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The Geometric Formula for affine Weyl groups

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Abstract

Let W be an affine Weyl group with corresponding finite Weyl group W_f. For each λ, a dominant coweight, corresponds an element $\theta(\lambda) \in W$. With N. Libedinsky and D. Plaza, we produce a conjecture called the *Geometric Formula* predicting the following: the cardinality of the set of elements in W that are lesser or equal to $\theta(\lambda)$ in the Bruhat order, is a linear combination (with coefficients not depending on λ) of the volumes of the faces of the polytope $\mathrm{Conv}(\lambda)$, constructed as the convex hull of the set $W_f \cdot \lambda$. We prove the geometric formula for type $\widetilde{A_3}$, by giving general algebraic and geometric constructions for the set $\leq \theta(\lambda)$. We study the polytope $\mathrm{Conv}(\lambda)$, its faces, and give some formulas to compute their volumes of the corresponding dimension.

Contents

Global conventions and notations

For $n \in \mathbb{N}$, let I_n denote the finite set $\{1, \dots, n\}$. Let W be a Coxeter group with simple reflections $S = \{s_1, \dots, s_n\}$. The set S is indexed by I_n, so the subsets of I_n are in bijection with the subsets of S. If W has rank $n \geq 4$ and $J = \{1, 2, 4\} \subset I_n$, then W_J denotes the parabolic subgroup of W generated by s_1, s_2 and s_4. If W is an affine Weyl group, regarded as a Coxeter group its simple reflections are be denoted by $\{s_0, s_1, \dots, s_n\}$, which we identify with $I_n \cup \{0\}$.

The identity element of a group W will be denoted by id. If W acts on a set A, and $B \subset A$, $W \cdot B$ denotes the collection of sets wB such that $w \in W$, unless explicitly stated otherwise. The cardinality of A will be denoted by $|A|$, and its topological closure (when there is an underlying topology) by $\overline{A}$.

Introduction

Bruhat Intervals on affine Weyl groups are widely studied objects, however, there still are some basic questions that remain unanswered. In this thesis we study the cardinality of these intervals by observing the underlying relationship between the alcovic geometry of the interval, given by the Coxeter complex of the affine Weyl group, and the Euclidean geometry of the interval, as the alcoves define a set in some Euclidean space. The main result of this thesis is the Geometric Formula which computes the cardinality of some of these intervals as a linear combination of volumes of faces of a permutahedron.

We will go quickly trough some notation and definitions. For more details, see Chapter 1. Let Φ be an irreducible (reduced, crystallographic) root system of rank n, and let V be the ambient (real) Euclidean space spanned by Φ, with positive definite symmetric bilinear form $(-,-)$. Fix a set $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots and a set $\Phi^+ \supset \Delta$ of positive roots. Denote by $\widetilde{\alpha}$ the highest root. Let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the coroot corresponding to $\alpha \in \Phi$. The *fundamental coweights* $\varpi_i^\vee$ are defined by the equations $(\varpi_i^\vee, \alpha_j) = \delta_{ij}$. They form a basis of V. A *coweight* is an integeral linear combination of the fundamental coweights, and a *dominant coweight* is a coweight whose coordinates in this basis are non-negative. For $\alpha_i \in \Delta$, let H_{α_i} be the hyperplane of V orthogonal to α_i and let s_i be the (simple) reflection through H_{α_i}. Let $S_f = \{s_1, \ldots, s_n\}$ be the collection of simple reflections. The subgroup W_f of the orthogonal transformations of V generated by S_f, is the *finite Weyl group*. The pair (W_f, S_f) is a Coxeter system.

Denote by w_0 the longest element of W_f in the Bruhat order.

The ***affine Weyl group*** W is the subgroup of affine transformations of V generated by W_f and translation by elements of $L^\vee = \mathbb{Z}\Phi^\vee$, where $\Phi^\vee$ is the coroot system. We have $W \cong W_f \ltimes L^\vee$. For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, let

$$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\}.$$

The group W can also be realized as the group generated by the affine reflections $s_{\alpha,k}$ along the hyperplanes $H_{\alpha,k}$, for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Remove all these hyperplanes from V. The connected components of the resulting set are called alcoves; they are in bijection with W. We will take

$$A_{\text{id}} = \{\lambda \in V \mid -1 < (\lambda, \alpha) < 0 \ \forall \alpha \in \Phi^+\}$$

to be the ***fundamental alcove***. Let $s_0 := s_{\widetilde{\alpha},-1}$ and $S := \{s_0, s_1, \ldots, s_n\}$, then (W, S) is a Coxeter system. Let $\leq$ correspond to the Bruhat order and let $l(-)$ be the length funciton. For $x, y \in W$, denote by $[x, y]$ the Bruhat interval consisting of the elements $z \in W$ such that $x \leq z \leq y$.

We want to study the intervals $[x, y]$. We are particularly interested in giving an Euclidean geometric interpretation of the cardinality of those sets. Computing the cardinality of these intervals is by no means an easy task. For example, the main result in [BE09] by Anders Björner and Torsten Ekedahl, published in 2009 in Annals of Mathematics, is that if f_i denotes the number of elements of length i in $\leq w := [\text{id}, w]$, then

$$0 \leq i < j \leq l(w) - i \qquad \text{implies} \qquad f_i \leq f_j.$$

There is a way to associate to each dominant coweight λ, an element $\theta(\lambda) \in W$. These elements can be determined by checking their respective alcoves. They lie in the dominant cone and are a translation by λ of the alcove corresponding to w_0. These are very important elements for representation theory. The intervals $\leq \theta(\lambda)$ will be the main focus of the present thesis. At the time of writing this thesis, to the knowledge of the author, there is only one result involving the cardinality of these intervals. In 2012, Waldeck Schützer gave a general formula for $|\leq \theta(\lambda)|$. We will briefly explain his result. Define the k^{th} Todd polynomial T_k in the variables $x_1, x_2, \ldots$ as the coefficient of t^k in

$$\prod_{i=1}^{n} \frac{tx_i}{1 - e^{-tx_i}} = \prod_{i=1}^{n} \sum_{j=0}^{\infty} \frac{B_j}{j!} (tx_i)^j,$$

where

$$B_j = \sum_{s=0}^{j} \sum_{l=0}^{s} (-1)^l \binom{s}{l} \frac{(l+1)^j}{s+1}$$

are the Bernoulli numbers. In [Sch12], Schützer proved that

$$|\leq \theta(\lambda)| = |W_f| \sum_{j=0}^{n} \sum_{w \in W_f} \frac{(\rho, w\lambda)^j \, T_{n-j}\left((\rho, w\alpha_1), \ldots, (\rho, w\alpha_n)\right)}{j! \prod_{i=1}^{n} (\rho, w\alpha_i)}.$$

Furthermore, if $\lambda = (m_1, \ldots, m_n)$ is expressed in the fundamental coweight basis, then $|\leq \theta(\lambda)|$ is a polynomial of degree n in $m_1, \ldots, m_n$. Fixing j in the first sum, gives the homogeneous part of degree j of the polynomial.

Although this achievement is remarkable (and underrated), it still leaves a bitter taste. The formula seems extremely complicated and does not provide much insight on what is going on. As a matter of fact, the author of this thesis wrote a program in SageMath for the Schützer formula, and the case of type $\widetilde{A}_7$ takes about 7-9 hours to run.

There is a more recent paper [LP20] by Leonardo Patimo and Nicolás Libedinsky, in which $|\leq \theta(\lambda)|$ is computed for type $\widetilde{A}_2$. What is interesting about this, is the way the formula is computed. Among other things, it relies on a geometric construction of the set $\leq \theta(\lambda)$ as a union of suited hexagons (see figure 2.3). The main result of this thesis is the Geometric Formula, that is a joint work with my master's thesis advisor and David Plaza, which computes $|\leq \theta(\lambda)|$ for any type. In the way of proving it, we generalize the geometric construction mentioned above. At present, we haven't yet been able to prove the geometric formula in full generality, but we are confident that we will provide an answer in the near future. Therefore, we will prove it for type $\widetilde{A_3}$, in a generalizable way. Before we can properly state the formula, we first need to introduce some notation.

Let λ be a dominant coweight, corresponding to the irreducible root system Φ of rank n, and let X_n be the type of (W_f, S_f). For any $J \subset I_n$, let W_J be the parabolic subgroup generated by $\{s_j \mid j \in J\}$. Note that since $0 \notin I_n$, W_J is a subgroup of W_f. We define $V_J^{X_n}(\lambda)$ as the $|J|$-dimensional volume of $\mathrm{Conv}(W_J \cdot \lambda)$, the convex hull of the W_J-orbit of λ. For example, there is a volume $V_{\{1,2,4\}}^{A_7}(\lambda)$. There are some results involving these volumes (see [Pos05]). Among other things, these are homogeneous polynomials fairly easy to compute.

The *geometric formula* for type $\widetilde{X_n}$, conjectures that there exists (unique) $\vartheta_J \in \mathbb{R}$ such that for any dominant coweight λ,

$$|\leq \theta(\lambda)| = \sum_{J \subset I_n} \vartheta_J V_J^{X_n}(\lambda). \tag{1}$$

This implies the result mentioned above, that if $\lambda = (m_i)_{i \in I_n}$ in the coweight basis, then $|\leq \theta(\lambda)|$ is a polynomial of degree n in the $m_1, \ldots, m_n$. Taking the sum over a fixed rank $|J| = d$ gives the degree part d of the polynomial. We call the coefficients ϑ_J the *geometric coefficients.*

It is very important that the reader keeps in mind throughout the present thesis, that the geometric coefficients do not depend on the choice of the dominant coweight λ. If they did, the geometric formula would be trivial. The geometric coefficients only depend on $J \subset I_n$ and, of course, the type of Φ. If one removes the ^{X_n} from the volume $V_J^{X_n}$, one can be misled to think that these do not depend on the type, hence the emphasis. In Chapter 4 it will be explained that they in fact do not have the same "degree of dependence". For example, fix any $J \subset I_3 \subset I_4$. The geometric coefficients corresponding to J in types $\widetilde{A_3}$ and $\widetilde{A_4}$ are not equal. However, $V_J^{A_3} = V_J^{A_4}$. This is why, later on, this notation will be slightly modified.

Not only this formula looks nice and compact, it also establishes a bridge between Euclidean and alcovic geometry. The polytope $\mathrm{Conv}(\lambda)$ whose vertex set is the W_f-orbit of λ has volume $V_{I_n}^{X_n}(\lambda)$, and an i-face of this polytope has volume $V_J^{X_n}(\lambda)$ for some $J \subset I_n$ with $|J| = i$. Thus, the geometric formula conjectures that $|\leq \theta(\lambda)|$ can be computed as a linear combination of the volumes (in their respective dimensions) of the faces of this polytope. In type A_n, $\mathrm{Conv}(\lambda)$ defines a permutahedron.

Let $A(\leq \theta(\lambda))$ be the closure of the alcoves corresponding to $\leq \theta(\lambda)$. To prove the geometric formula, we will show two ways to construct the set $\leq \theta(\lambda)$. One is an algebraic construction which uses various nice properties of $\theta(\lambda)$, and the other one is a geometric partition of $A(\leq \theta(\lambda))$, starting from the polytope $\mathrm{Conv}(\lambda)$. The proof is focused on studying in-depth their properties. The former construction, which we call the Polytope Construction, will be done in full generality, that is, with no restrictions on the root system Φ. The latter, called the Geometric Partition, will only be illustrated in type $\widetilde{A_3}$.

The reason for this, is that it uses that the polytope $\mathrm{Conv}(\lambda)$ is completely contained in $A(\leq \theta(\lambda))$. We were strongly convinced that this was a general fact. Unfortunately, in the way of trying to prove it, we found a counterexample. The containment is still true in type $\widetilde{A_3}$, and the counterexample was found in type $\widetilde{A_4}$. This terrible, cold and dark day for mathematics, was very recent. However, hope still remains. With the help of SageMath, besides computing Schützer formula, the author of this thesis was able to compute the volumes $V_J^{A_n}$ (see appendix A). Using these tools, we have been able to compute the geometric coefficients in types $\widetilde{A_1}, \ldots, \widetilde{A_7}$ so that the geometric formula holds in all of these cases (and no counterexample has been found). Thus, although the containment does not hold in $\widetilde{A_4}$, the geometric formula does. With this in mind, recent work suggests that we can "fix" the geometric partition, so that the proof of the geometric formula given in this thesis can still be extended to full generality, with some adjustments. This is currently an on-going work.

As for the geometric coefficients, at the time of writing this thesis, we have not yet been able to compute them in a satisfying way. We will provide tables with these coefficients for the cases mentioned above (see appendix B). We have nice conjectures that would make their computation extremely easy. Let $\mathcal{P}_{W_f}$ be the closure of the set of alcoves corresponding to the subgroup W_f of W. Roughly speaking, we believe that ϑ_J is completely determined by computing the volume of a certain section of $\mathcal{P}_{W_f}$. This section is determined by a finite collection of hyperplanes slicing through $\mathcal{P}_{W_f}$. In turn, this collection is given by J (see the numbers defined in Corollary 5.5). We are currently working on computing the volumes of these sections.

Finally, we give some results on the volumes $V_J^{X_n}$, and other objects, that were not needed to prove the geometric formula.

Structure of the thesis

This work is partitioned in two parts. Part I establishes the conventions and notations that we will use throughout this thesis. In this part we will also give some insights on the geometric formula, in a small case. Part II is where we will lay the groundwork and prove the geometric formula. This is the most important and extensive part of this thesis. At the end of this part, there are appendices with some volumes $V_J^{A_n}(\lambda)$ and tables showing the geometric coefficients in type $\widetilde{A_n}$, for the first few cases.

Part I consists of two chapters.

- Chapter 1 contains all the necessary background, conventions and notations that we will use in this thesis. The experienced reader can skip until Section 1.3.

- In Chapter 2 we will "prove" -without much rigor- the geometric formula in type $\widetilde{A_2}$, since most of the ideas arose from this case. This example will be constantly referenced to -for geometric intuition- in the subsequent chapters, but this chapter can be skipped without any loss of generality.

Part II has three chapters in it.

- In Chapter 3 we define and generalize the two geometric constructions that appeared in Chapter 2. We will discuss its properties and lay the groundwork for the proof of the geometric formula.

- Since the volumes $V_J^{X_n}(\lambda)$ are a big part of the geometric formula, in Chapter 4 we will give some insights on them. To prove the geometric formula, we will not make use of the results in this chapter.

- In Chapter 5 we give a proof of the geometric formula.

Part I

Preliminaries and the geometric formula in a baby example

1. Background and notations

The main purposes of this chapter are to briefly establish the necessary background, conventions and notations for the upcoming chapters, as well as to explain some known facts of particular interest to us. The elements $\theta(\lambda)$, which play an important role in this thesis, will be defined in Section 1.3, so the experienced reader can skip to that section.

1.1 Coxeter systems

Most of the definitions concerning Coxeter systems[1] can be found on [Bou02], [BB05] or [EMTW20].

A ***Coxeter system*** (W, S) is a group W, which we call ***Coxeter group*** together with a finite set $S \subset W$ of generators, such that W admits the presentation

$$W = \langle s \in S \mid (st)^{m_{st}} = id \quad \forall\, s, t \in S, \text{ with } m_{st} < \infty \rangle,$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \ldots \infty\}$ if $s \neq t \in S$. The ***rank*** of (W, S) is $|S|$ and the elements of S are called ***simple reflections***. One can prove that the order of st is exactly m_{st} (if $m_{st} = \infty$ then there is no corresponding relation between s and t).

Given a Coxeter system (W, S) and an element $w \in W$, we can write $w = s_1 \cdots s_k$

[1] A Coxeter group can often be equipped with the structure of a Coxeter system in multiple different ways. There are examples of Coxeter groups that can be described as two different Coxeter systems with different ranks (see [MR07], for example). For this reason, we focus our attention on the pair (W, S) rather than on the group W.

with $s_i \in S$. We say that $(s_1, \ldots, s_k)$ is an ***expression*** of w of length k. The ***length*** $l(w)$ of an element (not an expression) $w \in W$, is defined as the minimal k such that w has an expression of length k. A ***reduced expression*** of w is an expression of w of length $l(w)$. For a simple reflection $s \in S$, the numbers $l(sw)$ and $l(ws)$ are either $l(w)+1$ or $l(w)-1$. The ***left descent set*** $\mathcal{L}(w)$ of w is defined as the simple reflections $s \in S$ such that $l(sw) < l(w)$. Similarly, the ***right descent set*** of w is $\mathcal{R}(w) := \{s \in S \mid l(ws) < l(w)\}$. One can prove that W is finite if and only if there exist $w_0 \in W$ such that $\mathcal{R}(w_0) = S$. Such an element (if it exists) is unique and satisfies $l(w) < l(w_0)$ for all $w \neq w_0 \in W$. This important element w_0 is called the ***longest element*** of W and it also satisfies (and is determined by) $\mathcal{L}(w_0) = S$.

Let $u, w \in W$ and let $(s_1, \ldots, s_k)$ be a reduced expression of w. We write $u \leq w$ if there exists a reduced expression $(s_{i_1}, \ldots, s_{i_l})$ of u such that $1 \leq i_1 < \ldots < i_l \leq k$. This gives a partial order on W called the ***Bruhat order***. For $x, y \in W$ we define the ***Bruhat interval*** $[x, y]$ as the set of $w \in W$ such that $x \leq w \leq y$.

For $J \subset S$, the ***parabolic subgroup*** W_J of W is the subgroup generated by J. The pair (W_J, J) is a Coxeter system. Let $I \subset S$, it is not hard to check that $W_J \cap W_I = W_{J\cap I}$ and $\langle W_J \cup W_I \rangle = W_{J\cup I}$. We say that W_J is a ***maximal parabolic subgroup*** of W, if $|J| = |S| - 1$. Each left coset wW_J of the (not necessarily maximal) parabolic subgroup W_J has a unique representative of minimal length, and the set of all such ***minimal coset representatives*** is denoted by W^J. Similarly, we denote by ${}^JW$ the set of minimal coset representatives with respect to the parabolic W_J on the left. If W_J is finite, each left and right coset has a unique representative of maximal length.

For each $J \subset S$ there is an important decomposition of W. Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$. Also, for this factorization, $l(w) = l(w^J) + l(w_J)$. Similarly, for each $u \in W$ there exists unique $u_J \in W_J$, ${}^Ju \in {}^JW$ such that $u = u_J \cdot {}^Ju$, and $l(u) = l(u_J) + l({}^Ju)$ also holds. Furthermore, $u \leq w$ implies both $u^J \leq w^J$ and ${}^Ju \leq {}^Jw$.

There is a way to encode the presentation of (W, S) with a labeled graph. Its vertex set is S and the vertices s and t are joined with an edge if $m_{st} > 2$. The edges are labeled by m_{st}, if $m_{st} > 3$. Thus, $m_{st} = 2$ means that s, t are not connected by an edge, and $m_{st} = 3$ implies that there is an unlabeled edge joining s and t. This labeled graph is called the *Coxeter graph*.

1.2 Root systems and affine Weyl groups

The main references for the material covered in this section are [Bou02], [Hum90].

Let Φ be an irreducible reduced (crystallographic) root system in a (real) Euclidean space V, equipped whit a positive definite symmetric bilinear form $(-,-)$. Let $\Phi^+ \subset \Phi$ be the positive roots and $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$ the simple roots. The set Δ is a basis for the vector space V. The ***rank*** of Φ is $|\Delta| = n$. Given $\alpha \in \Phi$, Let H_α be the hyperplane of V orthogonal to α. The reflection s_α on V whose fixed hyperplane is H_α, is given by the formula

$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Let O(V) be the orthogonal group of V and let

$$S_f = \{s_\alpha \mid \alpha \in \Delta\}.$$

The subgroup W_f of O(V) generated by S_f is finite. The finite reflection group W_f is called the ***Weyl group*** attached to Φ. To emphasize its finitude, we may call it the ***finite*** Weyl group. The pair (W_f, S_f) is a Coxeter system. For $\alpha_i \in \Delta$ we write s_i instead of s_{α_i}. Every root $\alpha \in \Phi$ has a corresponding ***coroot*** and it is defined by $\alpha^\vee = 2\alpha/(\alpha, \alpha)$[2].

[2]Sometimes it is easier to define $\alpha^\vee$ in the dual space V^* of V, by $\alpha^\vee(\lambda) = 2(\alpha, \lambda)/(\alpha, \alpha)$. Identifying V with V^* gives the equivalence between these two definitions.

The set of all the coroots is denoted by $\Phi^\vee$, and the *simple coroots* by $\Delta^\vee$, which consists of the $\alpha^\vee$ with $\alpha \in \Delta$. Note that $(\Phi^\vee)^\vee = \Phi$. The set $\Phi^\vee$ is a root system itself with simple roots $\Delta^\vee$.

Removing all the (finite) hyperplanes H_α from V, with $\alpha \in \Phi$, leaves an open set whose $|W_f|$ connected components are called *Weyl chambers*. We denote by C^+ the *dominant Weyl chamber* which consists of the $\lambda \in V$ such that $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta$. This chamber is also called the *dominant cone*. The closure of the dominant cone will still be denoted by C^+. It will be explicitly stated whenever we use this notation.

A similar construction can be done using affine reflections. V has an underlying affine space, which we still denote by V as no confusion is possible. For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, let $H_{\alpha,k}$ be the affine hyperplane consisting of those $\lambda \in V$ such that $(\lambda, \alpha) = k$. The affine reflection $s_{\alpha,k}$ whose fixed hyperplane is $H_{\alpha,k}$, is given by the formula

$$s_{\alpha,k}(\lambda) = \lambda - ((\lambda, \alpha) - k)\alpha^\vee.$$

Consider the affine group Aff(V), which is the semidirect product of GL(V) and the group of translations by elements of V. The subgroup W of Aff(V) generated by all the $s_{\alpha,k}$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called the *affine Weyl group* attached to Φ. It is clear that W_f is the subgroup of W generated by the $s_\alpha := s_{\alpha,0}$. Although some authors write W_a or W_{aff} instead of W, as we will always write W_f for the finite Weyl group, there should be no confusion.

The *root lattice* L is the subgroup of V generated by Φ, that is, $L = \mathbb{Z}\Phi$. Similarly, the *coroot lattice* is $L^\vee = \mathbb{Z}\Phi^\vee$. Since W_f normalizes $L^\vee$, one way to understand the affine Weyl group is as a semidirect product $W = L^\vee \rtimes W_f$. The *fundamental weights* are defined to be the dual basis of the simple coroot basis, that is, the fundamental weight ϖ_i is determined by the equations $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$, where $\alpha_j \in \Delta$ and δ_{ij} is the Kronecker delta. One can check that the $\mathbb{R}_{\geq 0}$-span of the fundamental weights is the closure of the dominant cone. Likewise, the *fundamental coweights* $\varpi_i^\vee$ are defined by the formulas: $(\varpi_i^\vee, \alpha_j) = \delta_{ij}$ (they are the fundamental weights of $\Phi^\vee$). Define ρ as the sum of the fundamental weights, and $\rho^\vee$ as the sum of the fundamental coweights. The *weight lattice* Λ is the $\mathbb{Z}$-span of the fundamental weights, and the *coweight lattice* $\Lambda^\vee$ is the $\mathbb{Z}$-span of the fundamental coweights. The elements of Λ and $\Lambda^\vee$ are called *weights* and *coweights*, respectively. The sets obtained by taking the non-negative integer span, instead of the whole $\mathbb{Z}$-span, are called the *dominant weights* Λ^+ and the *dominant coweights* $(\Lambda^\vee)^+$. Note that W_f also normalizes $\Lambda^\vee$, and we can form their semidirect product W_e which we call *extended affine Weyl group.* The group W_e contains W as a normal subgroup of finite index and W_e/W is isomorphic to $\Lambda^\vee/L^\vee$. In general, W_e is not a Coxeter group, but one can still define the length $l(w)$ of $w \in W_e$ by counting the hyperplanes separating A_{id} and wA_{id}.

For $\lambda, \mu \in V$, write $\mu \leq \lambda$ if $\lambda - \mu$ can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of Δ. This is called the *dominance order.* There exists a unique maximal root with respect to the dominance order, called the *highest root* which we denote by $\widetilde{\alpha} \in \Phi$.

Remove all the hyperplanes $H_{\alpha,k}$ from V. The connected components $\mathcal{A}$ of the resulting set are called *alcoves.* Fix an alcove (it does not matter the choice) A_{id} and call it the *fundamental alcove.* The affine Weyl group W acts simply transitively on $\mathcal{A}$ so we can identify each element w of W with its respective alcove $w \cdot A_{\mathrm{id}}$. It is also worth noticing that if $\overline{\mathcal{A}} = \{\overline{A} \mid A \in \mathcal{A}\}$, then W also acts simply transitively on $\overline{\mathcal{A}}$. For the rest of this thesis, our choice of fundamental alcove is

$$A_{\mathrm{id}} \coloneqq \{\lambda \in V \mid -1 < (\lambda, \alpha) < 0 \ \forall \alpha \in \Phi^+\}.$$

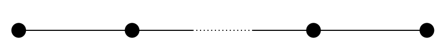
One can prove that $w_0 A_{\mathrm{id}}$ lies in the dominant cone, where w_0 is the longest element of W_f. The *walls* of A_{id} are the hyperplanes H_α, with $\alpha \in \Delta$, together with $H_{\widetilde{\alpha},-1}$ and the walls of an alcove $A = wA_{\mathrm{id}} \in \mathcal{A}$, with $w \in W$ (recall that each alcove is of this form) are the image of those hyperplanes under w. Let

$$S = \{s_\alpha \mid \alpha \in \Delta\} \cup \{s_{\widetilde{\alpha},-1}\},$$

then (W, S) is a Coxeter system. The extra generator $s_{\widetilde{\alpha},-1}$ will be denoted by s_0 so that if (W_f, S_f) has rank n, then $S = S_f \cup \{s_0\} = \{s_0, s_1, \ldots, s_n\}$ and therefore (W, S) has rank $n + 1$.

Recall that we started with an irreducible (reduced, crystallographic) root system. Those types of root systems can be classified, and belong to a finite collection of families. These families are named with a letter and a natural number. If a root system belongs to one of these families, for example A_2, we say that it is of ***type*** A_2. Every reducible root system is the direct sum of root systems belonging to these families. This classification agrees with the classification of finite crystallographic Coxeter groups, hence it can be stated in terms of the Coxeter graphs. These are called the ***Dynkin diagrams***. When we consider the corresponding affine Weyl group, as it still is a Coxeter group with one extra generator s_0, we can form its Coxeter graph by adding an extra vertex. These are called ***completed Dynkin diagrams***, which classify the affine Weyl groups. The families in the affine classification are denoted by adding a $\widetilde{\ }$ on top of the finite one, for example, there is an affine Weyl group of type $\widetilde{A_2}$.

Let $n \in \mathbb{N}$. One of the most important families in the finite (non-affine) classification, and of particular interest to us, is the family of type A_n. The Dynkin diagram corresponding to this type is



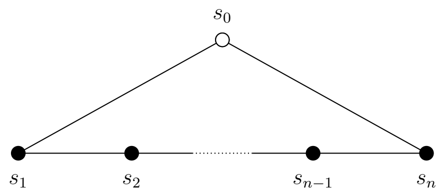
Thus, if a Coxeter system is of this type, it has generating set $S_f = \{s_1, \ldots, s_n\}$ and relations

$$s_i^2 = \text{id}, \qquad s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}, \qquad s_i s_j = s_j s_i \text{ if } |i-j| > 1.$$

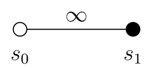
Let Φ be the root system of type A_n and let W_f be the corresponding finite Weyl group with simple reflections S_f. We also say that the Coxeter system (W_f, S_f) has type A_n. As a group, W_f is isomorphic to the symmetric group S_{n+1}, and as a Weyl group, it is self-dual, that is, $\alpha^\vee = \alpha$ for every root α. This means that, in this type, the distinction between roots and coroots, as well as weights and coweights, is non-existent. The ambient space V spanned by the simple roots Δ, is the hyperplane of $\mathbb{R}^{n+1}$ of vectors whose coordinate sum is zero. Let $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$. The simple roots α_i are $\varepsilon_i - \varepsilon_{i+1}$ with $1 \le i \le n$. The positive roots are of the form $\varepsilon_i - \varepsilon_j \in \Phi^+$, with $1 \le i < j \le n+1$. The highest root is $\widetilde{\alpha} = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \cdots + \alpha_n$. The ***Cartan matrix*** is the matrix whose j^{th} column is the coordinate vector of α_j in the fundamental weight basis. This $n \times n$ matrix is, in this case,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

It is known that the inverse of the Cartan matrix has positive entries. Let W be the corresponding affine Weyl group and s_0 the extra generator. As before, put $S = \{s_0\} \cup S_f$. The Coxeter system (W, S) has type $\widetilde{A_n}$. If $n \geq 2$, the corresponding completed Dynkin diagram is



We have labeled and colored its vertices according to S, for emphasis. The completed Dynkin diagram of $\widetilde{A_1}$ is



1.3 The elements $\theta(\lambda)$ and the group Ω

The purpose of this section is to explain some not so well-known facts about affine Weyl groups.

For the rest of this section, fix an irreducible root system Φ of rank n, let W_f be the finite Weyl group, and W the affine Weyl group, as in last section.

The subgroup of W_e of length 0 elements is denoted by Ω. This is the same as the subgroup consisting of the $\sigma \in W_e$ such that $\sigma A_{\text{id}} = A_{\text{id}}$. Thus, conjugation by Ω permutes the simple reflections[3]. We define $W_\sigma := \sigma W_f \sigma^{-1}$, which is isomorphic to W_f. Let $s_\sigma := \sigma s_0 \sigma^{-1}$, so that W_σ is the maximal (finite) parabolic subgroup of W, generated by $S \setminus \{s_\sigma\}$. Other equivalent realization is as a quotient (see [LPP21][4])

$$\Lambda^\vee / L^\vee \cong \Omega$$

Now we will define a specific system of representatives of $\Lambda^\vee / L^\vee$. Write the highest root as a combination of simple roots:

$$\widetilde{\alpha} = \eta_1 \alpha_1 + \cdots + \eta_n \alpha_n.$$

[3]In this way, Ω can be sen as a group of automorphisms of the completed Dynkin diagram.

[4]As one can check from their definitions, they are working with the dual root system so the result has been translated to our conventions.

One has that $\eta_i \in \mathbb{N}$. For each maximal (finite) parabolic subgroup of W, consider its generating set $S \setminus \{s_i\}$. It is not hard to check that the intersection of the corresponding reflecting hyperplanes is $v_i = -\varpi_i^\vee/\eta_i$, for $i \neq 0$, and $v_0 = O$. The set $\{O, -\varpi_i^\vee/\eta_i\}$ is precisely the set of vertices of the fundamental alcove A_{id}. A fundamental coweight $\varpi_i^\vee$ is called ***minuscule*** if $(\varpi_i^\vee, \widetilde{\alpha}) = 1$. This is equivalent to saying that $\eta_i = 1$. Let $M \subset I_n$ be the index set of the minuscule fundamental coweights. Both $\{O, -\varpi_i^\vee \mid i \in M\}$ and $\{O, \varpi_i^\vee \mid i \in M\}$ are complete systems of representatives of $\Lambda^\vee/L^\vee$.

It is known that for every $\sigma \in \Omega \setminus \{\text{id}\}$, $-\sigma(O)$ is a minuscule fundamental coweight. Furthermore, $\sigma \mapsto \sigma(O)$ is a bijection from Ω to the representatives $\{O, -\varpi_i^\vee \mid i \in M\}$ of $\Lambda^\vee/L^\vee$ (see [Bou02, Prop VI.2.3.6]). We will use this identification and put σ instead of $\sigma(O)$, by abuse of notation.

Let λ be a dominant coweight. We define $\theta(\lambda) \in W$ as the only element in the affine Weyl group such that $\theta(\lambda)A_{\text{id}} = w_0 A_{\text{id}} + \lambda$. These special elements can equivalently be described as the maximal length coset representatives of

$$\bigsqcup_{\sigma \in \Omega} W_f \backslash W / W_\sigma.$$

An important property of $\theta(\lambda)$ is that it is ***maximal in its double coset***, in the following sense. Let $\sigma \in \Omega$ such that $\lambda \in \sigma + L^\vee$, under the identification we just described. Then $\theta(\lambda)$ is the maximal element of its double coset $W_f\, \theta(\lambda) W_\sigma$.

Finally, let $\lambda \in \Lambda^\vee$. It is easy to check that $W \cdot \lambda = \lambda + L^\vee$. If A is any alcove then $A + \lambda$ is also an alcove, as one can easily see (we already used this in the definition of $\theta(\lambda)$). A direct computation shows that if $\lambda = m_1\varpi_1^\vee + \cdots + m_n\varpi_n^\vee$, then $\lambda - s_i(\lambda) = m_i\alpha_i^\vee$, where s_i is a simple reflection with $i > 0$.

2. The baby example, type $\widetilde{A_2}$

The vast majority of the results presented in this thesis, arose from furiously observing the situation occurring in type $\widetilde{A_2}$. In this chapter, we will illustrate this case -in a non-rigorous way- so that the generalizations in the upcoming chapters become more transparent. This particular case is interesting since it is small enough to be easy to handle, but large enough so that one can actually visualize what is going on. In type $\widetilde{A_1}$ the geometric formula is extremely easy to prove, but provides little insight.

Let Φ be the root system of type $\widetilde{A_2}$, and let W_f, W be the finite and affine Weyl groups attached to Φ. Recall that this type is self-dual, i.e. $\Phi = \Phi^\vee$. We identify the group W with its alcoves, which are equilateral triangles. Pick a color for each generator s_0, s_1, s_2 of W, for example, blue, red and green, respectively. Let $A = wA_{\mathrm{id}}$ be any alcove, with $w \in W$. Note that each alcove having a common edge with A is of the form ws_iA_{id}. Color that edge of A according to the simple reflection s_i. By doing this to all the edges of all the alcoves, one gets what is called the *Coxeter complex* of W. Put $(a,b) = a\varpi_1 + b\varpi_2 \in \Lambda^+$ ($a, b \in \mathbb{Z}_{\geq 0}$). In the following figure, we illustrate the Coxeter complex together with some key objects. The triangle with the big black dot is the identity triangle, its upper vertex O is the origin of the ambient space, the black arrows are the simple roots and the pink arrows are the fundamental weights. The triangle with the orange dot corresponds to the longest element $w_0 = \theta(0,0)$ of W_f, and the triangles with the turquoise dots correspond to all the $\theta(a,b)$ (for $(a,b) \neq (0,0)$) that fit in the figure. The special elements θ were defined in Section 1.3, but in this case, $\theta(a,b)$ is the triangle "pointing down" with lower vertex $a\varpi_1 + b\varpi_2$.

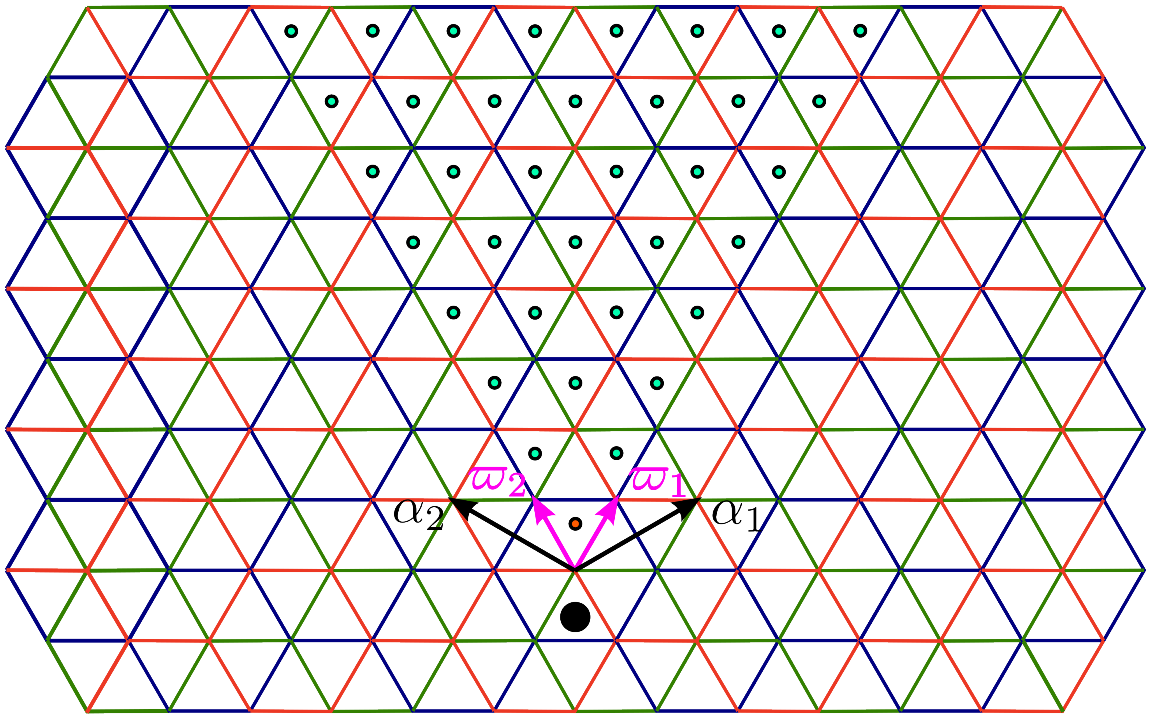


Figure 2.1: Coxeter complex of W

The subgroup W_f of W, generated by $\{s_1, s_2\}$, is isomorphic to the symmetric group S_3. The longest element w_0 of W_f has the reduced expressions

$$w_0 = s_1 s_2 s_1 = s_2 s_1 s_2.$$

The alcoves corresponding to W_f are

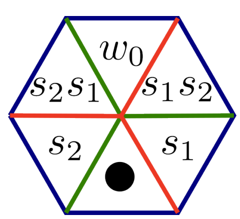


Figure 2.2: Alcoves of $W_f \cong S_3$

Write $\leq \theta(a,b) = [\mathrm{id}, \theta(a,b)]$. Figure 2.3 illustrates the set $\leq \theta(a,b)$. The yellow dots, from bottom to top, correspond to the elements $\theta(3,0)$ and $\theta(4,1)$ of W. The light grey area is the set $\leq \theta(3,0)$ and the darker area -together with the light grey area- corresponds to $\leq \theta(4,1)$.

The following general construction for the set $\leq \theta(a,b)$[1] is proven in [LP20]. The set $\leq \theta(a,0)$ (resp. $\leq \theta(0,b)$) is the "equilateral triangle" (with zigzag sides instead of straight lines) whose center is the origin O, its three medians are the three reflecting hyperplanes passing through O and it is the minimal triangle containing $\theta(a,0)$ (resp. $\theta(0,b)$) that also satisfies these two properties. The set $\leq \theta(a+1,b+1)$ is obtained from $\leq \theta(a,b)$ by adding the corresponding surrounding hexagons.

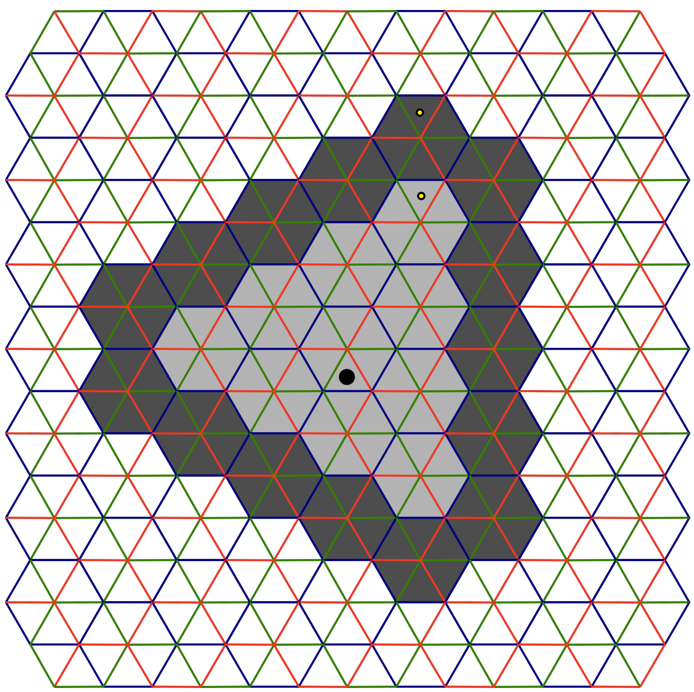


Figure 2.3: Alcoves of $\leq \theta(4,1)$

For instance, $\leq \theta(4,1)$ is the union of $\leq \theta(3,0)$, the light grey area, together with all the hexagons surrounding it, which are the ones painted in dark grey. These are the

[1]As we said, we are identifying W with its alcoves so by the set $\leq \theta(a,b)$ we actually mean the (topological) closure of the alcoves corresponding to $\leq \theta(a,b)$.

blue hexagons having a common blue edge with $\leq \theta(3,0)$.

Now, for $J \subset I_2$, and $\lambda = (a,b) \in \Lambda^+$, we will compute the volumes $V_J(\lambda) = V_J^{A_2}(\lambda)$ geometrically (for a rigorous computation, see Chapter 4). By definition,

$$V_\emptyset(\lambda) = 1.$$

For $J = \{1\}$, $V_{\{1\}}(\lambda)$ is just the 1-dimensional volume of $\mathrm{Conv}(W_{\{1\}} \cdot \lambda)$ which is $\|\lambda - s_1 \cdot \lambda\| = \|a\alpha_1\|$. The case $J = \{2\}$ is analogous. We have

$$V_{\{1\}}(\lambda) = a\sqrt{2}, \qquad\qquad V_{\{2\}}(\lambda) = b\sqrt{2}.$$

Finally, we compute the (2-dimensional) volume $V_{\{1,2\}}(\lambda)$. By definition, this is the volume of the convex hull of $W_f \cdot \lambda$, which is an hexagon (that may be degenerated if at least one of a, b are zero). By drawing lines from O to each $w \cdot \lambda$ with $w \in W_f$, we compute this volume as the area of 6 triangles (some of them will be degenerated if the hexagon is), three of which have base $V_{\{1\}}(\lambda)$. The height of these triangles is easy to determine by noticing that ϖ_2 must be orthogonal to the line containing $W_{\{1\}} \cdot \lambda$. The remaining 3 triangles all have base $V_{\{2\}}(\lambda)$ and the height can be computed similarly. We get

$$V_{\{1,2\}}(\lambda) = \frac{\sqrt{3}}{2}\left(a^2 + b^2 + 4ab\right).$$

In [LP20], by using the general construction of the set $\leq \theta(\lambda)$ explained above, the authors were able to show that

$$| \leq \theta(\lambda)| = 3a^2 + 3b^2 + 12ab + 9a + 9b + 6. \tag{2.1}$$

It follows that the Geometric Formula (1) is satisfied with $\vartheta_\emptyset = 6$, $\vartheta_{\{1\}} = \vartheta_{\{2\}} = 9\sqrt{2}/2$ and $\vartheta_{\{1,2\}} = 2\sqrt{3}$.

Now let us look at this situation from another angle. Let X be the closure of the alcoves corresponding to $\leq \theta(\lambda)$. Note that the hexagon $P^2 = \text{Conv}(W_f \cdot \lambda)$ is completely contained in X. We can construct X starting from P^2. For each edge E of P^2 take the infinite rectangle with base E, in the opposite direction of P^2. Let P^1 be the intersection of these rectangles with X and let P^0 be what is left of X. We have just constructed a partition[2] $X = P^0 \sqcup P^1 \sqcup P^2$. Figure 2.4 shows this partition for $\leq \theta(4,1)$. The big turquoise hexagon is P^2, P^1 is the orange region, and P^0 is the union of the magenta triangles. The yellow big dot is $\lambda = (4,1)$ and the magenta triangle containing the small yellow dot is $\theta(4,1)$.

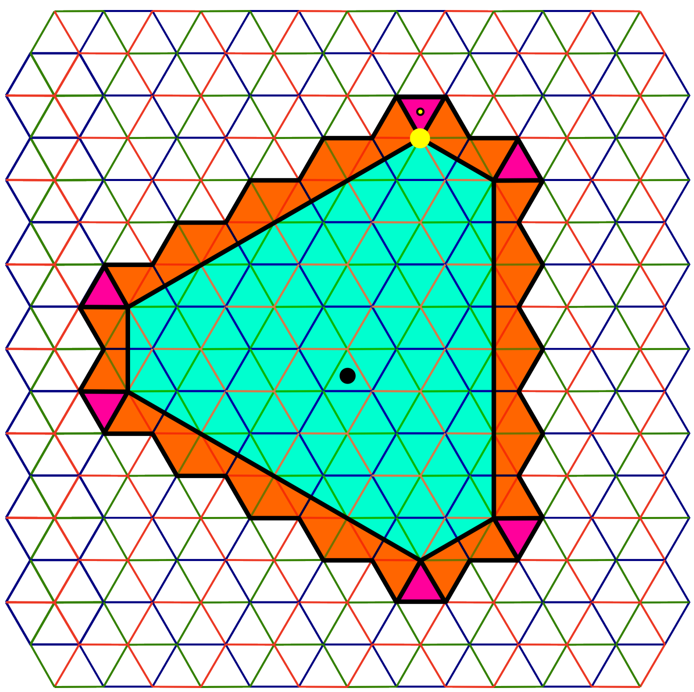


Figure 2.4: Partition of $\leq \theta(4,1)$

Let $\text{Vol}(A_{\text{id}})$ be the 2-dimensional volume of the fundamental alcove (or any alcove).

[2]Actually, P^1 still intersects the boundary of P^2 but this is not important as the intersection has volume zero. This can be easily fixed by removing the base E of the rectangles in the definition of P^1.

It follows that

$$\mathrm{Vol}(A_{\mathrm{id}})\,|\leq\theta(\lambda)| = \mathrm{Vol}(P^0)+\mathrm{Vol}(P^1)+\mathrm{Vol}(P^2).$$

In type $\widetilde{A_2}$ one can easily check that $\mathrm{Vol}(A_{\mathrm{id}})=\sqrt{3}/6$. Hence,

$$\frac{\mathrm{Vol}(P^0)}{\mathrm{Vol}(A_{\mathrm{id}})}=6.$$

Note that $\mathrm{Vol}(P^2)=V_{\{1,2\}}(\lambda)$, so that

$$\frac{\mathrm{Vol}(P^2)}{\mathrm{Vol}(A_{\mathrm{id}})}=3a^2+3b^2+12ab.$$

Looking at equation (2.1) we notice that, for $i=0,2$, $\mathrm{Vol}(P^i)/\mathrm{Vol}(A_{\mathrm{id}})$ is the homogeneous part of degree i of the polynomial $|\leq\theta(\lambda)|$. Therefore, it must be that $\mathrm{Vol}(P^1)/\mathrm{Vol}(A_{\mathrm{id}})$ is the homogeneous part of degree 1, so that the number of triangles (including fractions of triangles) contained in each color, in figure 2.4, is the corresponding homogeneous part of the polynomial $|\leq\theta(\lambda)|$.

Observe that P^1 still contains a lot of information about $|\leq\theta(\lambda)|$. For instance, from figure 2.4 one can guess that $\mathrm{Vol}(P^1)/\mathrm{Vol}(A_{\mathrm{id}})$ is a linear combination of $V_{\{1\}}(\lambda)$ and $V_{\{2\}}(\lambda)$, in fact,

$$\frac{\mathrm{Vol}(P^1)}{\mathrm{Vol}(A_{\mathrm{id}})}=\frac{9\sqrt{2}}{2}\left(V_{\{1\}}(\lambda)+V_{\{2\}}(\lambda)\right).$$

This suggests another way to prove the geometric formula...

Part II

The general case

3. Geometric constructions

In this chapter we will generalize and study the geometric constructions shown in Chapter 2 of the Bruhat interval $\leq \theta(\lambda) = [\mathrm{id}, \theta(\lambda)]$ for a dominant coweight λ. We will go from full generality, to the particular case of type $\widetilde{A_3}$.

Let Φ be the an irreducible root system of rank $n \in \mathbb{N}$, let W_f be the corresponding Weyl group and W the affine Weyl group. Let V be the Euclidean space spanned by Φ. For a subset $X \subset W$, let $A(X)$ be the closure (in V) of the alcoves corresponding to X, that is,

$$A(X) = \bigsqcup_{x \in X} x\overline{A}_{\mathrm{id}}.$$

Although the union is not disjoint, the intersection of the alcoves occurs in their walls which is a set of (n-dimensional) volume zero. Whenever the objects of any union are either disjoint or intersect each other in their boundaries, by abuse of notation we will write it as a disjoint union regardless. The reason for this, is that we only care about the volume of the involved objects (see Chapter 5).

By geometric constructions we mean the two constructions given for the set $A(\leq \theta(a,b))$ (in type $\widetilde{A_2}$) in Chapter 2. One is the partition $\{P^0, P^1, P^2\}$, shown in figure 2.4, and the other one is the general construction given in [LP20]. We will call the former the *Geometric Partition*, and the latter the *Polytope Construction*. The polytope construction can be generalized in a bigger picture and is mostly algebraic, whereas the geometric partition is purely geometric and depends completely on the polytope construction.

3.1 Polytope construction

Consider the set $A(W_f) = A(\leq \theta(0))$ consisting of the closure of $|W_f|$ alcoves intersecting in the common point O, the origin of V. Note that every alcove whose closure contains this point, is contained in $A(W_f)$. This set defines a polytope[1] which we call the W_f*-polytope* and denote by $\mathcal{P}_{W_f}$.

Remark 3.1. By $\mathcal{P}_{W_f}$ we mean both the polytope, as a geometric object, and the set $A(W_f)$ defined by this polytope on V. With this in mind, for $t \in V$, the set $\mathcal{P}_{W_f} + t \subset V$ is a W_f-polytope whose ***center*** is t, and if $t \in \Lambda^\vee$ then $\mathcal{P}_{W_f}$ is composed by a set of alcoves (since the translation by a coweight of an alcove is still an alcove). Also, a nice description for the W_f-polytope centered at the origin (see [LP12]) is

$$\mathcal{P}_{W_f} = \{v \in V \mid -1 \leq (v, \alpha) \leq 1, \text{ for all } \alpha \in \Phi^+\}. \tag{3.1}$$

Using this formula, it is not difficult to check that

$$s_{\alpha,k}\left(\mathcal{P}_{W_f} + t\right) = \mathcal{P}_{W_f} + s_{\alpha,k}(t).$$

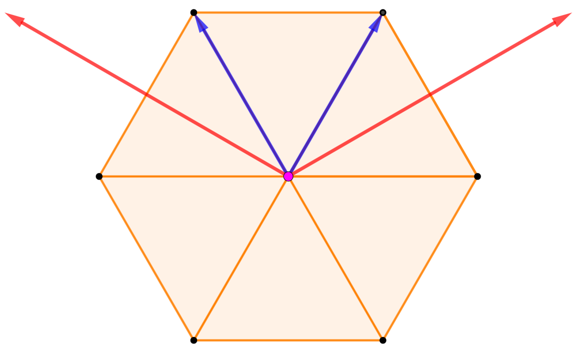
This implies that the alcoves corresponding to each coset wW_f define a W_f-polytope whose center is the image of O under w, in formulae, $A(wW_f) = \mathcal{P}_{W_f} + w(O)$. It is readily seen that $\mathcal{P}_{W_f}$ tessellates V, as

$$V = A(W) = \bigsqcup_{w \in W^f} A(wW_f) = \bigsqcup_{w \in W^f} \mathcal{P}_{W_f} + w(O), \tag{3.2}$$

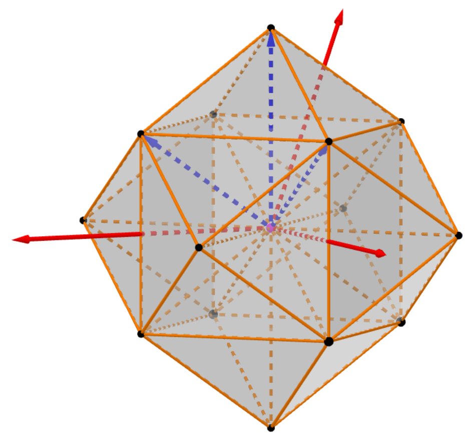
where W^f is the minimal length coset representatives of W/W_f (it does not matter the choice of representatives).

[1] In the literature, convex polytopes defined by the closure of a set of alcoves are called *alcoved polytopes*.

Example 3.2. Consider $\mathcal{P}_{W_f}$ centered at O. The alcove corresponding to $\theta(0) = w_0$ is contained in this this polytope. In type $\widetilde{A_n}$, the vertices of this alcove are precisely O together with the fundamental weights ϖ_i. Recall that this type is self-dual and that the finite Weyl group (of type A_n) is isomorphic to the symmetric group S_{n+1}. The following figure contains the S_3-polytope (also shown in figure 2.2) and the S_4-polytope. The blue arrows are the fundamental weights and the red arrows the simple roots. The (very small) pink dot is the origin.



S_3-polytope



S_4-polytope

Figure 3.1: W_f-polytope for types $\widetilde{A_2}$ and $\widetilde{A_3}$

There are distinct ways to tessellate V with $\mathcal{P}_{W_f}$ without slicing any alcove. Recall the group Ω of length 0 elements of the extended affine Weyl group W_e. The closure of the alcoves contained in $A(W_\sigma)$ intersect each other in a common point. The identification discussed in Section 1.3, allows us to identify σ with that point. Thus, $A(W_\sigma) = \mathcal{P}_{W_f} + \sigma$ is another W_f-polytope and so is every coset wW_σ. Thus, every $\sigma \in \Omega$ gives an alcovic tessellation of V

$$V = \bigsqcup_{w \in W^\sigma} \mathcal{P}_{W_f} + w(\sigma), \tag{3.3}$$

where W^σ is the minimal length coset representatives of W/W_σ. The identity element in Ω, gives the tessellation described in (3.2).

In type $\widetilde{A_2}$, the W_f-polytope is an hexagon (see example 3.2) and the group Ω is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, so there are exactly three σ-tessellations of V. These are the blue, red and green tessellations by hexagons shown in figure 2.1, corresponding to $O, -\varpi_1$ and $-\varpi_2$, respectively. The following lemma should further clarify the geometric role played by the group Ω.

Lemma 3.3. *For $\lambda \in (\Lambda^\vee)^+$, let $\sigma \in \Omega$ be such that $\lambda \in \sigma + L^\vee$. Then, in the σ-tessellation of V, the W_f-polytope containing the alcove of $\theta(\lambda)$, has center λ.*

Proof. Write $\lambda = \sigma + x$, and denote by $t_x \in W$ the translation by x. Note that the closure of the alcove corresponding to $t_x^{-1}\theta(\lambda)$, is

$$t_x^{-1}\left(w_0\overline{A}_{\text{id}} + \lambda\right) = \left(w_0\overline{A}_{\text{id}} + \lambda\right) - x = w_0\overline{A}_{\text{id}} + \sigma,$$

which obviously has σ as one of its vertices. Thus, we have that $t_x^{-1}\theta(\lambda) \in W_\sigma$, so that $t_xW_\sigma = \theta(\lambda)W_\sigma$. Equivalently, the W_f-polytope corresponding to $\theta(\lambda)W_\sigma$ is

$$\begin{aligned} A(\theta(\lambda)W_\sigma) &= A(t_xW_\sigma) \\ &= t_xA(W_\sigma) \\ &= \left(\mathcal{P}_{W_f} + \sigma\right) + x \\ &= \mathcal{P}_{W_f} + \lambda. \end{aligned}$$

In the notation of equation (3.3), $\theta(\lambda)(\sigma) = \lambda$. □

For $\lambda \in (\Lambda^\vee)^+$, define $\operatorname{Conv}(\lambda)$ as the convex hull of the orbit $W_f \cdot \lambda$. The following theorem generalizes the polytope construction shown in Chapter 2 for type $\widetilde{A_2}$.

Theorem 3.4 (Polytope construction theorem). *Let λ be a dominant coweight. The set $A(\leq \theta(\lambda))$ is tessellated by $\mathcal{P}_{W_f}$. Furthermore, let $\mathcal{C}_\lambda$ be the set consisting of the centers of the W_f-polytopes involved in the tessellation. Then $\mathcal{C}_\lambda = \mathit{Conv}(\lambda) \cap (\lambda + L^\vee)$.*

Remark 3.5. This result is implicit in the literature, although we did not find it explicitly. It is the first emergence of the relationship between the alcovic geometry and the Euclidean geometry.

It is important to note that, as a consequence of the theorem, all of the W_f-polytopes that compose $A(\leq \theta(\lambda))$, must have their centers lying in Conv(λ). Also, we can write

$$A(\leq \theta(\lambda)) = \bigsqcup_{t \in \mathcal{C}_\lambda} \mathcal{P}_{W_f} + t.$$

In fact, the formula Schützer provided in [Sch12] is for $|\mathcal{C}_\lambda|$. From this equation, it follows that

$$|\leq \theta(\lambda)| = \frac{\mathrm{Vol}(A(\leq \theta(\lambda)))}{\mathrm{Vol}(A_{\mathrm{id}})} = |W_f||\mathcal{C}_\lambda|,$$

since the volume of $\mathcal{P}_{W_f}$ is $|W_f|\mathrm{Vol}(A_{\mathrm{id}})$.

For type $\widetilde{A_2}$, the tessellation part is readily seen in figure 2.3 since, in this figure, $A(\leq \theta(a,b))$ is completely composed by "blue" W_f-polytopes (the color blue corresponds to the identity in Ω). The rest of the statement can be visualized in figure 2.4.

Proof of Theorem 3.4. Let $\sigma \in \Omega$ be such that $\lambda \in \sigma + L^\vee$. Let w_0, w_0^σ be the longest elements of the maximal parabolic subgroups W_f, W_σ of W, respectively.

Claim 3.4.1. *There exists $x \in W$ such that $\theta(\lambda) = w_0 x$. Similarly, there exists $y \in W$ such that $\theta(\lambda) = y w_0^\sigma$.*

Proof of Claim 3.4.1. There is a unique factorization $\theta(\lambda) = wx$, such that $w \in W_f$, $x \in {}^fW := {}^{I_n}W$ and $l(\theta(\lambda)) = l(w) + l(x)$. It suffices to show that $w = w_0$, or equivalently, that the left descent set of w is S_f. Let $s \in S_f$ and note that $sw \in W_f$ so that the unique factorization $s\theta(\lambda) = swx = (sw)x$ implies $l(s\theta(\lambda)) = l(sw) + l(x)$. Since $\theta(\lambda)$ is maximal in its coset $W_f\,\theta(\lambda)$, we have that $l(s\theta(\lambda)) < l(\theta(\lambda))$. It follows $l(sx) < l(x)$ and thus $x = w_0$. Similarly, decompose $\theta(\lambda)$ in $W^\sigma \cdot W_\sigma$ and use the maximality of $\theta(\lambda)$ in $\theta(\lambda)W_\sigma$ to get the desired claim. ■

Let x, y be the elements of W in the claim. Note that $\leq \theta(\lambda) = (\leq w_0) \cdot (\leq x)$. Indeed $\supset$ is clear, and for the inclusion $\subset$ let $u \leq \theta(\lambda)$. Consider the factorization $u = u_f \cdot {}^f u$. It is clear that $u_f \leq w_0$, and that ${}^f u \leq {}^f\theta(\lambda)$ since this projection is order-preserving. In the proof of the claim we saw that ${}^f\theta(\lambda) = x$. Analogously, $\leq \theta(\lambda) = (\leq y) \cdot (\leq w_0^\sigma)$. Therefore, the set $\leq \theta(\lambda)$ is W_f-invariant on the left and W_σ-invariant on the right, as $\leq w_0 = W_f$ and $\leq w_0^\sigma = W_\sigma$.

The following beautiful fact is in [LPP21]. For every $\sigma' \in \Omega$, the map θ defines a bijection

$$\theta : (\sigma' + L^\vee) \cap (\Lambda^\vee)^+ \xrightarrow{\sim} W_f \backslash W / W_{\sigma'},$$

which intertwines the dominance order[2] on the left with the Bruhat order on the right. Moreover, its image is the set of maximal length representatives of $W_f \backslash W / W_{\sigma'}$.

For brevity, put $X_\lambda = \{\mu \in (\Lambda^\vee)^+ \mid \mu \leq \lambda\}$. The bijection shows that $\theta(\mu) \leq \theta(\lambda)$ if and only if $\mu \in X_\lambda$. By the invariance of $\leq \theta(\lambda)$ mentioned above, we have that $W_f\,\theta(\mu)W_\sigma$ is contained in the set $\leq \theta(\lambda)$, for every $\mu \in X_\lambda$. Conversely, let $u \leq \theta(\lambda)$. The maximal element of $W_f\,uW_\sigma$ is $\theta(\mu)$ for some μ. Again, the double coset $W_f\,uW_\sigma$ is contained in $\leq \theta(\lambda)$, so that $\theta(\mu) \leq \theta(\lambda)$ and thus $\mu \in X_\lambda$. It follows that

$$\begin{aligned} \leq \theta(\lambda) &= \bigsqcup_{\mu \in X_\lambda} W_f \theta(\mu) W_\sigma \\ &= W_f \cdot \left(\bigsqcup_{\mu \in X_\lambda} \theta(\mu) W_\sigma \right). \end{aligned}$$

[2]Here, the dominance order is taken with the coroot lattice instead of the root lattice.

Looking at the corresponding alcoves and using Lemma 3.3, we get

$$\begin{aligned} A(\leq \theta(\lambda)) &= W_f \cdot A\left(\bigsqcup_{\mu \in X_\lambda} \theta(\mu) W_\sigma\right) \\ &= W_f \cdot \left(\bigsqcup_{\mu \in X_\lambda} \mathcal{P}_{W_f} + \mu\right) \\ &= \bigsqcup_{\mu \in W_f \cdot X_\lambda} \mathcal{P}_{W_f} + \mu, \end{aligned}$$

which establishes the tessellation part of the theorem.

Now let C^+ be the closure of the dominant Weyl chamber. Note that $\text{Conv}(\lambda) \cap C^+$ consists precisely of the elements $\lambda - (x_1\alpha_1^\vee + \cdots + x_n\alpha_n^\vee) \in C^+$ with all the $x_i \geq 0$. It follows that $X_\lambda = \text{Conv}(\lambda) \cap (\lambda + L^\vee) \cap C^+$. Since $\text{Conv}(\lambda)$, $\lambda + L^\vee$ are W_f-invariant and $V = W_f \cdot C^+$, we obtain

$$\mathcal{C}_\lambda \coloneqq W_f \cdot X_\lambda = \text{Conv}(\lambda) \cap (\lambda + L^\vee),$$

where the action is point-wise. □

3.2 Geometric partition

In this section, we will do some general work concerning the polytope $\text{Conv}(\lambda)$ and then we will illustrate the geometric partition in type $\widetilde{A_3}$. The reason for this, is that some recent work shows that $\text{Conv}(\lambda) \subset A(\leq \theta(\lambda))$ does not hold in general. It is true in type $\widetilde{A_2}$ (see figure 2.4) and we strongly believed this was a general fact -and easy to prove, using the polytope construction theorem 3.4-. The inclusion is still true in type $\widetilde{A_3}$ and the geometric partition can be replied whenever $\text{Conv}(\lambda) \subset A(\leq \theta(\lambda))$ (but some minor details will need to be addressed).

For the moment, Φ still is any irreducible root system of rank n. The set $\text{Conv}(\lambda)$

defines a polytope, whose vertex set is $W_f \cdot \lambda$. In order for this polytope to be non-degenerated, it must be that λ is not on a wall of the dominant chamber C^+. That is, if $\lambda = m_1\varpi_1^\vee + \cdots m_n\varpi_n^\vee = (m_1, \ldots, m_n)$, with $m_i \in \mathbb{Z}_{\geq 0}$, then it is necessary (and sufficient) that $m_i > 0$ for all i. When this is the case, we say that λ is *generic*. This implies that for any $w \in W_f$, $w\lambda$ lies in the dominant cone if and only if $w = \mathrm{id}$. For example, in type $\widetilde{A_2}$, Conv$(4, 1)$ is an hexagon -and so is every Conv(a, b) with $a, b > 0$-, but Conv$(3, 0)$ is a triangle (see figure 2.3 and draw the corresponding convex hull).

Let λ be a generic dominant coweight. We will describe the faces -of any dimension- of this polytope. Let $J \subset I_n$ such that $|J| = n - 1$, and consider the hyperplane H_J of V, generated by $\{\alpha_j \mid j \in J\}$. Let $t \in I_n \setminus J$ and define the closed half-space

$$H_J^- \coloneqq \{v \in V \mid (v, \varpi_t) \leq 0\}.$$

It is not hard to prove that

$$\mathrm{Conv}(\lambda) = \bigcap_{\substack{J \subset I_n \\ |J| = n-1}} W_f \cdot \left(\lambda + H_J^-\right),$$

where the action is point-wise. For $0 \leq i \leq n$, let $\mathcal{F}_i(\lambda)$ be the collection of i-faces of Conv(λ). For $J \subset I_n$, define $F_J(\lambda) = \mathrm{Conv}(W_J \cdot \lambda)$ (W_J is a subgroup of W_f since $0 \notin J$) and let $i = |J|$. The equation above implies that $F_J(\lambda) \in \mathcal{F}_i(\lambda)$ is a face with vertices $W_J \cdot \lambda$. In the dominant cone,

$$F_J(\lambda) \bigcap C^+ = \Big(\lambda - \langle \alpha_j \mid j \in J\rangle_{\geq 0}\Big) \bigcap C^+, \tag{3.4}$$

where $\langle \cdot \rangle_{\geq 0}$ means the $\mathbb{R}_{\geq 0}$-span (see [Hal15]). We say that a face F of Conv(λ) is a *J-face*, if $F \in W_f \cdot F_J(\lambda)$. All the J-faces are i-faces, but the converse is not true in general. It is important to keep in mind that the face $F_J(\lambda)$ is *the* J-face that touches the vertex λ, for example, $F_\emptyset(\lambda) = \{\lambda\}$. One can prove that all the i-faces that contain the vertex λ are $F_K(\lambda)$ for some $K \subset I_n$ with $|K| = i$. Similarly, for an i-face F, there exists $w \in W_f$ such that λ is a vertex of wF. Hence,

$$\mathcal{F}_i(\lambda) = \{W_f \cdot F_J(\lambda) \mid |J| = i\}.$$

It is natural to ask oneself about the stabilizer of $F_J(\lambda)$ as a set. By definition of this J-face, it is clear that

$$\mathrm{Stab}_{W_f}(F_J(\lambda)) = W_J, \tag{3.5}$$

since $w\mathrm{Conv}(W_J \cdot \lambda) = \mathrm{Conv}(wW_J \cdot \lambda)$. In particular, $\mathrm{Conv}(\lambda)$ is W_f-invariant (on the left). If λ is not generic, only $\supset$ holds in general.

Regarding the intersection of faces of $\mathrm{Conv}(\lambda)$, the common vertices of $F_J(\lambda)$ and $F_K(\lambda)$ are $W_{J\cap K} \cdot \lambda$. Hence, the face obtained from their intersection is

$$F_J(\lambda) \bigcap F_K(\lambda) = F_{J\cap K}(\lambda). \tag{3.6}$$

To describe the intersection of two arbitrary faces of $\mathrm{Conv}(\lambda)$, it suffices to understand the face $F_J(\lambda) \cap wF_K(\lambda)$[3] for any $w \in W_f$. This comes from

$$xF_J(\lambda) \bigcap yF_K(\lambda) = x\left(F_J(\lambda) \bigcap x^{-1}yF_K(\lambda)\right),$$

with $x, y \in W_f$. A common vertex of $F_J(\lambda)$ and $wF_K(\lambda)$, if it exists, is of the form $w_J\lambda = ww_K\lambda$, with $w_J \in W_J$, $w_K \in W_K$. Since λ is generic, it follows that $w = w_J w_K^{-1}$.

[3]The empty set is considered a face of dimension -1, by convention.

This assures that $F_J(\lambda) \cap wF_K(\lambda) \neq \emptyset$ if and only if $w \in W_J W_K$. Hence,

$$F_J(\lambda) \bigcap wF_K(\lambda) = \begin{cases} xF_{J\cap K}(\lambda) & \text{if } w \in W_J W_K \\ \emptyset & \text{if } w \notin W_J W_K \end{cases} \tag{3.7}$$

for some $x \in W_J$.

Now let us work in type $\widetilde{A_3}$, which is self-dual. Some general facts concerning this type can be found at the end of Section 1.2. First, we will establish a geometric relation between the polytope $\mathrm{Conv}(\lambda)$ and the set $A(\leq \theta(\lambda))$, which will let us dive into the geometric partition. Fix a generic dominant weight λ.

Proposition 3.6. *In type $\widetilde{A_3}$, Conv$(\lambda) \subset A(\leq \theta(\lambda))$ holds. Furthermore, let $\mu \in \lambda + L$. If the interior of $\mathcal{P}_{W_f} + \mu$ intersects some face F (of any dimension) of the polytope Conv(λ), then $\mu \in F$.*

This proposition, together with the polytope construction theorem, tells us that $\mathrm{Conv}(\lambda)$ is covered by the W_f-polytopes with centers in $\mathcal{C}_\lambda$. Furthermore, let F be an i-face and define $\mathcal{C}_\lambda^F := \mathcal{C}_\lambda \cap F$. The proposition can be restated as

$$F \subset \bigsqcup_{\mu \in \mathcal{C}_\lambda^F} \mathcal{P}_{W_f} + \mu, \tag{3.8}$$

for every face F of $\mathrm{Conv}(\lambda)$. We write $\mathcal{C}_\lambda^J = \mathcal{C}_\lambda^{F_J(\lambda)}$. Note that if $J = I_3$, by Theorem 3.4 equation (3.8) becomes $\mathrm{Conv}(\lambda) \subset A(\leq \theta(\lambda))$, since $F_{I_3}(\lambda) = \mathrm{Conv}(\lambda)$, so that $\mathcal{C}_\lambda^{I_3} = \mathcal{C}_\lambda$.

Before going into the proof, let us briefly show some basic properties of $\mathcal{C}_\lambda^J$. Define L_J^+ as the non-negative $\mathbb{Z}$-span of the simple roots α_j, with $j \in J$. By equation (3.4),

$$\mathcal{C}_\lambda^J \cap C^+ = \left(\lambda - L_J^+\right) \cap C^+.$$

Finally, suppose that there is some $w \in W_f$ such that $\mathcal{C}_\lambda^J = w\mathcal{C}_\lambda^J$. By definition, this implies that $\lambda \in \mathcal{C}_\lambda^J \subset F_J(\lambda) \cap wF_J(\lambda)$. By (3.7), we have that $w \in W_J$. Therefore,

$$\mathrm{Stab}_{W_f}(\mathcal{C}_\lambda^J) = W_J, \tag{3.9}$$

since the inclusion $\supset$ is clear.

Proof of Proposition 3.6. Since λ is generic, we know that the i-faces of $\mathrm{Conv}(\lambda)$ intersecting this chamber are of the form $F_J(\lambda)$, with $J \subset I_3$ and $|J| = i$. It suffices to show that

$$F_J(\lambda) \subset \bigsqcup_{\mu \in \mathcal{C}_\lambda^J} \mathcal{P}_{W_f} + \mu, \tag{3.10}$$

for every $J \subset I_3$.

The only vertex of $\mathrm{Conv}(\lambda)$ in this chamber is λ. It is clear that this vertex is covered by $\mathcal{P}_{W_f} + \lambda$, and that $\lambda \in \mathcal{C}_\lambda$. Therefore, equation (3.10) holds for $J = \emptyset$. We will now consider the case $|J| = 1$.

Claim 3.6.1. *We claim that if α is a simple root and μ is a weight, then the segment joining μ and $\mu + \alpha$ is covered by*

$$\left(\mathcal{P}_{W_f} + \mu\right) \bigcup \left(\mathcal{P}_{W_f} + \mu + \alpha\right).$$

Proof of Claim 3.6.1. This segment is $\mathrm{Conv}(\mu, \mu + \alpha)$. It is a consequence of formula (3.1) that $\alpha/2$ and $-\alpha/2$ belong to $\mathcal{P}_{W_f}$. As $\mathcal{P}_{W_f}$ is convex, it follows that

$$\mathrm{Conv}\left(\mu, \mu + \frac{\alpha}{2}\right) \subset \mathcal{P}_{W_f} + \mu,$$

and

$$\mathrm{Conv}\left(\mu + \frac{\alpha}{2}, \mu + \alpha\right) \subset \mathcal{P}_{W_f} + \mu + \alpha,$$

which proves our claim. $\blacksquare$

Now, the elements of $F_{\{i\}}(\lambda)$ are of the form $\lambda - x_i\alpha_i$, with $x_i \in \mathbb{R}_{\geq 0}$. Also, the elements of $\mathcal{C}_\lambda^{\{i\}} = \mathcal{C}_\lambda \cap F_{\{i\}}(\lambda)$ are of the form $\lambda - m_i\alpha_i$, with $m_i \in \mathbb{Z}_{\geq 0}$. By the claim above, we see that the W_f-polytopes corresponding to $\mathcal{C}_\lambda^{\{i\}}$ cover the edge $F_{\{i\}}(\lambda)$, which establishes (3.10) for $|J| = 1$.

We will prove (3.10) for $|J| = 3$. Let $\mu \in (\lambda + L) \cap C^+ \setminus \mathcal{C}_\lambda$. Write $\lambda - \mu$ in the simple root basis: $\lambda - \mu = a\alpha_1 + b\alpha_2 + c\alpha_3$. Then $a, b, c \in \mathbb{Z}$ and one of them must be negative. On the other hand, let ν_{id} be the set of vertices of the fundamental alcove minus the origin. Let $\nu_{\mathcal{P}} = W_f \cdot \nu_{\text{id}}$. We can think of $\mathcal{P}_{W_f}$ as the convex hull of $\nu_{\mathcal{P}}$. In this type, $\nu_{\text{id}} = \{-\varpi_1, -\varpi_2, -\varpi_3\}$ and $\nu_{\mathcal{P}}$ consists of $\pm\varpi_i$ with $1 \leq i \leq 3$, $\varpi_i - \varpi_j$ with $i \neq j$, and $\pm(\varpi_2 - \varpi_1 - \varpi_3)$. Use the inverse of the Cartan matrix

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

to write the vertices $\nu_{\mathcal{P}}$ of $\mathcal{P}_{W_f}$ in the simple root basis. For example, taking the vertex $v = \varpi_1 - \varpi_2 + \varpi_3 \in \nu_{\mathcal{P}}$ gives $v = \alpha_1/2 + \alpha_3/2$. Note that

$$\lambda - \mu - v = \left(a - \frac{1}{2}\right)\alpha_1 + b\alpha_2 + \left(c - \frac{1}{2}\right)\alpha_3.$$

Going through all the cases, shows that for all $v \in \nu_{\mathcal{P}}$, the coordinates of $\lambda - \mu - v$ -in the simple root basis- can not be all ≥ 0. This tells us that the vertices of $\mathcal{P}_{W_f} + \mu$, which are $\nu_{\mathcal{P}} + \mu$, do not belong to $\text{Conv}(\lambda)$. We lied a bit, since there is an exception when $v = -\varpi_2$. The vertex $v + \mu$ can lie in the face $F_{\{1,3\}}(\lambda)$ (but not in an edge), for a suitable μ. This comes from the 1 in the inverse of the Cartan matrix. Regardless, the interior of $\mathcal{P}_{W_f} + \mu$ can not intersect an edge (touching λ) of $\mathrm{Conv}(\lambda)$, since we already covered them with the W_f-polytopes corresponding to $\mathcal{C}_\lambda^{\{i\}}$. Thus, the interior of $\mathcal{P}_{W_f} + \mu$ can not intersect $\mathrm{Conv}(\lambda) \cap C^+$. By the polytope construction theorem, we conclude that $\mathrm{Conv}(\lambda) \cap C^+ \subset A(\leq \theta(\lambda))$. This proves (3.10) for $J = I_3$, since $\leq \theta(\lambda)$ is W_f-invariant.

Finally, let $F_J(\lambda)$ be a 2-face. If a W_f-polytope with center $\mu \in (\lambda + L) \cap C^+$ intersects this face, then μ is either on the face itself or it is in the interior of $\mathrm{Conv}(\lambda)$. With a similar reasoning as in the case $|J| = 3$, it is easy to check that the latter does not occur, as the interior of $\mathcal{P}_{W_f} + \mu$ is also contained in the interior of $\mathrm{Conv}(\lambda)$, since $\nu_{\mathcal{P}} + \mu$ is contained in it as well (with the possible exception of the vertex $\varpi_2 + \mu$). It follows that (3.10) holds for $|J| = 2$.

$\square$

Using this proposition, we will construct a "partition" $P \coloneqq \{P^0(\lambda), \ldots, P^3(\lambda)\}$ of $A(\leq \theta(\lambda))$, which we call the *Geometric Partition.* The sets in P will not be disjoint, but their intersection will be a set of zero-volume in V. For $J \subset I_3$, let us define

$$P_J(\lambda) = \big(F_J(\lambda) + \langle \varpi_i \mid i \notin J \rangle_{\geq 0}\big) \bigcap A(\leq \theta(\lambda)).$$

By Proposition 3.6, we have that $P_{I_3}(\lambda) = \mathrm{Conv}(\lambda)$. Also, $P_\emptyset(\lambda) = \theta(\lambda)\overline{A}_{\mathrm{id}}$, as one can check. Let H be the subspace of V obtained from translating the affine subspace generated by $F_J(\lambda)$, to the origin. Note that every ϖ_i with $i \notin J$ is orthogonal to H. In fact they generate its orthogonal complement, since H is generated by α_j with $j \in J$. Therefore, if $F_J(\lambda)$ is a face of codimension 1, $F_J(\lambda) + \langle \varpi_i \mid i \notin J \rangle_{\geq 0}$ is a "half infinite cylinder" starting from its base $F_J(\lambda)$ and "growing orthogonally away" from $\mathrm{Conv}(\lambda)$. Using these definitions in type $\widetilde{A_2}$, the infinite rectangles mentioned in Chapter 2 -in the construction shown in figure 2.4- are $W_f \cdot \big(F_{\{i\}}(4,1) + \langle \varpi_j \rangle_{\geq 0}\big)$, with $i \neq j$.

It is clear that W_J fixes $\langle \varpi_i \mid i \notin J \rangle_{\geq 0}$ point-wise. Since W_J is a subgroup of W_f, its elements are orthogonal transformations of V. By equation (3.5), it follows that, as a set,

$$\mathrm{Stab}_{W_f}(P_J(\lambda)) = W_J. \tag{3.11}$$

Again, if λ is not generic, only $\supset$ holds.

Let $0 \leq i \leq 3$. Define

$$P^i(\lambda) = \bigsqcup_{\substack{J \subset I_3 \\ |J|=i}} W_f \cdot P_J(\lambda).$$

In this definition, the action is understood point-wise, that is, the elements of $W_f \cdot P_J(\lambda)$ are $w(x)$, with $w \in W_f$ and $x \in P_J(\lambda)$. As explained in the description of the faces of Conv(λ), we take the W_f-orbit so we can reach all its faces and not just the ones intersecting the dominant cone. To see that the union of $P^0(\lambda), \ldots, P^3(\lambda)$ actually gives $A(\leq \theta(\lambda))$, it suffices to check it in the dominant cone. That is,

Lemma 3.7. *Let $J \subset I_3$ and define $Y_J(\lambda) \coloneqq F_J(\lambda) + \langle \varpi_i \mid i \notin J \rangle_{\geq 0}$. Then*

$$C^+ \subset \bigcup_{J \subset I_3} Y_J(\lambda).$$

Proof. Define

$$Y_J \coloneqq \langle \alpha_i \mid i \in J \rangle_{\leq 0} + \langle \varpi_i \mid i \notin J \rangle_{\geq 0}.$$

We have that

$$Y_J(\lambda) \subset Y_J + \lambda \qquad \text{and} \qquad Y_J(\lambda) \cap C^+ = (Y_J + \lambda) \cap C^+.$$

The inclusion is clear and the equality requires a bit of work, but is not hard to prove. We will describe Y_J for all J. Let M be the Cartan matrix, and Id the 3×3 identity matrix. Denote by X_j the j^{th} column of the matrix X. Consider the 3×3 matrix M_J defined by $(M_J)_j = M_j$ if $j \in J$, and $(M_J)_j = \mathrm{Id}_j$ if $j \notin J$. Let $\mathcal{B}_J = \{b_1, b_2, b_3\}$, where

$b_j = \alpha_j$, if $j \in J$, and $b_j = \varpi_j$ otherwise. The column $(M_J)_j$ is the coordinate vector of b_j in the fundamental weight basis. It is easy to check that M_J is invertible[4], so that

$$\mathcal{B}_J = \{\alpha_j, \varpi_i \mid j \in J,\ i \notin J\}$$

is a basis of V, for every $J \subset I_3$. We call this basis the ***mixed basis*** corresponding to J. Let $(x, y, z) \coloneqq x\varpi_1 + y\varpi_2 + z\varpi_3$. The matrix M_J^{-1} is the change of basis matrix from the fundamental weight basis to $\mathcal{B}_J$, so M_J^{-1} determines exactly when $(x, y, z) \in Y_J$. For example,

$$M_{\{1,2\}}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

implies that $(x, y, z) \in Y_{\{1,2\}}$ if and only if $2x + y \leq 0$, $x + 2y \leq 0$ and $x + 2y + 3z \geq 0$. In general,

- $J = \emptyset$: $\quad x, y, z \geq 0.$
- $J = \{1\}$: $\quad x \leq 0, \quad x + 2y \geq 0, \quad z \geq 0.$
- $J = \{2\}$: $\quad 2x + y \geq 0, \quad y \leq 0, \quad y + 2z \geq 0.$
- $J = \{3\}$: $\quad x \geq 0, \quad 2y + z \geq 0, \quad z \leq 0.$
- $J = \{1, 2\}$: $\quad 2x + y \leq 0, \quad x + 2y \leq 0, \quad x + 2y + 3z \geq 0.$
- $J = \{1, 3\}$: $\quad x \leq 0, \quad x + 2y + z \geq 0, \quad z \leq 0.$
- $J = \{2, 3\}$: $\quad 3x + 2y + z \geq 0, \quad 2y + z \leq 0, \quad y + 2z \leq 0.$
- $J = \{1, 2, 3\}$: $\quad 3x + 2y + z \leq 0, \quad x + 2y + z \leq 0, \quad x + 2y + 3z \leq 0.$

[4] We have a solid idea for a general proof of this fact (in any type). For the moment, it can be checked case by case.

One can check that this exhausts all of the options for (x, y, z)[5]. Therefore, $V = \bigcup_{J \subset I_3} Y_J$.

This is the same as $V = \bigcup_{J \subset I_3} Y_J + \lambda$. It follows

$$\begin{aligned} C^+ &= \bigcup_{J \subset I_3} (Y_J + \lambda) \cap C^+ \\ &= \bigcup_{J \subset I_3} \left(Y_J(\lambda) \cap C^+ \right). \end{aligned}$$

□

Using the inequalities presented in the proof above, the following result can be concluded case by case.

Corollary 3.8. *Let $J, K \subset I_3$, then*

$$Y_J(\lambda) \bigcap Y_K(\lambda) = F_{J \cap K}(\lambda) + \langle \varpi_i \mid i \notin J \cup K \rangle_{\geq 0}$$

□

Let $J, K \subset I_3$. Using the notation of Lemma 3.7, $P_J(\lambda) = Y_J(\lambda) \cap A(\leq \theta(\lambda))$. On the one hand, Corollary 3.8 tells us that if $J \neq K$, then the affine space of V generated by $P_J(\lambda) \cap P_K(\lambda)$ has codimension

$$3 - |J \cap K| - |(J \cup K)^c| = |J \cup K| - |J \cap K| \geq 1.$$

Thus, the intersection has volume zero in V. In turn, this implies that $P^i(\lambda) \cap P^j(\lambda)$

[5] We did this with the help of SageMath (and professor A. Behn).

also has volume zero, for $i \neq j$. On the other hand, Lemma 3.7 implies

$$\begin{aligned} C^+ \bigcap A(\leq \theta(\lambda)) &= \left(\bigcup_{J \subset I_3} Y_J(\lambda) \right) \bigcap C^+ \bigcap A(\leq \theta(\lambda)) \\ &= \left(\bigsqcup_{J \subset I_3} P_J(\lambda) \right) \bigcap C^+. \end{aligned}$$

Taking the W_f-orbit, we get

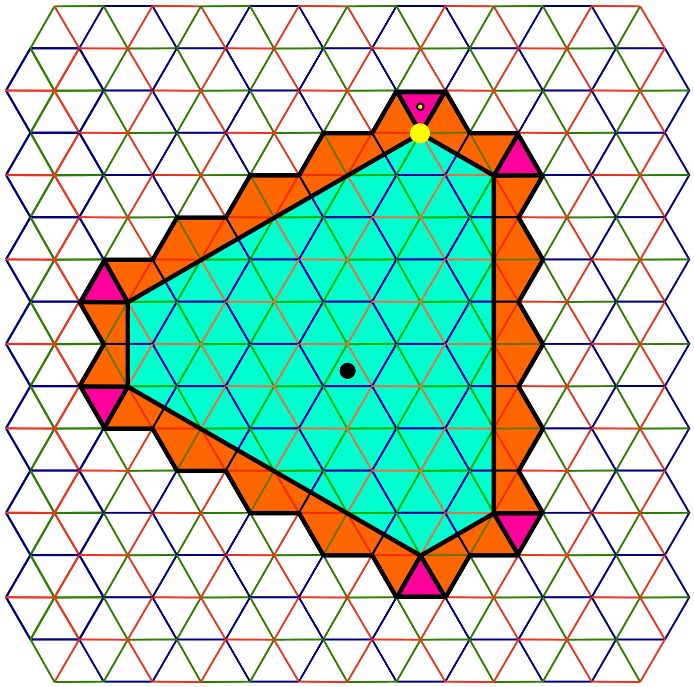
$$A(\leq \theta(\lambda)) = \bigsqcup_{i=0}^{3} P^i(\lambda).$$

As mentioned before, $P^0(\lambda)$ is the union of $|W_f| = 24$ alcoves (which are non-regular tetrahedra) and $P^3(\lambda)$ is the permutahedron $\mathrm{Conv}(\lambda)$.

For $|J| = 2$, $F_J(\lambda)$ is either a rectangle or a hexagon, and $P_J(\lambda)$ is a region contained in the "half infinite cylinder" $Y_J(\lambda)$ with base $F_J(\lambda)$. In fact, it is delimited by this cylinder (except "on the top"). The set $P_J(\lambda)$ looks like small spikes growing from $F_J(\lambda)$. Thus, $P^2(\lambda)$ is the union of those spiked regions (under the image of W_f).

For $|J| = 1$, $Y_J(\lambda)$ is an infinite triangular prism which has one of its rectangular 2-faces removed, so it grows infinitely in that direction. The only remaining finite edge of this prism is the face $F_J(\lambda)$ of $\mathrm{Conv}(\lambda)$, so the set $P_J(\lambda)$ resembles a (finite) triangular prism which has small spikes growing from one of its rectangular faces. Reflecting these sets under W_f, gives $P^1(\lambda)$ (it is the union of those).

It is much easier to check that the geometric partition can be done in type $\widetilde{A_2}$. Let us briefly see how the geometric partition looks like. Recall figure 2.4, in Chapter 2:



This figure illustrates the geometric partition of $A(\leq \theta(\lambda))$. The turquoise hexagon is the set $F_2(\lambda) = \mathrm{Conv}(\lambda)$, and the face $F_\emptyset(\lambda)$ is the vertex λ represented by the big yellow dot. As for the edges $F_{\{i\}}(\lambda)$ of $\mathrm{Conv}(\lambda)$, these are the ones touching the big yellow dot; $F_{\{1\}}(\lambda)$ is the left one, and $F_{\{2\}}(\lambda)$ the right one.

The set $P^2(\lambda)$ is the whole hexagon $\mathrm{Conv}(\lambda)$. The triangle with a small yellow dot is $P_\emptyset(\lambda) = \theta(\lambda)\overline{A}_{\mathrm{id}}$, so that $P^0(\lambda) = W_f \cdot P_\emptyset(\lambda)$ is the union of the $|W_f| = 6$ magenta triangles.

The set $P_{\{1\}}(\lambda)$ is the larger orange "crown" above the edge $F_{\{1\}}(\lambda)$ and $P_{\{2\}}(\lambda)$ is the shorter crown touching the big yellow dot. Hence, $P^1(\lambda)$ is the orange region and it is composed by larger and shorter crowns. There are exactly

$$\left[W_f : \mathrm{Stab}_{W_f}(P_{\{i\}}(\lambda))\right] = \frac{W_f}{W_{\{i\}}} = 3$$

crowns of each type, as we can see in the figure.

4. On the Volumes $V_J^{X_n}$

In this chapter, we show some properties of these volumes, and give formulas to compute them. In the literature, there are some results for $V_J^{A_n}$, however most of them are computed by considering the action of S_{n+1} on $\mathbb{R}^{n+1}$ (see [Pos05] for example), which is not the ambient space we are considering[1].

Let Φ be an irreducible root system of type X_n (of rank n), let λ be a dominant coweight and $J \subset I_n$. In the introduction we defined $V_J^{X_n}(\lambda)$ as the $|J|$-dimensional volume of $\mathrm{Conv}(W_J \cdot \lambda)$. Recall that this convex hull is the face $F_J(\lambda)$ of the polytope $\mathrm{Conv}(\lambda)$. For $J \subset I_n$, consider the connected components J_c of J, in the obvious way. For instance, $\{1,2,4\}_c = \{\{1,2\},\{4\}\}$. Here are some basic properties of $V_J^{X_n}$.

Lemma 4.1. *Let* $\lambda = (m_i)_{i \in I_n} = m_1\varpi_1^\vee + \ldots + m_n\varpi_n^\vee$ *be a dominant coweight and let* $J \subset I_n$. *Define* $\lambda_J := (m_j)_{j \in J} = \sum_{j \in J} m_j \varpi_j^\vee$.

(i) The function $V_J^{X_n}$ *only depends on the coordinates of* $\varpi_j^\vee$, *with* $j \in J$. *That is,*

$$V_J^{X_n}(\lambda) = V_J^{X_n}(\lambda_J).$$

(ii) For types $X = A, B, C, F$, *the volume* $V_J^{X_n}(\lambda)$ *can be computed as the product of the volumes corresponding to the connected components of* J. *That is,*

$$V_J^{X_n}(\lambda) = \prod_{K \in J_c} V_K^{X_n}(\lambda).$$

[1] One can still translate the results to our conventions, but it is not as easy -or satisfying- as it seems.

Proof.

(i) Note that $\lambda = \lambda_J + \lambda_{J^c}$, where J^c it the complement of J in I_n. Let $w \in W_J$ and write $w = s_{j_1} \dots s_{j_k}$, for some $j_i \in J$. It is easy to see that $s_{j_i}\lambda_{J^c} = \lambda_{J^c}$, so that $w\lambda = w\lambda_J + \lambda_{J^c}$. Hence, $W_J \cdot \lambda = W_J \cdot \lambda_J + \lambda_{J^c}$. It follows that

$$\text{Conv}(W_J \cdot \lambda) = \text{Conv}(W_J \cdot \lambda_J) + \lambda_{J^c}.$$

(ii) Similar to the equations above, one can check that

$$\text{Conv}(W_J \cdot \lambda) - \lambda = \sum_{K \in J_c} \left(\text{Conv}(W_K \cdot \lambda) - \lambda\right).$$

The hypothesis on the types implies that for any i, j, $|i - j| > 1$ implies that α_i, α_j are orthogonal. That is, if $i, j \in J$, two simple roots α_i, α_j are orthogonal if both i, j do not belong to the same connected component of J[2]. Thus, the sets $(\text{Conv}(W_K \cdot \lambda) - \lambda)$ are pairwise orthogonal and their sum can be thought as a Cartesian product. It follows that

$$\begin{aligned}
V_J^X(\lambda) &= \text{Vol}_{|J|}\left(\lambda + \sum_{K \in J_c} \left(\text{Conv}(W_K \cdot \lambda) - \lambda\right)\right) \\
&= \text{Vol}_{|J|}\left(\sum_{K \in J_c} \left(\text{Conv}(W_K \cdot \lambda) - \lambda\right)\right) \\
&= \prod_{K \in J_c} \text{Vol}_{|K|}\left(\text{Conv}(W_K \cdot \lambda) - \lambda\right) \\
&= \prod_{K \in J_c} V_K^{X_n}(\lambda),
\end{aligned}$$

[2] In other types. this does not hold for some J.

where Vol_i denotes the i-dimensional volume.

□

From now on, we will use the convention that the "weird simple root" is always denoted by α_1. In most types, it is standard to call it α_n. Since we do not want to re-label the corresponding Dynkin diagram, we are forced to put α_2 instead of α_{n-1} and so on, as we will explain. This is not an issue for types F_4 and G_2. Let us see some examples.

There are no weird roots in type A_n, and types E_6, E_7, E_8 are already in the desired form. In type B_n, $n \geq 2$, the standard root system is the following. The ambient space is $V = \mathbb{R}^n$, with standard basis ε_i^n. The roots are $\pm\varepsilon_i^n$ for $1 \leq i \leq n$, together with $\pm\varepsilon_i^n \pm \varepsilon_j^n$ for $1 \leq i < j \leq n$. The simple roots are $\alpha_i^n = \varepsilon_i^n - \varepsilon_{i+1}^n$ for $1 \leq i < n$, and $\alpha_n^n = \varepsilon_n^n$. Note that

$$1 = (\alpha_n^n, \alpha_n^n) \neq (\alpha_n^{n+1}, \alpha_n^{n+1}) = 2.$$

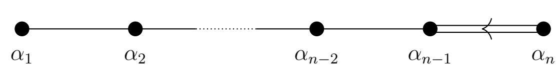
Consider a dominant coweight $\lambda^n = (m_i)_{i \in I_n}$ in the fundamental coweight basis of B_n. If $m_n \neq 0$, the bad thing about this convention is that $V_{\{n\}}^{B_n}(\lambda^n) \neq V_{\{n\}}^{B_{n+1}}(\lambda^{n+1})$ (we only care about the n^{th} coordinate, by Lemma 4.1).

To fix this, we put $\alpha_i^n = \varepsilon_{n-i}^n - \varepsilon_{n-i+1}^n$, for $1 < i \leq n$, and $\alpha_1^n = \varepsilon_n^n$, so that

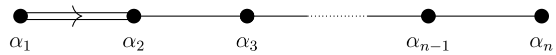
$$(\alpha_i^n, \alpha_j^n) = (\alpha_i^{n+1}, \alpha_j^{n+1}),$$

for all $i, j \in I_n$.

One consequence is that some Dynkin diagrams are flipped. For example in type C_n, $n \geq 3$, the standard corresponding Dynkin diagram



becomes



Thus, another way to state our convention is that we flip Dykin diagrams of types B_n, C_n and D_n, while keeping the other ones unchanged[3]. Using this convention,

Lemma 4.2. *Let Φ^n be a root system of type X_n, α_i^n the simple roots, W_f^n the finite Weyl group, and so on. Let $\lambda^n = m_1(\varpi_1^\vee)^n + \cdots + m_n(\varpi_n^\vee)^n$ be a dominant coweight. Then, for any $J \subset I_n$,*

$$V_J^{X_n}(\lambda^n) = V_J^{X_{n+1}}(\lambda^{n+1}).$$

This formula only makes sense if X_{n+1} actually exists. This might not be the case if $X = E$.

Proof. For brevity, put $\mathcal{O}^n = W_J^n \cdot \lambda^n$. We will construct a function that maps $\mathrm{Conv}(\mathcal{O}^n)$ to $\mathrm{Conv}(\mathcal{O}^{n+1})$, while preserving the $|J|$-dimensional volume.

Define H_J^n as the subspace (of the ambient space of Φ^n) generated by $\{\alpha_j^n \mid j \in J\}$. It is easy to see that $\mathcal{O}^n \subset H_J^n + \lambda^n$, so that $\mathrm{Conv}(\mathcal{O}^n) \subset H_J^n + \lambda^n$ as well. Consider the isomorphism

$$\begin{aligned} T : H_J^n &\longrightarrow H_J^{n+1} \\ \alpha_j^n &\longmapsto \alpha_j^{n+1} \end{aligned}$$

By our convention, T is an inner product preserving map. Let $w^n = s_{j_1}^n \dots s_{j_k}^n \in W_J^n$. Note that

$$\begin{aligned} T(s_{j_i}^n \lambda^n - \lambda^n) &= T(-m_{j_i}(\alpha_{j_i}^\vee)^n) \\ &= -m_{j_i}(\alpha_{j_i}^\vee)^{n+1} \\ &= s_{j_i}^{n+1}\lambda^{n+1} - \lambda^{n+1}. \end{aligned}$$

[3] A third way to put it, is that we demand that the Dynkin diagrams “grow to the right”.

It is not difficult to check that $T(w^n\lambda^n - \lambda^n) = w^{n+1}\lambda^{n+1} - \lambda^{n+1}$ also holds. This implies that $T(\mathcal{O}^n - \lambda^n) = \mathcal{O}^{n+1} - \lambda^{n+1}$. Denote by t_x the translation by x. We have

$$\begin{array}{ccccccc} H_J^n + \lambda^n & \xrightarrow{t_{\lambda^n}^{-1}} & H_J^n & \xrightarrow{T} & H_J^{n+1} & \xrightarrow{t_{\lambda^{n+1}}} & H_J^{n+1} + \lambda^{n+1} \\ \mathcal{O}^n & \longmapsto & \mathcal{O}^n - \lambda^n & \longmapsto & \mathcal{O}^{n+1} - \lambda^{n+1} & \longmapsto & \mathcal{O}^{n+1} \end{array}$$

It is easy to check that the above also holds for the corresponding convex hulls and that the volume is preserved. $\square$

As a consequence of this lemma, it is natural to slightly modify the notation from $V_J^{X_n}$ to V_J^X. Furthermore, let $m = (m_i)_{i\in\mathbb{N}}$ be a sequence of positive integers, then $V_J^X(m) = V_J^X((m_i)_{i\in J})$ is well-defined, for any finite $J \subset \mathbb{N}$[4], by Lemma 4.1. For example volume $V_{\{1,2,4\}}^{A_7}$ becomes $V_{\{1,2,4\}}^{A}$. Let $\lambda = m_1\varpi_1 + m_2\varpi_2 + m_4\varpi_4$ where ϖ_i are the fundamental (co)weights of type A_4 (or A_n, for $n \geq 4$). Then $V_{\{1,2,4\}}^{A}(m)$ is $V_{\{1,2,4\}}^{A}(\lambda)$.

Now we give an inductive formula for $V_{I_n}^X$. Let Φ be of type X_n and $\lambda = (m_i)_{i\in I_n}$ a generic dominant coweight.

Lemma 4.3. *For $J \subset I_n$ with $|J| = n-1$, let $t \in I_n \setminus J$ and define $\varpi_J^\vee := \varpi_t^\vee$. Then*

$$V_{I_n}^X(\lambda) = \frac{1}{n} \sum_{\substack{J\subset I_n \\ |J|=n-1}} [W_f : W_J] \, \frac{(\lambda, \varpi_J^\vee)}{\|\varpi_J^\vee\|} \, V_J^X(\lambda)$$

Proof. By definition, $V_{I_n}^X$ is the n-dimensional volume of $\mathrm{Conv}(\lambda) = \mathrm{Conv}(W_f \cdot \lambda)$. Draw segments from the origin O to every vertex of $\mathrm{Conv}(\lambda)$. This partitions $\mathrm{Conv}(\lambda)$ into several hyperpyramids, each one of them with an $(n-1)$-face of $\mathrm{Conv}(\lambda)$ as its base and O as its cusp. Recall that the $(n-1)$-faces of $\mathrm{Conv}(\lambda)$ are given by

$$\{wF_J(\lambda) \mid |J| = n-1, w \in W_f\}.$$

[4]This is only for infinite families, that is, for types A, B, C, D.

For brevity, let $\triangle$ be the hyperpyramid with base $F_J(\lambda)$. Since $\mathrm{Stab}_{W_f}(F_J(\lambda)) = W_J$, it follows that there are exactly $[W_f : W_J]$ hyperpyramids having a J-face as its base. It is clear that all of these hyperpyramids have the same n-dimensional volume as $\triangle$. The result follows from

$$\mathrm{Vol}_n(\triangle) = \frac{(\lambda, \varpi_J^\vee)}{n\,\|\varpi_J^\vee\|}\, V_J^X(\lambda).$$

This is easy to check. The $(n-1)$-dimensional volume of the base $F_J(\lambda)$ of $\triangle$, is $V_J^X(\lambda)$. Let H be the affine hyperplane of V generated by $F_J(\lambda)$. The height of $\triangle$ is the distance from O to H, which is given by

$$\frac{(\lambda, \varpi_J^\vee)}{\|\varpi_J^\vee\|},$$

since $\varpi_J^\vee$ is normal to H, and $\lambda \in F_J(\lambda) \subset H$. □

We can go a little bit further by giving a recursive formula for V_J^X, for connected J (recall part (ii) of Lemma 4.1). We will do this for type A_n, in which all the simple roots have the same length (independently of n). Let $J \subset I_n$ be connected, with $|J| = \ell$. There is a unique non-negative integer u such that $J = I_\ell + u$. We can obtain V_J^A from $V_{I_\ell}^A$ in the following way.

Lemma 4.4. *Let $J = I_\ell + u \subset \mathbb{N}$ and let $\lambda = (m_i)_{i \in I_{\ell+u}}$ be a dominant (co)weight of type $A_{\ell+u}$. Define*

$$\lambda_{+u} := (m_{i+u})_{i \in I_\ell} = m_{1+u}\varpi_1^\ell + \cdots + m_{\ell+u}\varpi_\ell^\ell,$$

where ϖ_i^ℓ is the i^{th} fundamental weight of A_ℓ. Then, $V_J^A(\lambda) = V_{I_\ell}^A(\lambda_{+u})$.

Proof. We know that $V_J^A(\lambda) = V_J^{A_{\ell+u}}(\lambda_J)$. Our aim is to map the set $\mathrm{Conv}(W_J^{\ell+u} \cdot \lambda_J)$ to $\mathrm{Conv}(W_f^\ell \cdot \lambda_{+u})$, while preserving the ℓ-dimensional volume. Replicate the proof of

Lemma 4.2, by considering the isomorphism

$$\begin{aligned} \widetilde{T} : H_J^{\ell+u} &\longrightarrow H_{I_\ell}^{\ell} \\ \alpha_j^{\ell+u} &\longmapsto \alpha_{j-u}^{\ell} \end{aligned}$$

which preserves inner product (here we are using type A). $\square$

Hence, to compute $V_J^A(\lambda)$, with $J = I_\ell + u$, we just need to compute $V_{I_\ell}^A(\lambda)$ and then shift by u the variables m_i. Explicitly, Lemma 4.3 gives

$$V_J^A(\lambda) = \frac{1}{\ell} \sum_{\substack{K \subset J \\ |K| = \ell - 1}} \left[W_f^\ell : W_{K-u}\right] \frac{\left(\lambda_{+u}, \varpi_{K-u}^\ell\right)}{\left\|\varpi_{K-u}^\ell\right\|} V_K^A(\lambda), \tag{4.1}$$

for generic λ_{+u}. Note that $K \subset J$ is the same as $K - u \subset I_\ell$.

Remark 4.5. Consider a sequence of positive integers $m = (m_i)_{i \in \mathbb{N}}$. Equation (4.1) shows that $V_J^A(m)$ is a homogeneous polynomial of degree $\ell = |J|$ in the variables m_j, $j \in J$. Indeed, it is clear that $\left(\lambda_{+u}, \varpi_{K-u}^\ell\right)$ is a homogeneous linear polynomial for every $K \subset J$ with $|K| = \ell - 1$. The result follows by a straightforward induction on ℓ, and by Lemma 4.1.

One can still do this for other types than A, but is not as easy. Lemma 4.4 greatly simplified the computation. Equation (4.1) implies the linear independence of the polynomials V_J^A, as we will see.

Lemma 4.6. *Consider the polynomial ring $\mathbb{R}[m_1, m_2, \ldots]$. For any finite $J \subset \mathbb{N}$, let $m_J := \prod_{j \in J} m_j$. Then, the coefficient c_J of the monomial m_J in the polynomial $V_J^A(m)$, is non-zero. Furthermore, let $K \subset \mathbb{N}$ be finite such that $K \neq J$. Then the coefficient of m_J in $V_K^A(m)$ is zero.*

Proof. On the one hand, we have that $c_\emptyset = 1$. Suppose J is connected, put $J = I_\ell + u$ and let $K' \subset J$ with $|K'| = \ell - 1$. Denote by t the unique element of $J \setminus K'$, and let $\lambda = (m_j)_{j \in J}$. Note that the coefficient of m_J in $(\lambda_{+u}, \varpi^\ell_{K'-u})V^A_{K'}(m)$ is $(\varpi^\ell_{t-u}, \varpi^\ell_{t-u})c_{K'}$. Therefore, equation (4.1) shows that

$$c_J = \frac{1}{\ell} \sum_{\substack{K \subset J \\ |K'| = \ell - 1}} \left[W^\ell_f : W_{K'-u}\right] \left\|\varpi^\ell_{t-u}\right\| \ c_{K'}.$$

If J is not connected, Lemma 4.1 gives

$$c_J = \prod_{K^* \in J_c} c_{K^*}.$$

The first part of the lemma follows by induction on $|J|$. On the other hand, let Υ be the coefficient of m_J in $V^A_K(m)$. Since $V^A_K(m)$ is a polynomial in the variables m_k, with $k \in K$, and m_J is a polynomial in the variables m_j, with $j \in J$, $\Upsilon \neq 0$ implies that $J \subset K$. However, m_J has degree $|J|$ and $V^A_K(m)$ is a homogeneous polynomial of degree $|K|$, so that $|J| = |K|$. This would imply that $J = K$, a contradiction. $\square$

Corollary 4.7. *The polynomials $V^A_J(m)$, with finite $J \subset \mathbb{N}$ are linearly independent.*

Proof. Take any finite sum of these polynomials and equate it to zero. Look at the coefficient of m_J in the sum. $\square$

5. Proof of the geometric formula

In this chapter, we will prove the geometric formula conjecture for type $\widetilde{A_3}$, using the geometric constructions of Chapter 3. That is, let Φ be the root system of type A_3. We will show

Theorem 5.1 (Geometric formula for type $\widetilde{A_3}$). *There exists $\vartheta_J \in \mathbb{R}$ such that, for any dominant (co)weight λ,*

$$|\leq \theta(\lambda)| = \sum_{J \subset I_3} \vartheta_J V_J^A(\lambda). \tag{5.1}$$

The importance of the partition $P = \{P^0(\lambda), \ldots, P^3(\lambda)\}$ of $A(\leq \theta(\lambda))$ is that we now can compute

$$|\leq \theta(\lambda)| = \frac{1}{\mathrm{Vol}(A_{\mathrm{id}})} \sum_{i=0}^{3} \mathrm{Vol}(P^i(\lambda)),$$

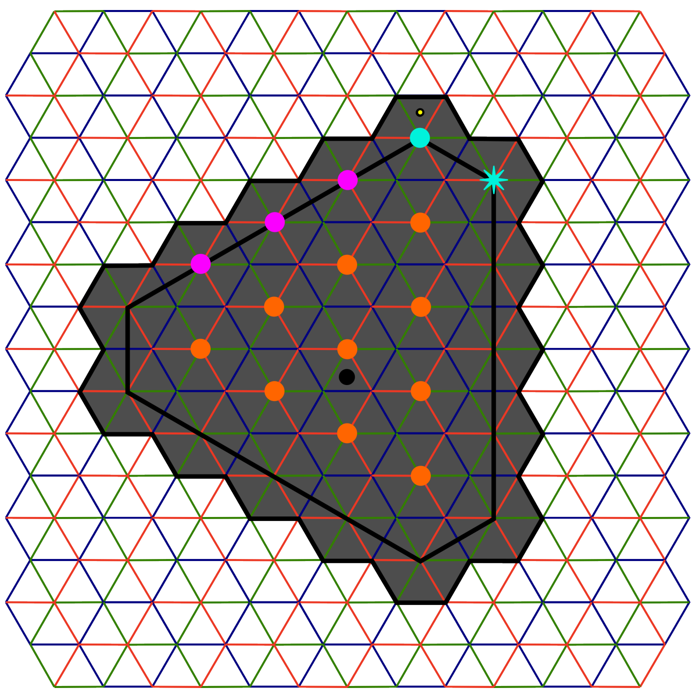
where Vol stands for the 3-dimensional volume. Before going into the proof of Theorem 5.1, we will prove some lemmas that will allow us to handle $\mathrm{Vol}(P^i(\lambda))$.

Let λ be a dominant generic weight. Recall the set $\mathcal{C}_\lambda$ from Theorem 3.4 and the set $\mathcal{C}_\lambda^F = \mathcal{C}_\lambda \cap F$, with F a face of $\mathrm{Conv}(\lambda)$. If $K \subset J$ and $w \in W_J$, then $wW_K \subset W_J$. This implies that $F' = wF_K(\lambda) \subset F_J(\lambda)$, so that $w\mathcal{C}_\lambda^K = \mathcal{C}_\lambda^{F'} \subset \mathcal{C}_\lambda^J$. Let us define

$$\widetilde{\mathcal{C}_\lambda^J} \coloneqq \mathcal{C}_\lambda^J \setminus \bigcup_{K \subsetneq J} W_J \cdot \mathcal{C}_\lambda^K = W_J \cdot \left(\mathcal{C}_\lambda^J \setminus \bigcup_{K \subsetneq J} \mathcal{C}_\lambda^K \right),$$

where the action is understood point-wise (the equality comes from equation (3.9)). That is, $\widetilde{\mathcal{C}_\lambda^J}$ is the set of centers of the W_f-polytopes that compose $A(\leq \theta(\lambda))$, that belong to the face $F_J(\lambda)$ but are not contained in any K-face, for $K \subsetneq J$.

Remark 5.2. Note that $\mathcal{C}_\lambda^\emptyset = \{\lambda\}$. For $J \neq \emptyset$, it is important to observe that $\widetilde{\mathcal{C}_\lambda^J}$ can be empty, even assuming that λ is generic. The following figure illustrates these sets in type $\widetilde{A_2}$ for $\lambda = (4,1)$, which is generic. The grey area is the set $A(\leq \theta(\lambda))$. The triangle with the small yellow dot is $\theta(\lambda)A_{\mathrm{id}}$. The set $\widetilde{\mathcal{C}_\lambda^{\{1,2\}}}$ is the collection of orange dots, $\widetilde{\mathcal{C}_\lambda^{\{1\}}}$ corresponds to the magenta dots, and $\widetilde{\mathcal{C}_\lambda^\emptyset}$ is the cyan dot (as a singleton), which is λ. The cyan asterisk $*$ is $s_2\lambda$, so that $s_2\widetilde{\mathcal{C}_\lambda^\emptyset} = \{*\}$. The set $\mathcal{C}_\lambda^{\{2\}}$ is the union of the cyan dot and $*$. That is, $\mathcal{C}_\lambda^{\{2\}} = \widetilde{\mathcal{C}_\lambda^\emptyset} \sqcup s_2\widetilde{\mathcal{C}_\lambda^\emptyset}$. Thus, there are no colored dots corresponding to $\widetilde{\mathcal{C}_\lambda^{\{2\}}} = \emptyset$. Note that if we were to take $\lambda = (4,2)$, then we would have $\widetilde{\mathcal{C}_\lambda^{\{2\}}} \neq \emptyset$.



One can prove that for "dominant enough" λ, all the sets $\widetilde{\mathcal{C}_\lambda^J}$ are non-empty. In type $\widetilde{A_2}$, this is the case for $\lambda = (a,b)$ with $a,b \geq 2$. In type $\widetilde{A_3}$, the sets $\widetilde{\mathcal{C}_\lambda^J}$ are non-empty whenever $\lambda = (a,b,c)$ with $a,b,c \geq 2$.

Lemma 5.3. *The following propositions hold, for any generic dominant weight λ.*

(i) (Stabilizer of $\widetilde{\mathcal{C}_\lambda^J}$) For every $J \subset I_3$,

$$\mathit{Stab}_{W_f}\left(\widetilde{\mathcal{C}_\lambda^J}\right) = W_J.$$

(ii) (Disjointness) Let $x, y \in W_f$ and $J \neq K \subset I_3$. Then

$$x\widetilde{\mathcal{C}_\lambda^J} \bigcap y\widetilde{\mathcal{C}_\lambda^K} = \emptyset.$$

(iii) (Partition of $\mathcal{C}_\lambda$)

$$\bigsqcup_{J \subset I_3} W_f \cdot \widetilde{\mathcal{C}_\lambda^J} = \mathcal{C}_\lambda.$$

(iv) (Partition of $\mathcal{C}_\lambda^J$) For every $J \subset I_3$,

$$\bigsqcup_{K \subset J} W_J \cdot \widetilde{\mathcal{C}_\lambda^K} = \mathcal{C}_\lambda^J.$$

Proof.

(i) The inclusion $\supset$ is clear. If $\widetilde{\mathcal{C}_\lambda^J}$ is empty, there is nothing to prove. Suppose $w\widetilde{\mathcal{C}_\lambda^J} = \widetilde{\mathcal{C}_\lambda^J}$, for some $w \in W_f$, and let $x = wy$, for $x, y \in \widetilde{\mathcal{C}_\lambda^J}$. Note that both x, y belong to the face $F_J(\lambda)$ so that $x \in F_J(\lambda) \cap wF_J(\lambda)$. Using (3.7), we have that $x \in W_J$.

(ii) It suffices to check that $\widetilde{\mathcal{C}_\lambda^J} \bigcap w\widetilde{\mathcal{C}_\lambda^K} = \emptyset$, for all $w \in W_f$. Suppose not. Note that

$$\widetilde{\mathcal{C}_\lambda^J} \bigcap w\widetilde{\mathcal{C}_\lambda^K} \subset F_J(\lambda) \bigcap wF_K(\lambda). \tag{5.2}$$

By (3.7), $w = w_1 w_2$, with $w_1 \in W_J$ and $w_2 \in W_K$. Therefore, using (i),

$$\widetilde{\mathcal{C}_\lambda^J} \bigcap w\widetilde{\mathcal{C}_\lambda^K} = w_1 \left(\widetilde{\mathcal{C}_\lambda^J} \bigcap \widetilde{\mathcal{C}_\lambda^K} \right).$$

Let $\mu' \in \widetilde{\mathcal{C}_\lambda^J} \bigcap \widetilde{\mathcal{C}_\lambda^K}$. Then, $\mu' \in F_{J\cap K}(\lambda)$ and $\mu' \in \mathcal{C}_\lambda$. That is, $\mu' \in \mathcal{C}_\lambda^{J\cap K}$. Since $J \neq K$, we have that at least one of $J \cap K \subsetneq J$ or $J \cap K \subsetneq K$ holds. The former implies that $\mu' \notin \widetilde{\mathcal{C}_\lambda^J}$, and the latter gives $\mu' \notin \widetilde{\mathcal{C}_\lambda^K}$, a contradiction.

(iii) The union is disjoint by (ii). The containment $\subset$ is easy to see. We will show that

$$C^+ \cap \mathcal{C}_\lambda \subset \bigsqcup_{J\subset I_3} \left(\mathcal{C}_\lambda^J \setminus \bigcup_{K \subsetneq J} \mathcal{C}_\lambda^K \right).$$

Taking the W_f-orbit would give the desired result. Let $\mu \in C^+ \cap \mathcal{C}_\lambda$. The above equation is equivalent to the existence of some $J \subset I_3$ such that $\mu \in \mathcal{C}_\lambda^J$ and $\mu \notin \mathcal{C}_\lambda^K$, for every $K \subsetneq J$. Suppose the negation of this proposition, that is, for every $J \subset I_3$,

$$\left(\mu \notin \mathcal{C}_\lambda^J\right) \qquad \text{or} \qquad \left(\exists K \subsetneq J \text{ such that } \mu \in \mathcal{C}_\lambda^K\right).$$

Since $\mu \in \mathcal{C}_\lambda = \mathcal{C}_\lambda^{I_3}$, it follows that there is $K \subsetneq J$ such that $\mu \in \mathcal{C}_\lambda^K$, so that there is some $K' \subsetneq K$ such that $\mu \in \mathcal{C}_\lambda^{K'}$, and so on. Eventually, we get that $\mu \in \mathcal{C}_\lambda^{K^*}$ with $K^* \subsetneq \emptyset$.

(iv) Let $J, K \subset I_3$ and $w \in W_f$. Suppose $F_J(\lambda) \bigcap w\widetilde{\mathcal{C}_\lambda^K} \neq \emptyset$, then

$$F_J(\lambda) \bigcap w\widetilde{\mathcal{C}_\lambda^K} \subset F_J(\lambda) \bigcap wF_K(\lambda) \neq \emptyset.$$

By (3.7), we get $w \in W_J W_K$. That is,

$$w \notin W_J W_K \implies F_J(\lambda) \bigcap w\widetilde{\mathcal{C}_\lambda^K} = \emptyset.$$

Next, suppose $K \not\subset J$. Note that $F_J(\lambda) \cap \widetilde{\mathcal{C}_\lambda^K} = \emptyset$. Indeed, if $x \in F_J(\lambda) \cap \widetilde{\mathcal{C}_\lambda^K}$, then $x \in F_J(\lambda) \cap \mathcal{C}_\lambda^K = \mathcal{C}_\lambda^{J \cap K}$. This is a contradiction, since $x \in \widetilde{\mathcal{C}_\lambda^K}$ and $J \cap K \subsetneq K$.

Finally, using (i) and (iii),

$$\begin{aligned}
\mathcal{C}_\lambda^J &= \bigsqcup_{K \subset I_3} \left(F_J(\lambda) \bigcap W_f \cdot \widetilde{\mathcal{C}_\lambda^K} \right) \\
&= \bigsqcup_{K \subset I_3} \left(F_J(\lambda) \bigcap W_J W_K \cdot \widetilde{\mathcal{C}_\lambda^K} \right) \\
&= \bigsqcup_{K \subset I_3} W_J \cdot \left(F_J(\lambda) \bigcap \widetilde{\mathcal{C}_\lambda^K} \right) \\
&= \bigsqcup_{K \subset J} W_J \cdot \widetilde{\mathcal{C}_\lambda^K},
\end{aligned}$$

since $\widetilde{\mathcal{C}_\lambda^K} \subset F_K(\lambda) \subset F_J(\lambda)$, for $K \subset J$.

□

Let $K \subset J \subset I_3$. Note that $W_J \cdot \widetilde{\mathcal{C}_\lambda^K} \subset F_J(\lambda) \subset Y_J(\lambda)$. It follows that if there exists $\mu \in W_J \cdot \widetilde{\mathcal{C}_\lambda^K}$ (see remark 5.2), then the sets $(\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda)$ and $(\mathcal{P}_{W_f} + \mu) \cap F_J(\lambda)$ are not empty. The following lemma is the key ingredient to handle the "any dominant (co)weight" of the geometric formula.

Lemma 5.4. *Fix $K \subset J \subset I_3$ and let λ be a generic dominant weight. Pick $\mu \in W_J \cdot \widetilde{\mathcal{C}_\lambda^K}$. Then, the numbers*

$$\textit{Vol}\left((\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda)\right) \qquad \textit{and} \qquad \textit{Vol}_{|J|}\left((\mathcal{P}_{W_f} + \mu) \cap F_J(\lambda)\right),$$

do not depend on the choice of λ nor on the choice of μ, as long as $\widetilde{\mathcal{C}_\lambda^K}$ is not empty.

Proof. Write $\mu = w\mu'$, with $w \in W_J$ and $\mu' \in \widetilde{\mathcal{C}_\lambda^K}$. Then,

$$(\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda) = w\left((\mathcal{P}_{W_f} + \mu') \cap Y_J(\lambda)\right).$$

As w is an orthogonal transformation, we can assume that $\mu \in \widetilde{\mathcal{C}_\lambda^K}$. Recall Lemma 3.7 and note that

$$(\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda) = (\mathcal{P}_{W_f} + \mu) \cap (Y_J + \lambda).$$

Let $\mathcal{B}_J = \{b_1, b_2, b_3\}$ be the mixed basis, that is, $b_j = \alpha_j$ if $j \in J$, and $b_j = \varpi_j$ otherwise. For $t \in I_3$, let $H(J,t)$ be the hyperplane spanned by $\mathcal{B}_J \setminus \{b_t\}$. Consider the vectors u_i^J defined by the equations $(u_i^J, b_j) = \delta_{ij}$ and consider the (closed) half-spaces

$$H(J,t)^+ = \{v \in V \mid (v, u_t^J) \geq 0\},$$
$$H(J,t)^- = \{v \in V \mid (v, u_t^J) \leq 0\}.$$

We have that

$$\bigcap_{t \in I_3} H(J,t)^{\operatorname{sgn}(J,t)} = \langle \alpha_j \mid j \in J \rangle_{\leq 0} + \langle \varpi_i \mid i \notin J \rangle_{\geq 0} = Y_J,$$

where $\operatorname{sgn}(J,t)$ is $+$ if $t \notin J$, and $-$ if $t \in J$.

Note that $\mu \in \widetilde{\mathcal{C}_\lambda^K}$ implies that $\mu - \lambda = -\sum_{j \in K} a_j \alpha_j$, with a_j positive integers, since μ lies exclusively in the face $F_K(\lambda)$. That is,

$$(\mu - \lambda, u_t^J) \begin{cases} \in \mathbb{Z}^- & \text{if } t \in K \\ = 0 & \text{if } t \notin K \end{cases} \tag{5.3}$$

Similar to the proof of Proposition 3.6, let $\nu_{\mathcal{P}}$ be the vertices of the $\mathcal{P}_{W_f}$ and let $v \in \nu_{\mathcal{P}}$. Use the matrix M_J^{-1} to get the coordinates of v in the mixed basis $\mathcal{B}_J$. One can check that $(v, u_t^J) - 1 \leq 0$, for each $t \in J$[1]. This tells us that

$$\mu + \nu_{\mathcal{P}} \subset \lambda + H(J,t)^{\mathrm{sgn}(J,t)},$$

for all $t \in K$, so that none of the affine hyperplanes $H(J,t) + \lambda$, with $t \in K$, slices through $\mathcal{P}_{W_f} + \mu$. It follows that

$$\begin{aligned}
\mathrm{Vol}\left((\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda)\right) &= \mathrm{Vol}\left((\mathcal{P}_{W_f} + \mu) \bigcap \left(\bigcap_{t \in I_3 \setminus K} H(J,t)^{\mathrm{sgn}(J,t)} + \lambda\right)\right) \\
&= \mathrm{Vol}\left(\mathcal{P}_{W_f} \bigcap \left(\bigcap_{t \in I_3 \setminus K} H(J,t)^{\mathrm{sgn}(J,t)} + \lambda - \mu\right)\right) \\
&= \mathrm{Vol}\left(\mathcal{P}_{W_f} \bigcap \left(\bigcap_{t \in I_3 \setminus K} H(J,t)^{\mathrm{sgn}(J,t)}\right)\right),
\end{aligned}$$

since we know that $\lambda - \mu \in H(J,t)$, for all $t \notin K$, by (5.3).

Similarly, let H_J be the subspace generated by $\{\alpha_j \mid j \in J\}$. Then,

$$\mathrm{Vol}_{|J|}\left((\mathcal{P}_{W_f} + \mu) \cap F_J(\lambda)\right) = \mathrm{Vol}_{|J|}\left(\mathcal{P}_{W_f} \bigcap H_J \bigcap \left(\bigcap_{t \in J \setminus K} H(J,t)^{\mathrm{sgn}(J,t)}\right)\right),$$

since $H_J = \bigcap_{t \notin J} H(J,t)^{\mathrm{sgn}(J,t)}$. $\square$

The following corollary follows directly from the proof above.

[1]In fact, $(v, u_t^J) - 1 = 0$ only for $J = I_3$, $t = 2$, and $v = \varpi_2$. Geometrically, as explained in Proposition 3.6, this means that for suitable λ and $\mu \in \widetilde{\mathcal{C}_\lambda^{I_3}}$, $\mathcal{P}_{W_f} + \mu$ is completely contained in $\mathrm{Conv}(\lambda)$, but its vertex $\varpi_2 + \mu$ lies in the face $F_{\{1,3\}}(\lambda)$.

Corollary 5.5. *Let $K \subset J \subset I_3$. Define*

$$\Gamma_J(K) := \textbf{\textit{Vol}}\left(\mathcal{P}_{W_f} \bigcap \left(\bigcap_{t \in I_3 \setminus K} H(J,t)^{sgn(J,t)}\right)\right),$$

$$\gamma_J(K) := \textbf{\textit{Vol}}_{|J|}\left(\mathcal{P}_{W_f} \bigcap H_J \bigcap \left(\bigcap_{t \in J \setminus K} H(J,t)^{sgn(J,t)}\right)\right).$$

Let λ be any generic dominant weight. Then, whenever $\mu \in W_J \cdot \widetilde{\mathcal{C}_\lambda^K}$ exists,

$$\Gamma_J(K) = \textbf{\textit{Vol}}\left((\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda)\right) \quad \textit{and} \quad \gamma_J(K) = \textbf{\textit{Vol}}_{|J|}\left((\mathcal{P}_{W_f} + \mu) \cap F_J(\lambda)\right).$$

□

One can prove that the numbers $\Gamma_J(K), \gamma_J(K)$ are non-zero. At a first glance, these numbers might not seem relevant. The following lemma tells us to glance twice.

Lemma 5.6. *Let λ be a generic dominant weight and $J \subset I_3$. Define $q_J(\lambda) := |\widetilde{\mathcal{C}_\lambda^J}|$. Then*

$$\begin{aligned} \textbf{\textit{Vol}}(P_J(\lambda)) &= \sum_{K \subset J} [W_J : W_K]\, \Gamma_J(K)\, q_K(\lambda), \\ V_J^A(\lambda) &= \sum_{K \subset J} [W_J : W_K]\, \gamma_J(K)\, q_K(\lambda). \end{aligned}$$

Proof. From the proof of Lemma 5.4, we get

$$\begin{aligned} P_J(\lambda) &= Y_J(\lambda) \bigcap A(\leq \theta(\lambda)) \\ &= Y_J(\lambda) \bigcap \left(\bigsqcup_{\mu \in \mathcal{C}_\lambda} \mathcal{P}_{W_f} + \mu\right) \\ &= \bigsqcup_{\mu \in \mathcal{C}_\lambda^J} (\mathcal{P}_{W_f} + \mu) \cap Y_J(\lambda). \end{aligned}$$

Using Lemmas 5.3 and 5.4, we obtain

$$\begin{aligned}
\mathrm{Vol}\left(P_J(\lambda)\right) &= \sum_{\mu\in\mathcal{C}_\lambda^J} \mathrm{Vol}\left((\mathcal{P}_{W_f}+\mu)\cap Y_J(\lambda)\right)\\
&= \sum_{K\subset J}\left(\sum_{\mu\in W_J\cdot\widetilde{C}_\lambda^K} \mathrm{Vol}\left((\mathcal{P}_{W_f}+\mu)\cap Y_J(\lambda)\right)\right)\\
&= \sum_{K\subset J} |W_J\cdot \widetilde{C}_\lambda^K|\,\Gamma_J(K)\\
&= \sum_{K\subset J} [W_J:W_K]\,\Gamma_J(K)\,q_K(\lambda).
\end{aligned}$$

The equation for V_J^A follows analogously from

$$F_J(\lambda) = \bigsqcup_{\mu\in\mathcal{C}_\lambda^J} \left(\mathcal{P}_{W_f}+\mu\right)\cap F_J(\lambda),$$

which is implied by Proposition 3.6. □

Think of $q_K(\lambda)$ as the "building blocks" of the volumes $V_J^A(\lambda)$ and $\mathrm{Vol}(P_J(\lambda))$. The following lemma is the last one we will need before the proof of the geometric formula.

Corollary 5.7. *For any generic dominant weight λ, $q_J(\lambda)$ is a linear combination (with coefficients not depending on λ) of the volumes $V_K^A(\lambda)$ such that $K\subset J$.*

Proof. Note that $q_\emptyset(\lambda) = |\{\lambda\}| = 1$. By Lemma 5.6, we have that

$$q_J(\lambda) = \frac{1}{\gamma_J(J)}\left(V_J^A(\lambda) - \sum_{K\subsetneq J}[W_J:W_K]\,\gamma_J(K)\,q_K(\lambda)\right).$$

The result follows by induction on $|J|$. □

This implies that $q_J(\lambda)$ is a polynomial of degree $|J|$. In general, these polynomials are not homogeneous. In turn, this shows that $\mathrm{Vol}(P_J(\lambda))$ is a polynomial of degree

$|J|$. Now, we give the much-anticipated proof.

Proof of Theorem 5.1. Let λ be a generic dominant weight. By Lemma 5.6, we know that for all $J \subset I_3$,

$$\operatorname{Vol}(P_J(\lambda)) = \sum_{K \subset J} [W_J : W_K]\, \Gamma_J(K)\, q_K(\lambda).$$

Therefore, Corollary 5.7 implies that there exists $\beta_{J,K} \in \mathbb{R}$ such that

$$\operatorname{Vol}(P_J(\lambda)) = \sum_{K \subset J} \beta_{J,K} V_K^A(\lambda).$$

The coefficients $\beta_{J,K}$ do not depend on the choice of λ. For brevity, let $\triangle = \operatorname{Vol}(A_{\mathrm{id}})$. Since

$$A(\leq \theta(\lambda)) = \bigsqcup_{J \subset I_3} W_f \cdot P_J(\lambda),$$

equation (3.11) gives

$$\begin{aligned} |\leq \theta(\lambda)| &= \frac{1}{\triangle} \sum_{J \subset I_3} [W_f : W_J] \operatorname{Vol}(P_J(\lambda)) \\ &= \sum_{J \subset I_3} \left(\sum_{K \subset J} \frac{[W_f : W_f]}{\triangle} \beta_{J,K} V_K^A(\lambda) \right) \\ &= \sum_{J \subset I_3} \vartheta_J V_J^A(\lambda), \end{aligned}$$

with

$$\vartheta_J = \sum_{J \subset K \subset I_3} \frac{[W_f : W_K]}{\triangle} \beta_{K,J},$$

where the sum is over K. This shows that the geometric formula holds in the generic case.

We can extend the geometric formula for the non-generic (dominant) case. Since we have determined the polynomial $|\leq \theta(a,b,c)|$ (Schützer proved that it is in fact a polynomial) for $a,b,c \in \mathbb{Z}^+$, we have determined it everywhere. This comes from the general fact that a polynomial $h \in \mathbb{R}[x_1,\ldots,x_d]$ that vanishes on $(\mathbb{Z}^+)^d$, must vanish everywhere.[2] □

Note that Chapter 4 was not needed at all. We can use Corollary 4.7 to deduce the uniqueness of the geometric coefficients ϑ_J. There is more to be said about the polynomials q_J. Let $m = (m_1, m_2, m_3)$. Recall Lemma 4.6.

Lemma 5.8. *Let $J \subset I_3$ and define c_J^* as the coefficient of m_J in the polynomial $q_J(m)$. Then c_J^* is non-zero. Furthermore, the set $\{q_J(m) \mid J \subset I_3\}$ is linearly independent.*

Proof. Lemma 5.6 shows that

$$c_J = [W_J : W_K]\,\gamma_J(J)\,c_J^*,$$

since $q_J(m)$ is a polynomial of degree $|J|$. By Lemma 4.6, it follows that $c_J^* \neq 0$. The linear independence is a bit trickier to show than it was for V_J^A, as $q_J(m)$ is not homogeneous in general. Let $K \subset I_3$, and denote by $c(K,J)$ the coefficient of m_K in $q_J(m)$. Note that if $K \not\subset J$, then $c(K,J) = 0$, as $q_J(m)$ is a polynomial in the variables m_j, $j \in J$ and m_K is not. Then for $c(K,J) \neq 0$ to be true, it would be necessary that $K \subset J$. Suppose there exists $\beta_J \in \mathbb{R}$ such that

$$\sum_{J \subset I_3} \beta_J q_J(m) = 0. \tag{5.4}$$

We will show that $\beta_J = 0$ by induction on $3 - |J|$ (J is not fixed anymore). If $|J| = 3$, then $J = I_3$. Looking at the coefficient of m_{I_3} in both sides of (5.4) gives $\beta_{I_3} c_{I_3}^* = 0$, by the argument above. Since $c_{I_3}^* \neq 0$, we have that $\beta_{I_3} = 0$. Now suppose $\beta_K = 0$ for all

[2] This can be proven by induction. The case $d = 1$ is clear. The inductive step comes from noticing that the univariate polynomial $h(z_1, \ldots, z_{d-1}, x)$ vanishes on $\mathbb{Z}^+$, for every $z_i \in \mathbb{Z}^+$.

$K \subset I_3$ such that $|K| > 3 - \ell > 0$, for some ℓ. Let $J \subset I_3$ with $|J| = 3 - \ell$. Note that $c(J, K')$ can be nonzero only for $J \subset K' \subset I_3$. Thus, the coefficient of m_J in equation (5.4) gives

$$\sum_{J \subset K' \subset I_3} \beta_{K'}\, c(J, K') = \beta_J c_J^* \;\; + \sum_{J \subsetneq K' \subset I_3} \beta_{K'}\, c(J, K') = 0.$$

Therefore, $\beta_J c_J^* = 0$ by the inductive hypothesis, and hence $\beta_J = 0$, as $c_J^* \neq 0$. $\square$

Appendices

A. Some volumes V_J^A

Let λ be any vector, instead of a (generic) dominant weight. Note that going back trough the definitions, there is no problem when we are dealing with Conv(λ) or $V_J^X(\lambda)$. The reason we work with dominant weights, is that our aim was to study the set $\leq \theta(\lambda)$. Thus, the following polynomials have a geometric meaning even if we evaluate them in non-integers numbers.

Let $m = (m_1, m_2, \ldots)$ be a sequence of real numbers. Formula (4.1) is a powerful tool to compute $V_J^A(m)$ for connected J. If J is not connected, we just take the product over its connected components. By Lemma 4.4, we only need to know $V_{I_n}^A(m)$ together with some unconnected J. For example, $V_{\{1,3\}}(m) = 2m_1m_3$ so that $V_{\{2,4\}}(m) = 2m_2m_4$. We show $V_{I_n}^A(m)$ for $n \leq 4$ ($V_{I_5}^A$ is already too big to fit in one page), together with $V_J^A(m)$ for some unconnected $J \subset I_4$.

$$V_{\emptyset}^A(m) = 1,$$

$$V_{I_1}^A(m) = \sqrt{2}m_1,$$

$$V_{I_2}^A(m) = \frac{1}{2}\sqrt{3}m_1^2 + 2\sqrt{3}m_1m_2 + \frac{1}{2}\sqrt{3}m_2^2,$$

$$V_{\{1,3\}}^A(m) = 2\,m_1m_3,$$

$$V_{\{1,2,4\}}^A(m) = \frac{1}{2}\sqrt{3}\sqrt{2}m_1^2m_4 + 2\sqrt{3}\sqrt{2}m_1m_2m_4 + \frac{1}{2}\sqrt{3}\sqrt{2}m_2^2m_4,$$

$$
\begin{aligned}
V_{I_3}(m) = {} & \frac{1}{3}\, m_1^3 + 2\, m_1^2 m_2 + 4\, m_1 m_2^2 + \frac{4}{3}\, m_2^3 + 3\, m_1^2 m_3 + 12\, m_1 m_2 m_3 + 4\, m_2^2 m_3 \\
& + 3\, m_1 m_3^2 + 2\, m_2 m_3^2 + \frac{1}{3}\, m_3^3,
\end{aligned}
$$

$$
V^A_{\{1,3,4\}}(m) = \frac{1}{2}\, \sqrt{3}\sqrt{2} m_1 m_3^2 + 2\, \sqrt{3}\sqrt{2} m_1 m_3 m_4 + \frac{1}{2}\, \sqrt{3}\sqrt{2} m_1 m_4^2,
$$

$$
\begin{aligned}
V^A_{I_4}(m) = {} & \frac{1}{24}\, \sqrt{5} m_1^4 + \frac{1}{3}\, \sqrt{5} m_1^3 m_2 + \sqrt{5} m_1^2 m_2^2 + \frac{4}{3}\, \sqrt{5} m_1 m_2^3 + \frac{11}{24}\, \sqrt{5} m_2^4 \\
& + \frac{1}{2}\, \sqrt{5} m_1^3 m_3 + 3\, \sqrt{5} m_1^2 m_2 m_3 + 6\, \sqrt{5} m_1 m_2^2 m_3 + \frac{7}{3}\, \sqrt{5} m_2^3 m_3 \\
& + \frac{9}{4}\, \sqrt{5} m_1^2 m_3^2 + 9\, \sqrt{5} m_1 m_2 m_3^2 + 4\, \sqrt{5} m_2^2 m_3^2 + \frac{17}{6}\, \sqrt{5} m_1 m_3^3 \\
& + \frac{7}{3}\, \sqrt{5} m_2 m_3^3 + \frac{11}{24}\, \sqrt{5} m_3^4 + \frac{2}{3}\, \sqrt{5} m_1^3 m_4 + 4\, \sqrt{5} m_1^2 m_2 m_4 \\
& + 8\, \sqrt{5} m_1 m_2^2 m_4 + \frac{17}{6}\, \sqrt{5} m_2^3 m_4 + 6\, \sqrt{5} m_1^2 m_3 m_4 + 24\, \sqrt{5} m_1 m_2 m_3 m_4 \\
& + 9\, \sqrt{5} m_2^2 m_3 m_4 + 8\, \sqrt{5} m_1 m_3^2 m_4 + 6\, \sqrt{5} m_2 m_3^2 m_4 + \frac{4}{3}\, \sqrt{5} m_3^3 m_4 \\
& + \frac{3}{2}\, \sqrt{5} m_1^2 m_4^2 + 6\, \sqrt{5} m_1 m_2 m_4^2 + \frac{9}{4}\, \sqrt{5} m_2^2 m_4^2 + 4\, \sqrt{5} m_1 m_3 m_4^2 \\
& + 3\, \sqrt{5} m_2 m_3 m_4^2 + \sqrt{5} m_3^2 m_4^2 + \frac{2}{3}\, \sqrt{5} m_1 m_4^3 + \frac{1}{2}\, \sqrt{5} m_2 m_4^3 \\
& + \frac{1}{3}\, \sqrt{5} m_3 m_4^3 + \frac{1}{24}\, \sqrt{5} m_4^4.
\end{aligned}
$$

B. Geometric coefficients tables

The following tables show the geometric coefficients ϑ_J of the geometric formula, for types $\widetilde{A_1}, \ldots, \widetilde{A_7}$. These have been computed using the Schützer formula for $\mathcal{C}_\lambda$, and using Lemma 4.6. The author of this thesis would be happy to share the codes to anyone intereseted.

In order to simplify the geometric coefficients, we will show them in different ways so that the numbers are more appealing. Let $J \subset I_n$ and define

$$\sqrt{J} := \prod_{K \in J_c} |K| + 1,$$

where J_c are the connected components of J. Put $\vartheta_J^R = \vartheta_J \sqrt{J}$. This will make the coefficients to be integers (at least in these cases). Let A_0 be the fundamental alcove (in type $\widetilde{A_n}$). It is natural to look at the numbers

$$\vartheta'_J = \frac{\mathrm{Vol}(A_0)}{[W_f : W_J]} \vartheta_J,$$

as we want to normalize by the volume of the fundamental alcove, and there are $[W_f : W_J]$ J-faces. One can prove that

$$\mathrm{Vol}(A_0) = \frac{\sqrt{n+1}}{(n+1)!}.$$

Type $\widetilde{A}_1$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	2	2	$\frac{1}{2}\sqrt{2}$
$\{1\}$	$\sqrt{2}$	2	1

Type $\widetilde{A}_2$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	6	6	$\frac{1}{6}\sqrt{3}$
$\{1\}$	$\frac{9}{2}\sqrt{2}$	9	$\frac{1}{4}\sqrt{3}\sqrt{2}$
$\{2\}$	$\frac{9}{2}\sqrt{2}$	9	$\frac{1}{4}\sqrt{3}\sqrt{2}$
$\{1,2\}$	$2\sqrt{3}$	6	1

Type $\widetilde{A}_3$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	24	24	$\frac{1}{12}$
$\{1\}$	$22\sqrt{2}$	44	$\frac{11}{72}\sqrt{2}$
$\{2\}$	$28\sqrt{2}$	56	$\frac{7}{36}\sqrt{2}$
$\{3\}$	$22\sqrt{2}$	44	$\frac{11}{72}\sqrt{2}$
$\{1,2\}$	$16\sqrt{3}$	48	$\frac{1}{3}\sqrt{3}$
$\{1,3\}$	36	72	$\frac{1}{2}$
$\{2,3\}$	$16\sqrt{3}$	48	$\frac{1}{3}\sqrt{3}$
$\{1,2,3\}$	12	24	1

Type $\widetilde{A}_4$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	120	120	$\frac{1}{120}\sqrt{5}$
$\{1\}$	$125\sqrt{2}$	250	$\frac{5}{288}\sqrt{5}\sqrt{2}$
$\{2\}$	$175\sqrt{2}$	350	$\frac{7}{288}\sqrt{5}\sqrt{2}$
$\{3\}$	$175\sqrt{2}$	350	$\frac{7}{288}\sqrt{5}\sqrt{2}$
$\{4\}$	$125\sqrt{2}$	250	$\frac{5}{288}\sqrt{5}\sqrt{2}$
$\{1,2\}$	$\frac{350}{3}\sqrt{3}$	350	$\frac{7}{144}\sqrt{5}\sqrt{3}$
$\{1,3\}$	325	650	$\frac{13}{144}\sqrt{5}$
$\{1,4\}$	250	500	$\frac{5}{72}\sqrt{5}$
$\{2,3\}$	$\frac{500}{3}\sqrt{3}$	500	$\frac{5}{72}\sqrt{5}\sqrt{3}$
$\{2,4\}$	325	650	$\frac{13}{144}\sqrt{5}$
$\{3,4\}$	$\frac{350}{3}\sqrt{3}$	350	$\frac{7}{144}\sqrt{5}\sqrt{3}$
$\{1,2,3\}$	150	300	$\frac{1}{4}\sqrt{5}$
$\{1,2,4\}$	$100\sqrt{3}\sqrt{2}$	600	$\frac{1}{12}\sqrt{5}\sqrt{3}\sqrt{2}$
$\{1,3,4\}$	$100\sqrt{3}\sqrt{2}$	600	$\frac{1}{12}\sqrt{5}\sqrt{3}\sqrt{2}$
$\{2,3,4\}$	150	300	$\frac{1}{4}\sqrt{5}$

$\{1,2,3,4\}$	$24\sqrt{5}$	120	1

Type $\widetilde{A}_5$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	720	720	$\frac{1}{720}\sqrt{3}\sqrt{2}$
$\{1\}$	$822\sqrt{2}$	1644	$\frac{137}{21600}\sqrt{3}$
$\{2\}$	$1212\sqrt{2}$	2424	$\frac{101}{10800}\sqrt{3}$
$\{3\}$	$1332\sqrt{2}$	2664	$\frac{37}{3600}\sqrt{3}$
$\{4\}$	$1212\sqrt{2}$	2424	$\frac{101}{10800}\sqrt{3}$
$\{5\}$	$822\sqrt{2}$	1644	$\frac{137}{21600}\sqrt{3}$
$\{1,2\}$	$900\sqrt{3}$	2700	$\frac{1}{32}\sqrt{2}$
$\{1,3\}$	2700	5400	$\frac{1}{48}\sqrt{3}\sqrt{2}$
$\{1,4\}$	2700	5400	$\frac{1}{48}\sqrt{3}\sqrt{2}$
$\{1,5\}$	1800	3600	$\frac{1}{72}\sqrt{3}\sqrt{2}$
$\{2,3\}$	$1500\sqrt{3}$	4500	$\frac{5}{96}\sqrt{2}$
$\{2,4\}$	3600	7200	$\frac{1}{36}\sqrt{3}\sqrt{2}$
$\{2,5\}$	2700	5400	$\frac{1}{48}\sqrt{3}\sqrt{2}$
$\{3,4\}$	$1500\sqrt{3}$	4500	$\frac{5}{96}\sqrt{2}$
$\{3,5\}$	2700	5400	$\frac{1}{48}\sqrt{3}\sqrt{2}$

$\{4,5\}$	$900\sqrt{3}$	2700	$\frac{1}{32}\sqrt{2}$
$\{1,2,3\}$	1530	3060	$\frac{17}{240}\sqrt{3}\sqrt{2}$
$\{1,2,4\}$	$1240\sqrt{3}\sqrt{2}$	7440	$\frac{31}{180}$
$\{1,2,5\}$	$950\sqrt{3}\sqrt{2}$	5700	$\frac{19}{144}$
$\{1,3,4\}$	$1410\sqrt{3}\sqrt{2}$	8460	$\frac{47}{240}$
$\{1,3,5\}$	$2700\sqrt{2}$	10800	$\frac{1}{12}\sqrt{3}$
$\{1,4,5\}$	$950\sqrt{3}\sqrt{2}$	5700	$\frac{19}{144}$
$\{2,3,4\}$	2340	4680	$\frac{13}{120}\sqrt{3}\sqrt{2}$
$\{2,3,5\}$	$1410\sqrt{3}\sqrt{2}$	8460	$\frac{47}{240}$
$\{2,4,5\}$	$1240\sqrt{3}\sqrt{2}$	7440	$\frac{31}{180}$
$\{3,4,5\}$	1530	3060	$\frac{17}{240}\sqrt{3}\sqrt{2}$
$\{1,2,3,4\}$	$432\sqrt{5}$	2160	$\frac{1}{10}\sqrt{5}\sqrt{3}\sqrt{2}$
$\{1,2,3,5\}$	$1350\sqrt{2}$	5400	$\frac{1}{4}\sqrt{3}$
$\{1,2,4,5\}$	2400	7200	$\frac{1}{6}\sqrt{3}\sqrt{2}$
$\{1,3,4,5\}$	$1350\sqrt{2}$	5400	$\frac{1}{4}\sqrt{3}$
$\{2,3,4,5\}$	$432\sqrt{5}$	2160	$\frac{1}{10}\sqrt{5}\sqrt{3}\sqrt{2}$
$\{1,2,3,4,5\}$	$120\sqrt{3}\sqrt{2}$	720	1

Type $\widetilde{A}_6$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	5040	5040	$\frac{1}{5040}\sqrt{7}$
$\{1\}$	$6174\sqrt{2}$	12348	$\frac{7}{14400}\sqrt{7}\sqrt{2}$
$\{2\}$	$9408\sqrt{2}$	18816	$\frac{1}{1350}\sqrt{7}\sqrt{2}$
$\{3\}$	$10878\sqrt{2}$	21756	$\frac{37}{43200}\sqrt{7}\sqrt{2}$
$\{4\}$	$10878\sqrt{2}$	21756	$\frac{37}{43200}\sqrt{7}\sqrt{2}$
$\{5\}$	$9408\sqrt{2}$	18816	$\frac{1}{1350}\sqrt{7}\sqrt{2}$
$\{6\}$	$6174\sqrt{2}$	12348	$\frac{7}{14400}\sqrt{7}\sqrt{2}$
$\{1,2\}$	$\frac{22736}{3}\sqrt{3}$	22736	$\frac{29}{16200}\sqrt{7}\sqrt{3}$
$\{1,3\}$	23520	47040	$\frac{1}{270}\sqrt{7}$
$\{1,4\}$	26509	53018	$\frac{541}{129600}\sqrt{7}$
$\{1,5\}$	21903	43806	$\frac{149}{43200}\sqrt{7}$
$\{1,6\}$	14798	29596	$\frac{151}{64800}\sqrt{7}$
$\{2,3\}$	$\frac{41356}{3}\sqrt{3}$	41356	$\frac{211}{64800}\sqrt{7}\sqrt{3}$
$\{2,4\}$	35231	70462	$\frac{719}{129600}\sqrt{7}$
$\{2,5\}$	35476	70952	$\frac{181}{32400}\sqrt{7}$

$\{2,6\}$	21903	43806	$\frac{149}{43200}\sqrt{7}$
$\{3,4\}$	$16072\sqrt{3}$	48216	$\frac{41}{10800}\sqrt{7}\sqrt{3}$
$\{3,5\}$	35231	70462	$\frac{719}{129600}\sqrt{7}$
$\{3,6\}$	26509	53018	$\frac{541}{129600}\sqrt{7}$
$\{4,5\}$	$\frac{41356}{3}\sqrt{3}$	41356	$\frac{211}{64800}\sqrt{7}\sqrt{3}$
$\{4,6\}$	23520	47040	$\frac{1}{270}\sqrt{7}$
$\{5,6\}$	$\frac{22736}{3}\sqrt{3}$	22736	$\frac{29}{16200}\sqrt{7}\sqrt{3}$
$\{1,2,3\}$	15435	30870	$\frac{7}{480}\sqrt{7}$
$\{1,2,4\}$	$13230\sqrt{3}\sqrt{2}$	79380	$\frac{1}{160}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,2,5\}$	$13475\sqrt{3}\sqrt{2}$	80850	$\frac{11}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,2,6\}$	$8575\sqrt{3}\sqrt{2}$	51450	$\frac{7}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,3,4\}$	$16170\sqrt{3}\sqrt{2}$	97020	$\frac{11}{1440}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,3,5\}$	$\frac{77175}{2}\sqrt{2}$	154350	$\frac{7}{576}\sqrt{7}\sqrt{2}$
$\{1,3,6\}$	$\frac{55125}{2}\sqrt{2}$	110250	$\frac{5}{576}\sqrt{7}\sqrt{2}$
$\{1,4,5\}$	$15925\sqrt{3}\sqrt{2}$	95550	$\frac{13}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,4,6\}$	$\frac{55125}{2}\sqrt{2}$	110250	$\frac{5}{576}\sqrt{7}\sqrt{2}$
$\{1,5,6\}$	$8575\sqrt{3}\sqrt{2}$	51450	$\frac{7}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{2,3,4\}$	28665	57330	$\frac{13}{480}\sqrt{7}$

$\{2,3,5\}$	$20825\sqrt{3}\sqrt{2}$	124950	$\frac{17}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{2,3,6\}$	$15925\sqrt{3}\sqrt{2}$	95550	$\frac{13}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{2,4,5\}$	$20825\sqrt{3}\sqrt{2}$	124950	$\frac{17}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{2,4,6\}$	$\frac{77175}{2}\sqrt{2}$	154350	$\frac{7}{576}\sqrt{7}\sqrt{2}$
$\{2,5,6\}$	$13475\sqrt{3}\sqrt{2}$	80850	$\frac{11}{1728}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{3,4,5\}$	28665	57330	$\frac{13}{480}\sqrt{7}$
$\{3,4,6\}$	$16170\sqrt{3}\sqrt{2}$	97020	$\frac{11}{1440}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{3,5,6\}$	$13230\sqrt{3}\sqrt{2}$	79380	$\frac{1}{160}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{4,5,6\}$	15435	30870	$\frac{7}{480}\sqrt{7}$
$\{1,2,3,4\}$	$5880\sqrt{5}$	29400	$\frac{1}{36}\sqrt{7}\sqrt{5}$
$\{1,2,3,5\}$	$22050\sqrt{2}$	88200	$\frac{1}{24}\sqrt{7}\sqrt{2}$
$\{1,2,3,6\}$	$16905\sqrt{2}$	67620	$\frac{23}{720}\sqrt{7}\sqrt{2}$
$\{1,2,4,5\}$	44100	132300	$\frac{1}{16}\sqrt{7}$
$\{1,2,4,6\}$	$27930\sqrt{3}$	167580	$\frac{19}{720}\sqrt{7}\sqrt{3}$
$\{1,2,5,6\}$	29400	88200	$\frac{1}{24}\sqrt{7}$
$\{1,3,4,5\}$	$27195\sqrt{2}$	108780	$\frac{37}{720}\sqrt{7}\sqrt{2}$
$\{1,3,4,6\}$	$32340\sqrt{3}$	194040	$\frac{11}{360}\sqrt{7}\sqrt{3}$
$\{1,3,5,6\}$	$27930\sqrt{3}$	167580	$\frac{19}{720}\sqrt{7}\sqrt{3}$

$\{1,4,5,6\}$	$16905\sqrt{2}$	67620	$\frac{23}{720}\sqrt{7}\sqrt{2}$
$\{2,3,4,5\}$	$9408\sqrt{5}$	47040	$\frac{2}{45}\sqrt{7}\sqrt{5}$
$\{2,3,4,6\}$	$27195\sqrt{2}$	108780	$\frac{37}{720}\sqrt{7}\sqrt{2}$
$\{2,3,5,6\}$	44100	132300	$\frac{1}{16}\sqrt{7}$
$\{2,4,5,6\}$	$22050\sqrt{2}$	88200	$\frac{1}{24}\sqrt{7}\sqrt{2}$
$\{3,4,5,6\}$	$5880\sqrt{5}$	29400	$\frac{1}{36}\sqrt{7}\sqrt{5}$
$\{1,2,3,4,5\}$	$2940\sqrt{3}\sqrt{2}$	17640	$\frac{1}{12}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,2,3,4,6\}$	$5292\sqrt{5}\sqrt{2}$	52920	$\frac{1}{20}\sqrt{7}\sqrt{5}\sqrt{2}$
$\{1,2,3,5,6\}$	$14700\sqrt{3}$	88200	$\frac{1}{12}\sqrt{7}\sqrt{3}$
$\{1,2,4,5,6\}$	$14700\sqrt{3}$	88200	$\frac{1}{12}\sqrt{7}\sqrt{3}$
$\{1,3,4,5,6\}$	$5292\sqrt{5}\sqrt{2}$	52920	$\frac{1}{20}\sqrt{7}\sqrt{5}\sqrt{2}$
$\{2,3,4,5,6\}$	$2940\sqrt{3}\sqrt{2}$	17640	$\frac{1}{12}\sqrt{7}\sqrt{3}\sqrt{2}$
$\{1,2,3,4,5,6\}$	$720\sqrt{7}$	5040	1

Type $\widetilde{A}_7$

J	ϑ_J	ϑ_J^R	ϑ'_J
$\emptyset$	40320	40320	$\frac{1}{20160}\sqrt{2}$
$\{1\}$	$52272\sqrt{2}$	104544	$\frac{121}{470400}$
$\{2\}$	$81504\sqrt{2}$	163008	$\frac{283}{705600}$
$\{3\}$	$97296\sqrt{2}$	194592	$\frac{2027}{4233600}$
$\{4\}$	$102336\sqrt{2}$	204672	$\frac{533}{1058400}$
$\{5\}$	$97296\sqrt{2}$	194592	$\frac{2027}{4233600}$
$\{6\}$	$81504\sqrt{2}$	163008	$\frac{283}{705600}$
$\{7\}$	$52272\sqrt{2}$	104544	$\frac{121}{470400}$
$\{1,2\}$	$\frac{210112}{3}\sqrt{3}$	210112	$\frac{67}{129600}\sqrt{3}\sqrt{2}$
$\{1,3\}$	221088	442176	$\frac{47}{43200}\sqrt{2}$
$\{1,4\}$	269696	539392	$\frac{43}{32400}\sqrt{2}$
$\{1,5\}$	235984	471968	$\frac{301}{259200}\sqrt{2}$
$\{1,6\}$	208544	417088	$\frac{133}{129600}\sqrt{2}$
$\{1,7\}$	132496	264992	$\frac{169}{259200}\sqrt{2}$
$\{2,3\}$	$134848\sqrt{3}$	404544	$\frac{43}{43200}\sqrt{3}\sqrt{2}$

$\{2,4\}$	351232	702464	$\frac{7}{4050}\sqrt{2}$
$\{2,5\}$	415520	831040	$\frac{53}{25920}\sqrt{2}$
$\{2,6\}$	304192	608384	$\frac{97}{64800}\sqrt{2}$
$\{2,7\}$	208544	417088	$\frac{133}{129600}\sqrt{2}$
$\{3,4\}$	$\frac{514304}{3}\sqrt{3}$	514304	$\frac{41}{32400}\sqrt{3}\sqrt{2}$
$\{3,5\}$	392784	785568	$\frac{167}{86400}\sqrt{2}$
$\{3,6\}$	415520	831040	$\frac{53}{25920}\sqrt{2}$
$\{3,7\}$	235984	471968	$\frac{301}{259200}\sqrt{2}$
$\{4,5\}$	$\frac{514304}{3}\sqrt{3}$	514304	$\frac{41}{32400}\sqrt{3}\sqrt{2}$
$\{4,6\}$	351232	702464	$\frac{7}{4050}\sqrt{2}$
$\{4,7\}$	269696	539392	$\frac{43}{32400}\sqrt{2}$
$\{5,6\}$	$134848\sqrt{3}$	404544	$\frac{43}{43200}\sqrt{3}\sqrt{2}$
$\{5,7\}$	221088	442176	$\frac{47}{43200}\sqrt{2}$
$\{6,7\}$	$\frac{210112}{3}\sqrt{3}$	210112	$\frac{67}{129600}\sqrt{3}\sqrt{2}$
$\{1,2,3\}$	162456	324912	$\frac{967}{201600}\sqrt{2}$
$\{1,2,4\}$	$141344\sqrt{3}\sqrt{2}$	848064	$\frac{631}{151200}\sqrt{3}$
$\{1,2,5\}$	$167272\sqrt{3}\sqrt{2}$	1003632	$\frac{2987}{604800}\sqrt{3}$
$\{1,2,6\}$	$\frac{385168}{3}\sqrt{3}\sqrt{2}$	770336	$\frac{3439}{907200}\sqrt{3}$

$\{1,2,7\}$	$\frac{260680}{3}\sqrt{3}\sqrt{2}$	521360	$\frac{133}{51840}\sqrt{3}$
$\{1,3,4\}$	$\frac{545104}{3}\sqrt{3}\sqrt{2}$	1090208	$\frac{4867}{907200}\sqrt{3}$
$\{1,3,5\}$	$468132\sqrt{2}$	1872528	$\frac{5573}{604800}$
$\{1,3,6\}$	$444472\sqrt{2}$	1777888	$\frac{7937}{907200}$
$\{1,3,7\}$	$267932\sqrt{2}$	1071728	$\frac{1367}{259200}$
$\{1,4,5\}$	$\frac{642992}{3}\sqrt{3}\sqrt{2}$	1285984	$\frac{5741}{907200}\sqrt{3}$
$\{1,4,6\}$	$444192\sqrt{2}$	1776768	$\frac{661}{75600}$
$\{1,4,7\}$	$339472\sqrt{2}$	1357888	$\frac{433}{64800}$
$\{1,5,6\}$	$\frac{479192}{3}\sqrt{3}\sqrt{2}$	958384	$\frac{8557}{1814400}\sqrt{3}$
$\{1,5,7\}$	$267932\sqrt{2}$	1071728	$\frac{1367}{259200}$
$\{1,6,7\}$	$\frac{260680}{3}\sqrt{3}\sqrt{2}$	521360	$\frac{133}{51840}\sqrt{3}$
$\{2,3,4\}$	338016	676032	$\frac{503}{50400}\sqrt{2}$
$\{2,3,5\}$	$255416\sqrt{3}\sqrt{2}$	1532496	$\frac{4561}{604800}\sqrt{3}$
$\{2,3,6\}$	$\frac{813232}{3}\sqrt{3}\sqrt{2}$	1626464	$\frac{7261}{907200}\sqrt{3}$
$\{2,3,7\}$	$\frac{479192}{3}\sqrt{3}\sqrt{2}$	958384	$\frac{8557}{1814400}\sqrt{3}$
$\{2,4,5\}$	$\frac{826336}{3}\sqrt{3}\sqrt{2}$	1652672	$\frac{527}{64800}\sqrt{3}$
$\{2,4,6\}$	$644672\sqrt{2}$	2578688	$\frac{1439}{113400}$
$\{2,4,7\}$	$444192\sqrt{2}$	1776768	$\frac{661}{75600}$

$\{2,5,6\}$	$\frac{813232}{3}\sqrt{3}\sqrt{2}$	1626464	$\frac{7261}{907200}\sqrt{3}$
$\{2,5,7\}$	$444472\sqrt{2}$	1777888	$\frac{7937}{907200}$
$\{2,6,7\}$	$\frac{385168}{3}\sqrt{3}\sqrt{2}$	770336	$\frac{3439}{907200}\sqrt{3}$
$\{3,4,5\}$	410256	820512	$\frac{407}{33600}\sqrt{2}$
$\{3,4,6\}$	$\frac{826336}{3}\sqrt{3}\sqrt{2}$	1652672	$\frac{527}{64800}\sqrt{3}$
$\{3,4,7\}$	$\frac{642992}{3}\sqrt{3}\sqrt{2}$	1285984	$\frac{5741}{907200}\sqrt{3}$
$\{3,5,6\}$	$255416\sqrt{3}\sqrt{2}$	1532496	$\frac{4561}{604800}\sqrt{3}$
$\{3,5,7\}$	$468132\sqrt{2}$	1872528	$\frac{5573}{604800}$
$\{3,6,7\}$	$167272\sqrt{3}\sqrt{2}$	1003632	$\frac{2987}{604800}\sqrt{3}$
$\{4,5,6\}$	338016	676032	$\frac{503}{50400}\sqrt{2}$
$\{4,5,7\}$	$\frac{545104}{3}\sqrt{3}\sqrt{2}$	1090208	$\frac{4867}{907200}\sqrt{3}$
$\{4,6,7\}$	$141344\sqrt{3}\sqrt{2}$	848064	$\frac{631}{151200}\sqrt{3}$
$\{5,6,7\}$	162456	324912	$\frac{967}{201600}\sqrt{2}$
$\{1,2,3,4\}$	$75264\sqrt{5}$	376320	$\frac{1}{90}\sqrt{5}\sqrt{2}$
$\{1,2,3,5\}$	$294000\sqrt{2}$	1176000	$\frac{5}{144}$
$\{1,2,3,6\}$	$305760\sqrt{2}$	1223040	$\frac{13}{360}$
$\{1,2,3,7\}$	$188160\sqrt{2}$	752640	$\frac{1}{45}$
$\{1,2,4,5\}$	627200	1881600	$\frac{1}{36}\sqrt{2}$

$\{1,2,4,6\}$	$501760\sqrt{3}$	3010560	$\frac{2}{135}\sqrt{3}\sqrt{2}$
$\{1,2,4,7\}$	$344960\sqrt{3}$	2069760	$\frac{11}{1080}\sqrt{3}\sqrt{2}$
$\{1,2,5,6\}$	627200	1881600	$\frac{1}{36}\sqrt{2}$
$\{1,2,5,7\}$	$352800\sqrt{3}$	2116800	$\frac{1}{96}\sqrt{3}\sqrt{2}$
$\{1,2,6,7\}$	313600	940800	$\frac{1}{72}\sqrt{2}$
$\{1,3,4,5\}$	$411600\sqrt{2}$	1646400	$\frac{7}{144}$
$\{1,3,4,6\}$	$595840\sqrt{3}$	3575040	$\frac{19}{1080}\sqrt{3}\sqrt{2}$
$\{1,3,4,7\}$	$439040\sqrt{3}$	2634240	$\frac{7}{540}\sqrt{3}\sqrt{2}$
$\{1,3,5,6\}$	$588000\sqrt{3}$	3528000	$\frac{5}{288}\sqrt{3}\sqrt{2}$
$\{1,3,5,7\}$	1058400	4233600	$\frac{1}{48}\sqrt{2}$
$\{1,3,6,7\}$	$352800\sqrt{3}$	2116800	$\frac{1}{96}\sqrt{3}\sqrt{2}$
$\{1,4,5,6\}$	$399840\sqrt{2}$	1599360	$\frac{17}{360}$
$\{1,4,5,7\}$	$439040\sqrt{3}$	2634240	$\frac{7}{540}\sqrt{3}\sqrt{2}$
$\{1,4,6,7\}$	$344960\sqrt{3}$	2069760	$\frac{11}{1080}\sqrt{3}\sqrt{2}$
$\{1,5,6,7\}$	$188160\sqrt{2}$	752640	$\frac{1}{45}$
$\{2,3,4,5\}$	$150528\sqrt{5}$	752640	$\frac{1}{45}\sqrt{5}\sqrt{2}$
$\{2,3,4,6\}$	$517440\sqrt{2}$	2069760	$\frac{11}{180}$
$\{2,3,4,7\}$	$399840\sqrt{2}$	1599360	$\frac{17}{360}$

$\{2,3,5,6\}$	940800	2822400	$\frac{1}{24}\sqrt{2}$
$\{2,3,5,7\}$	$588000\sqrt{3}$	3528000	$\frac{5}{288}\sqrt{3}\sqrt{2}$
$\{2,3,6,7\}$	627200	1881600	$\frac{1}{36}\sqrt{2}$
$\{2,4,5,6\}$	$517440\sqrt{2}$	2069760	$\frac{11}{180}$
$\{2,4,5,7\}$	$595840\sqrt{3}$	3575040	$\frac{19}{1080}\sqrt{3}\sqrt{2}$
$\{2,4,6,7\}$	$501760\sqrt{3}$	3010560	$\frac{2}{135}\sqrt{3}\sqrt{2}$
$\{2,5,6,7\}$	$305760\sqrt{2}$	1223040	$\frac{13}{360}$
$\{3,4,5,6\}$	$150528\sqrt{5}$	752640	$\frac{1}{45}\sqrt{5}\sqrt{2}$
$\{3,4,5,7\}$	$411600\sqrt{2}$	1646400	$\frac{7}{144}$
$\{3,4,6,7\}$	627200	1881600	$\frac{1}{36}\sqrt{2}$
$\{3,5,6,7\}$	$294000\sqrt{2}$	1176000	$\frac{5}{144}$
$\{4,5,6,7\}$	$75264\sqrt{5}$	376320	$\frac{1}{90}\sqrt{5}\sqrt{2}$
$\{1,2,3,4,5\}$	$51520\sqrt{3}\sqrt{2}$	309120	$\frac{23}{252}\sqrt{3}$
$\{1,2,3,4,6\}$	$110208\sqrt{5}\sqrt{2}$	1102080	$\frac{41}{630}\sqrt{5}$
$\{1,2,3,4,7\}$	$84672\sqrt{5}\sqrt{2}$	846720	$\frac{1}{20}\sqrt{5}$
$\{1,2,3,5,6\}$	$341600\sqrt{3}$	2049600	$\frac{61}{1008}\sqrt{3}\sqrt{2}$
$\{1,2,3,5,7\}$	646800	2587200	$\frac{11}{144}\sqrt{2}$
$\{1,2,3,6,7\}$	$227360\sqrt{3}$	1364160	$\frac{29}{720}\sqrt{3}\sqrt{2}$

$\{1,2,4,5,6\}$	$371840\sqrt{3}$	2231040	$\frac{83}{1260}\sqrt{3}\sqrt{2}$
$\{1,2,4,5,7\}$	$658560\sqrt{2}$	3951360	$\frac{7}{60}$
$\{1,2,4,6,7\}$	$564480\sqrt{2}$	3386880	$\frac{1}{10}$
$\{1,2,5,6,7\}$	$227360\sqrt{3}$	1364160	$\frac{29}{720}\sqrt{3}\sqrt{2}$
$\{1,3,4,5,6\}$	$143808\sqrt{5}\sqrt{2}$	1438080	$\frac{107}{1260}\sqrt{5}$
$\{1,3,4,5,7\}$	823200	3292800	$\frac{7}{72}\sqrt{2}$
$\{1,3,4,6,7\}$	$658560\sqrt{2}$	3951360	$\frac{7}{60}$
$\{1,3,5,6,7\}$	646800	2587200	$\frac{11}{144}\sqrt{2}$
$\{1,4,5,6,7\}$	$84672\sqrt{5}\sqrt{2}$	846720	$\frac{1}{20}\sqrt{5}$
$\{2,3,4,5,6\}$	$85120\sqrt{3}\sqrt{2}$	510720	$\frac{19}{126}\sqrt{3}$
$\{2,3,4,5,7\}$	$143808\sqrt{5}\sqrt{2}$	1438080	$\frac{107}{1260}\sqrt{5}$
$\{2,3,4,6,7\}$	$371840\sqrt{3}$	2231040	$\frac{83}{1260}\sqrt{3}\sqrt{2}$
$\{2,3,5,6,7\}$	$341600\sqrt{3}$	2049600	$\frac{61}{1008}\sqrt{3}\sqrt{2}$
$\{2,4,5,6,7\}$	$110208\sqrt{5}\sqrt{2}$	1102080	$\frac{41}{630}\sqrt{5}$
$\{3,4,5,6,7\}$	$51520\sqrt{3}\sqrt{2}$	309120	$\frac{23}{252}\sqrt{3}$
$\{1,2,3,4,5,6\}$	$23040\sqrt{7}$	161280	$\frac{1}{7}\sqrt{7}\sqrt{2}$
$\{1,2,3,4,5,7\}$	$94080\sqrt{3}$	564480	$\frac{1}{6}\sqrt{3}\sqrt{2}$
$\{1,2,3,4,6,7\}$	$75264\sqrt{5}\sqrt{3}$	1128960	$\frac{1}{15}\sqrt{5}\sqrt{3}\sqrt{2}$

$\{1,2,3,5,6,7\}$	352800	1411200	$\frac{1}{4}\sqrt{2}$
$\{1,2,4,5,6,7\}$	$75264\sqrt{5}\sqrt{3}$	1128960	$\frac{1}{15}\sqrt{5}\sqrt{3}\sqrt{2}$
$\{1,3,4,5,6,7\}$	$94080\sqrt{3}$	564480	$\frac{1}{6}\sqrt{3}\sqrt{2}$
$\{2,3,4,5,6,7\}$	$23040\sqrt{7}$	161280	$\frac{1}{7}\sqrt{7}\sqrt{2}$
$\{1,2,3,4,5,6,7\}$	$10080\sqrt{2}$	40320	1

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