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Secants to the Kummer Variety

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Abstract

In this thesis we mainly prove two results in an algebro-geometric way: If one has a curve Γ of (honest) quadrisecant planes to the Kummer variety of an indecomposable principally abelian variety (X, Θ) then the curve Γ is twice the minimal class, under certain technical geometric conditions. By previous analytic results (see [20]), this will imply that X is a Prym variety. As a generalization of this results, adding one geometric condition we get that having a curve of $(m+2)$-secants (for a minimal m) implies that the abelian variety has a curve that is m-times the minimal cohomological class.

The second result of this thesis is a an answer to a natural generalization of a question Welters asked about trisecants (see [35]) and is as follows: Under certain geometric conditions, does the existence of m different $(m+2)$-secant m-planes imply that one has a curve of honest $(m+2)$-secant (m-)planes? We show that under certain conditions, this question has a positive answer (see Theorem 4.4.4).

Layout of the thesis

We first introduce the basic notion of Abelian varieties in the first chapter, trying to be as complete as possible. We justify why we can just work with complex tori with a certain hermitian form. This was done with two ideas in mind: To start studying Abelian Varieties in a way such that there is no need to start navigating with different books simultaneously and to fully cover the contents required to understand this work.

In Chapter 2 we cover the fundamentals of Jacobian varieties aiming at Fay's trisecant formula, as a motivation for this work.

In Chapter 3 one can find the latest research in the area surrounding the main problem of this thesis.

In Chapter 4, there is the main content which contains the main results of this thesis on the topic of the m-secant property for the Kummer variety.

Lastly, in Chapter 5 there are some words of future work and open problems.

CHAPTER

1 Abelian varieties

First we will define *Abelian Varieties* in a completely algebraic way, and justify by *GAGA* that we can work analytically if needed when working over a field of characteristic 0.

1.1 Algebraic Introduction

We will asumme most of the schematic material to be known. The main reference that we use for this chapter is [24]. We will follow mainly [32] for the algebraic theory and [8] for the complex theory.

We will assume that k is an algebraically closed field.

Definition 1.1.1. An *abelian variety* X is a complete algebraic variety over a field k with a group law $m : X \times X \to X$ such that m and the inverse map are both morphisms of varieties.

We will denote by $m(x,y) = x+y$ and the inverse of an element $y \in X$ by $-y$, *a priori* without assuming that the group operation is commutative. Also, denote by $t_a : X \to X$ the translation $x \mapsto x+a$.

Lemma 1.1.2 (Rigidity lemma). *Let X be a complete variety, Y and Z any varieties, and $f : X \times Y \to Z$ a morphism such that for some $y_0 \in Y$, $f(X \times \{y_o\})$ is a single point z_0 of Z.*

Then, there is a morphism $g : Y \to Z$ such that if $p_2 : X \times Y \to Y$ is the projection, $f = g \circ p_2$.

Proof. [32, p. 40] Choose any point $x_0 \in X$, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. Since $X \times Y$ is a (irreducible) variety, to show that $f = g \circ p_2$, it is enough to show that these morphisms coincide on some non-empty open subset of $X \times Y$. Let U be an affine open neighbourhood of z_0 in Z, $F = Z \setminus U$, and $G = p_2\left(f^{-1}(F)\right)$; then G is closed in Y since X is complete and hence p_2 is a closed map. Further $y_0 \notin G$ since $f(X \times \{y_0\}) = \{z_0\}$. Therefore $Y \setminus G = V$ is a non-empty open subset of Y. For each $y \in$, the complete variety $X \times \{y\}$ gets mapped by f into the affine variety U, and hence to a single point of U. But this means that for any $x \in X$, $y \in V$, $f(x,y) = f(x_0,y) = g \circ p_2(x,y)$, and this proves the assertion. □

Corollary 1.1.3. *If X and Y are abelian varieties and $f : X \to Y$ is any morphism, there exists a(n algebraic) group homomorphism $h : X \to Y$ such that $f(x) = t_{f(0)}h(x)$.*

Proof. [32, p. 41] We have that $h = t_{-f(0)}f$ vanishes at 0.
Consider the morphism $X \times X \xrightarrow{\phi} X$ defined by $\phi(x,y) = h(x+y) - h(y) - h(x)$. Then $\phi(X \times \{0\}) = \phi(\{0\} \times X) = 0$, so that it follows by the above lemma that $\phi \equiv 0$ on $X \times X$, or equivalently, h is a homomorphism. □

Note that we have not used that X is commutative in the previous proof, but in fact we have:

Corollary 1.1.4. *An abelian variety X is a commutative group.*

Proof. [32, p. 41] By the previous corollary, the inverse map of X that maps $x \mapsto -x$ is a homomorphism. Hence, for $x, y \in X$, $-(x+y) = (-x) + (-y) = (-y) + (-x)$, and X is commutative. □

An interesting result is the following.

Theorem 1.1.5. *Let X be a complete variety, $e \in X$ a point, and*

$$m : X \times X \to X$$

a morphism such that $m(x,e) = m(e,x) = x$ for all $x \in X$. Then X is an abelian variety with group law m and identity e.

Proof. [32, p. 42] □

Now, if we want to say anything more explicit or do any computations in a general setting it can become quite hard to work, if not imposible due to the complications that usually appear when working with positive characteristic. First, due to the Lefschetz principle if we work over a field of characteristic 0, we can simply work over the complex numbers. One can find a short proof and statement in [14] and a generalization in [16]. On the other hand we have another algebraic geometry miracle. There is an equivalence when working over the complex numbers: Analytic Geometry and Algebraic Geometry. This is a collection of several results related to topology, morphisms and cohomology as one can review in [18, Exposé XII], better known as *GAGA* from the French *Géométrie Algébrique et Géométrie Analytique*. It is usual to reference it as the following theorem, but note that the whole chapter where Grothedieck explains *GAGA* is not exactly summarized here:

Theorem 1.1.6. *Let X be a proper $\mathbb{C}$-scheme. The functor that associates a $\mathscr{O}_X$-module F to its inverse image F^{an} on X^{an} is an equivalence of categories.*

1.2 Complex Tori

When the ground field[1] k has characteristic 0, by the Lefschetz principle we can reduce it to the case $k = \mathbb{C}$. We can be much more specific and explicit when working with abelian varieties. In this case an abelian variety will be a complex torus which is projective.

1.2.1 Preliminaries

Lemma 1.2.1. *Any connected compact complex Lie group X of dimension g is a complex torus.*

Proof. See [8, p. 8]. □

Corollary 1.2.2. *A complex abelian variety X of dimension g is a complex torus.*

[1]Recall that we are assuming that k is algebraically closed

Let $X = V/\Lambda$ be a complex torus, where V is a g-dimensional $\mathbb{C}$-vector space. We have that

$$\pi_1(X) = H_1(X,\mathbb{Z}) = \Lambda.$$

Also, as X is locally isomorphic to V, this implies that V may be considered as the tangent space T_0X of X at 0. From the Lie theoretic point of view, the universal covering map is just the exponential map.

In order to describe a complex torus $X = V/\Lambda$, choose a basis $e_1,\ldots,e_g$ of V and a basis $\lambda_1,\ldots,\lambda_{2g}$ of the lattice Λ. Write λ_j in terms of the basis $e_1,\ldots,e_g$, that is: $\lambda_j = \sum_{l=1}^{g} \lambda_{lj} e_l$. The matrix

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}$$

is called a *period matrix* for X. The period matrix Π determines the complex torus X completely, although it depends on the choice of the bases for V and Λ. Conversely, given a matrix $\Pi \in \mathbb{M}(g \times 2g,\mathbb{C})$, one may ask if Π is the period matrix for some complex torus. The following result gives us an answer.

Proposition 1.2.3. $\Pi \in \mathbb{M}(g \times 2g,\mathbb{C})$ *is the period matrix of a complex torus if and only if the matrix* $P = \binom{\Pi}{\overline{\Pi}} \in \mathbb{M}(2g \times 2g,\mathbb{C})$ *is nonsingular.*

Proposition 1.2.4. *See [8, p. 9].*

1.2.2 Homomorphisms

For what follows in this subsection, let $X = V/\Lambda$ and $X' = V'/\Lambda'$ be two complex tori of dimensions g and g' respectively.

Definition 1.2.5. A homomorphism of X to X' is a holomorphic map $f : X \to X'$, that is also a homomorphism of groups.

Remark 1.2.6. Recall that the translation $t_a : X \to X, x \mapsto x + a$ is a holomorphic map, but not a homomorphism except when $a = 0$.

Proposition 1.2.7. *Let $f : X \to X'$ be a holomorphic map.*

a) *There is a unique homomorphism $g : X \to X'$ such that $f = t_{f(0)}g$.*

b) *There is a unique $\mathbb{C}$-linear map $G : V \to V'$ with $G(\Lambda) \subset \Lambda'$ inducing the homomorphism g.*

Proof. Define $g = t_{-f(0)}f$. We can lift the composed map

$$V \xrightarrow{\pi} X \xrightarrow{g} X'$$

to a holomorphic map G from V to the universal covering V' of X':

$$\begin{tikzcd} V \arrow[rr, "G"] \arrow[dr, "g\pi"'] & & V' \arrow[dl, "\pi'"] \\ & X' & \end{tikzcd}$$

in such a way that $G(0) = 0$. The commutative diagram implies that for all $\lambda \in \Lambda$ and $v \in V$ we have $G(v+\lambda) - G(v) \in \Lambda'$. So, the continuous map $v \mapsto G(v+\lambda) - G(v)$ is constant and we get $G(v+\lambda) = F(v) + G(\lambda)$ for all $\lambda \in \Lambda$ and $v \in V$. Hence, the partial derivatives of G are $2g$-fold periodic and thus constant by Liouville's theorem. It follows that G is $\mathbb{C}$-linear and g is a homomorphism. The uniqueness is clear. $\square$

Under the addition operation, the set of homomorphism from X to X' defines an abelian group denoted by $\mathrm{Hom}(X, X')$. Proposition 1.2.7 gives us:

Definition 1.2.8. The map

$$\rho_a : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$$
$$g \mapsto G$$

is an injective homomorphism of abelian groups which is called the *analytic representation* of $\mathrm{Hom}(X, X')$.

The restriction G_Λ of G to the lattice Λ is $\mathbb{Z}$-linear. Note that G_Λ determines G and g completely.

Definition 1.2.9. The injective homomorphism

$$\rho_r : \mathrm{Hom}\,(X,X') \to \mathrm{Hom}_{\mathbb{Z}}\,(\Lambda,\Lambda')$$
$$g \mapsto G_\Lambda,$$

is called the *rational representation* of $\mathrm{Hom}\,(X,X')$.

We denote the extensions of ρ_a and ρ_r to $\mathrm{Hom}_{\mathbb{Q}}\,(X,X') := \mathrm{Hom}\,(X,X') \otimes_{\mathbb{Z}} \mathbb{Q}$ by the same letters. These will also be refered to as the analytic and rational representations.

Proposition 1.2.10. *$Hom\,(X,X') \simeq \mathbb{Z}^m$ for some $m \leq 4gg'$.*

Proof. Any subgroup of $\mathrm{Hom}_{\mathbb{Z}}\,(\Lambda,\Lambda') \simeq \mathbb{Z}^{4gg'}$ is isomorphic to $\mathbb{Z}^m$ for some m. The injectivity of ρ_r finishes the proof. □

Remark 1.2.11. Let $X'' = V''/\Lambda''$ be a third complex torus. For $f \in \mathrm{Hom}\,(X,X')$ and $f' \in \mathrm{Hom}\,(X',X'')$ we have $\rho_a\,(fg) = \rho_a(f)\rho_a(g)$. In particular, if $X = X'$, ρ_a and ρ_r are representations of the ring $\mathrm{End}(X)$ respectively $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The following proposition shows how the analytic and the rational representations are related:

Proposition 1.2.12. *The extended rational representation*

$$\rho_r \otimes 1 : End_{\mathbb{Q}}(X) \otimes \mathbb{C} \to End_{\mathbb{C}}\,(\Lambda \otimes \mathbb{C}) \simeq End_{\mathbb{C}}\,(V \times V)$$

is equivalent to the direct sum of the analytic representation and its complex conjugate: $\rho_r \otimes 1 \simeq \rho_a \oplus \overline{\rho_a}$.

Proof. See [8, p. 11]. □

Now we study the image $\mathrm{im} f$ and the kernel $\ker f$ of a homomorphism between complex tori.

Proposition 1.2.13. *Let $f : X \to Y$ be a homomorphism between complex tori. Then*

a) $\mathrm{im} f$ is a subtorus of X'.

b) $\ker f$ is a closed subgroup of X. The connected component $(\ker f)_0$ of $\ker f$ containing 0 is a subtorus of X of finite index in $\ker f$.

Proof. This is a direct consequence of Lemma 1.2.1. □

Next we define isogenies which play an important role in the sequel.

Definition 1.2.14. An *isogeny* of a complex torus X to a a complex torus X' is a surjective homomorphism $X \to X'$ with finite kernel.

Remark 1.2.15. Note that a homomorphism $X \to X'$ is an isogeny if and only if it is surjective and $\dim X = \dim X'$.

Remark 1.2.16. If Γ is a finite subgroup of X, the quotient X/Γ is a complex torus and the natural projection $X \to X/\Gamma$ is an isogeny. To see this: note that $\pi^{-1}(\Gamma) \subset V$ is a lattice containing Λ and $X/\Gamma = V/\pi^{-1}(\Gamma)$.

Suppose that $f : X \to X'$ is a surjective homomorphism of complex tori, by Proposition 1.2.13 it factors canonically into a surjective homomorphism g with a complex torus as kernel and an isogeny h. This is the *Stein factorization* of the homomorphism f:

$$\begin{tikzcd} & X/(\ker f)_0 \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & X' \end{tikzcd}$$

Definition 1.2.17. The *degree* $\deg f$ of a homomorphism $f : X \to X'$ is defined to be the order of the group $\ker f$ if it is finite and 0 otherwise.

Remark 1.2.18. Therefore, for an isogeny we have

$$\deg f = [\Lambda' : \rho_r(f)(\Lambda)] .$$

If moreover f is an endomorphism of X,

$$\deg f = \det \rho_r(f).$$

Remark 1.2.19. Note that if f and g are isogenies so is their composition fg. This is because $\deg(fg) = \deg f \cdot \deg g$.

For any integer n we define the homomorphism $n_X : X \to X$ given by $x \mapsto nx$. If $n \neq 0$ its kernel $X[n]$ is called the group of n-division points of X (or n-torsion points).

Proposition 1.2.20. $X[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Proof. $\ker n_X = \frac{1}{n}\Lambda/\Lambda \simeq \Lambda/n\Lambda \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$. □

In particular, any complex torus is a divisible group. Another consequence is that $\mathrm{Hom}(X,X')$ is torsion free as an abelian group. Hence, $\mathrm{Hom}(X,X')$ can be considered as a subgroup of $\mathrm{Hom}_{\mathbb{Q}}(X,X')$. Moreover, the definition of the degree of a homomorphism extends to $\mathrm{Hom}_{\mathbb{Q}}(X,X')$ by

$$\deg(rf) = r^{2g} \deg f$$

for any $r \in \mathbb{Q}$ and $f \in \mathrm{Hom}(X,X')$.

Definition 1.2.21. The *exponent* $e = e(f)$ of an isogeny f is defined to be the exponent of the finite group $\ker f$. In other words, $e(f)$ is the smallest positive integer n such that $nx = 0$ for all $x \in \ker f$.

Proposition 1.2.22. *For any isogeny $f : X \to X'$ of exponent e there exists an isogeny $g : X' \to X$, unique up to isomorphisms, such that $gf = e_X$ and $fg = e_{X'}$.*

Proof. As $\ker f$ is contained in $\ker e_X = X[e]$, there is a unique map $g : X' \to X$ such that $gf = e_X$. Since e_X and f are isogenies, so is g. The kernel of g is contained in the kernel $X'[e_{X'}]$ of $e_{X'}$, since for every $x' \in \ker g$ there is an $x \in \ker e_x$ with $f(x) = x'$ and $ex' = ef(x) = f(ex) = 0$. Thus, $e_{X'} = f'g$ for some isogeny $f' : X \to X'$ and we get $f'e_X = f'gf = e_{X'}f = fe_x$. This implies that $f = f'$ since e_X is surjective. □

Corollary 1.2.23. *i) Isogenies define an equivalence relation on the set of complex tori.*

ii) An element in End(X) is an isogeny if and only if it is invertible in End$_{\mathbb{Q}}(X)$.

Now the following definition makes sense:

Definition 1.2.24. We will say that two complex torus are *isogenous*, if there is an isogeny between them.

1.3 Complex Abelian Varieties

Now we will define an abelian variety over the complex numbers. This will help us be more explicit and will bring some tools that are simply not available when $\mathrm{char}(k) \neq 0$. Recall the *GAGA Theorem*[18, Exposé XII]; this will be equivalent to working in the algebraic category.

Definition 1.3.1. An *Abelian variety* is a complex torus admitting an ample line bundle[2].

We should clarify this definition and the relation between the line bundle with a certain hermitian form.

We want to define the *first Chern class*, but first we need some cohomological results.

Lemma 1.3.2. *The canonical map $\bigwedge^n H^1(X,\mathbb{Z}) \to H^n(X,\mathbb{Z})$ induced by the cup product is an isomorphism for every $n \geq 1$.*

Proof. See [8], Lemma 1.3.1. □

The following corollary will be useful in a few pages.

Corollary 1.3.3. *There is a canonical isomorphism $H^n(X,\mathbb{Z}) \simeq Alt^n(\Lambda,\mathbb{Z})$ for every $n \geq 1$.*

Theorem 1.3.4 (The Hodge decomposition). *a) For every $n \geq 0$ the de Rham and the Dolbeault isomorphisms induce an isomorphism*

$$H^n(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q\left(\Omega_X^p\right)$$

[2]This is equivalent to X admiting a positive definite line bundle.

with Ω_X^p the sheaf of holomorphic p-forms on X.

b) For every pair (p,q) there is a natural isomorphism

$$H^q\left(\Omega_X^p\right) \simeq \bigwedge^p \Omega \otimes \bigwedge^q \bar{\Omega}$$

with $\Omega := Hom_{\mathbb{C}}(V,\mathbb{C})$ and $\bar{\Omega} := Hom_{\bar{\mathbb{C}}}(V,\mathbb{C})$, the group of $\mathbb{C}$-antilinear forms on C.

Proof. See [17, pp. 116-117] for a compact Kähler manifold X, and for a simplier proof for complex tori see [8, p. 16]. □

Observe that we get that $H^q\left(\Omega_X^p\right) \simeq H^{p,q}(X)$.

Lemma 1.3.5. *Every holomorphic line bundle on a complex vector space is trivial.*

Proof. From the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_V \overset{\exp(2\pi i \cdot)}{\to} \mathcal{O}_V^* \to 1$$

we obtain the exact sequence

$$H^1(V,\mathcal{O}_V) \to H^1(V,\mathcal{O}_V^*) \to H^2(V,\mathcal{O}_V).$$

But $H^1(V,\mathcal{O}_V) = 0$ by the $\bar{\partial}$-Poincaré Lemma (see [17, p. 46]), whereas one knows from Algebraic Topology that $H^2(V,\mathbb{Z}) = 0$. This implies the assertion. □

Proposition 1.3.6. *Let $\pi : V \to X$ be the universal covering. There is a canonical isomorphism*

$$\phi_1 : H^1\left(\pi_1(X), H^0(V,\mathcal{O}_V^*)\right) \to \ker\left(H^1(X,\mathcal{O}_X^*) \overset{\pi^*}{\to} H^1(V,\mathcal{O}_V^*)\right).$$

Proof. See Proposition B.1 in [8]. □

Now we have an isomorphism $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*) \simeq H^1\left(\Lambda, H^0(V,\mathcal{O}_V^*)\right)$. That is, any holomorphic line bundle on X can be described by a factor of automorphy. Consider now the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

and its long exact sequence in cohomology

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \cdots$$

Definition 1.3.7. Let $L \in Pic(X)$ be a line bundle on X. The *first Chern class* of L is the image $c_1(L)$ of L in $H^2(X,\mathbb{Z})$.

Since $H^2(X,\mathbb{Z})$ is canonically isomorphic to $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ by 1.3.3, we may consider $c_1(L)$ as an alternating $\mathbb{Z}$-valued form on the lattice Λ.

The following theorem shows how to compute the first Chern class in terms of a factor of automorphy f of L. Note that any such f can be written in the form $f = e^{2\pi i g}$ with a map $g : \Lambda \times V \to \mathbb{C}$ which is holomorphic in the second variable.

Theorem 1.3.8. *There is a canonical isomorphism $H^2(X,\mathbb{Z}) \to Alt^2(\Lambda,\mathbb{Z})$, which maps $c_1(L)$ of a line bundle L on X with factor of automorphy $f = e^{2\pi i g}$ to the alternating form*

$$E_L(\lambda,\mu) = g(\mu, v+\lambda) + g(\lambda, v) - g(\lambda, v+\mu) - g(\mu, v)$$

for all $\lambda,\mu \in \Lambda$ and $v \in V$.

Proof. See [8, pp. 24-28]. □

Now, we characterize alternating forms which come from line bundles via c_1.

Proposition 1.3.9. *For an alternating form $E : V \times V \to \mathbb{R}$ the following conditions are equivalent:*

(i) There is a holomorphic line bundle L on X such that E represents the first Chern class $c_1(L)$.

(ii) $E(\Lambda,\Lambda) \subset \mathbb{Z}$ and $E(iv,iw) = E(v,w)$ for all $v,w \in V$.

Proof. Consider the following diagram

$$\begin{tikzcd} H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "\iota"] & H^2(X,\mathcal{O}_X) \arrow[d, equal] \\ & H^2(X,\mathbb{C}) \arrow[r, "p"] \arrow[d, "\beta_2^{-1}", "\simeq"'] & H^{0,2}(X) \arrow[d, "\gamma_2^{-1}", "\simeq"'] \\ & \bigwedge^2\Omega\oplus(\Omega\otimes\bar{\Omega})\oplus\bigwedge^2\bar{\Omega} \arrow[r, "p"] & \bigwedge^2\bar{\Omega} \end{tikzcd}$$

where the upper line is part of the long exact cohomology sequence of $0\to\mathbb{Z}\to\mathcal{O}_X\to\mathcal{O}_X^*$, the map ι is the natural embedding, p denotes the projection map, and β_2, γ_2 are the isomorphisms of the following diagram

$$\begin{tikzcd} \mathrm{Alt}^2(\Lambda,\mathbb{C}) \arrow[r, equal] \arrow[d, "\beta_2"] & \mathrm{Alt}^2_{\mathbb{R}}(V,\mathbb{C}) \arrow[r, "\simeq"] & \bigwedge^2\Omega\oplus(\Omega\otimes\overline{\Omega})\oplus\bigwedge^2\overline{\Omega} \arrow[d, "\gamma_2"] \\ H^2(X,\mathbb{C}) \arrow[r, "\simeq"] & H^2_{\mathrm{DR}}(X) \arrow[r, equal] & H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X) \end{tikzcd}$$

which is commutative (this is due to the proof of 1.3.8). We claim that the diagram commutes. To see this, it suffices to show that the homomorphism $H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O}_X)$ factors via the natural projection map

$$H^2(X,\mathbb{C})=H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)\to H^{0,2}(X)$$

But this is an immediate consequence of the fact that this projection is induced by the natural projection map from the de Rham complex to the Dolbeault complex.

Now suppose that $L\in\mathrm{Pic}(X)=H^1(X,\mathcal{O}_X^*)$, and $\beta_2^{-1}c_1(L)\doteq E=E_1+E_2+E_3$ with $E_1\in\bigwedge^2\Omega$, $E_2\in\Omega\otimes\overline{\Omega}$ and $E_3\in\bigwedge\overline{\Omega}$. But $E_1=\overline{E_3}$ since E has values in $\mathbb{R}$, whereas according to the first diagram $E_3=0$. Therefore, $E=E_2$ since E satisfies the second condition. The converse is analogous. $\square$

Lemma 1.3.10. *There is a* 1-1-*correspondence between the set of hermitian forms H on V and the set of real values alternating forms E on V satisfying $E(iv,iw)=E(v,w)$, given by*

$$E(v,w)=\Im H(v,w)\quad\textit{and}\quad H(v,w)=E(iv,w)+iE(v,w)$$

for all $v, w \in V$.

Proof. [8, p. 29] Given E, the form H is hermitian, since

$$\begin{aligned} H(v,w) &= E(iv,w) + iE(v,w) \\ &= -E(iw,-v) - iE(w,v) \\ &= \overline{H(w,v)}. \end{aligned}$$

Conversely, given H, the form $E = \Im H$ is alternating and $E(iv,iw) = \Im H(iv,iw) = \Im H(v,w) = E(v,w)$.

□

Now we can consider the first Chen class of $L \in \mathrm{Pic}(X)$ as an alternating from or as a hermitian form on V.

Definition 1.3.11. The *Néron-Severi* group of X is defined as

$$\mathrm{NS}(X) = \mathrm{im}\left(c_1 : H^1(X, \mathscr{O}_x^*) \to H^2(X,\mathbb{Z})\right).$$

1.4 Appel-Humbert Theorem

We have seen that any holomorphic line bundle on X can be given by a factor of automorphy since we have the isomorphism $\mathrm{Pic}(X) \simeq H^1\left(\Lambda, H^0\left(V, \mathscr{O}_V^*\right)\right)$. Now we will show that there is a canonical way to distinguish a factor in every class of $H^1\left(\Lambda, H^0\left(\mathscr{O}_V^*\right)\right)$.

A *semicharacter* for a hermitian form H is a map $\chi : \Lambda \to S^1$ such that

$$\chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)\exp\left(\pi i \Im H(\lambda,\mu)\right)$$

for all $\lambda, \mu \in \Lambda$. By definition the characters on Λ with values in S^1 are exactly the semicharacters for $0 \in \mathrm{NS}(X)$. We will denote by $\mathscr{P}(\Lambda)$ the set of pairs (H,χ) where $H \in \mathrm{NS}(X)$ and χ is a semicharacter for H. Note that $\mathscr{P}(\Lambda)$ is a group with respect to the composition

$$(H_1,\chi_1) \circ (H_2,\chi_2) = (H_1 + H_2, \chi_1\chi_2)$$

and we get the following exact sequence:

$$1 \to \operatorname{Hom}\left(\Lambda, S^1\right) \xrightarrow{\iota} \mathscr{P}(\Lambda) \xrightarrow{p} \mathrm{NS}(X).$$

Here $\iota(\chi) = (0,\chi)$ and $p(H,\chi) = H$. We will show that p is surjective but we need to define the map $\mathscr{P}(\Lambda) \to \mathrm{Pic}(X)$: For $(H,\chi) \in \mathscr{P}(\Lambda)$ define $a = a_{(H,\chi)} : \Lambda \times V \to \mathbb{C}^*$ by

$$a(\lambda, v) = \chi(\lambda) \exp\left(\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)\right).$$

Using [8, p. 30] and [8, Proposition B.1], we get that the cocycle a determines a line bundle on X, which we define by $L(H,\chi)$. By construction

$$L(H,\chi) \simeq V \times \mathbb{C}/\Lambda$$

where Λ acts on $V \times \mathbb{C}$ by $\lambda \circ (v,t) = \left(v+\lambda, a_{(H,\chi)}(\lambda,v)t\right)$.

The aim is to show that for every line bundle L on X there is a unique pair $(H,\chi) \in \mathscr{P}(\Lambda)$ such that $L \simeq L(H,\chi)$.

Lemma 1.4.1. *The map $\mathscr{P}(\Lambda) \to Pic(X)$, $(H,\chi) \mapsto L(H,\chi)$ is a homomorphism of groups and the following diagram commutes:*

$$\begin{tikzcd} \mathscr{P}(\Lambda) \arrow[dd] \arrow[dr, "p"] & \\ & NS(X) \\ Pic(X) \arrow[ur, "c_1"] & \end{tikzcd}$$

Proof. See [8, pp. 30-31]. □

The lemma implies in particular that p is surjective. In order to show that $\mathscr{P}(\Lambda) \to \mathrm{Pic}(X)$ is an isomorphism, consider its restrition to $\operatorname{Hom}\left(\Lambda, S^1\right)$. Define $\mathrm{Pic}^0(X)$ as the kernel of $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$.

Proposition 1.4.2. *The map $\mathscr{P}(\Lambda) \to Pic(X)$ induces an isomorphism*

$$Hom\left(\Lambda, S^1\right) \xrightarrow{\ \sim\ } Pic^0(X)$$

Proof. See [8, pp. 31-32]. □

Summing up we get the *Appell-Humbert Theorem*, which tells us that there is a canonical way to associate a factor of automorphy to any line bundle on X.

Theorem 1.4.3 (Appell-Humbert). *Let $X = V/\Lambda$ be a complex torus. There is a canonical isomorphism of exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Hom(\Lambda, S^1) & \longrightarrow & \mathscr{P}(\Lambda) & \longrightarrow & NS(X) & \longrightarrow & 0 \\
 & & \simeq\big\downarrow & & \simeq\big\downarrow & & \big\| & & \\
1 & \longrightarrow & Pic^0(X) & \longrightarrow & Pic(X) & \longrightarrow & NS(X) & \longrightarrow & 0.
\end{array}$$

In particular, if $L \in \mathrm{Pic}(X)$ with hermitian form $c_1(L) = H$, we can distinguish a semicharacter χ for H such that $L \simeq L(H, \chi)$.

1.5 Canonical factors

Let $X = V/\Lambda$ be a complex torus. Let $L \in \mathrm{Pic}(X)$ be a line bundle with hermitian form H. Thanks to the Appell-Humbert Theorem one can distinguish a semicharacter χ for H such that $L \simeq L(H, \chi)$.

Definition 1.5.1. The cocycle $a_L = a_{L(H,\chi)} \in Z^1\left(\Lambda, H^0\left(V, \mathscr{O}_V^*\right)\right)$ defined by

$$a_L(\lambda, v) = \chi(\lambda)\exp\left(\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)\right) \tag{1.1}$$

for all $(\lambda, v) \in \Lambda \times V$ is called the *canonical factor (of automorphy) for L.*

The following results follow easily from the definitions:

Lemma 1.5.2. *Let a_L be the canonical factor of $L = L(H, \chi) \in Pic(X)$. For all $\lambda \in \Lambda$, $v, w \in V$ and $n \in \mathbb{Z}$ we have*

a) $\chi(n\lambda) = \chi(\lambda)^n$.

b) $a_L(\lambda, v+w) = a_L(\lambda, v)\exp(\pi H(w,\lambda))$.

c) $a_L(\lambda, v)^{-1} = a_L(-\lambda, v)\exp(-\pi H(w,\lambda))$.

Using characteristics we get, in particular, the theorem of the square which give us a relation for line bundles under pullbacks by holomorphic maps of complex tori. This result will be useful when one uses the short exact sequence given by the Koszul complex.

Lemma 1.5.3. *For any $L = L(H\chi) \in Pic(X)$ and $\bar{v} \in X$ with representative $v \in V$*

$$t_{\bar{v}}^* L(H,\chi) \simeq L(H, \chi\exp(2\pi i \Im H(,v,\cdot))).$$

Proof. The translation t_v on V induces the translation $t_{\bar{v}}$ on X. Moreover the induced map $t_{\bar{v}}^*$ on $\pi(X) = \Lambda$ is the identity. So, if a_L denotes the canonical factor of L, $(\mathrm{id}\times t_v)^* a_L$ is a factor for $t_{\bar{v}}^*$, but not yet the canonical one. For $g(w) = \exp(-\pi(w,v))$ the factor

$$\begin{aligned}(\mathrm{id}\times t_v)^* a_L(\lambda,w)g(w+\lambda)g(w)^{-1} &= \chi(\lambda)\exp\left(\pi H(w+v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)\right)\\ &\quad\cdot\exp(-\pi H(w+\lambda,v)+\pi H(w,v))\\ &= \chi(\lambda)\exp(2\pi i\Im H(v,\lambda))\exp\left(\pi H(w,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)\right)\end{aligned}$$

is equivalent to $(\mathrm{id}_\Lambda \times t_v)^* a_L$. Moreover, it is the canonical factor for $t_{\bar{v}}^* L$, since $\chi(\cdot)\exp(2\pi i\Im H(v,\cdot))$ is a semicharacter for H. □

This leads immediately to

Theorem 1.5.4 (Theorem of the Square). *For all $\bar{v}, \bar{w} \in X$ and $L \in Pic(X)$ we have that*

$$t_{\bar{v}+\bar{w}}^* L \simeq t_{\bar{v}}^* L \otimes t_{\bar{w}}^* L \otimes L^{-1}.$$

Now we end this section with some nice consequences.
We denote $L^n := L^{\otimes n}$. Recall the homomorphism $n_X : X \to X$.

Proposition 1.5.5. *For every $L \in Pic(X)$ and $n \in \mathbb{Z}$ we have that*

$$n_X^* L \simeq L^{\frac{n^2+n}{2}} \otimes (-1)_X^* L^{\frac{n^2-n}{2}}.$$

Proof. Recall the analytic representation $n_V : V \to V, v \mapsto nv$ of n_X. For $L = L(H,\chi)$ we have that:

$$\begin{aligned} L^{\frac{n^2+n}{2}} \otimes (-1)_X^* L^{\frac{n^2-n}{2}} &= L\left(\frac{n^2+n}{2}H + \frac{n^2-n}{2}(-1)_V^* H, \chi^{\frac{n^2+n}{2}} \cdot (-1)_V^* \chi^{\frac{n^2-n}{2}}\right) \\ &= L\left(n^2 H, \chi(n\cdot)\right) \\ &= L\left(n_V^* H, n_\Lambda^* \chi\right) \\ &= n_X^* L(H,\chi). \end{aligned}$$

□

Definition 1.5.6. A line bundle L is called *symmetric* if $(-1)_X^* L \simeq L$.

Corollary 1.5.7. *For every symmetric line bundle $L \in Pic(X)$ and $n \in \mathbb{Z}$ we have*

$$n_X^* \simeq L^{n^2}.$$

Corollary 1.5.8. *The line bundle $L = L(H,\chi)$ is symmetric if and only if $\chi(\Lambda) \subset \{\pm 1\}$.*

1.6 Dual complex torus

Now we present the dual complex torus $\hat{X}$ of a complex torus X, and in particular define the map ϕ_L and the groups $K(L)$ and $\Lambda(L)$ that will be technical tools used later.

Consider the $\mathbb{C}$-vector space $\bar{\Omega} = \mathrm{Hom}_{\bar{\mathbb{C}}}(V,\mathbb{C})$ of $\mathbb{C}$-antilinear forms $l : V \to \mathbb{C}$. The underlying vector space of $\bar{\Omega}$ is canonically isomorphic to $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$, in fact, the isomorphism is given by $l \mapsto \Im l$ with inverse map $j \mapsto l(v) = -j(iv) + ij(v)$. Hence, the canonical $\mathbb{R}$-bilinear form

$$\langle\cdot,\cdot\rangle : \bar{\Omega} \times V \to \mathbb{R} : \langle l, v\rangle = \Im l(v)$$

is nondegenerate. As a consequence

$$\hat{\Lambda} = \left\{ l \in \bar{\Omega} \mid \langle l, \Lambda\rangle \subset \mathbb{Z} \right\}$$

is a lattice in $\bar{\Omega}$, which is called the *dual lattice of* Λ.

Definition 1.6.1. The complex torus of dimension g

$$\hat{X} = \bar{\Omega}/\hat{\Lambda}$$

is the *dual complex torus* of X.

Remark 1.6.2. Identifying V with the space of $\mathbb{C}$-antilinear forms $\bar{\Omega} \to \mathbb{C}$ by double antiduality, the nondegeneracy of $\langle\cdot,\cdot\rangle$ implies that Λ is just the lattice in V dual to $\hat{\Lambda}$. From this, it follows that

$$\hat{\hat{X}} = X.$$

Proposition 1.6.3. *The canonical homomorphism* $\bar{\Omega} \to Hom(\Lambda, S^1)$, $l \mapsto \exp(2\pi i \langle l, \cdot\rangle)$ *induces an isomorphism* $\hat{X} \to Pic^0(X)$.

This shows that $\mathrm{Pic}^0(X)$ has a natural structure of a complex torus. It is customary to identify $\mathrm{Pic}^0(X)$ and $\hat{X}$.

Proof. The nondegeneracy of the form $\langle\cdot,\cdot\rangle$ implies that the map $\bar{\Omega} \to \mathrm{Hom}(\Lambda, S^1)$ is surjective. By definition $\hat{\lambda}$ is precisely the kernel of this homomorphism. □

Remark 1.6.4. Let $X_j = V_j/\Lambda_j$ be two complex tori, with $j = 1,2$, and $f : X_1 \to X_2$ a homomorphism with analytic representation $F : V_1 \to V_2$. The (anti-)dual map $F^* : \bar{\Omega}_2 \to \bar{\Omega}_1$ which associates to an antilinear form $l_2 \in \bar{\Omega}_2$ the antilinear form $l_2 \circ F \in \bar{\Omega}_1$ induces a homomorphism $\hat{f} : \hat{X}_2 \to \hat{X}_1$, since $F^*\hat{\Lambda}_2 \subset \hat{\Lambda}_1$. Furthermore, by Proposition 1.6.3, the following diagram commutes:

$$\begin{array}{ccc}
\hat{X}_2 & \xrightarrow{\sim} & \mathrm{Pic}^0(X_2) \\
\downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle f^*} \\
\hat{X}_1 & \xrightarrow{\sim} & \mathrm{Pic}^0(X_1).
\end{array}$$

If $g : X_2 \to X_1$ is another homomorphism of complex tori, then

$$\widehat{gf} = \hat{f}\hat{g}.$$

Moreover $\widehat{\mathrm{id}_X} = \mathrm{id}_{\hat{X}}$ and $\hat{\hat{f}} = f$. Hence, $\hat{}$ is a functor from the category of complex tori to itself. In fact, it is an exact functor.

Proposition 1.6.5. *If $0 \to X_1 \to X_2 \to X_3 \to 0$ is an exact sequence of complex tori, the dual sequence $0 \to \hat{X}_3 \to \hat{X}_2 \to \hat{X}_1 \to 0$ is also exact.*

Proof. See [8, p. 35]. □

Proposition 1.6.6. *If $f : X_1 \to X_2$ is an isogeny of complex tori, the dual map $\hat{f} : \hat{X}_2 \to \hat{X}_1$ is also an isogeny and its kernel is isomorphic to Hom $\left(\ker f, S^1\right)$. In particular, $\deg \hat{f} = \deg f$.*

Proof. See [8, p. 35]. □

Corollary 1.6.7. *Let $f : X_1 \to X_2$ be an isogeny of complex tori $X_j = V/\Lambda_j$ with analytic representation F. For a line bundle $L = L(H,\chi) \in Pic(X_1)$ the following statements are equivalent:*

i) $L = f^*M$ for some line bundle $M \in Pic(X_2)$.

ii) $\Im H\left(F^{-1}\Lambda_2, F^{-1}\Lambda_2\right) \subset \mathbb{Z}$.

Proof. See [8, p. 36]. □

Let $L \in \mathrm{Pic}(X)$ be a line bundle. For any point $x \in X$, the line bundle $t_x^*L \otimes L^{-1}$ has first Chern class zero. So, identifying $\hat{X} = \mathrm{Pic}^0(X)$ we get a map

$$\begin{aligned} \phi_L : X &\to \hat{X} \\ x &\mapsto t_x^*L \otimes L^{-1}. \end{aligned}$$

According to the Theorem of the Square 1.5.4, ϕ_L is an homomorphism. Let's compute its analytic representation.

Lemma 1.6.8. *Let $L = L(H,\chi)$ be a line bundle on $X = V/\Lambda$. The map $\phi_H : V \to \bar{\Omega}$, $\phi_H(v) = H(v,\cdot)$ is the analytic representation of ϕ_L.*

Proof. Using Lemma 1.5.3 we get

$$t_{\bar{v}}^{*}L\otimes L^{-1}=L\left(0,\exp\left(2\pi i\Im H(\cdot,\cdot)\right)\right)=L\left(0,\exp\left(2\pi i\left\langle\phi_H(v),\cdot\right\rangle\right)\right).$$

Now consider the isomorphism $\widehat{X}\xrightarrow{\sim}\mathrm{Pic}^0(X)=\mathrm{Hom}(\Lambda,S^1)$; by comparing we can finish the proof. □

Corollary 1.6.9. *a) ϕ_L depends only on the first Chen class H of L.*

b) $\phi_{L\otimes M}=\phi_L+\phi_M$ for all $L,M\in Pic(X)$.

c) $\hat{\phi}_L=\phi_L$ under the natural identification $\hat{\hat{X}}=X$.

d) For any homomorphism $f:Y\to X$ of complex tori, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi_L} & \hat{X} \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \hat{f}} \\ Y & \xrightarrow{\phi_{f^*L}} & \hat{Y} \end{array}$$

Proof. a), b) and *d)* follow immediately. For *c)*: recall that ϕ_H^* is the analytic representation of $\widehat{\phi_L}$. So the assertion follows from the fact that $\phi_H^*=\phi_H$ under the natural identification $\mathrm{Hom}_{\overline{\mathbb{C}}}\left(\overline{\Omega},\mathbb{C}\right)=V$. □

Definition 1.6.10. Denote by $K(L)$ the kernel of ϕ_L. Also define

$$\Lambda(L)=\{v\in V\mid \Im H(v,\Lambda)\subset\mathbb{Z}\}.$$

Since $\Lambda(L)=\phi_H^{-1}\left(\hat{\Lambda}\right)$, we have that

$$K(L)=\Lambda(L)/\Lambda. \tag{1.2}$$

Note that $K(L)$ and $\Lambda(L)$ depend only on the hermitian form H, and so we can write $K(H)$ and $\Lambda(H)$ respectively for these groups. Let's see some elementary properties:

Lemma 1.6.11. *For any line bundle $L\in Pic(X)$,*

a) $K(L \otimes P) = K(L)$ *for any* $P \in Pic^0(X)$.

b) $K(L) = X$ *if and only if* $L \in Pic^0(X)$.

c) $K(L^n) = n_X^{-1} K(L)$ *for any* $n \in \mathbb{Z}$.

d) $K(L) = n_X K(L^n)$ *for any* $n \in \mathbb{Z}$, $n \neq 0$.

Proof. [8, p. 37]. a) and b) are immediate from the definition and the Appell-Humbert Theorem 1.4.3. Suppose $L = L(H, \chi)$, then $L^n = L(nH, \chi^n)$ and thus

$$\Lambda(L^n) = \{v \in V \mid \Im H(nv, \Lambda) \subset \mathbb{Z}\} = \left\{ \frac{1}{n} v \in V \mid v \in \Lambda(L) \right\}.$$

This implies c) and d). □

Now, the homomorphism ϕ_L associated to the line bundle L is an isogeny of complex tori if and only if the hermitian form $c_1(L)$ is nondegenerate. This motivates the following definition:

Definition 1.6.12. A line bundle L on X is called a *nondegenerate line bundle* if the associated hermitian form $H = c_1(L)$ is nondegenerate; equivalently, if the alternating form $\Im H$ is nondegenerate.

Proposition 1.6.13. *A line bundle L on X is nondegenerate if and only if $K(L)$ is finite.*

Recall from Lemma 1.3.10 that we will see $E = \Im H$ real valued. In other words, $\Im H$ is written in a real basis.

Proposition 1.6.14. $\deg \phi_L = \det \Im H$.

Proof. We may assume that H is nondegenerate, otherwise $\deg \phi_L = \det \Im H = 0$. Then,

$$\deg \phi_L = \left(\phi_H^{-1} \left(\widehat{\Lambda} \right) : \Lambda \right) = (\Lambda(L) : \Lambda) = \det \Im H.$$

□

1.7 Characteristics

Let $X = V/\Lambda$ be a complex torus of dimension g and $L \in \mathrm{Pic}(X)$ with first Chern class H. Recall that $E = \Im H$ is integer-valued on the lattice Λ. According to the Elementary Divisor Theorem [29, Th. 7.8], there is a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ, with respect to which E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$ with non-negative integers d_ν satisfying $d_\nu | d_{\nu+1}$ for $\nu = 1, \ldots g-1$.

Definition 1.7.1. The *elementary divisors* $d_1, \ldots, d_g$ are uniquely determined by E and Λ and thus by L. The vector $(d_1 \ldots, d_g)$ as well as the matrix D are called the *type* of the line bundle L. The basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ is called a *symplectic* (or *canonical*) *basis* of Λ for L (or H or E respectively).

Recall that the line bundle L is *nondegenerate* if the form H, and thus E, is nondegenerate.

Remark 1.7.2. Let L be a line bundle on X. L being a nondegenerate line bundle on X is equivalent to $d_\nu > 0$ for $\nu = 1, \ldots, g$.

Definition 1.7.3. A direct sum decomposition

$$\Lambda = \Lambda_1 \oplus \Lambda_2$$

is called a *decomposition* for L (or H or E respectively) if Λ_1 and Λ_2 are isotropic with respect to the alternating form E.

Remark 1.7.4. Such a decomposition always exists since if $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ is a symplectic basis of Λ for L, then

$$\Lambda = \langle \lambda_1, \ldots, \lambda_g \rangle \oplus \langle \mu_1, \ldots, \mu_g \rangle$$

is a decomposition for L. Conversely, it follows immediately from the proof of the elementary divisor theorem that for every decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ (with Λ_1 and Λ_2 isotropic with respect to E) for L there exists a symplectic basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ for L with $\Lambda_1 = \langle \lambda_1, \dots, \lambda_g \rangle$ and $\Lambda_2 = \langle \mu_1, \dots, \mu_g \rangle$

Definition 1.7.5. A decomposition

$$V = V_1 \oplus V_2$$

with real subvector spaces V_1 and V_2, is called a *decomposition* for L (or H or E respectively) if $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a decomposition of Λ for L.

According to the above definitions the decompositions of Λ for L are in 1-1-correspondence with the decompositions of V for L.

Let $H \in \mathrm{NS}(X)$ be nondegenerate. A decomposition $V = V_1 \oplus V_2$ for H leads to an explicit description of all line bundles $L \in \mathrm{Pic}^H(X) := c_1^{-1}(H)$. For this we define a map

$$\begin{aligned} \chi_0 : V &\to S^1 \\ v &\mapsto \exp\left(i\pi E(v_1, v_2)\right), \end{aligned}$$

where $v = v_1 + v_2$ with $v_j \in V_j$. Since V_1 and V_2 are isotropic with respect to E we get the following result:

Lemma 1.7.6. *For every $v = v_1 + v_2, w = w_1 + w_2 \in V_1 \oplus V_2$ we have that*

$$\chi_0(v+w) = \chi_0(v)\chi_0(w)\exp\left(i\pi E(v,w)\right)\exp\left(-2\pi i E(v_2, w_1)\right).$$

Thus, χ_Λ is a semicharacter for H. Denote by $L_0 = L(H, \chi_0)$ the corresponding line bundle. The data χ_0 and L_0 are uniquely determined by H and the chosen decomposition.

Lemma 1.7.7. *Suppose H is a nondegenerate hermitian form on V and $V = V_1 \oplus V_2$ a decomposition for H.*

a) $L_0 = L(H, \chi_0)$ *is the unique line bundle in* $Pic^H(X)$ *whose semicharacter is trivial on* $\Lambda_j = V_j \cap \Lambda$ *for* $j = 1, 2$.

b) *For every* $L = L(H, \chi)$ *on* X *there is a point* $c \in V$, *uniquely determined up to translation by elements of* $\Lambda(L)$, *such that* $L \simeq t_{\bar{c}}^* L_0$ *or equivalently,* $\chi = \chi_0 \exp(2\pi i E(c, \cdot))$.

Proof. See [8, p. 47]. $\square$

We will call c a *characteristic of the line bundle L with respect to the chosen decomposition for L*. If we speak only of a characteristic c of L, we mean that a decomposition for L is fixed and that c is the characteristic of L with respect to it.

Observe that a characteristic is only defined for nondegenerate line bundles and determined only up to translations by elements of $\Lambda(L)$. Note that 0 is a characteristic of L_0.

1.8 Theta functions

We start by defining the *Riemann theta function* which is more classical and analytic, and then we define theta functions in a modern way that is not much different. This simple strategy will allow us to understand the language used in some older papers on this subject.

1.8.1 Riemann Theta Functions

Recall that the *Siegel upper half-space* of genus g is defined[3] by

$$\mathscr{H}_g = \left\{ \tau \in \mathbb{M}(g \times g, \mathbb{C}) \mid \tau = \tau^T, \, \Im(\tau) > 0 \right\}.$$

To each point $\tau \in \mathscr{H}$ we can associate the lattice $\Lambda_\tau \subset \mathbb{C}^g$ spanned by the columns of the $g \times 2g$ matrix $\Omega_\tau = (\tau \; 1_g)$.

We can consider the complex torus

$$A_\tau = \mathbb{C}^g / \Lambda_\tau.$$

[3]When we write $\Im(\tau) > 0$, we mean that the imaginary part of τ is positive definite

Later we will see that this complex torus is in fact an (principally polarized) abelian variety. Theta functions play an important role in the study of X, but for now we will just define Riemann theta functions by themselves.

Definition 1.8.1. Let $\tau \in \mathscr{H}_g$ be a point. We define the *Riemann theta function* $\theta(\tau,\cdot) : \mathbb{C}^g \to \mathbb{C}$ by

$$\theta(\tau,z) = \sum_{n\in\mathbb{Z}^g} \exp\left(\frac{1}{2}n^T\tau n + n^T z\right),$$

This series converges absolutely and uniformly on compact sets in $\mathbb{C}^g$.

Let's fix τ; the theta function transforms as follows with respect to the lattice Λ_τ: for $m,n \in \mathbb{Z}^g$ we have

$$\theta(\tau, z+n+\tau m) = \exp\left(-\frac{1}{2}m\tau^T m - m^T z\right)\theta(\tau,z).$$

Moreover, the theta function is even in z;

$$\theta(\tau,-z) = \theta(\tau,z). \tag{1.3}$$

Remark 1.8.2. Later on, working over characteristic 0, one may ask for a universal polarization divisor when working with the moduli stack of principally polarized abelian varieties up to biholomorphisms preserving the polarization $\mathscr{A}_g$. Some technical difficulties make it necessary to define generalized theta functions. For each $\varepsilon,\delta \in \mathbb{R}^{2g}$ we define the function

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z) = \sum_{n\in\mathbb{Z}^g} \exp\left(\frac{1}{2}(n+\varepsilon)^T\tau(n+\varepsilon) + (n+\varepsilon)^T(z+\delta)\right).$$

Note that for $\varepsilon = \delta = 0$ this is just $\theta(\tau,z)$ as before.

1.8.2 Theta functions for a factor of automorphy

Let $X = V/\Lambda$ be a complex torus. Denote by $\pi : V \to X$ the canonical projection map. We have that the lattice Λ acts naturally on π^*L, but according to Lemma 1.3.5 the pullback π^*L of a line bundle L on X is trivial. Thus, $H^0(X,L) \simeq H^0(V,\mathscr{O}_V)^\Lambda$, the subspace of holomorphic functions on V invariant under the action of Λ. This isomorphism depends on the choice of a factor of automorphy for L. To be precise, let f be a factor of automorphy for L. Then, $H^0(X,L)$ can be identified with the set of holomorphic functions

$$\theta : V \to \mathbb{C}$$

such that

$$\theta(v+\lambda) = f(\lambda, v)\theta(v)$$

for all $v \in V$ and $\lambda \in \Lambda$.

Definition 1.8.3. We will call the functions in $H^0(X,L)$ *theta functions for the factor* f, where f is a factor of automorphy for L. Correspondingly[4] we call the theta functions for the factor of automorphy a_L *canonical theta functions for L.*

We want to determine the theta functions for L and compute $h^0(L)$ in the case that the hermitian form H of L is positive definite, which is the precise case of an abelian variety. First, we will define the *classical* factor of automorphy.

Suppose $L = L(H,\chi)$ is a nondegenerate line bundle on X, and that the first Chern class H of L is a positive definite hermitian form. Such a line bundle is called *positive definite line bundle* or just *positive line bundle.* Denote $E = \Im H$ and fix a decomposition $V = V_1 \oplus V_2$ for L.

Lemma 1.8.4. *V_2 generates V as a $\mathbb{C}$-vector space.*

Proof. $V_2 \cap iV_2$ is a $\mathbb{C}$-subvector space of V on which E vanishes. By Lemma 1.3.10 we get that H also vanishes on $V_2 \cap iV_2$. Since H is positive definite, $V_2 \cap iV_2 = \{0\}$ and thus $V = V_2 + iV_2$. □

The hermitian form H is symmetric on V_2 since E vanishes there. Denote by B the $\mathbb{C}$-bilinear extension of the symmetric form $H|_{V_2 \times V_2}$. According to Lemma 1.8.4 the symmetric

[4] Recall a_L from Section 1.5

form B is defined on the whole of V. The following properties of H and B are frequently applied:

Lemma 1.8.5. *a)* $(H-B)(v,w) = \begin{cases} 0 & \text{if } (v,w) \in V \times V_2 \\ 2iE(v,w) & \text{if } (v,w) \in V_2 \times V. \end{cases}$

b) $\Re(H-B)$ *is positive definite on* V_1.

Proof. a) We have that $H-B=0$ on $V \times V_2$, since H is $\mathbb{C}$-linear on its first component. Thus, for $v \in V_2$ and $w \in V$ we have:

$$\begin{aligned}(H-B)(v,w) &= \overline{H(w,v)} - B(w,v)\\ &= (H-B)(w,v) - 2iE(w,v)\\ &= 2iE(v,w).\end{aligned}$$

b) Since $V = V_2 + iV_2$, a vector $v \in V_1$, $v \neq 0$ can be uniquely written in the form $v = v_2 + iu_2$, for some $v_2, u_2 \in V_2$, with $u_2 \neq 0$. Using the previous item we have

$$\begin{aligned}\Re(H-B)(v,v) &= \Re(2iE(v_2,v) - 2E(u_2,v))\\ &= 2E(iu_2,u_2) + 2iR(u_2,u_2)\\ &= 2H(u_2,u_2).\end{aligned}$$

□

The bilinear form B enables us to introduce the classical factor of automorphy for L in a coordinate free way.

Definition 1.8.6. We define the *classical factor (of automorphy)* for L: $e_L : \Lambda \times V \to \mathbb{C}^*$ by

$$e_L(\lambda, v) = \chi(\lambda)\exp\left(\pi (H-B)(v,\lambda) + \frac{\pi}{2}(H-B)(\lambda,\lambda)\right).$$

Equivalently,

$$e_L(\lambda,v) = a_L(\lambda,v)\exp\left(\frac{\pi}{2}B(v,v)\right)\exp\left(\frac{\pi}{2}B(v+\lambda,v+\lambda)\right)^{-1}.$$

Remark 1.8.7. The theta functions for the factor e_L are called *classical theta functions for* L. This terminology came from the fact that the classical Riemann theta function with (real) characteristic $\left[\begin{smallmatrix} c^1 \\ c^2 \end{smallmatrix}\right]$ coincides with the classical notation as we have seen in the previous subsection. In fact, we have that the classical Riemann theta function with (real) characteristic $\left[\begin{smallmatrix} c^1 \\ c^2 \end{smallmatrix}\right]$ is defined by

$$\theta\left[\begin{smallmatrix} c^1 \\ c^2 \end{smallmatrix}\right](\tau,v) = \sum_{n\in\mathbb{Z}^g}\exp\left(\pi i(n+c^1)^T\tau(n+c^1)+2\pi i(v+c^2)(n+c^1)\right)$$

for all $v\in\mathbb{C}^g$. In fact, we have the following relation:

Let $H\in \mathrm{NS}(X)$ be positive definite and $L=L(H,\chi)$ a line bundle on X of type[5] $(1,\ldots,1)$; denote by c the characteristic associated to L. Denote the canonical theta function for L as $\theta^c=\theta^c_\tau$. Then we have the following:

Lemma 1.8.8.

$$\theta_\tau^{\tau c^1+c^2} = \exp\left(\frac{\pi}{2}B(\cdot,\cdot)-\pi i^T c^1c^2\right)\theta\left[\begin{smallmatrix} c^1 \\ c^2 \end{smallmatrix}\right](\tau,\cdot).$$

Proof. See [8, pp. 222-224]. □

Now, consider $L\in\mathrm{Pic}(X)$, and suppose that L is of type $D=\mathrm{diag}(d_1,\ldots,d_g)$. Recall that $E=\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ is the matrix of E with respect to a symplectic basis of the lattice Λ.

Definition 1.8.9. Let L be a line bundle of type D. We define the *Pfaffian of* E as $\mathrm{Pf}(E)=\det D$.

Note that $\mathrm{Pf}(E)$ is independent of the choice of the symplectic basis. Using this notation we get some useful results.

Lemma 1.8.10. *Let $L\in Pic(X)$ be a nondegenerate line bundle. Then, $h^0(X,L)\leq Pf(E)$.*

[5] i.e. principally polarized

Proof. See [8, p. 51]. □

We will construct an explicit basis of canonical theta functions of $H^0(X,L)$. Suppose $c \in V$ is a characteristic of L with respect to the decomposition $V = V_1 \oplus V_2$. Define $\theta^c : V \to C$ by

$$\theta^c(v) = \exp\left(-\pi H(v,c) - \frac{\pi}{2}H(c,c) + \frac{\pi}{2}B(v+c,v+c)\right) \cdot \sum_{\lambda\in\Lambda_1} \exp\left(\pi(H-B)(v+c,\lambda) - \frac{\pi}{2}(H-B)(\lambda,\lambda)\right) \tag{1.4}$$

Lemma 1.8.11. *θ^c is a canonical theta function for $L = t_{\bar{c}}^* L_0$.*

Proof. See [8, p. 52]. □

We can use θ^c to construct other canonical theta functions for L as follows: for any $\overline{w} \in K(L)$, define

$$\theta_{\bar{w}}^c = a_L(w,\cdot)^{-1}\theta^c(\cdot + w)$$

where w denotes some representative[6] of $\bar{w}$ in $\Lambda(L)$. It is easy to see that this definition does not depend on the choice of a representative w of $\bar{w}$. Note that $\theta_0^c = \theta^c$.

Corollary 1.8.12. *$\theta_{\bar{w}}^c$ is a canonical theta function for L for every $\bar{w} \in K(L)$.*

Proof. See [8, p. 53]. □

Now we state the main result of this section. We want to give a basis of the vector space $H^0(X,L)$, when L is a positive definite line bundle.Denote by $K(L)_1 := K(L) \cap V_1$.

Theorem 1.8.13. *Let $L = L(H,\chi)$ be a positive definite line bundle on X and let c be a characteristic with respect to a decomposition $V = V_1 \oplus V_2$ for L. Then, the set $\{\theta_{\bar{w}}^c \mid \bar{w} \in K(L)_1\}$ is a basis of the vector space $H^0(X,L)$ of canonical theta functions for L.*

Proof. See [8, pp. 53-54]. □

[6] Recall that $K(L) = \Lambda(L)/\Lambda$.

Remark 1.8.14. The basis $\{\theta^c_{\bar{w}} \mid \bar{w} \in K(L)_1\}$ is uniquely determined by the choice of a decomposition for L and the characteristic c. Moreover, using that $\#K(L)_1 = \det D = \mathrm{Pf}(E)$, so we get the following result:

Corollary 1.8.15. *$h^0(X,L) = Pf(E)$ for any positive definite $L = L(H,\chi)$.*

1.8.3 Riemann's Addition Formula

In this short subsection we show a useful relation for theta functions that is heavily used later. For a reference see [8, Ch. 7].

Theorem 1.8.13 gives bases of canonical theta functions

$$\begin{aligned}
\left\{\theta^L_x : x \in 2K_1\right\} &\text{ for } H^0(L),\\
\left\{\theta^{L'}_x : x \in 2K_1\right\} &\text{ for } H^0\left(L'\right),\\
\left\{\theta^{L\otimes L'}_x : x \in 2K_1\right\} &\text{ for } H^0\left(L\otimes L'\right), \text{ and}\\
\left\{\theta^{L\otimes(-1)^*L'}_x : x \in 2K_1\right\} &\text{ for } H^0\left(L\otimes(-1)^*_X L'\right)
\end{aligned}$$

Using this notation, we get the following:

Theorem 1.8.16 (Riemann's Addition Formula). *Let L and L' be algebraically equivalent ample line bundles on the abelian variety $X = V/\Lambda$ and $\alpha : X\times X \to X\times X$, given by $\alpha(x_1,x_2) = (x_1+x_2, x_1-x_2)$. Then,*

$$\theta^L_{x_1}(v_1+v_2)\,\theta^{L'}_{x_2}(v_1-v_2) = \sum_{\substack{(z_1,z_2)\in K_1\times K_1\\ \alpha(z_1,z_2)=(x_1,x_2)}} \theta^{L\otimes L'}_{z_1}(v_1)\,\theta^{L\otimes(-1)^*_X L'}_{z_2}(v_2) \tag{1.5}$$

for all $v_1,v_2 \in V$ and $x_1,x_2 \in 2K_1$. In particular, if L defines a principal polarization of X and $L = L'$ is of characteristic zero, then

$$\theta^L_0(v_1+v_2)\,\theta^L_0(v_1-v_2) = \sum_{z\in K_1} \theta^{L^2}_z(v_1)\,\theta^{L^2}_z(v_2) \tag{1.6}$$

for all $v_1,v_2 \in V$.

Proof. See [8, Ch. 7]. □

1.9 The Riemann-Roch Theorem

For a line bundle L on the g-dimensional complex torus $X = V/\Lambda$, the *Euler-Poincaré characteristic* of L is denoted by

$$\chi(L) = \sum_{\nu=0}^{g} (-1)^{\nu} h^{\nu}(X, L).$$

The *Riemann-Roch Theorem* is a formula for $\chi(L)$.

Theorem 1.9.1 (Analytic Riemann-Roch Theorem). *Let L be a line bundle on X whose first Chern class H has s negative eigenvalues. Then*[7]

$$\chi(L) = (-1)^{s} Pf(E).$$

In other words, if L is of type $D = \text{diag}(d_1, \ldots, d_g)$ and s is the number of eigenvalues of $c_1(L)$, then $\chi(L) = (-1)^s d_1 \cdot \ldots \cdot d_g$. In particular, for a degenerate L we have that $d_g = 0$, and so $\chi(L) = 0$.

Proof. See [8, p. 64]. □

Since $\deg \phi_L = \det E = \text{Pf}(E)^2$, we get an immediate corollary:

Corollary 1.9.2. $\chi(L)^2 = \deg \phi_L$ *for every* $L \in Pic(X)$.

Recall the *self-intersection number* (L^g) of a line bundle L on X

$$(L^g) = \int_X \bigwedge^{g} c_1(L).$$

In this integral the first Chern class is considered to be a 2-form on X via the de Rham isomorphism $H^2(X, \mathbb{C}) \simeq H^2_{\text{DR}}(X)$.

[7]Recall that we denote the alternating form $\Im H$ as E.

Theorem 1.9.3 (Geometric Riemann-Roch Theorem). *For any $L \in Pic(X)$ we have that*

$$\chi(L) = \frac{1}{g!}(L^g).$$

Proof. See [8, pp. 65-66]. □

Remark 1.9.4. In the case of a positive (definite) line bundle L, we have that $\dim H^i(X,L) = 0$, $\forall i > 0$. Therefore, $\dim H^0(X,L) = \chi(L) = \frac{(L^g)}{g!}$. For more details see [8, Section 3.5].

1.10 Lefschetz Theorem

We will leave all of the following results without proof. For a reference consider [8, pp. 84-88].

Let $X = V/\Lambda$ be an abelian variety of dimension g and $L \in \mathrm{Pic}(X)$ a polarization. The line bundle L induces a meromorphic map $\varphi_L : X \to \mathbb{P}^N$ defined as follows: if $\eta_0, \ldots, \eta_N$ is a basis of $H^0(X,L)$, then

$$\varphi_L(x) = [\eta_0(x) : \cdots : \eta_N(x)],$$

whenever $\eta_j(x) \neq 0$ for some j. Choosing a factor of automorphy f for L we may consider $H^0(X,L)$ as the vector space of theta functions on V with respect to f. Let $\theta_0, \ldots, \theta_N$ denote the basis of theta functions for the factor f. Then, the map φ_L is given by

$$\varphi_L(\bar{v}) = [\theta_0(v) : \cdots : \theta_N(v)].$$

Note that φ_L does not depend on the choice of f, although it depends on the choice of the basis of $H^0(X,L)$. We will see that one can find a useful choice of this basis in the *Riemann addition formula*.

Theorem 1.10.1 (Lefschetz Theorem). *If L is a positive definite line bundle on X of type $(d_1, \ldots, d_g)$ with $d_1 \geq 3$, then $\varphi_L : X \to \mathbb{P}^N$ is an embedding.*

Recall that a line bundle L is *ample* if L^n is very ample for some $n \geq 1$.

Proposition 1.10.2. *For a line bundle L on X the following statements are equivalent:*

(i) L is ample.

(ii) L is positive definite.

(iii) $H^0(L) \neq 0$ and $K(L)$ is finite.

(iv) $H^0(L) \neq 0$ and $(L^g) > 0$.

Corollary 1.10.3. *If $L \in Pic(X)$ is ample, then L^n is very ample for $n \geq 3$.*

Recall that the trascendence degree of the field of meromorphic functions on X is called the *algebraic dimension* $a(X)$ of X. We have that $a(X) \leq \dim X$ for any connected compact complex manifold. Proposition 1.10.2 leads to the following criterion for a complex torus to be an abelian varitety.

Theorem 1.10.4. *For a complex torus X, the following conditions are equivalent:*

1. *X is an abelian variety.*

2. *X admits the structure of a projective algebraic variety.*

3. $a(X) = \dim X$.

1.11 Kummer Varieties

We want to define the Kummer variety of an Abelian variety. This will be used later on to describe geometrically one characterization of Jacobian varieties, and furthermore we will use it to give a similar condition to describe Prym varieties and abelian varieties that contain a curve that is twice the minimal class which is the main topic of this thesis. The geometric condition will be secancy conditions on the Kummer variety in projective space.

Definition 1.11.1. Let $X = V/\Lambda$ be an abelian variety of dimension g. The *Kummer variety* associated to X is defined to be the quotient

$$K(X) = X/(-1)_X.$$

We have that $K(X)$ is an algebraic variety of dimension g over $\mathbb{C}$, smooth except at 2^{2g} singular points of multiplicity 2^{g-1} which are the images of the 2-torsion points of X under the natural map $p : X \to K(X)$.

Let $L = L(H,\chi)$ be an ample symmetric line bundle on X defining an indecomposable principal polarization. That is, any Theta divisor defining L is irreducible. Since $\chi(\Lambda) \subset \{\pm 1\}$, we have by Corollary 1.5.8 that the semicharacter χ^2 of L^2 is identically 1 on Λ. This implies that L^2 is of characteristic 0 with respect to any decomposition of L. By [8, Corollary 4.6.6] all theta functions in $H^0(L^2)$ are even. Hence, we have that there exists a map $\psi : K(X) \to \mathbb{P}^{2^g-1}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "\varphi=\varphi_{L^2}"] \arrow[dr, "p"'] & & \mathbb{P}^{2^g-1} \\ & K(X) \arrow[ur, "\psi"'] & \end{tikzcd}$$

Theorem 1.11.2. *Let $L \in Pic(X)$ be a symmetric line bundle that defines an indecomposable principal polarization on X, then $\psi : K(X) \to \mathbb{P}^{2^g-1}$ is an embedding.*

Proof. See [8, pp. 98-99]. □

This result implies that on an indecomposable principally polarized abelian variety, we can always consider $K(X)$ as canonically embedded in $\mathbb{P}^{2^g-1}$.

1.12 Endomorphisms Associated to Cycles

Let X be an abelian variety. The objective of this section is to show that to every pair of a divisor and a curve on an abelian variety, we can associate an endomorphism.

We will consider only *algebraic cycles* V with coefficients in $\mathbb{Z}$, *i.e.* finite formal sums

$$V = \sum r_i V_i$$

with integers r_i and algebraic subvarieties V_i of X, which we assume to be all of the same dimension. We will say that V is an *algebraic p-cicle* if $\dim V_i = p$ for all i.

Let $W = \sum s_i W_i$ be an algebraic q-cycle on X. The cycles V and W are said to *intersect properly* if $V_i \cap W_j$ is either of pure dimension $p+q-g$ or empty, whenever $r_i \neq 0 \neq s_j$.

Lemma 1.12.1 (Moving Lemma). *Let V be an algebraic p-cycle and W an algebraic q-cycle on X. There is an open dense subset U in X such that V and t_x^*W intersect properly for all $x \in U$.*

Proof. Without loss of generality we assume that V and W are subvarieties of X. Consider the difference map

$$d : V \times W \to X : (v, w) \mapsto w - v.$$

The fibre of d over any $x \in X$ is

$$d^{-1}(x) \simeq V \cap t_x^* W.$$

Since d is a closed morphism, there is[8] an open dense subset U of X such that $d^{-1}(x)$ is either of dimension $p+q-g$ or empty for all $x \in U$. □

We will need a Moving Lemma with parameters.

Lemma 1.12.2. *Let T be an algebraic variety and Z an algebraic cycle on $T \times X$ intersecting $\{t\} \times X$ properly for any $t \in T$. Let $Z(t)$ be the cycle on X defined by $Z \cdot (\{t\} \times X)$. For any algebraic cycle W on X there is an open dense subset $U \subset T \times X$ such that $Z(t)$ and t_x^*W intersect properly for all $(t,x) \in U$.*

Proof. We may assume that Z is an algebraic variety on $T \times X$. Consider the map

$$g : W \times Z \to T \times X : (w, (t,x)) \mapsto (t, w - x).$$

Then,

$$g^{-1}(t,x) \simeq Z(t) \cap t_x^* W.$$

As g is a closed morphism, we finish the proof using the same steps as the Moving Lemma 1.12.1. □

[8] See [24, II, ex. 3.22 c)]

Let V and W be two algebraic cycles on X of complementary dimension. Suppose that V and W intersect properly, then the usual intersection product $V \cdot W$ is a 0-cycle on X. That is, $V \cdot W = \sum_{j=1}^{n} r_j x_j$, with $x_j \in X$ and integers r_j. We define $S(V \cdot W)$ to be the sum of these points on X, say

$$S(V \cdot W) = r_1 x_1 + \cdots + r_n x_n \in X.$$

Note that S is symmetric and bilinear.

Let now (V, W) be an arbitrary pair of algebraic cycles of complementary dimension on X. The pair (V, W) induces an endomorphism $\delta(V, W)$ of X in the following way:

According to the Moving Lemma 1.12.1 the cycle V intersects $t_x^* W$ properly for all x of an open dense subset of X. So we have a rational map

$$\begin{aligned} \tilde{\delta} : X &\dashrightarrow X \\ x &\mapsto S(V \cdot t_x^* W). \end{aligned}$$

But, a rational map from a smooth variety into an abelian variety extends to a morphism (see [8, p. 101]). Since a morphism between complex tori is a homomorphism plus a constant, i.e. there is an endomorphism $\delta(V, W)$ of X and a point $c \in X$, both uniquely determined by $\tilde{\delta}$, such that $\delta(V, W) = \tilde{\delta}(V, W) - c$. So, we have

$$\begin{aligned} \delta(V, W) : X &\to X \\ x &\mapsto S(V \cdot t_x^* W) - c \end{aligned}$$

whenever V intersects $t_x^* W$ properly. The bilinearity of S implies that

$$\delta\left(V + V', W\right) = \delta\left(V, W\right) + \delta\left(V', W\right) \text{ and}$$
$$\delta\left(V, W + W'\right) = \delta\left(V, W\right) + \delta\left(V, W'\right)$$

for all algebraic cycles V, V' and W, W' of complementary dimension on X.

Remark 1.12.3. If V intersects W properly we have that $c = S(V \cdot W)$, that is

$$\delta(V,W)(x) = S(V \cdot (t_x^* W - W))$$

whenever defined.

We may always assume that V intersects W properly thanks to the following result.

Proposition 1.12.4. $\delta(V,W) = \delta(V',W)$ *for any algebraically equivalent algebraic p-cycles V and V' and any algebraic $(g-p)$-cycle W on X.*

Proof. See [8, pp. 129-130]. □

Lemma 1.12.5. $\delta(V,W) + \delta(W,V) = -(V \cdot W)\, id_X$.

Proof. See [8, p. 130]. □

Corollary 1.12.6. *The homomorphism $\delta(V,W)$ depends only on the algebraic equivalence classes of V and W.*

In particular, to every curve and divisor on an abelian variety we can associate an endomorphism. This will be explored further in the upcoming chapters.

CHAPTER

2 Jacobian varieties

Jacobian varieties are the historic origin of the study of abelian varieties [12, chap. VII, 57]. In this chapter we present the basic material, and we end by showing that a jacobian variety admits a 4-dimensional family of trisecants lines to its Kummer variety.

We will work over a field of characteristic 0. The basic material will be based on [8].

2.1 Jacobians

We start by defining the jacobian variety. Let C be a smooth projective curve of genus $g > 0$. Recall the g-dimensional $\mathbb{C}$-vector space $H^0(\omega_C)$ of holomorphic 1-forms on C. The homology group $H_1(C,\mathbb{Z})$ is a free abelian group of rank $2g$. By Stokes' theorem, any element $\gamma \in H_1(C,\mathbb{Z})$ yields in a canonical way a linear form on the vector space $H^0(\omega_C)$, which we also denote by γ:

$$\gamma : H^0(\omega_C) \to \mathbb{C} : \omega \mapsto \int_\gamma \omega.$$

Lemma 2.1.1. *The map*

$$H^1(C,\mathbb{Z}) \to H^0(\omega_C)^* = Hom\left(H^0(\omega_C),\mathbb{C}\right)$$

is injective.

Proof. See [8, pp. 316-317]. □

Hence, $H_1(C,\mathbb{Z})$ is a lattice in $H^0(\omega_C)^*$, and we get the complex torus of dimension g

$$J(C) := H^0(\omega_C)^*/H_1(C,\mathbb{Z}) \tag{2.1}$$

Definition 2.1.2. Let C be a smooth projective curve of genus g over $\mathbb{C}$. The *jacobian variety* (or *jacobian*) is the complex torus $J(C)$.

Remark 2.1.3. Since $J(C) = 0$ if $g = 0$, we will assumme in general that $g \geq 1$.

Remark 2.1.4. Using the intersection form on $H_1(C,\mathbb{Z})$ we can extend it to a hermitian form H on $H^0(\omega_C)^*$. In fact, one has

Proposition 2.1.5. *The hermitian form H defines a principal polarization on* $J(C)$.

This polarization H is called the *canonical polarization of* $J(C)$. Any effective divisor Θ on $J(C)$ such that the line bundle $\mathcal{O}_{J(C)}(\Theta)$ defines the canonical polarization is called a *theta divisor* of the jacobian $J(C)$. It is usual to write $(J(C),\Theta)$ for the canonically polarized jacobian.

There is another way to define the jacobian variety of C, namely as the group $\mathrm{Pic}^0(C)$. Recall that $\mathrm{Pic}^0(C)$ is the quotient

$$\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\mathrm{PDiv}(C)$$

of divisors of degree zero on C modulo the subgroup of principal divisors. We have a canonical map

$$\mathrm{Div}^0(C) \to J(C) = H^0(\omega_C)^*/H_1(C,\mathbb{Z})$$

defined as follows: any divisor $D \in \mathrm{Div}^0(C)$ can be written as a finite sum $D = \sum_{j=1}^n (p_j - q_j)$ for some point $p_j, q_j \in C$. The class of the linear form $\omega \mapsto \sum_{j=1}^n \int_{q_j}^{p_j} \omega$ in $J(C)$ depends only on the divisor D, but not on its special representation as a sum of differences of points. So

$$D \mapsto \left(\omega \mapsto \sum_{j=1}^{n} \int_{q_j}^{p_j} \omega\right) (\operatorname{mod} H_1(C,\mathbb{Z}))$$

gives a well defined map $Div^0(C) \to J(C)$, which is clearly a homomorphism of groups.

Definition 2.1.6. The map $Div^0(C) \to J(C)$ previously defined is called the *Abel-Jacobi map* of C.

Jacobi's Inversion Theorem (see [1, p. 19]) states that the Abel-Jacobi map is surjective. On the other hand, by a theorem of Abel (see [1, p. 18] or [31, ch. VIII]) its kernel is the subgroup of principal divisors in $\mathrm{Div}^0(C)$. Hence, we obtain the following theorem

Theorem 2.1.7 (Abel-Jacobi Theorem). *The Abel-Jacobi map induces a canonical isomorphism $Pic^0(C) \xrightarrow{\sim} J(C)$.*

Proof. See [31, ch. VIII]. □

Remark 2.1.8. Via this isomorphism $\mathrm{Pic}^0(C)$ inherits the structure of a principally polarized abelian variety.

In what follows we will identify $J(C) = \mathrm{Pic}^0(C)$ via the canonical isomophism. We will use $+$ for the group law and sometimes for line bundles on C we will write $\otimes$.

For any $n \in \mathbb{Z}$ denote by $\mathrm{Div}^n(C)$ the set of divisors of degree n on C. It is a principal homogeneous space for the group $\mathrm{Div}^0(C)$. We can define a more general Abel-Jacobi map but, it will be not canonical for $n \neq 0$:

Fix a divisor $D_n \in \mathrm{Div}^n(C)$ and define

$$\begin{aligned} \mathrm{Div}^n(C) &\to J(C) \\ D &\mapsto \mathscr{O}_C(D - D_n). \end{aligned} \tag{2.2}$$

The most important case is $D_n = nc$ for some point $c \in C$. In this case, the previous map can be written as

$$D = \sum r_j p_j \mapsto \left(\omega \mapsto \sum r_j \int_c^{p_j} \omega\right) \quad (\mathrm{mod}\, H_1(C,\mathbb{Z}))\,.$$

Restricting the map $\mathrm{Div}^n(C) \to J(C)$ to the symmetric product[1] $C^{(n)}$ (as we can consider the elements of $C^{(n)}$ as effective divisors of degree n on C), we get the following map

$$\alpha_{D_n} : C^{(n)} \to J(C)$$

which is also called the *Abel-Jacobi map*.

Let $\mathrm{Pic}^n(C)$ denote the set of line bundles of degree n on C. It is a principal homogeneous space for the group $\mathrm{Pic}^0(C)$. Given a line bundle L_n of degree n on C. We have that the map

$$\begin{aligned} \alpha_{L_n} : \mathrm{Pic}^n(C) &\to J(C) \\ L &\mapsto L \otimes L_n^{-1} \end{aligned} \tag{2.3}$$

is bijective. Finally, consider the canonical map $\rho : C^{(n)} \to \mathrm{Pic}^n(C)$ sending an effective divisor D in $C^{(n)}$ to its class $\mathscr{O}_C(D)$ in $\mathrm{Pic}^n(C)$. So we obtain the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Pic}^n(C) \arrow[d, "\alpha_{\mathscr{O}(D_n)}"] \\ C^{(n)} \arrow[ur, "\rho"] \arrow[r, "\alpha_{D_n}"'] & J(C) \end{tikzcd}$$

For any line bundle $L \in \mathrm{Pic}^n(C)$ the fibre $\rho^{-1}(L)$ is by definition the complete linear system $|L|$ on C. So for any $M \in \mathrm{Pic}^0(C)$ the fibre $\alpha_{D_n}^{-1}(M)$ is the complete linear system $|M \otimes \mathscr{O}_C(D_n)|$.

[1]Recall that $C^{(n)}$ is the n-fold that is the symmetric product of C. This is, $C^{(n)}$ is the quotient of the cartesian product C^n by the natural action of the symmetric group S_n.

Fix a point $c \in C$. We will show that the Abel-Jacobi map $\alpha = \alpha_c : C \to J = J(C)$ is an embedding. First recall that the differential $d\alpha$ is a holomorphic map from the tangent bundle $\mathscr{T}_C$ of C to the tangent bundle $\mathscr{T}_J$ of J. According to Lemma 1.3.5 the tangent bundle of J is trivial, i.e. $\mathscr{T}_J = J \times \mathbb{C}^g$. The projectivization of the composed map $\mathscr{T}_C \to \mathscr{T}_J \simeq J \times \mathbb{C}^g \to \mathbb{C}^g$ is a priori a rational map $C \to \mathbb{P}^{g-1}$ called the projectivized differential of α.

Proposition 2.1.9. *The projectivized differential of the Abel-Jacobi $\alpha : C \to J$ is the canonical map $\varphi_{\omega_C} : C \to \mathbb{P}^{g-1}$.*

Proof. See [8, p. 321]. □

Corollary 2.1.10. *For any $g \geq 1$ the Abel-Jacobi map $\alpha : C \to J(C)$ is an embedding.*

Proof. The map α is injective, since for every line bundle L of degree 1 on a curve of genus $g \geq 1$ we have that $h^0(L) \leq 1$. From Proposition 2.1.9 we conclude that $d\alpha$ is injective at every point $p \in C$, since the canonical line bundle ω_C on C is base point free. □

We finish this section with the famous Theorem of Torelli.

Theorem 2.1.11 (Torelli's Theorem). *Suppose C and C' are compact Riemann surfaces of genus g. If their jacobians $(J(C), \Theta)$ and $(J(C'), \Theta')$ are isomorphic as polarized abelian varieties, then C is isomorphic to C'.*

Proof. See [1, pp. 245-249]. □

2.2 The Theta Divisor

We will find an intrinsic way of defining a theta divisor in $\mathrm{Pic}^{g-1}(C)$. The study of the geometry of Θ will help to understand the geometry of the curve C as well.

Recall the canonical map $\rho : C^{(n)} \to \mathrm{Pic}^n(C)$ for $n \geq 1$. Its image $W_n = \rho\left(C^{(n)}\right)$ is the subset of $\mathrm{Pic}^n(C)$ of line bundles with nonempty linear system. For $n \geq g$ we have that $W_n = \mathrm{Pic}^n(C)$ as consequence of the Riemann-Roch Theorem for curves. For $1 \leq n \leq g$ it is well known that $h^0\left(\mathscr{O}_C(D)\right) = 1$ for a general divisor $D \in C^{(n)}$. This means that the map

$\rho : C^{(n)} \to \mathrm{Pic}^n(C)$ is birational onto its image W_n. Moreover, since ρ is a proper morphism, W_n is an irreducible closed subvariety of $\mathrm{Pic}^n(C)$ of dimension n. In particular, W_{g-1} is a divisor in $\mathrm{Pic}^{g-1}(C)$. We want to see the relation between W_{g-1} and the theta divisor since this will help in some computations later.

Recall that the *fundamental class* $[Y]$ of an n-dimensional subvariety Y of a smooth projective variety X of dimension g is the element it induces in $H^{2g-2n}(X,\mathbb{Z})$, Poincaré dual to the homology class $\{Y\}$ of Y in $H_{2n}(X,\mathbb{Z})$.

Theorem 2.2.1 (Poincaré Formula). *Let $c \in C$ be a point. Denote by $\tilde{W}_n$ the image of W_n in $J(C)$ under the bijection $\alpha_{\mathscr{O}(nc)}$. Then,*

$$[\tilde{W}_n] = \frac{1}{(g-n)!} \bigwedge^{g-n} [\Theta]$$

for any $1 \leq n \leq g$.

Proof. See [8, pp. 322-323]. □

Remark 2.2.2. The formula also holds for $n = 0$, if we define $\tilde{W}_0$ to be a point. By definition $[\Theta] = c_1(\Theta)$. Thus, $\bigwedge^g [\Theta]$ equals the intersection number (Θ^g) times the class of a point. So, for $n = 0$, the formula is equivalent to $(\Theta^g) = g!$ which is a consequence of Riemann-Roch.

Corollary 2.2.3. $(C \cdot \Theta) = g$.

Proof. According to the Poincaré Formula we have:

$$\begin{aligned}[C] \wedge [\Theta] &= \frac{1}{(g-1)!} \bigwedge^{g-1} [\Theta] \wedge [\Theta] \\ &= \frac{1}{(g-1)!} \bigwedge^{g} [\Theta].\end{aligned}$$

So, $(C \cdot \Theta) = \frac{1}{(g-1)!}(\Theta^g) = g$ by Riemann-Roch and the definition of the intersection number.

□

Corollary 2.2.4. *There is an $\eta \in Pic^{g-1}(C)$ such that $W_{g-1} = \alpha_\eta^* \Theta$.*

Proof. See [8, p 324]. □

Recall that a *theta characteristic* on C is a line bundle κ on C with $\kappa^2 = \omega_C$.

Theorem 2.2.5 (Riemann's Theorem). *For any symmetric theta divisor Θ there is a theta characteristic κ on C such that $W_{g-1} = \alpha_\kappa^* \Theta$.*

Proof. See [8, p 324]. □

Remark 2.2.6. This makes it natural to call W_{g-1} the *canonical theta divisor* of C.

2.3 The Universal Property of the Jacobian

Theorem 2.3.1 (Universal Property of the Jacobian). *Let X be an abelian variety and $\varphi : C \dashrightarrow X$ a rational map. Then, there exists a unique homomorphism $\tilde{\varphi} : J(C) \to X$ such that for every $c \in C$ the following diagram is commutative:*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\alpha_c"'] & X \arrow[d, "t_{-\varphi(c)}"] \\ J(C) \arrow[r, "\tilde{\varphi}"] & X. \end{tikzcd}$$

Proof. A rational map from a smooth variety into an abelian variety extends to a morphism (see [8, p. 101]), so the map φ is everywhere defined.

Consider the morphism $\left(t_{-\varphi(c)}\varphi\right)^{(g)} : C^{(g)} \to X$ defined by

$$\left(t_{-\varphi(c)}\varphi\right)^{(g)}(p_1 + \cdots + p_g) = \varphi(p_1) + \cdots + \varphi(p_g) - g\varphi(c).$$

Since $\alpha_{gc} : C^{(g)} \to J(C)$ is birational, there is a rational map $\tilde{\varphi}_c : J(C) \to X$ such that

$$\left(t_{-\varphi(c)}\varphi\right)^{(g)} = \tilde{\varphi}_c \alpha_{gc} \tag{2.4}$$

on an open dense set of $C^{(g)}$; again, since the map $\tilde{\varphi}_c$ is a rational map into an abelian variety, it extends to a morphism, so Equation 2.4 holds everywhere. Now, $\tilde{\varphi}_c(0) = \left(t_{-\varphi(c)}\varphi\right)^{(g)}(gc) = 0$ implies that $\tilde{\varphi}_c$ is a homomorphism (since it's a morphism between complex tori). Moreover, the diagram commutes since $\alpha_c(p) = \alpha_{gc}(p + (g-1)c)$ and $\varphi(p) - \varphi(c) = \left(t_{-\varphi(c)}\varphi\right)^{(g)}(p + (g-1)c)$

for all $p \in C$. We have that $\tilde{\varphi}_c$ is unique since $\alpha_c(C)$ generates $J(C)$ as a group. It remains to show that $\tilde{\varphi}_c = \tilde{\varphi}_{c'}$ for any $c, c' \in C$. In fact,

$$
\begin{aligned}
(\tilde{\varphi}_c - \tilde{\varphi}_{c'})\,\alpha_c(p) &= \tilde{\varphi}_c \alpha_c(p) - \tilde{\varphi}_{c'}\left(\alpha_{c'}(p) - \alpha_{c'}(c)\right) \\
&= t_{-\varphi(c)}\varphi(p) - t_{-\varphi(c')}\varphi(p) + t_{-\varphi(c')}\varphi(c) \\
&= 0
\end{aligned}
$$

for all $p \in C$. Again, since $\alpha_c(C)$ generates $J(C)$, we have finished the proof. □

2.4 Matsusaka-Ran Criterion

Recall that a curve C on an abelian variety X is said to generate X, if X is the smallest abelian subvariety containing C. More generally, an effective algebraic 1-cycle $\sum r_j C_j$ on X generates X if the union of the curves C_j generates X.

Theorem 2.4.1 (Matsusaka-Ran Criterion). *Let (X, L) be a polarized abelian variety of dimension g and suppose that $C = \sum_{j=1}^{n} r_j C_j$ is an effective 1-cycle generating X with $(C \cdot L) = g$. Then $r_1 = \cdots = r_n = 1$, the curves C_j are smooth, and (X, L) is isomorphic to the product of the canonically polarized jacobians of the C_j:*

$$(X, L) \simeq (J(C_1), \Theta_1) \times \cdots \times (J(C_n), \Theta_n).$$

In particular, if C is an irreducible curve generating X with $(C \cdot L) = g$, then C is smooth and (X, L) is the jacobian of C.

Proof. See [8, pp. 341-344]. □

Corollary 2.4.2. *a) A principally polarized abelian surface is either the jacobian of a smooth curve of genus 2 or the canonically polarized product of two elliptic curves.*

b) A principally polarized abelian threefold is either the jacobian of a smooth curve of genus 3 or the principally polarized product of an abelian surface with an elliptic curve or the principally polarized product of three elliptic curves.

Proof. See [8, p. 341]. □

2.5 Fay's Trisecant Property

Jacobian varieties have a interesting geometric property. Namely, the Kummer variety of a jacobian admits a 4-dimensional family of trisecant lines, thus its points are very far from being in general position in $\mathbb{P}^{2^g-1}$. The aim of this section is to show this property.

Let C be a smooth algebraic curve of genus $g \geq 2$ and (J,Θ) its jacobian variety. Assume that the line bundle $\mathscr{O}_J(\Theta)$ is of characteristic zero with respect to some symplectic basis of $H_1(C,\mathbb{Z})$. Recall the Kummer variety $K = K(X)$ of $X = J(C)$ from Section 1.11, it is the image of the map $\phi_{2\Theta} : J(C) \to \mathbb{P}^{2^g-1}$. First we want to show that having a(n honest) trisecant is equivalent to a certain schematic inclusion and a certain equation in terms of theta functions. Recall that on a principally polarized abelian variety (X,Θ) of dimension g one has $H^0(X,\mathscr{O}_X(\Theta)) = \langle\theta\rangle$, and by Riemann's Addition Formula 1.8.16 there is a basis $\theta_0,\dots,\theta_{2^g-1}$ of $H^0(X,\mathscr{O}_X(2\Theta))$ such that

$$\sum_{j=0}^{2^g-1} \theta_j(\zeta)\theta_j(\omega) = \theta(\zeta+\omega)\theta(\zeta-\omega)$$

for any $\zeta,\omega \in \mathbb{C}^g$. For $z \in \mathbb{C}^g$, we will denote by $\bar{z} \in \mathbb{C}^g$ its image via the universal covering map. For the translation t_ζ on Θ for any $\zeta \in X$, we will denote $t_{\bar{\zeta}}\Theta = \Theta + \bar{\zeta} = \Theta_{\bar{\zeta}}$, as well as $\theta_\zeta(z) = \theta(z-\zeta)$; usually we denote Θ_ζ for $\Theta_{\bar{\zeta}}$.

Theorem 2.5.1. *Let (X,Θ) be a principally polarized abelian variety, and consider the Kummer map $K : X \to \mathbb{P}^{2^g-1}$ whose image is the Kummer variety $K(X)$, given by*

$$K(\bar{x}) = [\theta_0(x) : \cdots : \theta_{2^g-1}(x)].$$

For any $\bar{a},\bar{b},\bar{c} \in X$ pairwise distinct and nonzero, the following conditions are equivalent:

i) The points $K(\bar{a})$, $K(\bar{b})$, and $K(\bar{c})$ are collinear in $\mathbb{P}^{2^g-1}$.

ii) There exist $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero such that

$$\alpha\theta_a\theta_{-a} + \beta\theta_b\theta_{-b} + \gamma\theta_c\theta_{-c} = 0.$$

iii) We have the schematic inclusion $\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c}$.

Proof. First we prove the equivalence between *i)* and *ii)*. Since the points $K(\bar{a})$, $K(\bar{b})$, and $K(\bar{c})$ are collinear, we have equivalently that the vectors

$$\begin{gathered}(\theta_0(a), \ldots, \theta_{2^g-1}(a)), \\ (\theta_0(b), \ldots, \theta_{2^g-1}(b)), \\ (\theta_0(c), \ldots, \theta_{2^g-1}(c))\end{gathered}$$

are $\mathbb{C}$-linearly dependent. This is equivalent to having $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero such that

$$\alpha\theta_j(a) + \beta\theta_j(b) + \gamma\theta_j(c) = 0$$

for all j, $0 \leq j \leq 2^g - 1$. We multiply by $\theta_j(z)$ and sum over j, getting

$$\sum_{j=0}^{2^g-1} \left(\alpha\theta_j(z)\theta_j(a) + \beta\theta_j(b) + \gamma\theta_j(z)\theta_j(c)\right) = 0.$$

Using Riemann's Addition Formula, this is equivalent to the equation

$$\alpha\theta(z-a)\theta(z+a) + \beta\theta(z-b)\theta(z+b) + \gamma\theta(z-c)\theta(z+c) = 0.$$

To finish the proof, we now prove the equivalence between *ii)* and *iii)*. Assuming condition *ii)*, restricting to $\Theta_a \cdot \Theta_b$ we get that $\theta_c\theta_{-c} = 0$, so that $\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c}$. On the other hand, if we suppose that $\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c}$, this implies that $\theta_c\theta_{-c} = 0$ on $\Theta_a \cdot \Theta_b$. Note that since $a \neq b$ we have that $\Theta_a \cdot \Theta_b$ is of pure codimesion 2 on X, in particular it is a divisor on Θ_a.

Consider the following exact sequence of sheaves[2]:

$$0 \to \mathcal{O}_{\Theta_a}(-\Theta_a) \stackrel{\cdot\theta_a}{\to} \mathcal{O}_{\Theta_a} \to \mathcal{O}_{\Theta_a \cdot \Theta_b} \to 0.$$

Applying $\otimes \mathcal{O}_{\Theta_a}(2\Theta)$, and using the Theorem[3] of the Square 1.5.4 we get the exact sequence (see Section A.1):

$$0 \to \mathcal{O}_{\Theta_a}(\Theta_{-a}) \stackrel{\cdot\theta_a}{\to} \mathcal{O}_{\Theta_a}(2\Theta) \to \mathcal{O}_{\Theta_a \cdot \Theta_b}(2\Theta) \to 0.$$

We have the long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(\Theta_a, \mathcal{O}_{\Theta_a}(\Theta_{-a})) & \xrightarrow{\cdot\theta_a} & H^0(\Theta_a, \mathcal{O}_{\Theta_a}(2\Theta)) & \longrightarrow & H^0\left(\Theta_a, \mathcal{O}_{\Theta_a \cdot \Theta_b}(2\Theta)\right) & \longrightarrow \cdots \\ & \| & & & & & \\ & \langle \theta_{-a} \rangle & & & & & \end{array}$$

By the previous comment, we get that $\theta_c \theta_{-c} \in H^0(\Theta_a, \mathcal{O}_{\Theta_a}(2\Theta))$ maps to zero, so the long exact sequence of cohomology shows that $\theta_c \theta_{-c}|_{\Theta_b} = \tilde{\alpha} \theta_a \theta_{-a}$ for some $\tilde{\alpha} \in \mathbb{C}$. To finish the proof consider the long exact sequence in cohomology of the following exact sequence of sheaves:

$$0 \to \mathcal{O}_X(\Theta_{-b}) \stackrel{\cdot\theta_b}{\to} \mathcal{O}_X(2\Theta) \to \mathcal{O}_{\Theta_b}(2\Theta) \to 0.$$

that is

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(X, \mathcal{O}_X(\Theta_{-b})) & \xrightarrow{\cdot\theta_b} & H^0(X, \mathcal{O}_X(2\Theta)) & \longrightarrow & H^0\left(X, \mathcal{O}_{\Theta_b}(2\Theta)\right) & \longrightarrow \cdots \\ & \| & & & & & \\ & \langle \theta_{-b} \rangle & & & & & \end{array}$$

Since $\theta_c \theta_{-c} - \tilde{\alpha} \theta_a \theta_{-a} \in H^0(X, \mathcal{O}_X(2\Theta))$ vanishes when restricted to Θ_b, we get that

$$\theta_c \theta_{-c} - \tilde{\alpha} \theta_a \theta_{-a} = \tilde{\beta} \theta_b \theta_{-b}$$

in X for some $\tilde{\beta} \in \mathbb{C}$. □

[2]For a reference see A.1

[3]In particular, $2\Theta - \Theta_a \sim \Theta_{-a}$, where $\sim$ refers to linear equivalence.

Now, we go back to jacobians. We will denote as usual by $p-q$ the image of (p,q) under the difference map $\delta : C\times C \to J(C)$. If C is hyperelliptic, we denote by ι the hyperelliptic involution.

Proposition 2.5.2. *Let $X = J(C)$ be the jacobian of a smooth curve of genus $g \geq 2$. Then*

a) $\Theta\cap\Theta_{q-p} \subset \Theta_{r-p}\cup\Theta_{q-s}$ for all $p\neq q$, $r,s\in C$.

b) The intersection $\Theta\cap\Theta_{q-p}$ is reducible, except if C is hyperelliptic and $q=\iota(p)$. In the remaining case, $\Theta\cap\Theta_{\iota(p)-p}$ is irreducible.

Proof. a) According to Riemann's Theorem 2.2.5 there is a theta characteristic $\kappa \in \mathrm{Pic}^{g-1}(C)$ with $\alpha_\kappa^*\Theta = W_{g-1}$. In terms of W_{g-1} the assertion reads

$$W_{g-1}\cap\left(W_{g-1}+(q-p)\right) \subset \left(W_{g-1}+(r-p)\right)\cup\left(W_{g-1}+(q-s)\right). \tag{2.5}$$

Consider the sets $V_q = \left\{L\in\mathrm{Pic}^{g-1}(C) \mid h^0(L(-q))\geq 1\right\}$ and $V_p' = \left\{L\in\mathrm{Pic}^{g-1}(C) \mid h^0(L(p))\geq 2\right\}$. It is enough to show that

$$W_{g-1}\cap\left(W_{g-1}+(q-p)\right)\subset V_q\cup V_p',$$

since $V_q\subset\left(W_{g-1}+(q-s)\right)$ for any $s\in C$ and $V_p'\subset\left(W_{g-1}+(r-p)\right)$ for any $r\in C$. Suppose $L\in\mathrm{Pic}^{g-1}(C)$ is contained in $W_{g-1}\cap\left(W_{g-1}+(q-p)\right)$ but not in V_p'. This implies that

$$h^0(L(p)) = h^0(L) = h^0(l(p-q)) = 1.$$

This means that p and q are base points of the linear system $|L(p)|$. Hence q is a base point of $|L|$, that is $h^0(L(-q))>0$.

b) Suppose first that C is non-hyperelliptic or that C is hyperelliptic with $q\neq\iota(p)$. It suffices to show that $W_{g-1}\cap\left(W_{g-1}+(q-p)\right)$ is contained neither in V_q nor V_p'. For this we apply the geometric Riemann-Roch Theorem (see [31, p. 209]) for the canonical map $\varphi=\varphi_{\omega_c} : C\to\mathbb{P}^{g-1}$. Choose points $p_1,\dots,p_{g-2}\in C$ such that the g points $\varphi(p_1),\dots,\varphi(p_{g-2}),\varphi(p),\varphi(q)$ span $\mathbb{P}^{g-1}$. Then $L=\mathscr{O}_C(p_1+\dots+p_{g-2}+q)$ satisfies

$h^0(L)=1$, $h^0(L(p))=2$ and thus $h^0(L(-q))=0$ and $h^0(L(p-q))=1$ so $L\in W_{g-1}\cap (W_{g-1}+(q-p))$ but $L\notin V_q$.

Finally suppose that C is hyperelliptic and $q=\iota(p)$. It suffices to show that $W_{g-1}\cap (W_{g-1}+(q-\iota(q)))=V_q$. This implies that the intersection is irreducible, since V_q is the image of the irreducible variety $C^{(g-2)}$ under the morphism

$$\begin{aligned} C^{(g-2)} &\to W_{g-1} \\ p_1+\cdots+p_{g-2} &\mapsto \mathscr{O}(p_1+\cdots+p_{g-2}+q). \end{aligned}$$

We only show that $W_{g-1}\cap (W_{g-1}+(q-\iota(p))\subset V_q$, the converse implication being obvious. So suppose that $L\in \mathrm{Pic}^{g-1}(C)$ is a line bundle with $h^0(L)>0$ and $h^0(L(\iota(q)-q))>0$. We have to show that $h^0(L(-q))>0$, but this is true since $h^0(L)\geq 2$. On the other hand, $h^0(L)=1$ if and only if we have $L=\mathscr{O}_C(p_1+\cdots+p_{g-1})$ with uniquely determined points p_i with $p_i\neq\iota(p_j)$ for $i\neq j$. The assumption of $h^0(L(\iota(q)-q))>0$ implies that $p_i=q$ for some i. This is clear if $h^0(L(\iota(q)))=1$, but also if $h^0(L(\iota(q)))=2$. Therefore, $h^0(L(-q))>0$.

□

Corollary 2.5.3. *The jacobian variety of any curve C admits a 4-dimensional family of trisecant lines to its Kummer variety.*

Proof. It follows from Theorem 2.5.2 that

$$\Theta\cap\Theta_{q-p}\subset\Theta_{r-p}\cup\Theta_{q-s}$$

for general points $p,q,r,s\in C$, and so by Theorem 2.5.1 we are done. □

CHAPTER 3 Jacobian varieties and trisecant lines

In this chapter we show known work that motivates this thesis. One strong solution to the Schottky problem is the characterization of Jacobian varieties with trisecant lines to its Kummer variety.

3.1 A Curve of trisecants implies $X = J(C)$

We will work over an algebraically closed field k of characteristic 0. Now we will show the basic settings for what follows.

Let (X,L) be an indecomposable principally polarized abelian variety over k, of dimension g. Since $h^0(X,L) = 1$, we can assume that Θ is symmetric. Note that the linear system $|2\Theta|$ is independent of the choice of Θ. By Riemann's Addition Formula 1.8.16 there is a basis of $H^0(X,\mathscr{O}_X(2\Theta))$, namely $\theta_0,\dots,\theta_{2^g-1}$, such that

$$\theta(z+w)\theta(z-w) = \sum_{j=0}^{2^g-2} \theta_j(z)\theta_j(w).$$

The corresponding map into the projective space

$$K : X \to \mathbb{P}^{2^g-1}$$
$$z \mapsto [\theta_0(z) : \cdots : \theta_{2^g-1}(z)]$$

is a 2-1-correspondence onto its image, which is the Kummer variety $K(X)$ of X (see Section 2.5).

Gunning in his paper [22] has shown how to use the trisecant property to the Kummer variety in order to characterize jacobians.

Theorem 3.1.1. *Let X be an indecomposable principally polarized abelian variety over k, and let $Y \subset X$ a reduced artinian ring of length 3, say $|Y| = \{a,b,c\}$. Set $s = a+b+c \in X$. Consider the scheme*

$$V = 2\left\{\zeta \mid \zeta + Y \subset K^{-1}(l)\ \textit{for some line}\ l \subset \mathbb{P}^{2^g-1}\right\}$$

Suppose that V contains some curve Γ and let α be the endomorphism defined[1] *by Γ and Θ. Then*

i) If $-s+Y$ does not meet Γ, then $\alpha|_Y$ is constant.

ii) If $-s+Y$ meets Γ, then $-s+Y \subset \Gamma$, and Γ is smooth at these points; furthermore, $(\alpha - id_X)|_Y$ is constant.

He uses the Matsusaka-Ran Criterion (recall 2.4.1) and some facts about the loci V for jacobians ([23]). This theorem implies:

Corollary 3.1.2 (Gunning [22]). *Let X be an indecomposable principally polarized abelian variety over k, and let $Y \subset X$ be a reduced artinian scheme of length 3, namely $|Y| = \{a,b,c\}$. Suppose that no endomorphism $\beta \neq 0$ exists such that $\beta(b-a) = \beta(c-a) = 0$. If $\dim(V) > 0$ at some point, then V is a smooth irreducible curve and X is the polarized jacobian of V.*

By the same criterion, Welters obtained ([37]):

[1] See Section 1.12.

Corollary 3.1.3. *Let X be an indecomposable principally polarized abelian variety over k. Assume that there exists a complete irreducible curve C in X such that, for some fixed $b, c \in C$, $b \neq c$, $a \in C$ general and $\zeta \in \frac{1}{2}(C - a - b - c)$,*

$$K(\zeta + a), K(\zeta + a) \text{ and } K(\zeta + a)$$

are collinear in $\mathbb{P}^{2^g-1}$. Then C is smooth and X is the polarized jacobian of C.

Welters also extended this result to the case of one dimensional degenerate trisecant families. The main result of Welters that will be used, states that having a one dimensional family of trisecants to the Kummer variety implies that the abelian variety is a jacobian.

Theorem 3.1.4 (Welters [35]). *Let X be an indecomposable principally polarized abelian variety over k, and let $Y \subset X$ be an artinian subscheme of length 3. Assume that the subscheme of X*

$$V = 2\left\{\zeta \in X \mid \zeta + Y \subset K^{-1}(l) \text{ for some line } l \subset \mathbb{P}^{2^g-1}\right\}$$

has positive dimension at some point. Then V is a smooth irreducible curve and X is the polarized jacobian of V.

In the proof of this theorem, there are just three cases for Y which have to be treated separately, namely

$$Y \simeq \sum_{j=1}^{3} \text{Spec}\,(k),$$

$$Y \simeq \text{Spec}\,\left(k[\varepsilon]/(\varepsilon^2)\right) + \text{Spec}\,(k)$$

and

$$Y \simeq \text{Spec}\,\left(k[\varepsilon]/(\varepsilon^3)\right).$$

Welters conjectures the following in [35]: If $K(X)$ has one trisecant, does it follow that X is a jacobian?

Welters' conjecture can be divided into three cases: an honest trisecant, a degenerate trisecant (two intersection points, one of them tangent) or a flex trisecant (total degeneration, that is one point of intersection with multiplicity 3).

3.2 The scheme Σ

In the study of the problem of trisecant lines, from both the algebraic and analytic approaches, there are certain schemes which are usually called *singular loci* Σ, which are a maximal subschemes of the Theta divisor (or a translate of the Theta divisor) invariant under certain translations depending on the case. Krichever in [28] proved, using hard analytic machinery, that Σ is empty in the cases that he studies. On the other hand, in the algebro-geometric approaches the technical complications that envolve this scheme are solved by showing that it is reduced.

Definition 3.2.1. Let (X,Θ) be a principally polarized abelian variety of dimension g and let G be a closed algebraic subgroup of X. We define

$$\Sigma(X,\Theta,G)=\bigcap_{g\in G}\Theta_g.$$

It is the maximal G-invariant subscheme of Θ.

Theorem 3.2.2 (Arbarello-Codogni-Pareschi [3]). *The scheme $\Sigma(X,\Theta,G)$ is reduced for every G.*

Let's review some interesting but not yet useful facts about Σ.

Proposition 3.2.3. *For each integer j, with $0\leq j\leq g-2$, there exist triples (X,Θ,G) such that (X,Θ) is an indecomposable principally polarized abelian variety, G is an abelian subvariety of X and* $\dim\Sigma(X,\Theta,G)=j$.

Proof. See [3]. $\square$

Remark 3.2.4. Note that $\dim\Sigma(X,\Theta,G)\leq g-2$ if G is not trivial. This is because, if $a\in G$ is not the identity, then $\Theta\cap\Theta_a$ has dimension $g-2$ since $\Theta\neq\Theta_a$ if $a\neq 0$.

Arbarello-Codogni-Pareschi in [3] studied some particular cases of Σ:

i) The first case is when G is the Zariski closure of $\langle a\rangle$. In this case, note that the ideal of $\Sigma(X,\Theta,G)$ is generated by $\{\theta_{na} \mid n\in\mathbb{Z}\}$.

ii) The second case is when G is the abelian subvariety generated by a constant vector field D. In this case, G is the smallest abelian subvariety A_D whose tangent space at the origin contains D. In the analytic topology, D corresponds to a non-zero vector U in the universal cover $\pi:\mathbb{C}^g\to X$, and A_D is the closure of the image of the line $U\cdot\mathbb{C}$ in X.

For the last case, we have the following fact:

Proposition 3.2.5 (Arbarello-Codogni-Pareschi [3]). *At each point of $\Sigma=\Sigma(X,\Theta,A_D)$ the ideal of Σ is generated by the functions $\{D^n\theta \mid \text{for every } n\geq 0\}$*

The notation $\Sigma(X,\Theta,D)$ is usually used for $\Sigma(X,\Theta,A_D)$.

Remark 3.2.6. Kempf, in his work in [26], constructed an example of a genus 3 Jacobian $(J(C),\Theta)$ whose theta divisor contains an elliptic curve E. By the work of Beauville-Debarre [6], if D is the tangent vector to the origin of E, we have that $\Sigma(X,\Theta,D)=E$.

Using algebro-geometric methods, we are interested in the following result:

Conjecture 1. Let (X,Θ) be an indecomposable principally polarized abelian variety and $a,b,c\in X$ three different points, with $a,b,c\notin X[2]$, and consider $G=\langle a-b,a-c\rangle^{-}$. Suppose that $K(a),K(b),K(c)$ are collinear in $\mathbb{P}^{2^g-1}$. Then

$$\operatorname{codim}_X\Sigma(X,\Theta,G)\geq 3.$$

If this is true, via algebro-geometric methods, one immediately solves the honest case of Welters' conjecture using the following result:

Theorem 3.2.7 (Debarre [10]). *Let (X,λ) be an indecomposable principally polarized abelian variety, let Θ be the symmetric representative of the polarization λ. Suppose that there exist* *points $a, b, c \in X$ such that $K(a), K(b), K(c)$ are collinear and that*

$$\text{codim} \bigcap_{\substack{p,q,r \in \mathbb{Z} \\ p+q+r=0}} \Theta_{pa+qb+rc} \geq 3.$$

Then, (X, λ) is isomorphic to the jacobian of a smooth algebraic curve.

We end this section, showing an example from [3].

Example 3.2.8. Let C be a smooth algebraic curve of genus g, and $X = J(C)$ its jacobian variety. Recall the Abel-Jacobi map $u : C \to X$ which is an immersion; set $\Gamma = u(C)$. Fix $p, q, r, s \in C$, and set:

$$\alpha = u(p), \beta = u(q), \gamma = u(r), x = 2\zeta = u(s) - u(p) - u(q) - u(r)$$

meaning that $\zeta \in \frac{1}{2}(\Gamma - \alpha - \beta - \gamma)$, *i.e.* ζ is in the preimage of $\Gamma - \alpha - \beta - \gamma$ under the multiplication by 2 map. Also set

$$a = \zeta + \alpha, b = \zeta + \beta, c = \zeta + \gamma.$$

If $K : X \to K(X)$ is the Kummer morphism, then $K(a), K(b), K(c)$ are collinear, and we have for some not all zero constants A, B, C:

$$A\theta_a\theta_{-a} + B\theta_b\theta_{-b} + C\theta_c\theta_{-c} = 0.$$

To have a corresponding notation that Krichever uses in his papers (for instance [27], [28]), and from the publication that is the origin of this example [3], we set:

$$U = a - c = u(p) - u(r), V = b - c = u(q) - u(r), U - V = a - b = u(p) - u(q), W = u(s) - u(r)$$

so that

$$a = \frac{1}{2}(U + W - V), b = \frac{1}{2}(V - U + W), c = \frac{1}{2}(W - U - V).$$

Set

$$G = \langle U - V \rangle, G' = \langle U \rangle, G'' = \langle V \rangle$$

We show that, for an appropiate choice of C, and of the points p, q, r, s, we get:

$$\dim \Sigma(\Theta, G) > 0, \quad \Sigma(\Theta, G') = \Sigma(\Theta, G'') = \emptyset \tag{3.1}$$

Note that we implicitly assumed that G is not finite.

Let C be a genus g curve that can be expressed as an n-sheeted cover of $\mathbb{P}^1$ with at least two points of total ramification: p and q, and assume that $n < g$. Hence:

$$n(U - V) = n[u(p) - u(q)] = 0.$$

By taking general points of C, we get that both U and V generate $J(C)$. It follows that $\Sigma(\Theta, G') = \Sigma(\Theta, G'') = \emptyset$. On the other hand, $\Sigma(\Theta, G)$ is the interection of n translated of Θ so that $\dim \Sigma(\Theta, G) \geq g - n > 0$.

3.3 One trisecant implies $X = J(C)$

All of the three cases of secancy that must be dealt with in Welters' conjecture have some issue with a bad locus Σ. The two degenerate ones were solved in [3] by Arbarello, Codogni and Pareschi using algebraic methods; while the honest case (this is when Y is reduced) was solved in [28] by Krichever using complicated analytic methods. We start showing the results that precede the results in [3], particularly the case of $Y_{\text{red}} = \{u, v\}$. We note that there is still no algebro-geometric proof of Welters' conjecture for the honest case.

3.3.1 An inflectionary tangent to the Kummer variety

The case of $Y \simeq \text{Spec}\left(k[\varepsilon]/(\varepsilon^3)\right)$ was solved by Marini in [30] under an extra hypothesis; *loc. cit.* requires that a subscheme $D_1\Theta$ of Θ does not contain set-theorical D_1-invariant components. This is an improvement of Arbarello's previous work [2]. Later, this extra hypothesis was removed as we will see in this section.

Marini gives a characterization of the trisecant condition in this case, as follows:

Proposition 3.3.1 (Marini [30]). *Let (X,L) be an indecomposable principally polarized abelian variety of dimension $g>1$. The following conditions are equivalent:*

i) The Kummer variety $K(X)$ has an inflectionary trisecant, i.e. *there exists a smooth point $K(u)$ in $K(X)$, where $K : X \to |2\Theta|^*$ is the Kummer morphism, and a line l in the projective space $P^N = |2\Theta|^*$ which meets $K(X)$ at $K(u)$ with at least multiplicity* 3.

ii) There exist invariant vector fields $D_1 \neq 0$, $D_2 \in \mathbb{C}^g$, a complex number c and a point $u \in X$ such that

$$D_1^2\theta \cdot \theta_{2u} - 2D_1\theta \cdot D_1\theta_{2u} + \theta \cdot D_1^2\theta_{2u} + D_2\theta \cdot \theta_{2u} - \theta \cdot D_2\theta_{2u} + c\theta\theta_{2u} = 0. \quad (3.2)$$

iii) There exists a constant vector field $D_1 \neq 0$ and a point $u \in X$ such that the following inclusion holds

$$\Theta \cap \Theta_{2u} \subset D_1\Theta \cup D_1\Theta_{2u}, \quad (3.3)$$

where $D_1\Theta$ is the locus of zeroes of the section $D_1\theta \in H^0(\Theta, \mathscr{O}_\Theta(\Theta))$.

Recall that $K(u)$ is a smooth point of $K(X)$ if and only if $2u \neq 0$. Hence, the main result of Marini is the following:

Theorem 3.3.2. *Let X be an indecomposable principally polarized abelian variety. Assume that the Kummer variety $K(X)$ has an inflectionary trisecant (see the Proposition 3.3.1) and that the scheme $D_1\Theta$ does not contain set-theoretical D_1-invariant components. Then, X is the Jacobian of a smooth curve C.*

Years later, the inflectionary tangent case was solved using analytic methods in [27] by Krichever removing the hypothesis of $\langle a \rangle$ being irreducible for $a \neq 0$. We present his result as follows:

Theorem 3.3.3 (Arbarello-Krichever-Marini [4]). *Let (X,Θ) be an indecomposable, principally polarized abelian variety. Then X is the Jacobian of a curve of genus g if and only if there exist vectors $U \neq 0, V \in \mathbb{C}^g$ (or equivalently constant vector fields $D_1 \neq 0$, D_2, on X) and a point $a \in X \setminus \{0\}$, with $\langle a \rangle$ irreducible, such that: There is a point $b \in X$ with $2b \neq a \neq 0$ such that $K(b)$ is a flex of the Kummer variety.*

Remark 3.3.4. From [30] and [2] recall that $K(b)$ being a flex of the Kummer variety is equivalent by definition to having $2b \in V_Y$, where

$$V_Y = \left\{2b \in X \mid b+Y \subset K^{-1}(l) \text{ for some line } l \subset \mathbb{P}^{2^g-1}\right\}$$

and Y the artinian subscheme of length 3 of X associated to the second order germ:

$$Y : \varepsilon \mapsto \varepsilon 2U + \varepsilon^2 2V \subset X.$$

Now, simply removing the extra hypothesis, Krichever got:

Theorem 3.3.5 (Krichever [27]). *Let (X,Θ) be an indecomposable, principally polarized abelian variety. Then X is the Jacobian of a curve of genus g if and only if there exist vectors $U \neq 0, V \in \mathbb{C}^g$ (or equivalently constant vector fields $D_1 \neq 0$, D_2, on X) such that there is a point $b \in X$ with $2b \neq 0$ such that $K(b)$ is a flex of the Kummer variety.*

An algebro-geometric proof of this result is due to Arbarello, Codogni and Pareschi in [3]. They proved and used that Σ is reduced (see Theorem 3.2.2) to remove the extra condition that Debarre had in [9] using some ideas due to Marini (see [30]).

3.3.2 A degenerate trisecant of the Kummer Variety

Now we review the case $Y \simeq \operatorname{Spec}\left(k \times k[\varepsilon]/(\varepsilon^2)\right)$. The ideas that were used to solve this case can be generalized later in Chapter 4 (see [9]).

First let's see the equivalence between the trisecant conditions. See [9] for more details.

Proposition 3.3.6. *Let (X,L) be an indecomposable principally polarized abelian variety of* *dimension $g \geq 1$, and let Θ be a symmetric representative of the polarization L. The following conditions are equivalent:*

i) The Kummer variety $K(X)$ has a degenerate trisecant, in other words: there exist smooth points $K(u)$ and $K(v)$ in $K(X)$, and a line l in the projective space $\mathbb{P}^{2^g-1}$ which meets $K(X)$ passing through $K(v)$ and being tangent at $K(u)$.

ii) There exists an invariant vector field $D_1 \neq 0$ in $\mathbb{C}^g$ and complex numbers η, ν and points $u, v \in X$ such that

$$\eta\,\theta_u\theta_{-u} + (D_1\theta_{-u})\cdot\theta_u - (D_1\theta_u)\cdot\theta_{-u} + \nu\theta_v\theta_{-v} = 0.$$

iii) There is a constant vector field $D_1 \neq 0$ in $\mathbb{C}^g$ and points $u, v \in X$ such that the following inclusion holds

$$\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}.$$

Theorem 3.3.7 (Debarre [9]). *Let X be a complex indecomposable principally polarized abelian variety and let Θ be a symmetric representative of the polarization. Then, X is a jacobian if and only if there exist points $u, v \in X$ with $u, -u, v, -v$ all distinct, such that*

$$\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}$$

and $\Theta_u \cdot \Theta_{-u}$ reduced.

Krichever in [27] using analytic methods and Arbarello, Codogni and Pareschi in [3] using algebro-geomtric methods proved Theorem [9] without requiere that $\Theta_u \cdot \Theta_{-u}$ is reduced.

3.3.3 The Honest case

As mentioned in Section 3.2, if one can prove that $\Sigma = \operatorname{codim} \bigcap_{\substack{p,q,r \in \mathbb{Z} \\ p+q+r=0}} \Theta_{pa+qb+rc} \geq 3$ in the context of 3.2.7, then one has solved Welters' conjecture for the case of honest trisecants.

Krichever has gone much further using analytic theory that has been developed in the last 20 years, including soliton equations, τ functions, commuting difference operators, etc. He proved in [28] that Σ is empty in the case of

$$\Sigma = \bigcap_{t \in \mathbb{Z}} \Theta_{tu}.$$

We reformulate the theorem to be in the language of this thesis, which is of course the algebro-geometric setting.

Theorem 3.3.8 (Krichever [28]). *Let (X, Θ) be an indecomposable principally polarized abelian variety, with Θ symmetric. Suppose that X has points a, b, c with $2a, 2b, 2c, a-b, a-c, b-c$ non-zero, such that $K(a), K(b), K(c)$ are collinear in $\mathbb{P}^{2^g-1}$. Then X is a jacobian variety.*

CHAPTER 4 $(m+2)$-secant m-planes

4.1 History and motivation

The origin of the problem of secants to the Kummer variety comes from the Schottky problem, that is to determine which complex principally polarized abelian varieties arise as jacobian varieties of complex curves. One known geometrical solution is presented in Chapter 3. There are two possible paths to follow in order to generalize this work: One is to try to characterize jacobian varieties by higher secancy conditions as in Grushevsky's approach in [19] or Pareschi and Popa's in [34].

The other path is to follow Debarre's question in [11]: He asks about the relation between the following conditions for a p.p.a.v. (X,Θ):

1) The existence of a curve of m-planes $(m+2)$-secant to the Kummer variety.

2) The existence of one m-plane $(m+2)$-secant to the Kummer variety of X.

3) The existence of a curve in X that is m times the minimal class.

4) The singular locus of Θ is of dimension at least $g-2m-2$.

5) The variety (X,Θ) is the Prym-Tyurin variety associated to a symmetric effective correspondence D without fixed points that verifies $(D-1)(D+m-1)=0$.

Due to Welters in [36], Bertram in [7] and Debarre in [11], under certain geometrical conditions one has the following relations:

$$\begin{array}{ccccc} 1) & \Longrightarrow & 2) & \Longrightarrow & 4) \\ \Downarrow & & & & \Uparrow \\ 3) & & \Longleftarrow & & 5) \end{array}$$

Our motivation is to improve Debarre's work on 1) $\Rightarrow$ 3) by removing some geometric conditions, and to give a partial answer to 2) $\Rightarrow$ 1).

In this chapter we present the *minimal cohomological class*, in Theorem 4.3.2 we improve Debarre's Theorem 4.1 in [11] removing the general position condition and the hypothesis of the ring of endomorphisms of the p.p.a.v. to be $\mathbb{Z}$ (that is a proof of 1) $\Rightarrow$ 3)). Also, in Theorem 4.4.4 we give a similar condition to 2) that under certain geometric conditions imply 1). Finally, we show the particular case of quadrisecant planes: in Theorem 4.5.1 we have an improvement of Debarre's Theorem 5.1 in [11] and in Theorem 4.5.4 we see the particular case of two quadrisecant planes implying the existence of a curve of quadrisecant planes under certain geometrical conditions.

4.2 The minimal cohomological class

To understand condition 3) shown in Section 4.1 we briefly present the *minimal cohomological class*.

Recall from Chapter 2 that for a Jacobian $J(C)$ one can naturally map the symmetric product $\mathrm{Sym}^d(C)$ to $J(C)=\mathrm{Pic}^{g-1}(C)$ (for $1\leq d<g$, where g is the genus of C) by fixing a divisor $D\in\mathrm{Pic}^{g-1-d}(C)$ and mapping $(p_1,\ldots,p_d)\mapsto D+\sum_{i=1}^{d}p_i$. The image of such a map, denoted $W^d\subset J(C)$, is independent of D up to a translation and one can compute its cohomology class. By the Geometrical Riemann-Roch Theorem 1.9.3 we have:

$$\left[W^d\right]=\frac{[\Theta]^{g-d}}{(g-d)!}\in H^{2g-2d}(J(C)), \tag{4.1}$$

where by $[\Theta]$ we denote the cohomology class of the polarization. One can show that this cohomology class is indivisible in cohomology with coefficients in $\mathbb{Z}$ (see [1, p. 25]). Since, in particular $W^1 \simeq C$, one can ask whether the existence of a curve that is the minimal class is a special property of Jacobians, and one has one implication by the Matsusaka-Ran Criterion 2.4.1 and another summarized in Chapter 3.

This motivates the following definition:

Definition 4.2.1. Let (X,Θ) be a principally polarized abelian variety. We call

$$\frac{[\Theta]^{g-1}}{(g-1)!}$$

the *minimal cohomology class* of Θ (or (X,Θ)).

4.3 A curve of $(m+2)$-secant m-planes implies that X has a curve m times the minimal class

Let (X,λ) be an indecomposable principally polarized abelian variety of dimension g, let Θ be a symmetric representative of the polarization λ. We can take a basis $\{\theta\}$ of $H^0(X,\mathscr{O}_X(\Theta))$ and a basis $\{\theta_0,\ldots,\theta_N\}$ (with $N=2^g-1$) of the 1-dimensional space $H^0(X,\mathscr{O}_X(2\Theta))$ which satisfies the addition formula:

$$\theta(z+w)\theta(z-w)=\sum_{j=0}^{N}\theta_j(z)\theta_j(w),$$

for all z,w. Now we identify $K=[\theta_0:\cdots:\theta_N]$. For any $x\in X$, we write Θ_x for the divisor $\Theta+x$ and θ_x for the section $z\mapsto\theta(z-x)$ of $H^0(X,\mathscr{O}_X(\Theta_x))$.

Recall that an Artinian scheme of length n is the spectrum of an artinian ring of length n.

Definition 4.3.1. Let $Y\subset X$ be an Artinian subscheme of length $m+2$. Define the scheme:

$$V_Y=2\left\{\zeta\ :\ \exists W\in\mathbb{G}(m,2^g-1)\ \text{such that}\ \zeta+Y\subset K^{-1}(W)\right\}.$$

We will say that X satisfies the $(m+2)$-secant (m-planes) condition if there exists Y and $\zeta \in V_Y$ such that K restricted to $\zeta + Y$ is an embedding.
Also, we say that X has a curve of $(m+2)$-secant (m-planes) if V_Y contains a curve of points whose satisfy the $(m+2)$-secant (m-planes) condition with m minimal. That is, for a point 2ζ inside such curve in, X satisfies the $(m+2)$-secant condition but does not satisfy the $(n+2)$-secant condition for $n < m$.

The purpose of this section is to prove the following result:

Theorem 4.3.2. *Let (X, λ) be an indecomposable principally polarized abelian variety of dimension g, let Θ be a symmetric representative of the polarization λ. Let $Y = \{a_1, \dots, a_{m+2}\}$ be a reduced subscheme of points of X that do not have order 2. Suppose that X has a curve of $(m+2)$-secant m-planes and assume that $\Theta_{a_1} \cap \cdots \cap \Theta_{a_{m+2}}$ is a complete intersection. Suppose further that $(a_1 - a_j)_j$ are $\mathbb{Z}$-linearly independent. Then, the class of Γ is m times the minimal cohomology class of Θ.*

Proof. Since we have that the points $K(\zeta + a_1), \dots, K(\zeta + a_{m+2})$ lie on an m-plane, we have that there exist constants α_j such that

$$\sum_{i=1}^{m+2} \alpha_i \theta_j(\zeta + a_i) = 0$$

Multiplying by $\theta_j(z+\zeta)$ and taking the sum over j we get

$$\sum_{i=1}^{m+2} \alpha_i \theta(z + 2\zeta + a_i)\theta(z - a_i).$$

In particular, restricting to $\Theta_{a_3} \cap \cdots \cap \Theta_{a_{m+2}}$ we obtain

$$\alpha_1 \theta_{-2\zeta - a_1} \theta_{a_1} + \alpha_2 \theta_{-2\zeta - a_2} \theta_{a_2} = 0,$$

and since this is true for all $\zeta \in \frac{1}{2}\Gamma$, it is true for $\zeta' \in \frac{1}{2}\Gamma$:

$$\alpha_1' \theta_{-2\zeta' - a_1} \theta_{a_1} + \alpha_2' \theta_{-2\zeta' - a_2} \theta_{a_2} = 0.$$

Multiplying the first equation by $\alpha_2'\theta_{-2\zeta'-a_2}\theta_{a_2}$ and the second one by $\alpha_2\theta_{-2\zeta-a_2}\theta_{a_2}$ and substracting we get

$$\alpha_1\alpha_2'\theta_{-2\zeta-a_1}\theta_{-2\zeta'-a_2}\theta_{a_1}\theta_{a_2} - \alpha_1'\alpha_2\theta_{-2\zeta'-a_1}\theta_{-2\zeta-a_2}\theta_{a_1}\theta_{a_2} = 0.$$

Factorize by $\theta_{a_1}\theta_{a_2}$, denote the remaining term as A. We get that

$$\Theta_{a_3}\cap\cdots\cap\Theta_{a_{m+2}} \subset \{A=0\}\cup\Theta_{a_1}\cup\Theta_{a_2}.$$

If $W\subset\Theta_{a_3}\cap\cdots\cap\Theta_{a_{m+2}}$ is an irreducible component such that W is contained in either Θ_{a_1} or Θ_{a_2} we will contradict that $\Theta_{a_1}\cap\cdots\cap\Theta_{a_{m+2}}$ is a complete intersection, since $\mathrm{codim}_X W = m$ by the first contention, while $\mathrm{codim}_X W = m+1$ by the second one. Therefore,

$$\Theta_{a_3}\cap\cdots\cap\Theta_{a_{m+2}} \subset \{A=0\},$$

and so,

$$\Theta_{a_3}\cap\cdots\cap\Theta_{a_{m+2}}\cap\Theta_{-2\zeta-a_2} \subset \Theta_{-2\zeta-a_1}\cup\Theta_{-2\zeta'-a_2}.$$

Equivalently, this means that the points

$$K(\zeta+b_1),\cdots,K(\zeta+b_{m+2})$$

lie on an m-plane, where

$$2b_1 = 2\zeta'+2a_3+a_1+a_2$$
$$b_j = b_1-a_3+a_{j+2}$$
$$b_{m+1} = a_2+a_3-b_1$$
$$b_{m+2} = a_1+a_3-b_1$$

for $2\le j\le m$.

In particular, we have a two dimensional family of secant m-planes, since now the b_i

depend on a one dimensional parameter ζ'.

Let α be the endomorphism associated to Γ and Θ. We will prove under the above hypotheses that $\alpha = m \cdot \mathrm{id}$, and so Γ is m times the minimal class. In what comes next we will follow [11, Section 4]. First we will show that $\alpha \in \mathbb{Z}$ and then we compute that $\alpha = m$. Let N be the normalization of Γ, $\tilde{\Gamma} \subset 2^{-1}(\Gamma)$ an irreducible complete curve and $\tilde{N}$ its normalization. We have the following commutative diagram

$$\begin{tikzcd} \tilde{N} \arrow[r, "\psi"] \arrow[d, "\tilde{\pi}"'] & N \arrow[r] \arrow[d, "\pi"'] & J(N) \arrow[d, "\gamma"] \\ \tilde{\Gamma} \arrow[r, "2"'] & \Gamma \arrow[r] & X \end{tikzcd}$$

where γ is the homomorphism induced by π. We observe that ψ is a Galois morphism with Galois group isomorphic to those 2-torsion points of X that leave $\tilde{\Gamma}$ invariant.

By [11, Lemme 3.7] there exist sections

$$\lambda_j \in H^0\left(\tilde{N}, \tilde{\pi}^* \mathscr{O}_X\left(2\Theta_{-b_1} + \cdots + 2\Theta_{-b_{m+2}} - 2\Theta_{-b_j}\right)\right)$$

such that for all $j = 1, \ldots, 2^g$, $\sum_{i=1}^{m+2} \lambda_i \pi^* t_{b_i}^* \theta_j = 0$ on $\tilde{N}$. Now each section λ_i is stable by the Galois group of ψ and so $\mathrm{div}(\lambda_i) = \psi^* E_i$ for some divisor E_i on N. Note that by the Theorem of the Square, $\mathscr{O}_\Gamma\left(\Theta_{b_i - b_j}\right) \simeq \mathscr{O}_\Gamma\left(E_i - E_j\right)$. Hence, if $\mathscr{S} : \mathrm{Div}(N) \to X$ is the pushforward by π composed with the addition map of 0-cycles, we have that

$$\alpha_\Gamma\left(b_i - b_j\right) = \mathscr{S}\left(E_i - E_j\right).$$

Moreover, λ_i vanishes at a point ζ only if the points belong to a trisecant, unless they coincide, *i.e.* $\zeta + b_p = \zeta_q + b_q$ for $j \notin \{p, q\}$, that is, the image in X is $2\zeta = -b_p - b_q$. Hence

$$\mathscr{S}(E_i - E_j) = \sum_{\substack{p,q \neq i \\ p<q}} \delta_{pq}\left(\zeta'\right)\left(-b_p - b_q\right) - \sum_{\substack{p,q \neq j \\ p<q}} \delta_{pq}\left(\zeta'\right)\left(-b_p - b_q\right)$$

for certain integers $\delta_{pq}\left(\zeta'\right) \geq 0$. Since the points $b_p - b_q$ for $1 \leq p < q \leq m$ and $-b_{m+1} - b_{m+2}$ depend on ζ', we can choose ζ' so that $b_p - b_q$ and $-b_{m+1} - b_{m+2}$ are either not on Γ or are smooth points of Γ. This way, we get that $\delta_{pq} \in \{0,1\}$.

Note that $-b_p - b_q$ is constant if $p < m+1$ and $q \geq m+1$. So that,

$$\alpha\left(2\zeta'\right) = 2\zeta' M + \text{constant}$$

where[1] $M = \left(\sum_{\substack{1\leq p<q\leq m \\ p \text{ or } q \text{ equals } j}} \delta_{pq} + \sum_{\substack{1\leq p<q\leq m \\ p \text{ or } q \text{ equals } i}} \delta_{pq}\right)$.

In particular, for a general $z \in \Gamma$ we have that $\alpha(z) = Mz$. Since Γ generates X we get $\alpha = M \in \mathbb{Z}$, and so α is M times the identity. Now, we show that $M = m$ using an argument due to Debarre [11].

Denote by M the cardinality of the set:

$$\left\{p \neq j \,:\, (-a_j - a_p) \in \Gamma\right\}.$$

Analogous to the previous computations, since $K(\zeta + a_1), \ldots, K(\zeta + a_{m+2})$ lie on an m-plane, we have that

$$\begin{aligned}
\alpha(a_i - a_j) &= \sum_{\substack{p,q\neq i \\ p<q \\ -a_p-a_q\in\Gamma}} (-a_p - a_q) - \sum_{\substack{p,q\neq j \\ p<q \\ -a_p-a_q\in\Gamma}} (-a_p - a_q) \\
&= -\sum_{\substack{p\neq i,j \\ -a_j-a_p\in\Gamma}} (a_j + a_p) + \sum_{\substack{q\neq i,j \\ -a_i-a_p\in\Gamma}} (a_i + a_q).
\end{aligned}$$

Since $\alpha = M \in \mathbb{Z}$, then

$$N(a_i - a_j) = \begin{cases} M(a_i - a_j) + \sum_{\substack{p\neq i,j \\ -a_i-a_p}} a_p - \sum_{\substack{q\neq i,j \\ -a_j-a_q}} a_q & \text{if } -a_i - a_j \notin \Gamma \\ (M-1)(a_i - a_j) + \sum_{\substack{p\neq i,j \\ -a_i-a_p}} a_p - \sum_{\substack{q\neq i,j \\ -a_j-a_q}} a_q & \text{if } -a_i - a_j \in \Gamma. \end{cases}$$

Since $(a_1 - a_j)$ are $\mathbb{Z}$-linearly independent, this leads to two cases:

- Let $N = M$ and no $(a_i - a_j)$ lie on Γ. This implies that M is null, and so is α which is absurd.

[1] For simplicity one can think of $i = 1$ and $j = m$.

- Let $N = M-1$ and $(a_i - a_j) \in \Gamma$ for all i, j. Then, $M = m+1$ and $\alpha = m$.

Therefore, Γ is m times the minimal class.

□

4.4 Finite $(m+2)$-secant m-planes

One may ask if the existence of a finite number of $(m+2)$-secant m-planes is enough to have a curve of $(m+2)$-secant m-planes as in the case of Jacobians for $m=1$ (recall Chapter 3). The answer is positive under certain geometric conditions. In this section we show that the existence of one degenerate $(m+2)$-secant m-plane and $(m-1)$ honest (reduced) $(m+2)$-secant m-planes implies the existence of a curve of $(m+2)$-secant m-planes.

Before showing the main theorem of this section we will treat some preliminaries. We base this on Debarre's work in [9].

4.4.1 Preliminaries

Let (X, λ) be an indecomposable principally polarized abelian variety of dimension g, let Θ be a symmetric representative of the polarization λ. We can take a basis $\{\theta\}$ of $H^0(X, \mathscr{O}_X(\Theta))$ and a basis $\{\theta_0, \ldots, \theta_N\}$ (with $N = 2^g - 1$) of $H^0(X, \mathscr{O}_X(2\Theta))$ which satisfy the addition formula:

$$\theta(z+w)\theta(z-w) = \sum_{j=0}^{N} \theta_j(z)\theta_j(w),$$

for all z, w. Now we identify $K = [\theta_0 : \cdots : \theta_N]$. For any $x \in X$, we write Θ_x for the divisor $\Theta + x$ and θ_x for the section $z \mapsto \theta(z-x)$ of $H^0(X, \mathscr{O}_X(\Theta_x))$.

Recall from Section 4.3 the scheme V_Y. In particular, if $Y = \{a_1, \ldots, a_{m+2}\}$ has reduced structure we can write

$$V_Y = \{2\zeta \,:\, K(\zeta + a_1) \wedge \cdots \wedge K(\zeta + a_{m+2}) = 0\}. \tag{4.2}$$

Since K is an even function[2] we have that $-a_1 - a_2 \in V_Y$.

[2]One way to see this is that $K : X \to \mathbb{P}^{2^g-1}$ can be defined as $x \mapsto \Theta_x + \Theta_{-x}$, from X to $|2\Theta|^* = \mathbb{P}^{2^g-1}$.

4.4.2 Hierarchy

Our intention is to translate the condition $\dim_{-a_1-a_2} V_Y > 0$ into an infinite set of equations. We will look for a smooth germ with $D_1 \neq 0$.

The condition $\dim_{-a_1-a_2} V_Y > 0$ equals the existence of a formal curve

$$2\zeta(\varepsilon) = -a_1 - a_2 + C(\varepsilon), \quad \text{with } C(\varepsilon) = \sum_{j\geq 1} W^{(j)}\varepsilon^j$$

contained in V_Y with $W^{(j)} = \left(W_1^{(j)}, \dots, W_g^{(j)}\right) \in \mathbb{C}^g$. We define the differential operators

$$D_j = \sum_{i=1}^{g} W_i^{(j)} \frac{\partial}{\partial z_i}$$

and set

$$D(\varepsilon) = \sum_{j\geq 0} D_j \varepsilon^j.$$

Also set,

$$\Delta_s = \sum_{i_1+2i_2+\cdots+si_s=s} \frac{1}{i_1!\cdots i_s!} D_1^{i_1}\cdots D_s^{i_s}.$$

Then, for a C^∞ function $f(z)$ in $\mathbb{C}^g$ one has

$$f(C_0 + C(\varepsilon)) = \sum_{s\geq 0} \Delta_s(f)|_{\zeta=C_0}\varepsilon^s. \tag{4.3}$$

So that

$$e^{D(\varepsilon)} = \sum_{s\geq 0} \Delta_s(\cdot)|_{\zeta=C_0}\varepsilon^s. \tag{4.4}$$

Therefore, the existence of such a formal curve turns into the following equivalent relation:

$$\sum_{j=1}^{m+2} \alpha_j(\varepsilon) K\left(a_j + \zeta(\varepsilon)\right) = 0. \tag{4.5}$$

Here $\alpha_j(\varepsilon)$, $1 \le j \le m$ are relatively prime elements in $\mathbb{C}[[\varepsilon]]$. Set $u = a_1 - \frac{1}{2}(a_1 + a_2)$ and $b_j = a_{j+2} - \frac{1}{2}(a_1 + a_2)$ for $1 \le j \le m$. Now, equation 4.5 becomes:

$$\alpha_1(\varepsilon) K\left(u + \frac{1}{2}C(\varepsilon)\right) + \alpha_2(\varepsilon) K\left(-u + \frac{1}{2}C(\varepsilon)\right) + \sum_{j=1}^{m} \alpha_{j+2}(\varepsilon) K\left(b_j + \frac{1}{2}C(\varepsilon)\right) = 0 \tag{4.6}$$

Taking dot product with the left side of 4.6, using the Riemann addition formula we get:

$$\alpha_1(\varepsilon)\theta\left(z + u + \frac{1}{2}C(\varepsilon)\right)\theta\left(z - u - \frac{1}{2}C(\varepsilon)\right) + \alpha_2(\varepsilon)\theta\left(z - u + \frac{1}{2}C(\varepsilon)\right)\theta\left(z + u - \frac{1}{2}C(\varepsilon)\right)$$
$$+ \sum_{j=1}^{m} \alpha_{j+2}(\varepsilon)\theta\left(z + b_j + \frac{1}{2}C(\varepsilon)\right)\theta\left(z - b_j - \frac{1}{2}C(\varepsilon)\right) = 0 \tag{4.7}$$

Write $P(z,\varepsilon) = \sum_{s \ge 0} P_s(z)\varepsilon^s$ for the left side of the equation. Note that $P_s \in H^0(X, \mathscr{O}_X(2\Theta))$. We have that

$$P_0 = (\alpha_1(0) + \alpha_2(0))\,\theta_u\theta_{-u} + \sum_{j=1}^{m} \alpha_{j+2}(0)\theta_{b_j}\theta_{-b_j}.$$

Since there are no $(n+2)$-secant n-planes, with $n < m$, P_0 vanishes if and only if $\alpha_1(0) + \alpha_2(0) = 0$ and $\alpha_j(0) =$ for any $3 \le j \le m+2$. Given that $\alpha_1(\varepsilon), \ldots, \alpha_{m+2}(\varepsilon)$ are relatively prime, we get that $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ are units. We may assume:

$$\alpha_1(\varepsilon) = 1 + \sum_{i \ge 1} \alpha_{1,i}\varepsilon^i, \qquad \alpha_2(\varepsilon) = -1,$$

$$\alpha_j(\varepsilon) = \sum_{i \ge 1} \alpha_{j,i}\varepsilon^i, \text{ for } j \ge 2.$$

Observe that

$$P_1 = \alpha_{1,1}\theta_u\theta_{-u} + D_1\theta_{-u}\cdot\theta_u - D_1\theta_u\theta_{-u} + \sum_{j=1}^{m} \alpha_{j+2,1}\theta_{\beta_j}\theta_{-\beta_j}$$

where $D_1 \ne 0$ and $2u \ne 0$. Recall that we are assuming that there are no $(n+2)$-secant n-planes, with $n < m$, so $\alpha_{j+2,1} \ne 0$ for any $3 \le j \le m+2$. Allowing linear changes in the D_i

operators, we may assume that $\alpha_{m+2}(\varepsilon)=\varepsilon$. We state this procedure as follows:

Theorem 4.4.1. *The abelian variety X satisfies $\dim_{-a_1-a_2} V_Y > 0$ if and only if there exist complex numbers $\alpha_{j,i}$, with $1 \leq j \leq m+1$, $j \neq 2$, $i \geq 1$; and constant vector fields $D_1 \neq 0, D_2, \ldots$ on X such that the sections P_s vanish for all positive integers s.*

Remark 4.4.2. Note that P_s only depends on the corresponding $\alpha_{j,i}$, and D_i with $i \leq s$. Hence, we can write P_s as:

$$P_s = Q_s + \alpha_{1,s}\theta_u\theta_{-u} + D_s\theta_{-u}\cdot\theta_u - D_s\theta_u\cdot\theta_{-u} + \sum_{j=1}^{m-1}\alpha_{j+2,s}\theta_{b_j}\theta_{-b_j}$$

where Q_s does not depend on $\alpha_{j,s}$ nor D_s. Furthermore, $Q_s \in H^0(X, \mathscr{O}_X(2\Theta))$.

Lemma 4.4.3. *Suppose that $\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\ldots\cdot\Theta_{b_{m-1}}$ is a complete intersection. Then, the section $P_s \in H^0(X, \mathscr{O}_X(2\Theta))$ vanishes for some choice of α_j, s and D_s for $j \in \{1,3,4,\ldots,m+1\}$ if and only if the restriction of Q_s to $\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\ldots\cdot\Theta_{b_{m-1}}$ does.*

Note that $\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\ldots\cdot\Theta_{b_{m-1}}$ is just a translate of $\Theta_{a_1}\cdot\Theta_{a_2}\cdot\ldots\cdot\Theta_{a_{m+1}}$.

Proof. Consider the ideal sheaf exact sequence

$$0 \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}}(-\Theta_{b_{m-1}}) \xrightarrow{\cdot\theta_{b_{m-1}}} \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}} \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_w\cdot\Theta_{b_1}\cdots\Theta_{b_{m-1}}} \longrightarrow 0$$

we apply $\otimes\mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}}(2\Theta)$, and using the Theorem of the Square 1.5.4 we get $2\Theta \sim \Theta_w + \Theta_{-w}$, so that

$$0 \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}}(\Theta_{-b_{m-1}}) \xrightarrow{\cdot\theta_{b_{m-1}}} \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}}(2\Theta) \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_w\cdot\Theta_{b_1}\cdots\Theta_{b_{m-1}}}(2\Theta) \longrightarrow 0$$

Then, using the cohomology of this exact sequence (see Section A.1) we get that Q_s vanishes on $\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-1}}$ if and only if there exists a complex number $\alpha_{m+1,s}$ such that

$$\left(Q_s + \alpha_{m+1,s}\theta_{b_{m-1}}\theta_{-b_{m-1}}\right)\big|_{\Theta_u\cdot\Theta_{-u}\cdot\Theta_{b_1}\cdots\Theta_{b_{m-2}}} = 0.$$

We repeat the process until we get that

$$\left(Q_s + \sum_{j=1}^{m-1} \alpha_{j+2,s}\theta_{b_j}\theta_{-b_j}\right)\Bigg|_{\Theta_u\cdot\Theta_{-u}} = 0.$$

Now, as before we consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{\Theta_u}(-\Theta_{-u}) \xrightarrow{\cdot\theta_{-u}} \mathscr{O}_{\Theta_u} \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}} \longrightarrow 0,$$

we apply $\otimes\mathscr{O}_{\Theta_u}(2\Theta)$, and using the Theorem of the Square 1.5.4 we get $2\Theta \sim \Theta_u + \Theta_{-u}$, so that

$$0 \longrightarrow \mathscr{O}_{\Theta_u}(\Theta_u) \xrightarrow{\cdot\theta_{-u}} \mathscr{O}_{\Theta_u}(2\Theta) \longrightarrow \mathscr{O}_{\Theta_u\cdot\Theta_{-u}}(2\Theta) \longrightarrow 0.$$

Take the long exact sequence in cohomology[3]

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(\Theta_u, \mathscr{O}_{\Theta_u}(\Theta_u)) & \xrightarrow{\cdot\theta_{-u}} & H^0(\Theta_u, \mathscr{O}_{\Theta_u}(2\Theta)) & \longrightarrow & H^0\left(\Theta_u, \mathscr{O}_{\Theta_u\cdot\Theta_{-u}}(2\Theta)\right) & \longrightarrow \cdots \\ & \Vert & & & & & \\ & \langle D\theta_u\rangle & & & & & \end{array}$$

implies that $\left(Q_s + \sum_{j=1}^{m-1} \alpha_{j+2,s}\theta_{b_j}\theta_{-b_j}\right)$ vanishes on $\Theta_u\cdot\Theta_{-u}$ if and only if

$$\left(Q_s + \sum_{j=1}^{m-1} \alpha_{j+2,s}\theta_{b_j}\theta_{-b_j} - D\theta_u\cdot\theta_{-u}\right)\Bigg|_{\Theta_u} = 0.$$

The section

$$\left(Q_s + \sum_{j=1}^{m-1} \alpha_{j+2,s}\theta_{b_j}\theta_{-b_j} - D\theta_u\cdot\theta_{-u} + D\theta_{-u}\cdot\theta_u\right) \in H^0(X, \mathscr{O}_X(2\Theta))$$

vanishes on Θ_u and one concludes by the following the same steps as before. We get the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X(\Theta_u) \longrightarrow \mathscr{O}_X(2\Theta) \longrightarrow \mathscr{O}_{\Theta_u}(2\Theta) \longrightarrow 0.$$

and consider its exact sequence in cohomology

[3]Note that $H^0(\Theta_u, \mathscr{O}_{\Theta_u}(\Theta_u))$ is generated by $D\theta_u$, where D is a constant vector field.

$$0 \longrightarrow H^0(X, \mathscr{O}_X(\Theta_u)) \longrightarrow H^0(X, \mathscr{O}_X(2\Theta)) \longrightarrow H^0(X, \mathscr{O}_{\Theta_u}(2\Theta)) \longrightarrow 0$$

to conclude that there exists a constant $\alpha_{1,s} \in \mathbb{C}$ such that

$$Q_s + \alpha_{1,s}\theta_u\theta_{-u} + D_s\theta_{-u} \cdot \theta_u - D_s\theta_u \cdot \theta_{-u} + \sum_{j=1}^{m-1} \alpha_{j+2,s}\theta_{b_j}\theta_{-b_j} = 0.$$

□

4.4.3 Main result

Now we show the main result. Note that having a degenerate $(m+2)$-secant m-plane is equivalent to supposing that $P_1 = 0$. Recall that the reduced scheme $Y = \{a_1, \dots, a_{m+2}\}$.

Theorem 4.4.4. *Let (X, λ) be an indecomposable principally polarized abelian variety of dimension g that has no $(n+2)$-secant n-planes for $n < m$, let Θ be a symmetric representative of the polarization λ. Suppose that $\Theta_u \cdot \Theta_{-u} \cdot \Theta_{b_1} \dots \cdot \Theta_{b_{m-1}}$ is a complete intersection and reduced, and that one has the $(m+2)$-secant m-planes:*

$$\Theta_u \cdot \Theta_{-u} \cdot \Theta_{b_1} \dots \cdot \Theta_{b_{m-1}} \subset \Theta_{b_m} \cup \Theta_\mu$$

for $\mu \in \{-b_1, \dots, -b_{m-1}, -b_m\}$. Then, $\dim V_Y > 0$.

Note that this implies that from finite $(m+2)$-secant m-plane one gets infinite honest $(m+2)$-secant planes.

Proof. By 4.4.1 and 4.4.3 it is enough to prove that for any integer $s \geq 2$ one has that

$$P_1 = \dots = P_{s-1} = 0 \Rightarrow P_s \text{ (or } Q_s\text{) vanishes on } \Theta_u \cdot \Theta_{-u} \cdot \Theta_{b_1} \dots \cdot \Theta_{b_{m-1}}.$$

Set

$$R(z, \varepsilon) = \sum_{s \geq 0} R_s(z)\varepsilon^s := P\left(z + \frac{1}{2}C(\varepsilon), \varepsilon\right).$$

Assume that $P_1 = \cdots = P_{s-1} = 0$, then $R_1 = \cdots = R_{s-1} = 0$ and that $P_s = R_s$. Since:

$$\begin{aligned} R(z,\varepsilon) &= P\left(z + \tfrac{1}{2}\zeta(\varepsilon)\right) \\ &= e^{\frac{1}{2}\sum_{r\geq 0}\Delta_r} P(z,\varepsilon) \\ &= \sum_{s\geq 0} \varepsilon^s \sum_{j=0}^{s} \Delta_j P_{s-j}. \end{aligned}$$

Hence, we will show that R_s vanishes on $G = \Theta_u \cdot \Theta_{-u} \cdot \Theta_{b_1} \cdots \cdot \Theta_{b_{m-1}}$.

We have that

$$R_s|_G = \Delta_{s-1}\theta_{-b_m} \cdot \theta_{b_m}. \tag{4.8}$$

On the other hand, we set

$$T(z,\varepsilon) = \sum_{s\geq 0} T_s(z)\varepsilon^s := P\left(z - \frac{1}{2}C(\varepsilon), \varepsilon\right).$$

and in a similar way, under the vanishing of $P_1, \ldots, P_{s-1}$ hypothesis, we get that $T_s = P_s = R_s$. In particular,

$$T_s|_G = \Delta^-_{s-1}\theta_{b_m}\theta_{-b_m} + \sum_{j=1}^{m-1}\sum_{l=1}^{s} \alpha_{j+2,l}\theta_{-b_j} \cdot \left(\Delta^-_{s-j}\theta_{b_j}\right) \tag{4.9}$$

Hence, $R_s^2 = R_s T_s$ so that we compute

$$\begin{aligned} R_s^2| &= \left(\Delta^-_{s-1}\theta_{b_m}\right) \cdot \theta_{b_m}\theta_{-b_m} \cdot \left(\Delta_{s-1}\theta_{-b_m}\right) \\ &+ \theta_{b_m}\theta_{-b_j} \cdot \sum_{j=1}^{m-1}\sum_{l=1}^{s} \alpha_{j+2,l}\left(\Delta^-_{s-j}\theta_{b_j}\right) \cdot \left(\Delta_{s-1}\theta_{-b_m}\right) \end{aligned}$$

which is zero by the $m-1$ different honest $(m+2)$-secant m-planes and one degenerate $(m+2)$-secant m-plane. Since the intersection is reduced, $R_s = 0$, and we conclude the proof. $\square$

4.5 Quadrisecant planes

In this section we view the particular case $m = 2$ in light of what has been studied in the previous chapter.

4.5.1 A curve of quadrisecant planes

This is an improvement of Debarre's Theorem 5.1 in [11] and a variant of Theorem 4.3.2.

Theorem 4.5.1. *Let (X, λ) be an indecomposable principally polarized abelian variety of dimension $g \geq 4$, let Θ be a symmetric representative of the polarization λ. Suppose that X does not possess the trisecant property, and let $Y = \{a_1, a_2, a_3, a_4\} \subset X$ be a subscheme with reduced structure such that V_Y contains an irreducible complete curve Γ that generates X and $\Theta_{a_1} \cap \Theta_{a_2} \cap \Theta_{a_3} \cap \Theta_{a_4}$ is a complete intersection. Then, Γ is twice the minimal cohomology class of Θ.*

Proof. As in the proof of Theorem 4.3.2, we get that the endomorphism $\alpha = \delta_{12} + \delta_{34}$ where $\delta_{12}, \delta_{34} \in \{0, 1\}$. So we have three cases:

- $\alpha = 0$. This case is discarted immediately.
- $\alpha = 1$. Then, Γ is the minimal class. By Matsusaka-Ran Criterion 2.4.1 this implies that X is a Jacobian variety, which contradicts our hypothesis.
- $\alpha = 2$. Then, Γ is twice the minimal class. This proves the theorem.

□

Corollary 4.5.2. *If $Y = \{a_1, a_2, a_3, a_4\}$ consists of four reduced points of X, $s = a_1 + a_2 + a_3 + a_4$ and $\Gamma \subset V_Y$ is an irreducible complete curve that generates X, then the involution $z \mapsto -z - s$ on X restricts to an involution of Γ.*

Proof. The previous proof shows that $\delta_{12} = 1$ for a generic $\zeta' \in \frac{1}{2}\Gamma$. Therefore, this means that for a generic $\zeta' \in \frac{1}{2}\Gamma$, $-b_1 - b_2 \in \Gamma$. However,

$$-b_1 - b_2 = -2\zeta' - a_1 - a_2 - a_3 - a_4.$$

$\square$

Remark 4.5.3. We note that if we take $x \in \frac{1}{2}s$, then the involution $z \mapsto -z$ restricts to the curve $C := \Gamma + x$. Here we have that $C \subset V_{t^*Y}$, so if $\eta \in C$ and we write $q_i := a_i - x_i$, then

$$\Pi_\eta = \operatorname{span}\{K(\eta + q_1), K(\eta + q_2), K(\eta + q_3), K(\eta + q_4)\}$$
$$\Pi_{-\eta} = \operatorname{span}\{K(\eta - q_1), K(\eta - q_2), K(\eta - q_3), K(\eta - q_4)\}$$

are planes in $\mathbb{P}^{2^g-1}$. This pair, by [21] is necessary and sufficient for Γ to be an Abel-Prym curve. Although, they use complicated analytic methods, there is work in progress to give a fully algebro-geometric proof.

4.5.2 A pair of quadrisecant planes implies the existence of a curve of quadrisecant planes

Now we view a particular case of Theorem 4.4.4. Under certain geometric conditions, having one degenerate quadrisecant plane and one honest quadrisecant plane on the Kummer variety implies the existence of a curve of quadrisecant planes on the Kummer variety.

Recall that the scheme V_Y in this particular case equals

$$V_Y = \{2\zeta \ : \ K(\zeta + a_1) \wedge K(\zeta + a_2) \wedge K(\zeta + a_3) \wedge K(\zeta + a_4) = 0\}.$$

Based on Section 4.4 we denote $u = a_1 + \frac{1}{2}(-a_1 - a_2)$, $v = a_3 + \frac{1}{2}(-a_1 - a_2)$ and $w = a_4 + \frac{1}{2}(-a_1 - a_2)$. Recall the reduced scheme $Y = \{a_1, a_2, a_3, a_4\}$.

Theorem 4.5.4. *Let (X, λ) be an indecomposable principally polarized abelian variety of dimension g that has no trisecant lines, let Θ be a symmetric representative of the polarization λ. Suppose that $\Theta_u \cdot \Theta_{-u} \cdot \Theta_w$ is a reduced complete intersection and that we have the quadrisecant conditions:*

a) $\Theta_u \cdot \Theta_{-u} \cdot \Theta_w \subset \Theta_v \cup \Theta_{-v}$

b) $\Theta_u \cdot \Theta_{-u} \cdot \Theta_w \subset \Theta_v \cup \Theta_{-w}$.

Then, $\dim V_Y > 0$.

Proof. This is an particular case, with $m = 2$ of Theorem 4.4.4 $\square$

This is an interesting result since by having two quadrisecant planes one obtains infinite honest quadrisecant planes.

Note that we have a symmetry as in the case studied by Grushevsky-Krichever in [20] with the difference that they studied a pair of honest quadrisecant planes with no further conditions.

CHAPTER

5 Further work

In this chapter we show some problems to work after this thesis, and some ideas.

5.1 A curve of quadrisecant: Degenerate cases.

Since we will be dealing with artinian schemes of length 4, we need to determine exactly which ones can appear in the context we are studying. This is given precisely by the following lemma.

Lemma 5.1.1. *If Y is an artinian subscheme of X of length 4 such that V_Y contains a curve and (X,Θ) does not possess the trisecant property, then Y must be isomorphic to the spectrum of one of the following rings:*

i) k^4

ii) $k[\varepsilon]/(\varepsilon)^2\times k^2$

iii) $k[\varepsilon]/(\varepsilon)^2\times k[\varepsilon]/(\varepsilon)^2$

iv) $k[\varepsilon]/(\varepsilon)^3\times k$

v) $k[\varepsilon]/(\varepsilon)^4$

vi) $k[x,y]/(x^3,xy,y^2)$

vii) $k[x,y]/(x^2,y^2)$

Proof. Taking $\zeta\in V_Y$ general, we have that the intersection of the plane generated by $K(\zeta+Y)$ with $K(X)$ is isomorphic to Y, hence Y is an artinian subscheme of $\mathbb{A}^2$ of length

4. Since every artinian ring is isomorphic to a product of local artinian rings, we only need to look at products of artinian schemes supported at $0 \in \mathbb{A}^2$, such that the total length is 4. This gives us immediatetly eight different schemes. The only scheme that is missing from the above list is the spectrum of

$$k[x,y]/(x^2,xy,y^2) \times k.$$

Its reduced scheme consists of two points and the tangent space at one of the points is 2-dimensional. Therefore, if the intersection of a plane with $K(X)$ containing this scheme, we can take a line on the tangent plane that passes through the two points. This line will be tangent to $K(X)$ at the point of higher multiplicity and therefore (X,Θ) would possess the trisecant property. □

The remaining problem is to prove in all the remaining cases *ii)-vii)* that are missing in Theorem 4.5.1. In other words, we want to prove the following result:

Conjecture 2. Let (X,λ) be an indecomposable principally polarized abelian variety of dimension $g \geq 4$, let Θ be a symmetric representative of the polarization λ. Suppose that X does not possess the trisecant property, and let $Y \subset X$ be an artinian subscheme of length 4 such that V_Y contains an irreducible complete curve Γ that generates X. Then, Γ is twice the minimal cohomology class of Θ.

When we work with $(m+2)$-secant m-planes there is not a clear path to follow. Both analytic and algebro-geometric aproaches requiere studying each case individually.

Other problems we would like to work on in the future are:

1. Is there an involution in the case of $(m+2)$-secants? (see 4.5.2)

2. Let $\Gamma \subset V_Y$ be a curve, with $Y \subset X$ an artinian subscheme of length m. Is Γ m-times the minimal class?

3. In a general setting, does the existence of finite $(m+2)$-secant m-planes imply the existence of a curve of $(m+2)$-secant m-planes?

4. Is a Prym-Tyurin variety of exponent m characterized by being an i.p.p.a.v. containing a curve which is m-times the minimal class under certain geometric conditions?

5. What if we require a n-dimensional surface that parametrizes $(m+2)$-secant m-planes? Are m and n related if we look for a more general characterization of Jacobians (or Pryms)?

6. Can we degenerate $(m+2)$-secants or get the honest case from the degenerate case? This would give an interesting solution to the honest case of trisecants when working in a algebro-geometric setting.

7. Can we use Theorem 3.2.2 in order to remove the complete intersection condition in Theorem 4.3.2 and further the reduced condition in Theorem 4.4.4?

CHAPTER

A Algebraic background

A.1 The Koszul Complex

One can find more about this topic in [13, ch. 17] and [25, p. 252].

We will develop the theory of the Koszul complex for complete intersection of theta divisors on abelian varieties. A ring is commutative with $1 \neq 0$.

Definition A.1.1. Let R be a ring. A sequence of elements $a_1, \dots, a_n \in R$ is called a *regular sequence* if

i) $(a_1, \dots, a_n) \neq R$.

ii) For every j the image of a_j in $R/(a_1, \dots, a_n)$ is not a zero divisor.

Remark A.1.2. Let R be a ring, $a_1, \dots, a_n \in R$ a regular sequence and $R^{\binom{n}{m}}$ be the free R-module with basis $\{e_{i_1, \dots, i_m} : 1 \leq i_1 < \dots < i_m \leq n\}$. The notation $e_{i_1, \dots, \hat{i}_s, \dots, i_m}$ stands for the element of $R^{\binom{n}{m-1}}$ that is obtained by removing the index i_s.

The Koszul Lemma gives us a technical yet useful tool, we state it as a theorem.

Theorem A.1.3. *The sequence:*

$$0 \longrightarrow R^{\binom{n}{n}} \xrightarrow{\partial_{n-1}} R^{\binom{n}{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} R^{\binom{n}{1}} \xrightarrow{\partial_0} (a_1, \dots, a_n) \longrightarrow 0$$

is exact, where

$$\partial_r\left(e_{i_1,\ldots,e_{r+1}}\right)=\sum_{s=1}^{r+1}(-1)^{s-1}a_{i_s}e_{i_1,\ldots,\hat{i}_s,\ldots,i_{r+1}}$$

for $r\geq 1$ *and* $\delta_0(e_i)=a_i$.

Consider now a principally polarized abelian variety (X,Θ). If $x\in X$, we write θ_x to represent the theta function $\mapsto z\theta(z-x)$. Let $x_1,\ldots,x_n\in X$ and consider the sheaves

$$\mathscr{F}_r:=\bigoplus_{1\leq i_1<\cdots<i_r\leq n}\mathscr{O}_X\left(-\Theta_{x_{i_1}}-\cdots-\Theta_{x_{i_r}}\right).$$

If $g\in\mathscr{O}_X\left(-\Theta_{x_{i_1}}-\cdots-\Theta_{x_{i_r}}\right)(U)$ for some open subset $U\subset X$, we write $ge_{i_1,\ldots,i_r}$ for the element of $\mathscr{F}_r(U)$ that has g in the $(i_1,\ldots,i_r)$-th coordinate and zeroes in the rest. We define the map of sheaves

$$\partial_r:\mathscr{F}_{r+1}\to\mathscr{F}_r$$

where locally

$$g_{i_1,\ldots,i_{r+1}}\theta_{x_{i_1}}\cdots\theta_{x_{i_{r+1}}}\cdot e_{i_1,\ldots,i_{r+1}}\mapsto\sum_{s=1}^{r+1}(-1)^{s-1}g_{i_1,\ldots,i_{r+1}}\theta_{x_{i_1}}\cdots\theta_{x_{i_{r+1}}}\cdot e_{i_1,\ldots,\hat{i}_s,\ldots,i_{r+1}}$$

for $r\geq 1$ and

$$\partial_0:\mathscr{F}\to\mathscr{O}_X$$

where locally

$$g_i\theta_ie_i\mapsto g_i\theta_i.$$

The following results is necessary for what follows.

Proposition A.1.4. *Let* $x_1,\ldots,x_n\in X$ *be points such that* $\Theta_{x_1}\cap\cdots\cap\Theta_{x_n}$ *is a complete intersection. Then, if* $\mathscr{I}$ *is the ideal sheaf that defines this complete intersection, we have that the following exact sequence*

$$0\longrightarrow\mathscr{F}_n\xrightarrow{\partial_{n-1}}\mathscr{F}_{n-1}\xrightarrow{\partial_{n-2}}\cdots\xrightarrow{\partial_1}\mathscr{F}_1\xrightarrow{\partial_0}\mathscr{I}\longrightarrow 0$$

is exact.

Proof. We only need to show that this sequence is exact after taking the stalks at an arbitrary point $x \in X$. In this case, the complex tat emerges is the Koszul complex, so it is exact; this is since the intersection is complete, the local generators of the sheaves form a regular sequence. $\square$

Now consider two elements $y, z \in X$. If we tensor the previous exact sequence by the invertible sheaf $\mathscr{O}_X(\Theta_y + \Theta_z)$, we obtain again an exact sequence of sheaves. The new maps we obtain will be written $\partial_r(y,z)$.
Assume that $\dim X = g$.

Lemma A.1.5. *If $y + z \neq x_i + x_j$ for all i, j then for every $1 \leq k \leq g-2$, we have that*

$$H^k(im\, \partial_r(y,z)) \simeq H^{k+1}(im\, \partial_{r+1}(y,z))$$

Proof. For all r we have the following exact sequence:

$$0 \longrightarrow \operatorname{im} \partial_{r+1}(y,z) \longrightarrow \mathscr{F}_{r+1} \otimes \mathscr{O}_X(\Theta_y + \Theta_z) \longrightarrow \operatorname{im} \partial_r(y,z) \longrightarrow 0.$$

Now

$$\mathscr{F}_{r+1} \otimes \mathscr{O}_X(\Theta_y + \Theta_z) \simeq \bigoplus_{1 \leq i_1 < \cdots < i_{r+1} \leq n} \mathscr{O}_X\left(\Theta_y + \Theta_z - \left(\Theta_{x_{i_1}} + \cdots + \Theta_{i_{r+1}}\right)\right).$$

We can see that for $r \geq 2$, $\Theta_y + \Theta_z - \left(\Theta_{x_{i_1}} + \cdots + \Theta_{x_{i_{r+1}}}\right)$ is the inverse of an ample invertible sheaf, and by the Kodaira Vanishing Theorem, we have that its cohomology vanishes up until the $(g-1)$-th cohomology group. For $r = 1$, we have that $\Theta_y - \Theta_z - \Theta_{x_{i_1}} - \Theta_{x_{i_2}}$ is algebraically equivalent to zero but not linearly equivalent to zero. Then we get that its cohomology vanishes. By taking the long exact sequence of cohomology from the short exact sequence above, we conclude the proof. $\square$

If we take the last terms of the exact sequence from the Proposition A.1.4 tensored with $\mathscr{O}_X(\Theta_y + \Theta_z)$, we can take the short exact sequence:

$$0 \longrightarrow \operatorname{im} \partial_1(y,z) \longrightarrow \mathscr{F}_1 \otimes \mathscr{O}_X(\Theta_y + \Theta_z) \longrightarrow \mathscr{I} \otimes \mathscr{O}_X(\Theta_y + \Theta_z) \longrightarrow 0.$$

By taking part of the long exact sequence of cohomology, we have that the sequence

$$H^0(\mathscr{F}_1 \otimes \mathscr{O}_X(\Theta_y + \Theta_z)) \longrightarrow H^0(\mathscr{I} \otimes \mathscr{O}_X(\Theta_y + \Theta_z)) \longrightarrow H^0(\operatorname{im} \partial_1(y,z))$$

is exact. By the previous Lemma, we have that $H^1(\operatorname{im} \partial_1(y,z)) \simeq H^r(\operatorname{im} \partial_r(y,z))$ for all $r \leq g-2$. If $n \leq g-2$, then we have that $H^n(\operatorname{im} \partial_n(y,z)) = 0$, since $\partial_n(y,z)$ is just the zero map.

From this we can conclude that

Proposition A.1.6. *Let $x_1, \ldots, x_n \in X$ be points such that $\Theta_{x_1} \cap \cdots \cap \Theta_{x_n}$ is a complete intersection, and assume that $n \geq g-2$. Then, the map*

$$H^0(\mathscr{F}_1 \otimes \mathscr{O}_X(\Theta_y + \Theta_z)) \longrightarrow H^0(\mathscr{I} \otimes \mathscr{O}_X(\Theta_y + \Theta_z))$$

is surjective.

To finish this section and to remark its usage on the secancy conditions, suppose that $y, z \in X$ are points such that

$$\Theta_{x_1} \cap \cdots \cap \Theta_{x_n} \subset \Theta_y \cup \Theta_z.$$

We see that locally $\theta_y \theta_z$ can be considered as a section of $\mathscr{I}$. By the Theorem of the Square, we have that $\Theta_y + \Theta_z - \Theta_{x_i} \sim \Theta_{y+z-x_i}$, and So

$$\mathscr{F}_1 \otimes \mathscr{O}_X(\Theta_y + \Theta_z) \simeq \bigoplus_{i=1}^{n} \mathscr{O}_X(\Theta_{y+z-x_i}).$$

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