Research Article

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Recovery of a Lamé parameter from displacement fields in nonlinear elasticity models

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Abstract: We study some inverse problems involving elasticity models by assuming the knowledge of measurements of a function of the displaced field. In the first case, we have a linear model of elasticity with a semi-linear type forcing term in the solution. Under the hypothesis the fluid is incompressible, we recover the displaced field and the second Lamé parameter from power density measurements in two dimensions. A stability estimate is shown to hold for small displacement fields, under some natural hypotheses on the direction of the displacement, with the background pressure fixed. On the other hand, we prove in dimensions two and three a stability result for the second Lamé parameter when the displacement field follows the (nonlinear) Saint-Venant model when we add the knowledge of displaced field solution measurements. The Saint-Venant model is the most basic model of a hyperelastic material. The use of over-determined elliptic systems is new in the analysis of linearization of nonlinear inverse elasticity problems.

Keywords: Stability analysis, shear modulus reconstruction, magnetic resonance elastography, biological tissues, optimal control

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1 Introduction

We consider models of isotropic elastic wave equations in a bounded domain Ω. The stress the material is undergoing is described by the Lamé parameters λ, μ and ρ. We study the following problem: is it possible to determine the Lamé parameters λ, μ and ρ from the knowledge of Neumann data of the solution on the boundary? We are interested in the global recovery problem of the displacement of the parameters.

Our main motivation is twofold. In the case of external forcing terms, the current models for linear elasticity are not equipped to cope with any type of power nonlinearity. Moreover, the structure of hyperelastic materials are not accurately described by linear elastic models. A hyperelastic model is one for an ideally elastic material in which the stress-strain relationship is derived from the strain energy density function. This type of model is often known as Green's model which was made rigorous by Ogden [28], in the case of constant coefficients. Hyperelastic models accurately describe the stress-strain behavior of materials such as rubber [26]. Unfilled vulcanized elastomers almost always conform to the hyperelastic ideal. Filled elastomers and biological tissues are also modelled via the hyperelastic idealization [13]. In the linear elasticity case, for reconstruction of the Lamé coefficients concerning biological tissues, one can see [2] for example. Our focus is on some nonlinear mathematical models, and the reduction of the amount of required data to recover the coefficients uniquely. Of the three parameters required to recover the material structure, it is often the most

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natural to recover the parameter μ which encodes more about possible disease in patients than the other parameters. Several diseases involve changes in the mechanical properties of tissue and normal function of tissue, for example in skeletal muscle, heart, lungs and gut [15, 17, 24].

We will consider nonlinear partial differential equations coming from elasticity coupled with the equation of the measurement in the interior of a domain, and we will equip these systems with appropriate boundary conditions. Linearization of the differential operator seen as acting on μ, λ and u creates a linear (system of) PDE(s) in the variables $\delta\mu$, $\delta\lambda$ and δu. This creates some confusion in the nomenclature since the unknowns now are $\delta\mu$, $\delta\lambda$ and δu, where finding $\delta\mu$, $\delta\lambda$ solves the well-known inverse problem of recovery of elastic parameters, and finding δu solves the direct problem. Of course, we assume the point of linearization as given, and it provides an estimate of the true values. The linearized problem can then be solved by using the theory of over-determined elliptic systems, a technique which has been used to successfully analyze linear models of elasticity after linearization in the sense described above. This comes at the caveat of having to use multiple sets of boundary excitations. The first model we consider consists of a linear elasticity operator plus a semi-linear forcing term in the solution u. From power density measurements, we are able to prove a stability estimate for the linearized problem bounding both the displacement δu and the displacement parameter $\delta\mu$ in terms of the change in power density measurements. Even in the model case of linear elasticity without the addition of a forcing term, this has not been shown before in the literature.

For each of the corresponding elasticity models, the closest works in two and three dimensions are for the anisotropic conductivity problem [9] and for full solution measurements in [6, 10, 34]. However, this list is not exhaustive as there are numerous results on recovering the parameters μ and sometimes λ from knowledge of the solution u in a domain for the linear problem [20, 21, 27, 30, 32]. As such, the significant contribution of this article is the extension to nonlinear mathematical models involving elasticity.

For the latter part of the article, in Section 8, we consider the Saint-Venant model of hyperelasticity with solution measurements. The Saint-Venant model provides a nonlinear PDE with appropriate boundary conditions written as

$$\begin{aligned} \nabla(\lambda\nabla\cdot u) + 2\nabla\cdot\mu(\nabla^S u + c_\tau \nabla u^\intercal \nabla u) + \nabla(\lambda|\nabla u|^2) + \omega^2 u = 0 \qquad & \text{in } \Omega, \\ u = g \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where c_τ is a constant in x. Regularity and existence and uniqueness results are discussed in the next section. This model (in specific the PDE) contains the Lamé coefficients (μ, λ), and they induce the solution u when equipped with appropriate boundary conditions g on a domain. When the curl operator is applied to the model, the λ terms disappear. Because the (nonlinear) Saint-Venant model depends on the parameter λ and this in practice is large, we also prove convergence of the linearized Saint-Venant model in two and three dimensions using a differential operator (the curl) which removes the parameter λ. The size of the parameter λ adversely affects the size of the class of solutions which can be considered in the linearized Saint-Venant model, unless we apply the curl. Furthermore, if we linearize equation (1.1), we will have the extra terms containing λ complicating the symbol computations.

The outline of this article is as follows. We remind the reader of some technical notation in Section 2. We present the main theorems in Section 3, which is followed by a subsection on their relationship to inverse problems. In Section 4, we present necessary preliminaries on over-determined systems. In Section 5, we use these over-determined systems to recover $\delta\mu$ and δu from power density measurements in the case of the linear elasticity without a forcing term. In Section 6, we add forcing terms $f(u)$ to the model, which are semi-linear in the solution variable u and also recover $\delta\mu$ and δu from power density measurements. We provide uniqueness in the recovery of an unknown μ which is a perturbation of the background for the linear elasticity model with a nonlinear forcing term in the process of proving iterative algorithms and convergence results in Sections 6.4 and 6.5 for the two cases mentioned above. This solves the inverse problem.

The latter parts of the paper switch to elasticity and hyperelasticity models with solution measurements. In Section 7, we prove stability for $\delta\mu$ from measurement of δu, which represents a simplification of the symbol computations in the literature. We give a brief derivation of the Saint-Venant model for hyperelastic materials in Section 8 and then use Section 7 along with some difficult symbol computations to expand the stability results for the corresponding $\delta\mu$ in this case. We also provide local uniqueness results for the linearized problem of the Saint-Venant model in Section 8.2. Main tools in this article come from the theory of over-determined elliptic boundary-value problems. The displacement terms δu are treated explicitly both in the theorems are not just considered perturbations.

2 Notation

In this paper, we use the Einstein summation convention. For two vectors a and b, the exterior product is denoted by $a \otimes b = ab^\intercal$, i.e., $a \otimes b$ is a matrix with entries $(a \otimes b)_{ij} = a_i b_j$. More generally, the exterior product between a tensor A of order m and B a tensor of order n is a new tensor $A \otimes B$ of order $m + n$ with entries $(A \otimes B)_{i_1 \ldots i_m j_1 \ldots j_n} = A_{i_1 \ldots i_m} B_{j_1 \ldots j_n}$. For two matrices A and B of the same size, the inner product is denoted by $A : B = a_{ij} b_{ji}$, and we write $|A|^2 = A : A$. Let $\Omega \subset \mathbb{R}^d$ be a simply connected bounded domain in $\mathbb{R}^d$ which is $\mathcal{C}^5$. For vector-valued functions

$$f(x) = (f_1(x), f_2(x), \ldots, f_d(x)) \colon \Omega \to \mathbb{R}^d,$$

the Hilbert space $H_m(\Omega)^d$, $m \in \mathbb{N}$, is defined as the completion of the space $\mathcal{C}_c^\infty(\Omega)^d$ with respect to the norm

$$\|f\|_m^2 = \|f\|_{m,\Omega}^2 = \sum_{|i|=1}^{m} \int_\Omega |\nabla^i f(x)|^2 + |f(x)|^2 \, dx,$$

where we write $\nabla^i = \partial^{i_1} \ldots \partial^{i_d}$ for $i = (i_1, \ldots, i_d)$ for the higher-order derivative. Let E be the symmetric gradient acting on $u \in H_0^1(\Omega)^d$ as

$$Eu = \frac{1}{2}(\nabla u + (\nabla u)^\intercal) = \nabla^S u.$$

In general, we assume the Lamé coefficients are $C^3(\overline{\Omega})$, where $\overline{\Omega}$ denotes the closure of Ω, and that they satisfy the following conditions:

$$\begin{aligned} \lambda(x) &\geq \lambda_{\min} = \min\{\lambda(x) : x \in \overline{\Omega}\} > 0, \\ \mu(x) &\geq \mu_{\min} = \min\{\mu(x) : x \in \overline{\Omega}\} > 0, \end{aligned} \tag{2.1}$$

We consider the density $\rho(x)$ to be fixed for this article, and as such, we remove it from the symbol computations. We will also need the following lemma.

Lemma 1 (Korn's inequality). *Let Ω be as above. Let $u \in H_0^1(\Omega)^d$. Then*

$$\int_\Omega |\nabla u|^2 \, dx \leq 2 \int_\Omega |\nabla^S u|^2 \, dx;$$

cf. for instance [3].

We now review the existence and uniqueness results for the elasticity system. We consider the following boundary-value problem for the elasticity equations:

$$\begin{cases} \nabla(\lambda(x)\nabla \cdot u_\lambda) + \omega^2 u_\lambda(x) + 2\nabla \cdot \mu(x)\nabla^S u_\lambda(x) = 0 & \text{in } \Omega, \\ u_\lambda(x) = g(x) & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

with $\mu(x), \lambda(x) \in \mathcal{C}^1(\overline{\Omega})$ the Lamé coefficients.

The solution $u_\lambda(x)$ is such that $u_\lambda(x) \colon \Omega \to \mathbb{R}^d$. It is known that the solution $u_\lambda(x)$ exists and is unique. In particular, $\nabla^S u_\lambda(x) \in L^2(\Omega)^d$ if $g(x) \in H^{\frac{1}{2}}(\partial\Omega)$, $\lambda, \mu \in L^\infty(\Omega)$ satisfy (2.1) and $\nabla^S u_\lambda(x) \in H^4(\Omega)^d$ under the additional assumptions that $\mu(x), \lambda(x) \in \mathcal{C}^4(\overline{\Omega})$, $g \in H^{\frac{9}{2}}(\partial\Omega)^d$. We need the latter regularity assumption for later stability estimates.

The Poisson ratio σ of the anomaly is given in terms of the Lamé coefficients by

$$\sigma = \frac{\lambda/\mu}{1 + 2\lambda/\mu}.$$

It is known in soft tissues $\sigma \approx \frac{1}{2}$ or equivalently $\lambda \gg \mu$. This makes it difficult to reconstruct both parameters μ and λ simultaneously [16, 23]. Therefore, we first construct asymptotic solutions to problem (2.2) when $\lambda_{\min} \to \infty$. We recall that, in the limit, the elasticity equations (2.2) reduce to the following Stokes system:

$$\begin{cases} \omega^2 u(x) + 2\nabla \cdot \mu(x)\nabla^S u(x) + \nabla p(x) = 0 & \text{in } \Omega, \\ \nabla \cdot u(x) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega, \\ \int_\Omega p(x)\,dx = 0. \end{cases} \tag{2.3}$$

The relation between the pressure p in (2.3) and u_λ in (2.2) is that p is the limit of $\lambda\nabla \cdot u_\lambda$ as $\lambda_{\min} \to \infty$. This is a result of [6]. We also consider the associated nonlinear problem

$$\begin{cases} \omega^2 u(x) + 2\nabla \cdot \mu(x)\nabla^S u(x) + \nabla p(x) + f(u) = 0 & \text{in } \Omega, \\ \nabla \cdot u(x) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega, \\ \int_\Omega p(x)\,dx = 0, \end{cases} \tag{2.4}$$

with $f \in C^3(H^3(\Omega)^d, L^2(\Omega)^d)$. This corresponds to a large λ limit of (2.3) with a nonlinear forcing term depending on u. The second half of the paper focuses on the nonlinear Saint-Venant model in two and three dimensions

$$\begin{aligned} \nabla(\lambda\nabla \cdot u) + 2\nabla \cdot \mu(\nabla^S u + c_\tau \nabla u^\intercal \nabla u) + \nabla(\lambda|\nabla u|^2) + \omega^2 u = 0 \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where c_τ is a constant in x coming from the fact that we cannot obtain a time-independent equation by applying a periodic force in time. As mentioned in the introduction, this model is arguably the simplest nonlinear model for hyperelastic materials. It is a result of [19] for the same regularity coefficients and $\|g\|_{H^{\frac{9}{2}}(\Omega)} < \varepsilon$ that the corresponding time-dependent equations are well posed on bounded domains for short times T proportional to $|\log(\varepsilon^{-1})|$ (cf. also [33, Appendix] for a more modern formulation). Since these are the time stationary versions of those found in [19], we therefore assume when analyzing the nonlinear problem that this additional assumption on g holds.

3 Statement of the main theorems

Let $f \in \mathcal{C}^3(H^3(\mathbb{R}^d)^d, L^2(\mathbb{R}^d)^d)$ be a function whose symbol contains at most one power of ξ. The model studied in the first half of this article is

$$\begin{cases} 2\nabla \cdot \mu\nabla^S u_j + \omega^2 u_j - f(u_j) = -\nabla p_j & \text{in } \Omega, \\ \frac{\mu}{2}|\nabla^S u_j|^2 - f(u_j) \cdot u_j = H_j & \text{in } \Omega, \\ \nabla \cdot u_j = 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $j = 1, \dots, J$. The various subscripts j correspond to different measurement functionals H_j with a fixed μ and p, with different boundary excitations g_j. The motivation for considering the term $f(u_j)$ is to have a first intuition on more general nonlinear elasticity models in dimension $d = 2$. In [33], a simplified nonlinear elasticity model is studied in dimension $d = 3$ with scalar-valued functions. If $f = 0$, this corresponds to (2.3), and if $f \neq 0$, this corresponds to (2.4), respectively, with power density measurements, and as such, we assume that the functions u_j, μ_j and g_j have the regularity properties assumed in the previous section for all j. We consider the background pressure ∇p to be fixed. The stability estimates given here then would allow us to go back and solve for p as soon as u and μ are known since, by applying divergence, we can determine Δp and then obtain an elliptic equation in p. We do not perform this calculation here, but it is the motivation behind our choice of model in the earlier sections.

For each j, we consider a problem with a different μ which we denote as μ_1 and μ_2. As such, we let $\delta u_j = u_{1j} - u_{2j}$ and $\delta\mu = \mu_1 - \mu_2$. We analyze the following linearized version of linear or nonlinear elasticity:

$$\begin{cases} 2\nabla \cdot \delta\mu\nabla^S u_j + 2\nabla \cdot \mu\nabla^S \delta u_j + \omega^2 \delta u_j + Df(u_j)\delta u_j = 0 & \text{in } \Omega, \\ \frac{\delta\mu}{2}|\nabla^S u_j|^2 + \mu\nabla^S u_j : \nabla^S \delta u_j + f(u_j)\cdot u_j : \nabla^S \delta u_j = \delta H_j & \text{in } \Omega, \\ \nabla \cdot \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = \delta g_j & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Naturally, for linearization of linear elasticity, $Df(u_j) \equiv 0$. We provide a general criterion on system (3.2) for arbitrary J to be elliptic; however, we focus on the case $J = 2$.

Theorem 1. *Assume we have that*

$$\left|\frac{\nabla^S u_{1j}}{|\nabla^S u_{1j}|} : \frac{\nabla^S u_{2j}}{|\nabla^S u_{2j}|}\right| \neq 1, \quad j = 1, 2.$$

Let $d = 2$. Then there exists constants C_1 and C_2 depending on $\|f\|_{C^3}$, $\|\mu_2\|_{C^2(\Omega)}$ (C_2 may also depend on ω) such that

$$\|\delta\mu\|_{H^3(\Omega)} + \sum_{j=1}^{2}\|\delta u_j\|_{H^4(\Omega)^2} \le C_1 \sum_{j=1}^{2}(\|\delta H_j\|_{H^3(\Omega)} + \|\delta g_j\|_{H^{\frac{5}{2}}(\Omega)^2}) + C_2\left(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2}\|\delta u_j\|_{L^2(\Omega)^2}\right) \tag{3.3}$$

Corollary 1. *For all ω sufficiently large, the linearized system is injective, that is, we can find a C_1 such that $C_2 = 0$ in Theorem 1, provided δg_j is zero.*

Using the stability estimates, we develop an iteration scheme which is convergent. The result of this scheme which is interesting in its own right is the following existence and uniqueness theorem.

Theorem 2. *The solution $w = (\delta\mu, \{\delta u_j\}_{j=1}^2)$ to (3.2) exists as a limit of an explicit sequence of Duhamel iterates and is unique in $H^3(\Omega) \times (H^4(\Omega)^2)^2$ for all ω sufficiently large and $\delta g_j = 0$.*

The second half of the paper focuses on the model

$$\begin{aligned} \nabla(\lambda \cdot \nabla u_j) + 2\nabla \cdot \mu(\nabla^S u_j + ac_\tau \nabla u_j^\intercal \nabla u_j) + a\nabla(\lambda|\nabla u_j|^2) + \omega^2 u_j = 0 \quad & \text{in } \Omega, \\ \delta u_j = H_j \quad & \text{in } \Omega, \\ u_j = g_j \quad & \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

where $j = 1, \dots, J$ and c_τ is a constant in x coming from the fact that we cannot obtain a time-independent equation by applying a periodic force in time. The model is derived in the text. The number $a = 0, 1$ corresponding to the linear elasticity problem or the Saint-Venant model (first-order nonlinear elasticity model), respectively.

We assume the background (μ_1, λ_1, u_1) is known and solves (1.1) so that, for the Saint-Venant model operator with the curl applied to it, we have, say for shorthand, $P(\mu_1, u_1) = 0$. Then we consider δu which is a solution to the linearized nonlinear model. The displaced field, which we measure as $u_2 = \delta u + u_1$ for the same boundary conditions corresponds to a $\mu_2 = \mu_1 + \delta\mu$ which is unknown and λ_1 which is fixed and large. The linearized operator corresponding to $P(\mu_1, u_1)$, say $L(\mu_1, u_1)$, acts on $(\delta\mu, \delta u)$ and can be split into two parts L_1 and L_2 acting on $\delta\mu$ and δu, respectively (given explicitly in (3.6)). By Fréchet differentiability, we then have

$$P(\mu_1, u_1) = P(\mu_2, u_2) + L(\mu_1, u_1)(\delta\mu, \delta u) + o((\delta\mu), (\delta u)). \tag{3.5}$$

Then, assuming μ_1, u_1 and μ_2, u_2 are actually solutions to the original equation, we also have

$$L_1(\mu_1, u_1)\delta\mu + L_2(\mu_1, u_1)\delta u \approx 0.$$

The operator L_1 is invertible using the theory of over-determined elliptic systems, provided we repeat this process to add extra boundary conditions and corresponding measurements. Then we use elliptic regularity to provide a stability estimate in terms of a finite collection of δu for $\delta\mu$. This stability estimate in Theorem 3 holds up to the order terms (3.5). When the nonlinear terms in the Saint-Venant model are set to zero ($c_\tau = 0, \lambda|\nabla u|^2 = 0$), we cover a case in the linearization of linear elasticity which is not covered in [18], where the λ parameter must not be too large, and their algorithm has a possibly infinite-dimensional kernel.

In the case of power density measurements, the perturbation of the local energy density is known, and we consider the background pressure fixed. A similar procedure is used to find a stability result for instead $(\delta u, \delta\mu)$ in terms of power density measurements; see Theorem 1. In the case of power density measurements, we also give a fixed-point algorithm including the lower-order terms from the linearization which allows for unique and stable reconstruction of $\delta\mu$ (and hence μ_2), a more powerful result in this case where the nonlinearity does not affect the symbol computation. Furthermore, the stability estimates in Theorem 1 have no kernel (they are injective) for all ω sufficiently large on the entirety of the domain with two measurements. This is the first time global injectivity with a single fixed ω has been shown under any conditions. The only known theorems similar, involving a similar contraction argument principle for the linearized *linear* elasticity model, is in [18] (the terms $f(u) = 0$ in their model, and they use solution measurements). Power density measurements are different as they are a measure of local energy density.

It is important to emphasize that we linearize the genuinely *nonlinear* Saint-Venant model and prove stability of perturbations of the Lamé parameter in terms of a difference of the solutions δu. The Saint-Venant model is perhaps the most simple of the hyperelastic models. For variable coefficients, it has not been discussed in the inverse problems literature. It is very difficult due to the symbol computations involved. We use solution measurements in the linearized Saint-Venant model since power density measurements do not work well when using the annihilation (curl) operator.

In the case $a = 0$, we proved a more relaxed criterion than in [6] for the properties of over-determined elliptic systems to hold; however, this is not the main theorem. The main theorem is more difficult because the derivatives on u when $a = 1$ change the properties of the principal symbol when linearized. Briefly suppressing the subscript j, after applying the curl operator to remove the λ terms, the linearized system from (3.4) with internal measurements is

$$\begin{cases} D\tilde{L}(\mu_j, u_j)[\delta\mu_j, \delta u_j] + D\tilde{N}(\mu_j, u_j)[\delta\mu_j, \delta u_j] + \omega^2\nabla\times\delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = \delta H_j & \text{in } \Omega, \\ \delta u_j = \delta g_j & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

where $D\tilde{L}$ and $D\tilde{N}$ are the Fréchet derivatives of $\tilde{L}$ and $\tilde{N}$, respectively, given by

$$\begin{aligned} D\tilde{L}(\mu, u)[\delta\mu, \delta u] &= 2\nabla\times\nabla\cdot\delta\mu\nabla^S u + 2\nabla\times\nabla\cdot\mu\nabla^S\delta u, \\ D\tilde{N}(\mu, u)[\delta\mu, \delta u] &= a(2c_\tau\nabla\times\nabla\cdot(\delta\mu\nabla u^\intercal\nabla u) + 2\nabla\times\nabla\cdot(\mu\nabla\delta u^\intercal\nabla u) + 2\nabla\times\nabla\cdot(\mu\nabla u^\intercal\nabla\delta u)). \end{aligned}$$

For the theorem below, the case $a = 0$ was essentially established in [6], and local injectivity in [14] in dimension 3; the small error in dimension 2 in these articles we correct. The case $a = 1$ is not considered anywhere in the literature for variable coefficients.

Theorem 3. *Let $a = 1$, $d = 2, 3$. Assume, for $j = 1, 2$,*

$$|((\nabla^S u_{1j} + c_\tau\nabla u_{1j}^\intercal\nabla u_{1j})\xi)\times\xi| + |((\nabla^S u_{2j} + c_\tau\nabla u_{2j}^\intercal\nabla u_{2j})\xi)\times\xi| \neq 0 \quad \text{for all } \xi \neq 0.$$

Let C_1, C_2 depend on $\|\mu_2\|_{C^4(\Omega)}$, and C_2 also depends on ω^2. Then we have the following stability estimate:

$$\|\delta\mu\|_{H^5(\Omega)} \leq C_1\bigg(\sum_{j=1}^{2}\|\delta u_j\|_{H^4(\Omega)^d} + \|\delta g_j\|_{H^{\frac{5}{2}}(\partial\Omega)^d}\bigg) + C_2\bigg(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2}\|\delta u_j\|_{L^2(\Omega)^d}\bigg).$$

Corollary 2. *The constant C_2 can be absorbed into the constant C_1 if*

$$(\nabla^S u_{1j} + c_\tau\nabla u_{1j}^\intercal\nabla u_{1j}) \neq \alpha(\nabla^S u_{2j} + c_\tau\nabla u_{2j}^\intercal\nabla u_{2j})$$

for $j = 1, 2$ and all $\alpha \in \mathbb{R}$.

3.1 Short comparison with previous literature on nonlinear inverse problems

While there are many known mathematical and engineering articles on linear elasticity which are mentioned in the introduction, not so much is written about the problem of nonlinear elasticity. Some of our motivation comes from the results on inverse problems in mathematical physics. The problem of local metric recovery for general Lorentzian manifolds (M, g) in $3 + 1$ dimensions and the semi-linear wave equation $\partial_t^2 u - \Delta_g u = |u|^2 + h$ was analyzed in [22]. Here h is a highly oscillatory source term, with small $H^{\frac{9}{2}}(M)$ norm, and Δ_g is the Laplace–Beltrami operator. In the case of general time-independent metrics g, locally, the authors can recover metric perturbations uniquely from an infinite number of oscillatory source terms h in the manifold, and solution measurements everywhere in a local neighborhood of the manifold. In [33], this amount of data was reduced to codimension 1 source terms to the vector-valued Dirichlet-to-Neumann map and a coupled system of simple metrics. The coupled system of metrics in [33] is a toy model for the nonlinear elasticity problem. However, the issue with these articles is that the number of excitation states/source terms required to recover the solutions is *infinite*. Furthermore, they are based on the boundary control method, a purely theoretical technique, and the X-ray transform, respectively. The aim of the main theorems here is the reduction of the number of source terms (two only!) for models of nonlinear elasticity with non-constant coefficients. An open question is if it is possible to reduce the required solution measurements further to just boundary data. The arguments on over-determined elliptic systems should also be applicable to other nonlinear systems.

4 Preliminaries on over-determined elliptic boundary-value problems

In this section, we present some basic properties about over-determined elliptic boundary-value problems which play a key role in our stability estimates in the next sections. The presentation follows closely the ones in [29, 34]. We present it here for the convenience of the reader.

We first recall the definition of ellipticity in the sense of Douglis–Nirenberg. Consider the (possibly) redundant system of linear partial differential equations

$$\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)y = \mathcal{S}, \quad \mathcal{B}\Big(x, \frac{\partial}{\partial x}\Big)y = \phi \tag{4.1}$$

for m unknown functions $y = (y_1, \ldots, y_m)$ comprising in total of M equations. Here $\mathcal{L}(x, \frac{\partial}{\partial x})$ is a matrix differential operator of dimension $M \times m$ with entries $L_{ij}(x, \frac{\partial}{\partial x})$. For each $1 \le i \le M$, $1 \le j \le m$ and for each point x, the entry $L_{ij}(x, \frac{\partial}{\partial x})$ is a polynomial in $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$. If the system is redundant, then there are possibly more equations than unknowns, $M \ge m$. The matrix $\mathcal{B}(x, \frac{\partial}{\partial x})$ has entries $B_{ij}(x, \frac{\partial}{\partial x})$ for $1 \le k \le Q$, $1 \le j \le m$ consisting of Q equations at the boundary. The operators are also polynomial in the partials of x. Naturally, the vector $\mathcal{S}$ is a vector of length M, and ϕ is a vector of length Q.

Definition 1 (cf. [1, 12]). Let integers $s_i, t_j \in \mathbb{Z}$ be given for each row $1 \le i \le M$ and column $1 \le j \le m$ with the following property: for $s_i + t_j \ge 0$, the order of L_{ij} does not exceed $s_i + t_j$. For $s_i + t_j < 0$, one has $L_{ij} = 0$. Furthermore, the numbers are normalized so that, for all i, one has $s_i \le 0$. The numbers s_i, t_j are known as Douglis–Nirenberg numbers.

The principal part of $\mathcal{L}$ for this choice of numbers s_i, t_j is defined as the matrix operator $\mathcal{L}^0$ whose entries are composed of those terms in L_{ij} which are exactly of order $s_i + t_j$.

The principal part $\mathcal{B}^0$ of $\mathcal{B}$ is composed of the entries which are composed of those terms in B_{kj} which are exactly of order $\sigma_k + t_j$. The numbers σ_k, $1 \le k \le Q$, are computed as $\sigma_k = \max_{1 \le j \le m}(b_{kj} - t_j)$ with b_{kj} denoting the order of B_{kj}. Real directions with $\xi \ne 0$ and rank $\mathcal{L}^0(x, i\xi) < m$ are called characteristic directions of $\mathcal{L}$ at x. The operator $\mathcal{L}$ is said to be (possibly) over-determined elliptic in Ω if, for all $x \in \overline{\Omega}$ and for all real nonzero vectors ξ, one has rank $\mathcal{L}^0(x, i\xi) = m$.

We next recall the following Lopatinskii boundary condition.

Definition 2. Fix $x \in \partial\Omega$, and let ν be the inward unit normal vector at x. Let ζ be any nonzero tangential vector to Ω at x. We consider the line $\{x + z\nu,\ z > 0\}$ in the upper half plane and the following system of ODEs:

$$\mathcal{L}^0\Big(x, i\zeta + \nu\frac{d}{dz}\Big)\tilde{y}(z) = 0, \quad z > 0, \tag{4.2}$$

$$\mathcal{B}^0\Big(x, i\zeta + \nu\frac{d}{dz}\Big)\tilde{y}(z) = 0, \quad z = 0. \tag{4.3}$$

We define the vector space V of all solutions to system (4.2)–(4.3) which are such that $\tilde{y}(z) \to 0$ as $z \to \infty$. If $V = \{0\}$, then we say that the Lopatinskii condition is fulfilled for the pair $(\mathcal{L}, \mathcal{B})$ at x.

Now let $\mathcal{A}$ be the operator defined by $\mathcal{A} = (\mathcal{L}, \mathcal{B})$. Then equations (4.1) read as $\mathcal{A}y = (\mathcal{S}, \phi)$.

Let $\mathcal{A}$ act on the space $D(p, l) = W_p^{l+t_1}(\Omega) \times \cdots \times W_p^{l+t_m}(\Omega)$ with $l \geq 0$, $p > 1$. Here W_p^α denotes the standard Sobolev space with order α partial derivatives in the L^p space. With some regularity assumptions on the coefficients of $\mathcal{L}$ and $\mathcal{B}$, $\mathcal{A}$ is bounded with range in the space

$$R(p, l) = W_p^{l-s_1}(\Omega) \times \cdots \times W_p^{l-s_m}(\Omega) \times W_p^{l-\sigma_1-\frac{1}{p}} \times \cdots \times W_p^{l-\sigma_q-\frac{1}{p}}(\partial\Omega).$$

We have the following result; see [34, Theorem 1].

Theorem 4. *Let the integers $l \geq 0$, $p > 1$ be given. Let $(\mathcal{S}, \phi) \in \mathcal{R}(p, l)$. Let the Douglis–Nirenberg numbers s_i and t_j be given for $\mathcal{L}$, and let σ_k be as in Definition 1. Let Ω be a bounded domain with boundary in $\mathcal{C}^{l+\max t_j}$. Also assume that $p(l - s_i) > d$ and $p(l - \sigma_k) > d$ for all i and k. Let the coefficients L_{ij} be in $W_p^{l-s_i}(\Omega)$, and let the coefficients of B_{kj} be in $W^{l-\sigma_k-\frac{1}{p}}$. The following statements are equivalent.*

(1) *$\mathcal{L}$ is over-determined elliptic, and the Lopatinskii condition is fulfilled for $(\mathcal{L}, \mathcal{B})$ on $\partial\Omega$.*
(2) *There exists a left regularizer $\mathcal{R}$ for the operator $\mathcal{A} = \mathcal{L} \times \mathcal{B}$ such that $\mathcal{R}\mathcal{A} = \mathcal{I} - \mathcal{T}$ with $\mathcal{T}$ compact from $R(p, l)$ to $D(p, l)$.*
(3) *The following a priori estimate holds:*

$$\sum_{j=1}^{m} \|y_j\|_{W_p^{l+t_j}(\Omega)} \leq C_1\Big(\sum_{i=1}^{M} \|\mathcal{S}_i\|_{W_p^{l-s_i}(\Omega)} + \sum_{k=1}^{Q} \|\phi_k\|_{W_p^{l-\sigma_j-\frac{1}{p}}(\partial\Omega)}\Big) + C_2 \sum_{t_j>0} \|y_j\|_{L^p(\Omega)},$$

where y_j is the j-th component of the solution y.

5 Recovery of $\delta\mu$ and δu in dimension two from power density measurements for the linear elasticity model

In dimension $d = 2$, notice that $\xi \in \mathbb{R}^2$ can be written as

$$\xi = |\xi|\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$$

for some $\theta \in\ ]-\pi, \pi]$. Moreover, the symmetric gradient of a incompressible vector-valued function u satisfies $\nabla^S u = (\nabla^S u)^\intercal$, $\operatorname{tr}(\nabla^S u) = 0$. Then $\nabla^S u$ can be written as

$$\nabla^S u(x) = \frac{|\nabla^S u(x)|}{\sqrt{2}}\begin{bmatrix}\cos(\alpha(x)) & \sin(\alpha(x))\\ \sin(\alpha(x)) & -\cos(\alpha(x))\end{bmatrix}$$

for some $\alpha(x) \in\ ]-\pi, \pi]$. We will use these structures along the section. We also use the notation $\hat{F} = \frac{F}{|F|}$, where F is a vector or a matrix.

5.1 One measurement, lack of invertibility

We consider the case of dimension $d = 2$ only in this section. Consider the case $J = 1$, that is, only one measurement. Let us define $F_j = \nabla^S u_j$, and assume that $|F_j| > 0$ for all $x \in \Omega$. From equation (3.1), we obtain $\mu = \frac{2H_j}{|F_j|^2}$, and then we can replace μ in equation (3.1) to obtain the following lemma.

Lemma 2. *We have*

$$\frac{\omega^2|F_j|^2}{4H_j}u_j + \frac{1}{2}\nabla^S u_j \nabla \ln(H_j) + (\mathbb{I} - 2\hat{F}_j \otimes \hat{F}_j)\nabla \otimes \nabla^S u_j = -\frac{|F_j|^2}{4H_j}\nabla p_j, \tag{5.1}$$

where $\mathbb{I}$ *is a fourth-order tensor whose entries are defined as*

$$\mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}).$$

Proof. Dropping the subscript j, we compute

$$\nabla\Big(\frac{H}{|F|^2}\Big) = \frac{1}{|F|^4}(|F|^2\nabla H - H\nabla|F|^2) \quad \text{with} \quad \nabla|F|^2 = \frac{\partial|F|^2}{\partial x_k}\hat{e}_k = 2F_{ij}\frac{\partial F_{ij}}{\partial x_k}\hat{e}_k = 2(\nabla \otimes F)F.$$

Therefore,

$$2\nabla\cdot\mu\nabla^S u = \frac{2H}{|F|^2}(\Delta u + \nabla(\nabla\cdot u)) + 4\nabla^S u\Big(\frac{\nabla H}{|F|^2} - \frac{2H(\nabla\otimes F)F}{|F|^4}\Big),$$

and then

$$\frac{2H}{|F|^2}(\Delta u + \nabla(\nabla\cdot u)) + \frac{4}{|F|^2}\nabla^S u\nabla H - \frac{8H\nabla^S u}{|F|^4}(\nabla\otimes F)F + \omega^2 u + \nabla p = 0.$$

Finally, by definition of $\mathbb{I}$ and $\otimes$, we see that

$$\frac{1}{2}(\Delta u + \nabla(\nabla\cdot u)) = \mathbb{I}\nabla\otimes\nabla^S u, \quad 2\hat{F}(\nabla\otimes F)\hat{F} = 2(\hat{F}\otimes\hat{F})\nabla\otimes\nabla^S u;$$

hence we obtain

$$\frac{1}{2}(\Delta u + \nabla(\nabla\cdot u)) - 2\hat{F}(\nabla\otimes F)\hat{F} = (\mathbb{I} - 2\hat{F}\otimes\hat{F})\nabla\otimes\nabla^S u,$$

and so we obtain (5.1). This computation of the principal symbol is fairly standard but is included for completeness since it does not appear in the literature for power density measurements. □

Now, identifying the leading term of (5.1), we define the operator

$$P_j(x, D) = (\mathbb{I} - 2\hat{F}_j\otimes\hat{F}_j)\nabla\otimes\nabla^S, \tag{5.2}$$

and it has the symbol

$$q_j(x,\xi) = 2(\hat{F}_j\xi)\otimes(\hat{F}_j\xi) - \frac{1}{2}(|\xi|^2 I_d + (\xi\otimes\xi)). \tag{5.3}$$

By the definition of the product operation between a fourth- and a third-order tensor and the symmetry of $\hat{F}_j$, we see that $2(\hat{F}_j\otimes\hat{F}_j)\nabla\otimes\nabla^S u$ and $2(\hat{F}_j\nabla)\otimes(\hat{F}_j\nabla)u$ have the same principal symbol. The latter is easier to calculate as $-2(\hat{F}_j\xi\otimes\hat{F}_j\xi)$.

Lemma 3. *In dimension* $d = 2$, *let*

$$\xi = |\xi|\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}, \quad \hat{F}_j(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos(\alpha(x)) & \sin(\alpha(x))\\ \sin(\alpha(x)) & -\cos(\alpha(x))\end{bmatrix}. \tag{5.4}$$

Computing, we have that

$$\det(q_j(x,\xi)) = -\frac{|\xi|^4}{2}\sin^2(2\theta - \alpha(x)).$$

The conclusion is the operator is not elliptic for only one set of measurements given by (3.1) *with* $J = 1$.

Proof. In this case, we have

$$q_j(x,\xi) = -\frac{1}{2}\left(\begin{bmatrix}|\xi|^2 & 0\\ 0 & |\xi|^2\end{bmatrix} + \begin{bmatrix}\xi_1^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2^2\end{bmatrix}\right) + 2\begin{bmatrix}A^2 & AB\\ AB & B^2\end{bmatrix},$$

where $A = (\hat{F}_j\xi)_1$, $B = (\hat{F}_j\xi)_2$. As a result of a short computation, we have that

$$\det(q_j(x,\xi)) = \frac{|\xi|^4}{2} - |\xi|^2(A^2 + B^2) - (A\xi_2 - B\xi_1)^2.$$

In addition, notice that, using representation (5.4), we have

$$A = (F_{11}\xi_1 + F_{12}\xi_2) = \frac{|\xi|}{\sqrt{2}}(\cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta)) = \frac{|\xi|}{\sqrt{2}}\cos(\alpha-\theta),$$
$$B = (F_{21}\xi_1 + F_{22}\xi_2) = \frac{|\xi|}{\sqrt{2}}(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)) = \frac{|\xi|}{\sqrt{2}}\sin(\alpha-\theta),$$

which results in

$$\det(q_j(x,\xi)) = \frac{|\xi|^4}{2}\sin^2(2\theta-\alpha),$$

and we conclude the proof of the estimate on the principal symbol. Notice that, for all $\hat{F}_j(x)$ with the structure given in equation (5.4), the operator $P_j(x,D)$ is not elliptic since, for all $x \in \Omega$ and for all $\hat{F}_j(x)$, it is possible to find $\xi = (\cos(\frac{\alpha(x)}{2}), \sin(\frac{\alpha(x)}{2})) \in \mathbb{S}^1$ such that $\det(q_j(x,\xi)) = 0$, i.e., $q_j(x,\xi)$ is not of full rank. □

Remark 5.1. Observe that, in the differential operator (5.2), $\hat{F}_j$ depends on the solution u_j, but only on the "direction" of $\nabla^S u_j$. The possible directions are described by the angle $\alpha(x)$ in Lemma 3, and we see that there is no ellipticity for all possible $\alpha(x)$ and so for all possible direction of $\nabla^S u_j(x)$.

Remark 5.2. Although this result gives us an idea about the ellipticity for the equation, this is a result of the ellipticity for the operator $P_j(x,D)$. Similar problems have been studied in [7, 32], where a result says that an analogue system (in scalar case) is in fact hyperbolic. It seems natural to linearize in nonlinear models, since the problem is reduced to a linear one, and better mathematical results are known to hold. In the remaining part of the article, we show results concerning the linearization of the models in study.

5.2 Linearization of the model problem for *J* measurements

We consider the background pressure to be fixed and let d be the dimension which is arbitrary for this system. The linearized problem for $j \in \{1,\dots,J\}$ is given by

$$\begin{cases} 2\nabla\cdot\delta\mu\nabla^S u_j + 2\nabla\cdot\mu\nabla^S\delta u_j + \omega^2\delta u_j + Df(u_j)\delta u_j = 0 & \text{in } \Omega,\\ \frac{\delta\mu}{2}|\nabla^S u_j|^2 + \mu\nabla^S u_j : \nabla^S\delta u_j + f(u_j)\cdot u_j : \nabla^S\delta u_j = \delta H_j & \text{in } \Omega,\\ \nabla\cdot\delta u_j = 0 & \text{in } \Omega,\\ \delta u_j = \delta g_j & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We make the definition $w = (\delta\mu, \{\delta u_j\}_{j=1}^J)$ which allows us to re-write the system as

$$\begin{cases} \mathcal{L}w = \mathcal{S} & \text{in } \Omega,\\ \mathcal{B}w = g & \text{on } \partial\Omega. \end{cases}$$

The principal symbol associated to (3.2) is, rearranging rows, the following:

$$\mathcal{P}_J(x,\xi) = \begin{bmatrix} \frac{|F_1|^2}{2} & i\mu(F_1\xi)^\intercal & 0 & \dots & 0\\ 2iF_1\xi & -\mu(|\xi|^2 I_d + (\xi\otimes\xi)) & 0 & \dots & 0\\ 0 & i\xi^\intercal & 0 & \dots & 0\\ \frac{|F_2|^2}{2} & 0 & i\mu(F_2\xi)^\intercal & \dots & 0\\ 2iF_2\xi & 0 & -\mu(|\xi|^2 I_d + (\xi\otimes\xi)) & \dots & 0\\ 0 & 0 & i\xi^\intercal & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{|F_J|^2}{2} & 0 & 0 & \dots & i\mu(F_J\xi)^\intercal\\ 2iF_J\xi & 0 & 0 & \dots & -\mu(|\xi|^2 I_d + (\xi\otimes\xi))\\ 0 & 0 & 0 & \dots & i\xi^\intercal \end{bmatrix}$$

which is a matrix of size $J(d+2) \times (Jd+1)$. We can recognize the following family of submatrices:

$$\rho_j(x,\xi) = \begin{bmatrix} \frac{|F_j|^2}{2} & i\mu(F_j\xi)^\intercal \\ 2iF_j\xi & -\mu(|\xi|^2 I_d + (\xi \otimes \xi)) \end{bmatrix},$$

and we have from the formulas for the determinant of block matrices (see, for example, [25, Section 6.2]) that

$$\det(\rho_j(x,\xi)) = 2^{d-1}\mu^d |F_j|^2 \det(q_j(x,\xi)), \tag{5.5}$$

where q_j is defined in (5.3). Note that Lemma 3 now says that the linearized operator $\mathcal{L}$ is not elliptic.

On the other hand, if we take determinant for the submatrices with the rows containing the highest power of ξ in $\mathcal{P}_j$, we obtain, by applying properties for determinant of block matrices, the following:

$$(-1)^{(J-1)d}\frac{\mu^{Jd}}{2^{(J-1)d}}|F_j|^2 \det(|\xi|^2 I_d + \xi \otimes \xi)^{J-1}\det(q_j(x,\xi)).$$

Definition 3. We say that a family $\{\mathrm{Op}(\rho_j(x,\xi))\}_{j=1}^J$ of operators is elliptic in $x \in \Omega$ if $\rho_j(x,\xi)$ being invertible for all $j = 1,\dots,J$ implies $\xi = 0$. Moreover, we say that $\{\mathrm{Op}(\rho_j(x,\xi))\}_{j=1}^J$ is elliptic in Ω if the family is elliptic for all $x \in \Omega$.

This definition is inspired by the one in [8, Definition 2.1].

Lemma 4. *If $\{\rho_j\}$ forms an elliptic family and $|F_j| > 0$ for all $x \in \Omega$ and $j = 1,\dots,J$, then the full linearized operator $\mathcal{L}(x,\xi)$ is elliptic.*

Proof. Let C_0 and $\{C_j\}_{j=1}^J$ be the submatrices of $\mathcal{P}_J$ defined by

$$C_0 = \begin{pmatrix} |F_1|^2 \\ 2iF_1\xi \\ 0 \\ |F_2|^2 \\ 2iF_2\xi \\ 0 \\ \vdots \\ |F_J|^2 \\ 2iF_J\xi \\ 0 \end{pmatrix}, \qquad C_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2i\mu(F_j\xi)^\intercal \\ -\mu(|\xi|^2 I_d + \xi\otimes\xi) \\ i\xi^\intercal \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} \\ \\ \\ \leftarrow \text{row } ((j-1)(d+2)+1) \\ \\ \\ \\ \\ \\ \end{matrix}$$

where $C_0 \in \mathcal{M}_{J(d+2)\times 1}(\mathbb{C})$ and $C_j \in \mathcal{M}_{J(d+2)\times d}(\mathbb{C})$ for $j = 1,\dots,J$.

Let $\xi \neq 0$. Then we can see easily that $-\mu(|\xi|^2 + \xi \otimes \xi)$ is invertible; hence C_j has complete column rank. In addition, if $j_1 \neq j_2$, then C_{j_1} and C_{j_2} do not have the same nonzero rows.

If $\mathcal{L}(x,\xi)$ is not full rank, then it is clear that there exist j_0 and $\alpha_{j_0} \in \mathbb{R}^d \setminus \{0\}$ such that, in the nonzero rows of C_{j_0}, we have

$$\begin{pmatrix} |F_{j_0}|^2 \\ 2i\mu F_{j_0}\xi \\ 0 \end{pmatrix} = \begin{pmatrix} 2i\mu(F_{j_0}\xi)^\intercal \\ -\mu(|\xi|^2 I_d + \xi\otimes\xi) \\ i\xi^\intercal \end{pmatrix} \begin{pmatrix} \alpha_{j_0 1} \\ \vdots \\ \alpha_{j_0 d} \end{pmatrix},$$

and then we have that $\xi^\intercal \alpha_{j_0} = 0$ and

$$\begin{pmatrix} |F_{j_0}|^2 & 2i\mu(F_{j_0}\xi)^\intercal \\ 2i\mu F_{j_0}\xi & -\mu(|\xi|^2 I_d + \xi\otimes\xi) \end{pmatrix} \begin{pmatrix} -1 \\ \alpha_{j_0 1} \\ \vdots \\ \alpha_{j_0 d} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

That is, $\rho_{j_0}(x,\xi)$ is not invertible. □

Proposition 1. *For $J = 2$, $d = 2$, if $\alpha_2(x) \neq \alpha_1(x) + k\pi$ for all $k \in \mathbb{Z}$ and for all $x \in \Omega$, then the differential operator corresponding to system* (3.2) *is elliptic.*

Proof. This proof was inspired by the one in [8] for the Calderon problem. We have to prove that

$$\det(q_j(x,\xi)) = 0 \text{ for all } j \implies \xi = 0$$

since equation (5.5) establishes that $\rho_j(x,\xi)$ is invertible if and only if $q_j(x,\xi)$ is invertible.

If $\det(q_j(x,\xi)) = 0$ for $j = 1, 2$, then we have

$$\sin(2\theta - \alpha_1(x)) = 0 \quad \text{and} \quad \sin(2\theta - \alpha_2(x)) = 0 \tag{5.6}$$

or $\xi = 0$, but (5.6) implies $\alpha_2 = \alpha_1 + k\pi$ for some $k \in \mathbb{Z}$, which is false by hypothesis. So we conclude that $\xi = 0$. That is, (q_1, q_2) forms an elliptic family. We conclude the proof using Lemma 4. □

5.3 Lopatinskii condition

We prove now the following in dimension $d = 2$.

Lemma 5. *Consider $w = (\delta\mu, \{\delta u_j\}_{j=1,\dots,J})$. Let $x \in \partial\Omega$, ν the outward unit normal to Ω at x, and $\zeta \in \mathbb{S}^{d-1}$ satisfying $\zeta \cdot \nu = 0$. Define $\tilde{w}(z) = w(x - \nu z)$, $\delta\tilde{u}_j = \delta u_j(x - \nu z)$ and $\delta\tilde{\mu} = \delta\mu(x - \nu z)$. Then the only solution of the system of ODEs*

$$\begin{cases} \mathcal{P}_J(x, i\zeta + \nu\partial_z)\tilde{w} = 0, & z > 0, \\ \mathcal{B}\tilde{w} = 0, & z = 0, \end{cases}$$

such that $\tilde{w}(z) \to 0$ as $z \to \infty$ is $\tilde{w} \equiv 0$.

Proof. The system can be seen as the following:

$$\begin{cases} |F_j|^2\delta\tilde{\mu} + 2\mu(F_j[i\zeta + \nu\partial_z])^\intercal\delta\tilde{u}_j = 0, & z > 0, \\ F_j[i\zeta + \nu\partial_z]\delta\tilde{\mu} - \frac{\mu}{2}((i\zeta + \nu\partial_z)^2 I_d + (i\zeta + \nu\partial_z) \otimes (i\zeta + \nu\partial_z))\delta\tilde{u}_j = 0, & z > 0, \\ i(i\eta + \nu\partial_z)^\intercal\delta\tilde{u}_j = 0, & z > 0, \\ \delta\tilde{u}_j = 0, & z = 0, \end{cases}$$

for all $j = 1, \dots, J$.

We can eliminate $\delta\tilde{\mu}$ using the first equation,

$$\delta\tilde{\mu} = -\frac{2\mu}{|F_j|^2}(F_j[i\zeta + \nu\partial_z])^\intercal\delta\tilde{u}_j. \tag{5.7}$$

Replacing it in the second equation, after some calculations, we have

$$q_j(x,\nu)\partial_z^2\delta\tilde{u}_j + ir_j(x,\nu,\zeta)\partial_z\delta\tilde{u}_j + s_j(x,\zeta)\delta\tilde{u}_j = 0 \tag{5.8}$$

for all $j = 1, \dots, J$, where q_j is the same matrix of previous sections and r_j, s_j are real matrices given by

$$r_j(x,\nu,\zeta) = 2(\hat{F}_j\nu \otimes \hat{F}_j\zeta + \hat{F}_j\zeta \otimes \hat{F}_j\nu) - \frac{1}{2}(\nu \otimes \zeta + \zeta \otimes \nu), \quad s_j(x,\zeta) = -q_j(x,\zeta).$$

We look at the imaginary part of (5.8), $r_j\partial_z\delta\tilde{u}_j = 0$, $z > 0$. After some calculations (see Lemma 6), we have $\det(r_j) \neq 0$, so we have $\partial_z\delta\tilde{u}_j = 0$, and this implies $\delta\tilde{u}_j \equiv 0$ since $\delta\tilde{u}(0) = 0$. Then using (5.7), we obtain $\delta\tilde{\mu} \equiv 0$. Therefore, we conclude $\tilde{w} \equiv 0$. □

Lemma 6. *In dimension $d = 2$, we have* $\det(r_j(x,\nu,\zeta)) \neq 0$.

Proof. We have $r_j(x,\nu,\zeta) = M + N$, where

$$M = \begin{bmatrix} 2AC & AD + BC \\ AD + BC & 2BD \end{bmatrix}, \qquad N = -\frac{1}{2}\begin{bmatrix} 2\nu_1\zeta_1 & \nu_1\zeta_2 + \zeta_1\nu_2 \\ \nu_1\zeta_2 + \zeta_1\nu_2 & 2\nu_2\zeta_2 \end{bmatrix}$$

and $A = (\hat{F}\nu)_1$, $B = (\hat{F}\nu)_2$, $C = (\hat{F}\zeta)_1$, $D = (\hat{F}\zeta)_2$.

Since $\nu \cdot \zeta = 0$, without loss of generality, we can take $\zeta_1 = -\nu_2$ and $\zeta_2 = \nu_1$, and using the properties of $\hat{F}_j$, we have

$$C = (\hat{F}_j)_{11}\zeta_1 + (\hat{F}_j)_{12}\zeta_2 = -(\hat{F}_j)_{11}\nu_2 + (\hat{F}_j)_{12}\nu_1 = B,$$
$$D = (\hat{F}_j)_{21}\zeta_1 + (\hat{F}_j)_{22}\zeta_2 = -(\hat{F}_j)_{21}\nu_2 + (\hat{F}_j)_{22}\nu_1 = -A.$$

Then

$$r_j = \begin{bmatrix} 4AB + \nu_1\nu_2 & 2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2) \\ 2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2) & -(4AB + \nu_1\nu_2) \end{bmatrix},$$

and we can compute the determinant

$$-\det(r_j) = (4AB + \nu_1\nu_2)^2 + \Big(2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2)\Big)^2. \tag{5.9}$$

Using the fact that $\nabla^S u_j$ are divergence free, we have

$$A = \frac{1}{\sqrt{2}}\cos(\alpha_j - \theta), \quad B = \frac{1}{\sqrt{2}}\sin(\alpha_j - \theta),$$

where $\theta = \arg(\nu)$ so that $\nu = (\cos(\theta), \sin(\theta))$. Then

$$-\det(r_j) = \frac{5}{4} + \cos(2\alpha_j - 3\theta) \neq 0 \quad \text{for all } x, \nu, \zeta. \qquad \square$$

Remark 5.3. It should be possible to show the theorem holds under weaker assumptions given the form of the determinant (5.9).

6 Recovery of the parameters $\delta\mu$ and δu with the modified model with generic forcing term $f(u)$

System (3.1) can be written as

$$\begin{cases} \mathcal{F}_{\mathrm{FT}}\nu = \mathcal{H} & \text{in } \Omega, \\ \mathcal{B}\nu = g & \text{on } \partial\Omega, \end{cases}$$

where $\nu = (\mu, \{u_j\}_{j=1}^J)$ with

$$\mathcal{F}(\nu_j) = \begin{pmatrix} \frac{\mu}{2}|\nabla^S u_j|^2 \\ 2\nabla \cdot \mu\nabla^S u_j + \omega^2 u_j \\ \nabla \cdot u_j \end{pmatrix}, \quad \mathcal{H}_j = \begin{pmatrix} H_j \\ \nabla p \\ 0 \end{pmatrix}, \quad \mathcal{B}\nu_j = g_j,$$

$$\mathcal{F}_{\mathrm{FT}} = \mathcal{F} + \mathcal{F}_{\mathrm{add}} \quad \text{with } \mathcal{F}_{\mathrm{add}}\nu_j = \begin{pmatrix} -f(u_j) \cdot u_j \\ -f(u_j) \\ 0 \end{pmatrix}.$$

The linearized problem for $j \in \{1, \dots, J\}$ is then given by (3.2) with $w = (\delta\mu, \{\delta u_j\}_{j=1}^J)$ and can be rewritten as

$$\begin{cases} \mathcal{L}_{\mathrm{FT}}w = \mathcal{S} & \text{in } \Omega, \\ \mathcal{B}w = g & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathcal{L}_j w_j = \mathcal{F}'(\nu_{0j})w_j = \begin{pmatrix} \frac{\delta\mu}{2}|\nabla^S u_{0j}|^2 + \mu\nabla^S u_{0j} : \nabla^S \delta u_{0j} \\ 2\nabla \cdot \delta\mu\nabla^S u_{0j} + 2\nabla \cdot \mu\nabla^S \delta u_{0j} + \omega^2 \delta u_{0j} \\ \nabla \cdot \delta u_{0j} \end{pmatrix}, \quad \mathcal{S}_j = \begin{pmatrix} \delta H_j \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{L}w = \{\mathcal{L}_j w_j\}_{j=1}^J, \quad \mathcal{S} = \{\mathcal{S}_j\}_{j=1}^J \quad \text{and} \quad \mathcal{L}_{\mathrm{FT}} = \mathcal{L} + \mathcal{L}_{\mathrm{add}} \quad \text{with } \mathcal{L}_{j,\mathrm{add}}\nu_j = \begin{pmatrix} -(Df(u_j)\delta u_j) \cdot u_j - f(u_j) \cdot \delta u_j \\ -Df(u_j)\delta u_j \\ 0 \end{pmatrix}.$$

It can be seen as the equation $\mathcal{A}_{\mathrm{FT}}w = \binom{\mathcal{S}}{g}$.

6.1 Stability estimates

In any dimension d with J measurements, we can see problem (3.2) in the framework of Section 4. The Douglis–Nirenberg numbers are

$$s_i = \begin{cases} -1 & \text{if } i = k' \cdot (d+2) + k'',\ k' = 0, 1, \dots, J,\ k'' = 0, 1, \\ 0 & \text{otherwise}, \end{cases}$$
$$t_j = \begin{cases} 1 & \text{if } j = 1, \\ 2 & \text{otherwise}, \end{cases}$$
$$\sigma_k = -1, \quad k = 1, \dots, Jd,$$

where $i = 1, \dots, J(d+2)$ and $j = 1, \dots, Jd + 1$. The operator $\mathcal{A} = (\mathcal{L}, \mathcal{B})$ is defined from

$$\mathcal{X} = \prod_{j=1}^{Jd+1} H^{l+t_j}(\Omega) \quad \text{to} \quad \mathcal{Y} = \prod_{i=1}^{J(d+2)} H^{l-s_i}(\Omega) \times \prod_{j=1}^{Jd} H^{l-\sigma_j-\frac{1}{2}}(\partial\Omega),$$

where we choose l such that $2(l - s_i) > d$, $2(l - \sigma_k) > d$. In dimension $d = 2$, we can choose $l = 2$.

Moreover, if $d = 2$ and $J = 2$, then we have

$$\mathcal{X} = H^3(\Omega) \times (H^4(\Omega)^2)^2,$$
$$\mathcal{Y} = (H^3(\Omega) \times H^2(\Omega)^2 \times H^3(\Omega))^2 \times (H^{\frac{5}{2}}(\partial\Omega)^2)^2.$$

6.2 Ellipticity and Lopatinskii condition

The principal symbol associated to (3.2) measurements is exactly $\mathcal{P}_J(x, \xi)$ given in Section 5.2. That is, for $J = 2$ measurements,

$$\mathcal{P}_J(x, \xi) = \begin{bmatrix} \frac{|F_1|^2}{\mu} & i\mu(F_1\xi)^\intercal & 0 \\ 2iF_1\xi & -\mu(|\xi|^2 + (\xi \otimes \xi)) & 0 \\ 0 & i\xi^\intercal & 0 \\ \frac{|F_2|^2}{2} & 0 & i\mu(F_2\xi)^\intercal \\ 2iF_2\xi & 0 & -\mu(|\xi|^2 + (\xi \otimes \xi)) \\ 0 & 0 & i\xi^\intercal \end{bmatrix},$$

which is a matrix of size $J(d+2) \times (Jd+1)$.

We finally prove the main theorem of this section, that is, Theorem 1, which we recall.

Theorem 1. *Assume we have that*

$$\left| \frac{\nabla^S u_{1j}}{|\nabla^S u_{1j}|} : \frac{\nabla^S u_{2j}}{|\nabla^S u_{2j}|} \right| \neq 1, \quad j = 1, 2.$$

Let $d = 2$. Then there exists constants C_1 and C_2 depending on $\|f\|_{C^3}$, $\|\mu_2\|_{C^2(\Omega)}$ (C_2 may also depend on ω) such that

$$\|\delta\mu\|_{H^3(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{H^4(\Omega)^2} \leq C_1 \sum_{j=1}^{2} (\|\delta H_j\|_{H^3(\Omega)} + \|\delta g_j\|_{H^{\frac{5}{2}}(\Omega)^2}) + C_2\left(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{L^2(\Omega)^2} \right) \tag{3.3}$$

Proof of Theorem 1. Since $(\mathcal{L}, \mathcal{B})$ satisfies the Lopatinskii condition, by Theorem 4, we have the estimate

$$\|w\|_{\mathcal{X}} \leq C\|(\mathcal{S}, g)\|_{\mathcal{Y}} + C_2\|w\|_{L^2(\Omega)^{d \cdot J}}.$$

We remark that, in dimension 2, we can choose $l = 2$. Let $d = 2$, $J = 2$. Then the operator $\mathcal{L}_{\mathrm{FT}}$ is elliptic, and $\mathcal{B}$ covers $\mathcal{L}_{\mathrm{FT}}$. Moreover, we have for $w = (\delta\mu, \{\delta u_j\}_{j=1}^2)$ a solution to (3.2) the estimate

$$\begin{aligned}\|\delta\mu\|_{H^3(\Omega)} + \sum_{j=1}^{J}\|\delta u_j\|_{H^4(\Omega)^2} \le C\sum_{j=1}^{J}(&\|\mathcal{L}^{\mathrm{ec}}_{\mathrm{FT},j}(\delta\mu,\delta u_j)\|_{H^2(\Omega)^2} + \|\mathcal{L}^{\mathrm{pd}}_{\mathrm{FT},j}(\delta\mu,\delta u_j)\|_{H^3(\Omega)} \\ &+ \|\mathcal{L}^{\mathrm{div}}_{\mathrm{FT}}(\delta\mu,\delta u_j)\|_{H^3(\Omega)} + \|\mathcal{B}\delta u_j\|_{H^{\frac{5}{2}}(\Omega)^2}) \\ &+ C_2\biggl(\|\delta\mu\|_{L^2(\Omega)^2} + \sum_{j=1}^{2}\|\delta u_j\|_{L^2(\Omega)^2}\biggr),\end{aligned} \tag{6.1}$$

where $\mathcal{L}^{\mathrm{ec}}_{\mathrm{FT},j}, \mathcal{L}^{\mathrm{pd}}_{\mathrm{FT},j}, \mathcal{L}^{\mathrm{div}}_{\mathrm{FT}}$ are the parts of $\mathcal{L}_{\mathrm{FT}}$ coming from the elasticity equations, the power density measurements and the divergence condition, respectively. In particular, we have that

$$\mathcal{L}^{\mathrm{ec}}_{\mathrm{FT},j} = -\omega^2\delta u_j + Df(u_j)\delta u_j, \quad \mathcal{L}^{\mathrm{pd}}_{\mathrm{FT},j} = \delta H_j, \quad \mathcal{L}^{\mathrm{div}}_{\mathrm{FT}} = 0.$$

We then remark that there exists a constant C depending only on ω such that

$$\|\mathcal{L}^{\mathrm{ec}}_{\mathrm{FT},j}(\delta\mu,\delta u_j)\|_{H^2(\Omega)^2} \le C\|\delta H_j\|_{H^3(\Omega)},$$

which completes the proof of inequality (3.3) and the theorem. □

6.3 Injectivity

Lemma 7. *Let $J = 2$. The following boundary problem is elliptic:*

$$\begin{cases} L_{j,FT}[\delta u_j] = 0 & \text{in } \Omega, \\ \nabla\cdot\delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

for $j = 1, 2$, $d = 2$, where

$$L_{j,FT}[\delta u_j] = 2\nabla\cdot\Bigl(\Bigl[-\frac{2\mu}{|F_j|^2}(F_j : \nabla^S\delta u_j) + h(u_j)\delta u_j\Bigr]F_j\Bigr) + 2\nabla\cdot\mu\nabla^S\delta u_j + \omega^2\delta u_j - Df(u_j)\delta u_j$$

and

$$h(u_j) = \frac{2}{|F_j|^2}(u_j^\top Df(u_j) - f(u_j)^\top).$$

Furthermore, we have the following estimate:

$$\sum_{j=1}^{J}\|\delta u_j\|_{H^4(\Omega)^2} \le C\sum_{j=1}^{J}(\|\mathcal{L}_{\mathrm{FT}}\delta u_j\|_{H^2(\Omega)^2} + \|\mathcal{B}_{\mathrm{FT}}\delta u_j\|_{H^{\frac{5}{2}}(\Omega)^2}) + C_2\sum_{j=1}^{J}\|\delta u_j\|_{L^2(\Omega)^2}.$$

Proof. In fact, since the symbol of f is a polynomial with degree at most 1, we notice that the principal symbol for system (6.2) is given by the principal symbol associated to (3.2). The Lopatinskii condition is satisfied because it depends only on the principal symbol. Therefore, we conclude the ellipticity and the estimate. □

We recall the following "matrix orthogonality" identities. Let $F_j^\perp$ be such that $F_j : F_j^\perp = 0$ and $|F_j^\perp| = |F_j|$. Then $\nabla^S\delta u_j$ can be expressed as

$$\nabla^S\delta u_j = (\nabla^S\delta u_j : \hat{F}_j)\hat{F}_j + (\nabla^S\delta u_j : \hat{F}_j^\perp)\hat{F}_j^\perp,$$

and then

$$\int_\Omega \mu|\nabla^S\delta u_j|^2 = \int_\Omega \mu(|\nabla^S\delta u_j : \hat{F}_j|^2 + |\nabla^S\delta u_j : \hat{F}_j^\perp|^2)\,dx. \tag{6.3}$$

We can use these to prove the following lemma.

Lemma 8. *Let $\tilde{A}_{\mathrm{FT}}$ be the operator corresponding to the equation given in the previous lemma. In dimension two, if $\{\delta u_j\} \in \ker(\tilde{A}_{\mathrm{FT}})$, then*

$$\int_\Omega |\delta u_j|^2 \le \tilde{C}(\omega^2)^2 \int_\Omega |D\delta u_j|^2 \tag{6.4}$$

where

$$\tilde{C}(\omega^2) = \sqrt{\frac{1 + 2\|\mu\|_{L^\infty}}{\omega^2 - (\|Df(u_j)\|_{\mathcal{L}(H^1,L^2)} + \|h(u_j)\|_{\mathcal{L}(H^1,L^2)})}}.$$

Proof. If $\delta u_j \in \ker(\tilde{A}_{\mathrm{FT}})$, then

$$\begin{cases} L_{j,FT}[\delta\mu, \delta u_j] = 0 & \text{in } \Omega, \\ \nabla \cdot \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Note that

$$\frac{1}{|F_j|^2}(Df(u_j)\delta u_j) \cdot u_j = \frac{1}{|F_j|^2}(u_j^\intercal Df(u_j))\delta u_j.$$

From the second equation in (6.5), we obtain

$$\delta\mu = -\frac{2\mu}{|F_j|^2}(F_j : \nabla^S \delta u_j) + h(u_j)\delta u_j.$$

On the other hand, multiplying the first equation of (6.5) by δu_j and integrating, we obtain

$$\omega^2 \int_\Omega |\delta u_j|^2 = \int_\Omega (Df(u_j)\delta u_j) \cdot \delta u_j - 4\int_\Omega \mu|\hat{F}_j : \nabla^S \delta u_j|^2 + 2\int_\Omega (h(u_j)\delta u_j)(\hat{F}_j : \nabla^S \delta u_j) + 2\int_\Omega \mu|\nabla^S \delta u_j|^2,$$

and considering the identity (6.3),

$$\begin{aligned}\omega^2 \int_\Omega |\delta u_j|^2 &= \int_\Omega (Df(u_j)\delta u_j) \cdot \delta u_j + 2\int_\Omega (h(u_j)\delta u_j)(\hat{F}_j : \nabla^S \delta u_j) + 2\int_\Omega \mu|\hat{F}_j^\perp : \nabla^S \delta u_j|^2 - 2\int_\Omega \mu|\hat{F}_j : \nabla^S \delta u_j|^2 \\ &\le (\|Df(u_j)\|_{L(H^1,L^2)} + \|h(u_j)\|_{L(H^1,L^2)})\|\delta u_j\|_{L^2}^2 + (1 + 2\|\mu\|_{L^\infty})\int_\Omega |\nabla^S \delta u_j|^2.\end{aligned}$$

Therefore, we obtain the desired result

$$\int_\Omega |\delta u_j|^2 \le \frac{1 + 2\|\mu\|_{L^\infty}}{\omega^2 - (\|Df(u_j)\|_{\mathcal{L}(H^3,L^2)} + \|h(u_j)\|_{\mathcal{L}(H^3,L^2)})}\int_\Omega |\nabla u_j|^2.$$ □

Lemma 9. *In dimension 2, there exists $\omega_0 > 0$ such that, for all $\omega \ge \omega_0$, we have $\ker(\tilde{A}_{\mathrm{FT}}) = \{0\}$. In other words, the operator is injective.*

Proof. From Theorem 1, taking $\tilde{A}_{\mathrm{FT}} w = (0, 0)$, using the previous lemma, we have

$$\sum_j \|\delta u_j\|_{H^4(\Omega)^2} \le C_2 \sum_j \|\delta u_j\|_{L^2(\Omega)^2} \le C_2 \tilde{C}(\omega) \sum_j \|\nabla \delta u_j\|_{L^2},$$

where $\tilde{C}(\omega^2)$ is given in (6.4). If we take ω large enough such that $C_2\tilde{C}(\omega^2) < 1$, we can absorb the right side of the estimate. So we conclude that $\delta u_j = 0$. □

The main corollary now follows.

Corollary 1. *For all ω sufficiently large, the linearized system is injective, that is, we can find a C_1 such that $C_2 = 0$ in Theorem 1, provided δg_j is zero.*

Proof. Considering equation (3.2) with the terms not depending on u_j equal to zero, we can take the second equation and obtain

$$\delta\mu = \frac{1}{|F_j|^2}[(f(u_j) + u_j^\intercal Df(u_j)) \cdot \delta u_j - 2\mu\nabla^S u_j : \nabla^S \delta u_j]. \tag{6.6}$$

Then we replace $\delta\mu$ in the first equation, so we obtain equation (6.2). By Lemma 9, we obtain $\delta u_j = 0$, and using equation (6.6), we conclude $\delta\mu = 0$. Hence, we can eliminate the terms multiplying C_2 in equation (6.1) for sufficiently large ω^2. □

6.4 Fixed-point algorithm: Preliminaries

We introduce the general fixed-point lemmas which are needed to solve nonlinear PDEs with small data. Let J be a linear operator and N a power nonlinearity. We view the nonlinear PDE as

$$\begin{aligned} J(w) &= N(w) && \text{in } \Omega,\\ w &= f && \text{in } \Omega,\\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We assume that f is a generic function in a Banach space $\mathcal{N}$. The solution then looks like $w = w_{\text{lin}} + J^{-1}N(w)$. We also have the following abstract iteration result.

Lemma 10 ([31, Proposition 1.38]). *Let $\mathcal{N}, \mathcal{S}$ be two Banach spaces, and suppose we are given an invertible linear operator $J\colon \mathcal{N} \to \mathcal{S}$ with the bound $\|J^{-1}F\|_{\mathcal{S}} \le C_0\|F\|_{\mathcal{N}}$ for all $F \in \mathcal{N}$ and some $C_0 > 0$. Suppose that we are given a nonlinear operator $N\colon \mathcal{S} \to \mathcal{N}$ which is a sum of a u-dependent part and a u-independent part. Assume the u-dependent part N_u is such that $N_u(0) = 0$ and obeys the following Lipschitz bounds:*

$$\|N(u) - N(v)\|_{\mathcal{N}} \le \frac{1}{2C_0}\|u - v\|_{\mathcal{S}}$$

for all $u, v \in B_\varepsilon = \{u \in \mathcal{S} : \|u\|_{\mathcal{S}} \le \varepsilon\}$ for some $\varepsilon > 0$. In other words, we have that $\|N\|_{\dot{C}^{0,1}(B_\varepsilon \to \mathcal{N})} \le \frac{1}{2C_0}$. Then, for all $u_{\text{lin}} \in B_{\frac{\varepsilon}{2}}$, there exists a unique solution $u \in B_\varepsilon$ with the map $u_{\text{lin}} \mapsto u$ Lipschitz with constant at most 2. In particular, we have that $\|u\|_{\mathcal{S}} \le 2\|u_{\text{lin}}\|_{\mathcal{S}}$.

Remark 6.1. The proof of Lemma 10 consists in establishing the convergence of the iterative sequence

$$u^{(n)} = \begin{cases} u_{\text{lin}} & \text{if } n = 0,\\ u_{\text{lin}} + J^{-1}N(u^{(n-1)}) & \text{if } n \ge 1. \end{cases}$$

Therefore, Lemma 10 also establishes the convergence of this kind of sequences.

Given the abstract convergence lemma above, we want to apply this to the linearized linear and subsequently nonlinear elasticity problem to give a direct proof of existence and uniqueness to system (3.2).

6.5 Fixed-point algorithm for the recovery of μ

We first focus on the case of linearized linear elasticity, that is, with $f(u_j) = 0$ in (3.2). We set the following notation:

$$v_j = (\mu, \{u_j\}_j) \quad \text{and} \quad v = \{v_j\}_{j=1}^J, \quad \text{also,} \quad v = v_0 + \delta v, \quad \text{where } v_0 = (\mu_0, \{u_{0,j}\}_{j=1}^J) = \{v_j\}_{j=1}^J,$$

$$\delta v = (\delta\mu, \{\delta u_j\}_j) = \{w_j\}_{j=1}^J = w,$$

$$\mathcal{F}(v_j) = \begin{pmatrix} \frac{\mu}{2}|\nabla^S u_j|^2 \\ 2\nabla \cdot \mu\nabla^S u_j + \omega^2 u_j \\ \nabla \cdot u_j \end{pmatrix}, \quad \mathcal{H}_j = \begin{pmatrix} H_j \\ \nabla p \\ 0 \end{pmatrix}, \quad \mathcal{B}v_j = g_j,$$

$$\mathcal{F}v = \{\mathcal{F}v_j\}_{j=1}^J, \quad \mathcal{H} = \{\mathcal{H}_j\}_{j=1}^J, \quad \mathcal{B}v = \{\mathcal{B}v_j\}_{j=1}^J,$$

$$\mathcal{L}_j = \mathcal{F}'(v_{0j}), \quad \text{that is,} \quad \mathcal{L}_j w_j = \mathcal{F}'(v_{0j})w_j = \begin{pmatrix} \frac{\delta\mu}{2}|\nabla^S u_{0j}|^2 + \mu\nabla^S u_{0j} : \nabla^S \delta u_{0j} \\ 2\nabla \cdot \delta\mu\nabla^S u_{0j} + 2\nabla \cdot \mu\nabla^S \delta u_{0j} + \omega^2 \delta u_{0j} \\ \nabla \cdot \delta u_{0j} \end{pmatrix},$$

$$\mathcal{S}_j = \begin{pmatrix} \delta H_j \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{L}w = \{\mathcal{L}_j w_j\}_{j=1}^J, \quad \mathcal{S} = \{\mathcal{S}_j\}_{j=1}^J, \quad \mathcal{H}_0 := \mathcal{F}(v_{0j}), \quad g_0 = \mathcal{B}v_0,$$

and consider the nonlinear problem

$$\begin{cases} \mathcal{F}(v_0 + w) = \mathcal{H} & \text{in } \Omega,\\ \qquad \mathcal{B}w = g - g_0 & \text{on } \partial\Omega, \end{cases}$$

and the linear problem

$$\begin{cases} \mathcal{L}w = \mathcal{S} & \text{in } \Omega, \\ \mathcal{B}w = g - g_0 & \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

System (6.7) can be written as $\mathcal{A}w = \binom{\mathcal{S}}{g-g_0}$. Note that

$$\mathcal{F}(v_0 + w) = \mathcal{F}(v_0) + \mathcal{F}'(v_0)w + \mathcal{G}(w; v_0),$$

where $\mathcal{G}(w; v_0)$ is given by

$$\mathcal{G}_j(w; v_0) = \begin{pmatrix} \delta\mu\nabla^S u_{0j} : \nabla^S \delta u_j + \frac{(\mu_0+\delta\mu)}{2}|\nabla^S \delta u_j|^2 \\ 2\nabla \cdot \delta\mu\nabla^S \delta u_j \\ 0 \end{pmatrix}, \tag{6.8}$$

is such that $\|(\mathcal{G}(w; v_0)\|_{\mathcal{Y}} \le C\|w\|_{\mathcal{X}}^2$, where the constant C depends only on the $L^\infty(\Omega)$ norm of $|\nabla^S u_j|$ and μ for $j = 1, 2$ so that we can write the problem as

$$\begin{cases} \mathcal{L}w = \mathcal{H} - \mathcal{H}_0 - \mathcal{G}(w; v_0) & \text{in } \Omega, \\ \mathcal{B}w = g - g_0 & \text{on } \partial\Omega. \end{cases}$$

We define the following fixed-point algorithm.

Algorithm 1.
Input:
- a function $v_0 = (\mu_0, \{u_{0j}\})$, where μ_0 is given and then $u_{0,j}$ is the solution of the system

$$\begin{cases} 2\nabla \cdot \mu_0\nabla^S u_j + \omega^2 u_j = -\nabla p & \text{in } \Omega, \\ \nabla \cdot u_j = 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega, \end{cases}$$

- observations $\mathcal{H}$ in Ω and boundary information g on $\partial\Omega$, i.e., $\mathcal{H} = \mathcal{F}(v_0 + w_{\text{true}})$ and $g = g_0 + \mathcal{B}w_{\text{true}}$,
- a tolerance $\varepsilon > 0$.

Steps:
- compute $\mathcal{H}_0$ via the formula $\mathcal{H}_0 = \mathcal{F}(v_0)$;
- define $w^0 = 0$;
- iterate, from k to $k + 1$,
  - $w^{k+1} = \mathcal{I}(w^k) := \mathcal{A}^{-1}(\mathcal{H} - \mathcal{H}_0 - \mathcal{G}(w^k; v_0), g - g_0)$,
  - stop if $\|w^{k+1} - w^k\| < \varepsilon$;
- define $v = v_0 + w^{k+1}$.

Return v.

Lemma 11. *There exists a constant $c_1 = c_1(\varepsilon) > 0$ such that*

$$\|\mathcal{G}(w; v_0) - \mathcal{G}(\tilde{w}; v_0)\|_{\mathcal{Y}} \le c_1\Big(\|\delta\mu - \delta\tilde{\mu}\|_{H^3(\Omega)} + \sum_j \|\delta u_j - \delta\tilde{u}_j\|_{(H^4(\Omega))^2}\Big),$$

provided $\|\delta\mu\|_{H^3(\Omega)}, \|\delta u_j\|_{H^4(\Omega)^2} \le \varepsilon$ for some $\varepsilon > 0$. Such a constant satisfies $c_1(\varepsilon) \to 0$ whenever $\varepsilon \to 0$.

Proof. The definition of $\mathcal{G}_j(w, v_0)$ in (6.8) implies $\mathcal{G}_j(w, v_0)$ is a differentiable function of w. The mean value theorem gives the result. Alternatively, using that $H^2(\Omega)^d$ and $H^3(\Omega)^d$ are Banach algebras gives a bound for c_1,

$$c_1 \le C_{\text{BA}}\varepsilon\big(JC_{\text{BA}} \max_j \|u_{0j}\|_{H^4(\Omega)^d} + J\|\mu_0\|_{H^3(\Omega)} + 5\varepsilon\big)$$

with $C_{\text{BA}} > 0$ the constant from the bound given by the fact that $H^2(\Omega)$ and $H^3(\Omega)$ are Banach algebras, cf. [11, Theorem 6.1-4]. □

Proposition 2. *If $\varepsilon > 0$ is sufficiently small so that $c_1(\varepsilon)\|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} < \frac{1}{2}$, where $c_1(\varepsilon)$ is given by the previous lemma, then the algorithm converges if in addition we have $\|(\mathcal{H} - \mathcal{H}_0, g - g_0)\|_{\mathcal{X}} \le \frac{\varepsilon}{2}$, and we obtain $\|w\|_{\mathcal{X}} < \varepsilon$.*

Proof. We take

$$J = \mathcal{A}, \quad N(w) = (\mathcal{G}(w; v_0), 0), \quad w_{\text{lin}} = (\mathcal{H} - \mathcal{H}_0, g - g_0).$$

Because the nonlinearity satisfies the conditions for the fixed-point iteration by Lemma 11, application of the previous convergence lemma, Lemma 10, gives the desired result. □

We note that $\mathcal{F}_{\text{FT}} = \mathcal{F} + \mathcal{F}_{\text{add}}$ and $\mathcal{L}_{\text{FT}} = \mathcal{L} + \mathcal{L}_{\text{add}}$ with $\mathcal{F}$ and $\mathcal{L}$ given in the previous case whenever $f(u_j)$ is nonzero and

$$\mathcal{F}_{\text{add}} v_j = \begin{pmatrix} -f(u_j) \cdot u_j \\ -f(u_j) \\ 0 \end{pmatrix}, \qquad \mathcal{L}_{j,\text{add}} v_j = \begin{pmatrix} -(Df(u_j)\delta u_j) \cdot u_j - f(u_j) \cdot \delta u_j \\ -Df(u_j)\delta u_j \\ 0 \end{pmatrix}.$$

In addition, we define $\mathcal{G}_{\text{FT}}(w; v) = \mathcal{F}(v + w) - \mathcal{F}v - \mathcal{L}w$. It is clear that $\mathcal{G}_{\text{FT}}(w; v) = \mathcal{G}(w; v) + \mathcal{G}_{\text{add}}(w; v)$ with $\mathcal{G}$ defined as before and

$$\mathcal{G}_{j,\text{add}}(w; v) = \begin{pmatrix} o(\delta u_j) \cdot (u_j + \delta u_j) - (Df(u_j)\delta u_j) \cdot u_j \\ o(\delta u_j) \\ 0 \end{pmatrix},$$

where

$$o(\delta u_j) = \int_0^1 (1 - t) D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j]\, dt$$

comes from Taylor's formula

$$f(u_j + \delta u_j) = f(u_j) + Df(u_j)\delta u_j + \int_0^1 (1 - t) D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j]\, dt.$$

The fixed-point algorithm for the case of linearized nonlinear elasticity is the same as Algorithm 1, with the following changes:

- instead of $\mathcal{F}, \mathcal{L}, \mathcal{G}, \mathcal{A}$, we use $\mathcal{F}_{\text{FT}}, \mathcal{L}_{\text{FT}}, \mathcal{G}_{\text{FT}}, \mathcal{A}_{\text{FT}}$;
- in the step of solving equation, we solve

$$\begin{cases} 2\nabla \cdot \mu_0 \nabla^S u_j + \omega^2 u_j - f(u_j) = -\nabla p & \text{in } \Omega, \\ \nabla \cdot u_j = 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega. \end{cases}$$

Lemma 12. *There exists a constant $c_2 = c_2(\varepsilon) > 0$ such that*

$$\|\mathcal{G}_{\text{FT}}(w; v_0) - \mathcal{G}_{\text{FT}}(\tilde{w}; v_0)\|_{\mathcal{Y}} \le c_2 \Big(\|\delta\mu - \delta\tilde{\mu}\|_{H^3(\Omega)} + \sum_j \|\delta u_j - \delta\tilde{u}_j\|_{(H^4(\Omega))^2} \Big),$$

provided $\|\delta\mu\|_{H^3(\Omega)}, \|\delta u_j\|_{H^4(\Omega)^2} \le \varepsilon$ for some $\varepsilon > 0$.

Proof. Let

$$\begin{aligned} \psi(\delta u_j, \delta\tilde{u}_j) &= D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j] - D^2 f(u + t\delta\tilde{u}_j)[\delta\tilde{u}_j, \delta\tilde{u}_j] \\ &= D^2 f(u_j + t\delta u_j)[\delta u_j - \delta\tilde{u}_j, \delta u_j] + D^2 f(u_j + t\delta u_j)[\delta\tilde{u}_j, \delta u_j - \delta\tilde{u}_j] \\ &\quad + (D^2 f(u_j + t\delta u_j) - D^2 f(u_j + t\delta\tilde{u}_j))[\delta\tilde{u}_j, \delta\tilde{u}_j]; \end{aligned}$$

hence

$$\|\psi(\delta u_j, \delta\tilde{u}_j)\|_{L^2(\Omega)^2} \le c_3 \varepsilon \|\delta u_j - \delta\tilde{u}_j\|_{H^1(\Omega)^2},$$

with c_3 being the maximum between

$$\begin{gathered} 2 \sup\{\|D^2 f(h)\|_{\mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, L^2(\Omega)^2))};\ \|u_j - h\|_{H^3(\Omega)^2} \le \varepsilon\}, \\ 2\varepsilon^2 \sup\{\|D^3 f(h)\|_{\mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, L^2(\Omega)^2)))};\ \|u_j - h\|_{H^4(\Omega)^2} \le \varepsilon\} \end{gathered}$$

given by the mean value theorem over $D^2 f$. Then

$$\|o(\delta u_j) - o(\delta\tilde{u}_j)\|_{L^2(\Omega)^2} = \int_0^1 |1 - t| c_3 \varepsilon \|\delta u_j - \delta\tilde{u}_j\|_{H^4(\Omega)^2}\, dt \le c_3 \varepsilon \|\delta u_j - \delta\tilde{u}_j\|_{H^4(\Omega)^2}.$$

Then the conclusion is direct from Lemma 11 and the definition of $\mathcal{G}_{\text{add}}$. □

Corollary 4. *If $\varepsilon > 0$ is sufficiently small so that $c_2(\varepsilon)\|\mathcal{A}_{\mathrm{FT}}^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} < \frac{1}{2}$, then the linearization for the nonlinear elasticity problem converges if in addition we have $\|(\mathcal{H} - \mathcal{H}_0, g - g_0)\|_{\mathcal{X}} \leq \frac{\varepsilon}{2}$, and we obtain $\|w\|_{\mathcal{X}} < \varepsilon$.*

Finally, we can conclude the proof of the main theorem of this section.

Theorem 2. *The solution $w = (\delta\mu, \{\delta u_j\}_{j=1}^2)$ to (3.2) exists as a limit of an explicit sequence of Duhamel iterates and is unique in $H^3(\Omega) \times (H^4(\Omega)^2)^2$ for all ω sufficiently large and $\delta g_j = 0$.*

Proof. The proof of Theorem 2 follows from the algorithms themselves combined with Proposition 2 and Corollary 4. □

7 Simplification of recovery of $\delta\mu$ for linear elasticity with internal measurements

The model considered in this section is given by

$$\begin{cases} 2\nabla \cdot \mu\nabla^S u_j + \omega^2 u_j - f(u_j) = -\nabla p_j & \text{in } \Omega, \\ \frac{\mu}{2}|\nabla^S u_j|^2 - f(u_j)\cdot u_j = H_j & \text{in } \Omega, \\ \nabla \cdot u_j = 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $f_j = 0$, but with internal measurements of u_j, i.e., $u_j = H_j$ in Ω. In [34, Proposition 1 c)], the authors proved that there is no ellipticity for the joint recovery of μ and p. Therefore, we must either apply the curl to the operator to remove ∇p or we must hold ∇p fixed. This last case is studied in [34], establishing the ellipticity and Lopatinskii condition with at least one measurement, but null kernel with two measurements. If we are to use the model with ∇p fixed, then we know that λ is large. This causes some convergence problems when considering the Saint-Venant model of nonlinear elasticity, for example with results like in Sections 5 and 6, where we need to have a contraction map, so we choose to apply the curl operator, which eliminates the λ terms.

Hence, we consider the model

$$\begin{cases} \omega^2\nabla \times u_j + 2\nabla \times \nabla \cdot \mu\nabla^S u_j = 0 & \text{in } \Omega, \\ u_j = H_j & \text{in } \Omega, \\ \nabla \cdot u_j = 0 & \text{in } \Omega, \\ u_j = g_j & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

The linearization of (7.1) gives

$$\begin{cases} \omega^2\nabla \times \delta u_j + 2\nabla \times \nabla \cdot \mu\nabla^S \delta u_j + 2\nabla \times \nabla \cdot \delta\mu\nabla^S u_j = 0 & \text{in } \Omega, \\ \delta u_j = \delta H_j & \text{in } \Omega, \\ \nabla \cdot \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

7.1 Ellipticity

Let $\Sigma_{\mathrm{curl}}(\xi)$ be the symbol of the curl operator, that is $\Sigma_{\mathrm{curl}}(\xi) = i(-\xi_2\ \xi_1)$ in dimension two, and

$$\Sigma_{\mathrm{curl}}(\xi) = i\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}$$

in dimension three. Note that if $b \in \mathbb{R}^d$, then $\Sigma_{\mathrm{curl}}(\xi)b = ib \times \xi$.

The linearized system then has the following principal symbol:

$$\mathcal{P}(x,\xi) = \begin{bmatrix} 2(\nabla^S u\xi)\times\xi & -2\mu\Sigma_{\mathrm{curl}}(\xi)(|\xi|^2 I_d + \xi\otimes\xi) \\ 0 & I_d \\ 0 & i\xi^\intercal \end{bmatrix}$$

which is a matrix with size $(2d+1)\times(d+1)$. Let $\xi \neq 0$, and let $\boldsymbol{C}_1,\dots,\boldsymbol{C}_{d+1}$ be the columns of that matrix. Let $\alpha_1,\dots,\alpha_{d+1}\in\mathbb{C}$ be such that

$$\sum_{i=1}^{d+1}\alpha_i\boldsymbol{C}_i = \mathbf{0}.$$

We see that, because of the identity matrix, necessarily, $\alpha_2=\cdots=\alpha_{d+1}=0$, so we have to analyze the equation $\alpha_1\boldsymbol{C}_1=\mathbf{0}$. This last equation can be reduced to the case studied in [6], giving the nonellipticity for one measurement. If we consider the augmented system for two measurements, we obtain the ellipticity as in [6] for three dimensions. Notice that this computation in two dimensions corrects a mistake in the original computations presented there. In particular, the curl in two dimensions is defined as

$$f:\Omega\to\mathbb{R}^d,\quad \nabla\times f: \partial_1 f_2 - \partial_2 f_1,$$

and the computations in [6] flip the order of the partial derivatives. The results there then only hold for symmetric pressure gradients, those for which $\nabla p(x_1,x_2)=\nabla p(x_2,x_1)$.

The symbol for the augmented system is

$$\mathcal{P}_2(x,\xi) = \begin{bmatrix} 2(\nabla^S u_1\,\xi)\times\xi & P(\xi) & 0 \\ 0 & I_d & 0 \\ 0 & i\xi^\intercal & 0 \\ 2(\nabla^S u_2\,\xi)\times\xi & 0 & P(\xi) \\ 0 & 0 & I_d \\ 0 & 0 & i\xi^\intercal \end{bmatrix},$$

where $P(\xi) = -2\mu\Sigma_{\mathrm{curl}}(\xi)(|\xi|^2 I_d + \xi\otimes\xi)$.

Lemma 13. *For ellipticity of system* (7.2)*, in other words, for $\mathcal{P}_2(x,\xi)$ being column rank, we need that he following condition holds:*

$$|(\nabla^S u_1\xi)\times\xi| + |(\nabla^S u_2\xi)\times\xi| \neq 0 \quad \text{for all } |\xi|\neq 0. \tag{7.3}$$

This is slightly different to the case in [6] where the following is considered:

$$|(\nabla^S u_1\xi)\times\xi| + |(\nabla^S u_2\xi)\times\xi| \geq |\xi|^2. \tag{7.4}$$

The first condition is more relaxed and does not require ellipticity of the added symbols, only that they be nonzero simultaneously.

Proof. Condition (7.3) is equivalent to the following: let $A^{(j)} = \nabla^S u_j$, and let the matrices $B^{(j)}$ be defined in two dimensions by

$$B^{(j)} = \begin{pmatrix} a_{11}^{(j)} - a_{22}^{(j)} & 2(a_{12}^{(j)} + a_{21}^{(j)}) \end{pmatrix} \tag{7.5}$$

and in three dimensions by

$$B^{(j)} = \begin{pmatrix} a_{23}^{(j)} & 0 & 0 & a_{22}^{(j)} - a_{33}^{(j)} & a_{12}^{(j)} & -a_{13}^{(j)} \\ 0 & -a_{13}^{(j)} & 0 & -a_{12}^{(j)} & a_{33}^{(j)} - a_{11}^{(j)} & a_{23}^{(j)} \\ 0 & 0 & a_{12}^{(j)} & a_{13}^{(j)} & -a_{23}^{(j)} & a_{11}^{(j)} - a_{22}^{(j)} \end{pmatrix}. \tag{7.6}$$

A condition in dimension $d=2,3$ for having ellipticity is that the $d\times d$ matrix

$$\begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \quad \text{must be invertible.} \tag{7.7}$$

The equivalence between (7.3) and (7.7) comes from the equality

$$\begin{pmatrix} ((A^{(1)}\xi)\times\xi)^{\mathsf{T}} \\ ((A^{(1)}\xi)\times\xi)^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \begin{pmatrix} \xi_2^2-\xi_1^2 \\ \xi_1\xi_2 \end{pmatrix}$$

in dimension two and

$$\begin{pmatrix} ((A^{(1)}\xi)\times\xi)^{\mathsf{T}} \\ ((A^{(1)}\xi)\times\xi)^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \begin{pmatrix} \xi_3^2-\xi_2^2 \\ \xi_3^2-\xi_1^2 \\ \xi_2^2-\xi_1^2 \\ \xi_2\xi_3 \\ \xi_1\xi_3 \\ \xi_1\xi_2 \end{pmatrix}$$

in dimension three. Note that condition (7.7) is ensured when $\nabla^S u_1 \neq \alpha \nabla^S u_2$ for all $\alpha \in \mathbb{R}$. □

7.2 Lopatinskii condition

The Lopatinskii condition we show is based on [6]. The analysis is the same, but in a certain step, we consider condition (7.3) instead of (7.4).

Lemma 14. *The Lopatinskii condition holds for* (7.2) *under assumption* (7.3).

Proof. If $\mathcal{P}_2(x, i\eta + \nu\partial_z)(\tilde{\mu}, \tilde{u}) = 0$, then we easily see that $\tilde{u} \equiv 0$, due to the identity blocks. Then consider $A^{(j)} = \nabla^S u_j$. Then we have the equation

$$(A^{(j)}\nu \times \nu)\partial_z^2\tilde{\mu} + i(A^{(j)}\eta \times \nu + A^{(j)}\nu \times \eta)\partial_z\tilde{\mu} - (A^{(j)}\eta \times \eta)\tilde{\mu} = 0, \quad j = 1, 2.$$

If, in each equation, we apply the dot product with $A^{(j)}\nu \times \nu$ and then we sum both equations, we obtain

$$a\partial_z^2\tilde{\mu} + b\partial_z\tilde{\mu} + c\tilde{\mu} = 0 \tag{7.8}$$

with $a = \sum_j |A^{(j)}\nu \times \nu|^2$, which is nonzero by (7.3). Then let

$$\lambda_{1,2} = \frac{-ib \pm \sqrt{-b^2 - 4ac}}{2a}$$

be the roots of the characteristic polynomial related to equation (7.8). The solutions have the structure

$$\tilde{\mu}(z) = \alpha(\exp(\lambda_1 z) - \exp(\lambda_2 z))$$

since $\tilde{\mu}(0) = 0$. If $\lambda_{1,2}$ is purely imaginary, the only option for $\tilde{\mu}$ going to 0 when $z \to \infty$ is when $\alpha = 0$. If $\lambda_{1,2}$ has a real part, then one of the exponentials goes to infinity and the other goes to zero when $z \to \infty$, so the only option we have is $\alpha = 0$. That is, we have the Lopatinskii condition. □

The Douglis numbers are

$$s_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, d+1, 2d+2, \dots, 3d+1\}, \\ -2 & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 2 & \text{if } i = 1, \\ 3 & \text{otherwise,} \end{cases}$$

$$\sigma_k = -1, \quad k = 1, \dots, 2d.$$

Then the operator over $(\delta\mu, \{\delta u_j\}_{j=1}^J)$ given by equation (7.2) with two measurements is defined from

$$\mathcal{X} = H^{l+2}(\Omega) \times H^{l+3}(\Omega)^d \times H^{l+3}(\Omega)^d \quad \text{to} \quad \mathcal{Y} = (H^l(\Omega)^d \times H^{l+2}(\Omega)^d \times H^{l+2}(\Omega) \times H^{l+\frac{1}{2}}(\Omega)^d)^2,$$

where we can take $l = 2$ in dimension two and dimension three.

Proposition 3. *We have the following stability estimate:*

$$\begin{aligned}\|\delta\mu\|_{H^{l+2}(\Omega)} + \sum_{j=1}^{2}\|\delta u_j\|_{H^{l+3}(\Omega)^d} \le C\sum_{j=1}^{2}\Big(\|\mathcal{L}_j^{\mathrm{ec}}(\delta\mu,\delta u_j)\|_{H^{l}(\Omega)^d} + \|\mathcal{L}_j^{\mathrm{int}}(\delta\mu,\delta u_j)\|_{H^{l+2}(\Omega)^d} \\ + \|\mathcal{L}_j^{\mathrm{div}}\delta u_j\|_{H^{l+2}(\Omega)} + \|\delta u_j\|_{H^{l+\frac{1}{2}}(\partial\Omega)^d}\Big) \\ + C_2\bigg(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2}\|\delta u_j\|_{L^2(\Omega)^d}\bigg).\end{aligned}$$

Here $\mathcal{L}^{\mathrm{ec}}$, $\mathcal{L}_j^{\mathrm{int}}$ and $\mathcal{L}^{\mathrm{div}}$ are coming from the elasticity, the solution measurements and the divergence condition, respectively.

The proof follows directly from Theorem 4 and the verification of the Lopatanskii condition.

7.3 Local injectivity

The results in [14] prove local injectivity and the convergence of an algorithm for the recovery of μ. They use unique continuation properties assuming $\delta\mu|_{\partial\Omega} = 0$ (in our notation). In this section, we show another injectivity argument, based on [8].

If we consider the right-hand side of (7.1) being 0, then we have $\nabla\times\nabla\cdot(\delta\mu A^{(j)}) = 0$, $j = 1, 2$. Let $\rho(x,\xi)$ be the principal symbol for this last equation. Then

$$\rho(x,\xi) = \begin{pmatrix}(A^{(1)}\xi)\times\xi \\ (A^{(2)}\xi)\times\xi\end{pmatrix}.$$

In dimension two, we need to assume that $A_{12}^{(j)} \neq 0$ to obtain that $(0,1)$ is non-characteristic at the origin since

$$A^{(j)}\begin{pmatrix}0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\end{pmatrix} = A_{12}^{(j)}.$$

In dimension three, we need to assume that $a_{13}^{(j)}, a_{23}^{(j)} \neq 0$ to obtain that $(0,0,1)$ is non-characteristic at the origin since

$$A^{(j)}(0,0,1)\times(0,0,1) = (a_{23}^{(j)}, -a_{13}^{(j)}, 0).$$

Condition (7.3) provides the hypothesis for [8, Theorem 3.6] since there are not real roots, and then, due to the fundamental algebra theorem, we have two different complex roots. Therefore, we have a unique continuation principle for μ, and we can take $C_2 = 0$ in the last estimate above.

8 Nonlinear elasticity (Saint-Venant model) with internal measurements

The Saint-Venant model is the first nonlinear model in elasticity that is studied in the literature. It is a generalization of the linear model studied before, and it comes from the simplification of the Green strain tensor

$$Eu = \nabla^S u + \frac{1}{2}\nabla u^\intercal \nabla u. \tag{8.1}$$

In linear elasticity, it is assumed that the displacements are sufficiently small for neglecting the term $\nabla u^\intercal\nabla u$, considering the *small strain tensor*

$$\varepsilon u = \nabla^S u, \tag{8.2}$$

cf. [28] for the constant coefficient calculations. The Saint-Venant–Kirchhoff model considers (8.1) instead of (8.2), since it is assumed that the deformations are not so small, and Eu plays the role of εu in the constitutive equations of linear elasticity.

In this section, we consider the Saint-Venant model under a periodic force with frequency ω, which can be written as a "steady state" equation by

$$\begin{cases} (L_{\mu,\lambda} + N_{\mu,\lambda})u + \omega^2 u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{8.3}$$

where

$$L_{\mu,\lambda}u = 2\nabla \cdot \mu\nabla^S u + \nabla(\lambda\nabla \cdot u), \quad N_{\mu,\lambda}u = 2c_\tau\nabla \cdot (\mu\nabla u^\intercal \nabla u) + \nabla(\lambda|\nabla u|^2),$$

and c_τ is a constant in x coming from the fact that we cannot obtain a time-independent equation by applying a periodic force in time, as in the previous cases, since they are linear in u. So our model is considered for a fixed time τ. For two sets of measurements and boundary conditions, this is system (3.4), which we recall:

$$\begin{aligned} \nabla(\lambda \cdot \nabla u_j) + 2\nabla \cdot \mu(\nabla^S u_j + ac_\tau\nabla u_j^\intercal \nabla u_j) + a\nabla(\lambda|\nabla u_j|^2) + \omega^2 u_j = 0 &\quad \text{in } \Omega, \\ \delta u_j = H_j &\quad \text{in } \Omega, \\ u_j = g_j &\quad \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

when the measurements are $u_j = H_j$ in Ω for $j = 1, 2$. Applying the curl operator to (8.3), we obtain

$$\begin{cases} (\tilde{L}_\mu + \tilde{N}_\mu)u_j + \omega^2\nabla \times u_j = 0 & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.4}$$

where

$$\tilde{L}_\mu u_j = 2\nabla \times \nabla \cdot \mu\nabla^S u_j, \quad \tilde{N}_\mu u_j = 2c_\tau\nabla \times \nabla \cdot (\mu\nabla u_j^\intercal \nabla u_j).$$

The linearized system from (8.4) with two internal measurements is then equation (3.6), which we recall:

$$\begin{cases} D\tilde{L}(\mu_j, u_j)[\delta\mu_j, \delta u_j] + D\tilde{N}(\mu_j, u_j)[\delta\mu_j, \delta u_j] + \omega^2\nabla \times \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = \delta H_j & \text{in } \Omega, \\ \delta u_j = \delta g_j & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

8.1 Ellipticity

Lemma 15. *System* (3.6) *has elliptic principal symbol if*

$$|((\nabla^S u_1 + c_\tau\nabla u_1^\intercal \nabla u_1)\xi) \times \xi| + |((\nabla^S u_2 + c_\tau\nabla u_2^\intercal \nabla u_2)\xi) \times \xi| \neq 0 \quad \textit{for all } \xi \neq 0. \tag{8.5}$$

Proof. The symbol of the linearized operator is

$$\mathcal{P}(x, \xi) = \begin{bmatrix} 2((\nabla^S u + c_\tau\nabla u^\intercal \nabla u)\xi) \times \xi & P(\xi) \\ 0 & I_d \end{bmatrix},$$

where

$$P(\xi) = -2\mu\Sigma_{\text{curl}}(\xi)(|\xi|^2 I_d + \xi \otimes \xi)(I_d + \nabla u^\intercal).$$

We see that $\mathrm{Op}(\mathcal{P}(x, \xi))$ is not elliptic. If we add a measurement, we will have the symbol

$$\mathcal{P}_2(x, \xi) = \begin{bmatrix} 2((\nabla^S u_1 + c_\tau\nabla u_1^\intercal \nabla u_1)\xi) \times \xi & P(\xi) & 0 \\ 0 & I_d & 0 \\ 2((\nabla^S u_2 + c_\tau\nabla u_2^\intercal \nabla u_2)\xi) \times \xi & 0 & P(\xi) \\ 0 & 0 & I_d \end{bmatrix},$$

and we see that the linearized operator is elliptic if (8.5) holds. □

Let $A^{(j)} = \nabla^S u_j + c_\tau\nabla u_j^\intercal \nabla u_j$, and let the matrices $B^{(j)}$ be defined as in (7.5)–(7.6). Then a condition for having ellipticity is (7.7). Notice if this fails, we can simply add more measurements.

8.2 Lopatinskii condition and local injectivity

The deduction of the Lopatinskii condition (Lemma 14) and local injectivity are the same as the ones presented in Section 7, with the change

$$A^{(j)} = \nabla^S u_j + c_\tau \nabla u_j^\intercal \nabla u_j, \quad j = 1, 2.$$

The Douglis numbers are

$$s_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, d, 2d+1, \dots, 3d\}, \\ -2 & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 2 & \text{if } i = 1, \\ 3 & \text{otherwise,} \end{cases}$$

$$\sigma_k = -1, \quad k = 1, \dots, 2d.$$

Then the operator over $(\delta\mu, \{\delta u_j\}_{j=1}^J)$ given by equation (3.6) with two measurements is defined from

$$\mathcal{X} = H^{l+2}(\Omega) \times H^{l+3}(\Omega)^d \times H^{l+3}(\Omega)^d \quad \text{to} \quad \mathcal{Y} = (H^l(\Omega)^d \times H^{l+2}(\Omega)^d \times H^{l+\frac{1}{2}}(\Omega)^d)^2$$

with $l = 2$ in dimensions two and three.

Now we can prove the main theorem and corollary of this section.

Theorem 3. *Let $a = 1$, $d = 2, 3$. Assume, for $j = 1, 2$,*

$$|((\nabla^S u_{1j} + c_\tau \nabla u_{1j}^\intercal \nabla u_{1j})\xi) \times \xi| + |((\nabla^S u_{2j} + c_\tau \nabla u_{2j}^\intercal \nabla u_{2j})\xi) \times \xi| \neq 0 \quad \textit{for all } \xi \neq 0.$$

Let C_1, C_2 depend on $\|\mu_2\|_{C^4(\Omega)}$, and C_2 also depends on ω^2. Then we have the following stability estimate:

$$\|\delta\mu\|_{H^5(\Omega)} \leq C_1\bigg(\sum_{j=1}^{2} \|\delta u_j\|_{H^4(\Omega)^d} + \|\delta g_j\|_{H^{\frac{5}{2}}(\partial\Omega)^d}\bigg) + C_2\bigg(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{L^2(\Omega)^d}\bigg).$$

Proof. We have the following estimate after applying Theorem 4 and condition (8.5):

$$\begin{aligned} \|\delta\mu\|_{H^{l+2}(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{H^{l+3}(\Omega)^d} \leq C \sum_{j=1}^{2} (\|\mathcal{L}_j^{\mathrm{ec}}(\delta\mu, \delta u_j)\|_{H^l(\Omega)^d} + \|\mathcal{L}_j^{\mathrm{int}}(\delta\mu, \delta u_j)\|_{H^{l+2}(\Omega)^d} + \|\delta u_j\|_{H^{l+\frac{1}{2}}(\partial\Omega)^d}) \\ + C_2\bigg(\|\delta\mu\|_{L^2(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{L^2(\Omega)^d}\bigg), \end{aligned} \tag{8.6}$$

where $\mathcal{L}_j^{\mathrm{ec}}, \mathcal{L}_j^{\mathrm{int}}$ are the parts of the linearization coming from the elasticity equations and the solution measurements, respectively. These are as follows:

$$\mathcal{L}_j^{\mathrm{ec}}(\delta\mu, \delta u_j) = (D\tilde{L} + D\tilde{N})(\mu, u_j)[\delta\mu, \delta u_j] + \omega^2 \nabla \times \delta u_j, \quad \mathcal{L}_j^{\mathrm{ec}}(\delta\mu, \delta u_j) = \delta u_j.$$

Rearranging, we have the desired result. □

Corollary 2. *The constant C_2 can be absorbed into the constant C_1 if*

$$(\nabla^S u_{1j} + c_\tau \nabla u_{1j}^\intercal \nabla u_{1j}) \neq \alpha(\nabla^S u_{2j} + c_\tau \nabla u_{2j}^\intercal \nabla u_{2j})$$

for $j = 1, 2$ and all $\alpha \in \mathbb{R}$.

Proof. Since we have local injectivity by Section 8.1, we can take $C_2 = 0$ in inequality (8.6). That is, we have

$$\|\delta\mu\|_{H^{l+2}(\Omega)} + \sum_{j=1}^{2} \|\delta u_j\|_{H^{l+3}(\Omega)^d} \leq C \sum_{j=1}^{2} (\|\mathcal{L}_j^{\mathrm{ec}}(\delta\mu, \delta u_j)\|_{H^l(\Omega)^d} + \|\mathcal{L}_j^{\mathrm{int}}(\delta\mu, \delta u_j)\|_{H^{l+2}(\Omega)^d} + \|\delta u_j\|_{H^{l+\frac{1}{2}}(\partial\Omega)^d}).$$

Re-arranging using the definitions of $\mathcal{L}_j^{\mathrm{ec}}$ and $\mathcal{L}_j^{\mathrm{int}}$ in the previous theorem gives the desired result. □

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