (Jacobian varieties). The *Jacobian variety* of a compact Riemann surface X or simply the *Jacobian of* X is

$$JX := \frac{H^{1,0}(X, \mathbb{C})^*}{\varphi(H_1(X, \mathbb{Z}))}.$$

Note that

- If $g = 0$, then $JX = 0$, and
- If $g = 1$, then JX is a complex torus of dimension 1. Moreover, we have $JX \cong X$ in this case [15] [23].

Remark A.4.3. In all that follows, we assume $g \geq 2$ in order to avoid trivialities.

Let X be a compact Riemann surface of genus g. To describe JX in terms of period matrices, choose bases $\{l_1, ..., l_{2g}\}$ of $H_1(X, \mathbb{Z})$ and $\omega_1, ..., \omega_g$ of $H^{1,0}(X, \mathbb{C})$.

Let $t_1, ..., t_g$ denote the basis of $H^{1,0}(X, \mathbb{C})^*$ dual to $\omega_1, ..., \omega_g$, i.e, $t_i(\omega_j) = \delta_{ij}$ for all i, j.

We have

$$l_i = \sum_{j=1}^{g} (\int_{l_i} \omega_j) t_j.$$

Hence

$$\Pi = \begin{pmatrix} \int_{l_1} \omega_1 & \dots & \dots & \int_{l_{2g}} \omega_g \\ \vdots & & & \vdots \\ \int_{l_1} \omega_g & \dots & \dots & \int_{l_{2g}} \omega_g \end{pmatrix}$$

is a period matrix for JX respect to these bases.

Fix $\{l_1 = \alpha_1, ..., l_g = \alpha_g, l_{g+1} = \beta_1, ..., l_{2g} = \beta_g\}$ a basis of $H_1(X, \mathbb{Z})$ with intersection matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$, as indicated in the following figure (which is called a symplectic basis).

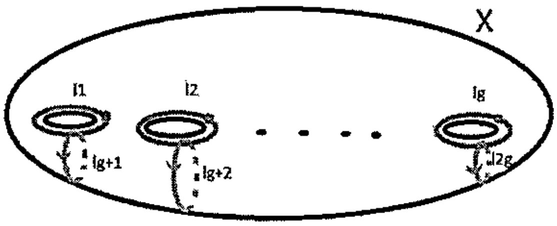

Figure A.4

Denoted by E the alternating form on $H^{1,0}(X, \mathbb{C})^*$ with matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ with respect to these base and define

$$H(v, w) = E(iv, w) + iE(v, w),$$

which defines a principal polarization on JX, called the canonical polarization on JX.

A.5. Group action on JX

Every time that we have a Riemann Surface X with $G-$action, we have associated an action of G on the corresponding Jacobian [4]. Hence, by Remark A.3.3, we obtain two representations for the action of G on its Jacobian variety JX:

(1) The Rational Representation, $\rho_r : G \to GL(H_1(X, \mathbb{Z}) \otimes \mathbb{Q})$.
(2) The Analytical Representation, $\rho_a : G \to GL(H^{1,0*}(X, \mathbb{C}))$.

Where X denotes the corresponding Riemann surface.

Both are related by,

$$\rho_r \otimes \mathbb{C} \cong \rho_a \oplus \overline{\rho_a}$$

EXAMPLE A.5.1 (S_3 on $H_1(X,\mathbb{Z})$). Let us recall Example A.2.10. If we choose a symplectic basis $\beta = \{\alpha_1, ..., \alpha_3, \beta_1, ..., \beta_3\}$ of $H_1(X,\mathbb{Z})$ then the action of S_3 on X is represented by

$$[a]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$[b]_\beta = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

A.6. Singular locus

Now we introduce the moduli space of principally polarized abelian varieties, some of its properties and its singular locus [[4], Chapter 8].

DEFINITION A.6.1 (Moduli). The set of isomorphism classes of principally polarized abelian varieties of dimension g is called the moduli space of principally polarized abelian varieties of dimension g. It is denoted by $\mathcal{A}_g$.

Suppose that $(T = \mathbb{C}^g/L, H)$ is a principally polarized abelian variety. Then there exists a symplectic basis $\gamma = \{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$ of L for H such that the alternating form $E = \Im(H)$ is given by the matrix

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

In fact, γ can be chosen such that, if we define $e_\nu = \alpha_\nu$ for $\nu \in \{1, ..., g\}$, then $\{e_1, ..., e_g\}$ is a $\mathbb{C}$−basis for $\mathbb{C}^g$. Therefore, with respect to these bases, the period matrix of T is of the form

$$\begin{pmatrix} I_g & Z \end{pmatrix}$$

for some $Z \in M_g(\mathbb{C})$. The matrix Z satisfies

- $Z^t = Z$ and $\Im Z > 0$.
- $(\Im Z)^{-1}$ is the matrix of the Hermitian form H with respect to the basis $\{e_1, ..., e_g\}$ [4, Prop. 8.1.1]

DEFINITION A.6.2 (Siegel upper half-space). The set

$$\mathcal{H}_g := \{Z \in M_g(\mathbb{C}) : Z^t = Z, \Im Z > 0\}$$

is called the Siegel upper half-space. It is a $\frac{1}{2}g(g+1)$-dimensional open sub-manifold of the vector space of symmetric matrices in $M_g(\mathbb{C})$.

We have seen that a principally polarized abelian variety determines a point $Z \in \mathcal{H}_g$. Conversely, any $Z \in \mathcal{H}_g$ determines a principally polarized abelian variety

$$T_Z = \mathbb{C}^g / L_Z$$

where L_Z is the lattice generated by the columns of $\Pi = (I_g\ Z)$. We can define the polarization $H_Z = (\Im Z)^{-1}$ with respect to the standard basis of $\mathbb{C}^g$.

Then we have

$$\mathcal{A}_g \cong \mathcal{H}_g / \sim$$

where $\sim$ denotes the equivalence relation

$$Z \sim Z' \text{ if and only if } Z' = M(Z),$$

where $M(Z) = (A + ZC)^{-1}(B + ZD)$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $A, B, C, D \in M_g(\mathbb{Z})$. $M \in Sp_{2g}(\mathbb{Z})$, the symplectic group. This comes from Equation (9), with M being the rational representation of the isomorphism between the variety with Riemann matrices Z and $M(Z)$.

REMARK A.6.3. It is known that the singular locus $\text{Sing}(\mathcal{A}_g)$ of $\mathcal{A}_g$ is the subset which contains the principally polarized abelian varieties with non trivial automorphisms.

In particular, the loci (singular locus) of Jacobian varieties of dimension g contains the Jacobian varieties with non trivial group of automorphisms.

A.7. Representations of a finite group

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DEFINITION A.7.1 (Representation of a finite group). Let G a finite group, K a field and V a finite dimensional K-vector space.

A *representation* of G with space of representation V is a homomorphism $r : G \to GL(V)$ which sends g to a linear transformation r_g of V. We say the *degree of* r is $\dim_K V$.

Once a basis for V is chosen, we have a matrix representation for every $g \in G$. We say that the representation r is given in matrix form.

An important concept in group representation theory is the concept of irreducible representations over K.

DEFINITION A.7.2. (Sub-representation) Let $r : G \to GL(V)$ be a linear representation and let W be a vector subspace of V. Suppose that W is *stable* under the action of G (we also say *invariant*), i.e.

$$x \in W \text{ implies } r_g(x) \in W \text{ for all } g \in G,$$

then $r|_W : G \to GL(W)$ is a linear representation of G in W and W is said to be a *subrepresentation* of V.

DEFINITION A.7.3. (Irreducible representation) Let $r : G \to GL(V)$ be a linear representation of G, we say that r is *irreducible* if the only G-invariant subspaces of V are V y $\{0\}$. We denote the set of $K-$irreducible representations of G by $\mathrm{Irr}_K(G)$.

DEFINITION A.7.4. (Equivalent representations) Let $r : G \to GL(V)$ and $r' : G \to GL(V')$ be two linear representations of the same group G. These representations are said to be equivalent (similar, isomorphic or equal) if there exists a linear isomorphism $\tau : V \to V'$ such that

$$\tau \circ r(s) = r'(s)\tau \text{ for all } s \in G.$$

If r and r' are given in matrix form, the above definition means that there exists an invertible matrix which conjugates r_s to $r_{s'}$ for all $s \in G$.

THEOREM A.7.5 (Maschke, Section 10.8, [**11**]). *Every representation over a field of characteristic* 0 *is a direct sum of irreducible representations.*

We concentrate along this work in to study $\mathbb{C}-$irreducible representations and the $\mathbb{Q}-$irreducible representation of a given group, see [**34**] and [**11**] for references.

EXAMPLE A.7.6 (Representations of S_3). Again we write

$$S_3 = < a, b : a^3, b^2, abab >$$

The $\mathbb{Q}-$irreducible representations of S_3 are 3, two of degree 1 and one of degree 2. We give them here in its matrix form:

- the trivial one $r_0 : g \to r_{0_g} = Id_{\mathbb{C}}$,
- the sign representation r, i.e $r_a = 1$ and $r_b = -1$, and
- the geometric 2-dimensional representation t with space of representation $W = \mathbb{C}^2$.

  This representation is defined by

$$t_a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \text{ and } t_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $\omega = e^{\frac{2\pi i}{3}}$ is a primitive cube root of the unity.

It is a known fact that the $\mathbb{C}$−irreducible representations of the group S_3 can be defined, in fact, over the field of the rational numbers $\mathbb{Q}$ [34]. In fact, this happens to every group $S_n; n \geq 2$.

We have that t is equivalent to the following representation.

$$t'_a = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}; t'_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

THEOREM A.7.7 ([34]). *The number of complex irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.*

THEOREM A.7.8 (Chapter 13, [34]). *The number of rational irreducible representations of G (up to isomorphism) is equal to the number of cyclic subgroups of G up to conjugation.*

On the other hand, it is possible to study the representations of a group using the following tool (see [34, 2.2.3]).

DEFINITION A.7.9. (Character of a representation) Let $r : G \to GL(V)$ be a linear representation of a finite group G in the vector space V. We choose a basis on V and for each $g \in G$ we define *the character of* r *on* g as

$$\chi_r(g) = \text{Trace}(r_g)$$

EXAMPLE A.7.10. Recall that the character table of S_3 is

Elements of G	χ_0	χ_1	χ_2
e	1	1	2
a, a^2	1	1	-1
a^2b, ab, b	1	-1	0

DEFINITION A.7.11. (Scalar product) Let χ and χ' be two characters of a group G. The scalar product between them is defined as

$$(\chi, \chi') = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

In all that follows K is a field of characteristic 0, in fact K is $\mathbb{C}$ or $\mathbb{Q}$.

THEOREM A.7.12. *(Orthogonality relations for characters)*

1) If χ is a character of a complex irreducible representation, we have $(\chi, \chi) = 1$.

2) If χ and χ' are characters of two non isomorphic complex irreducible representations, we have $(\chi, \chi') = 0$.

THEOREM A.7.13. [**11**, Thm. 30.15] *Let U, V be absolutely irreducible representations of a finite group G with characters χ, ν respectively. Then U is equivalent to V (see Definition A.7.4) if and only if $\chi = \nu$.*

THEOREM A.7.14 (multiplicity). *Let V be a linear representation of G with character ϕ, and suppose V decomposes into a direct sum of irreducible representations:*

$$V = W_1 \oplus \cdots \oplus W_s.$$

Then, if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product (ϕ, χ).

COROLLARY A.7.15. *The number of W_i isomorphic to W does not depend on the chosen decomposition.*

This allows us to make the following definition.

DEFINITION A.7.16 (Isotypical decomposition). Let V be a K–linear representation of a finite group G, with character χ, and $W_1, \ldots, W_s$ all the K–irreducible representations of G, up to equivalence. We call

$$V = W_1^{(\chi,\chi_1)} \oplus \cdots \oplus W_s^{(\chi,\chi_s)}$$

the isotypical decomposition of V, where the scalar products (χ, χ_j) can be 0.

DEFINITION A.7.17. Given $r \in \mathrm{Irr}_{\mathbb{C}}(G)$, $r : G \to GL(V)$:

- L_V denotes the field of definition of V and

  $$K_V = \mathbb{Q}(\chi_V(g) : g \in G)$$

  denotes the field obtained by adjoining to the rational numbers $\mathbb{Q}$ the values of the character χ_V.
- $m_V = m_{\mathbb{Q}}(V) = [L_V : K_V]$ is the Schur index of V.

Note that we obtain the following extensions of fields over $\mathbb{Q}$.

$$\begin{array}{c} L_V \\ m_V \Big| \\ K_V = \mathbb{Q}(\chi_V(g) : g \in G) \\ \Big| \\ \mathbb{Q} \end{array}$$

Figure A.7

Note that, for all r (or V), all the field extensions in Figure A.7 are Galois extensions; this is because L_V, K_V are subfields of $\mathbb{Q}(\xi_n)$, where ξ_n is a primitive n root of unity, for n equals to the exponent of the group G [**11**, Thm. 41.1]. Denote by $\mathrm{Gal}(L_V/\mathbb{Q}), \mathrm{Gal}(L_V/K_V), \mathrm{Gal}(K_V/\mathbb{Q})$ the respective Galois groups.

For each complex irreducible representation V of G (we denote the representation r by its underlying vector space V), we call the set

$$\mathcal{G}(V) := \{V^\sigma : \sigma \in \mathrm{Gal}(L_V/\mathbb{Q})\}$$

the Galois class of V, where V^σ is the complex irreducible representation given as follows:

for each $g \in G$, let $V^\sigma(g)$ denote the matrix obtained by letting g act on all the coefficients of the matrix $V(g)$. Note that the representation V^σ is also defined over L_V and both V and V^σ share the same field K_V. Furthermore, due to Theorem A.7.13, V^σ is equivalent to V if and only if σ is in $\mathrm{Gal}(L_V/K_V)$.

The rational and complex irreducible representations of a group G are related by the following theorem [**11**, Thm. 70.15].

THEOREM A.7.18. [**11**, Thm. 70.15] *Let $\{V_1, ..., V_t\}$ be a full set of representatives of Galois classes from the set $\mathrm{Irr}_{\mathbb{C}}(G)$ and let $K_j := K_{V_j}, L_j = L_{V_j}$. Then for each rational irreducible representation $\mathcal{W}$ of G there exists precisely one V_j satisfying*

$$\mathcal{W} \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in Gal(L_j/\mathbb{Q})} V_j^\sigma =: \bigoplus_{\sigma \in Gal(K_j:\mathbb{Q})} m_{V_j} V_j^\sigma.$$

Now we recall results regarding subgroups and induced representations [**34**, 3.3.3].

Let $r \in \mathrm{Irr}_{\mathbb{C}}(G)$ and let r_H be its restriction to H. Denote by $\theta : H \to GL(W)$ a subrepresentation of r_H. For $s \in G$; the vector space $r_s W$ depends only on the left coset sH of s. In fact, if we replace s by st with $t \in H$, we have

$$r_{st}W = r_s r_t W = r_s W$$

since $r_t W = W$.

If σ is a left coset of H, we can thus define a space W_σ of V to be $r_s W$ for any $s \in \sigma$. It is clear that the W_σ are permuted among themselves by the r_s for $s \in G$. Their sum

$$\sum_{\sigma \in G/H} W_\sigma$$

is then a subrepresentation of V.

DEFINITION A.7.19 (Induced representation). We say that the representation r of G in V is induced by the representation $\theta : H \to GL(W)$ of H, if

$$V = \sum_{\sigma \in G/H} W_\sigma.$$

We can reformulate this definition in the following way:

If R is a set of representatives of G/H, the vector space V is the direct sum of the r_tW with $t \in R$. In particular, we have

$$\dim V = \sum_{t\in R} r_tW = [G : H] \dim W.$$

We will use specially the induced representation by the trivial representation of a subgroup H of G.

DEFINITION A.7.20 (Subspace of V fixed by H). Let $V \in \mathrm{Irr}_{\mathbb{C}}(G)$, and $H \leq G$. We define $\mathrm{Fix}_H V = V^H$ as the set of fixed points of V under the action of all $h \in H$.

Due to the Frobenius reciprocity theorem [**34**, Chapter 7, Thm. 13] we have

PROPOSITION A.7.21. *Given V a $\mathbb{C}$−irreducible representation of G, we have*

$$\dim_{\mathbb{C}} V^H = (Ind_H^G 1_H, V),$$

where $\mathrm{Ind}_{1_H^G}$ *is the representation of G induced by the trivial representation in H, and $(\ ,\ )$ denotes the scalar product defined in Def. A.7.11.*

We will use more known facts about representation theory such as: the regular representation of a group, relations among the degrees of the representations, etc. For references see [**34**] or [**11**].

A.8. The group algebra $\mathbb{Q}[G]$.

DEFINITION A.8.1. The set of the formal sums $\sum_{g\in G} \alpha_g g$, where $\alpha_g \in \mathbb{Q}$ form an *algebra* over $\mathbb{Q}$ which is called *the group algebra of G over* $\mathbb{Q}$ [**11**].

This is a finite dimensional vector space over $\mathbb{Q}$, with the natural operations, and a ring with unity, with the product extended from the product in G. $\mathbb{Q}[G]$ is a semisimple algebra [**34**, Chapter 6, Prop. 9], so it decomposes as a product of simple algebras

$$\mathbb{Q}_0 \times \cdots \times \mathbb{Q}_r$$

where $r + 1$ is the number of rational irreducible representations of G, every $\mathbb{Q}_i$ is a simple algebra generated by a central idempotent element e_i. This is, $\mathbb{Q}_i = \mathbb{Q}[G]e_i$ where e_i is a central element (uniquely defined) of $\mathbb{Q}[G]$ which satisfies $e_i^2 = e_i$.
Hence, we can write

$$1 = e_0 + ... + e_r.$$

EXAMPLE A.8.2. We have

$$\mathbb{Q}[S_3] = \mathbb{Q}_0 \times \mathbb{Q}_1 \times \mathbb{Q}_2$$

where every $\mathbb{Q}_i = \mathbb{Q}[S_3]e_i$ with

$$e_i = \frac{\dim \chi_i}{|G|} \sum_{g\in G} \chi_i(g^{-1})g$$

Then we have

$$\begin{aligned} e_0 &= \frac{1}{6}\sum_{g\in G} g \\ e_1 &= \frac{1}{6}(e + a + a^2 + (-1)b + (-1)ab + (-1)a^2b) \\ e_2 &= \frac{2}{6}(2e + (-1)a + (-1)a^2) \end{aligned}$$

It is not difficult verify that $1 = e_0 + e_1 + e_2$.

APPENDIX B

Rest of the proof of Theorem 3.3.1

In this appendix we complete the proof of Theorem 3.3.1. We use the method of Chapter 2 running on the computational program MAGMA [6] and the algorithm given in [3]. A nice explanation and examples of this algorithm can be found in [33].

The notation is coming from Section 1.3 and Theorem 2.1.8. In particular, we denote the factors in the isogeny $\nu_\times$ by $B_\times = B_{H_{ij}}$, where i corresponds to the irreducible rational representation W_i acting on A_i associated to one of the irreducible complex representations V_j for $j \in \{1, ..., n_i\}$ that corresponds to W_i under the Galois group action (as in theorem A.7.18). The numbering and notation follow the MAGMA database.

We divide our study into sections, according to the dimension of the Jacobian.

B.1. Dimension 2:

This case was presented in Example 2.1.11, we obtain an isomorphism $\nu_\times$. We use this case to show that if we choose a different pair of subgroups in the same conjugacy class, we obtain an isogeny with non trivial kernel. Set

$$H_1' = H^{ba^{-2}b^{-2}a^2b^{-1}} = < a^2ba^3b^{-2} >$$

and

$$H_2' = H^{b^{-1}} = < ba >,$$

where $H =< ab >$. If $B_{41}' = \mathrm{Im}(p_{H_1'})$ and $B_{42}' = \mathrm{Im}(p_{H_2'})$ then $|Ker(\nu)| = 3$. This due to

$$\rho_s(p_{H_1'}) = \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ -1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\rho_s(p_{H_2'}) = \begin{pmatrix} 0 & -1/2 & 0 & 1/2 \\ 1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 1 \end{pmatrix}$$

The matrix coordinate of the lattice of the product $B'_{41} \times B'_{42}$ is given in this case

$$L_{\{H'_1,H'_2\}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Continuing with the effective set for this group found in Example 2.1.11, the matrix $L_{\{H_1,H_2\}}$ provides the coordinate matrices of the lattices of B_{41} and B_{42}. Then, if we denote by $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ the symplectic basis of $\Lambda = H_1(X, \mathbb{Z})$ we have that a basis of the lattice of B_{4j} is given by γ_{1j}, γ_{2j} for $j \in \{1,2\}$. Where

$$\gamma_{11} = \alpha_2, \quad \gamma_{21} = \beta_1 + 2\beta_2; \; \gamma_{12} = \alpha_1 + \alpha_2, \quad \gamma_{22} = \beta_1 + \beta_2.$$

Moreover, we have obtained, using the algorithm from **[3]**, the Riemann matrix of JX:

$$Z = \begin{pmatrix} z & -z/2 \\ -z/2 & z \end{pmatrix},$$

where $z^2 - 4/3z + 4/3 = 0$.

Therefore we obtain $\gamma_{21} = 3/2z\gamma_{11}$ and $\gamma_{22} = z/2\gamma_{12}$.

Then if we consider $\{\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}\}$ as the basis of the lattice of the product $B_{41} \times B_{42}$ and $\{\gamma_{11}, \gamma_{12}\}$ as basis of the complex vector space which defines it as a complex tori, we may compute its period matrix (see [[**4**], Chapter 1]) which is

$$\Pi_D = \begin{pmatrix} 1 & 0 & 3\tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix},$$

where $\tau = z/2$.

We obtain the induced polarization of each of the factors B_{41} and B_{42} with respect to the chosen basis for the lattice of $B_{41} \times B_{42}$. They are given by

$$E_{41} = E_{42} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover the matrix of the rational representation of $B_{41} \times B_{42}$ and the matrix of its analytic representation must be (see [[**4**], Chapter 1])

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}; \; A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

We verify the equation for the isogeny's matrix form $A\Pi_D = (I_2\ Z)R$ (see [**4**], [**30**] or equation 9 in this tesis).

Finally, we note that the Riemann matrix of JX depends on one parameter $z \in \mathbb{C}$ such that $z^2 - (4/3)z + 4/3 = 0$. Then we have a priori two Jacobians of dimension

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Size	1	1	6	4	4	1	1	6	4	4	4	4	4	4
Order	1	2	2	3	3	4	4	4	6	6	12	12	12	12
V_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	-1
V_3	1	1	1	-1-J	J	1	1	1	J	-1-J	-1-J	J	-1-J	J
V_4	1	1	-1	J	-1-J	-1	-1	1	-1-J	J	-J	1+J	-J	1+J
V_5	1	1	-1	-1-J	J	-1	-1	1	J	-1-J	1+J	-J	1+J	-J
V_6	1	1	1	J	-1-J	1	1	1	-1-J	J	J	-1-J	J	-1-J
V_7	2	-2	0	-1	-1	-2i	2i	0	1	1	i	i	-i	-i
V_8	2	-2	0	-1	-1	2i	-2i	0	1	1	-i	-i	i	i
V_9	2	-2	0	1+J	-J	-2i	2i	0	J	-1-J	Z1	$Z1^5$	-Z1	$-Z1^5$
V_{10}	2	-2	0	-J	1+J	2i	-2i	0	-1-J	J	$-Z1^5$	-Z1	$Z1^5$	Z1
V_{11}	2	-2	0	-J	1+J	-2i	2i	0	-1-J	J	$Z1^5$	Z1	$-Z1^5$	-Z1
V_{12}	2	-2	0	1+J	-J	2i	-2i	0	J	-1-J	-Z1	$-Z1^5$	Z1	$Z1^5$
V_{13}	3	3	1	0	0	-3	-3	-1	0	0	0	0	0	0
V_{14}	3	3	-1	0	0	3	3	-1	0	0	0	0	0	0

TABLE 1. Character table of $(C_4 \times SL(2,3))/\mathrm{CD}$, where $Z1 = \frac{\sqrt{3}}{2} - \frac{i}{2}$, $J = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

2 with action of $GL(2,3)$. However, only when $z = 2/3(1+\sqrt{2}i)$ we obtain that the imaginary part of Z is positive definite.

Then the Jacobian of the Bolza's curve has Riemann matrix

$$Z = \begin{pmatrix} 2/3(1+\sqrt{2}i) & -1/3(1+\sqrt{2}i) \\ -1/3(1+\sqrt{2}i) & 2/3(1+\sqrt{2}i) \end{pmatrix}.$$

B.2. Dimension 3

The genus 3 surface on Table 2 corresponds to the curve with plane model (see [**22**, Table 2]).

$$X = \{(x,y) : y^4 = x^3 - 1\}$$

The full automorphism group of X is $G = (C_4 \times SL(2,3))/\mathrm{CD}$ acting with signature $(0; 12, 3, 2)$

Using the formula for the dimensions of the factors in the group algebra decomposition of JX (Theorem 1.4.1), we obtain that the Jacobian variety of X is completely decomposable. In fact,

$$\tilde{\nu} : B_1 \times B_2^2 \sim JX,$$

where B_1 and B_2 are complex elliptic curves.

The irreducible rational representations of degree 1 and 2 of G, V_3 and V_7 respectively in Table 1, act on the factors B_1 and B_2^2 respectively.

To compute the minimal kernel for the isogeny ν, we consider three subgroups H_1, H_2 and H_3 of $G(48,33)$ satisfying the conditions of Lemma 2.1.1.

Subgroup classes	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}
Identity element	1	1	1	1	1	1	2	2	2	2	2	2	3	3
order 2, length 1*	1	1	1	1	1	1	0	0	0	0	0	0	3	3
order 2, length 6**	1	0	0	1	1	0	1	1	1	1	1	1	2	1
order 3, length 4	1	1	0	0	0	0	0	0	1	1	1	1	1	1
order 4, length 1	1	0	0	1	1	0	0	0	0	0	0	0	0	3
order 4, length 3	1	0	0	1	1	0	0	0	0	0	0	0	2	1
order 4, length 3*	1	1	1	1	1	1	0	0	0	0	0	0	1	1
order 6, length 4	1	1	0	0	0	0	0	0	0	0	0	0	1	1
order 8, length 1*	1	1	1	1	1	1	0	0	0	0	0	0	0	0
order 8, length 3	1	0	0	1	1	0	0	0	0	0	0	0	0	1
order 8, length 3	1	0	0	1	1	0	0	0	0	0	0	0	1	0
order 12, length 4	1	0	0	0	0	0	0	0	0	0	0	0	0	1
order 16, length 1	1	0	0	1	1	0	0	0	0	0	0	0	0	0
order 24, length 1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
order 48, length 1	1	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 2. Decomposition of $\mathrm{Ind}_H^G 1_H$ in complex irreducible representations.

To find such subgroups H_1, H_2, H_3 we use the symplectic representation of the action given in [3]. This allows us to write every element of $G(48, 29)$ as a square symplectic matrix of size 6.

We determine the complex irreducible representation decomposition of the induced representation $\mathrm{Ind}_H^G 1_H$ in G by the trivial representation in H for each $H \leq G$.

Table 2 shows the multiplicity of the complex irreducible representations in $\mathrm{Ind}_H^G 1_H$ for all $H \leq G$ up to conjugacy.

We get from Table 2 that the following subgroups satisfy both conditions of Lemma 2.1.1, hence the hypothesis of Theorem 2.1.8.

- For V_4: the classes of subgroups are given by subgroups of order 2, 4 and 8 (marked with *).
- For V_{11}: the subgroups of order 2 and is of length 6 (marked with **).

We now move inside these classes to find the subgroups H_j corresponding to the smallest order for the kernel of $\nu_\times$. To describe them, consider the presentation for G

$$G =< a, b : a^{12} = b^3 = (ab)^2 = a^{11}b^{-1}aba^{-1}ba^{-7} = 1 >$$

Consider

$$H_1 = sub < G : ba^3b^{-1}a^{-3} >,$$

$$H_2 = sub < G : ab >,$$

$$H_3 = sub < G : ba >,$$

subgroups of order 2 of G where H_2 and H_3 are conjugates and H_1 is the unique subgroup in its conjugacy class.

Then an effective set S for G is $\{H_1, H_2, H_3\}$ Define $B_{H_i} = \mathrm{Im}(p_{H_i})$, for $i = 1..3$. Then $|\mathrm{Ker}(\nu_\times)| = 4$. This due to

$$\rho_s(p_{H_1}) = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$$\rho_s(p_{H_2}) = \begin{pmatrix} 0 & 0 & -1/2 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & -1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\rho_s(p_{H_3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 & 1/2 \\ -1/2 & 0 & -1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 & -1/2 & 1/2 \end{pmatrix}$$

Then the coordinate matrices of the lattices are

$$\Lambda_{H_1} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Lambda_{H_2} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Lambda_{H_3} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

and the matrix coordinate of the lattice of the product $B_{31} \times B_{71} \times B_{72}$ is given by

$$L_{\{H_1,H_2,H_3\}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Therefore the isogeny

$$\nu_\times : B_{H_1} \times B_{H_2} \times B_{H_3} \to JX$$

has kernel of order 4. Finally, using Theorem 2.2.1, we obtain the induced polarization on each factor is (2).

On the other side, the algorithm of [3] allows us to verify that the groups in Table 3 do not act on genus 3.

Moreover, working as in subsection 2, we find that the possible Riemann Matrices of JX depends on one parameter $z \in \mathbb{C}$ such that $z^4+4z^3+2z^2-4z+13=0$. Then we have a priori four Jacobians of dimension 3 with action of G with signature $(0;2,3,12)$. However, we know from [22] that only one of this solutions correspond to a Riemann matrix of JX.

The Jacobian of the trigonal curve of genus 3 is given by

$$Z = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & z \end{pmatrix}$$

where $z = -1 + \sqrt{2+2i\sqrt{3}})$, $a = 1/4z^2 + 1/2z + 1/4$, $b = -1/2z + 1/2$, $c = 1/8z^3 + 1/8z^2 - 5/8z + 3/8$, $d = 1/4z^3 + 1/2z^2 + 1/4z + 1$ y $e = 1/8z^3 + 1/8z^2 + 3/8z + 3/8$.

B.3. Dimension 4

We divide this subsection in two parts.

B.3.1. Non-normal trigonal curves. For this curves the groups acting are $G_1 = D_3 \times D_3$ and $G_2 = (C_3 \times C_3) \rtimes D_4$.

We start the proof of this case with trigonal curves admitting a G_1-action. We call X_1 to the curves with action of this group, which is with signature $(0;3,2,2,2)$. This curves form a one dimensional family with planar model

$$X_1 = \{(x,y) \in \mathbb{C}^2 : ax^3y^3 - (x^3+y^3) + a = 0\}$$

where $a \notin \{0, \pm 1, \infty\}$. As we explained in Remark 3.2.3, this family contains the surface with action of G_2.

We determine as in the previous cases that the Jacobian variety of a curve X_1 in this family is completely decomposable. In fact the group algebra decomposition for its Jacobian is

$$\tilde{\nu} : B_1^2 \times B_2^2 \sim JX_1,$$

where B_1, B_2 are elliptic curves with the action of the irreducible rational representations V_5 and V_7 of G_1, both of degree 2 with character shown in Table 3).

We follow the procedure as before. We search four subgroups H_1, H_2, H_3 and H_4 of G satisfying the conditions in the Lemma 2.1.1.

We determine the dimension of the V_i^H for all the subgroups and all the representations, although we are interested on V_5 and V_7. As before this is equivalent to compute the complex irreducible representation decomposition of the induced representation $\mathrm{Ind}_H^{G_1} 1_H$ in G_1. This information is in Table 4.

Class	1	2	3	4	5	6	7	8	9
Size	1	3	3	9	2	2	4	6	6
Order	1	2	2	2	3	3	3	6	6
V_1	1	1	1	1	1	1	1	1	1
V_2	1	-1	1	-1	1	1	1	-1	1
V_3	1	-1	-1	1	1	1	1	-1	-1
V_4	1	1	-1	-1	1	1	1	1	-1
V_5	2	-2	0	0	-1	2	-1	1	0
V_6	2	2	0	0	-1	2	-1	-1	0
V_7	2	0	-2	0	2	-1	-1	0	1
V_8	2	0	2	0	2	-1	-1	0	-1
V_9	4	0	0	0	-2	-2	1	0	0

TABLE 3. Character table of $G_1 = D_3 \times D_3$.

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
Identity element	1	1	1	1	2	2	2	2	4
Order 2, length 3**	1	0	0	1	0	2	1	1	2
Order 2, length 3*	1	1	0	0	1	1	0	2	2
Order 2, length 9	1	0	1	0	1	1	1	1	2
Order 3, length 1	1	1	1	1	0	0	2	2	0
Order 3, length 1	1	1	1	1	2	2	0	0	0
Order 3, length 2	1	1	1	1	0	0	0	0	2
Order 4, length 9	1	0	0	0	0	1	0	1	1
Order 6, length 1	1	0	0	1	0	2	0	0	0
Order 6, length 1	1	1	0	0	0	0	0	2	0
Order 6, length 3*	1	0	1	0	1	1	0	0	0
Order 6, length 3**	1	0	0	1	0	0	1	1	0
Order 6, length 3*	1	1	0	0	1	1	0	0	0
Order 6, length 3**	1	0	1	0	0	0	1	1	0
Order 6, length 6	1	0	1	0	0	0	0	0	1
Order 9, length 1	1	1	1	0	0	0	0	0	0
Order 12, length 3	1	0	0	0	0	1	0	0	0
Order 12, length 3	1	0	0	0	0	0	0	1	0
Order 18, length 1	1	1	0	0	0	0	0	0	0
Order 18, length 1	1	0	0	1	0	0	0	0	0
Order 18, length 1	1	0	1	0	0	0	0	0	0
Order 36, length 1	1	0	0	0	0	0	0	0	0

TABLE 4. Decomposition of $\mathrm{Ind}_H^G 1_H$.

We observe that the classes of subgroups of G_1 satisfying the conditions of Lemma 2.1.1 for the representations we need are

(1) For V_5: the classes of subgroups of order 2, 6 and 6 , all of them of length 3, marked with *.
(2) For V_7: the clases of subgroups of order 2, 6, 6, all of them of length 3, marked with **.

To give a description of the subgroups of G_1 which allow us to find the isogeny $\nu_\times$ with smallest kernel, we consider the following presentation of G_1.

$G_1 =< a, b, c | a^3 = b^2 = c^2 = (abc)^2 = a^2ca^{-1}bca^{-1}b^{-1}a^{-2} = a^2ca^{-1}bab^{-1}ac^{-1}a^{-1} = 1 >$

Consider

$$\begin{aligned} H_1 &= < c > \\ H_2 &= < cb, (ab)^2 > \\ H_3 &= < a^{-1}bab, c > \\ H_4 &= < aba^{-1} > \end{aligned}$$

where H_1, H_4 are subgroups of order 2 of G_1 in different conjugacy classes, and H_2, H_3 are subgroups of order 6 in different conjugacy classes.

If we write $B_{51} = \mathrm{Im}(p_{H_1})$, $B_{52} = \mathrm{Im}(p_{H_2})$, $B_{71} = \mathrm{Im}(p_{H_3})$ and $B_{72} = \mathrm{Im}(p_{H_4})$ then $|\mathrm{Ker}(\nu)| = 9$. This is because

$$\rho_s(p_{H_1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \end{pmatrix}$$

$$\rho_s(p_{H_2}) = \begin{pmatrix} 1/6 & 1/2 & 2/3 & -1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 3/2 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1/6 & -1/2 & -2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 1/2 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 3/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 2 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & -1/3 & -1 & 1/3 & 0 \end{pmatrix}$$

$$\rho_s(p_{H_3}) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_s(p_{H_4}) = \begin{pmatrix} 1/3 & 0 & 1/3 & -1/6 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1/2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2/3 & -1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & -1/6 & 1/2 & -1/3 & 0 \end{pmatrix}$$

Then the coordinate matrices of the lattices are

$$\Lambda_{H_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & -1 \end{pmatrix}$$

$$\Lambda_{H_2} = \begin{pmatrix} 1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 4 & -2 \end{pmatrix}$$

$$\Lambda_{H_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_{H_4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix coordinate of the lattice of the product $B_{51} \times B_{52} \times B_{71} \times B_{72}$ is

$$L_{\{H_1,\dots,H_4\}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & -1 \\ 1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 4 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We obtain the polarizations on each factor using Theorem 2.2.1, they are (2), (6), (2) and (6), respectively.

Consider now trigonal curves with G_2-action. An affine model [27] for the unique surface X_2 of genus 4 with action of G_2 is

$$X_2 := \{y^3 = (x^3-1)(x^3+1)^2\}.$$

Class	1	2	3	4	5	6	7	8	9
Size	1	6	6	9	4	4	18	12	12
Order	1	2	2	2	3	3	4	6	6
V_1	1	1	1	1	1	1	1	1	1
V_2	1	-1	1	1	1	1	-1	-1	1
V_3	1	-1	-1	1	1	1	1	-1	-1
V_4	1	1	-1	1	1	1	-1	1	-1
V_5	2	0	0	-2	2	2	0	0	0
V_6	4	0	2	0	1	-2	0	0	-1
V_7	4	0	-2	0	1	-2	0	0	1
V_8	4	2	0	0	-2	1	0	-1	0
V_9	4	-2	0	0	-2	1	0	1	0

TABLE 5. Character table of $G_2 = (C_3 \times C_3) \rtimes D_4$.

The Jacobian variety of X_2 with this action is completely decomposable

$$\tilde{\nu} : B^4 \sim JX_2,$$

where B is a complex elliptic curve with action of one of the irreducible rational representations of degree 4 of G_2, V_9 in Table 5).

Proceeding as before we compute the dimension of the fixed subspaces, these are contained in Table 6.

As the dimension of the subvarieties corresponding to representations different from V_9 are 0, any class of subgroups with 1 in the column of V_9 in Table 6 will contain subgroups to be taken to construct the effective set.

We consider the class of subgroups of order 2 which has length 6. Then the subgroups H_1, H_2, H_3 and H_4 of G_2 will be in this class. We obtain

$$\nu_\times : JX/H_1 \times JX/H_2 \times JX/H_3 \times JX/H_4 \sim JX,$$

where its kernel depends on the choice of the subgroups H_j.

To give a description of the subgroups of G_2 which allow us to find the isogeny $\nu_\times$ yielding in a kernel with smallest order, we consider the following presentation of G_2.

$$G_2 =< a, b : a^6 = b^4 = (ab)^2 = a^5ba^{-2}bab^{-1}a^2b^{-1}a^{-4} = 1 >$$

If we consider any subgroup H_1, H_2, H_3, H_4 in the conjugacy class of

$$H =< ab >$$

and we write $B_{91} = \text{Im}(p_{H_1})$, $B_{92} = \text{Im}(p_{H_2})$, $B_{93} = \text{Im}(p_{H_3})$ and $B_{94} = \text{Im}(p_{H_4})$ then $|Ker(\nu_\times)| = 9$ which is the smallest possible order. In this case, we obtain the same

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
Identity element	1	1	1	1	2	4	4	4	4
Order 2, length 6	1	1	0	0	1	3	1	2	2
Order 2, length 6	1	0	0	1	1	2	2	3	1
Order 2, length 9	1	1	1	1	0	2	2	2	2
Order 3, length 2	1	1	1	1	2	2	2	0	0
Order 3, length 2	1	1	1	1	2	0	0	2	2
Order 4, length 9	1	0	0	1	0	1	1	2	0
Order 4, length 9	1	0	1	0	0	1	1	1	1
Order 4, length 9	1	1	0	0	0	2	0	1	1
Order 6, length 2	1	0	0	1	1	0	0	2	0
Order 6, length 2	1	1	0	0	1	2	0	0	0
Order 6, length 6	1	1	0	0	1	1	1	0	0
Order 6, length 6	1	0	0	1	1	0	0	1	1
Order 6, length 6	1	1	1	1	0	0	0	1	1
Order 6, length 6	1	1	1	1	0	1	1	0	0
Order 9, length 1	1	1	1	1	2	0	0	0	0
Order 8, length 9	1	0	0	0	0	1	0	1	0
Order 12, length 6	1	1	0	0	0	1	0	0	0
Order 12, length 6	1	0	0	1	1	0	0	0	0
Order 18, length 1	1	1	1	1	0	0	0	0	0
Order 18, length 2	1	0	0	1	1	0	0	0	0
Order 18, length 2	1	1	0	0	1	0	0	0	0
Order 36, length 1	1	1	0	0	0	0	0	0	0
Order 36, length 1	1	0	1	0	0	0	0	0	0
Order 36, length 1	0	0	1	0	0	0	0	0	0
Order 72, length 1	0	0	0	0	0	0	0	0	0

TABLE 6. Decomposition of $\mathrm{Ind}_H^{G_2} 1_H$ in complex irreducible representations.

order of the kernel of $\nu_\times$ considering any of the subgroups in the conjugacy class of H.

If we take, for example, the following image by ρ_s of idempotents elements $p_{H'}$ with H' some element in the conjugacy class of H, we obtain

$$\rho_s(p_{H_1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 & -1/2 \\ 0 & 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$$\rho_s(p_{H_2}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -1/2 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_s(H_3) = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_s(p_{H_4}) = \begin{pmatrix} 1 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and then the matrix coordinate of the lattice of the product $B_{91} \times B_{92} \times B_{93} \times B_{94}$ is

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size	1	3	6	1	1	8	8	8	6	3	3	6	6	6	6
V_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
V_2	1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	-1	-1
V_3	1	1	-1	-1-J	J	1	-1-J	J	-1	-1-J	J	1+J	-J	1+J	-J
V_4	1	1	1	-1-J	J	1	-1-J	J	1	-1-J	J	-1-J	J	-1-J	J
V_5	1	1	1	J	-1-J	1	J	-1-J	1	J	-1-J	J	-1-J	J	-1-J
V_6	1	1	-1	J	-1-J	1	J	-1-J	-1	J	-1-J	-J	1+J	-J	1+J
V_7	2	2	0	2	2	-1	-1	-1	0	2	2	0	0	0	0
V_8	2	2	0	-2-2J	2J	-1	1+J	-J	0	-2-2J	2J	0	0	0	0
V_9	2	2	0	2J	-2-2J	-1	-J	1+J	0	2J	-2-2J	0	0	0	0
V_10	3	-1	-1	3	3	0	0	0	1	-1	-1	-1	-1	1	1
V_11	3	-1	1	3	3	0	0	0	-1	-1	-1	1	1	-1	-1
V_{12}	3	-1	-1	-3-3J	3J	0	0	0	1	1+J	-J	1+J	-J	-1-J	J
V_{13}	3	-1	-1	3J	-3-3J	0	0	0	1	-J	1+J	-J	1+J	J	-1-J
V_{14}	3	-1	1	-3-3J	3J	0	0	0	-1	1+J	-J	-1-J	J	1+J	-J
V_{15}	3	-1	1	3J	-3-3J	0	0	0	-1	-J	1+J	J	-1-J	-J	1+J

TABLE 7. Character table of $G(72,42)$, where $J = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

$$L_{\{H_1,\dots,H_4\}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 & 0 & 0 \end{pmatrix}$$

We obtain the polarizations of the factors using Theorem 2.2.1, all of the are (2).

B.3.2. Normal trigonal curves with reduced group A_4, S_4 and A_5. Now we continue with the study of trigonal curves with full group of automorphisms $G_3 = C_3 \times S_4$.

Let X be a surface of genus 4 with action of G_3, it acts with signature $(0; 12, 3, 2)$. In [27] was determined a planar model for this curve

$$X = \{(x, y) : y^3 = x^5 - x\}.$$

Using the same process followed in the above subsections, we obtain that

$$B_1 \times B_2^3 \sim JX,$$

where B_1, B_2 are elliptic curves with action of one of the irreducible rational representations of degree 1 and 3; V_3 and V_{12} in Table 7.

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}	V_{15}
order 2, length 6**	1	0	0	1	1	0	1	1	1	1	2	1	1	2	2
order 3, length 8**	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
order 4, length 1*	1	1	1	1	1	1	2	2	2	0	0	0	0	0	0
order 12, length 1*	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

TABLE 8. Decomposition of $\mathrm{Ind}_H^{G_3} 1_H$ in complex irreducible representations for G_3.

Proceeding as before, we consider four subgroups H_1, H_2, H_3 and H_4 of G_3 such that they satisfy the conditions of Lemma 2.1.1.

In the same way that in the previous subsection, we observe that there are two classes (*) of subgroups of G_3 satisfying the desired conditions for V_3, and two classes of subgroups (**) of G_3 satisfying them for V_{12}. We show just these classes in Table 8, for the sake of space (there are 24 conjugacy classes of subgroups in $G(72, 42)$).

To give a description of the subgroups of G_3 which allow us to find $\nu_\times$ with smallest kernel, we consider the following presentation of G_3.

$$G_3 =< a, b | a^{12} = b^3 = (ab)^2 = a^{11}ba^{-1}bab^{-1}ab^{-1}a^{-6} = 1 >$$

Consider

$$\begin{aligned}
H_1 &= < ba^4ba^3b^2a^3ba^2b^2(aba^4)^2ba, \\
&\quad (a^2b^{-1})^3, a^{-2}b^{-1}a^{-8}b^{-1}a^{-4}b^{-1} > \\
H_2 &= < (a^2b^{-1})^3, ba^4ba^3b^2a^3ba^2b^2a > \\
H_3 &= < (a^2b^{-1})^3, ba^4ba^3b^2a^3ba^2b^2a(ba^4)^2 > \\
H_4 &= < (a^2b^{-1})^3, ba^4ba^5ba^{-4}b^{-1}a^{-4}b^{-1} >
\end{aligned}$$

Where the last 3 subgroups, which are of order 2, belong to the same class of conjugation. The first group is a normal subgroup of order 4.

We write $B_{31} = \mathrm{Im}(p_{H_1})$, $B_{121} = \mathrm{Im}(p_{H_2})$, $B_{122} = \mathrm{Im}(p_{H_3})$ and $B_{123} = \mathrm{Im}(p_{H_4})$ then $|Ker(\nu_\times)| = 16$.

The image of the p_H's is given by

$$\rho_s(p_{H_1}) = \begin{pmatrix} 1/4 & 1/2 & 1/2 & -1/4 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 0 & 0 & -1/4 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 0 & 0 & -1/4 \\ -1/4 & -1/2 & -1/2 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 0 & 0 & -1/4 \\ -1/4 & 0 & 0 & 1/4 & 1/2 & 1/4 & 1/4 & -1/2 \\ -1/4 & 0 & 0 & 1/4 & 1/2 & 1/4 & 1/4 & -1/2 \\ 0 & -1/4 & -1/4 & 0 & -1/4 & 0 & 0 & 1/4 \end{pmatrix}$$

$$\rho_s(p_{H_2}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & 1/2 & 0 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \end{pmatrix}$$

$$\rho_s(p_{H_3}) = \begin{pmatrix} 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\rho_s(p_{H_4}) = \begin{pmatrix} 0 & -1/2 & 0 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 1/2 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 0 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 1/2 \end{pmatrix}$$

We calculate the coordinate matrices of the lattices Λ_{H_i} in the same way that we did before.

Then, the coordinate matrix of the lattice of the product $B_{31} \times B_{121} \times B_{122} \times B_{123}$ is

$$L_{\{H_1,\dots,H_4\}} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We obtain the polarizations of the factors B_{31} and B_{12j} using the Theorem 2.2.1. They are all equal to (4).

Moreover, we have that the Riemann Matrix for JX depends on one parameter $z \in \mathbb{C}$ such that $-z^4 + 37/14z^3 - 117/28z^2 + 89/28z + 43/28 = 0$. Then we have a priori four abelian varieties of dimension 3 with action of G_3. However, only one of this solutions define the Riemann matrix of JX.

Then the Jacobian of this unique trigonal curve of genus 3 is given by

$$Z = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & j \\ d & g & j & z \end{pmatrix}$$

where $z = 1 + \sqrt{3}i$ and

$$\begin{aligned}
a &= z - 1 \\
b &= -364/465z^3 + 316/155z^2 - 471/155z + 1049/465 \\
c &= 448/465z^3 - 222/155z^2 + 377/155z - 218/465 \\
d &= -1372/1395z^3 + 1358/465z^2 - 1823/465z + 2702/1395 \\
e &= 1064/1395z^3 - 256/465z^2 + 256/465z + 1691/1395 \\
f &= 2128/1395z^3 - 512/465z^2 + 512/465z + 1987/1395 \\
g &= 812/1395z^3 - 538/465z^2 + 538/465z - 802/1395 \\
h &= 56/279z^3 - 82/93z^2 - 82/93z + 190/279 \\
i &= 1372/1395z^3 - 1358/465z^2 + 1823/465z - 2702/1395
\end{aligned}$$

B.4. Dimension $g \geq 5$

Due to Corollary 3.2.2 any trigonal curve with genus $g \geq 5$ is normal. As we said before, we restrict to study normal trigonal curves with reduced group A_4, S_4 and A_5 up to genus 10. As said in Remark 3 any trigonal curve with reduced group A_4, S_4 and A_5 has even genus. Then, to finish our goal of studying Jacobians of trigonal curves up to genus 10, it remains to study trigonal curves of genus 6 because there are not these kind of actions in genus 8, as explained in Remark 3.

Class	1	2	3	4	5	6	7	8	9
Size	1	3	18	2	18	6	8	8	8
Order	1	2	2	3	4	6	9	9	9
V_1	1	1	1	1	1	1	1	1	1
V_2	1	1	-1	1	-1	1	1	1	1
V_3	2	2	0	2	0	2	-1	-1	-1
V_4	2	2	0	-1	0	-1	Z1	$Z1^2$	$Z1^4$
V_5	2	2	0	-1	0	-1	$Z1^2$	$Z1^4$	Z1
V_6	2	2	0	-1	0	-1	$Z1^4$	Z1	$Z1^2$
V_7	3	-1	1	3	-1	-1	0	0	0
V_8	3	-1	-1	3	1	-1	0	0	0
V_9	6	-2	0	-3	0	1	0	0	0

TABLE 9. Character table of G_4. Where $Z1 = 2\cos(\frac{8\pi}{9})$.

Classes of subgroups	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
order 4, length 9	1	0	1	1	1	1	1	0	1
order 4, length 9	1	0	1	1	1	1	0	1	1

TABLE 10. Decomposition of $\mathrm{Ind}_H^{G_4} 1_H$ in complex irreducible representations.

B.4.1. Dimension 6. Now we continue with the study of trigonal curves of genus 6 which have the group $G_4 = ((C_2 \times C_2) \rtimes C_9) \rtimes S_2)$ as their full group of automorphism. Let X be any of these curves, the group G_4 acts on X with signature $(0; 9, 4, 2)$.

The Jacobian variety of X with such action is completely decomposable. Moreover,

$$\tilde{\nu} : B^6 \sim JX,$$

where B is a complex elliptic curve with the action of the irreducible rational representation of degree 6 of G_4; V_9 in Table 9.

Proceeding as before we consider six subgroups $H_1, ..., H_5$ and H_6 of G such that they satisfy the conditions of the Lemma 2.1.1. To do that, we use the symplectic representation given by [**3**].

Now, we write the decomposition of the induced representation by the trivial one for every conjugacy class of subgroup in G_4 satisfying the conditions of Lemma 2.1.1.

We observe these are given by two classes of subgroups of order 4 and length 9. Then the subgroups $H_1, ..., H_6$ of G must be in these classes.

To give a description of the subgroups of G which allow us to find the isogeny $\nu_\times$ with kernel of smallest order, we consider the following presentation of G_4.

$$G(72, 15) =< a, b, c, d, e | a^2 = b^9 = (ab)^4 = ab^3ab^3 = 1 >$$

Consider

$$\begin{aligned} H_1 &= \langle ab, c \rangle \\ H_2 &= \langle ad^{-1}bd, ce \rangle \\ H_3 &= \langle dce^{-1}, ad^{-1}b \rangle \\ H_4 &= \langle d^{-1}ce^{-1}, abd \rangle \\ H_5 &= \langle d^{-1}ce^{-1}, abd \rangle \\ H_6 &= \langle d^{-1}ce^{-1}, abd \rangle \end{aligned}$$

The first three are conjugated among each other.

If we write $B_{9j} = \mathrm{Im}(p_{H_j})$ for $l \in \{1, ..., 6\}$ then $|Ker(\nu_\times)| = 64$.

This because the matrix coordinate of the lattice of the product $B_{91} \times B_{92} \times B_{93} \times B_{94} \times B_{95} \times B_{96}$ is

$$L_{\{H_1,\dots,H_6\}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & -1 & 2 & 2 & -2 & 2 & 0 & 2 \\ 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & -2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 2 & -2 & -2 \end{pmatrix}$$

which has determinant equal to 64.

We obtain the polarizations of the factors using the Theorem 2.2.1. They are all equal to (4). We complete in this way the proof of Theorem 3.3.1.

APPENDIX C

Help on MAGMA commands.

To perform some of the computations needed in the applications, we use the software MAGMA [**6**]. We give in Table 1 a list of the groups we study with the identifier in the Small Group Magma Data Base.

In the following we introduce some of the notations that we use to run our method in MAGMA.

- **SmallGroup** Used for denote the group that we want to use. For example, if $g = C_3 \times A_4$ we write $g :=$ SmallGroup$(36, 11)$. MAGMA give us the order of the generators of g and the non-trivial relations of them in g.
- **PermutationRepresentation** Used to convert the group g in a permutation group. We write
  G:=PermutationRepresentation(g);
  In the case of $g = C_3 \times A_4$, we have
  G;
  Permutation group G acting on a set of cardinality 12
  (2, 4, 3)(6, 8, 7)(10, 12, 11)
  (1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8)
  (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)
  (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)
- **Subgroups** Give the complete list of the conjugation classes of a group. We write

Group	ID
$GL(2,3)$	(48,29)
$SL(2,3)/CD$	(48,33)
$D_3 \times D_3$	(36,10)
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes D_4$	(72,40)
$C_3 \times A_4$	(36,11)
$(C_2 \times C_2) \rtimes C_9$	(36,3)
$C_3 \times S_4$	(72,42)
$((C_2 \times C_2) \rtimes C_9) \rtimes C_2$	(72,15)
$C_3 \times A_5$	(180,19)

TABLE 1. Groups and their respective MAGMA ID.

s:=Subgroups(G);

- **Setseq(Class())**We use this command to get a complete list of each class of subgroups of G. We write
  s1:=Setseq(Class(G,rs[1]));
  where rs[1] is the list of all subgroups in the first conjugation class of G and rs:=[x'subgroup:x in s];
- **Group<>** Used to write the presentation of a group. For example, in the case of the group $G = C_3 \times A_4$ we write
  G:=Group$< a, b|a^6 = b^3 = (a*b)^3 = b*a^2*b^-1*a^4 = 1 >$;
- **IdentifyGroup** We use this command to identify the MAGMA ID of a group given by its presentation. In MAGMA we put
  IdentifyGroup(G);
  As before, if $G = C_3 \times A_4$ we receive
  $< 36, 11 >$;