Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile
Marzo 2013, Santiago, Chile

REGULARITY AND QUALITATIVE PROPERTIES FOR SOLUTIONS OF SOME EVOLUTION EQUATIONS

Author : Juan Carlos Pozo Vera

ADVISOR 1 : VERÓNICA POBLETE OVIEDO

ADVISOR 2 : CARLOS LIZAMA YAÑEZ

Thesis submitted to the Department of Mathematics of Universidad de Chile in partial fulfillment of requirements for the degree of Doctor in Mathematic

Author's e-mail: jpozo@ug.uchile.cl

FACULTAD DE CIENCIAS
UNIVERSIDAD DE CHILE

INFORME DE APROBACIÓN
TESIS DE DOCTORADO

Se informa a la escuela de Postgrado de la Facultad de Ciencias que la Tesis de Doctorado presentada por el candidato

Juan Carlos Pozo Vera

ha sido aprobada por la Comisión de Evaluación de la tesis como requisito parcial para optar al grado de Doctor en Ciencias con mención en Matemática en el examen de Defensa de Tesis rendido el día 8 de Marzo de 2013.

Director De Tesis
Dra. Verónica Poblete Oviedo. ....

Codirector De Tesis
Dr. Carlos Lizama Yañez. ....

Comisión de Evaluación de Tesis

Dr. Manuel Pinto.
Presidente de la Comisíon ....

Dr. Hernán Henríquez. ....

Dr. Claudio Cuevas. ....

Biografía

Nací el 24 de Octubre del año 1984 en la comuna de Independencia en la ciudad de Santiago. Desde kinder hasta sexto básico estudié en la escuela Cornelia Olivares n°18. Pese a ser una escuela modesta, tuve la suerte de tener muy buenos profesores que me motivaron a tratar a ser siempre el primero de la clase. En el año 1997 ingresé a estudiar al Instituto Miguel León Prado para cursar séptimo básico. Gracias a los profesores que tuve en este estableciemiento, especialmente Consuelo Godoy, Hector Farías y Margot Briones, asomó en mí un gusto especial por las matemáticas. De hecho, al finalizar mi Enseñaza Media en el año 2002, fui premiado por ser el mejor alumno matemático de mi generación. El año 2003 comencé a estudiar Ingeniería en la Universidad de Santiago de Chile, sin embargo, estando en esta carrera me di cuenta que quería estudiar las ciencias matemáticas con mucha más profundidad. Fue así que el año 2005 ingresé a la Universidad de Chile a estudiar Licenciatura en Ciencias con mención en Matemáticas. Y el año 2009 ingresé al programa de Doctorado de Matemáticas en la misma Universidad de Chile. Hoy, al egresar, está terminando la etapa en que soy alumno de matemáticas, no obstante siempre seré un motivado estudiante de esta hermosa ciencia, en la cuál he encontrado mi vocación...

Agradecimientos

Quiero expresar mi más sincera gratitud a todas aquellos que de una u otra manera influyeron en la realización de este trabajo.

En primer lugar, mis agradecimientos están dirigidos a MECESUP, en particular al proyecto PUC 0711, por confiar en mí y haber financiado mis estudios durante mi permanencia en el programa de doctorado, sin este apoyo hubiese sido bastante difícil terminar. También estoy enormemente agradecido del proyecto FONDECYT 1110090 por financiar mi asistencia a varias conferencias y congresos.

Agradezco a la doctora Verónica Poblete, quien no sólo me introdujo en el tema de Ecuaciones de Evolución y aceptó ser mi tutora de tesis, sino que también tuvo la generosidad de compartir sus conocimientos y provechosas opiniones en muchas horas de charla y motivantes discusiones. Además me incentivó a realizar varias estadías en el extranjero para expandir mis horizontes.

Agradezco a los doctores Carlos Lizama y Hernán Henríquez, por todos los valiosos comentarios e importantes sugerencias que hicieron al momento de trabajar con ellos. Sus oportunas y acertadas críticas, siempre desde una perspectiva positiva y constructiva, permitieron que mejorara sustancialmente mi trabajo.

Deseo agradecer a todo el personal, tanto académico como administrativo, del departamento de Matemática de la Facultad de Ciencias de la Universidad de Chile, por entregarme todas las herramientas necesarias para afrontar de manera existosa un trabajo de investigación como lo es una tesis doctoral.

Siempre es importante recibir el apoyo de tus compañeros de trabajo más cercanos y en ese sentido yo he tenido la suerte de haber formadodo parte de un departamento donde existe un maravilloso clima de compañerismo y amistad. Gracias Luchín, Alán, Pablo, Pato, Leslie, Natu, Eugenio, por nombrar solo unos pocos con quienes compartí.

Finalmente he reservado unas palabras a lo más importante que tengo en la vida, mi familia. Quiero agradecer profundamente a mis padres, Juana Vera y Carlos Pozo, y a mi hermano Eduardo Pozo, a quiénes durante todo este tiempo he sentido junto a mí apoyándome, ya sea estando a unos pocos metros o a miles de kilómetros de distancia. Con todo mi corazón le dedico esta tesis a mi amorcito Sofía, quien ha estado junto a mí en esta larga travesía, ha compartido mis ilusiones, disfrutado de mis pequeños logros y sufrido con mis tropiezos. A ella le debo haber tenido que soportarme todo este tiempo con infinita paciencia.

Contents

Resumen

En muchos casos una ecuación diferencial parcial puede ser reescrita como una ecuación diferencial ordinaria tomando valores en un espacio vectorial de dimensión infinita. Esto motiva el estudio de ecuaciones de evolución definidas sobre espacios abstractos, en especial sobre espacios de Banach. Esta tesis forma parte de esta teoría. De hecho, nosotros investigamos existencia, unicidad y propiedades cualitativas de las soluciones de algunas ecuaciones diferenciales abstractas con valores en un espacio de Banach.

En relación a las ecuaciones diferenciales en general, uno de los principales aspectos que se debe estudiar es la existencia de soluciones. Por este motivo, nosotros establecemos condiciones suficientes que garantizan la existencia de soluciones mild para dos ecuaciones de evolución. Específicamente, estudiamos la existencia de soluciones mild de una ecución integrodiferencial y de una ecuación diferencial no autónoma de segundo orden sometidas a condiciones iniciales no locales. Abordamos estos problema usando teoría de operadores de evolución, fórmulas de varación de parametros y teoremas de punto fijo asociados al concepto de medida de no compacidad.

Además, es un hecho conocido que en relación a ecuaciones diferenciales, otro tópico importante de estudio es el comportamiento cualitativo de las soluciones. Es por esto que estudiamos existencia y unicidad de soluciones periódicas clásicas de algunas ecuaciones de evolución. En concreto, hemos considerado una ecuación diferencial abstracta de tercer orden y una ecuación neutral de orden fraccionario con retardo finito. Además, estudiamos la propiedad de regularidad maximal de estas ecuaciones en algunos espacios de funciones, como por ejemplo espacios periódicos de Lebesgue, espacios periódicos de Besov y espacios periódicos de Triebel–Lizorkin. El método que usamos para lograr nuestro cometido es una versión operador–valuada del teorema de multiplicadores de Fourier de Miklhin. En el caso de espacios periódicos de Lebesgue nuestros resultados involucran la noción de espacios UMD y el concepto de R–acotamiento de familias de operadores. Por otro lado, en los casos de espacios periódicos de Besov y Triebel–Lizorkin nuestros resultados sólo involucran acotamiento de familias de operadores y no imponemos ninguna condición adicional sobre el espacio de Banach donde las ecuaciones estén definidas.

Abstract

In many cases a partial differential equation can be rewritten as an ordinary differential equation taking values in an infinite dimensional vector space. This motivates the study of evolution equations defined in abstract spaces, especially in Banach spaces. This thesis forms part of this theory. Indeed, we investigate the existence, uniqueness and qualitative properties of the solutions of some abstract differential equations with values in Banach spaces.

Regarding to general differential equations, one of the main subject of study is related with the existence of solutions. For this reason, we establish some conditions which guarantee the existence of mild solutions for two evolution equations. Specifically, we study the existence of mild solutions for an integrodifferential equation and a second order non–autonomous differential equation submitted to nonlocal initial conditions. Our approach is based on the theory of evolution operators, variation of parameters formulas and fixed–point theorems associated with the concept of measure of noncompactness.

Furthermore, it is a well known fact that concerning to differential equations, another important topic of research is the study of qualitative properties of their solutions. Motivated by this, we study the existence and uniqueness of periodic strong solutions for some evolution equations. Moreover, we analyse the property of maximal regularity of these equations in periodic Lebesgue spaces, periodic Besov spaces and periodic Triebel–Lizorkin spaces. In specific, we have studied a third–order abstract differential equation and a fractional order neutral equation with finite delay. The main tool which we have used to achieve our goal is an operator–valued version of Miklhin's Fourier multiplier theorem. In the case of periodic Lebesgue spaces our results involve the concept of UMD spaces and the notion of R–boundedness for families of operators. In the case of periodic Besov spaces and periodic Triebel–Lizorkin spaces our results only involve a boundedness conditions for some families of the operators. Moreover, we do not impose any additional condition for the Banach space where the equations are defined.

Introduction

Since many natural phenomena arising from applied sciences can be described by partial differential equations or their generalizations, the study of existence and another interesting properties of the solutions for these equations is a very important and active field of research in mathematics. In many situations, a partial differential equation can be transformed into an ordinary differential equation with values in an infinite dimensional vector space. This fact motivates the study of evolution equations in abstract spaces, especially in Banach spaces. An amazing progress of several powerful techniques for the study of this type of equations has been developed in the last decades. In this thesis, we apply some of these techniques for studying the existence, uniqueness and qualitative properties of the solutions for some abstract evolution equations. Specifically, most of the material of the thesis is based on the following two methods:

- Theory of evolution operators and variation of parameters formula.
- Maximal regularity property and operator–valued Fourier multipliers.

This work is the outcome of the author's research during his Math Ph.D. study at Universidad de Chile (March 2009 – March 2013). The main results that have been obtained in this period are available through the following four articles:

1) C. Lizama and J.C. Pozo. *"Existence of mild solutions for semilinear integrodifferential equations with nonlocal initial conditions"*. Abstract and Applied Analysis, Volume 2012 (2012), Article ID 647103, 15 pages doi:10.1155/2012/647103.

2) H. R. Henríquez, V. Poblete and J.C. Pozo. *"Existence of mild solutions for a non–autonomous second order Cauchy problem with nonlocal initial conditions"*. Submitted.

3) V. Poblete and J.C. Pozo. *"Periodic solutions of an abstract third–order differential equation"*. Submitted.

4) V. Poblete and J.C. Pozo. *"Periodic solutions for a fractional order abstract neutral differential equation with finite delay"*. Submitted.

It is well known that concerning to a general differential equation, one of the most important subject of study is the existence of solutions. However, there exist several notions of solution for an evolution equation. The concept of *strong solution* or *classical solution* includes the differentiability of the involved function, so it is a demanding notion. A weaker concept of solution is the concept of *mild solution*. We remark that strong solutions are also mild solutions which satisfy additional differentiability properties. Many authors prove the existence of strong solutions by proving the existence of mild solutions and giving smoothness conditions in the initial value. The reader can see the works [27, 52, 55, 91] and references therein. Therefore, the establishment of conditions which ensure the existence of mild solutions for evolution equations is a very important problem.

The theory of evolution operators has been subject of an increasing interest in past decades, because it is a central issue for studying existence of mild solutions of abstract differential equations. Indeed, the evolution operator is applied to the corresponding inhomogeneous equation to derive various variation of parameters formula. In this direction, we mention the works made by Agarwal, Cuevas and Dos Santos [2], de Andrade and Lizama [32], Grimmer and Prüss [45] Lizama and N'Guérékata [80] and Prüss [97]. Moreover, there exist several methods for proving existence theorems for the resolvent operators, for example operational calculus in Hilbert spaces, perturbation arguments, and Laplace transform method. For more information see [98].

Another important subject of research in the theory of evolution equations is the study of qualitative properties of their solutions. In particular, the existence of solutions having a periodicity property has been considered by many authors, the reader can see [7, 8, 15, 18, 53, 54, 66, 67, 79, 81, 82, 95] and references therein. In the same manner, the study of some regularity properties of solutions for evolution equations has been an active topic of research. In particular, *the maximal regularity property* has received much attention in recent years, because it is an important tool for studying of a lot of problems, such as:

- Existence and uniqueness of solutions of quasi–linear partial differential equations.
- Existence and uniqueness of solutions of Volterra integral equations.
- Existence and uniqueness of solutions of neutral equations.
- Existence and uniqueness of solutions of non-autonomous evolution equations.
- Uniqueness of mild solutions of the Navier–Stokes equation.

Usually maximal regularity property is used in these applications to reduce, for example, a non–autonomous or a nonlinear problem via a fixed–point argument to an autonomous or, respectively, a linear problem. In some cases, maximal regularity is needed to apply an implicit function theorem. (See [30]).

Several techniques are used to study the property of maximal regularity for evolution equations. One of these is the concept of Fourier multipliers or symbols. There exists an extensive literature about operator–valued Fourier theorems and concrete applications. The reader can see the works made by Amann [3], Arendt, Batty and Bu [5], Arendt and Bu [7, 8], Bu [16], Bu and Fang [17, 18], Bu and Kim [19, 20], Denk, Hieber and Prüss [34], Girardi and Weis [41, 42], Kalton and Lancien [62], Keyantuo, Lizama and Poblete [67], Lizama [79], Poblete [94, 95] and references therein.

This thesis consists in the study of four problems, two of them are related with the existence of mild solutions for an integrodifferential equation and a non–autonomous second order equation submitted to nonlocal conditions; the other two problems are associated with the existence and uniqueness of strong periodic solutions for a third order differential equation and an abstract fractional neutral equation having the maximal regularity property. In what follows, we will describe briefly each chapter included in this work.

The chapter 1 contains most of the notation which we have used. Furthermore, for reader's convenience we have summarized some relevant concepts and important theorems related to general evolution equation, as well as preliminary results and some background material which we have needed to establish our main results.

In chapter 2, we study the following problem. Find conditions that ensure the existence of a mild solution for the semi–linear integrodifferential equation with nonlocal initial conditions

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad t \in [0,1], \\ u(0) &= g(u). \end{aligned}\right\} \tag{1}$$

where $A : D(A) \subseteq X \to X$ and for every $t \in [0,1]$ the mappings $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0,1]$, and the functions $f : [0,1] \times X \to X$ and $g : C([0,1];X) \to X$ are X–valued functions which satisfy appropriate conditions, which we will describe later. Here $C([0,1];X)$ denotes the space of all continuous functions from $[0,1]$ to X, endowed with the norm of uniform convergence.

The initial condition involved in the equation (1) is known in the specialized literature as *nonlocal initial condition*. The evolution equations submitted to nonlocal initial conditions are more accurate for describing natural phenomena than the classical initial value problems, because additional information is taken into account as initial data. For this reason, in recent decades there has been a lot of interest in this type of problems and their applications. The interested reader can see [33, 90, 109] and the references cited therein.

The first investigations in this area were made by Byszewski [23, 24, 25]. Thenceforth, many authors have worked in evolution equations with this type of initial conditions, the reader can see [27, 57, 60, 73, 116, 117] for abstract results and concrete applications.

The classical initial valued problem (1), that is $u(0) = u_0$ for some $u_0 \in X$, has been the subject of many research papers in recent years, because it has many applications in different fields such as thermodynamics, electrodynamics, continuum mechanics, heat conduction in materials with memory, among others, see [98]. Therefore, the study of the existence and another properties of solutions for the equation (1) is a very interesting research problem.

The main tool that we have used for resolve this problem is the theory resolvent operators or evolution operators. We achieve our goal by employing a mixed method. Specifically, we have combined the existence of a family of operators denoted by $\{T(t)\}_{t\in[0,1]}$ and called evolution operator for the equation (1), a formula of variation of parameters and a fixed–point argument related with the concept of measure of noncompactness. Using this method, we are able to prove the existence of mild solutions of the equation (1) under conditions of compactness of the function g and continuity in operator topology of the function $t \mapsto T(t)$ for $t > 0$. Furthermore, in the particular case $B(t) = b(t)A$ for all $t \in [0,1]$, where the operator A is the infinitesimal generator of a C_0–semigroup defined in a Hilbert space $\mathcal{H}$, and the kernel b is a scalar map which satisfies appropriate hypotheses, we are able to give sufficient conditions for the existence of mild solutions only in terms of spectral properties of the operator A and regularity properties of the kernel b. We show that our abstract results can be applied to concrete situations. Indeed, we consider an example with a particular choice of the function b and the operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial\xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial\xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi),$$

where the coefficients a_1, b_1, $\overline{c}$ satisfy the usual uniform ellipticity conditions. We remark that the results of this chapter can be found in the joint work made by Lizama and Pozo [77].

The chapter 3 is devoted to study the following problem. Find sufficient conditions which guarantee the existence of mild solutions for the second order non–autonomous equation submitted to nonlocal initial conditions

$$\left.\begin{array}{rcl} u''(t) & = & A(t)u(t) + f(t, N(t)(u)), \quad t \in [0, a], \\ u(0) & = & g(u), \\ u'(0) & = & h(u). \end{array}\right\} \qquad (2)$$

In the equation (2), for each $t \in [0, a]$ the map $A(t) : D(A(t)) : X \to X$ is a linear closed operator defined in a Banach space X. Moreover, we suppose that $D(A(t)) = D$ for all $t \in [0, a]$. Further, as general conditions, we always assume that $g, h, N(t) : C([0, a]; X) \to X$ are continuous maps, the function $t \mapsto N(t)(u)$ is continuous for each $u \in C([0, a]; X)$, and $f : [0, a] \times X \to X$ is a function that satisfies Carathéodory type conditions. Where $C([0, a]; X)$ denotes the space of all continuous functions from $[0, a]$ to X provided with the norm of uniform convergence.

It is a well known fact that the behavior of the first and second order differential equations is different in many aspects, see [37]. For this reason the theory of second order functional differential equations has been the object of several works in the past decades. In the autonomous case, this is $A(t) = A$ for all $t \in [0, a]$, the existence of solutions of the second order abstract Cauchy problem is strongly related with the concept of cosine functions. We refer the reader to [37, 102, 103, 104, 105, 108] for basic concepts about the theory of cosine functions.

Similarly to what happens in the autonomous case, the existence of solutions for non–autonomous second order abstract Cauchy problems corresponding to the family $\{A(t)\}_{t \in [0,a]}$ is strongly related with the existence of a family of operators depending of two parameters denoted by $\{S(t, s) : t, s \in [0, a]\}$ and called evolution operator generated by the family $\{A(t) : t \in [0, a]\}$. The interested reader can consult the works [52, 70, 71, 75, 101, 111] and the references cited therein, for more information about evolution operators. Furthermore, in the literature there are various techniques for establishing the existence of an evolution operator $\{S(t, s) : t, s \in [0, a]\}$. In the chapter 3, we will adopt the concept of evolution operator introduced by Kozak [71]. Our main results are based on the properties of this evolution operator and measure of noncompactness. We apply our results to concrete situations, specifically we study some time dependent perturbations of the wave equation submitted to nonlocal initial conditions. All these results can be found in the joint work made by Henríquez, Poblete and Pozo [51].

In chapter 4 we find a characterization of maximal regularity property in several function spaces for an abstract third–order differential equation.

Recent investigations have demonstrated that third–order differential equations can describe several models arising from very interesting natural phenomena, such as wave propagation in viscous thermally relaxing fluids, high–intensity ultrasound and flexible structures with internal damping, for example a solar cell array, or a spacecraft with flexible attachments (cf. e.g., [12, 43, 61, 89]).

Motivated by this fact, many authors have worked in abstract third–order differential equations. In particular, the following equation has been widely studied

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A u'(t) + F(t), \quad for \ t \in \mathbb{R}^+, \qquad (3)$$

where A is a closed linear operator defined in a Banach space X, the function F is a X–valued map satisfying suitable conditions, and the constants $\alpha,\beta,\gamma \in \mathbb{R}^+$. The equation (3) has been studied in many aspects, we next just mention a few of them. Cuevas and Lizama [29] have obtained a characterization of its solutions belonging to Hölder spaces $C^s(\mathbb{R};X)$. Similarly, Fernández, Lizama and Poblete [38] characterize the well–posedness of this equation in Lebesgue spaces $L^p(\mathbb{R};X)$. In addition, the same authors [39] have studied some regularity properties and qualitative behaviour of mild and strong solutions in the space $L^p(\mathbb{R}^+;X)$ whenever the underlying space X is a Hilbert space. On the other hand, De Andrade and Lizama [32] have analysed the existence of asymptotically almost periodic solutions for the semi–linear version of the equation (3).

However, the existence of periodic strong solutions of the linear version of the equation (3) has not been addressed in the existing literature. With this purpose, the chapter 4 is devoted to study the existence of periodic strong solutions for the following abstract third–order equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \quad t \in [0, 2\pi], \tag{4}$$

with periodic boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, where the operators A and B are closed linear operators defined in a Banach space X satisfying $D(A) \subseteq D(B)$, the constants $\alpha,\beta,\gamma \in \mathbb{R}^+$, and f belongs to either periodic Lebesgue spaces , or periodic Besov spaces, or periodic Triebel–Lizorkin spaces. We remark that the study of the existence of solutions for equation (4) in the particular case $A \equiv B$ is a manner to study periodic strong solutions of the equation (3).

Our approach is based on the maximal regularity property for evolution equations and operator–valued Fourier multiplier theorems. In the case of periodic Lebesgue spaces, our results involve the key notions of UMD**–spaces** and R**–boundedness** for the families of operators

$$\{kB(i\alpha k^3 + k^2 + i\gamma kB + \beta A)^{-1})\}_{k\in\mathbb{Z}} \text{ and } \{ik^3(i\alpha k^3 + k^2 + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}.$$

In the case of periodic Besov spaces or periodic Triebel–Lizorkin spaces, it is remarkable that our results only involve a boundedness condition of the preceding families.

In general, it is not simple the verification of the R–boundedness or boundedness condition of a specific family of operators, especially when two different operators are involved. However, we verify our hypotheses in the particular case $B = A^{1/2}$ where A is a sectorial operator; the scalar values α,β, and γ related with the equation (3) play a crucial role in this proof. The results of this chapter are available in the joint work Poblete and Pozo [93].

The chapter 5 is devoted to find sufficient conditions for the existence of a periodic strong solution for a fractional neutral equation with finite delay.

The fractional calculus which allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of extensive study for analyzing not only anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials. See [4, 85] and references therein), but also fractional phenomena in optimal control (see, e.g., [87, 96, 100]). As indicated in [83, 86, 100] and the related references given there, the advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. One of the emerging branches of the study is the Cauchy problems for abstract differential equations involving fractional derivatives in time. In recent decades there has been a lot of interest in this type of problems, its applications and various generalizations (cf. e.g., [9, 28, 56] and references therein). It is significant to study this class of problems, because, in this way, one is more realistic to describe the memory and hereditary properties of various materials and processes (cf. [58, 68, 87, 96]).

There are several systems of great interest in science which are modeled by partial neutral functional differential equations, see [48, 113, 114]. Many of these equations can be written as an abstract neutral functional differential equations. Additionally, as we have mentioned, it is well known that one of the most interesting topics, both from a theoretical as practical point of view, of the qualitative theory of differential equations and functional differential equations is the existence of periodic solutions. In particular, the existence of periodic solutions of abstract neutral functional differential equations has been considered in several works [40, 50, 54] and references therein.

Let $0 < \beta < \alpha \leqslant 2$. In chapter 5 we study the existence and uniqueness of strong solutions for the following fractional order neutral differential equation with finite delay

$$D^{\alpha}(u(t) - Bu(t-r)) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \quad t \in [0, 2\pi], \tag{5}$$

where the fractional derivative is taken in sense of Liouville–Grünwald–Letnikov, $0 < r < 2\pi$ is a fixed number, $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ are closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. For each $t \in [0, 2\pi]$ the function u_t, given by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-2\pi, 0]$, denotes the history of the function $u(\cdot)$ at time t and $D^{\beta}u_t(\cdot)$ is defined by $D^{\beta}u_t(\cdot) = (D^{\beta}u)_t(\cdot)$. The delay operators F and G are bounded linear maps defined on an suitable space and f is an X–valued function which belongs to either periodic Besov spaces, or periodic Triebel–Lizorkin spaces.

Our approach is based on a mixed method. By proving and using the maximal regularity property on periodic Besov spaces and periodic Triebel–Lizorkin spaces of an auxiliary equation and a fixed–point argument, we demonstrate the existence and uniqueness of a periodic solution of the equation (5). Here the auxiliary equation is given by

$$D^{\alpha}u(t) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \quad t \in [0, 2\pi], \tag{6}$$

with boundary periodic conditions depending of the values of the numbers α and β. All terms in the equation (6) are defined in the same manner as in the equation (5).

Our main results involve, among other considerations, a boundedness condition for the family

$$\{(ik)^{\alpha}((ik)^{\alpha} - F_k - (ik)^{\beta}G_k - A)^{-1}\}_{k \in \mathbb{Z}}$$

and regularity properties for the families of bounded operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$, where for $k \in \mathbb{Z}$ the operators F_k and G_k are defined by

$$F_k x = F(e_k x) \text{ and } G_k x = G(e_k x), \text{ for } x \in X \text{ and } t \in [-2\pi, 0].$$

Here, for all $k \in \mathbb{Z}$ the X–valued function $e_k x$ is defined by $e_k x(t) = e^{-ikt}x$ for $t \in [-r, 0]$.

Several particular cases of the equation (6) have been studied in recent years. In fact, if $\alpha = 1$ and $F \equiv G \equiv 0$, Arendt and Bu [7, 8] have studied L^p–maximal regularity and $B^s_{p,q}$–maximal regularity, and Bu and Kim [19], have studied $F^s_{p,q}$–maximal regularity. On the other hand, Lizama [79] has obtained a characterization of the existence and uniqueness of strong L^p–solutions, and Lizama and Poblete [81] study C^s–maximal regularity of the corresponding equation on the real line. In the same manner, if $\alpha = 2$ and $\beta = 1$, Bu [15] characterizes C^s–maximal regularity on $\mathbb{R}$.

Furthermore, if $\alpha = 2$ and $\beta = 1$, Bu and Fang [18] have studied this equation simultaneosly in periodic Lebesgue spaces, periodic Besov spaces and periodic Triebel–Lizorkin spaces. Moreover, if $1 < \alpha < 2$ and $G \equiv 0$, Lizama and Poblete [82] study L^p–maximal regularity for this equation. The main results of this chapter have been established in the joint work Poblete and Pozo [92].

Chapter 1

Preliminaries

Most of the notation used throughout this work is standard. So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, integers, real and complex numbers respectively. In addition, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}_0^+ = [0,\infty)$.

In this thesis X and Y always are complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscript will be dropped when there is no danger of confusion. We denote the space of all bounded linear operators from X to Y by $\mathcal{B}(X,Y)$. In the case $X = Y$, we will write briefly $\mathcal{B}(X)$. Let A be an operator defined in X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Let X be a Banach space and $R \geqslant 0$. By $B_R[x_0, X]$ we denote the closed ball with center x_0 and radius R in the space X. When the space X is clearly determined from the context, we abbreviate this notation by $B_R[x_0]$. Similarly, $B_R(x_0, X)$ will denote the open ball with radius R and center x_0 in the space X.

Let $I = (a,b) \subseteq \mathbb{R}$ be an interval of real numbers, with $-\infty \leqslant a < b \leqslant +\infty$. For $1 \leqslant p < \infty$, we will denote by $L^p(I;X)$ the space of all (equivalent classes of) Bochner measurable functions $f : I \to X$ such that $\|f(t)\|_X^p$ is integrable in I. It is well known that this space is a Banach space with the norm

$$\|f\|_{L^p(I;X)} = \left(\int_I \|f(s)\|_X^p \, ds\right)^{\frac{1}{p}}.$$

The space $L^\infty(I;X)$ consists of all measurable functions with a finite norm

$$\|f\|_{L^\infty(I;X)} = \operatorname{ess\,sup}_{t\in I} \|f(t)\|_X.$$

We denote by $C(I;X)$ the space of all continuous functions $f : I \to X$. This space is a Banach space endowed with the norm

$$\|f\|_\infty = \sup_{t\in I} \|f(t)\|_X.$$

Let $0 < s < 1$, the Hölder continuous functions space of index s is denoted by $C^s(I;X)$ and it is defined by

$$C^s(I;X) = \left\{ f \in C(I;X) \ : \ \sup_{t,r\in I,\ t\neq r} \frac{\|f(t)-f(r)\|_X}{|t-r|^s} < \infty \right\}.$$

This space is a Banach space endowed with the norm

$$\|f\|_{C^s(I;X)} = \sup_{t\in I}\|f(t)\|_X + \sup_{t,s\in I,\ t\neq r}\frac{\|f(t)-f(r)\|_X}{|t-r|^s}.$$

We will identify $\mathbb{T}$ with the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ and we shall identify the spaces of vector or operator–valued functions defined on $[0,2\pi]$ to their periodic extensions to $\mathbb{R}$. Let $f \in L^1(\mathbb{T};X)$ and $k \in \mathbb{Z}$. We denote by $\widehat{f}(k)$ the k-th Fourier coefficient of the function f. This coefficient is defined by

$$\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}f(t)dt, \quad \text{for } k \in \mathbb{Z}.$$

1.1 Measure of noncompactness and fixed–point theorems

Let Y be a set and a function $F : Y \to Y$. We will say that F has a fixed–point in Y if there is $y_0 \in Y$ such that $F(y_0) = y_0$. The existence of solutions for non–linear differential equations is strongly related with the existence of fixed–points for some operators associated to these equations.

There exists a huge bibliography about fixed–point theorems, see [44] and references therein. Many of the most classics fixed–point theorems, for example Krasnosel'skiĭ fixed– point theorem and Leray–Schauder alternative, exploit the notion of compactness of some operators arising in the investigated equations. It is worthwhile mentioning that there are other methods which are performed exploiting some compactness conditions. Fixed–point theorems associated with the concept of measure of noncompactness conform the most important example of these methods. The reader can see [76, 99, 117] for some abstract results and applications.

In this thesis, mainly in chapter 2 and chapter 3, we apply some of the fixed–point theorems related with Hausdorff measure of noncompactness in the study of the existence of mild solutions for two evolution equations submitted to nonlocal initial conditions. With this purpose, we next include some preliminaries concerning to Hausdorff measure of noncompactness.

Definition 1.1. *Let S be a bounded subset of a metric space Y. The Hausdorff measure of noncompactness is defined by*

$$\eta(S) = \inf\Big\{\varepsilon > 0 \ :\ S \subseteq \bigcup_{i=1}^{p} B_\varepsilon(y_i, Y),\ y_i \in Y\Big\}.$$

***Remark* 1.1.** *We have summarized the most important properties of the Hausdorff measure of noncompactness, for more details see* [10]. *Let S_1, S_2 be bounded sets of a metric space Y.*

- *If $S_1 \subseteq S_2$ then $\eta(S_1) \leqslant \eta(S_2)$.*
- *$\eta(S_1) = \eta(\overline{S_1})$, where $\overline{S_1}$ denotes the closure of S_1.*
- *$\eta(S_1) = 0$ if and only if S_1 is totally bounded.*
- *$\eta(S_1 \cup S_2) = \max\{\eta(S_1), \eta(S_2)\}$.*
- *$\eta(S_1) = \eta(\overline{co}(S_1))$ where $\overline{co}(S_1)$ is the closed convex hull of S_1.*

Moreover, if Y is a normed space then the following properties hold.

- $\eta(\lambda S_1) = |\lambda|\eta(S_1)$ *with* $\lambda \in \mathbb{R}$.
- $\eta(S_1 + S_2) \leqslant \eta(S_1) + \eta(S_2)$, *where* $S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$.

The following definition is required for Theorem 1.1 and Theorem 1.2.

Definition 1.2. *Let Y be a normed space. A continuous map $F : Y \to Y$ is said to be an η–k–set contraction, for $0 < k < 1$, if for all bounded subsets S of Y, $\eta(F(S)) \leqslant k\eta(S)$, and F is said to be η–condensing if $\eta(F(S)) < \eta(S)$ for every bounded subset S of Y with $\eta(S) > 0$.*

The following two fixed–point theorems play a crucial role in the proof of our main results, specially in chapters 2 and 3. The first one was proved by Darbo [31] in 1955 for η–k–set contractions. In 1967 Sadovskiĭ [99] generalized Darbo's result to η–condensing maps. The second one is a sharpening of Sadovskiĭ's Theorem and it has been established in [76] by Guo et al.

Theorem 1.1. *Assume that S is a nonempty bounded closed and convex subset of a Banach space Y and suppose $F : S \to S$ is an η–condensing map. Then the operator F has a fixed point in S.*

Theorem 1.2. *Let S be a closed and convex subset of a complex Banach space Y, let $F : S \to S$ be a continuous operator such that $F(S)$ is bounded. For each bounded subset $C \subseteq S$, define*

$$F^1(C) = F(C) \ \text{and} \ F^n(C) = F(\overline{co}(F^{n-1}(C))), \quad n = 2, 3, \dots$$

If there exist a constant $0 \leqslant r < 1$ and $n_0 \in \mathbb{N}$ such that for each bounded subset $C \subseteq S$

$$\eta(F^{n_0}(C)) \leqslant r\eta(C)$$

then F has a fixed point in S.

1.2 $\mathcal{M}$–bounded families and n–regular sequences

In order to develop some conditions that we will need in chapter 4 and chapter 5, we introduce the following notation. Let X and Y be Banach spaces and let $\{L_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ be a family of bounded operators. For all $k \in \mathbb{Z}$ we define

$$(\Delta^0 L_k) = L_k, \quad (\Delta L_k) = (\Delta^1 L_k) = L_{k+1} - L_k$$

and for $n = 2, 3, \dots$, set

$$(\Delta^n L_k) = \Delta(\Delta^{n-1} L_k).$$

Definition 1.3. [65] *We say that a family of operators $\{L_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is $\mathcal{M}$–***bounded of order n** $(n \in \mathbb{N}_0)$ *if*

$$\sup_{0\leqslant r\leqslant n} \sup_{k\in\mathbb{Z}} \|k^r(\Delta^r L_k)\| < \infty. \tag{1.1}$$

Remark, for $j \in \mathbb{Z}$ fixed, $\sup_{0\leqslant r\leqslant n} \sup_{k\in\mathbb{Z}} \|k^r(\Delta^r L_k)\| < \infty$, if and only if $\sup_{0\leqslant r\leqslant n} \sup_{k\in\mathbb{Z}} \|k^r(\Delta^r L_{k+j})\| < \infty$. The statement follows directly from the binomial formula.

In the preceding definition when $n = 0$, the $\mathcal{M}$–boundedness of order n for $\{L_k\}_{k\in\mathbb{Z}}$ simply means that $\{L_k\}_{k\in\mathbb{Z}}$ is bounded.

When $n = 1$, this is equivalent to

$$\sup_{k\in\mathbb{Z}} \|L_k\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k\,(L_{k+1} - L_k)\| < \infty. \tag{1.2}$$

When $n = 2$, in addition to (1.2), we must have

$$\sup_{k\in\mathbb{Z}} \|k^2\,(L_{k+2} - 2L_{k+1} + L_k)\| < \infty\,. \tag{1.3}$$

When $n = 3$, in addition to (1.2) and (1.3), we must have

$$\sup_{k\in\mathbb{Z}} \|k^3\,(L_{k+3} - 3L_{k+2} + 3L_{k+1} - L_k)\| < \infty\,. \tag{1.4}$$

In the scalar case, that is, $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$, we will write $\Delta^n a_k = \Delta(\Delta^{n-1}a_k)$.

Definition 1.4. [63] *A sequence $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ is called*

a) ***1–regular*** *if the sequence $\left\{k\,\frac{(\Delta^1 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded;*

b) ***2–regular*** *if it is 1–regular and the sequence $\left\{k^2\frac{(\Delta^2 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded;*

c) ***3–regular*** *if it is 2–regular and the sequence $\left\{k^3\frac{(\Delta^3 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded.*

For useful properties and further details about n–regularity, see [67].

Remark 1.2. *Note that if the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is an 1–regular sequence then, for all $j \in \mathbb{Z}$ fixed, the sequence $\left\{k\frac{a_{k+j} - a_k}{a_{k+j}}\right\}_{k\in\mathbb{Z}}$ is bounded. In the cases $n = 2, 3$, analogous properties hold.*

1.3 Vector–valued periodic Besov and Triebel–Lizorkin spaces

Periodic Besov spaces and periodic Triebel-Lizorkin spaces form part of functions spaces which have a special interest in mathematics. They generalize a lot of important functions spaces and have many interesting properties. For example, if $0 < s < 1$, the periodic Hölder continuous functions space of index s, is a particular case of periodic Besov spaces, see [8] for more details. However, the main reason for working in these spaces is that a certain form of Mikhlin's multiplier theorem holds for operator–valued symbols defined in a general Banach space X. This is a dramatic contrast to Lebesgue spaces where the corresponding theorem merely holds for Hilbert spaces even when $p = 2$ (for more information [42]).

Let X be a Banach space. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the topology given by the seminorms $\|f\|_n = \sup_{t\in\mathbb{T}} |f^{(n)}(t)|$, where $n \in \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T};X) = \mathcal{B}(\mathcal{D}(\mathbb{T});X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. The elements of $\mathcal{D}'(\mathbb{T};X)$ are called X–valued distributions on $\mathbb{T}$. Let e_k be the function $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. For $x \in X$ and $k \in \mathbb{Z}$, we denote by $(e_k \otimes x)$ the X–valued function given by $(e_k \otimes x)(t) = e_k(t)x$. Consequently we have that $(e_k \otimes x) \in \mathcal{D}'(\mathbb{T};X)$.

In order to define the X–valued periodic Besov space, we denote by $\Phi(\mathbb{R})$ the set of all systems $\phi = \{\phi_j\}_{j\geqslant 0} \subseteq \mathcal{S}(\mathbb{R})$ such that $supp(\phi_0) \subseteq [-2, 2]$, and for all $j \in \mathbb{N}$

$$supp(\phi_j) \subseteq [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \qquad \sum_{j\geqslant 0} \phi_j(t) = 1, \; for \;\; t \in \mathbb{R},$$

and for $\alpha \in \mathbb{N} \cup \{0\}$, there is a $C_\alpha > 0$ such that $\sup\limits_{j\geqslant 0, x\in\mathbb{R}} 2^{\alpha j} \|\phi_j^{(\alpha)}(x)\| \leqslant C_\alpha$.
Where $supp(f)$ denotes the support of the function f.

Definition 1.5. [8] *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and $\phi = \{\phi_j\}_{j\geqslant 0} \in \Phi(\mathbb{R})$. The X–valued periodic Besov space is defined by*

$$B^{s,\phi}_{p,q}(\mathbb{T}; X) = \{f \in \mathcal{D}'(\mathbb{T}; X) : \|f\|_{B^{s,\phi}_{p,q}} < \infty\},$$

where

$$\|f\|_{B^{s,\phi}_{p,q}} = \Big(\sum_{j\geqslant 0} 2^{jsq} \Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k)\Big\|_p^q\Big)^{\frac{1}{q}},$$

with usual modifications when $q = \infty$. The space $B^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and different choices of $\phi \in \Phi(\mathbb{R})$ generate equivalent norms. As consequence, we will denote $\|\cdot\|_{B^{s,\phi}_{p,q}}$ simply by $\|\cdot\|_{B^s_{p,q}}$. Moreover, if $r \in \mathbb{T}$ is fixed, we say that a function $u : [r, r + 2\pi] \to X$ belongs to $B^s_{p,q}([r, r + 2\pi]; X)$ if and only if the periodic extension to $\mathbb{R}$ of the function u belongs to $B^s_{p,q}(\mathbb{T}; X)$.

We recall some important properties of these spaces:

- Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$ be fixed. The X–valued periodic space $B^s_{p,q}(\mathbb{T}; X)$ is a Banach space.
- Let $1 \leqslant p, q \leqslant \infty$ be fixed. If $s > 0$, the natural injection from $B^s_{p,q}(\mathbb{T}; X)$ into $L^p(\mathbb{T}; X)$ is a continuous linear operator.
- Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$ be fixed. For all $\varepsilon > 0$, we have that $B^{s+\varepsilon}_{p,q}(\mathbb{T}; X) \subseteq B^s_{p,q}(\mathbb{T}; X)$.
- (Lifting property) Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}, f \in \mathcal{D}'(\mathbb{T}; X)$ and $\alpha \in \mathbb{R}$ then $f \in B^s_{p,q}(\mathbb{T}; X)$ if and only if $\sum_{k\neq 0} e_k \otimes (ik)^\alpha \widehat{f}(k) \in B^{s-\alpha}_{p,q}(\mathbb{T}; X)$.

To define the X–valued periodic Triebel–Lizorkin space, we use the same notation for $\mathcal{S}(\mathbb{R})$, $\mathcal{D}(\mathbb{T})$, $\mathcal{D}'(\mathbb{T}; X)$ and $\Phi(\mathbb{R})$ as those which we used in the definition of X–valued periodic Besov spaces.

Definition 1.6. [20] *Let $\phi = \{\phi_j\}_{j\geqslant 0} \in \Phi(\mathbb{R})$ be fixed, for $1 \leqslant p,\ q \leqslant \infty$, and $s \in \mathbb{R}$. The X–valued periodic Triebel–Lizorkin space is defined by*

$$F^{s,\phi}_{p,q}(\mathbb{T}; X) = \{f \in \mathcal{D}'(\mathbb{T}; X) : \|f\|_{F^{s,\phi}_{p,q}} < \infty\},$$

where

$$\|f\|_{F^{s,\phi}_{p,q}} = \Big\|\Big(\sum_{j\geqslant 0} 2^{jsq} \Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k)\Big\|_X^q\Big)^{\frac{1}{q}}\Big\|_p,$$

with the usual modification when $q = \infty$. The space $F^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and different choices of $\phi \in \Phi(\mathbb{R})$ generate equivalent norms. Consequently, we simply denote $\|\cdot\|_{F^{s,\phi}_{p,q}}$ by $\|\cdot\|_{F^s_{p,q}}$. Moreover, if $r \in \mathbb{T}$ is fixed, we will say that the function $u : [r, r+2\pi] \to X$ belongs $F^s_{p,q}([r, r+2\pi]; X)$ if and only if the periodic extension to $\mathbb{R}$ of the function u belongs to $F^s_{p,q}(\mathbb{T}; X)$.

Note that the X–valued periodic Triebel–Lizorkin spaces have similar properties to those of X–valued periodic Besov spaces, the reader can see [20]. The following list summarizes the most elementary properties of Triebel–Lizorkin spaces.

- Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ be fixed. The X–valued periodic space $F^s_{p,q}(\mathbb{T}; X)$ is a Banach space.
- Let $1 \leqslant p, q \leqslant \infty$ be fixed. If $s > 0$, then the natural injection from $F^s_{p,q}(\mathbb{T}; X)$ into $L^p(\mathbb{T}; X)$ is a continuous linear operator.
- Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$ be fixed. For all $\varepsilon > 0$, we have that $F^{s+\varepsilon}_{p,q}(\mathbb{T}; X) \subseteq F^s_{p,q}(\mathbb{T}; X)$.
- (Lifting property) Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}, f \in \mathcal{D}'(\mathbb{T}; X)$ and $\alpha \in \mathbb{R}$ then $f \in F^s_{p,q}(\mathbb{T}; X)$ if and only if $\sum_{k\neq 0} e_k \otimes (ik)^\alpha \widehat{f}(k) \in F^{s-\alpha}_{p,q}(\mathbb{T}; X)$.

The following property is really important in the analysis of evolution equations with delay. We will apply it in chapter 5.

***Remark* 1.3.** *It is simple to verify from the definition that if $u \in B^s_{p,q}(\mathbb{T}; X)$ and $t_0 \in [0, 2\pi]$ is fixed, then the function u_{t_0} defined on $[-2\pi, 0]$ by the formula $u_{t_0}(\theta) = u(t_0 + \theta)$, is an element of the Besov space $B^s_{p,q}(\mathbb{T}; X)$, and $\|u_{t_0}\|_{B^s_{p,q}} = \|u\|_{B^s_{p,q}}$. For periodic Triebel–Lizorkin, we have a similar result.*

1.4 Operator–valued Fourier multipliers

In this section, we recall some operator–valued Fourier multipliers theorems, that we shall use for characterizing the maximal regularity property of the problems which we have studied in chapter 4 and chapter 5.

Let X be a Banach space. We denote the space consisting of all 2π–periodic and X–valued functions by $E(\mathbb{T}; X)$. The following definition will be used in this thesis with periodic Lebesgue spaces, periodic Besov spaces and periodic Triebel–Lizorkin spaces.

Definition 1.7. *Let X and Y be two Banach spaces. We say that the sequence $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X, Y)$ is an $(E(X), E(Y))$– multiplier if for each $f \in E(\mathbb{T}; X)$, there exists a function $u \in E(\mathbb{T}; Y)$ such that*

$$\widehat{u}(k) = L_k \widehat{f}(k)\,, \ \textit{for all } k \in \mathbb{Z}.$$

*In the case $X = Y$ we will abbreviate this terminology writing that $\{L_k\}_{k\in\mathbb{Z}}$ is an **E–multiplier**.*

The next theorem, proved by Arendt and Bu [7], provides a sufficient condition to guarantee when a family $\{L_k\}_{k\in\mathbb{Z}}$ is a L^p–multiplier. It is remarkable that for the demonstration of this theorem the key concepts of family of operators R–bounded and UMD–spaces are needed.

Theorem 1.3. *Let $p \in (1, \infty)$, and let X be a UMD–space. Assume that $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$. If the families of operators $\{L_k\}_{k\in\mathbb{Z}}$ and $\{k(\Delta^1 L_k)\}_{k\in\mathbb{Z}}$ are R–bounded, then $\{L_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier.*

The following theorem, proved by Arendt and Bu [8], establishes a sufficient condition ensuring if a family $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. We remark that this theorem impose stronger conditions than theorem 1.4 for family of operators $\{L_k\}_{k\in\mathbb{Z}}$, however it is valid in a general Banach space X.

Theorem 1.4. *Let $1 \leqslant p, q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$ is $\mathcal{M}$–bounded of order 2, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

The following theorem, proved by Bu and Kim [20] , establishes a sufficient condition that guarantees if a family of operators $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier. We remark, as well as in theorem 1.2, this theorem is valid for an arbitrary Banach space X, however additional conditions are imposed for the family of operators $\{L_k\}_{k\in\mathbb{Z}}$.

Theorem 1.5. *Let $1 \leqslant p, q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$ is $\mathcal{M}$–bounded of order 3, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Chapter 2

Mild solutions for an integrodifferential equation with nonlocal initial conditions

As we have already mentioned in the Introduction of this thesis, the evolution equations submitted to nonlocal initial conditions generalize the classical initial value problems. Moreover, this notion is more complete for describing nature phenomena than the classical one because additional information is taken into account. For the importance of nonlocal conditions in different fields of applied sciences see [33, 90, 109] and the references cited therein. For example, in [33] the author describes the diffusion phenomenon of a small amount of gas in a transparent tube by using a partial differential equation submitted to a nonlocal initial condition given by the formula $g(u) = \sum_{i=0}^{p} c_i u(t_i)$, where c_i, $i = 0, 1, \ldots, p$, are given constants and $0 < t_0 < t_1 < \cdots < t_p < 1$.

The earliest works related with nonlocal initial conditions were made by Byszewski [23,24,25]. In these works, using semigroup methods and the Banach fixed point theorem the author has proved the existence of mild and strong solutions for the first order Cauchy problem

$$\left.\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in [0,1], \\ u(0) &= g(u), \end{aligned}\right\} \tag{2.1}$$

where A is an operator defined in a Banach space X that generates a semigroup $\{S(t)\}_{t \geqslant 0}$, and the maps f and g are suitable X–valued functions.

Thenceforth, the equation (2.1) has been extensively studied in many works. We just mention a few of these works. Byszewski and Lakshmikantham [26] have studied the existence and uniqueness of mild solutions whenever f and g satisfy Lipschitz–type conditions. Ntouyas and Tsamatos [106] have studied this problem under conditions of compactness for the function g and the semigroup generated by A. Recently, Zhu, Song and Li [117], have treated this problem without conditions of compactness on the semigroup generated by A, or the function f.

On the other hand, the study of integrodifferential equations has been an active topic of research in recent years because it has many applications in different areas. In addition, there exists an extensive literature about integrodifferential equations with nonlocal initial conditions, (cf. e.g., [27, 57, 60, 73, 116, 117] and references therein). Our work is a contribution to this theory. Indeed, this chapter is devoted to study the existence of mild solutions for the following semi–linear integrodifferential evolution equation

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad t \in [0,1], \\ u(0) &= g(u), \end{aligned}\right\} \tag{2.2}$$

where $A : D(A) \subseteq X \to X$ and for every $t \in [0,1]$ the mappings $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0,1]$, and the functions $f : [0,1] \times X \to X$ and $g : C([0,1]; X) \to X$ are X–valued functions which satisfy appropriate conditions. In order to abbreviate the text of this chapter, henceforth we will denote by I to the interval $[0,1]$.

The initial valued version of the equation (2.2), this is $u(0) = u_0$ for some $u_0 \in X$, has been extensively studied by many researchers because has many important applications in different fields such as thermodynamics, electrodynamics, continuum mechanics, population biology, heat conduction in materials with memory, among others. For more information see [11] or [98]. For this reason the study of existence and other properties of mild solutions for the equation (2.2) is a very important problem.

2.1 Existence Results

Most of authors obtain the existence, uniqueness of solutions and well–posedness for the equation (2.2) by establishing the existence of an evolution operator $\{T(t)\}_{t\in I}$ and applying a variation of parameters formula (see [45, 97, 98]).

We next include some preliminaries concerning to the theory of the evolution operator $\{T(t)\}_{t\in I}$ for the equation (2.2).

Definition 2.1. *A family $\{T(t)\}_{t\in I}$ of bounded linear operators on X is called evolution operator for the equation* (2.2) *if the following conditions are fulfilled.*

(T1) For each $x \in X$, $T(0)x = x$ and $T(\cdot)x \in C(I; X)$.

(T2) The map $T : I \to \mathcal{B}([D(A)])$ is strongly continuous.

(T3) For each $y \in D(A)$, the function $t \mapsto T(t)y$ is continuously differentiable and

$$\begin{aligned} \frac{d}{dt}T(t)y &= AT(t)y + \int_0^t B(t-s)T(s)yds = \\ &= T(t)Ay + \int_0^t T(t-s)B(s)yds, \quad t \in I. \end{aligned} \tag{2.3}$$

In what follows we assume that there exists an evolution operator $\{T(t)\}_{t\in I}$ for the equation (2.2). In the literature there are several techniques for proving existence theorem of this evolution operator, for example operational calculus in Hilbert spaces, perturbation arguments, and Laplace transform method. For more information see [45, 97, 98].

As we have mentioned, the existence of mild solutions of the linear classical version of equation (2.2), this is

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t), \quad t \in I \\ u(0) &= u_0 \in X, \end{aligned}\right\} \tag{2.4}$$

has been studied by Grimmer and Prüss. Indeed, by assuming that the function $f \in L^1(I; X)$ they prove that the function u given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in I, \tag{2.5}$$

is a mild solution of the problem (2.4). Motivated by this result, we adopt the following concept of solution.

Definition 2.2. *A function $u \in C(I; X)$ is called a* **mild solution** *of the equation* (2.2) *if satisfies the equation*

$$u(t) = T(t)g(u) + \int_0^t T(t-s)f(s, u(s))ds, \qquad t \in I. \tag{2.6}$$

Obviously, a manner to get the existence of a mild solution of the equation (2.2) is by applying fixed–point arguments. We will use the fixed–point theorem 1.2 and theorem 1.3. These theorems are related with Hausdorff measure of noncompactness. For the rest of the thesis, mainly in chapter 2 and chapter 3, we have adopted the following consideration. Let X a Banach space and J any interval of real numbers. When we need to compare the measures of noncompactness in X and $C(J; X)$, we will use ζ to denote the Hausdorff measure of noncompactness defined in X and γ to denote the Hausdorff measure of noncompactness of $C(J; X)$.

Lemma 2.1. *Let $W \subseteq C(J; X)$. If W is bounded and equicontinuous, then the set $\overline{co}(W)$ is also bounded and equicontinuous.*

For the rest of the chapter we will use the following notation. Let W be a set of functions from J to X and $t \in J$ fixed, we denote by $W(t) = \{w(t) : w \in W\}$. The proof of the next Lemma can be found in [10].

Lemma 2.2. *Let $W \subseteq C(J; X)$ be a bounded set. Then $\zeta(W(t)) \leqslant \gamma(W)$ for all $t \in J$. Furthermore, if W is equicontinuous on J, then $\zeta(W(t))$ is continuous on J, and*

$$\gamma(W) = \sup\{\zeta(W(t)) \;:\; t \in J\}.$$

Definition 2.3. *A set $W \subseteq L^1(J; X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1(J; \mathbb{R}^+)$ such that $\|w(t)\| \leqslant \kappa(t)$ a.e. for all $w \in W$.*

The next lemma was established in [49, Theorem 3.1].

Lemma 2.3. *Assume that X is a separable Banach space. If $W \subseteq L^1(J; X)$ is uniformly integrable, then $\zeta(\{W(t)\}$ is measurable and*

$$\zeta\left(\left\{\int_0^a w(s)ds : w \in W\right\}\right) \leqslant \int_0^a \zeta(\{w(s) : w \in W\})ds.$$

The next property has been studied by several authors under different hypotheses, for example see [13, 116]. We establish it here both for references purposes and to unify the presentation and avoid some unnecessary hypotheses.

Lemma 2.4. *Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Then there exists a countable set $S_0 \subseteq S$ such that $= \zeta(S) \leqslant 2\zeta(S_0)$.*

Proof. Without lost of generality, we can assume that $\eta(D) > 0$. Let $n \in \mathbb{N}$ be a positive integer and define $r_n = (1 - \frac{1}{n+1})\eta(D) > 0$. Let $x_1^n \in D$, clearly there exists $x_2^n \in D \setminus B_{r_n}[x_1^n]$, and hence $d(x_1^n, x_2^n) > r_n$. Applying repeatedly this argument, we can construct inductively a sequence $\{x_k^n\}_{k\in\mathbb{N}} \subset D$ such that $x_{k+1}^n \in D \setminus \cup_{i=1}^k B_{r_n}[x_i^n]$ and $d(x_i^n, x_j^n) > r_n$ for all $i \neq j$. Define the sets $D_n = \{x_k^n : k \in \mathbb{N}\}$ and $D_0 = \cup_{n=1}^\infty D_n$. Evidently D_0 is a countable set and $\eta(D_n) \geqslant \frac{r_n}{2}$. Since $D_n \subset D_0$ for all $n \in \mathbb{N}$ we have that $\eta(D_0) \geqslant \frac{r_n}{2}$. Therefore,

$$2\eta(D_0) \geqslant (1 - \frac{1}{n+1})\eta(D),$$

and taking limit as $n \to \infty$ we infer that $\eta(D) \leqslant 2\eta(D_0)$. ∎

Corollary 2.1. *Let $a > 0$ and denote by J the interval $[0, a]$. Let $F : L^1(J; X) \to X$ be the map given by*

$$F(u) = \int_0^a u(s)ds.$$

If $W \subseteq L^1(J; X)$ is a uniformly integrable set of functions, then there exists a countable set $W_0 \subseteq W$ such that

$$\zeta(F(W)) \leqslant 2\int_0^a \zeta(W_0(s))ds. \tag{2.7}$$

Proof. Using the Lemma 2.4, we infer that there exists a countable set $W_0 = \{w_n : n \in \mathbb{N}\} \subseteq W$ such that

$$\zeta(F(W)) \leqslant 2\zeta(F(W_0)).$$

It follows from [84, Proposition 2.2.6] that there exist $Z_n \subseteq J$ with Lebesgue measure $\lambda(Z_n) = 0$ such that $w_n(J \setminus Z_n)$ is separable. Redefining w_n on a set of measure zero, which does not change the value $F(w_n)$, we can assume that the set $\bigcup_{n=1}^\infty w_n(J)$ is separable. Therefore, there exists a separable closed subspace X_0 of X such that $W_0(J) \subseteq X_0$.

We identify F with its restriction to $L^1(J, X_0)$. Since W_0 is uniformly integrable, using Lemma 2.2 we obtain that

$$\zeta(F(W_0)) \leqslant \int_J \zeta(W_0(s))ds,$$

which establishes the inequality (2.7). ∎

The main result of this chapter are theorem 2.1 and theorem 2.2. For its proof we impose some conditions on the resolvent operator associated to equation (2.2) and the functions f and g, which are listed bellow.

(HT) There exists an evolution operator $\{T(t)\}_{t\in I}$ for the equation (2.2) such that the function $t \mapsto T(t)$ is continuous for $t > 0$.

(Hf1) The function $f : I \times X \to X$ satisfies Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.

(Hf2) There are a function $m \in L^1(I; \mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \leqslant m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in I$.

(Hf3) There exists a function $H \in L^1(I;\mathbb{R}^+)$ such that for any subset of functions $S \subseteq X$, we have

$$\zeta(f(t,S)) \leqslant H(t)\zeta(S)$$

for almost all $t \in I$.

(Hg1) There exists a constant $L > 0$ such that $\zeta(g(W)) < L\gamma(W)$ for all bounded set $W \subseteq C(I;X)$.

(Hg2) The function $g : C(I;X) \to X$ is a compact map.

Remark* 2.1.** *Assuming that the function g satisfies the hypothesis* ***(Hg1) *or* ***(Hg2)****, it is clear that g takes bounded set into bounded sets. For this reason, for each* $R \geqslant 0$ *we will denote by* g_R *to the number*

$$g_R = \sup\{\|g(u)\| : \|u\|_\infty \leqslant R\}.$$

In addition, for the rest of the chapter we denote by K the number $K = \sup\{\|T(t)\| : t \in I\}$.

Theorem 2.1. *Suppose that all the hypotheses* ***(HT)****,* ***(Hf1)****,* ***(Hf2)****,* ***(Hf3)*** *and* ***(Hg1)*** *are satisfied. Assume further that there exists a constant* $R \geqslant 0$ *such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R. \tag{2.8}$$

If the following inequality holds

$$K\Big(L + 2\int_0^1 H(s)ds\Big) < 1, \tag{2.9}$$

then the equation (2.2) *has at least one mild solution.*

Proof. Define $F : C(I;X) \to C(I;X)$ by

$$(Fu)(t) = T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in I.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $C(I;X)$ such that $u_n \to u$, when $n \to \infty$, for the norm of uniform convergence. Since g is a continuous map,

$$g(u_n) \to g(u) \quad \text{as} \quad n \to \infty.$$

It follows from ***(Hf1)*** that $f(s,u_n(s)) \to f(s,u(s))$, whenever $n \to \infty$. Moreover, in view of condition ***(Hf2)*** we have that

$$\|f(s,u_n(s))\| \leqslant m(s)\Phi(\|u_n(s)\|) \leqslant m(s)\Phi(C),$$

where $C \geqslant 0$ is a constant such that $\|u_n\|_\infty \leqslant C$, applying the Lebesgue dominated convergence theorem we obtain that $F(u_n) \to F(u)$ as $n \to \infty$.

On the other hand, if $\|u\|_\infty \leqslant R$, we have that

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \|T(t)g(u)\| + \left\|\int_0^t T(t-s)f(s,u(s))ds\right\| \\ &\leqslant Kg_R + K\Phi(R)\int_0^1 (t-s)m(s)ds \\ &\leqslant R,\end{aligned}$$

which implies that $F(B_R[0]) \subseteq B_R[0]$.

Let now W be a bounded subset of $C(I;X)$ with $\gamma(W) > 0$. It follows directly from the definition of the measure of noncompactness that

$$\gamma(\{T(\cdot)g(u) : u \in W\}) \leqslant K\zeta(g(W)),$$

We define the map $F_1 : C(I;X) \to C(I;X)$ by the formula $(F_1 u)(t) = \int_0^t T(t-s)f(s,u(s))ds$. Clearly the set $\{f(\cdot,u(\cdot)) : u \in W\}$ is a uniformly integrable set of functions over I. Thus, applying the Corollary 2.1, there exists a countable set $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that

$$\begin{aligned}\gamma\Big(\{F_1 u \ : \ u \in W\}\Big) &\leqslant 2K \sup_{t\in I} \int_0^t \zeta\{f(s,u_n(s)) \ : \ n \in \mathbb{N}\}ds \\ &\leqslant 2K \int_0^1 H(s)ds\, \gamma(\{u_n\}_{n\in\mathbb{N}}) \\ &\leqslant 2K \int_0^1 H(s)ds\, \gamma(W).\end{aligned}$$

Combining this estimate we conclude that

$$\begin{aligned}\gamma(F(W)) &\leqslant K\zeta(g(W)) + 2K\int_0^1 H(s)ds\, \gamma(W) \\ &< \gamma(W),\end{aligned}$$

which implies that $F : B_R[0] \to B_R[0]$ is a γ–condensing map. The assertion is consequence of the Theorem 1.1 ■

The condition (2.9) used in the statements of Theorem 2.1 is some difficult to verify many practical situations. However, if the function g is a compact map this condition may be omitted. This motivates the following result.

Theorem 2.2. *Suppose that all the hypotheses (HT), (Hf1), (Hf2), (Hf3) and (Hg2) are satisfied. If there exists a constant $R \geqslant 0$ such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R, \tag{2.10}$$

then the equation (2.2) *has at least one mild solution.*

Proof. We define the map $F : C(I;X) \to C(I;X)$ by

$$(Fu)(t) = T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in I.$$

Proceeding as in the proof of Theorem 2.1 we obtain that F is continuous and $F(B_M[0]) \subseteq B_M[0]$. Since the function g is a compact map, we have that $T(\cdot)$ is a uniform continuous function on $g(B_R[0])$. Hence, the set $\{T(\cdot)g(u) \ : \ u \in B_R[0]\}$ is an equicontinuous set of functions. Moreover,

$$\begin{aligned}&\left\|\int_0^{t+h} T(t+h-s)f(s,u(s))ds - \int_0^t T(t-s)f(s,u(s))ds\right\| \\ &\qquad \leqslant \int_0^t \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\|ds + \int_t^{t+h} \|T(t+h-s)f(s,u(s))\|\, ds\end{aligned}$$

$$\leqslant \int_0^t \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds + K\Phi(R) \int_t^{t+h} m(s) ds.$$

Since $m \in L^1(I;\mathbb{R}^+)$ it follows that $\int_t^{t+h} m(s)ds \to 0$ as $h \to 0$ uniformly for $u \in B_R[0]$.
On the other hand, for each $\delta > 0$ there exists a measurable set $E \subseteq [0,1]$ and a constant $M > 0$ such that $\int_{[0,1]\setminus E} m(s)ds < \delta$ and $m(t) < M$ for all $t \in E$. Thus,

$$\int_0^t \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds \leqslant \int_0^{t-\delta} \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds + \int_{t-\delta}^{t} \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds$$

Therefore,

$$\begin{aligned}\int_0^t \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds \leqslant \Phi(R)M \int_{E\cap[0,t-\delta]} \|T(t+h-s) - T(t-s)\| ds \\ + 2K\Phi(R) \int_{E^c\cap[0,t-\delta]} m(s)ds \\ + 2K\Phi(R) \int_{t-\delta}^{t} m(s)ds\end{aligned}$$

where $E^c = [0,1] \setminus E$. Since $t - s > \delta$ and using the absolute continuity of the integral we have that $\int_0^{t-\delta} \|T(t+h-s) - T(t-s)\| \, \|f(s,u(s))\| ds \to 0$ uniformly on $u \in B_R[0]$ as $t \to 0$.
Furthermore, $\int_{t-\delta}^{t} m(s)ds \to 0$ as $\delta \to 0$. Therefore, $F(B_R[0])$ is an equicontinuous set of functions.
Define $\mathfrak{B} = \overline{co}(F(B_R[0]))$. Since $F(B_R[0])$ is an equicontinuous set of functions, it follows from Lemma 2.1 that $\mathfrak{B}$ is an equicontinuous set of functions. Let D be a bounded subset of $\mathfrak{B}$.

Using the fact that g is a compact map, by properties of Hausdorff measure of noncompactness we have that

$$\zeta\{(Fv)(t) \, : \, v \in D\} \leqslant \zeta\Big(\Big\{\int_0^t T(t-s)f(s,v(s))ds \, : \, v \in D\Big\}\Big).$$

Moreover,

$$\zeta\Big(\Big\{\int_0^t T(t-s)f(s,v(s))ds \, : \, v \in D\Big\}\Big) \leqslant K\zeta\Big(\Big\{\int_0^t f(s,v(s))ds \, : \, v \in D\Big\}\Big).$$

By *(Hf2)* there exist a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $m \in L^1(I;\mathbb{R}^+)$ such that $\|f(s,u(s))\| \leqslant K\Phi(R)m(s)$. Therefore the set of functions $\{f(\cdot,u(\cdot)) \, : \, u \in D\}$ is uniformly integrable in the interval I. By corollary 2.1 there exists a numerable subset $\{w_n\}_{n\in\mathbb{N}} \subseteq D$ such that

$$\zeta\{(Fv)(t) \, : \, v \in D\} \leqslant 2K \int_0^t \zeta\Big(\{f(s,w_n(s))ds \, : \, n \in \mathbb{N}\}\Big).$$

It follows from condition *(Hf3)* that

$$\begin{aligned}\zeta\{(Fv)(t) : v \in D\} &\leqslant 2K \int_0^t H(s)\zeta(\{w_n(s)\}_{n\in\mathbb{N}})ds \\ &\leqslant 2K \int_0^t H(s)ds\, \gamma(\{w_n\}_{n\in\mathbb{N}}) \\ &\leqslant 2K \int_0^t H(s)ds\, \gamma(D)\end{aligned}$$

Since D is an equicontinuous set of functions, we have that

$$\gamma(F(D)) \leqslant 2K \int_0^1 H(s)ds\, \gamma(D).$$

Using an inductive process we can conclude that

$$\begin{aligned}&\zeta(F^n(D)(t)) = \zeta(F(\overline{co}(F^{n-1}(D))(t)) \\ &\leqslant \ (2K)^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} H(s_n)\cdots H(s_2)H(s_1)ds_n \cdots ds_1\, \gamma(D) \\ &= \ \frac{(2K)^n}{n!}\left(\int_0^t H(s)ds\right)^n \gamma(D).\end{aligned}$$

Since for all $n \in \mathbb{N}$ the sets $F^n(D)$ are equicontinuous sets of functions, we conclude that

$$\gamma(F^n(D)) \leqslant \frac{(2K)^n}{n!}\left(\int_0^1 H(s)ds\right)^n \gamma(D).$$

It follows from the fact that $H \in L^1(I; X)$ that $\frac{(2K)^n}{n!}\left(\int_0^1 H(s)ds\right)^n \to 0$ as $n \to \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(2K)^{n_0}}{(n_0)!}\left(\int_0^1 H(s)ds\right)^{n_0} = r < 1,$$

and applying Theorem 1.2 it follows that F has a fixed point in $\mathfrak{B}$. This fixed point is a mild solution of equation (2.2). ■

The hypothesis *(HT)* involved in the statement of the theorems 2.1 and 2.2 is really difficult to verify. However, if for all $t \in I$ we have $B(t) = b(t)A$ this condition can be verified only in terms of spectral properties of the operator A and regularity of the scalar function b. Consider the following particular case of the equation (2.2)

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I \\ u(0) &= g(u). \end{aligned}\right\} \tag{2.11}$$

where A is a closed linear operator defined on a Hilbert space $\mathcal{H}$ and the kernel $b \in L^1(I; \mathbb{R})$.

To prove the existence of mild solutions for the equation (2.7), we will introduce the Laplace transform of a X–valued function. Furthermore, the following definitions introduced in [98] will be necessary.

Let X be a Banach space. The Laplace transform of a function $f \in L^1(\mathbb{R}^+; X)$ is defined by

$$\widetilde{f}(\lambda) = \int_0^\infty e^{-\omega t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

if the integral is absolute convergent for $\mathrm{Re}(\lambda) > \omega$. When the Laplace transform of a function f is well defined we say that the function f is Laplace transformable.

Definition 2.4. *Let $f \in L^1(\mathbb{R}^+; \mathbb{R})$ be Laplace transformable and $k \in \mathbb{N}$. We say that the map f is k–regular if there exists a constant $C > 0$ such that*

$$|\lambda^n \widetilde{f}^{(n)}(\lambda)| \leqslant C|\widetilde{f}(\lambda)|$$

for all $Re(\lambda) \geqslant \omega$, $0 < n \leqslant k$.

Convolutions of k–regular kernels are again k–regular. Moreover, integration and differentiation are operations which preserve k–regularity as well. See [98, pp. 70].

Definition 2.5. *Let $f \in C^\infty(\mathbb{R}^+; \mathbb{R})$. We will say that f is completely monotone if and only if $(-1)^n f^{(n)}(\lambda) \geqslant 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$.*

Definition 2.6. *Let $a \in L^1(\mathbb{R}^+; \mathbb{R})$ such that a is Laplace transformable. We say that a is completely positive if and only if*

$$\frac{1}{\lambda \widetilde{a}(\lambda)} \quad and \quad \frac{-\widetilde{a}'(\lambda)}{(\widetilde{a}(\lambda))^2}$$

are completely monotone.

Finally, we recall that one-parameter family $\{S(t)\}_{t \geqslant 0}$ of operators is said to be exponentially bounded of type (M, ω) if there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leqslant Me^{\omega t}, \quad \text{for all } t \geqslant 0.$$

The next Proposition establishes the existence of an evolution operator for the equation (2.7), furthermore the function $t \mapsto T(t)$ is continuous for $t > 0$ in the operator topology. For this purpose we will introduce the conditions *(C1)* and *(C2)*.

(C1) The kernel a defined by $a(t) = 1 + \displaystyle\int_0^t b(s)ds$, for all $t \in \mathbb{R}^+$, is 2–regular and completely positive.

(C2) The semigroup generated by A is exponentially bounded of type (M, ω) and there exists $\mu_0 > \omega$ such that

$$\lim_{|\mu| \to \infty} \left\| \frac{1}{\widetilde{b}(\mu_0 + i\mu) + 1} \left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A \right)^{-1} \right\| = 0$$

Proposition 2.1. *Suppose that the operator A is the generator of a C_0–semigroup of type (M, ω) in a Hilbert space $\mathcal{H}$. If the conditions (C1) and (C2) are satisfied then, there exists an evolution operator $\{T(t)\}_{t \in I}$ for the equation (2.7) such that the function $t \mapsto T(t)$ is continuous in the norm of operators for $t > 0$.*

Proof. Integrating in time the equation (2.11) we get

$$u(t) = \int_0^t a(t-s)Au(s)ds + \int_0^t f(s,u(s)) + g(u). \tag{2.12}$$

Since the scalar kernel a is completely positive and A generates a C_0–semigroup, it follows from [98, Theorem 4.2] that there exists a family of operators $\{T(t)\}_{t\in I}$ strongly continuous, exponentially bounded that commutes with A, satisfying

$$T(t)x = x + \int_0^t a(t-s)AT(s)xds, \qquad \text{for all } x \in D(A). \tag{2.13}$$

On the other hand, using the condition ***(C2)*** and since the scalar kernel a is 2–regular, it follows from [78, Theorem 2.2] that the function $t \mapsto T(t)$ is continuous for $t > 0$. Further, since $a \in C^1(\mathbb{R}^+;\mathbb{R})$, it follows from equation (2.13), that for all $x \in D(A)$ the map $T(\cdot)x$ is differentiable for all $t \geqslant 0$ and satisfies

$$\frac{d}{dt}T(t)x = AT(t)x + \int_0^t b(t-s)AT(s)xds, \quad t \in I. \tag{2.14}$$

From the equality (2.14), we conclude that $\{T(t)\}_{t\in I}$ is an evolution operator for the equation (2.11) such that $t \mapsto T(t)$ is continuous for the norm operator. ∎

Corollary 2.2. *Suppose that the operator A generates a C_0–semigroup of type (M,ω) in a Hilbert space $\mathcal{H}$. Assume further that the conditions **(C1)** and **(C2)** are fulfilled. If the hypothesis **(Hf1)**, **(Hf2)**, **(Hf3)** and **(Hg2)** are satisfied and there exists $R \geqslant 0$ such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R,$$

then equation (2.11) *has at least one mild solution.*

Proof. It follows from the Proposition 2.1 that the (2.11) admits an evolution operator $\{T(t)\}_{t\in I}$ such the function $t \mapsto T(t)$ is continuous for $t > 0$. Moreover, since the hypotheses ***(Hf1), (Hf2), (Hf3)*** and ***(Hg2)*** are satisfied, we apply the Theorem 2.2 and conclude that equation (2.11) has at least one mild solution. ∎

2.2 Applications

In this section we apply the abstract results which we have obtained in the preceding section to study the existence of solutions for a partial differential equation submitted to nonlocal initial conditions. This type of equations arises in the study of heat conduction in materials with memory (see [74, 88]). Specifically, we will study the following problem

$$\left.\begin{aligned}
\frac{\partial w(t,\xi)}{\partial t} &= Aw(t,\xi) + \int_0^t \beta e^{-\alpha(t-s)}Aw(s,\xi)ds + p_1(t)p_2(w(t,\xi)), \quad t \in I,\\
w(t,0) &= w(t,2\pi), \quad \text{for } t \in I,\\
w(0,\xi) &= \int_0^1\int_0^\xi qk(s,\xi)w(s,y)dsdy, \quad 0 \leqslant \xi \leqslant 2\pi,
\end{aligned}\right\} \tag{2.15}$$

where $k : I \times [0, 2\pi] \to \mathbb{R}^+$ is a continuous function such that $k(t, 2\pi) = 0$ for all $t \in I$, the constant $q \in \mathbb{R}^+$ and the constants α, β satisfy the relation $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$. The operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial \xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial \xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi),$$

where the coefficients a_1, b_1 and $\overline{c}$ satisfy the usual uniformly ellipticity conditions. The domain of A is defined by $D(A) = \{v \in L^2([0, 2\pi]; \mathbb{R}) : v'' \in L^2([0, 2\pi]; \mathbb{R})\}$. The functions $p_1 : I \to \mathbb{R}^+$ and $p_2 : \mathbb{R} \to \mathbb{R}$ satisfy appropriate conditions which will be specified later.

Identifying $u(t) = w(t, \cdot)$ we model this problem in the space $X = L^2(\mathbb{T}; \mathbb{R})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π–periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T}; \mathbb{R})$ the space of 2π–periodic 2–integrable functions from $\mathbb{R}$ into $\mathbb{R}$. Consequently, the equation (2.15) is rewritten as

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t b(t-s)Au(s)ds + f(t, u(t)), \quad t \in I, \\ u(0) &= g(u), \end{aligned}\right\} \tag{2.16}$$

where the function $g : C(I; X) \to X$ is defined by $g(w)(\xi) = \displaystyle\int_0^1 \int_0^\xi qk(s,\xi)w(s,y)dsdy$, and $f(t, u(t)) = p_1(t)p_2(u(t))$ where p_1 is integrable on I, and p_2 is a bounded function satisfying a Lipschitz type condition with Lipschitz constant L.

We will prove that there exists $q > 0$ sufficiently small such that equation (2.16) has a mild solution in $L^2(\mathbb{T}; \mathbb{R})$.

With this purpose, we begin noting that $\|g\| \leqslant q(2\pi)^{1/2} \left(\displaystyle\int_0^{2\pi} \int_0^1 k(s,\xi)^2 dsd\xi \right)^{1/2}$. Moreover, it is well known fact that the function g is a compact map.

Further, the function f satisfies $\|f(t, u(t))\| \leqslant p_1(t)\Phi(\|u(t)\|)$, with $\Phi(\|u(t)\|) \equiv \|p_2\|$ and $\|f(t, u_1(t)) - f(t, u_2(t))\| \leqslant Lp_1(t)\|u_1 - u_2\|$. Thus, the conditions *(Hf1), (Hf2), (Hf3)*and *(Hg2)* are fulfilled.

On the other hand, define $a(t) = 1 + \displaystyle\int_0^t \beta e^{-\alpha s} ds$, for all $t \in \mathbb{R}_0^+$. Since the kernel b defined by $b(t) = \beta e^{-\alpha t}$ is 2–regular, it follows that a is 2–regular. Furthermore, we claim that a is completely positive. Indeed, we have

$$\widetilde{a}(\lambda) = \frac{\lambda + \alpha + \beta}{\lambda(\lambda + \alpha)}.$$

Define the functions f_1 and f_2 by $f_1(\lambda) = \dfrac{1}{\lambda \widetilde{a}(\lambda)}$ and $f_2(\lambda) = \dfrac{-\widetilde{a}'(\lambda)}{[\widetilde{a}(\lambda)]^2}$ respectively. In another words

$$f_1(\lambda) = \frac{\lambda + \alpha}{\lambda + \alpha + \beta} \quad \text{and} \quad f_2(\lambda) = \frac{\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + \alpha^2}{(\lambda + \alpha + \beta)^2}.$$

A direct calculation shows that

$$f_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(n+1)!}{(\lambda + \alpha + \beta)^{n+1}} \quad \text{and} \quad f_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha + \beta)(n+1)!}{(\lambda + \alpha + \beta)^{n+2}} \quad \text{for } n \in \mathbb{N}.$$

Since $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$, we have that f_1 and f_2 are completely monotone. Thus, the kernel a is completely positive.

On the other hand, it follows from [35] that A generates an analytic, non compact semigroup $\{T(t)\}_{t\geqslant 0}$ on $L^2(\mathbb{T};\mathbb{R})$. In addition, there exists a constant $M > 0$ such that

$$M = \sup\{\|T(t)\| : t \geqslant 0\} < +\infty.$$

It follows from the preceding fact and the Hille–Yosida theorem that $z \in \rho(A)$ for all $z \in \mathbb{C}$ such that $Re(z) > 0$. Let $z = \mu_0 + i\mu$. By direct computation we have

$$Re\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1}\right) = \frac{\mu_0^3 + \mu_0^2\alpha + \mu_0^2(\alpha+\beta) + \mu_0\alpha(\alpha+\beta) + \mu_0\mu^2 - \mu^2\beta}{(\alpha+\beta)^2 + 2\mu_0(\alpha+\beta) + \mu_0^2 + \mu^2}.$$

Hence, $Re\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1}\right) > 0$ for all $z = \mu_0 + i\mu$, such that $\mu_0 > 0$. This implies that

$$\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1} \in \mathcal{B}(X), \text{ for all } \mu_0 > 0.$$

Since the semigroup generated by A is an analytic semigroup we have

$$\left\|\frac{1}{\widetilde{b}(\mu_0 + i\mu) + 1}\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1}\right\| \leqslant \left\|\frac{M}{\mu_0 + i\mu}\right\|$$

Therefore,

$$\lim_{|\mu|\to\infty}\left\|\frac{1}{\widetilde{b}(\mu_0 + i\mu) + 1}\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1}\right\| = 0.$$

It follows from Proposition 2.1 that the equation (2.16) admits an evolution operator $\{T(t)\}_{t\in I}$ such that $t \mapsto T(t)$ is continuous for $t > 0$.

Let $K = \sup\{\|T(t)\| : t \in I\}$ and $c = (2\pi)^{1/2}\left(\int_0^{2\pi}\int_0^1 k(s,\xi)^2 ds d\xi\right)^{1/2}$.
A direct computation shows that for each $R \geqslant 0$ the number g_R is equal to $g_R = qcR$.

Therefore the expression $\left(Kg_R + K\Phi(R)\int_0^1 m(s)ds\right)$, is equivalent to $(qcKR + \|p_1\|_1 LK)$.

Since, there exists $q > 0$ such that $qcK < 1$, we have that there exists $R \geqslant 0$ such that

$$qcKR + \|p_1\|_1 LK \leqslant R.$$

From the Corollary 2.2 we conclude that there exists a mild solution of the equation (2.16).

Chapter 3

Mild solutions of a second order non–autonomous Cauchy problem with nonlocal initial conditions

The present chapter is dedicated to study the existence of solutions for the following second order non–autonomous Cauchy problem submitted to nonlocal initial conditions.

$$\left.\begin{aligned} u''(t) &= A(t)u(t) + f(t, N(t)(u)), \ t \in J, \\ u(0) &= g(u), \\ u'(0) &= h(u). \end{aligned}\right\} \tag{3.1}$$

In this equation X is a Banach space and J is the interval $[0, a]$ with $a > 0$. Further, we assume that the operators $A(t) : D(A(t)) \subseteq X \to X$ for $t \in J$ are closed linear operators with domain $D(A(t)) = D$ for all $t \in J$. As general conditions, we always assume that $g, h, N(\cdot) : C(J; X) \to X$ are continuous maps, the function $t \mapsto N(t)(u)$ is continuous for each $u \in C(J; X)$, and the mapping $f : J \times X \to X$ is a function that satisfies Carathéodory type conditions.

The second order abstract Cauchy problem represents numerous concrete situation modeled by partial differential equations or functional differential equations with boundary conditions. Moreover, the behavior of first and second order Cauchy problems is different in many aspects. For this reason, in the last 50 years has had a remarkable progress of the theory of second order Cauchy problem. The development of this theory has followed a course parallel to the theory of strongly continuous semigroups of operators. As general references we refer the reader to [37, 69, 103, 104, 105, 108].

The existence of solutions for the autonomous second order Cauchy problem is closely related with the concept of cosine operator functions. Similarly, the existence of solutions to the non-autonomous second order abstract Cauchy problem corresponding to the family $\{A(t) : t \in J\}$ is directly related to the concept of evolution operator generated by the family $\{A(t) : t \in J\}$. In the literature can be found various techniques to establish the existence of an evolution operator $\{S(t, s) : t, s \in J\}$ generated by the family $\{A(t) : t \in J\}$. In particular, a widely studied situation is the case when the operators $A(t)$ are additive time perturbations of an operator A_0 which generates a cosine operator function.

We will present an outline of the general theory of cosine functions. We will review their most important properties to establish our main results.

Definition 3.1. *Let X be a Banach space. An operator–valued map $C_0 : \mathbb{R} \to \mathcal{B}(X)$ is called strongly continuous cosine function of operators is the following conditions hold.*

(a) $C_0(0) = I_X$.

(b) $C_0(t+s) + C_0(t-s) = 2C_0(t)C_0(s), \quad s, t \in \mathbb{R}$.

(c) For each $x \in X$ the function $t \mapsto C_0(t)x$ is continuous.

Let $\{C_0(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators in X. Define the sine function by

$$S_0(t)x = \int_0^t C_0(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{3.2}$$

We next mention some basic properties of cosine function and its associated sine function.

Proposition 3.1. *Let $\{C_0(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators in X and $\{S_0(t)\}_{t\in\mathbb{R}}$ the sine family associated. The following properties hold.*

(i) $C_0(t) = C_0(-t)$ *and* $S_0(t) = -S_0(t)$, *for all* $t \in \mathbb{R}$.

(ii) The function $S_0(\cdot)$ is continuous for the norm operator.

(iii) $S_0(t+s) = S_0(t)C_0(s) + S_0(s)C_0(t)$, *for all* $t, s \in \mathbb{R}$.

(iv) There are constants $M \geqslant 1$ and $\omega \geqslant 0$ such that

$$\|C_0(t)\| \leqslant Me^{\omega|t|}, \quad t \in \mathbb{R}.$$

(v) $\lim_{t\to 0} \frac{1}{t} S_0(t)x = x$ *for all* $x \in X$.

To relate the cosine function with abstract Cauchy problem we introduce the notion of infinitesimal generator of a cosine function and differentiable vector space. We call infinitesimal generator of $\{C(t)\}_{t\in\mathbb{R}}$ to the linear operator $A_0 : D(A_0) \subseteq X \to X$ defined by

$$A_0 x = C_0''(0)x = 2\lim_{t\to 0} \frac{C_0(t)x - x}{t^2}, \quad x \in D(A_0),$$

where $D(A_0)$ is the subspace of vectors $x \in X$ for which there exists the limit in the above expression. We denote by E the subspace consisting of elements $x \in X$ such the function $C_0(\cdot)x$ is continuously differentiable. The following proposition collect the most important relationships between these concepts.

Proposition 3.2. *Let $\{C_0(T)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A_0. The next properties hold.*

(i) A_0 is a closed linear operator and $D(A_0)$ is dense in X. Moreover, $D(A_0)$ is the subspace formed by all $x \in X$ for which $C_0(\cdot)x$ is a function of class C^2, and $D(A_0) \subseteq E$.

(ii) For all $x \in X$ and $t \in \mathbb{R}$, we have that $\int_0^t S_0(s)xds \in D(A_0)$, and

$$C_0(t)x - x = A_0 \int_0^t S_0(s)xds.$$

(iii) Let $x \in X$. Then $x \in D(A_0)$ if and only if there exists $y \in X$ such that $C_0(t)x - x = \int_0^t S_0(s)yds$. In this case $y = A_0x$.

(iv) For all $x \in X$ and $t \in \mathbb{R}$, $S_0(t)x \in E$.

(v) For all $x \in E$ and $t \in \mathbb{R}$, $C_0(t)x \in E$, $S_0(t)x \in D(A_0)$ and $\frac{d}{dt}C_0(t)x = A_0S_0(t)x$.

(vi) If $x \in D(A_0)$, then $C_0(t)x \in D(A_0)$, and $\frac{d^2}{dt^2}C_0(t)x = A_0C_0(t)x = C_0(t)A_0x$.

(vii) If $x \in E$, then $S_0(\cdot)x$ is a function of class C^2 and $\frac{d^2}{dt^2}S_0(t)x = A_0S_0(t)x$.

(viii) If $x \in D(A_0)$, then $S_0(t)x \in D(A_0)$ and $S_0(t)A_0x = A_0S_0(t)x$.

The following result has been proved by Kisińsky [69].

Theorem 3.1. *Let $\{C_0(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine functions with infinitesimal generator A_0. Then the space E endowed with the norm*

$$\|x\|_1 = \|x\| + \sup_{0\leqslant t\leqslant 1} \|A_0S_0(t)x\|, \quad x \in E,$$

is a Banach space. Moreover the operator–valued function

$$G(t) = \begin{bmatrix} C_0(t) & S_0(t) \\ A_0S_0(t) & C_0(t) \end{bmatrix}$$

is a strongly continuous group of linear operators in the space $E \times X$ with infinitesimal generator

$$\mathcal{A}_0 = \begin{bmatrix} 0 & I \\ A_0 & 0 \end{bmatrix}$$

defined in $D(A_0) \times E$.

For the rest of the chapter we consider E endowed with the norm $\|\cdot\|_1$. A fundamental subject is to decide when a linear operator generates a strongly continuous cosine function of operators. We begin by establishing a few properties of cosine functions.

Proposition 3.3. *Let $\{C_0(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine functions with infinitesimal generator A_0, such that $\|C_0(t)\| \leqslant Me^{\omega|t|}$. Let $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega$. Then $\lambda^2 \in \rho(A_0)$ and for all $x \in X$ the following properties are fulfilled.*

(i) $\lambda(\lambda^2 - A_0)^{-1}x = \int_0^\infty e^{-\lambda t}C_0(t)xdt$.

(ii) $(\lambda^2 - A_0)^{-1}x = \int_0^\infty e^{-\lambda t}S_0(t)xdt$.

(iii) $\left\|\frac{d^n}{d\lambda^n}\lambda(\lambda^2 - A_0)^{-1}\right\| \leqslant \frac{Mn!}{(Re(\lambda) - \omega)^{n+1}}, \quad n \in \mathbb{N}_0$.

It follows from this proposition that if $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine function of operators uniformly bounded, then $\sigma(A_0) \subseteq (-\infty, 0]$. The following result, analogous to the Hille–Yosida theorem for semigroups theory, has been established by Fattorini [37] and Sova [102].

Theorem 3.2. *Let A_0 be a linear operator with dense domain. Assume that there are constants $M \geqslant 0$ and $\omega \geqslant 0$ such that $\lambda^2 \in \rho(A_0)$ for all $\lambda > \omega$ and*

$$\left\| \frac{d^n}{d\lambda^n}\lambda(\lambda^2 - A_0)^1 \right\| \leqslant \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad n \in \mathbb{N}_0.$$

Then the operator A_0 is the infinitesimal generator of a strongly continuous cosine function of operators such that $\|C(t)\| \leqslant Me^{\omega|t|}$.

Next we return to the inhomogeneous abstract second order Cauchy problem. For this consider a Banach space X. Let $A_0 : D(A_0) \subseteq X \to X$ be a linear operator that generates a strongly cosine function of operators and $f \in L^1(I; X)$, where I is an interval of real numbers.

$$\left.\begin{array}{rcl} u''(t) &=& A_0u(t) + f(t), \quad t \in I, \\ u(0) &=& y, \\ u'(0) &=& z. \end{array}\right\} \tag{3.3}$$

We will say that a function $u \in C(I; X)$ is a mild solution of the equation (3.3) if it is defined by the expression

$$u(t) = C_0(t)y + S_0(t)z + \int_0^t S_0(t-\xi)f(\xi)d\xi, \quad t \in I. \tag{3.4}$$

Furthermore, if $y \in E$, the function $u(\cdot)$ given by (3.4) is continuously differentiable, and

$$u'(t) = A_0S_0(t)y + C_0(t)z + \int_0^t C_0(t-\xi)f(\xi)d\xi, \quad t \in I.$$

Moreover, if $y \in D(A)$, $z \in E$ and f is a continuously differentiable function, then the function $u(\cdot)$ is a classical solution of the problem (3.3).

Similarly, the existence of solutions and well–posedness for the inhomogeneous non–autonomous second order abstract Cauchy problem

$$\left.\begin{array}{rcl} u''(t) &=& A(t)u(t) + f(t), \quad t \in J, \\ u(s) &=& y, \\ u'(s) &=& z, \end{array}\right\} \tag{3.5}$$

is related with the existence of the evolution operator $\{S(t,s)\}_{t,s\in J}$ for the homogeneous equation

$$\left.\begin{array}{rcl} u''(t) &=& A(t)u(t), \quad t \in J, \\ u(s) &=& y, \\ u'(s) &=& z. \end{array}\right\} \tag{3.6}$$

In this chapter, we will use the concept of evolution operator $\{S(t,s)\}_{t,s\in J}$ associated with problem (3.6) introduced by Kozak in [70]. With this purpose, we assume that the domain of $A(t)$ is a subspace D dense in X and independent of $t \in J$, and for each $x \in D$ the function $t \mapsto A(t)x$ is continuous.

Definition 3.2. *Let $S : J \times J \to \mathcal{B}(X)$. The family $\{S(t,s)\}_{t,s\in J}$ is said to be an evolution operator generated by the family $\{A(t) : t \in J\}$ if the following conditions are fulfilled:*

(D1) For each $x \in X$ the map $(t,s) \mapsto S(t,s)x$ is continuously differentiable and

- *(a) For each $t \in J$, $S(t,t) = 0$.*
- *(b) For all $t,s \in J$ and each $x \in X$, $\frac{\partial}{\partial t}S(t,s)x\mid_{t=s} = x$ and $\frac{\partial}{\partial s}S(t,s)x\mid_{t=s} = -x$.*

(D2) For all $t,s \in J$, if $x \in D$, then $S(t,s)x \in D$, the map $(t,s) \mapsto S(t,s)x$ is of class C^2 and

- *(a) $\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$.*
- *(b) $\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$.*
- *(c) $\frac{\partial^2}{\partial s \partial t}S(t,s)x\mid_{t=s} = 0$.*

(D3) For all $s,t \in J$, if $x \in D$, then $\frac{\partial}{\partial t}S(t,s)x \in D$. Further, there exist $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x$ and $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x$, and

- *(a) $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$. Also, the map $(t,s) \mapsto A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous.*
- *(b) $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x = A(t)\frac{\partial}{\partial t}S(t,s)x$.*

Assuming that $f : J \to X$ is an integrable function, Kozak [70] has proved that the function $u : J \to X$ given by

$$u(t) = -\frac{\partial}{\partial s}S(t,s)y + S(t,s)z + \int_s^t S(t,\xi)f(\xi)d\xi,$$

is the mild solution of problem (3.5). Motivated by this result, we establish the following notion.

Definition 3.3. *A continuous function $u \in C(J;X)$ is said to be a mild solution of problem* (3.1) *if the equation*

$$u(t) = -\frac{\partial}{\partial s}S(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi,N(\xi)(u))d\xi, \ t \in J,$$

is verified

Henceforth, we assume that there exists an evolution operator $\{S(t,s)\}_{t,s\in J}$ associated with the family $\{A(t) : t \in J\}$. To abbreviate the text, we introduce the operator $C(t,s) = -\frac{\partial}{\partial s}S(t,s)$. With this notation, a mild solution of the problem (3.1) is a continuous function $u \in C(J;X)$ that satisfies the equation

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi,N(\xi)(u))d\xi, \ t \in J.$$

In addition, we set $K, K_1 > 0$ for constants such that

$$\begin{aligned} \|C(t,s)\| &\leqslant K, \\ \left\|\frac{\partial}{\partial t}S(t,s)\right\| &\leqslant K_1, \end{aligned}$$

for all $s, t \in J$. Since the operator valued map $C(t, \cdot)$ is strongly continuous, for $x \in X$, we have

$$S(t,s)x = -\int_s^t \frac{\partial}{\partial \xi} S(t,\xi)x d\xi = \int_s^t C(t,\xi)x d\xi,$$

which implies that

$$\|S(t,s)x\| \leqslant K|t-s|, \quad s,t \in J.$$

Moreover, it is clear that

$$\|S(t_2,s) - S(t_1,s)\| \leqslant K_1|t_2 - t_1|, \quad \text{for all } t_1, t_2, s \in J. \tag{3.7}$$

It is well known that, except in the case $dim(X) < \infty$, a cosine function $\{C_0(t)\}_{t\in\mathbb{R}}$ cannot be compact for all $t \in [t_1, t_2]$, with $t_2 - t_1 > 0$ ([103]). In contrast to the cosine functions that arise in specific applications, the sine function $\{S_0(t)\}_{t\in\mathbb{R}}$ is very often a compact operator for all $t \in \mathbb{R}$. A similar situation occurs for the evolution operator $\{S(t,s)\}_{t,s\in J}$ generated by a family $\{A(t) : t \in J\}$. We next consider a particular situation.

Assume that A_0 is the infinitesimal generator of a strongly continuous cosine function of operators $\{C_0(t)\}_{t\in\mathbb{R}}$. Let $A(t) = A_0 + P(t)$ for all $t \in J$, where $P : J \to \mathcal{B}(E;X)$ is a map such that the function $t \mapsto P(t)x$ is continuously differentiable in X for each $x \in E$. It has been established by Lin [75] and by Serizawa and Watanabe [101] that for each $(y,z) \in D(A_0) \times E$ the non–autonomous abstract Cauchy problem

$$\left.\begin{array}{rcl} u''(t) &=& (A_0 + P(t))u(t), \quad t \in J, \\ u(0) &=& y, \\ u'(0) &=& z \end{array}\right\}$$

has a unique solution $u(\cdot)$ such that the function $t \mapsto u(t)$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that equation

$$\left.\begin{array}{rcl} u''(t) &=& (A_0 + P(t))u(t), \quad t \in J, \\ u(s) &=& y, \\ u'(s) &=& z, \end{array}\right\}$$

has a unique solution $u(\cdot, s)$ such that the function $t \mapsto u(t,s)$ is continuously differentiable in E. It follows from (3.4) that

$$u(t,s) = C_0(t-s)y + S_0(t-s)z + \int_s^t S_0(t-\xi)P(\xi)u(\xi,s)d\xi.$$

In particular, for $y = 0$, we have

$$u(t,s) = S_0(t-s)z + \int_s^t S_0(t-\xi)P(\xi)u(\xi,s)d\xi. \tag{3.8}$$

Consequently,

$$\|u(t,s)\|_1 \leqslant \|S_0(t-s)\|_{\mathcal{B}(X,E)}\|z\| + \int_s^t \|S_0(t-\xi)\|_{\mathcal{B}(X,E)}\|P(\xi)\|_{\mathcal{B}(E,X)}\|u(\xi,s)\|_1 d\xi.$$

Applying the Gronwall–Bellman lemma, there is a constant $\widetilde{M} \geqslant 0$ such that $\|u(t,s)\|_1 \leqslant \widetilde{M}\|z\|$, for $s, t \in J$.

We define the operator $S(t,s)z = u(t,s)$. It follows from the previous estimate that $S(t,s)$ is a bounded linear map on E for the norm in X. Since E is dense in X, we can extend $S(t,s)$ to X. We keep the notation $S(t,s)$ for this extension. This motivates the following result established by Henríquez in [52].

Lemma 3.1. *Under the preceding conditions, $\{S(t,s)\}_{t,s\in J}$ is the evolution operator generated by the family $\{A(t) : t \in J\}$. Moreover, if $S_0(t)$ is compact for all $t \in \mathbb{R}$, then $S(t,s)$ is also compact for all $s,t \in J$.*

3.1 Existence Results

In this section we will present our main results. Initially we introduce some conditions related to function f.

(Cf1) The function $f : J \times X \to X$ satisfies Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t,\cdot)$ is continuous for almost all $t \in J$.

(Cf2) There are a function $m \in L^1(J;\mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leqslant m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in J$.

(Cf3) There exists a function $H \in L^1(J;\mathbb{R}^+)$ such that for any subset of functions $S \subseteq X$, we have

$$\zeta(f(t,S)) \leqslant H(t)\zeta(S)$$

for almost all $t \in J$.

Before continuing our development, it is important to note that in the context of infinite dimensional spaces conditions ***(Cf2)*** and ***(Cf3)*** are different. We will justify our claim exhibiting a few elementary examples.

***Example* 3.1.** *Let $f : C([0,1]) \to C([0,1])$ given by*

$$f(x)(\xi) = \sqrt{|x(\xi)|}, \; \xi \in [0,1].$$

*It is easy to see that f is continuous. In fact, if the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x for the norm of uniform convergence, then $\bigcup_{n=1}^{\infty} x_n([0,1]) \cup x([0,1])$ is a compact set. Since the function $\alpha(t) = \sqrt{|t|}$ is uniformly continuous on compact sets, then $f(x_n) = \alpha \circ x_n \to \alpha \circ x$ as $n \to \infty$ uniformly on $[0,1]$. Moreover, $\|f(x)\| \leqslant \Phi(\|x\|)$, where $\Phi(t) = \sqrt{t}$, for $t \geqslant 0$. Hence, the function f verifies condition **(Cf2)**. On the other hand, assume that*

$$\zeta(f(W)) \leqslant H\zeta(W), \tag{3.9}$$

for all bounded set $W \subseteq C([0,1])$ and certain constant $H > 0$. For each $n \in \mathbb{N}$, we take the constant function $x_n(t) = 1/n$ and the closed ball $W = B_{1/n^2}[x_n, C([0,1])]$. We know that $\zeta(W) = 1/n^2$. Furthermore, it follows from (3.9) that there exist $\varphi, \psi \in B_{1/n^2}[0, C([0,1])]$ and $s \in [0,1]$ such that $|\varphi(s) - \psi(s)| = 1/n^2$ and

$$\|f(x_n + \varphi) - f(x_n + \psi)\| \leqslant 2H\,\zeta(W) = \frac{2H}{n^2}.$$

Hence

$$\left|\sqrt{\frac{1}{n}+\varphi(s)}-\sqrt{\frac{1}{n}+\psi(s)}\right| = \frac{|\varphi(s)-\psi(s)|}{\sqrt{\frac{1}{n}+\varphi(s)}+\sqrt{\frac{1}{n}+\psi(s)}}$$
$$\leqslant \|f(x_n+\varphi)-f(x_n+\psi)\| \leqslant \frac{2H}{n^2}.$$

This implies that

$$1 \leqslant 2H\Big(\sqrt{\frac{1}{n}+\varphi(s)}+\sqrt{\frac{1}{n}+\psi(s)}\Big) \to 0,\ n\to\infty,$$

which is a contradiction.

Example 3.2. *If a function $f : X \to X$ satisfies **(Cf3)**, then f also satisfies **(Cf2)**. In fact, it follows from **(Cf3)** that f takes bounded sets into bounded sets. We define $\Psi : [0,\infty) \to [0,\infty)$ by $\Psi(\xi) = \sup_{\|x\|\leqslant\xi} \|f(x)\|$.*

It is clear that Ψ is an increasing function and $\|f(x)\| \leqslant \Psi(\|x\|)$. It is also easy to see that Ψ is left continuous. Now, we define $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(t) = \begin{cases} \Psi(t+1), & \text{if } t \in \mathbb{N}\cup\{0\} \\ \Psi(n+1)+[\Psi(n+2)-\Psi(n+1)](t-n), & \text{if } t\in[n,n+1],\ n\in\mathbb{N}. \end{cases}$$

*Clearly Φ is a continuous and nondecreasing map, and $\|f(x)\| \leqslant \Psi(\|x\|) \leqslant \Phi(\|x\|)$. Hence f satisfies **(Cf2)**.*

*However, for a function $f : J\times X \to X$ the assertion does not hold. In fact, let $f(t,x) = \alpha(t)f_0(x)$, where $f_0 : X \to X$ is a completely continuous function and $\alpha : J \to$ is a measurable function such that $\alpha \notin L^1(J;\mathbb{R}^+)$. In this case, clearly f verifies condition **(Cf3)** but not **(Cf2)**.*

Next, we consider the following condition for the family of functions $\{N(t) : t\in J\}$.

(CN1) There exists a constant $\nu > 0$ such that

$$\zeta(\{N(t)(u) : u\in W\}) \leqslant \nu\gamma(W),$$

for all $t\in J$ and every bounded set $W \subseteq C(J;X)$.

We point out that condition ***(CN1)*** implies that, for all $t\in J$, the functions $N(t)$ take bounded sets into bounded sets. Thus, in this case, for $R \geqslant 0$ we denote

$$N_R = \sup\{\|N(t)(u)\| : t\in J, u\in C(J;X), \|u\|_\infty \leqslant R\}.$$

Note that, if f satisfies conditions ***(Cf1)*** and ***(Cf2)***, and $u\in C(J;X)$, then the X–valued function $t\mapsto f(t,N(t)(u))$ is integrable on J.

We are in a position to establish the following essential result.

Theorem 3.3. *Assume that the function $f : J\times X \to X$ satisfies conditions **(Cf1)**, **(Cf2)**, **(Cf3)**, and that the family $\{N(t) : t\in J\}$ satisfies condition **(CN1)**. Let $F : C(J;X) \to C(J;X)$ be the map given by*

$$Fu(t) = \int_0^t f(s,N(s)(u))ds.$$

Let $W \subseteq C(J;X)$ be a bounded set. Then

$$\gamma(F(W)) \leqslant 2\nu\gamma(W)\int_0^a H(s)ds.$$

Proof. It is clear that the set of functions $\{f(\cdot, N(\cdot)(u)) : u \in W\}$ is uniformly integrable. Therefore, according to Corollary 2.1, for each fixed $t \in J$, there exists a countable set $W_0(t) \subseteq W(t)$ such that

$$\begin{aligned}\zeta(F(W)(t)) &\leqslant 2\int_0^t \zeta(f(s, N(s)(W_0(t))))ds \\ &\leqslant 2\nu\int_0^t H(s)ds\zeta(W_0(t)) \\ &\leqslant 2\nu\int_0^t H(s)ds\gamma(W). \end{aligned} \tag{3.10}$$

On the other hand, it follows from ***(Cf2)*** that $F(W)$ is equicontinuous. Consequently, using again the Lemma 2.1, we have that

$$\gamma(F(W)) \leqslant \sup_{t\in J}\zeta(F(W)(t)) \leqslant 2\nu\int_0^a H(s)ds\gamma(W),$$

which establishes the assertion. ∎

In what follows, we need a slightly extension of this result.

Corollary 3.1. *Assume that $p : J\times J\times X \to X$ is a function that satisfies the following conditions:*

(Cp1) *For each $t \in J$, the function $p(t,\cdot,\cdot)$ satisfies the Carathéodory conditions.*

(Cp2) *Let $S \subseteq X$ be a bounded set. The set $\{p(t,\cdot,x) : t \in J, x \in S\}$ is uniformly integrable.*

(Cp3) *Let $S \subseteq X$ be a bounded set. The set $\{p(\cdot,s,x) : s \in J, x \in S\}$ is equicontinuous.*

(Cp4) *There exists a positive function $\widetilde{H} : J \times J \to$ such that $\widetilde{H}(t,\cdot)$ is integrable on J and*

$$\zeta(\{p(t,s,x) : x \in S\}) \leqslant \widetilde{H}(t,s)\zeta(S),$$

for each bounded set $S \subseteq X$.

*Assume further that the family $\{N(t) : t \in J\}$ satisfies condition **(CN1)**. Let $F : C(J;X) \to C(J;X)$ be the map given by*

$$Fu(t) = \int_0^t p(t,s,N(s)(u))ds.$$

Let $W \subseteq C(J;X)$ be a bounded set. Then

$$\gamma(F(W)) \leqslant 2\nu\sup_{t\in J}\int_0^t \widetilde{H}(t,s)ds\gamma(W).$$

Furthermore, if the function $t \mapsto \zeta(N(t)(W))$ is measurable, then

$$\zeta(F(W)(t)) \leqslant 2\int_0^t \widetilde{H}(t,s)\zeta(N(s)(W))ds, \tag{3.11}$$

for $t \in J$.

Proof. Applying Corollary 2.1, there exists a countable set $W_0 = \{u_n : n \in \mathbb{N}\} \subseteq W$ such that

$$\begin{aligned}
\zeta(F(W)(t)) &\leqslant 2\zeta(F(W_0)(t)) \\
&= 2\zeta\left(\left\{\int_0^t p(t,s,N(s)(u_n))ds : n \in \mathbb{N}\right\}\right) \\
&\leqslant 2\int_0^t \zeta(\{p(t,s,N(s)(u_n)) : n \in \mathbb{N}\})ds \\
&\leqslant 2\int_0^t \widetilde{H}(t,s)\zeta(N(s)(W))ds,
\end{aligned}$$

which shows that the inequality (3.11) holds. Now, by using the condition *(CN1)*, we have

$$\zeta(F(W)(t)) \leqslant 2\nu\int_0^t \widetilde{H}(t,s)ds\gamma(W),$$

for $t \in J$. In addition, combining conditions ***(Cp2), (Cp3)*** and ***(CN1)*** with the equality

$$\begin{aligned}
&\int_0^{t+s} p(t+s,\xi,N(\xi)(u))d\xi - \int_0^t p(t,\xi,N(\xi)(u))d\xi \\
&= \int_0^t [p(t+s,\xi,N(\xi)(u)) - p(t,\xi,N(\xi)(u))]d\xi + \int_t^{t+s} p(t+s,\xi,N(\xi)(u))d\xi,
\end{aligned}$$

we deduce that $F(W)$ is an equicontinuous subset of $C(J;X)$. The assertion is now a consequence of Lemma 2.2. ■

In order to show the generality of our presentation, we exhibit below a pair of simple examples of maps that verify the condition *(CN1)*.

Example **3.3.** *Let $Q : J \to \mathcal{B}(X)$ be a strongly continuous operator–valued map. Then*

$$N(t)(u) = Q(t)u(t), \ t \in J,$$

*satisfies the condition **(CN1)**. In particular, this occurs for $Q(t) = I_X$. In this case, the differential equation* (3.1) *is reduced to the usual second order equation*

$$u''(t) = A(t)u(t) + f(t,u(t)).$$

Example **3.4.** *Let $k : J \times J \times X \to X$ be a continuous function. Assume that k takes bounded sets into bounded sets, and that there exists a positive function $\mu \in L^1(J;\mathbb{R}^+)$ such that*

$$\zeta(\{k(s,t,x) : x \in S\}) \leqslant \mu(s)\zeta(S),$$

for every bounded set $S \subseteq X$. Then

$$N(t)(u) = \int_0^a k(s,t,u(s))ds, \ t \in J,$$

*satisfies condition **(CN1)**. In fact, it is clear that $N(\cdot)(u)$ is continuous for each $u \in C(J;X)$. Moreover, applying again (3.10), we have*

$$\zeta(N(t)(W)) \leqslant 2\int_0^a \mu(s)ds\gamma(W),$$

for all bounded set $W \subseteq C(J;X)$. In this case, the differential equation (3.1) is reduced to the integro-differential equation

$$u''(t) = A(t)u(t) + f(t, \int_0^a k(s,t,u(s))ds).$$

We next consider the following condition for the functions g and h.

(Cgh) There exists $\beta > 0$ such that

$$\zeta(g(W)) + a\zeta(h(W)) \leqslant \beta\gamma(W),$$

for all bounded set $W \subseteq C(J;X)$.

If the condition ***(Cgh)*** is fulfilled, then the functions g and h take bounded sets into bounded sets. In a similar manner as in chapter 2, for $R \geqslant 0$, we use the following notation

$$\begin{aligned} g_R &= \sup\{\|g(u)\| : \|u\| \leqslant R\}, \\ h_R &= \sup\{\|h(u)\| : \|u\| \leqslant R\}. \end{aligned}$$

Theorem 3.4. *Assume that conditions **(Cf1)**, **(Cf2)**, **(Cf3)**, **(CN1)** and **(Cgh)** are fulfilled. If*

$$K(\beta + 2\nu \int_0^a (a-s)H(s)ds) < 1, \tag{3.12}$$

and there exists a constant $M \geqslant 0$ such that

$$K(g_M + ah_M + \Phi(N_M) \int_0^a (a-s)m(s)ds) \leqslant M, \tag{3.13}$$

then the problem (3.1) *has at least one mild solution.*

Proof. We define $F : C(J;X) \to C(J;X)$ by

$$(Fu)(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,s)f(s,N(s)(u))ds, \quad t \in J. \tag{3.14}$$

Since the function $s \mapsto f(s,N(s)(u))$ is integrable over J, we infer that F is well defined. We next show that F is a continuous map. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $C(J;X)$ such that $u_n \to u$, $n \to \infty$, for the norm of uniform convergence. Since g and h are continuous maps,

$$C(t,0)g(u_n) + S(t,0)h(u_n) \to C(t,0)g(u) + S(t,0)h(u), \; n \to \infty,$$

uniformly for $t \in J$. Similarly, since $N(s)(u_n) \to N(s)(u)$, $n \to \infty$, for each $s \in J$, it follows from ***(Cf1)*** that $f(s,N(s)(u_n)) \to f(s,N(s)(u))$ as $n \to \infty$. Moreover, in view of that

$$\|f(s,N(s)(u_n))\| \leqslant m(s)\Phi(\|N(s)(u_n)\|) \leqslant m(s)\Phi(N_R),$$

where $R \geqslant 0$ is a constant such that $\|u_n\|_\infty \leqslant R$, applying the Lebesgue dominated convergence theorem we obtain that $F(u_n) \to F(u) \to 0$ as $n \to \infty$.

On the other hand, if $\|u\|_\infty \leqslant M$, it follows from the inequality (3.13) that

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \|C(t,0)g(u)\| + \|S(t,0)h(u)\| + \left\|\int_0^t S(t,s)f(s,N(s)(u))ds\right\| \\ &\leqslant K(g_M + ah_M) + K\Phi(N_M)\int_0^t (t-s)m(s)ds \\ &\leqslant K(g_M + ah_M) + K\Phi(N_M)\int_0^a (a-s)m(s)ds \\ &\leqslant M,\end{aligned}$$

which implies that $F(B_M[0]) \subseteq B_M[0]$.

Let now W be a bounded subset of $C(J;X)$ with $\gamma(W) > 0$. It follows directly from the definition of the Hausdorff measure of noncompactness that

$$\begin{aligned}\gamma(\{C(\cdot,0)g(u) : u \in W\}) &\leqslant K\zeta(g(W)), \\ \gamma(\{S(\cdot,0)h(u) : u \in W\}) &\leqslant Ka\zeta(h(W)).\end{aligned}$$

We define the map $F_1 : C(J;X) \to C(J;X)$ given by

$$F_1(u)(t) = \int_0^t S(t,s)f(s,N(s)(u))ds. \tag{3.15}$$

Let $p(t,s,x) = S(t,s)f(s,x)$. It is easy to see that p satisfies the hypotheses of the Corollary 3.1. Furthermore, the function $\widetilde{H}$ involved in the statement of the Corollary 3.1 can be chosen as $\widetilde{H}(t,s) = K(t-s)H(s)$. Therefore, we get

$$\begin{aligned}\gamma(\{F_1(u) : u \in W\}) &\leqslant 2K\sup_{t\in J}\int_0^t (t-s)H(s)ds\gamma(W) \\ &= 2\nu K\int_0^a (a-s)H(s)ds\gamma(W),\end{aligned}$$

and combining the preceding estimates, we have

$$\begin{aligned}\gamma(F(W)) &\leqslant K(\zeta(g(W)) + a\zeta(h(W)) + 2\nu\int_0^a (a-s)H(s)ds\gamma(W)) \\ &< \gamma(W),\end{aligned}$$

which implies that $F : B_M[0] \to B_M[0]$ is a condensing map. The assertion is a consequence of Theorem 1.1. ∎

Corollary 3.2. *Assume that conditions (Cf1), (Cf2), (Cf3) and (CN1) are fulfilled and that $S(t,s)$ is compact for all $s,t \in J$. Assume further that the following conditions are satisfied:*

(a) The map $g : C(J;X) \to X$ is continuous and verifies

$$\zeta(g(W)) < \frac{1}{K}\gamma(W),$$

for all bounded set $W \subseteq C(J;X)$ such that $\gamma(W) \neq 0$.

(b) The map $h : C(J;X) \to X$ is continuous and takes bounded sets into bounded sets.

If there exists a constant $M \geqslant 0$ such that the inequality (3.13) *holds, then problem* (3.1) *has at least one mild solution.*

Proof. We define F as in the formula (3.14). Proceeding as in the proof of Theorem 3.4 we know that F is continuous and $F : B_M[0] \to B_M[0]$.

Let $W \subseteq B_M[0]$ with $\gamma(W) > 0$. It follows from *(a)* that

$$\gamma(\{C(\cdot,0)g(u) : u \in W\}) \leqslant K\zeta(\{g(u) : u \in W\}) < \gamma(W).$$

Moreover, since each operator $S(t,0)$ is compact and the operator-valued map $S(\cdot,0)$ is continuously differentiable, and h takes bounded sets into bounded sets, a direct application of the Arzelá–Ascoli theorem allows us to conclude that $\{S(\cdot,0)h(u) : u \in W\}$ is relatively compact in $C(J;X)$. This shows that condition *(Cgh)* holds with $\beta = 1/K$.

Let $p(t,s,x) = S(t,s)f(s,x)$. As was mentioned in the proof of Theorem 3.4 the function p satisfies the hypotheses of Corollary 3.1. In this case, the function $\widetilde{H}$ involved in the statement of Corollary 3.1 can be chosen as $\widetilde{H}(t,s) = 0$. Therefore, if F_1 is the map defined in Theorem (3.4) by the equation 3.15, we have

$$\gamma(F_1(W)) = 0,$$

and

$$\gamma(F(W)) \leqslant \gamma(\{C(\cdot,0)g(u) + S(\cdot,0)h(u) : u \in W\}) + \gamma(F_1(W)) < \gamma(W).$$

This implies that condition (3.12) is verified. The assertion is a consequence of the Theorem 3.4. ■

The condition *(a)* used in Corollary 3.2 is difficult to verify in concrete situations. For this reason, we modify slightly the statement of the Corollary 3.2 to get the following result.

Corollary 3.3. *Assume that conditions (Cf1), (Cf2), (Cf3) and (CN1) are fulfilled and that $S(t,s)$ is compact for all $s,t \in J$. Assume further that the following conditions are satisfied:*

(a) The map $g : C(J;X) \to X$ verifies the Lipschitz condition

$$\|g(u_2) - g(u_1)\| \leqslant L_g\|u_2 - u_1\|,$$

for all $u_1, u_2 \in C(J;X)$.

(b) The map $h : C(J;X) \to X$ is continuous and takes bounded sets into bounded sets.

If $KL_g < 1$ and there is a constant $M \geqslant 0$ such that

$$K(L_gM + \|g(0)\| + ah_M + \Phi(N_M)\int_0^a (a-s)m(s)ds) \leqslant M, \qquad (3.16)$$

then problem (3.1) *has at least one mild solution.*

Proof. It follows directly from the Definition of the Hausdorff measure of noncompactness that

$$\zeta(g(W)) \leqslant L_g\gamma(W),$$

for every bounded set $W \subseteq C(J;X)$. Moreover, since

$$\|g(u)\| \leqslant \|g(u) - g(0)\| + \|g(0)\| \leqslant L_g\|u\| + \|g(0)\|,$$

it follows that $g_M \leqslant L_gM + \|g(0)\|$. Proceeding as in the proof of Theorem 3.4, we obtain $F(B_M[0]) \subseteq B_M[0]$. The assertion is now a direct consequence of Corollary 3.2. ■

Corollary 3.4. *Assume that conditions (Cf1), (Cf2), (CN1) and (Cgh) are fulfilled and the set* $\{f(s, N(s)(u)) : u \in W\}$ *is relatively compact for each bounded set* $W \subseteq C(J; X)$. *If* $K\beta < 1$ *and there exists a constant* $M \geqslant 0$ *such that* (3.13) *holds, then the problem* (3.1) *has at least one mild solution.*

Proof. Initially we argue as in the proof of the Theorem 3.4 to obtain

$$\gamma(F(W)) \leqslant K\beta\gamma(W) + \gamma(F_1(W)).$$

Now, arguing as in the proof of Corollary 3.3, we define $p(t, s, x) = S(t, s)f(s, x)$. Since the set $\{S(t, s)f(s, N(s)(u)) : u \in W\}$ is relatively compact, we can take $\overline{H}(t, s) = 0$ in the statement of Corollary 3.2 to conclude that $\gamma(F_1(W)) = 0$. Hence, we get

$$\gamma(F(W)) \leqslant K\beta\gamma(W) < \gamma(W),$$

and we complete the proof as in Theorem 3.4. ∎

For our next results we need to strengthen the condition *(CN1)* for the family of functions $\{N(t) : t \in J\}$. For a bounded set $W \subseteq C(J; X)$ and $t \in J$, we denote

$$\gamma(W, [0, t]) = \gamma(\{w|_{[0,t]} : w \in W\}).$$

(CN2) There exists a constant $\nu > 0$ such that

$$\zeta(N(t)(W)) \leqslant \nu\gamma(\{w|_{[0,t]} : w \in W\})$$

for each bounded set $W \subseteq C(J; X)$.

***Example* 3.5.** *Let* $N(t)(u) = u(t)$ *be the map considered in Example 3.3. It is clear that th family* $\{N(t) : t \in J\}$ *satisfies condition (CN2).*

***Example* 3.6.** *Let*

$$N(t)(u) = \int_0^t k(t, s, u(s))ds, \ t \in J,$$

where $k : \{(t, s) : t \in J, 0 \leqslant s \leqslant t\} \times X \to X$ *is a continuous function. Assume that* k *takes bounded sets into bounded sets, and there exists a positive function* $\mu \in L^1(J; \mathbb{R}^+)$ *such that*

$$\zeta(\{k(t, s, x) : x \in S\}) \leqslant \mu(s)\zeta(S),$$

for every bounded set $S \subseteq X$. *Then* $\{N(t) : t \in J\}$ *satisfies condition (CN2). In fact, it is clear that* $N(\cdot)(u)$ *is continuous for each* $u \in C(J; X)$. *Moreover, proceeding as in Example 3.4,*

$$\zeta(N(t)(W)) \leqslant 2\int_0^t \mu(s)ds\gamma(W, [0, t]),$$

for all bounded set $W \subseteq C(J; X)$.

Theorem 3.5. *Assume that conditions (Cf1), (Cf2), (Cf3) and (CN2) are fulfilled, and that* g, h *are compact maps. If the inequality* (3.13) *holds, then the problem* (3.1) *has at least one mild solution.*

Proof. We define the map F as the expression (3.14). Proceeding as in the proof of Theorem 3.4 we obtain that F is continuous and $F(B_M[0]) \subseteq B_M[0]$.

Moreover, $F(B_M[0])$ is an equicontinuous set of functions. In fact, since $g(B_M[0])$ is relatively compact and $C(\cdot, 0)$ is strongly continuous, applying the Arzelá–Ascoli theorem, we infer that the set $\{C(\cdot, 0)g(u) : u \in B_M[0]\}$ is relatively compact in $C(J; X)$. Using the same argument we can establish that the set $\{S(\cdot, 0)h(u) : u \in B_M[0]\}$ is relatively compact in $C(J; X)$.

Let F_1 be the map given by the expression (3.15)). Proceeding as in the proof of Corollary 3.2 with $p(t, s, x) = S(t, s)f(s, x)$ we infer that the set $F_1(B_M[0])$ is equicontinuous.

We define $\mathfrak{B} = \overline{co}(F(B_M[0]))$. Since $\mathfrak{B} \subseteq B_M[0]$, then $F : \mathfrak{B} \to \mathfrak{B}$, and it follows from the previous assertions and Lemma 2.1 that the set $\mathfrak{B}$ is equicontinuous.

Let $D \subseteq \mathfrak{B}$. Since D is a equicontinuous set, it follows from Lemma 2.2 that

$$\gamma(D, [0, t]) = \sup_{0 \leqslant s \leqslant t} \zeta(D(s))$$

is a continuous function. By using the general properties of the Hausdorff measure of non-compactness and Corollary 3.2, we have that

$$\begin{aligned}
\zeta(F(D)(t)) &\leqslant \zeta(C(t, 0)g(D)) + \zeta(S(t, 0)h(D)) + \zeta\left(\int_0^t S(t, s)f(s, N(s)(D))ds\right) \\
&= \zeta\left(\int_0^t S(t, s)f(s, N(s)(D))ds\right) \\
&\leqslant 2\nu aK \int_0^t H(s)\gamma(D, [0, s])ds \\
&\leqslant 2\nu aK \int_0^t H(s)ds\gamma(D, [0, t])
\end{aligned}$$

and

$$\zeta(F(\overline{co}(D))(t)) \leqslant 2\nu aK \int_0^t H(s)\gamma(\overline{co}(D), [0, s])ds = 2\nu aK \int_0^t H(s)\gamma(D, [0, s])ds.$$

proceeding inductively, and arguing as above, we can show that

$$\begin{aligned}
&\zeta(F^n(D)(t)) = \zeta(F(F^{n-1}(\overline{co}(D)))(t)) \\
&\leqslant \ (2\nu aK)^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} H(s_n) \cdots H(s_2)H(s_1)ds_n \cdots ds_1 \, \gamma(D, [0, t]) \\
&= \ \frac{(2\nu aK)^n}{n!} \left(\int_0^t H(s)ds\right)^n \gamma(D, [0, t]).
\end{aligned}$$

Therefore,

$$\gamma(F^n(D)) = \sup_{t \in J} \zeta(F^n(D)(t)) \leqslant \frac{(2\nu aK)^n}{n!} \left(\int_0^t H(s)ds\right)^n \gamma(D).$$

Since $\frac{(\nu aK)^n}{n!}(\int_0^t H(s)ds)^n \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(2\nu aK)^{n_0}}{n_0!} \left(\int_0^t H(s)ds\right)^{n_0} = r < 1,$$

and applying Theorem 1.2 it follows that F has a fixed point in $\mathfrak{B}$. This fixed point is a mild solution of equation (3.1). ∎

The condition (3.13) is some difficult to verify. We next state a case where the verification of this hypothesis is immediate.

Corollary 3.5. *Assume that conditions (Cf1), (Cf2), (Cf3) and (CN2) are fulfilled, and that g, h are bounded and compact maps. If*

$$\int_0^a m(t)\sup\{\Phi(\|N(t)(u)\|) : u \in C(J;X)\}dt < \infty,$$

then the problem (3.1) *has at least one mild solution.*

Proof. In this case the condition (3.13)) is verified for any constant $M > 0$ such that

$$K(\sup_{u\in C(J;X)}\|g(u)\| + a\sup_{u\in C(J;X)}\|h(u)\| + a\int_0^a m(t)\sup_{u\in C(J;X)}\Phi(\|N(t)(u)\|)dt) \leqslant M.$$

The assertion is an immediate consequence of Theorem 3.5. ∎

3.2 Applications

The one–dimensional wave equation modeled as an abstract Cauchy problem has been studied extensively. See for example [115]. In this section, we apply the results established in preceding section to study the existence of solutions of the non–autonomous wave equation with nonlocal initial conditions. Initially, we will study the following problem

$$\left.\begin{aligned}
\frac{\partial^2 w(t,\xi)}{\partial t^2} &= \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + \tilde{f}(t, w(t,\xi)), \ t \in J, \\
w(t,0) &= w(t,2\pi), \ t \in J, \\
w(0,\xi) &= \sum_{i=0}^{m} g_i w(t_i,\xi), \\
\frac{\partial w(0,\xi)}{\partial t} &= \sum_{i=0}^{m} h_i w(t_i,\xi),
\end{aligned}\right\} \tag{3.17}$$

for $0 \leqslant \xi \leqslant 2\pi$. Here $b : J \to \mathbb{R}$ is a continuous function, $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ satisfies appropriate conditions which will be specified later, $0 < t_0 < \cdots < t_m \leqslant a$, and $g_i, h_i \in$, for $i = 0, 1, \ldots, m$.

We model this problem in the space $X = L^2(\mathbb{T}, \mathbb{R})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π–periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T}, \mathbb{R})$ the space of 2π–periodic 2–integrable functions from $\mathbb{R}$ into $\mathbb{R}$. Similarly, $H^2(\mathbb{T}, \mathbb{R})$ denotes the Sobolev space of 2π–periodic functions $u : \mathbb{R} \to \mathbb{R}$ such that $u'' \in L^2(\mathbb{T}, \mathbb{R})$.

We consider the operator A_0 defined by

$$A_0 z = \frac{d^2 z(\xi)}{d\xi^2} \text{ with domain } D(A_0) = H^2(\mathbb{T}, \mathbb{R}).$$

It is well known that A_0 is the infinitesimal generator of a strongly continuous cosine function $C_0(t)$ in X. Moreover, A_0 has discrete spectrum, the spectrum of A_0 consists of eigenvalues $-n^2$ for $n \in \mathbb{Z}$ with associated eigenvectors

$$z_n(\xi) = \frac{1}{\sqrt{2\pi}}e^{in\xi},\ n \in \mathbb{Z}.$$

Furthermore, the set $\{z_n \ : \ n \in \mathbb{Z}\}$ is an orthonormal basis of X. In particular,

$$A_0 z = \sum_{n\in\mathbb{Z}} -n^2\langle z, z_n\rangle z_n,$$

for $z \in D(A_0)$. The cosine function $\{C_0(t)\}_{t\in\mathbb{R}}$ is given by

$$C_0(t)z = \sum_{n\in\mathbb{Z}} \cos(nt)\langle z, z_n\rangle z_n,\quad t \in \mathbb{R},$$

with associated sine function

$$S_0(t)z = t\langle z, z_0\rangle z_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin(nt)}{n}\langle z, z_n\rangle z_n,\quad t \in \mathbb{R}.$$

It is clear that $\|C_0(t)\| \leqslant 1$ for all $t \in \mathbb{R}$. Thus, $C_0(\cdot)$ is uniformly bounded on $\mathbb{R}$. Hence, $\|S_0(t)\| \leqslant |t|$, for all $t \in$. Moreover, $S_0(t)$ is a compact operator.

For $t \in J$ the operators $P(t)$ are defined by

$$P(t)z = b(t)\frac{dz(\xi)}{d\xi} \ \text{ with domain } \ D(P(t)) = H^1(\mathbb{T}, \mathbb{R}).$$

Let $A(t) = A_0 + P(t)$, $t \in J$. It has been proved by Henríquez in [52] that the family $\{A(t) : t \in J\}$ generates an evolution operator $\{S(t,s)\}_{t,s\in J}$. From Lemma 3.1 we have that the operators $S(t,s)$ are compact. We now estimate the constant K involved in our statements. For $z \in E$, we abbreviate $x(t,s) = S(t,s)z$. We decompose $x(t,s) = \sum_{n\in\mathbb{Z}} x_n(t,s)z_n$, where $x_n(t,s) = \langle x(t,s), z_n\rangle$. It follows from 3.8 that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} x_n(t,s)z_n &= S_0(t-s)z + \int_s^t S_0(t-\tau)P(\tau)\sum_{n\in\mathbb{Z}} x_n(\tau,s)z_n d\tau \\
&= (t-s)\langle z, z_0\rangle z_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin n(t-s)}{n}\langle z, z_n\rangle z_n \\
&\quad + \int_s^t S_0(t-\tau)b(\tau)\sum_{n\in\mathbb{Z}} x_n(\tau,s)z_n' d\tau \\
&= (t-s)\langle z, z_0\rangle z_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin n(t-s)}{n}\langle z, z_n\rangle z_n \\
&\quad + \sum_{n\in\mathbb{Z}} in \int_s^t b(\tau)x_n(\tau,s)S_0(t-\tau)z_n d\tau \\
&= (t-s)\langle z, z_0\rangle z_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin n(t-s)}{n}\langle z, z_n\rangle z_n
\end{aligned}$$

$$+i\sum_{n\in\mathbb{Z}}\int_s^t b(\tau)x_n(\tau,s)\sin n(t-\tau)z_n d\tau,$$

which implies that

$$\begin{aligned} x_0(t,s) &= (t-s)\langle z,z_0\rangle \\ x_n(t,s) &= \frac{\sin n(t-s)}{n}\langle z,z_n\rangle + i\int_s^t b(\tau)\sin n(t-\tau)x_n(\tau,s)d\tau, \end{aligned}$$

for $n\in\mathbb{Z}$, $n\neq 0$. Since $C(t,s)z = -\dfrac{\partial S(t,s)z}{\partial s}$, we define $v_n(t,s) = -\dfrac{\partial x_n(t,s)}{\partial s}$ for $n\in\mathbb{Z}$. It follows from the above expressions that

$$\begin{aligned} v_0(t,s) &= \langle z,z_0\rangle \\ v_n(t,s) &= \cos n(t-s)\langle z,z_n\rangle + i\int_s^t b(\tau)\sin n(t-\tau)v_n(\tau,s)d\tau,\ n\neq 0. \end{aligned}$$

Hence we obtain that

$$|v_n(t,s)| \leqslant |\langle z,z_n\rangle| + \int_s^t |b(\tau)||v_n(\tau,s)|d\tau,\ 0\leqslant s\leqslant t, n\neq 0.$$

Applying the Gronwall-Bellman lemma, we obtain

$$|v_n(t,s)| \leqslant e^{\int_s^t |b(\tau)|d\tau}|\langle z,z_n\rangle|,$$

which implies that

$$\|C(t,s)z\| \leqslant e^{\int_s^t |b(\tau)|d\tau}\|z\|.$$

Therefore, since $t\in J$, we can take $K = e^{\int_0^a |b(\tau)|d\tau}$.

We assume that $\tilde{f} : J\times \to$ is continuous and

$$|\tilde{f}(t,r)| \leqslant m(t)|r|,\ t\in J,\ r\in,$$

where $m\in L^1(J:\mathbb{R}^+)$.

To complete our construction we define the functions f, N, g and h by

$$\begin{aligned} f(t,w)(\xi) &= \tilde{f}(t,w(t,\xi)), \\ N(t)(w)(\xi) &= w(t,\xi), \\ g(w)(\xi) &= \sum_{i=0}^m g_i w(t_i,\xi), \\ h(w)(\xi) &= \sum_{i=0}^m h_i w(t_i,\xi). \end{aligned}$$

Using this construction, and defining $u(t) = w(t,\cdot)\in X$, the problem (3.17) is modeled in the abstract form (3.1)). It is clear that f satisfies conditions *(Cf1)* and *(Cf2)*, with $\Phi(r) = r$; the family $\{N(t) : t\in J\}$ satisfies the condition *(CN1)*, with $\nu = 1$ and $N_R = R$, and the functions g and h are bounded linear maps with $\|g\| = \sum_{i=0}^m |g_i|$ and $\|h\| = \sum_{i=0}^m |h_i|$. Therefore, the following result is an easy consequence of Corollary 3.2.

Corollary 3.6. *Under the above conditions, assume further that*

$$K(\sum_{i=0}^{m}(|g_i| + a|h_i|) + \int_0^a (a-s)m(s)ds) < 1, \tag{3.18}$$

then problem (3.17)) *has at least one mild solution.*

Proof. It follows from our preceding considerations and Lemma 3.1 that $S(t,s)$ is compact. Moreover, condition (3.13)) is an immediate consequence of 3.18). Since g is a bounded linear map,

$$\zeta(g(W)) \leqslant \|g\|\gamma(W) \leqslant \sum_{i=0}^{m}|g_i|\gamma(W) < \frac{1}{K}\gamma(W),$$

for all bounded set $W \subseteq C(J;X)$. Therefore, the hypotheses of Corollary 3.2 are fulfilled. ∎

We now are concerned with the problem

$$\left.\begin{aligned}
\frac{\partial^2 w(t,\xi)}{\partial t^2} &= \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + \tilde{f}(t, \textstyle\int_0^t p(s)w(s,\xi)ds), \quad t \in J,\\
w(t,0) &= w(t,2\pi), \quad t \in J,\\
w(0,\xi) &= \int_0^a \int_0^\xi q_0(s,\xi)w(s,r)drds,\\
\frac{\partial w(0,\xi)}{\partial t} &= \int_0^a q_1(s)w(s,\xi)ds.
\end{aligned}\right\} \tag{3.19}$$

for $0 \leqslant \xi \leqslant 2\pi$. To study this problem we keep the notations and conditions introduced in the analysis of problem (3.17)). Additionally, we assume that $p, q_1 : J \to \mathbb{R}$ and $q_0 : J \times [0,2\pi]$ are continuous functions, and that $q_0(t,2\pi) = 0$ for all $t \in J$.

On the other hand, in this case, we define

$$\begin{aligned}
N(t)(w)(\xi) &= \int_0^t p(s)w(s,\xi)ds,\\
g(w)(\xi) &= \int_0^\xi \int_0^a q_0(s,\xi)w(s,\eta)dsd\eta,\\
h(w)(\xi) &= \int_0^a q_1(s)w(s,\xi)ds.
\end{aligned}$$

It is clear that $N(t), g, h$ are bounded linear maps with

$$\begin{aligned}
\|N(t)\| &= \int_0^t |p(s)|ds,\\
\|g\| &\leqslant (2\pi a)^{1/2}\left(\int_0^{2\pi}\int_0^a q_0(s,\xi)^2 dsd\xi\right)^{1/2},\\
\|h\| &= \int_0^a |q_1(s)|ds.
\end{aligned}$$

Moreover, the function g is a compact map. Therefore, using again the Corollary 3.2, and arguing as above, we can state the following result.

Corollary 3.7. *Under the above conditions, assume further that*

$$K\left(\|g\| + a\|h\| + \nu \int_0^a (a-s)m(s)ds\right) \leqslant 1,$$

where $\nu = \int_0^a |p(s)|ds$. *Then problem (3.19) has at least one mild solution.*

Chapter 4

Periodic solutions of an abstract third–order differential equation

Recent investigations have demonstrated that third–order differential equations describe several models arising from natural phenomena, such as wave propagation in viscous thermally relaxing fluids or flexible space structures with internal damping, for example, a thin uniform rectangular panel, like a solar cell array for more information a spacecraft with flexible attachments. For more information see [12, 43, 61, 89] and references therein.
Considering the influence of an external force, many of these equations take the abstract form

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + F(t, u(t)), \quad \textit{for} \ \ t \in \mathbb{R}^+, \tag{4.1}$$

where A is a closed linear operator defined in a Banach space X, the function F is an appropriate X–valued map, and the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$.

The equation (4.1) has been studied in many aspects, we next just mention a few of them. Cuevas and Lizama [29] have obtained a characterization of its solutions belonging to Hölder spaces $C^s(\mathbb{R}; X)$. Similarly, Fernández, Lizama and Poblete [38] characterize the well-posedness of this equation in Lebesgue spaces $L^p(\mathbb{R}; X)$. In addition, the same authors in [39] have studied some regularity properties and qualitative behaviour of the mild and strong solutions in the space $L^p(\mathbb{R}^+; X)$ whenever the underlying space X is a Hilbert space. On the other hand, De Andrade and Lizama [32] have analysed the existence of asymptotically almost periodic solutions for the equation (4.1).

As we have said in the Introduction of this thesis, concerning to general abstract evolution equations, the study of solutions having a periodicity property has yielded many research papers. However, for the abstract third–order differential equation (4.1) this aspect has not been addressed in the existing literature. For this reason, this chapter is dedicated to study the existence and uniqueness of a periodic strong solution for the abstract third–order equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \quad t \in [0, 2\pi], \tag{4.2}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$. Here A and B are closed linear operators defined in a Banach space X with $D(A) \subseteq D(B)$, the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$, and f belongs to either periodic Lebesgue spaces , or periodic Besov spaces, or periodic Triebel–Lizorkin spaces. Our approach is based on operator–valued Fourier theorems and the maximal regularity property. It is clear that the study of the existence of periodic solutions for equation (4.2), in the particular case when $A = B$, leads to the study of the existence of periodic solutions of equation (4.1).

With a specific norm, we will denote the Banach space consisting of all 2π–periodic, X–valued functions by $E(\mathbb{T}; X)$, and denote the space consisting of all functions in $E(\mathbb{T}; X)$ which are n–times differentiable with respect to the norm in $E(\mathbb{T}; X)$ by $E^n(\mathbb{T}; X)$. The following definitions will be used in subsequent sections with either periodic Lebesgue spaces, periodic Besov spaces or periodic Triebel–Lizorkin spaces.

Definition 4.1. *We will say that a function u is a **strong E–solution** of the equation* (4.2) *if* $u \in E^3(\mathbb{T}; X) \cap E^1(\mathbb{T}; [D(B)]) \cap E(\mathbb{T}; [D(A)])$ *and the equation* (4.2) *holds a.e. in* $[0, 2\pi]$.

Definition 4.2. *We will say that the equation* (4.2) *has **E–maximal regularity** if for each* $f \in E(\mathbb{T}; X)$, *there exists a unique strong E–solution for the equation* (4.2).

For the rest of this chapter we introduce the following notation. Let three constants $\alpha, \beta, \gamma > 0$, and two closed linear operators A and B defined in a Banach space X such that $D(A) \subseteq D(B)$. For $k \in \mathbb{Z}$, we will write

$$a_k = ik^3 \quad \text{and} \quad b_k = i\alpha k^3 + k^2, \tag{4.3}$$

and, if the inverses are well defined, the operators

$$N_k = (b_k + i\gamma k B + \beta A)^{-1} \quad \text{and} \quad M_k = a_k N_k\,. \tag{4.4}$$

Furthermore, we denote

$$\rho(A, B) = \{k \in \mathbb{Z} : N_k \text{ exists and is bounded}\} \quad \text{and} \quad \sigma(A, B) = \mathbb{Z} \setminus \rho(A, B)\,.$$

4.1 L^p–maximal regularity

The L^p–maximal regularity property is a very important topic of evolution equations because it is a fundamental tool for the study of non–linear problems. It is remarkable that classical theorems on L^p–multipliers are no longer valid for operator–valued functions unless the underlying space is isomorphic to a Hilbert space. However, Weis in [110] gives a characterization of L^p–maximal regularity in UMD–spaces using the key notion of R–boundedness and Fourier multipliers techniques. Thenceforth, many authors have used these concepts for studying L^p–maximal regularity property. The reader can see [7, 17, 18, 38, 41, 66, 95] and references cited therein. In this section we characterize L^p–maximal regularity of a third–order differential equation in UMD–spaces, using R–boundedness for some families of operators. We next state the necessary definitions.

Let $\mathcal{S}(\mathbb{R}; X)$ be the Schwartz space, consisting of all the rapidly decreasing X–valued functions. A Banach space will be called a UMD–space if the Hilbert transform defined on $\mathcal{S}(\mathbb{R}; X)$ can be extended to a bounded linear operator in $L^p(\mathbb{R}; X)$, for some (and hence for all) $p \in (1, \infty)$. The Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}; X)$ is defined by

$$(Hf)(s) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |t| < \frac{1}{\varepsilon}} \frac{f(t-s)}{t} dt.$$

Examples of UMD–spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, with $1 < p < \infty$, the Schatten–von Neumann classes $C_p(H)$ of operators on Hilbert spaces for $1 < p < \infty$, the Lebesgue spaces $L^p(\Omega, \mu)$ and $L^p(\Omega, \mu; X)$, with $1 < p < \infty$ and whenever X is a UMD–space. Moreover, every closed subspace of a UMD–space is a UMD–space. On the other hand, every UMD–space is reflexive, and therefore, $L^1(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ (if Ω is an unbounded set) and periodic Hölder spaces of index s with $0 < s < 1$, $C^s([0, 2\pi]; X)$ are not UMD–spaces. For further information about these spaces, see [14, 21].

The preliminary concepts for the definition and properties of R–boundedness that we will use may be found in [34, 62].

Let $j \in \mathbb{N}_0$, we denote by r_j the j–th Rademacher function on $[0, 1]$. These functions are defined by $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we will write $r_j x$ for the vector–valued function $t \mapsto r_j(t)x$. The definition of R–boundedness is given as follows.

Definition 4.3. *Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is called **R–bounded** if there exist a constant $C \geqslant 0$ and $p \in [1, \infty)$ such that for each $n \in \mathbb{N}$, $\{T_1, T_2, \cdots, T_n\} \subset \mathcal{T}$ and $\{x_1, x_2, \cdots, x_n\} \subset X$ the inequality*

$$\Big\|\sum_{j=1}^{n} r_j T_j x_j\Big\|_{L^p((0,1);Y)} \leqslant C \Big\|\sum_{j=1}^{n} r_j x_j\Big\|_{L^p((0,1);X)}$$

*holds. The smallest such $C \geqslant 0$ is called the **R–bound** of $\mathcal{T}$, and it is denoted by $R_p(\mathcal{T})$.*

***Remark* 4.1.** *We remark that large classes of operators are R–bounded, (the reader can see* [41, 59, 107] *and references therein). Several properties of R–bounded families can be found in the monograph of Denk-Hieber-Prüss* [34]. *For the reader's convenience we have summarized the most important of them.*

(i) If $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is R–bounded, then it is uniformly bounded with

$$\sup\{\|T\| : T \in \mathcal{T}\} \leqslant R_p(\mathcal{T}).$$

(ii) The definition of R–boundedness is independent of $p \in [1, \infty)$.

(iii) When X and Y are Hilbert spaces, $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is R–bounded if and only if $\mathcal{T}$ is uniformly bounded.

(iv) Let X, Y be Banach spaces and $\mathcal{T}$, $\mathcal{S} \subseteq \mathcal{B}(X, Y)$ be R–bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R–bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leqslant R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(v) Let X, Y and Z be Banach spaces and $\mathcal{T} \subseteq \mathcal{B}(X, Y)$, and $\mathcal{S} \subseteq \mathcal{B}(Y, Z)$ be R–bounded. Then

$$\mathcal{T}\mathcal{S} = \{TS : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R–bounded as well, and $R_p(\mathcal{T}\mathcal{S}) \leqslant R_p(\mathcal{T})R_p(\mathcal{S})$.

(vi) Let X, Y be Banach spaces and $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ be R–bounded. If $\{\alpha_k\}_{k \in \mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k \mathcal{T} : k \in \mathbb{Z}, T \in \mathcal{T}\}$ is R–bounded.

The next proposition proved in [7] relates L^p–multipliers and R–bounded families of operators.

Proposition 4.1. *Let $p \in (1,\infty)$, and let X be a UMD–space. Assume that $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$. If the family $\{L_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier, then $\{L_k\}_{k\in\mathbb{Z}}$ is R–bounded.*

In order to work with L^p–maximal regularity for evolution equations, various researchers introduce the following vector–valued spaces of functions. See [7, 65, 67].

Definition 4.4. *Let $p \in [1,\infty)$, and let $n \in \mathbb{N}$. Let X and Y be Banach spaces. We define the following vector–valued function spaces.*

$$H^{n,p}_{per}(X,Y) = \{u \in L^p(\mathbb{T};X) : \exists\ v \in L^p(\mathbb{T};Y) \text{ such that } \widehat{v}(k) = (ik)^n\widehat{u}(k), \text{ for all } k \in \mathbb{Z}\}.$$

In the case $X = Y$, we just write $H^{n,p}_{per}(X)$. We highlight two important properties of these spaces:

- Let $n, m \in \mathbb{N}$. If $n \leqslant m$, then $H^{m,p}_{per}(X,Y) \subseteq H^{n,p}_{per}(X,Y)$.
- If $u \in H^{n,p}_{per}(X)$, then for all $0 \leqslant k \leqslant n-1$, we have $u^{(k)}(0) = u^{(k)}(2\pi)$.

***Remark* 4.2.** *For $1 < p < \infty$, by* [7, Lemma 2.2], *for all $n \in \mathbb{N}$ the family of operators $\{k^n L_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier if and only if for each function $f \in L^p(X)$ there exists a function $u \in H^{n,p}_{per}(X)$ such that $\widehat{u}(k) = L_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$.*

Now present a characterization of L^p–maximal regularity for the equation (4.2). Our main result is theorem (4.1). The proof of this theorem is based on properties of R–bounded families of operators, which are presented below.

Lemma 4.1. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families of operators, then*

$$\{ka_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}} \text{ and } \{k^2 B(\Delta^1 N_k)\}_{k\in\mathbb{Z}}$$

are R–bounded families of operators.

Proof. We begin noting that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is R–bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is R–bounded. Furthermore, for all $j \in \mathbb{Z}$ fixed, we have $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and that $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are R–bounded families. For all $k \in \mathbb{Z}$, we have

$$(\Delta^1 N_k) = N_{k+1}(b_k - b_{k+1} - i\gamma B)N_k = -(\Delta^1 b_k)N_{k+1}N_k - i\gamma N_{k+1}BN_k. \tag{4.5}$$

Hence, for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$ka_k(\Delta^1 N_k) = -k\frac{(\Delta^1 b_k)}{b_k}\frac{b_k}{a_k}a_k N_{k+1}M_k - i\gamma a_k N_{k+1}kBN_k$$

and,

$$\begin{aligned} k^2 B(\Delta^1 N_k) &= -k(\Delta^1 b_k)kBN_{k+1}N_k - i\gamma kBN_{k+1}kBN_k \\ &= -k\frac{(\Delta^1 b_k)}{b_k}\frac{b_k}{a_k}kBN_{k+1}M_k - i\gamma kBN_{k+1}kBN_k. \end{aligned}$$

A direct computation shows that if $k = 0$ the operators $ka_k(\Delta^1 N_k)$ and $k^2 B(\Delta^1 N_k)$ are bounded. In addition, $\{b_k\}_{k\in\mathbb{Z}}$ is a 1–regular sequence and $\sup_{k\in\mathbb{Z}\setminus\{0\}} \left|b_k/a_k\right| < \infty$. The assertion follows from the properties of R–bounded families of operators. ∎

Lemma 4.2. *Let $p \in (1,\infty)$, and let X be a UMD–space. If $\alpha,\beta,\gamma > 0$, and A and B are closed linear operators defined in X such that $D(A) \subseteq D(B)$, then the following two assertions are equivalent.*

(i) The families of operators $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are R–bounded.

(ii) The families of operators $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are L^p–multipliers.

Proof. *(i)* $\Rightarrow$ *(ii)*. By hypotheses we have that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families of operators. According to Theorem 1.4, it suffices to show that the families of operators

$$\{k(\Delta^1 M_k)\}_{k\in\mathbb{Z}} \quad \text{and} \quad \{k(\Delta^1 kBN_k)\}_{k\in\mathbb{Z}}$$

are R–bounded. For this, note that for all $k \in \mathbb{Z}$ it holds

$$k(\Delta^1 M_k) = k\frac{(\Delta^1 a_k)}{a_k}a_k N_{k+1} + ka_k(\Delta^1 N_k)\,.$$

and

$$k(\Delta^1 kBN_k) = k^2 B(\Delta^1 N_k) + kBN_{k+1}.$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is a 1–regular sequence, statement *(ii)* follows from Lemma 4.1 and the properties of R–bounded families.

(ii) $\Rightarrow$ *(i)*. Statement *(i)* follows from the Proposition 4.1. ∎

Theorem 4.1. *Let $p \in (1,\infty)$, and let X be a UMD–space. The following two assertions are equivalent.*

(i) The equation (4.2) *has L^p–maximal regularity.*

(ii) The set $\sigma(A,B) = \emptyset$. The families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded.

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $k \in \mathbb{Z}$ and $x \in X$. Define $h(t) = e^{ikt}x$. A direct computation shows that $\widehat{h}(k) = x$. By the hypotheses, there exists a function $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X;[D(B)]) \cap L^p(\mathbb{T};[D(A)])$ such that, for almost all $t \in [0,2\pi]$, we have

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + h(t)\,.$$

Applying Fourier transform to both sides of the preceding equality, we obtain

$$(-i\alpha k^3 - k^2 - i\gamma kB - \beta A)\widehat{u}(k) = x.$$

Since x is arbitrary, we have that $(-b_k - i\gamma kB - \beta A)$ is surjective.

On the other hand, let $z \in D(A)$, and assume $(-b_k - i\gamma kB - \beta A)z = 0$. Substituting $u(t) = e^{ikt}z$ in the equation (4.2), we see that u is a periodic solution of this equation whenever $f \equiv 0$. The uniqueness of the solution implies that $z = 0$.

Now suppose $(b_k + i\gamma kB + \beta A)$ has no bounded inverse. Then for each $k \in \mathbb{Z}$, there exists a sequence $\{y_{k,n}\}_{n\in\mathbb{Z}} \subseteq X$ such that

$$\left\|y_{n,k}\right\| \leqslant 1 \quad \text{and} \quad \left\|N_k y_{k,n}\right\| \geqslant n^2, \quad \text{for all } n \in \mathbb{Z}\,.$$

Write $x_k = y_{k,k}$. We obtain $\|N_k x_k\| \geqslant k^2$, for all $k \in \mathbb{Z}$.
Let $g(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{x_k}{k^2} e^{ikt}$. Note that $g \in L^p(\mathbb{T}; X)$. By the assertion (i), there exists a unique strong L^p–solution $u \in L^p(\mathbb{T}; X)$. Applying Fourier transform to the equation (4.2), we have $\widehat{u}(k) = -N_k\widehat{g}(k)$, for all $k \in \mathbb{Z}$. We know

$$u(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} -\frac{x_k}{k^2} e^{ikt} N_k, \quad \text{for almost } t \in [0, 2\pi].$$

Since for all $k \in \mathbb{Z}$, we have that $\left\|\frac{x_k}{k^2} N_k\right\| \geqslant 1$. Therefore, $u \notin L^p(\mathbb{T}; X)$. Since u is a strong L^p–solution of equation (4.2), this is a contradiction. Hence $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. Thus, $\sigma(A, B) = \emptyset$.

Next let $f \in L^p(\mathbb{T}; X)$. It follows from the hypotheses that there exists a unique function $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X; [D(B)]) \cap L^p(\mathbb{T}; [D(A)])$ such that

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t)$$

for almost all $t \in [0, 2\pi]$. Applying Fourier transform to both sides of the preceding equation, we have

$$(-b_k - i\gamma kB - \beta A)\widehat{u}(k) = \widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Since $\sigma(A, B) = \emptyset$, we have

$$\widehat{u}(k) = (-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k), \quad \text{for all k} \in \mathbb{Z}.$$

Multiplying by $i\gamma k$ on both sides of the preceding equality, we obtain

$$i\gamma k\widehat{u}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k).$$

Since $u \in H^{1,p}_{per}(X; [D(B)])$, there is a function $v \in L^p(\mathbb{T}; [D(B)])$ satisfying $\widehat{v}(k) = i\gamma k\widehat{u}(k)$, for all $k \in \mathbb{Z}$. Therefore,

$$\widehat{v}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Define $w = Bv$. Since $v \in L^p(\mathbb{T}; [D(B)])$, we conclude $w \in L^p(\mathbb{T}; X)$. Since B is a closed linear operator, it follows from Lemma [7, Lemma 3.1] that,

$$\widehat{w}(k) = -i\gamma kBN_k\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

This implies that the family $\{kBN_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier.
On the other hand, since $u \in L^p(\mathbb{T}; [D(A)])$, we define $r = -\beta Au$, and we have $r \in L^p(\mathbb{T}; X)$. Since A is linear and closed, it follows from Lemma [7, Lemma 3.1] that

$$\widehat{r}(k) = -\beta AN_k\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Hence, the family $\{-\beta AN_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Now for all $k \in \mathbb{Z}$, we have

$$b_kN_k = I - i\gamma kBN_k - \beta AN_k.$$

Since the sum of L^p–multipliers is also an L^p–multiplier, we conclude that the family $\{b_k N_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Since, the sequence $\{a_k/b_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded and $\frac{a_k}{b_k} b_k N_k = M_k$, we have that $\{M_k\}_{k\in\mathbb{Z}}$ an L^p–multiplier. It now follows from Proposition 4.1 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families of operators.

(ii) $\Rightarrow$ *(i)*. By the hypotheses all the conditions of the Lemma 4.2 are fulfilled. Therefore, the families of operators $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are L^p–multipliers. From the Remark 1.1 we conclude that the family of operators $\{(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an $L^p(X)$–multiplier, such that for each function $f \in L^p(\mathbb{T}; X)$, there exists a function $u \in H^{3,p}_{per}(X)$ such that

$$\widehat{u}(k) = (-b_k - \beta A - i\gamma kB)^{-1}\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z} \tag{4.6}$$

Moreover, from Lemma [7, Lemma 3.1] we have that $u(t) \in D(A)$ for almost all $t \in [0, 2\pi]$. As we have shown that the family $\{ikB(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p–multiplier, there exists a function $v \in L^p(\mathbb{T}; X)$ satisfying

$$\widehat{v}(k) = ikB(-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k)$$

for all k. According to equality (4.6), we have $\widehat{v}(k) = ikB\widehat{u}(k)$, for all $k \in \mathbb{Z}$.
On the another hand, since $H^{3,p}_{per}(X) \subseteq H^{1,p}_{per}(X)$, there exists a function $w \in L^p(\mathbb{T}; X)$ such that $\widehat{w}(k) = ik\widehat{u}(k)$, for all $k \in \mathbb{Z}$. Since B is closed linear operator, we have

$$\widehat{v}(k) = B(ik\widehat{u}(k)) = B\widehat{w}(k) = \widehat{Bw}(k), \quad \text{for all } k \in \mathbb{Z}.$$

By the uniqueness of the Fourier coefficients, $v = Bw$. This implies that $w \in L^p(\mathbb{T}; [D(B)])$. Therefore, $u \in H^{1,p}_{per}(X; [D(B)])$. We claim that $u \in L^p(\mathbb{T}; [D(A)])$. Indeed, using the identity

$$\beta A(b_k + i\gamma kB + \beta A)^{-1} = I - b_k(b_k + i\gamma kB + \beta A)^{-1} - i\gamma kB(b_k + i\gamma kB + \beta A)^{-1}$$

we conclude that $\{\beta A(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Thus, there exists a function $h \in L^p(\mathbb{T}; X)$ satisfying

$$\widehat{h}(k) = A(b_k + i\gamma B + \beta A)^{-1}\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

It follows from identity (4.6) that $\widehat{h}(k) = A\widehat{u}(k)$, for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, we have $h = Au$. This implies that $u \in L^p(\mathbb{T}; [D(A)])$ as asserted. Therefore, $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X; [D(B)]) \cap L^p(\mathbb{T}; [D(A)])$.

We have shown that $u \in H^{3,p}_{per}(X)$. Thus, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, and $u''(0) = u''(2\pi)$. Since A and B are closed linear operators, it now follows from equality (4.3) that

$$\alpha\widehat{u'''}(k) + \widehat{u''}(k) = \beta\widehat{Au}(k) + \gamma\widehat{Bu}(k) + \widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

From the uniqueness of the Fourier coefficients we conclude that equation (4.2) holds a.e. in $[0, 2\pi]$. Therefore, u is a strong L^p–solution of equation (4.2). It remains to show that this solution is unique. Indeed, let $f \in L^p(\mathbb{T}; X)$. Suppose equation (4.2) has two strong L^p–solutions, u_1 and u_2. A direct computation shows that

$$(-b_k - i\gamma kB - \beta A)[\widehat{u_1}(k) - \widehat{u_2}(k)] = 0$$

for all $k \in \mathbb{Z}$. Since $(-b_k - i\gamma kB - \beta A)$ is invertible, we have $\widehat{u_1}(k) = \widehat{u_2}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $u_1 \equiv u_2$. Therefore, the equation (4.2) has L^p–maximal regularity. ∎

Although our next corollary imposes additional conditions on the operators A and B, these conditions are easier to verify than those included in the statement of theorem (4.1). For this purpose, consider $A : D(A) \subseteq X \to X$ and for $k \in \mathbb{Z}$ the operators $S_k = \left(-\frac{b_k}{\beta} - A\right)^{-1}$.

Corollary 4.1. *Let $1 < p < \infty$, and let X be a UMD–space. Suppose that for all $k \in \mathbb{Z}$ we have that $\frac{-b_k}{\beta} \in \rho(A)$. Assume further that the families of operators $\mathcal{F}_1 = \left\{a_k S_k : k \in \mathbb{Z}\right\}$ and $\mathcal{F}_2 = \left\{ik\frac{\gamma}{\beta}BS_k : k \in \mathbb{Z}\right\}$ are R–bounded. If $R_p(\mathcal{F}_2) < 1$, then equation (4.2) has L^p–maximal regularity.*

Proof. According to [50, Lemma 3.17], the family $\left\{\left(I - \frac{ik\gamma}{\beta}BS_k\right)^{-1}\right\}_{k\in\mathbb{Z}}$ is a R–bounded family of operators. Moreover, note that for $k \in \mathbb{Z}$ the operators

$$M_k = a_k S_k \left(I - \frac{ik\gamma}{\beta}BS_k\right)^{-1} \quad \text{and} \quad kBN_k = kBS_k\left(I - \frac{ik\gamma}{\beta}BS_k\right)^{-1}.$$

It follows from the properties of R–bounded families of operators that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded. The Corollary follows from Theorem 4.1. ∎

By using the following corollary we present an answer for the study of the existence of periodic solutions for the equation (4.1). With this purpose, we denote the complex sequence $\{d_k\}_{k\in\mathbb{Z}}$ given by

$$d_k = -\frac{i\alpha k^3 + k^2}{i\gamma k + \beta} \quad \text{for } k \in \mathbb{Z}.$$

Corollary 4.2. *Let $p \in (1, \infty)$, and let X be a UMD–space. The following two assertions are equivalent.*

(i) The equation (4.2), with $B \equiv A$, has L^p–maximal regularity.

(ii) The sequence $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the family of operators $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R–bounded.

Proof. $(i) \Rightarrow (ii)$. According to Theorem 4.1, we have that $\sigma(A, A) = \emptyset$ and for all $k \in \mathbb{Z}$ the operators $(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} \in \mathcal{B}(X)$. Furthermore, $\{ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is a R–bounded family of operators, then it is bounded. Hence there exists a constant $C \geqslant 0$ such that for all $k \in \mathbb{Z}$ $\left\|ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1}\right\| \leqslant C$. This implies

$$\|(d_k - A)^{-1}\| \leqslant \frac{|i\gamma k + \beta|}{|ik^3|}C, \quad \text{for all } k \in \mathbb{Z} \setminus \{0\}.$$

Since $0 \in \rho(A, A)$ if and only if $0 \in \rho(A)$, we have $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$. It follows from the properties of R–bounded families and the identity

$$d_k(d_k - A)^{-1} = \frac{i\alpha k^3 + k^2}{ik^3} ik^3(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}, \quad \text{for all } k \in \mathbb{Z},$$

that $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is a R-bounded family of operators.
$(ii) \Rightarrow (i)$. For this, note that assertion (ii) guarantees that condition (ii) of Theorem 4.1 is satisfied. In fact, we know that $d_k \in \rho(A)$ for all $k \in \mathbb{Z}$, which implies that $(d_k - A)^{-1}$ is well defined in $\mathcal{B}(X)$. Since $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded, there exists a constant $C \geqslant 0$ such that

$$\sup_{k\in\mathbb{Z}} \|d_k(d_k - A)^{-1}\| = \sup_{k\in\mathbb{Z}} |i\alpha k^3 + k^2| \, \|(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}\| \leqslant C\,.$$

Then, for all $k \in \mathbb{Z} \setminus \{0\}$, we obtain

$$\|(-i\alpha k^3 - k^2 - (i\gamma k + \beta)A)^{-1}\| \leqslant \frac{C}{|i\alpha k^3 + k^2|}\,.$$

Since $0 \in \rho(A)$ if and only if $0 \in \rho(A, A)$, we have $\sigma(A, A) = \emptyset$. We combine properties of R-bounded families with the identities

$$ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{ik^3}{i\alpha k^3 + k^2} d_k(d_k - A)^{-1}$$

and

$$kA(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{-k}{i\gamma k + \beta}(d_k(d_k - A)^{-1} - I)$$

to get that the families of operators $\{ik^3(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{kA(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded. ∎

4.2 $B^s_{p,q}$–maximal regularity

In this section, we present a characterization of the $B^s_{p,q}$–maximal regularity property of the equation (4.2). As we have mentioned in the Preliminaries of this thesis, in contrast to what happens for L^p–maximal regularity, there is not any restriction for the Banach space where the equation (4.2) is defined. The main result of this section is the theorem 4.2. To establish its proof, we need several properties of bounded families of operators, which are developed below.

Lemma 4.3. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families of operators, then*

$$\{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}} \quad \text{and} \quad \{k^3 B(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$$

are bounded families of operators.

Proof. Bellow we make similar considerations as those made in the proof of Lemma 4.1. We have that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j \in \mathbb{Z}$ fixed, we have that $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded families. For $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned} k^2 a_k(\Delta^2 N_k) &= i\gamma k a_k(N_k - N_{k+2})kBN_{k+1} - M_k\, k^2 \frac{(\Delta^2 b_k)}{b_k}\frac{b_k}{a_k} a_k N_{k+1} \\ &\quad + k a_k(N_{k+2} - N_k)k\frac{(\Delta^1 b_{k+1})}{b_k}\frac{b_k}{a_k} a_k N_{k+1} \end{aligned}$$

and

$$k^3B(\Delta^2N_k) = k^2B(N_k - N_{k+2})kBN_{k+1} - kBN_k\,k^2\frac{(\Delta^2b_k)}{b_k}\frac{b_k}{a_k}a_kN_{k+1}$$
$$- k^2B(N_{k+2} - N_k)\,k\frac{(\Delta^1b_{k+1})}{b_k}\frac{b_k}{a_k}a_kN_{k+1}.$$

If $k = 0$, a direct computation shows that the operators $k^2a_k(\Delta^2N_k)$ and $k^3B(\Delta^2N_k)$ are bounded. Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 2–regular sequence and by similar calculations as those made in the proof of Lemma 4.1 we obtain that the families of operators $\{k^2a_k(\Delta^2N_k)\}_{k\in\mathbb{Z}}$ and $\{k^3B(\Delta^2N_k)\}_{k\in\mathbb{Z}}$ are bounded. ∎

Lemma 4.4. *Let $1 \leqslant p, q \leqslant \infty$, and $s > 0$. Let $\alpha,\beta,\gamma \in \mathbb{R}_+$, and let A and B be closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. The following two assertions are equivalent.*

(i) The families of operators $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

(ii) The families of operators $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$–multiplier.

Proof. *(i)* ⇒ *(ii)*. According to Theorem 1.2, we need to show that the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathcal{M}$–bounded families of order 2. The same calculations made in the proof of Lemma (4.1) show that

$$ka_k(\Delta^1N_k) = -k\frac{(\Delta^1b_k)}{b_k}\frac{b_k}{a_k}a_kN_{k+1}M_k - i\gamma a_kN_{k+1}kBN_k,$$

and

$$k^2B(\Delta^1N_k) = -k\frac{(\Delta^1b_k)}{b_k}\frac{b_k}{a_k}kBN_{k+1}M_k - i\gamma kBN_{k+1}kBN_k.$$

Furthermore, the same computations made in the proof of Lemma 4.2 show that for all $k \in \mathbb{Z}\setminus\{0\}$

$$k(\Delta^1M_k) = k\frac{(\Delta^1a_k)}{a_k}a_kN_{k+1} + ka_k(\Delta^1N_k),$$

and

$$k(\Delta^1kBN_k) = k^2B(\Delta^1N_k) + kBN_{k+1}.$$

Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 1–regular sequence we conclude that $\{k(\Delta^1M_k)\}_{k\in\mathbb{Z}}$ and $\{k(\Delta^1kBN_k)\}_{k\in\mathbb{Z}}$ are bounded families of operators. Now we pursue by noting that for all $k \in \mathbb{Z}\setminus\{0\}$,

$$k^2(\Delta^2M_k) = k^2a_k(\Delta^2N_k) + k^2\frac{(\Delta^2a_k)}{a_k}a_kN_{k+1} + k\frac{(\Delta^1a_k)}{a_k}ka_k(\Delta^1N_{k+1} + \Delta^1N_k),$$

and for all $k \in \mathbb{Z}$

$$k^2(\Delta^2kBN_k) = k^3B(\Delta^2N_k) + k^2B(\Delta^1N_{k+1} + \Delta^1N_k).$$

A direct verification shows that if $k = 0$ the operator $k^2(\Delta^2 M_k)$ is bounded. Therefore, it follows from Lemma 4.1 and Lemma 4.3 that the families of operators $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathcal{M}$–bounded families of order 2.

$(ii) \Rightarrow (i)$. It follows from the closed graph theorem that there exists a $C \geqslant 0$ (independent of f) such that, for $f \in B^s_{p,q}(\mathbb{T}; X)$, we have

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes M_k\widehat{f}(k)\Big\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p,q}}.$$

Let $x \in X$, and define $f(t) = e^{ikt}x$ for $k \in \mathbb{Z}$ fixed. The preceding inequality implies

$$\|e_k\|_{B^s_{p,q}}\|M_k x\|_{B^s_{p,q}} = \|e_k M_k x\|_{B^s_{p,q}} \leqslant C\|e_k\|_{B^s_{p,q}}\|x\|_{B^s_{p,q}}.$$

Hence for all $k \in \mathbb{Z}$ we have $\|M_k\| \leqslant C$, and consequently $\sup_{k\in\mathbb{Z}}\|M_k\| < \infty$. Similarly we conclude that $\sup_{k\in\mathbb{Z}}\|kBN_k\| < \infty$. ∎

The following theorem characterizes the maximal regularity property on periodic Besov spaces for the equation (4.2). Its proof is very similar to that carried out to establish Theorem 4.1, so we omit it.

Theorem 4.2. *Let $1 \leqslant p,\ q \leqslant \infty$, and $s > 0$. Let X be a Banach space. The following two assertions are equivalent.*

(i) The equation (4.2) has $B^s_{p,q}$–maximal regularity.

(ii) The set $\sigma(A, B) = \emptyset$ and the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded.

Proceeding as in the previous section, our next corollary imposes additional conditions on the operators A and B, however these conditions are easier to verify than those included in the statement of Theorem 4.2 . We will omit its proof because it follows the same lines as those of the proof of the Corollary 4.1.

Corollary 4.3. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and X a Banach space. Suppose that for all $k \in \mathbb{Z}$ we have $\frac{-b_k}{\beta} \in \rho(A)$. Assume that the families $\left\{a_k S_k\right\}_{k\in\mathbb{Z}}$ and $\left\{\frac{i\gamma k}{\beta}BS_k\right\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}}\|a_k S_k\| < 1$, then the equation (4.2) has $B^s_{p,q}$–maximal regularity.*

The following corollary studies the equation (4.2) in the particular case $A = B$. Its proof follows the same lines as those of proof of Corollary 4.2, so we omit it.

Corollary 4.4. *Let X a Banach space and $1 \leqslant p, q \leqslant \infty$ and $s > 0$ The following assertions are equivalent.*

(i) The equation (4.2) with $B \equiv A$, has $B^s_{p,q}$–maximal regularity.

(ii) The sequence $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is a bounded family of operator.

4.3 $F^s_{p,q}$–maximal regularity

In this section, we establish a characterization of the $F^s_{p,q}$–maximal regularity property for the equation (4.2). In a similar manner to what happens for $B^s_{p,q}$–maximal regularity, the Banach space where the equation is defined does not have any additional condition. The principal result of this section is the Theorem 4.3. To prove it, we need the following results related with bounded families of operators, which we describe bellow.

Lemma 4.5. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined in X such that $D(A) \subseteq D(B)$. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families, then*

$$\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \{k^4 B(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$$

are bounded families.

Proof. Bellow we make the same considerations as those made in Lemma 4.1 and Lemma 4.3. The family of operators $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j \in \mathbb{Z}$ fixed, we have that $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded families.
For $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned} k^3 a_k(\Delta^3 N_k) &= k^2 a_k\left((\Delta^2 N_{k+1}) + (\Delta^2 N_k)\right) k\left(-(\Delta b_{k+2}) - i\gamma B\right) N_{k+2} \\ &\quad - k a_k (N_{k+2} - N_k)\, k^2 \frac{(\Delta^2 b_{k+1})}{b_k} \frac{b_k}{a_k} a_k N_{k+2} \\ &\quad + k a_k (N_{k+2} - N_k)\, k^2\left(-(\Delta b_{k+1}) - i\gamma B\right)(\Delta^1 N_{k+1}) \\ &\quad - k^3 \frac{(\Delta^3 b_k)}{b_k} a_k N_{k+1}\, b_k N_{k+2} - k^2 \frac{(\Delta^2 b_k)}{b_k} k a_k (\Delta^1 N_k)\, b_k N_{k+2} \\ &\quad - k^2 \frac{(\Delta^2 b_k)}{b_k} b_k N_k\, k a_k (N_{k+2} - N_k), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} k^4 B(\Delta^3 N_k) &= k^3 B(\Delta^2 N_{k+1})\, k\left(-(\Delta^1 b_{k+2}) - i\gamma B\right) N_{k+2} \\ &\quad + k^3 B(\Delta^2 N_k)\, k\left(-(\Delta^2 b_{k+2}) - i\gamma B\right) N_{k+2} \\ &\quad + k^2 B(N_{k+2} - N_k)\, k^2 \frac{(\Delta^2 b_{k+1})}{b_{k+2}} b_{k+2} N_{k+2} \\ &\quad - k^2 B(N_{k+2} - N_k)\, k^2\left(-(\Delta^2 b_{k+2}) - i\gamma B\right)(N_{k+2} - N_k) \\ &\quad + \frac{k^3(\Delta^3 b_k)}{b_k} a_k N_{k+1} b_k N_{k+2} - \frac{k^2(\Delta^2 b_k)}{b_k} k^2 B(\Delta^1 N_k) b_k N_{k+2} \\ &\quad - \frac{k^2(\Delta^2 b_k)}{b_k} b_k B N_k\, k^2 (N_{k+2} - N_k). \end{aligned} \tag{4.8}$$

A direct computation shows that if $k = 0$ the operators $k^3 a_k(\Delta^3 N_k)$ and $k^4 B(\Delta^3 N_k)$ are bounded. Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 3–regular sequence, it follows from Lemma 4.3 that all of the terms on the right hand side of identities (4.7) and (4.8) are uniformly bounded. Consequently, $\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ and $\{k^4 B(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ are bounded families of operators. ∎

Lemma 4.6. *Let $1 \leqslant p, q \leqslant \infty$, and $s > 0$, and let A and B be closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. The following two assertions are equivalent.*

(i) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

(ii) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are $F^s_{p,q}$–multiplier.

Proof. *(i)* $\Rightarrow$ *(ii)*. The proof of Lemma 4.4 shows that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$, are $\mathcal{M}$–bounded families of order 2. Moreover, we have

$$\begin{aligned} k^3(\Delta^3 M_k) &= k^3 a_k(\Delta^3 N_k) + k^3(a_{k+3} - a_k)(\Delta^2 N_{k+1}) + k^3(\Delta^2 a_{k+1})(\Delta^1 N_{k+1}) \\ &\quad - 2k^3(\Delta^2 a_k)(\Delta^1 N_{k+1}) + (\Delta^3 a_k)N_{k+2}, \end{aligned}$$

and

$$k^3(\Delta^3 kBN_k) = k^4 B(\Delta^3 N_k) + 3k^3 B(\Delta^2 N_{k+1}).$$

It follows from Lemma 4.3, and Lemma 4.5 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}$ are $\mathcal{M}$–bounded families of order 3. The statement *(ii)* now follows from Theorem 1.3.

(ii) $\Rightarrow$ *(i)*. The proof follows the same lines as that of Theorem 4.4. ∎

With the following theorem we establish a characterization of maximal regularity for solutions of equation (4.2) on periodic Triebel–Lizorkin spaces. The proof of this theorem is analogous to the proof of Theorem 4.2, so we will omit it.

Theorem 4.3. *Let $1 \leqslant p, q \leqslant \infty$. If $s > 0$ and X is a Banach space, then the following two assertions are equivalent.*

(i) The equation (4.2) *has $F^s_{p,q}$–maximal regularity.*

(ii) The set $\sigma(A, B) = \emptyset$ and the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded.

Following the same ideas of previous sections, the next corollary imposes additional conditions on operators A and B. However, these hypotheses are simpler to verify than statement described in Theorem 4.3. We will omit its proof because it is similar to the proof of Corollary 4.3.

Corollary 4.5. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and X be a Banach space. Supose that for all $k \in \mathbb{Z}$ we have $\dfrac{-b_k}{\beta} \in \rho(A)$. Assume that the families of operators $\{a_k S_k\}_{k\in\mathbb{Z}}$ and $\left\{\frac{i\gamma k}{\beta} BS_k\right\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}} \|a_k S_k\| < 1$, then the equation* (4.2) *has $F^s_{p,q}$–maximal regularity.*

With the next corollary we study the existence of periodic solutions of equation (4.2) in the particular case $A \equiv B$. We omit the proof because follows the same lines of the proof of Corollary (4.4).

Corollary 4.6. *Let X a Banach space and $1 \leqslant p, q \leqslant \infty$ and $s > 0$. The following two assertions are equivalent.*

(i) The equation (4.1) *with $B \equiv A$, has $F^s_{p,q}$–maximal regularity.*

(ii) The sequence $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ the family of operators is bounded.

4.4 Applications

To finish this chapter we apply our results to some interesting examples.

***Example* 4.1.** *Let $\alpha, \beta, \gamma \in \mathbb{R}^+$. Let $1 \leqslant p, q \leqslant \infty$, and $s > 0$. Consider the abstract equation*

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A u'(t) + f(t), \quad \text{for } t \in [0, 2\pi] \tag{4.9}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$. The operator A is a positive selfadjoint operator defined in a Hilbert space $\mathcal{H}$ such that $\inf_{\lambda \in \sigma(A)} \{\lambda\} \neq 0$. If $f \in B^s_{p,q}(\mathbb{T}; \mathcal{H})$, then the equation (4.9) has $B^s_{p,q}$–maximal regularity.

Proof. *Using the same notation of the Corollary 4.2, we have $d_k = \frac{-(\alpha\gamma k^4 + \beta k^2)}{(\gamma k)^2 + \beta^2} + i\frac{(\gamma - \alpha\beta)k^3}{(\gamma k)^2 + \beta^2}$. Since the operator A is a positive selfadjoint operator such that $\inf_{\lambda \in \sigma(A)} |\lambda| \neq 0$, we know that $\sigma(A) \subseteq [\varepsilon, +\infty)$, with $\varepsilon > 0$. This implies that the sequence $\{d_k\}_{k \in \mathbb{Z}} \subseteq \rho(A)$. Moreover, by* [64, Chapter 5, Section 3.5], *we know that for $k \in \mathbb{Z}$, $\|(d_k - A)^{-1}\| = \frac{1}{dist(d_k, \sigma(A))}$. Therefore, $\sup_{k \in \mathbb{Z}} \|d_k(d_k - A)^{-1}\| < \infty$. According to Corollary 4.4 the quation* (4.9) *has $B^s_{p,q}$–maximal regularity.* ∎

For the next example we need to introduce some preliminaries about sectorial operators. This type of operators form a very important class of closed but unbounded linear operators in analysis. Most of the closed linear operators appearing in applications are sectorial. The definitions and results which we use in our work are available in the work of Denk, Hieber and Prüss [34].

Denote by $\Sigma_\phi \subseteq \mathbb{C}$ the open sector

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} \ : \ |\arg \lambda| < \phi\}.$$

We denote

$$\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$$

and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded }\}.$$

The space $\mathcal{H}^\infty(\Sigma_\phi)$ is a Banach space endowed with the norm

$$\|f\|^\phi_\infty = \sup_{\lambda \in \Sigma_\phi} |f(\lambda)|.$$

We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows:

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha, \beta < 0} \left\{ f \in \mathcal{H}(\Sigma_\phi) : \|f\|^\infty_{\alpha,\beta} < \infty \right\},$$

where

$$\|f\|^\infty_{\alpha,\beta} = \sup_{|\lambda| \leqslant 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda| \geqslant 1} |\lambda^{-\beta} f(\lambda)|.$$

Definition 4.5. [34] *A closed linear operator A in X is called **sectorial** if the following two conditions hold.*

(S1) $\overline{D(A)} = X$, $\overline{R(A)} = X$, *and* $(-\infty, 0) \subseteq \rho(A)$.

(S2) $\sup_{t>0} \|t(t+A)^{-1}\| \leqslant M$, *for some* $M > 0$.

The operator A is called ***R–sectorial*** *if the family* $\{t(t+A)^{-1}\}_{t>0}$ *is R–bounded. We denote the class of sectorial operators (resp. R–sectorial operators) in X by* $\mathcal{S}(X)$ *(resp.* $\mathcal{RS}(X)$*).*

Note that if $A \in \mathcal{S}(X)$, then $\Sigma_\phi \subseteq \rho(-A)$, for some $\phi > 0$ and $\sup_{\lambda \in \Sigma_\phi} \|\lambda(\lambda + A)^{-1}\| < \infty$.
We denote the *spectral angle* of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\{\phi \ : \ \Sigma_{\pi-\phi} \subseteq \rho(-A)\,, \ \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty\}.$$

The next theorem proved by Denk, Hieber and Prüss [34] allows to us use the functional calculus for sectorial operators.

Theorem 4.4. *Let* $A \in \mathcal{S}(X)$*, fix any* $\phi \in (\phi_A, \pi]$ *and let* $\mathcal{H}_0(\Sigma_\phi)$ *be defined as above. Then, with* $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ *where* $\phi_A < \psi < \phi$*, the Dunford integral*

$$f(A) = \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \qquad f \in \mathcal{H}_0(\Sigma_\phi),$$

defines via $\Phi_A(f) = f(A)$ *a functional calculus* $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$ *which is a bounded algebra homomorphism. Moreover, we have*

$$\lim_{\varepsilon \to 0^+} f(A_\varepsilon) = f(A) \quad in\ \mathcal{B}(X),$$

and $\{f(A_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{B}(X)$ *is uniformly bounded for each* $f \in \mathcal{H}_0(\Sigma_\phi)$.

Definition 4.6. *Let A be a sectorial operator. If there exist* $\phi > \phi_A$ *and a constant* $K_\phi > 0$ *such that*

$$\|f(A)\| \leqslant K_\phi \|f\|_\infty^\phi, \quad for\ all\ f \in \mathcal{H}_0(\Sigma_\phi), \tag{4.10}$$

then we say that a sectorial operator A ***admits a bounded*** $\mathcal{H}^\infty$***–calculus.***

We denote the class of sectorial operators A which admit a bounded $\mathcal{H}^\infty$–calculus by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty$–angle is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : \ \text{Inequality (4.10) holds}\}.$$

Remark **4.3.** *Let A be a sectorial operator which admits a bounded* $\mathcal{H}^\infty$*–calculus. If the set*

$$\left\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta)\ ,\ \|h\|_\infty^\theta\} \leqslant 1\right\}$$

is R–bounded for some $\theta > 0$*, we say that A* ***admits an R–bounded*** $\mathcal{H}^\infty$***–calculus.*** *We denote the class of such operators by* $\mathcal{RH}^\infty(X)$*. The* $\mathcal{RH}^\infty$*–angle is analogous to the* $\mathcal{H}^\infty$*–angle, and is denoted* $\theta_A^{R\infty}$*. For further information about sectorial and R–sectorial operators see* [34].

We state the following proposition from functional calculus theory without proof (compare [34, Proposition 4.10]). The proof of Lemma of 4.7 depends on this result.

Proposition 4.2. *Let X be a Banach space and $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R_\infty}$, where Λ is an arbitrary index set. Then the family $\{h_\lambda(A)\}_{\lambda\in\Lambda}$ is R–bounded.*

Lemma 4.7. *Let $\alpha, \beta \in \mathbb{R}^+$. Assume that X is a UMD–space. Suppose that $A \in \mathcal{RH}^\infty(X)$, with $\mathcal{RH}^\infty$–angle $\theta_A^{R_\infty} < \frac{\pi}{3}$, then the families of operators*

$$\left\{ik^3\left(-\frac{i\alpha k^3+k^2}{\beta}-A\right)^{-1}\right\}_{k\in\mathbb{Z}} \text{ and } \left\{ikA^{1/2}\left(-\frac{i\alpha k^3+k^2}{\beta}-A\right)^{-1}\right\}_{k\in\mathbb{Z}}$$

are R–bounded.

Proof. For all $k\in\mathbb{Z}$ we define the functions $F_k^1 : \Sigma_{\pi/3} \to \mathbb{C}$ and $F_k^2 : \Sigma_{\pi/3} \to \mathbb{C}$ given by

$$F_k^1(z) = \frac{i\beta k^3}{-(i\alpha k^3+k^2+\beta z)} \quad\text{and}\quad F_k^2(z) = \frac{i\beta k z^{1/2}}{-(i\alpha k^3+k^2+\beta z)}.$$

where $z^{1/2}$ is defined in $\mathbb{C}\setminus\{0\}$ and it is holomorphic in the principal branch $\mathbb{C}\setminus(-\infty,0]$. Furthermore, for all $k\in\mathbb{Z}$ and for all $z\in\Sigma_{\pi/3}$ we have that $(i\alpha k^3+k^2+\beta z)\neq 0$. Therefore, for all $k\in\mathbb{Z}$ the functions F_k^1 and F_k^2 are holomorphic in the region $\Sigma_{\pi/3}$.
We claim that for $j\in\{1,2\}$ there exist a constant $M\geqslant 0$ such that $\sup_{k\in\mathbb{Z}}\left\|F_k^j\right\|_\infty^{\pi/3}\leqslant M$.
Indeed, note that for all $k\in\mathbb{Z}\setminus\{0\}$ we have

$$-(i\alpha k^3+k^2+\beta z) = -(i\alpha k^3+k^2)\left(1+\frac{\beta z}{i\alpha k^3+k^2}\right).$$

Since for all $k\in\mathbb{Z}\setminus\{0\}$ and for all $z\in\Sigma_{\pi/3}$ the fraction $\frac{\beta z}{i\alpha k^3+k^2}\in\sum_{\pi/3+\pi/2}$ and the distance of -1 to this sector is positive we have that $\sup_{k\in\mathbb{Z}\setminus\{0\}}\left\|F_k^1\right\|_\infty^{\pi/3}\leqslant M_1$ for some $M_1\geqslant 0$.
Note also that for all $k\in\mathbb{Z}\setminus\{0\}$ we have

$$-(i\alpha k^3+k^2+\beta z) = -\sqrt{i\alpha k^3+k^2}\,z^{1/2}\left(1+\frac{i\beta^{1/2}z^{1/2}}{\sqrt{i\alpha k^3+k^2}}\right)\left(\frac{\sqrt{i\alpha k^3+k^2}}{z^{1/2}}-i\beta^{1/2}\right).$$

For all $k\in\mathbb{Z}\setminus\{0\}$ and for all $z\in\Sigma_{\pi/3}$ we have that the fraction $\frac{i\beta^{1/2}z^{1/2}}{\sqrt{i\alpha k^3+k^2}}\in\sum_{\pi/2+\pi/6+\pi/4}$, and the fraction $\frac{\sqrt{i\alpha k^3+k^2}}{z^{1/2}}\in\sum_{\pi/6+\pi/4}$. Since the distance of -1 to the sector $\sum_{11\pi/12}$ is positive and the distance of i to the region $\sum_{5\pi/12}$ is positive we have that $\sup_{k\in\mathbb{Z}\setminus\{0\}}\left\|F_k^2\right\|_\infty^{\pi/3}\leqslant M_2$ for some $M_2\geqslant 0$.
In addition, for all $z\in\Sigma_{\pi/3}$ the functions $F_0^1(z)=0=F_0^2(z)$. Therefore, there exists $M\geqslant 0$ such that $\sup_{k\in\mathbb{Z}}\left\|F_k^j\right\|_\infty^{\pi/3}\leqslant M$, for $j=1,2$. With a direct computation for all $k\in\mathbb{Z}$ and $z\in\Sigma_{\pi/3}$ we have

$$F_k^1(z)=\frac{ik^3}{-\frac{i\alpha k^3+k^2}{\beta}-z} \quad\text{and}\quad F_k^2(z)=\frac{ikz^{1/2}}{\frac{-(i\alpha k^3+k^2)}{\beta}-z}.$$

Since $A\in\mathcal{RH}^\infty(X)$ we have that and for all $k\in\mathbb{Z}$ the operators $F_k^1(A)=ik^3\left(-\frac{i\alpha k^3+k^2}{\beta}-A\right)^{-1}$ and $F_k^2(A)=ikA^{1/2}\left(-\frac{i\alpha k^3+k^2}{\beta}-A\right)^{-1}$. It follows from proposition 4.2 that the families of operators $\{F_k^1(A)\}_{k\in\mathbb{Z}}$ and $\{F_k^2(A)\}_{k\in\mathbb{Z}}$ are R–bounded. ∎

***Example* 4.2.** *Let X be a UMD–space, and let $p \in (1, \infty)$. Suppose $A \in \mathcal{RH}^\infty(X)$, with $\mathcal{RH}^\infty$–angle $\theta_A^{R_\infty} < \frac{\pi}{3}$. Consider the family of operators*

$$\mathcal{F} = \left\{ ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} : k \in \mathbb{Z} \right\}$$

with $\alpha, \beta > 0$. If $\gamma > 0$ is such that $\frac{\gamma}{\beta}\mathcal{R}_p(\mathcal{F}) < 1$, then the equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma A^{1/2}u'(t) + f(t), \quad \text{for } t \in [0, 2\pi] \tag{4.11}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, has L^p–maximal regularity.

Proof. *According to Lemma 4.7, the families of operators*

$$\left\{ ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} \right\}_{k \in \mathbb{Z}} \quad \text{and} \quad \left\{ ik^3\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} \right\}_{k \in \mathbb{Z}}$$

are R–bounded. Since $\frac{\gamma}{\beta}R_p(\mathcal{F}) < 1$, it follows from Corollary 4.1 that the equation (4.11) *has L^p–maximal regularity.* ∎

Chapter 5

Periodic solutions of a fractional order neutral differential equation with finite delay

The fractional calculus which allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of an extensive study for analyzing not only anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials. See [4, 86] and references therein), but also fractional phenomena in optimal control (see, e.g., [87, 96, 100]). As indicated in [83, 85] and the related references given there, the advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. One of the emerging branches of the study is the Cauchy problems for abstract differential equations involving fractional derivatives in time. In recent decades there has been a lot of interest in this type of problems, its applications and various generalizations (cf. e.g., [9, 28, 56] and references therein). It is significant to study this class of problems, because, in this way, one is more realistic to describe the memory and hereditary properties of various materials and processes (cf. [58, 68, 87, 96]).

In the same manner, several systems of great interest in science are modeled by partial neutral functional differential equations. The reader can see [1, 47, 112]. Many of these equations can be written as an abstract neutral functional differential equations. Additionally, it is well known that one of the most interesting topics, both from a theoretical as practical point of view, of the qualitative theory of differential equations and functional differential equations is the existence of periodic solutions. In particular, the existence of periodic solutions of abstract neutral functional differential equation has been considered in several works [40, 50, 54] and papers cited therein.

Let $0 < \beta < \alpha \leqslant 2$. This chapter is devoted to the study of sufficient conditions that guarantee the existence and uniqueness of a periodic strong solution for the following fractional order abstract neutral differential equation with finite delay

$$D^{\alpha}(u(t) - Bu(t-r)) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \quad t \in [0, 2\pi], \tag{5.1}$$

where the fractional derivative is taken in sense of Liouville–Grünwald–Letnikov, the delay $r > 0$ is a fixed number, $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ are closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. The function u_t is given by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-2\pi, 0]$, and denotes the history of the function $u(\cdot)$ at t and $D^\beta u_t(\cdot)$ is defined by $D^\beta u_t(\cdot) = (D^\beta u)_t(\cdot)$. The operators F and G are called delay operators, and they belong to appropriate spaces, which will be described later. The map f is a X–valued function which belongs to either periodic Besov spaces, or periodic Triebel–Lizorkin spaces.

We prove the maximal regularity property of an auxiliary equation, on periodic Besov spaces and periodic Triebel–Lizorkin spaces, and using this result together with a fixed–point argument we show existence and uniqueness of periodic solution of the equation (5.1). Here the auxiliary equation is described by

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \qquad t \in [0, 2\pi], \tag{5.2}$$

with boundary periodic conditions depending of the values of the numbers α and β. The other terms in the equation (5.2) are defined in the same manner as in the equation (5.1).

In recent years, several particular cases of the equation (5.2) have been studied. If $\alpha = 1$ and $F \equiv G \equiv 0$, Arendt and Bu [7, 8] have studied L^p–maximal regularity and $B^s_{p,q}$–maximal regularity, and Bu and Kim [19], have studied $F^s_{p,q}$–maximal regularity. On the other hand, Lizama [79] has obtained a characterization of the existence and uniqueness of strong L^p–solutions, and Lizama and Poblete [81] study C^s–maximal regularity of the corresponding equation on the real line. In the same manner, if $\alpha = 2$ and $\beta = 1$, Bu [15] characterizes C^s–maximal regularity on $\mathbb{R}$. Furthermore, if $\alpha = 2$ and $\beta = 1$, Bu and Fang [18] have studied this equation simultaneosly in periodic Lebesgue spaces, periodic Besov spaces and periodic Triebel–Lizorkin spaces. Moreover, if $1 < \alpha < 2$ and $G \equiv 0$, Lizama and Poblete [82] study L^p–maximal regularity for this equation in the periodic case.

There exist several notions of fractional differentiation. In this chapter we use the fractional differentiation in sense of Liouville–Grünwald–Letnikov. This concept was introduced in [46, 72] and has been widely studied by several authors. In these works the fractional derivative is defined directly as a limit of a fractional difference quotient. In [22] the authors apply this approach based on fractional differences to study fractional differentiation of periodic scalar functions. This idea has been used to extend the definition of fractional differentiation to vector–valued functions, (see [66]). In the case of periodic functions this concept enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as provides a connection with the classical Weyl fractional derivative (see [100]). Next we present the most important preliminaries concerning to fractional derivative in sense of Liouville–Grünwald–Letnikov.

Let $\alpha > 0$ and $f \in L^p(\mathbb{T}; X)$ for $1 \leqslant p < \infty$, the Riemann difference of f is defined by

$$\Delta_t^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj),$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}$ is the binomial coefficient. The Riemann difference of the function f exists almost everywhere, (see [22]). Moreover, $\sum_{j=0}^{\infty} |\binom{\alpha}{j}| < \infty$, and

$$\|\Delta_t^\alpha f\|_{L^p(\mathbb{T};X)} \leqslant \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \|f\|_{L^p(\mathbb{T};X)}.$$

The following definition is a direct extension of [22, Definition 2.1] to the vector–valued case. See [66] for its connection with differential equations.

Definition 5.1. *Let X be a Banach space, $\alpha > 0$ and $1 \leqslant p < \infty$. Let $f \in L^p(\mathbb{T};X)$. If there is $g \in L^p(\mathbb{T};X)$ such that $\lim_{t\to 0^+} t^{-\alpha}\Delta_t^\alpha f = g$ in the $L^p(\mathbb{T};X)$ norm, then the funcion g is called the α^{th}–**Liouville–Grünwald–Letnikov derivative** of f in the mean of order p.*

We abbreviate this terminology by α^{th}–derivative and we denote it by $D^\alpha f = g$. We also mention here a few properties of this fractional derivative. The proof of the following proposition follows the same steps as in the scalar case given in [22, Proposition 4.1].

Proposition 5.1. *Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{T};X)$. For $\alpha,\beta > 0$ the following properties hold:*

- *If $D^\alpha f \in L^p(\mathbb{T};X)$, then $D^\beta f \in L^p(\mathbb{T};X)$ for all $0 < \beta < \alpha$.*
- *$D^\alpha D^\beta f = D^{\alpha+\beta} f$ whenever one of the two sides is well defined.*

***Remark* 5.1.** *Let $f \in L^p(\mathbb{T};X)$ and $\alpha > 0$. It has been proved by Butzer and Westphal* [22] *that $D^\alpha f \in L^p(\mathbb{T};X)$ if and only if there exists $g \in L^p(\mathbb{T};X)$ such that for all $k \in \mathbb{Z}$ it holds $(ik)^\alpha \widehat{f}(k) = \widehat{g}(k)$, where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2}\operatorname{sgn}(k)}$. In this case $D^\alpha f = g$.*

In order to abbreviate the text of the present chapter, we introduce the following notation. Let $1 \leqslant p,q \leqslant \infty$ and $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Assume that A is an operator defined in a Banach space X, and that $F,G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi,0];X);X)$ or $F,G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi,0];X);X)$ are linear, bounded operators. For $k \in \mathbb{Z}$, we will write

$$a_k = (ik)^\alpha \quad \text{and} \quad b_k = (ik)^\beta, \tag{5.3}$$

where $(ik)^\gamma = |k|^\gamma e^{\frac{\pi i \gamma}{2}\operatorname{sgn}(k)}$. Note that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 3–regular sequences. Moreover, if there exist the inverses, we will denote

$$N_k = (a_k I - F_k - b_k G_k - A)^{-1}, \tag{5.4}$$

and

$$M_k = a_k(a_k I - b_k G_k - F_k - A)^{-1} = a_k N_k. \tag{5.5}$$

Now, the bounded linear operators F_k and G_k are defined by $F_k x = F(e_k x)$ and $G_k x = G(e_k x)$, where $(e_k x)(t) = e^{ikt}x$ for all $t \in [-2\pi,0]$ and $x \in X$.

For reference purposes, we introduce the following conditions for the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$.

(F2) The family of operators $\left\{\frac{k}{a_k}\left(\Delta^1 F_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ and $\left\{\frac{k^2}{a_k}\left(\Delta^2 F_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ are bounded.

(F3) The family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(F2)*** and the family $\left\{\frac{k^3}{a_k}\left(\Delta^3 F_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

(G2) The families of operators $\left\{\frac{kb_k}{a_k}\left(\Delta^1 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ and $\left\{\frac{k^2 b_k}{a_k}\left(\Delta^2 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ are bounded.

(G3) The family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(G2)*** and the family $\left\{\frac{k^3 b_k}{a_k}\left(\Delta^3 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

5.1 Maximal regularity in periodic Besov spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. The first objective of this section is the study of $B^s_{p,q}$–maximal regularity of the fractional neutral equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \qquad t \in [0, 2\pi], \tag{5.6}$$

where the fractional derivative is taken in sense of Liouville–Grünwald–Letnikov. The operator $A : D(A) \subseteq X \to X$ is a closed linear operator defined in a Banach space X. The function u_t is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-2\pi, 0]$, and denotes the history of the function $u(\cdot)$ at t. Further, $D^\beta u_t(\cdot)$ is defined by $D^\beta u_t(\cdot) = (D^\beta u)_t(\cdot)$. We suppose that $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$. The mapping f is a X–valued function which belongs to the periodic Besov space $B^s_{p,q}(\mathbb{T}; X)$. Moreover, we assume that this equation has periodic boundary conditions depending of the numbers α and β. This conditions are

$$\left.\begin{array}{ll} u(0) = u(2\pi) & \text{if } 0 < \beta < \alpha \leqslant 1, \\ u(0) = u(2\pi) \text{ and } D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi) & \text{if } 0 < \beta \leqslant 1 < \alpha \leqslant 2, \\ u(0) = u(2\pi)\,, D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi)\,, D^{\beta-1}u(0) = D^{\beta-1}u(2\pi) & \text{if } 1 < \beta < \alpha \leqslant 2. \end{array}\right\}$$

Let $\alpha > 0$. We establish a characterization of the periodic Besov space $B^{s+\alpha}_{p,q}(\mathbb{T}; X)$ in terms of the fractional derivative.

Proposition 5.2. *Let X be a Banach space and $1 \leqslant p, q \leqslant \infty$ and $s > 0$. If $\alpha > 0$ then*

$$B^{s+\alpha}_{p,q}(\mathbb{T}; X) = \left\{u \in B^s_{p,q}(\mathbb{T}; X) : D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)\right\}.$$

Proof. Suppose that $u \in B^s_{p,q}(\mathbb{T}; X)$ and $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$. By the *lifting property* we have that

$$\sum_{k \neq 0} e_k \otimes \widehat{D^\alpha u}(k) \in B^s_{p,q}(\mathbb{T}; X).$$

Since $s > 0$, we have that $D^\alpha u \in L^p(\mathbb{T}; X)$ then $\widehat{D^\alpha u}(k) = (ik)^\alpha \hat{u}(k)$, for all $k \in \mathbb{Z}$, hence

$$\sum_{k \neq 0} e_k \otimes (ik)^\alpha \widehat{u}(k) \in B^s_{p,q}(\mathbb{T}; X).$$

Using again the *lifting property* we obtain that $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$

Reciprocally, let $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$, it is clear $u \in B^s_{p,q}(\mathbb{T}; X)$. Furthermore,

$$\sum_{k \neq 0} e_k \otimes (ik)^\alpha \widehat{u}(k) \in B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X) \tag{5.7}$$

It follows from [22, Theorem 4.1] that there exists $g \in L^p(\mathbb{T}; X)$ such that $\widehat{g}(k) = (ik)^\alpha \widehat{u}(k)$ for all $k \in \mathbb{Z}$. From (5.7) we have that $g \in B^s_{p,q}(\mathbb{T}; X)$. Therefore $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$ and $\widehat{D^\alpha u}(k) = (ik)^\alpha \widehat{u}(k)$ for all $k \in \mathbb{Z}$.

■

Let $s > 0$, using the previous characterization we define the concept of $B^s_{p,q}$–maximal regularity of the equation (5.6).

Definition 5.2. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and let $f \in B^s_{p,q}(\mathbb{T};X)$. A function u is called strong $B^s_{p,q}$–solution of the equation (5.6) if $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X) \cap B^s_{p,q}(\mathbb{T};[D(A)])$ and u satisfies the equation (5.6) for almost $t \in [0,2\pi]$ and the functions $t \mapsto Fu_t$, $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T};X)$. We say that the equation (5.6) has $B^s_{p,q}$–maximal regularity if, for each $f \in B^s_{p,q}(\mathbb{T};X)$ the equation (5.6) has unique strong $B^s_{p,q}$–solution.*

One of the main results of this paper is the Theorem 5.1. Its proof depends of our next results related with some bounded families of operators.

Lemma 5.1. *Let X be a Banach space. Consider $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let $G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. If the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition (G2) then*

$$\left\{\frac{k}{a_k}(\Delta^1 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \textit{and} \quad \left\{\frac{k^2}{a_k}(\Delta^2 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

are bounded families of operators.

Proof. Is clear that $(\Delta^1 b_k G_k) = (\Delta^1 b_k)G_{k+1} + b_k(\Delta^1 G_k)$, for all $k \in \mathbb{Z}$. Therefore,

$$\frac{k}{a_k}(\Delta^1 b_k G_k) = \frac{k(\Delta^1 b_k)}{b_k}\frac{b_k}{a_k}G_{k+1} + \frac{kb_k}{a_k}(\Delta^1 G_k), \text{ for all } k \in \mathbb{Z}\setminus\{0\}.$$

On the other hand, a direct computation shows that

$$(\Delta^2 b_k G_k) = (\Delta^1 b_{k+1})\left[(\Delta^1 G_{k+1}) + (\Delta^1 G_k)\right] + (\Delta^2 b_k)G_k + b_{k+1}(\Delta^2 G_k), \text{ for all } k \in \mathbb{Z}.$$

Now, for all $k \in \mathbb{Z}\setminus\{0\}$, we have

$$\frac{k^2}{a_k}(\Delta^2 b_k G_k) = \frac{k(\Delta^1 b_{k+1})}{b_k}\frac{kb_k}{a_k}\left[(\Delta^1 G_{k+1}) + (\Delta^1 G_k)\right] + \frac{k^2(\Delta^2 b_k)}{b_k}\frac{b_k}{a_k}G_k + \frac{k^2 b_{k+1}}{a_k}(\Delta^2 G_k).$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 2–regular and $\sup_{k\in\mathbb{Z}}\|G_k\| \leqslant C\|G\|$ for some $C \geqslant 0$, all terms included in the right hand side of the preceding identities are uniformly bounded. Hence, the families of operators

$$\left\{\frac{k}{a_k}(\Delta^1 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \text{and} \quad \left\{\frac{k^2}{a_k}(\Delta^2 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

are bounded. ∎

Lemma 5.2. *Consider $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi,0];X);X)$, and that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the condition (F2) and (G2) respectively, and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then*

$$\{ka_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}} \textit{ and } \{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$$

are bounded families of operators.

Proof. Observe that the equality

$$\begin{aligned}(\Delta^1 N_k) &= N_{k+1}(a_k - F_k - b_k G_k - a_{k+1} + F_{k+1} + b_{k+1}G_{k+1})N_k \\ &= (-\Delta^1 a_k)N_{k+1}N_k + N_{k+1}(\Delta^1 F_k)N_k + N_{k+1}(\Delta^1 b_k G_k)N_k.\end{aligned} \tag{5.8}$$

holds for all $k \in \mathbb{Z}$. Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$, we have

$$ka_k(\Delta^1 N_k) = k\frac{(-\Delta^1 a_k)}{a_k}a_k N_{k+1}M_k + \frac{k}{a_k}a_k N_{k+1}(\Delta^1 F_k)M_k + a_k N_{k+1}\frac{k}{a_k}(\Delta^1 b_k G_k)M_k.$$

Is obvious that if $k = 0$ the operator $ka_k(\Delta^1 N_k)$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular and the families of operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are bounded, it follows from Lemma 5.1 that $\{ka_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}}$ is bounded family of operators.
On the other hand, for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}(\Delta^2 N_k) &= [(\Delta^1 N_{k+1}) + (\Delta^1 N_k)][(-\Delta^1 a_{k+1}) + (\Delta^1 F_{k+1}) + (\Delta^1 b_{k+1}G_{k+1})]N_{k+1} \\ &+ N_k[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)]N_{k+1}.\end{aligned} \tag{5.9}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}k^2 a_k(\Delta^2 N_k) &= ka_k[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\frac{k}{a_k}\left[-(\Delta^1 a_{k+1}) + (\Delta^1 F_{k+1}) + (\Delta^1 b_{k+1}G_{k+1})\right]a_k N_{k+1} \\ &+ M_k\left[k^2\frac{(-\Delta^2 a_k)}{a_k} + \frac{k^2}{a_k}(\Delta^2 F_k) + \frac{k^2}{a_k}(\Delta^2 b_k G_k)\right]a_k N_{k+1}.\end{aligned}$$

Is clear that if $k = 0$ the operator $k^2 a_k(\Delta^2 N_k)$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular, the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are bounded and satisfy the conditions *(F2)* and *(G2)* respectively, and the family $\{ka_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}}$ is bounded, it follows from Lemma 5.1 that the family of operators $\{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$ is bounded. ∎

Lemma 5.3. *Consider $1 \leqslant p, q \leqslant \infty, s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$, and that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the condition* **(F2)** *and* **(G2)** *respectively, and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.4, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2. With this purpose, note that $\sup_{k\in\mathbb{Z}} \|F_k\| \leqslant C\|F\|$ for some $C \geqslant 0$, and $\sup_{k\in\mathbb{Z}} \|N_k\| < \infty$. Therefore the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is bounded.
On the other hand, for all $k \in \mathbb{Z}$ we have

$$k(\Delta^1 F_k N_k) = \frac{k}{a_k}(\Delta^1 F_k)a_k N_{k+1} + \frac{1}{a_k}F_k k a_k(\Delta^1 N_k),$$

and

$$k^2(\Delta^2 F_k N_k) = \frac{1}{a_k}F_{k+1}k^2 a_k(\Delta^2 N_k) + \frac{k^2}{a_k}(\Delta^2 F_k)M_k + \frac{k}{a_k}(\Delta^1 F_{k+1})ka_k((\Delta^1 N_{k+1}) + (\Delta^1 N_k)).$$

It follows from Lemma 5.1 and Lemma 5.2 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2. ∎

Lemma 5.4. *Consider $1 \leqslant p,q \leqslant \infty, s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume further that $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$. Suppose that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition (F2), $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition (G2), and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then the family $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. According to the Theorem 1.4 it suffices to show that the family $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of operators of order 2.
For this, note that $\sup_{k\in\mathbb{Z}} \|G_k\| \leqslant C\|G\|$ for some $C \geqslant 0$ and $\sup_{k\in\mathbb{Z}} \|b_k N_k\| \leqslant \sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Therefore the family of operators $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is bounded.

On the other hand, for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$k(\Delta^1 b_k G_k N_k) = \frac{k}{a_k}(\Delta^1 b_k G_k) a_k N_{k+1} + \frac{b_k}{a_k} G_k k a_k (\Delta^1 N_k)$$

and

$$\begin{aligned} k^2(\Delta^2 b_k G_k N_k) &= \frac{k}{a_k}(\Delta^1 b_{k+1} G_{k+1}) k a_k [(\Delta^1 N_{k+1}) + (\Delta^1 N_k)] + \frac{k^2}{a_k}(\Delta^2 b_k G_k) M_k \\ &+ \frac{b_{k+1}}{a_k} G_{k+1} k^2 a_k (\Delta^2 N_k). \end{aligned}$$

Writing in this manner the preceding families, it follows from Lemma 5.1 and Lemma 5.2 that the family $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2. ∎

Lemma 5.5. *Consider $1 \leqslant p,q \leqslant \infty, s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$, and that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the condition (F2) and (G2) respectively, then the following assertions are equivalent.*

(i) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

(ii) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. According to Theorem 1.4, it suffices to show that $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2. From the hypotheses we already know that $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Moreover, for all $k \in \mathbb{Z} \setminus \{0\}$ we have the identity

$$k(\Delta^1 M_k) = \frac{k(\Delta^1 a_k)}{a_k} a_k N_{k+1} + k a_k (\Delta^1 N_k).$$

On the other hand, we have

$$k^2(\Delta^2 M_k) = \frac{k(\Delta^1 a_{k+1})}{a_k} k a_k [(\Delta^1 N_{k+1}) + (\Delta^1 N_k)] + \frac{k^2(\Delta^2 a_k)}{a_k} M_k + k^2 a_{k+1} (\Delta^2 N_k).$$

Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular, it follows from Lemma 5.1 and Lemma 5.2 that $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2.

$(ii) \Rightarrow (i)$. It follows from closed graph theorem that there exists $C \geqslant 0$ (independent of f) such that for $f \in B^s_{p,q}(\mathbb{T}; X)$ we have,

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes M_k\widehat{f}(k)\Big\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p,q}}.$$

Let $x \in X$ and define $f(t) = e^{ikt}x$ for $k \in \mathbb{Z}$ fixed. Then the above inequality implies

$$\|e_k\|_{B^s_{p,q}}\|M_kx\|_{B^s_{p,q}} = \|e_kM_kx\|_{B^s_{p,q}} \leqslant C\|e_k\|_{B^s_{p,q}}\|x\|_{B^s_{p,q}}.$$

Hence for all $k \in \mathbb{Z}$ we have $\|M_k\| \leqslant C$. Thus $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. ■

The next theorem establishes a characterization of $B^s_{p,q}$–maximal regularity for the equation (5.6).

Theorem 5.1. *Consider $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F2)** and **(G2)** respectively, then the following assertions are equivalent.*

(i) The equation (5.6) *has $B^s_{p,q}$–maximal regularity.*

(ii) The families $\{N_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

Proof. $(i) \Rightarrow (ii)$. We show that for $k \in \mathbb{Z}$ the operators $((ik)^\alpha I - (ik)^\beta G_k - F_k - A)$ are invertible. For this, let $k \in \mathbb{Z}$ and $x \in X$, and define $h(t) = e^{ikt}x$. By the assertion (i) there exists $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ such that the functions $t \mapsto Fu_t$ and $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T}; X)$ and the function u satisfies the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + h(t). \tag{5.10}$$

Since the function $Fu_. \in B^s_{p,q}(\mathbb{T}; X)$ and $s > 0$, we have that $Fu_. \in L^p(\mathbb{T}; X)$. Hence, by Fejér's Theorem (see [6]), we have $\widehat{Fu_.}(k) = F_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$. In the same manner, we have that $\widehat{GD^\beta u_.}(k) = G_k\widehat{D^\beta u}(k)$ for all $k \in \mathbb{Z}$. It follows from $\alpha > \beta$ and $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$, that $u \in B^{s+\beta}_{p,q}(\mathbb{T}; X)$. Therefore $\widehat{D^\beta u}(k) = (ik)^\beta\widehat{u}(k)$ for all $k \in \mathbb{Z}$. Consequently, $\widehat{GD^\beta u_.}(k) = (ik)^\beta G_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$.

Applying the Fourier transform on both sides of the equation (5.10), we obtain

$$((ik)^\alpha - F_k - (ik)^\beta G_k - A)\widehat{u}(k) = \widehat{h}(k) = x,$$

since x is arbitrary, we have that for $k \in \mathbb{Z}$ the operators $((ik)^\alpha - F_k - (ik)^\beta G_k - A)$ are surjective.

On the other hand, let $z \in D(A)$, and assume that $((ik)^\alpha - F_k - (ik)^\beta G_k - A)z = 0$. Substituting $u(t) = e^{ikt}z$ in equation (5.6), we see that u is a periodic solution of this equation when $f \equiv 0$. The uniqueness of solution implies that $z = 0$.

Since for all $k \in \mathbb{Z}$ the linear operators N_k are closed defined in whole space X, it follows from closed graph theorem that $N_k \in \mathcal{B}(X)$. Thus $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$.

Let $f \in B^s_{p,q}(\mathbb{T}; X)$. By (i), there exists a function $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ such that the functions $t \mapsto Fu_t$ and $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T}; X)$ and u is the unique strong solution of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi].$$

Applying Fourier transform on the both sides of the preceding equation, we have

$$((ik)^\alpha - F_k - (ik)^\beta G_k - A)\widehat{u}(k) = \widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Since for all $k \in \mathbb{Z}$ the operators $\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)$ are invertible, we have

$$\widehat{u}(k) = ((ik)^\alpha - F_k - (ik)^\beta G_k - A)^{-1}\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $(ik)^\alpha\widehat{u}(k) = \widehat{D^\alpha u}(k) = (ik)^\alpha N_k\widehat{f}(k) = M_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$.
Since $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$, it follows from Proposition 5.2 that $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. Therefore, by definition the family $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. It follows from Lemma 5.5 that $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators.

$(ii) \Rightarrow (i)$. We are assuming that the hypothesis and (ii) condition of Lemma 5.5 are satisfied. Therefore, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Define the family of operator $\{I_k\}_{k\in\mathbb{Z}}$, by $I_k = \frac{1}{(ik)^\alpha}I$ when $k \neq 0$ and $I_0 = I$. It follows from Theorem 1.4 that $\{I_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Since $N_k = I_kM_k$ for all $k \in \mathbb{Z} \setminus \{0\}$ we have $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Accordingly, for an arbitrary function $f \in B^s_{p,q}(\mathbb{T};X)$ there are two functions $u, w \in B^s_{p,q}(\mathbb{T},X)$ such that

$$\widehat{u}(k) = N_k\widehat{f}(k) \text{ and } \widehat{w}(k) = (ik)^\alpha N_k\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.11}$$

Therefore, $\widehat{w}(k) = (ik)^\alpha\widehat{u}(k) = \widehat{D^\alpha u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $D^\alpha u = w$. This implies that $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. It follows from Proposition 5.2 that $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$.

On the other hand, it follows from Lemma 5.3 that $\{F_kN_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Consequently, there exists a function $g \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{g}(k) = F_kN_k\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By equality in (5.11) we have $\widehat{g}(k) = F_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$.
As we have shown, $\widehat{Fu_\cdot}(k) = F_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $Fu_\cdot = g$. This implies that that $Fu_\cdot \in B^s_{p,q}(\mathbb{T};X)$. Hence, the function $t \mapsto Fu_t$ belongs to $B^s_{p,q}(\mathbb{T};X)$.
In the same manner, it follows from Lemma 5.4 that $\{(ik)^\beta G_kN_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Hence there exists a function $h \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{h}(k) = (ik)^\beta G_kN_k\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Using again the equality (5.11) we have

$$\widehat{h}(k) = (ik)^\beta G_k\widehat{u}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Since $(ik)^\beta G_k\widehat{u}(k) = \widehat{GD^\beta u_\cdot}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients we have that $GD^\beta u_\cdot = h$. This implies that that $GD^\beta u_\cdot \in B^s_{p,q}(\mathbb{T};X)$, and the function $t \mapsto GD^\beta u_t$ belongs to $B^s_{p,q}(\mathbb{T};X)$. It follows from equality (5.11) that

$$\widehat{u}(k) = ((ik)^\alpha - F_k - (ik)^\beta G_k - A)^{-1}\widehat{f}(k).$$

Thus,

$$((ik)^\alpha - F_k - (ik)^\beta G_k - A)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Using the fact that A is a closed operator, from the fact that $B^s_{p,q}(\mathbb{T};X)$ is continuously embedded into $L^p(\mathbb{T};X)$ and [7, Lemma 3.1] it follows that $u(t) \in D(A)$ for almost $t \in [0,2\pi]$. Moreover, by uniqueness of Fourier coefficients we have

$$D^\alpha_t u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t)$$

for almost $t \in [0,2\pi]$. Since f, $Fu_.$, $GD^\beta u_.$ and $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$, we conclude that $Au \in B^s_{p,q}(\mathbb{T};X)$. This implies that $u \in B^s_{p,q}(\mathbb{T};[D(A)])$. Therefore, u is a strong $B^s_{p,q}$–solution of equation (5.6).

Since $((ik)^\alpha I - (ik)^\beta G_k - F_k - A)^{-1}$ is invertible for all $k \in \mathbb{Z}$, this strong $B^s_{p,q}$–solution is unique. Therefore the equation (5.6) has $B^s_{p,q}$–maximal regularity. ∎

When the operators A, F and G satisfy some additional conditions, our next corollary provides a simple criterion to verify that the family $\{N_k\}_{k\in\mathbb{Z}}$ is bounded. Let $\alpha > 0$, for $k \in \mathbb{Z}$, we define the operators $S_k = ((ik)^\alpha - A)^{-1}$.

Corollary 5.1. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let X be a Banach space. Assume further that the sequence $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F2)** and **(G2)** respectively. If the family of operators $\{(ik)^\alpha((ik)^\alpha - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded, and $\sup_{k\in\mathbb{Z}}\left\|((ik)^\beta G_k + F_k)((ik)^\alpha - A)^{-1}\right\| < 1$, then the equation (5.6) has $B^s_{p,q}$–maximal regularity.*

Proof. Since $\sup_{k\in\mathbb{Z}}\left\|((ik)^\beta G_k + F_k)((ik)^\alpha - A)^{-1}\right\| < 1$, we have that the family

$$\left\{\left(I - ((ik)^\beta G_k + F_k)S_k\right)^{-1}\right\}_{k\in\mathbb{Z}}$$

is bounded. In addition

$$N_k = \left[((ik)^\alpha - A)(I - ((ik)^\beta G_k + F_k)S_k)\right]^{-1} = (I - ((ik)^\beta G_k + F_k)S_k)^{-1}((ik)^\alpha - A)^{-1}.$$

Therefore the family $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is bounded. Since the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions ***(F2)*** and ***(G2)*** respectively, it follows from Theorem 5.1 that the equation (5.6) has $B^s_{p,q}$–maximal regularity. ∎

5.2 Existence and uniqueness of periodic strong solution of a neutral equation in Besov spaces

Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$, and $0 < r < 2\pi$. Consider $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ linear closed operators such that $D(A) \subseteq D(B)$, and the operators $F, G \in \mathcal{B}(B^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. In this section we use the results about $B^s_{p,q}$–maximal regularity of the equation (5.6) to prove that the abstract fractional neutral differential equation

$$D^\alpha(u(t) - Bu(t-r)) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0,2\pi], \tag{5.1}$$

has a unique periodic strong $B^s_{p,q}$–solution, provided that $f \in B^s_{p,q}(\mathbb{T};X)$.

Let $1 \leqslant p,q \leqslant \infty$ and $s > 0$. Suppose that the equation (5.6) have $B^s_{p,q}$–maximal regularity, hence for each $g \in B^s_{p,q}(\mathbb{T};X)$ there exists a unique strong $B^s_{p,q}$–solution v of the equation

$$D^\alpha v = Av + Fv_t + GD^\beta v_t + g(t). \tag{5.2}$$

Denote by Ψ the operator $\Psi : B^s_{p,q}(\mathbb{T};X) \to B^s_{p,q}(\mathbb{T};X)$ defined by the formula $\Psi(g) = D^\alpha v$, where v is the unique strong $B^s_{p,q}$–solution of the equation (5.2). This linear operator is well defined. Moreover, by the closed graph theorem there exists a constant $M \geqslant 0$ such that for all $f \in B^s_{p,q}(\mathbb{T};X)$ we have

$$\|D^\alpha u\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Fu_.\|_{B^s_{p,q}} + \|GD^\beta u_.\|_{B^s_{p,q}} \leqslant M\|f\|_{B^s_{p,q}}.$$

Lemma 5.6. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\,\|\Psi\| < 1$ and $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. Suppose further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions (F2) and (G2) respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then the family $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. Denote $R_k = (I - e^{-ikr}(ik)^\alpha BN_k)^{-1}$ for all $k \in \mathbb{Z}$. Since $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, the family of operators $\{R_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$. Let $f \in B^s_{p,q}(\mathbb{T};X)$ fixed. Define the map $\mathcal{P} : B^s_{p,q}(\mathbb{T};X) \to B^s_{p,q}(\mathbb{T};X)$ by

$$\mathcal{P}\varphi(t) = B\Psi(\varphi)(t-r) + f(t).$$

By Theorem 5.1 the map $\mathcal{P}$ is well defined. Moreover, this mapping is a contraction, thus there exists a function $g \in B^s_{p,q}(\mathbb{T};X)$ such that

$$g(t) \;=\; B\Psi(g)(t-r) + f(t) \;=\; BD^\alpha u(t-r) + f(t), \tag{5.3}$$

where u is the unique strong $B^s_{p,q}$–solution of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + g(t), \quad t \in [0,2\pi], \quad 0 < \beta < \alpha \leqslant 2. \tag{5.4}$$

Applying the Fourier transform to the both sides of equation (5.3) we have

$$\widehat{g}(k) = e^{-ikr}(ik)^\alpha B\widehat{u}(k) + \widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}. \tag{5.5}$$

On the other hand, applying the Fourier transform to the both sides of equation (5.4) we have

$$\widehat{u}(k) = N_k\widehat{g}(k), \quad \text{for all } k \in \mathbb{Z}. \tag{5.6}$$

Therefore, $\widehat{g}(k) = e^{-ikr}(ik)^\alpha BN_k\widehat{g}(k) + \widehat{f}(k)$, for all $k \in \mathbb{Z}$. This implies that $\widehat{g}(k) = R_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$. Hence, the family of operators $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. ■

The following theorem establishes the existence and uniqueness of a strong $B^s_{p,q}$–solution for the equation (5.1). We use the same notations introduced in the preceding lemma.

Theorem 5.1. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\,\|\Psi\| < 1$ and $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. Suppose further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions (F2) and (G2) respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then for each $f \in B^s_{p,q}(\mathbb{T}; X)$ there exists an unique strong $B^s_{p,q}$–solution of equation* (5.1).

Proof. It follows from Lemma 5.6 that the family of operators $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Denote $R_k = (I - e^{-ikr}(ik)^\alpha BN_k)^{-1}$. Let $f \in B^s_{p,q}(\mathbb{T}; X)$. Since $\{R_k\}_{k\in\mathbb{Z}}$ is $B^s_{p,q}$–multiplier, there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{g}(k) = R_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.7}$$

On the other hand, by Theorem 5.1, there exists a function $u \in B^s_{p,q}(\mathbb{T}; X)$ such that u is the unique strong $B^s_{p,q}$–solution of equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + g(t), \qquad t \in [0, 2\pi], \quad 0 < \beta < \alpha \leqslant 2. \tag{5.8}$$

Applying the Fourier transform to the both sides of the preceding equality we have $\widehat{u}(k) = N_k \widehat{g}(k)$ for all $k \in \mathbb{Z}$.

It follows from equality (5.7) that $\widehat{u}(k) = N_k R_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$. Note that

$$N_k R_k = ((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A)^{-1} \quad \text{for all } k \in \mathbb{Z}.$$

Thus, $((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A)\widehat{u}(k) = \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

Since A is a closed linear operator, it follows from uniqueness of Fourier coefficients that u satisfies the equation

$$D^\alpha (u(t) - Bu(t - r)) = Au(t) + Fu_t + GD^\beta u_t + f(t) \quad \text{for almost } t \in [0, 2\pi].$$

Hence u is a strong $B^s_{p,q}$–solution of equation (5.1). It only remains to show that the strong $B^s_{p,q}$–solution is unique. Indeed, let $f \in B^s_{p,q}(\mathbb{T}; X)$. Suppose that the equation (5.1) has two strong $B^s_{p,q}$–solutions, u_1 and u_2. A direct computation shows that

$$((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A)(\widehat{u_1}(k) - \widehat{u_2}(k)) = 0$$

for all $k \in \mathbb{Z}$. Since $((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A)$ is invertible, for all $k \in \mathbb{Z}$ we have that $\widehat{u_1}(k) = \widehat{u_2}(k)$. By the uniqueness of the Fourier coefficients we conclude that $u_1 \equiv u_2$. ∎

5.3 Maximal regularity on periodic Triebel–Lizorkin spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. In this section we study $F^s_{p,q}$–maximal regularity of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \qquad t \in [0, 2\pi], \tag{5.1}$$

where the mapping f is a X–valued function belonging to the periodic Triebel–Lizorkin space $F^s_{p,q}(\mathbb{T}; X)$ and the delay operators $F, G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi, 0]); X)$. The rest of the terms of this equation are defined as those of the equation (5.6). For this reason we present a characterization of the periodic X–valued Triebel–Lizorkin $F^{s+\alpha}_{p,q}(\mathbb{T}; X)$ using the fractional derivative of Liouville–Grünwald–Letnikov .

Proposition 5.3. *Let X be a Banach space, $1 \leqslant p,q \leqslant \infty$, and $s > 0$. If $\alpha > 0$ then*

$$F^{s+\alpha}_{p,q}(\mathbb{T};X) = \{u \in F^s_{p,q}(\mathbb{T};X) : D^\alpha u \in F^s_{p,q}(\mathbb{T};X)\}.$$

Proof. The proof follows the same lines as those made in the proof Proposition 5.2 ∎

Using this characterization we define the $F^s_{p,q}$–maximal regularity for the solutions of equation (5.1) in the particular case $s > 0$.

Definition 5.3. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and let $f \in F^s_{p,q}(\mathbb{T};X)$. A function u is called strong $F^s_{p,q}$–solution of equation* (5.1) *if $u \in F^{s+\alpha}_{p,q}(\mathbb{T};X) \cap F^s_{p,q}(\mathbb{T};[D(A)])$ and u satisfies the equation* (5.1) *for almost $t \in [0,2\pi]$ and the functions $t \mapsto Fu_t$, $t \mapsto GD^\beta u_t$ belongs to $F^s_{p,q}(\mathbb{T};X)$. We say that the equation* (5.1) *has $F^s_{p,q}$–maximal regularity if, for each $f \in F^s_{p,q}(\mathbb{T};X)$ the equation* (5.1) *has unique strong $F^s_{p,q}$–solution.*

One of the most important results of this chapter is the theorem 5.1. To prove it we need the following results which are related with bounded families of operators.

Lemma 5.7. *Let X be a Banach space. Consider $1 \leqslant p,q \leqslant \infty, s > 0$ and $0 < \beta < \alpha \leqslant 2$. Assume further $G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. If the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(G3)**, then*

$$\left\{\frac{k^3}{a_k}(\Delta^3 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators.

Proof. It is clear that, for all $k \in \mathbb{Z}$, we obtain

$$\begin{aligned}(\Delta^3 b_k G_k) &= b_k(\Delta^3 G_k) + (b_{k+3} - b_k)(\Delta^2 G_{k+1}) + (\Delta^2 b_{k+1})(\Delta^1 G_{k+1}) \\ &\quad + (\Delta^3 b_k) G_{k+2} - 2(\Delta^2 b_k)(\Delta^1 G_{k+1}).\end{aligned}$$

Now, for all $k \in \mathbb{Z} \setminus \{0\}$ we have the identity

$$\begin{aligned}\frac{k^3}{a_k}(\Delta^3 b_k G_k) &= \frac{k b_k}{a_k}(\Delta^3 G_k) + \frac{k(b_{k+3} - b_k)}{b_k}\frac{k^2 b_k}{a_k}(\Delta^2 G_{k+1}) + \frac{k^2(\Delta^2 b_{k+1})}{b_k}\frac{k b_k}{a_k}(\Delta^1 G_{k+1}) \\ &\quad + \frac{k^3(\Delta^3 b_k)}{b_k}\frac{b_k}{a_k} G_{k+2} - 2\frac{k^2(\Delta^2 b_k)}{b_k}\frac{k b_k}{a_k}(\Delta^1 G_{k+1}).\end{aligned}$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 3–regular and $\{G_k\}_{k\in\mathbb{Z}}$ is a bounded family satisfying condition ***(G3)***, it follows from Lemma 5.1 that

$$\left\{\frac{k^3}{a_k}(\Delta^3 b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators. ∎

Lemma 5.8. *Consider $1 \leqslant p,q \leqslant \infty, s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume further that $F, G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. Suppose that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$, and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively. If the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then*

$$\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$$

is a bounded family of operators.

Proof. Note that, for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}(\Delta^3 N_k) \;=\; & [(\Delta^2 N_{k+1}) + (\Delta^2 N_k)]\,[(-\Delta^1 a_{k+2}) + (\Delta^1 F_{k+2}) + (\Delta^1 b_{k+2} G_{k+2})]\,N_{k+1} \\ & +[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\,[(-\Delta^2 a_{k+1}) + (\Delta^2 F_k) + (\Delta^2 b_{k+1} G_{k+1})]\,N_{k+1} \\ & +[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\,[(-\Delta^1 a_{k+1}) + (\Delta^1 F_k) + (\Delta^1 b_{k+1} G_{k+1}](\Delta^1 N_k) \\ & +(\Delta^1 N_k)[(-\Delta^2 a_{k+1}) + (\Delta^2 F_{k+1}) + (\Delta^2 b_{k+1} G_{k+1})]\,N_{k+2} \\ & +N_k\,[(-\Delta^3 a_k) + (\Delta^3 F_k) + (\Delta^3 b_k G_k)]\,N_{k+2} \\ & +N_k\,[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)](\Delta^1 N_{k+1}).\end{aligned}$$

From the preceding identity, we conclude

$$\begin{aligned}k^3 a_k(\Delta^3 N_k) = {} & k^2 a_k\,[(\Delta^2 N_{k+1}) + (\Delta^2 N_k)]\,\frac{k}{a_k}\,[(-\Delta^1 a_{k+2}) + (\Delta^1 F_{k+2}) + (\Delta^1 b_{k+2} G_{k+2})]\,a_k N_{k+1} \\ & + k a_k\,[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\,\frac{k^2}{a_k}\,[(-\Delta^2 a_{k+1}) + (\Delta^2 F_k) + (\Delta^2 b_{k+1} G_{k+1})]\,a_k N_{k+1} \\ & + k a_k\,[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\,\frac{k}{a_k}\,[(-\Delta^1 a_{k+1}) + (\Delta^1 F_k) + (\Delta^1 b_{k+1} G_{k+1})]\,k a_k(\Delta^1 N_k) \\ & + k a_k(\Delta^1 N_k)\frac{k^2}{a_k}\,[(-\Delta^2 a_{k+1}) + (\Delta^2 F_{k+1}) + (\Delta^2 b_{k+1} G_{k+1})]\,a_k N_{k+2} \\ & + M_k\frac{k^3}{a_k}\,[(-\Delta^3 a_k) + (\Delta^3 F_k) + (\Delta^3 b_k G_k)]\,a_k N_{k+2} \\ & + M_k\frac{k^2}{a_k}\,[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)]\,k a_k(\Delta^1 N_{k+1}),\end{aligned}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is a 3–regular sequence, it follows from Lemma 5.1 and Lemma 5.2 that all the terms in the right hand of the preceding equality are uniformly bounded. Moreover, for $k = 0$ is clear that $k^3 a_k(\Delta^3 N_k)$ is a bounded operator. Therefore, $\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ is a bounded family of operators. ∎

Lemma 5.9. *Consider $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume further that $F, G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-\rho, 0]; X); X)$. Suppose that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Proof. According to the Theorem 1.5, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 3. It follows from Lemma 5.3 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 2. It remains to show that $\{k^3(\Delta^3 F_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. To prove this we first observe that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned}(\Delta^3 F_k N_k) = {} & F_k(\Delta^3 N_k) + (F_{k+3} - F_k)(\Delta^2 N_{k+1}) + (\Delta^2 F_{k+1})(\Delta^1 N_{k+1}) \\ & + (\Delta^3 F_k) N_{k+2} - 2(\Delta^2 F_k)(\Delta^1 N_{k+1}),\end{aligned}$$

Thus, for all $k \in \mathbb{Z} \setminus \{0\}$, it holds

$$\begin{aligned}k^3(\Delta^3 F_k N_k) = {} & \frac{1}{a_k} F_k k^3 a_k(\Delta^3 N_k) + \frac{k}{a_k}(F_{k+3} - F_k) k^2 a_k(\Delta^2 N_{k+1}) + \frac{k^2}{a_k}(\Delta^2 F_{k+1}) k a_k(\Delta^1 N_{k+1}) \\ & + \frac{k^3}{a_k}(\Delta^3 F_k) a_k N_{k+2} - 2\frac{k^2}{a_k}(\Delta^2 F_k) k a_k(\Delta^1 N_{k+1}),\end{aligned}$$

Since the family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(F3)***, and clearly, when $k = 0$ the operator $k^3(\Delta^3 F_k N_k)$ is bounded, the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 3. ∎

Lemma 5.10. *Consider $1 \leqslant p,q \leqslant \infty, s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume further that $F, G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. Suppose that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then the family $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.5, it suffices to show that the family of operators $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 3. It follows from Lemma 5.4 that $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 2. It remains to show that $\{k^3(\Delta^3 b_k G_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. Note that, for all $k \in \mathbb{Z}$,

$$\begin{aligned}(\Delta^3 b_k G_k N_k) &= b_k G_k(\Delta^3 N_k) + (b_{k+3}G_{k+3} - b_k G_k)(\Delta^2 N_{k+1}) + (\Delta^2 b_{k+1}G_{k+1})(\Delta^1 N_{k+1}) \\ &\quad + (\Delta^3 b_k G_k)N_{k+2} - 2(\Delta^2 b_k G_k)(\Delta^1 N_{k+1}).\end{aligned}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned}k^3(\Delta^3 b_k G_k N_k) &= \frac{b_k}{a_k} G_k k^3 a_k(\Delta^3 N_k) + \frac{k}{a_k}(b_{k+3}G_{k+3} - b_k G_k)k^2 a_k(\Delta^2 N_{k+1}) \\ &\quad + \frac{k^2}{a_k}(\Delta^2 b_{k+1}G_{k+1})k a_k(\Delta^1 N_{k+1}) + \frac{k^3}{a_k}(\Delta^3 b_k G_k)a_k N_{k+2} \\ &\quad - 2\frac{k^2}{a_k}(\Delta^2 b_k G_k)k a_k(\Delta^1 N_{k+1}).\end{aligned}$$

Since $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(G3)***, it follows from Lemma 5.1, Lemma 5.2, Lemma 5.7 and Lemma 5.8 that all the terms in the right hand of the preceding identity are uniformly bounded. In addition, $k^3(\Delta^3 b_k G_k N_k)$ is a bounded operator when $k = 0$. Consequently, the family $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 3. ∎

Lemma 5.11. *Let $1 \leqslant p,q \leqslant \infty, s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Suppose that $F, G \in \mathcal{B}(F^{s+\alpha}_{p,q}([-2\pi,0];X);X)$. Assume that the operators $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, then the following assertions are equivalent.*

(i) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

(ii) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. According Theorem 1.5 it suffices to show that $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 3. It follows from Lemma 5.5 that $\{M_k\}_{k\in\mathbb{Z}}$ is a family of operators $\mathcal{M}$–bounded of order 2. It remains to show that $\{k^3(\Delta^3 M_k)\}_{k\in\mathbb{Z}}$ is a bounded family of operators. For this we note

$$\begin{aligned}\Delta^3 M_k &= a_k(\Delta^3 N_k) + (a_{k+3} - a_k)(\Delta^2 N_{k+1}) + (\Delta^2 a_{k+1})(\Delta^1 N_{k+1}) \\ &\quad + (\Delta^3 a_k)N_{k+2} - 2(\Delta^2 a_k)(\Delta^1 N_{k+1}).\end{aligned}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}k^3(\Delta^3 M_k) &= k^3 a_k(\Delta^3 N_k) + \frac{k(a_{k+3} - a_k)}{a_k} k a_k(\Delta^2 N_{k+1}) + \frac{k^2}{a_k}(\Delta^2 a_{k+1}) k a_k(\Delta^1 N_{k+1}) \\ &\quad + \frac{k^3(\Delta^3 a_k)}{a_k} a_k N_{k+2} - \frac{2k^2(\Delta^2 a_k)}{a_k} k a_k(\Delta^1 N_{k+1}).\end{aligned}$$

Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3–regular, and all hypotheses of the Lemma 5.2 and Lemma 5.7 are fulfilled, we conclude that all the operators included in the right hand side of the equality above are uniformly bounded. Additionally, when $k = 0$ the operator $k^3(\Delta^3 M_k)$ is bounded. In consequence, the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$–bounded family of order 3.
$(ii) \Rightarrow (i)$ This proof is analogous to the proof of the implication $(ii) \Rightarrow (i)$ of the Lemma 5.5, so we omit it. ∎

We are now ready to prove the main results of this section. We omit their proof because are analogous to the proof of the Theorem 5.1 and Corollary 5.1, respectively.

Theorem 5.1. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$. Let be X a Banach space. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, then the following assertions are equivalent.*

(i) The equation (5.1) *has $F^s_{p,q}$–maximal regularity.*

(ii) The families $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X)$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

Our next objective is to give other conditions on the operators A, F and G that imply the hypotheses of Theorem 5.1 and are easier to verify in applications. With this purpose, for $k \in \mathbb{Z}$ we define the operators $S_k = ((ik)^\alpha - A)^{-1}$.

Corollary 5.2. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$. Let be X a Banach space. Assume that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively. If the family of operators $\{(ik)^\alpha S_k\}_{k\in\mathbb{Z}}$ is bounded, and $\sup_{k\in\mathbb{Z}}\left\|((ik)^\beta G_k + F_k)S_k\right\| < 1$, then the solution of equation* (5.1) *has $F^s_{p,q}$–maximal regularity.*

5.4 Existence and uniqueness of periodic strong solution of neutral equation in Triebel–Lizorkin spaces

Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ linear closed operators such that $D(A) \subseteq D(B)$. By using the results about $F^s_{p,q}$–maximal regularity of the equation (5.1) obtained in section 5.3, we prove that the fractional neutral differential equation

$$D^\alpha(u(t) - Bu(t-r)) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi], \tag{5.1}$$

has a unique periodic strong $F^s_{p,q}$–solution. Suppose that the equation (5.1) have $F^s_{p,q}$–maximal regularity, then for each $g \in F^s_{p,q}(\mathbb{T}; X)$ there exists a unique strong $F^s_{p,q}$–solution v of the equation

$$D^\alpha v = Av + Fv_\cdot + GD^\beta v_\cdot + g(t). \tag{5.2}$$

Denote by Ψ the operator $\Psi : F^s_{p,q}(\mathbb{T}; X) \to F^s_{p,q}(\mathbb{T}; X)$ defined by the formula $\Psi(g) = D^\alpha v$, where v is the unique strong $F^s_{p,q}$–solution of the equation (5.2). This linear operator is well defined. Moreover, by the closed graph theorem there exists a constant $M \geqslant 0$ such that for all $f \in F^s_{p,q}(\mathbb{T}; X)$ we have

$$\|D^\alpha u\|_{F^s_{p,q}} + \|Au\|_{F^s_{p,q}} + \|Fu_\cdot\|_{F^s_{p,q}} + \|GD^\beta u_\cdot\|_{F^s_{p,q}} \leqslant M\|f\|_{F^s_{p,q}}.$$

With the following two results we study the existence and uniqueness of a strong $F^s_{p,q}$–solution for the equation (5.1). We omit the details of their proofs because the are analogous to Lemma 5.6 and Theorem 5.1 respectively.

Lemma 5.12. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\,\|\Psi\| < 1$ and $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. Suppose further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions (F3) and (G3) respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then the family $\{(I - e^{-ikr}(ik)^\alpha B N_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Theorem 5.1. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\,\|\Psi\| < 1$ and $N_k \in \mathcal{B}(X)$, for all $k \in \mathbb{Z}$. Assume further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions (F3) and (G3) respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then for each $f \in F^s_{p,q}(\mathbb{T};X)$ there exists an unique strong $F^s_{p,q}$–solution of equation (5.1).*

5.5 Applications

In this last section we present an application of our results to partial neutral functional differential equations. As we have already mentioned, equations of type (5.1) and (5.2), have been studied by several authors to model important physical systems. Next we consider an integro–differential perturbation of the equation studied in [1,36].

***Example* 5.1.** *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and $1 < \beta < \alpha < 2$ and $0 < r < 2\pi$. Consider the following neutral fractional differential equation with finite delay*

$$\left.\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}[w(t,\xi) - bw(t-r,\xi)] &= \frac{\partial^2}{\partial \xi^2} w(t,\xi) + \int_{-2\pi}^{0} q_1\gamma(s) w(t+s,\xi)\,ds \\
&+ \int_{-2\pi}^{0} q_2\gamma(s)\frac{\partial^\beta}{\partial t^\beta} w(t+s,\xi)\,ds + \tilde{f}(t,\xi),\ t\in\mathbb{R},\ \xi\in[0,\pi], \\
w(t,\xi) - bw(t-r,\xi) &= 0, \quad \xi = 0,\pi, \quad t \in \mathbb{R}.
\end{aligned}\right\} \tag{5.3}$$

In order to rewrite the equation (5.3) in the abstract form of the equation (5.1), we consider X as the space $L^2([0,\pi];\mathbb{R})$. The operators A and B are defined by

$$A\varphi = \frac{\partial^2\varphi(\xi)}{\partial\xi^2} \text{ with domain } D(A) = \{\varphi \in L^2([0,\pi];\mathbb{R}) : \varphi'' \in L^2([0,\pi];\mathbb{R}),\ \varphi(0) = \varphi(\pi) = 0\},$$

$$B\varphi = b\varphi, \quad \text{where the constant } b \text{ is a positive number.}$$

We assume that the function $\gamma : [-2\pi, 0] \to \mathbb{R}$ is a function of class C^2, and the operators $F, G : B^{s+\alpha}_{p,q}([-2\pi,0]; L^2([0,\pi];\mathbb{R})) \to L^2([0,\pi];\mathbb{R})$ are described by the formula

$$(F\psi)(\xi) = \int_{-2\pi}^{0} q_1\gamma(s)\psi(s)(\xi)\,ds \text{ and } (G\psi)(\xi) = \int_{-2\pi}^{0} q_2\gamma(s)\psi(s)(\xi)\,ds.$$

It follows from Cauchy–Schwartz inequality that $F\psi$ and $G\psi$ are elements of $L^2([0,\pi];\mathbb{R})$. Moreover, since $B^{s+\alpha}_{p,q}([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$ is continuously embedded in $C([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$, the maps F and G define bounded linear operators from $B^{s+\alpha}_{p,q}([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$ to $L^2([0,\pi];\mathbb{R})$.

Let identify $f(t) = \widetilde{f}(t, \cdot)$, and assume that $\widetilde{f}(t, \xi)$ is 2π–periodic at the variable t.

With all these considerations the equation (5.3) takes the abstract form of the equation (5.2).

We will show that there exists $b > 0$ sufficiently small such that there exists a unique strong $B^s_{p,q}$–solution of equation (5.3), whenever $f \in B^s_{p,q}(\mathbb{T}; L^2([0,\pi]))$. For this purpose, we assume that q_1 and q_2 are positive numbers such that

$$\left|q_1 + q_2\cos\left(\frac{\beta\pi}{2}\right)\right| \leqslant \left|q_2\cos\left(\frac{\beta\pi}{2}\right)\right| \quad \text{and} \quad q_2 K < \sin\left(\frac{\alpha\pi}{2}\right),$$

where K is a constant satisfying $\|F\| \leqslant K$ and $\|G\| \leqslant K$.

Note that for $k \in \mathbb{Z}$ the operators F_k and G_k take the form

$$F_k\varphi = \int_{-2\pi}^{0} q_1\gamma(s)(e_k\varphi)(s)ds \quad \text{and} \quad G_k\varphi = \int_{-2\pi}^{0} q_2\gamma(s)(e_k\varphi)(s)ds.$$

By using the Cauchy–Schwartz inequality we conclude that $F_k \in \mathcal{B}(L^2([0,\pi];\mathbb{R}))$ and $G_k \in \mathcal{B}(L^2([0,\pi];\mathbb{R}))$ for all $k \in \mathbb{Z}$. Integrating by parts twice, we obtain the following representation for the operator F_k and G_k.

$$F_k\varphi = \frac{iq_1[\gamma(-2\pi) - \gamma(0)]\varphi}{k} + \frac{q_1[\gamma'(0) - \gamma'(-2\pi)]\varphi}{k^2} - \frac{iq_1}{k^2}\int_{-2\pi}^{0} \gamma''(s)e^{iks}\varphi ds,$$

and

$$G_k\varphi = \frac{iq_2[\gamma(-2\pi) - \gamma(0)]\varphi}{k} + \frac{q_1[\gamma'(0) - \gamma'(-2\pi)]\varphi}{k^2} - \frac{iq_2}{k^2}\int_{-2\pi}^{0} \gamma''(s)e^{iks}\varphi ds.$$

With this representation, by a direct computation, it follows that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions *(F2)* and *(G2)* respectively.

On another hand, the spectrum of A consists of eigenvalues $-n^2$, for $n \in \mathbb{N}$. Their associated eigenvectors are given by

$$x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

Moreover, the set $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2([0,\pi];\mathbb{R})$. In particular

$$A\varphi = \sum_{n=1}^{\infty} -n^2\langle\varphi, x_n\rangle x_n, \qquad \text{for all } \varphi \in D(A). \tag{5.4}$$

Therefore $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and

$$((ik)^\alpha I - A)^{-1}\varphi = \sum_{n\in\mathbb{N}} \frac{1}{(ik)^\alpha + n^2}\langle\varphi, x_n\rangle x_n. \tag{5.5}$$

Since $1 < \alpha < 2$ we have that $Re(ik)^\alpha < 0$ for $k \neq 0$. Thus, for $k \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$ we have

$$|(ik)^\alpha + n^2| \geqslant \left|Im((ik)^\alpha)\right| = |k|^\alpha \sin\left(\frac{\alpha\pi}{2}\right).$$

Hence, for $k \neq 0$ we have the following estimative

$$\left\|((ik)^\alpha I - A)^{-1}\right\| \leqslant \frac{1}{|k|^\alpha \sin(\frac{\alpha\pi}{2})}. \tag{5.6}$$

It is clear from equality (5.5) that $\left\|((ik)^\alpha I - A)^{-1}\right\| < \infty$, in the case $k = 0$. On the other hand, for all $k \neq 0$ we have that

$$\left\|(ik)^\beta G_k + F_k\right\| \leqslant |(ik)^\beta q_2 + q_1|K \leqslant q_2|(ik)^\beta|K = q_2|k|^\beta K. \qquad (5.7)$$

Hence, we have that

$$\sup_{k\in\mathbb{Z}} \|(ik)^\alpha ((ik)^\alpha I - A)^{-1}\| < \infty,$$

and

$$\left\|((ik)^\beta G_k + F_k)((ik)^\alpha I - A)^{-1}\right\| \leqslant \frac{q_2\,|k|^\beta K}{|k|^\alpha \sin(\frac{\alpha\pi}{2})}.$$

Since $q_2 K < \sin(\frac{\alpha\pi}{2})$ we have $\sup_{k\in\mathbb{Z}}\left\|((ik)^\beta G_k + F_k)S_k\right\| < 1$. From Corollary 5.1, it follows that fractional delay equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \qquad t \in [0, 2\pi], \qquad (5.8)$$

where the operators A, F and G are described as above, has $B^s_{p,q}$–maximal regularity. Thus the mapping $\Psi : B^s_{p,q}(\mathbb{T}; X) \to B^s_{p,q}(\mathbb{T}; X)$, defined by $\Psi(f) = D^\alpha u$ where u is the unique strong $B^s_{p,q}$–solution of equation (5.8), is a bounded linear operator. Therefore there exists $C_2 \geqslant 0$ such that $\|\Psi\| \leqslant C_2$.

Moreover, there exists $C_1 \geqslant 0$ such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|N_k\| \leqslant C_1$. If the constant $b > 0$ satisfies the condition $b < \min\left\{\frac{1}{C_1}, \frac{1}{C_2}\right\}$ we have

$$\sup_{k\in\mathbb{Z}} b|k|^\alpha \|N_k\| < 1 \quad \text{and} \quad b\|\Psi\| < 1.$$

It follows from Theorem 5.1 that equation (5.3) has a unique strong $B^s_{p,q}$–solution.

Bibliography

[1] M. Adimy, A. Elazzouzi, and K. Ezzinbi, *Bohr-Neugebauer type theorem for some partial neutral functional differential equations*, Nonlinear Anal. **66** (2007), no. 5, 1145–1160. MR2286625

[2] R. P. Agarwal, J. P. C. dos Santos, and C. Cuevas, *Analytic resolvent operator and existence results for fractional integro-differential equations*, J. Abstr. Differ. Equ. Appl. **2** (2012), no. 2, 26–47. MR2874361

[3] H. Amann, *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr. **186** (1997), 5–56. MR1461211

[4] V. V. Anh and N. N. Leonenko, *Spectral analysis of fractional kinetic equations with random data*, J. Statist. Phys. **104** (2001), no. 5-6, 1349–1387. MR1859007

[5] W. Arendt, C. Batty, and S. Bu, *Fourier multipliers for Hölder continuous functions and maximal regularity*, Studia Math. **160** (2004), no. 1, 23–51. MR2029738

[6] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Second, Monographs in Mathematics, vol. 96, Birkhäuser/Springer Basel AG, Basel, 2011. MR2798103

[7] W. Arendt and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), no. 2, 311–343. MR1900314

[8] W. Arendt and S. Bu, *Operator-valued Fourier multipliers on periodic Besov spaces and applications*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 15–33. MR2064734

[9] E. G. Bajlekova, *Fractional evolution equations in Banach spaces*, Ph.D. Thesis, 2001. Thesis (Dr.)–Technische Universiteit Eindhoven (The Netherlands). MR2715439

[10] J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker Inc., New York, 1980. MR591679

[11] A. Bátkai and S. Piazzera, *Semigroups for delay equations*, Research Notes in Mathematics, vol. 10, A K Peters Ltd., Wellesley, MA, 2005. MR2181405

[12] S. K. Bose and G. C. Gorain, *Stability of the boundary stabilised internally damped wave equation $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ in a bounded domain in $\mathbf{R}^n$*, Indian J. Math. 40 (1998), no. 1, 1–15. MR1654783

[13] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, Israel J. Math. **108** (1998), 109–138. MR1669396

[14] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. **21** (1983), no. 2, 163–168. MR727340

[15] S. Bu, *Maximal regularity of second order delay equations in Banach spaces*, Acta Math. Sin. (Engl. Ser.) **25** (2009), no. 1, 21–28. MR2465519

[16] S. Bu, *Well-posedness of equations with fractional derivative*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 7, 1223–1232. MR2657796

[17] S. Bu and Y. Fang, *Periodic solutions of delay equations in Besov spaces and Triebel-Lizorkin spaces*, Taiwanese J. Math. **13** (2009), no. 3, 1063–1076. MR2526359

[18] S. Bu and Y. Fang, *Maximal regularity of second order delay equations in Banach spaces*, Sci. China Math. **53** (2010), no. 1, 51–62. MR2594746

[19] S. Bu and J.-M. Kim, *A note on operator-valued Fourier multipliers on Besov spaces*, Math. Nachr. **278** (2005), no. 14, 1659–1664. MR2176269

[20] S. Bu and J.-M. Kim, *Operator-valued Fourier multiplier theorems on Triebel spaces*, Acta Math. Sci. Ser. B Engl. Ed. **25** (2005), no. 4, 599–609. MR2175925

[21] D. L. Burkholder, *Martingales and singular integrals in Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, 2001, pp. 233–269. MR1863694

[22] P. L. Butzer and U. Westphal, *An access to fractional differentiation via fractional difference quotients*, Fractional calculus and its applications (Proc. Internat. Conf., Univ. New Haven, West Haven, Conn., 1974), 1975, pp. 116–145. Lecture Notes in Math., Vol. 457. MR0492115

[23] L. Byszewski, *Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with arbitrary functionals*, J. Math. Anal. Appl. **156** (1991), no. 2, 457–470. MR1103024

[24] L. Byszewski, *Theorem about existence and uniqueness of continuous solution of nonlocal problem for nonlinear hyperbolic equation*, Appl. Anal. **40** (1991), no. 2-3, 173–180. MR1095412

[25] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), no. 2, 494–505. MR1137634

[26] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40** (1991), no. 1, 11–19. MR1121321

[27] L. Byszewski and T. Winiarska, *An abstract nonlocal second order evolution problem*, Opuscula Math. **32** (2012), no. 1, 75–82. MR2852470

[28] C. Cuevas and J. César de Souza, *Existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay*, Nonlinear Anal. **72** (2010), no. 3-4, 1683–1689. MR2577568

[29] C. Cuevas and C. Lizama, *Well posedness for a class of flexible structure in Hölder spaces*, Math. Probl. Eng. (2009), Art. ID 358329, 13. MR2530061

[30] G. Da Prato, P. C. Kunstmann, I. Lasiecka, A. Lunardi, R. Schnaubelt, and L. Weis, *Functional analytic methods for evolution equations*, Lecture Notes in Mathematics, vol. 1855, Springer-Verlag, Berlin, 2004. Edited by M. Iannelli, R. Nagel and S. Piazzera. MR2108957

[31] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92. MR0070164

[32] B. de Andrade and C. Lizama, *Existence of asymptotically almost periodic solutions for damped wave equations*, J. Math. Anal. Appl. **382** (2011), no. 2, 761–771. MR2810829

[33] K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, J. Math. Anal. Appl. **179** (1993), no. 2, 630–637. MR1249842

[34] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **166** (2003), no. 788, viii+114. MR2006641

[35] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR1721989

[36] K. Ezzinbi, S. Fatajou, and G. M. N'Guérékata, *C^n-almost automorphic solutions for partial neutral functional differential equations*, Appl. Anal. **86** (2007), no. 9, 1127–1146. MR2351637

[37] H. O. Fattorini, *Second order linear differential equations in Banach spaces*, North-Holland Mathematics Studies, vol. 108, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 99. MR797071

[38] C. Fernández, C. Lizama, and V. Poblete, *Maximal regularity for flexible structural systems in Lebesgue spaces*, Math. Probl. Eng. (2010), Art. ID 196956, 15. MR2610518

[39] C. Fernández, C. Lizama, and V. Poblete, *Regularity of solutions for a third order differential equation in Hilbert spaces*, Appl. Math. Comput. **217** (2011), no. 21, 8522–8533. MR2802263

[40] X. Fu, *Existence of solutions and periodic solutions for abstract neutral equations with unbounded delay*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **15** (2008), no. 1, 17–35. MR2387348

[41] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on Besov spaces*, Math. Nachr. **251** (2003), 34–51. MR1960803

[42] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces*, J. Funct. Anal. **204** (2003), no. 2, 320–354. MR2017318

[43] G. C. Gorain and S. K. Bose, *Exact controllability and boundary stabilization of flexural vibrations of an internally damped flexible space structure*, Appl. Math. Comput. **126** (2002), no. 2-3, 341–360. MR1879167

[44] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. MR1987179

[45] R. Grimmer and J. Prüss, *On linear Volterra equations in Banach spaces*, Comput. Math. Appl. **11** (1985), no. 1-3, 189–205. Hyperbolic partial differential equations, II. MR787436

[46] A. Grünwald, *Überbegrenzte derivationen und deren anwendung*, Z. Angew. Math. Phys. **12** (1867), 441–480.

[47] J. K. Hale, *Coupled oscillators on a circle*, Resenhas **1** (1994), no. 4, 441–457. Dynamical phase transitions (São Paulo, 1994). MR1357945

[48] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional-differential equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993. MR1243878

[49] H.-P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983), no. 12, 1351–1371. MR726478

[50] H. R. Henríquez and V. Poblete, *Periodic solutions of neutral fractional differential equations*. Submitted.

[51] H. R. Henríquez, V. Poblete, and J. C. Pozo, *Existence of mild solutions for a non–autonomous second order cauchy problem with nonlocal initial conditions*. Submitted.

[52] H. R. Henríquez, *Existence of solutions of non-autonomous second order functional differential equations with infinite delay*, Nonlinear Anal. **74** (2011), no. 10, 3333–3352. MR2793566

[53] H. R. Henríquez and C. Lizama, *Periodic solutions of abstract functional differential equations with infinite delay*, Nonlinear Anal. **75** (2012), no. 4, 2016–2023. MR2870895

[54] H. R. Henríquez, M. Pierri, and A. Prokopczyk, *Periodic solutions of abstract neutral functional differential equations*, J. Math. Anal. Appl. **385** (2012), no. 2, 608–621. MR2834838

[55] H. R. Henríquez and C. H. Vásquez, *Differentiabilty of solutions of the second order abstract Cauchy problem*, Semigroup Forum **64** (2002), no. 3, 472–488. MR1896382

[56] E. Hernández, D. O'Regan, and K. Balachandran, *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal. **73** (2010), no. 10, 3462–3471. MR2680040

[57] E. Hernández, J. Silvério dos Santos, and K. A. G. Azevedo, *On abstract differential equations with nonlocal conditions involving the temporal derivative of the solution*, Indag. Math. (N.S.) **23** (2012), no. 3, 401–422. MR2948636

[58] R. Hilfer (ed.), *Applications of fractional calculus in physics*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000. MR1890104

[59] T. Hytönen and M. Veraar, *R-boundedness of smooth operator-valued functions*, Integral Equations Operator Theory **63** (2009), no. 3, 373–402. MR2491037

[60] S. Ji and G. Li, *Existence results for impulsive differential inclusions with nonlocal conditions*, Comput. Math. Appl. **62** (2011), no. 4, 1908–1915. MR2834816

[61] B. Kaltenbacher, I. Lasiecka, and M. k Pospieszalska, *Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensisty ultrasound* **22** (2012).

[62] N. J. Kalton and G. Lancien, *A solution to the problem of L^p-maximal regularity*, Math. Z. **235** (2000), no. 3, 559–568. MR1800212

[63] N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), no. 2, 319–345. MR1866491

[64] T. Kato, *Perturbation theory for linear operators*, Second, Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132. MR0407617

[65] V. Keyantuo and C. Lizama, *Periodic solutions of second order differential equations in Banach spaces*, Math. Z. **253** (2006), no. 3, 489–514. MR2221083

[66] V. Keyantuo and C. Lizama, *A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications*, Math. Nachr. **284** (2011), no. 4, 494–506. MR2789193

[67] V. Keyantuo, C. Lizama, and V. Poblete, *Periodic solutions of integro-differential equations in vector-valued function spaces*, J. Differential Equations **246** (2009), no. 3, 1007–1037. MR2474584

[68] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006. MR2218073

[69] J. Kisyński, *On operator valued solutions of d'alembert's functional equation, ii*, Studia Mathematica 42 (1972), no. 1, 43–66 (eng).

[70] M. Kozak, *An abstract second-order temporally inhomogeneous linear differential equation. II*, Univ. Iagel. Acta Math. **32** (1995), 263–274. MR1345143

[71] M. Kozak, *A fundamental solution of a second-order differential equation in a Banach space*, Univ. Iagel. Acta Math. **32** (1995), 275–289. MR1345144

[72] A. Letnikov, *Theory and differentiation of fractional order*, Mat. Sb. **3** (1868), 1–66.

[73] F. Li, *Nonlocal Cauchy problem for delay fractional integrodifferential equations of neutral type*, Adv. Difference Equ. (2012), 2012:47, 23. MR2948727

[74] J. Liang and T.-J. Xiao, *Semilinear integrodifferential equations with nonlocal initial conditions*, Comput. Math. Appl. **47** (2004), no. 6-7, 863–875. MR2060321

[75] Y. Lin, *Time-dependent perturbation theory for abstract evolution equations of second order*, Studia Math. **130** (1998), no. 3, 263–274. MR1624819

[76] L. Liu, F. Guo, C. Wu, and Y. Wu, *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, J. Math. Anal. Appl. **309** (2005), no. 2, 638–649. MR2154141

[77] C. Lizama and J. C. Pozo, *Existence of mild solutions for semilinear integrodifferential equations with nonlocal initial conditions*, Abstract and Applied Analysis (2012), 15 pp.

[78] C. Lizama, *A characterization of uniform continuity for Volterra equations in Hilbert spaces*, Proc. Amer. Math. Soc. **126** (1998), no. 12, 3581–3587. MR1469423

[79] C. Lizama, *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl. **324** (2006), no. 2, 921–933. MR2265090

[80] C. Lizama and G. M. N'Guérékata, *Bounded mild solutions for semilinear integro differential equations in Banach spaces*, Integral Equations Operator Theory **68** (2010), no. 2, 207–227. MR2721083

[81] C. Lizama and V. Poblete, *Maximal regularity of delay equations in Banach spaces*, Studia Math. **175** (2006), no. 1, 91–102. MR2261702

[82] C. Lizama and V. Poblete, *Periodic solutions of fractional differential equations with delay*, J. Evol. Equ. **11** (2011), no. 1, 57–70. MR2780573

[83] F. Mainardi and R. Gorenflo, *On Mittag-Leffler-type functions in fractional evolution processes*, J. Comput. Appl. Math. **118** (2000), no. 1-2, 283–299. Higher transcendental functions and their applications. MR1765955

[84] C.-M. Marle, *Mesures et probabilités*, Hermann, Paris, 1974. Collection Enseignement des Sciences, No. 19. MR0486378

[85] R. Metzler and J. Klafter, *Boundary value problems for fractional diffusion equations*, Phys. A **278** (2000), no. 1-2, 107–125. MR1763650

[86] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77. MR1809268

[87] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1993. MR1219954

[88] R. K. Miller, *An integro-differential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. **66** (1978), no. 2, 313–332. MR515894

[89] F. K. Moore and W. E. Gibson, *Propagation of weak disturbances in a gas subject to relaxation effects*, Journal of Aerospace Science and Technologies **27** (1960), 117–127.

[90] W. E. Olmstead and C. A. Roberts, *The one-dimensional heat equation with a nonlocal initial condition*, Appl. Math. Lett. **10** (1997), no. 3, 89–94. MR1457646

[91] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR710486

[92] V. Poblete and J. C. Pozo, *Periodic solutions for a fractional order abstract neutral differential equation with finite delay*. Submitted.

[93] V. Poblete and J. C. Pozo, *Periodic solutions of an abstract third–order differential equation*. Submitted.

[94] V. Poblete, *Solutions of second-order integro-differential equations on periodic Besov spaces*, Proc. Edinb. Math. Soc. (2) **50** (2007), no. 2, 477–492. MR2334958

[95] V. Poblete, *Maximal regularity of second-order equations with delay*, J. Differential Equations **246** (2009), no. 1, 261–276. MR2467023

[96] I. Podlubny, I. Petráš, B. M. Vinagre, P. O'Leary, and L'. Dorčák, *Analogue realizations of fractional-order controllers*, Nonlinear Dynam. **29** (2002), no. 1-4, 281–296. Fractional order calculus and its applications. MR1926477

[97] J. Prüss, *Positivity and regularity of hyperbolic Volterra equations in Banach spaces*, Math. Ann. **279** (1987), no. 2, 317–344. MR919509

[98] J. Prüss, *Evolutionary integral equations and applications*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1993. [2012] reprint of the 1993 edition. MR2964432

[99] B. N. Sadovskiĭ, *On a fixed point principle*, Funkcional. Anal. i Priložen. **1** (1967), no. 2, 74–76. MR0211302

[100] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikol'skiĭ, Translated from the 1987 Russian original, Revised by the authors. MR1347689

[101] H. Serizawa and M. Watanabe, *Time-dependent perturbation for cosine families in Banach spaces*, Houston J. Math. **12** (1986), no. 4, 579–586. MR873652

[102] M. Sova, *Cosine operator functions*, Rozprawy Mat. **49** (1966), 47. MR0193525

[103] C. C. Travis and G. F. Webb, *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families*, Houston J. Math. **3** (1977), no. 4, 555–567. MR0500288

[104] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungar. **32** (1978), no. 1-2, 75–96. MR0499581

[105] C. C. Travis and G. F. Webb, *An abstract second-order semilinear Volterra integro-differential equation*, SIAM J. Math. Anal. **10** (1979), no. 2, 412–424. MR523855

[106] P. Ch. Tsamatos and S. K. Ntouyas, *Boundary value problems for differential equations with deviating arguments*, Czechoslovak Math. J. **47(122)** (1997), no. 1, 1–17. MR1435602

[107] O. van Gaans, *On R-boundedness of unions of sets of operators*, Partial differential equations and functional analysis, 2006, pp. 97–111. MR2240055

[108] V. V. Vasil'ev and S. I. Piskarev, *Differential equations in Banach spaces. II. Theory of cosine operator functions*, J. Math. Sci. (N. Y.) **122** (2004), no. 2, 3055–3174. Functional analysis. MR2084186

[109] J. R. L. Webb, *Optimal constants in a nonlocal boundary value problem*, Nonlinear Anal. **63** (2005), no. 5-7, 672–685. MR2188140

[110] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), no. 4, 735–758. MR1825406

[111] T. Winiarska, *Quasilinear evolution equations with operators dependent on t*, Mat. Stud. **21** (2004), no. 2, 170–178. MR2090817

[112] J. Wu and H. Xia, *Rotating waves in neutral partial functional-differential equations*, J. Dynam. Differential Equations **11** (1999), no. 2, 209–238. MR1695243

[113] J. Wu and H. Xia, *Self-sustained oscillations in a ring array of coupled lossless transmission lines*, J. Differential Equations **124** (1996), no. 1, 247–278. MR1368068

[114] J. Wu, H. Xia, and B. Zhang, *Topological transversality and periodic solutions of neutral functional-differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 1, 199–220. MR1669185

[115] T. J. Xiao and J. Liang, *The Cauchy problem for higher order abstract differential equations*, Chinese J. Contemp. Math. **14** (1993), no. 4, 305–321. MR1277672

[116] W.-X. Zhou and J. Peng, *Existence of solution to a second-order boundary value problem via noncompactness measures*, Discrete Dyn. Nat. Soc. (2012), Art. ID 786404, 16. MR2923387

[117] T. Zhu, C. Song, and G. Li, *Existence of mild solutions for abstract semilinear evolution equations in Banach spaces*, Nonlinear Anal. **75** (2012), no. 1, 177–181. MR2846791