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COUNTEREXAMPLES TO THE FORKING PATH CONJECTURE

THESIS SUBMITTED IN FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR

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ABSTRACT OF THE THESIS SUBMITTED
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In 2011, N. Libedinsky studied in [1] morphisms induced by paths in the reduced expression graph of extra-large Coxeter systems. He showed that morphisms induced by complete paths are idempotents that act as projectors. In 2016, B. Elias provides in [2] an extension of his work with M. Khovanov [3], where they gave a diagrammatic presentation of the category of Bott-Samelson bimodules $\mathbb{BS}$Bim. Here, morphisms can be translated into linear combinations of planar graphs, and stacking planar graphs can be interpreted as composing morphisms. Elias uses his diagrammatic calculus to construct an idempotent in the reduced expressions graph for the longest element w_0 in the symmetric group S_n, $Rex(w_{0,n})$. This idempotent can also be described by complete paths. These observations, plus a considerable number of computer comprobations, motivate Libedinsky to formulate the *Forking Path Conjecture* [4, Section 6.3].

Conjecture 1 (Forking Path Conjecture) Let $x \in S_n$, let p, q be two complete paths with the same starting points and the same ending points in the reduced expression graph of x. The morphisms induced by these paths are equal.

While trying to prove Libedinsky's conjecture, a counterexample was found. In this document we prove the conjecture for all but one element in S_4. The outstanding element is the one that sends 1 to 4, 2 to 2, 3 to 3, and 4 to 1.

You should enjoy the little detours
to the fullest,
because that's where you'll find
the things more important than what you want

ジン フリークス

Acknowledgements

A mi familia.
A mis amistades.
A mis profes.

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Chapter 1

Introduction

This chapter provides background to understand the Forking Path Conjecture. We will explore its origin and the main results that inspired it. This initial chapter is strongly based on the excellent books [5] and [6].

1.1. Coxeter systems

1.1.1. Coxeter groups basics

Definition 1.1 A **Coxeter System** (W, S) is a group W, a finite generator set $S \subset W$, and a matrix $(m_{s,t})_{s,t\in S}$ satisfying $m_{s,s} = 1$ for each $s \in S$, and $m_{s,t} = m_{t,s} \in \{2, 3, \dots\} \cup \{\infty\}$ for $s \neq t \in S$, such that

$$W = \langle s \in S \mid (st)^{m_{s,t}} = \mathrm{Id} \text{ for any } s, t \in S \text{ with } m_{s,t} < \infty\rangle \tag{1.1}$$

where Id is the identity element of W. The **rank** of the Coxeter system (W, S) is defined as $|S|$.

In particular, the relations for $s = t$ have the form

$$(ss)^1 = s^2 = \mathrm{Id} \tag{1.2}$$

and are called **quadratic relations**. The relation $(st)^{m_{s,t}} = \mathrm{Id}$, for $s \neq t \in S$ and $m_{s,t} \neq \infty$, is equivalent (under the quadratic relations) to the **braid relation**

$$\underbrace{sts\dots}_{m_{s,t}} = \underbrace{tst\dots}_{m_{s,t}} \tag{1.3}$$

It is not straightforward from the presentation, but it is possible to show that $m_{s,t}$ is precisely the order of the element st (see [6, Prop.4.1.2]). We emphasize that when $m_{s,t} = \infty$, there is no relation between s and t.
The elements of S are called **simple reflections**. Elements of W which are conjugate to elements of S are called **reflections**.

Observation 1.1 For a Coxeter system (W, S), the group W is called a **Coxeter group**. A Coxeter group can often be equipped with the structure of a Coxeter system in different ways. For example, any conjugate of S could also be used as a set of simple reflections. Even more, sometimes it is possible for the same group to be described as a Coxeter group using two Coxeter systems with different ranks.

Example 1.1 Note that D_6, the dihedral group of order 12, can be presented as $\langle p, q \mid (pq)^6 = p^2 = q^2 = \mathrm{Id}\rangle$ and as $\langle s, t, r \mid (st)^3 = (sr)^2 = (tr)^2 = s^2 = t^2 = r^2 = \mathrm{Id}\rangle$.

For each $w \in W$, we can write $w = s_1 \cdots s_k$ for some $s_1, \ldots, s_k \in S$. We call the sequence $(s_1, \ldots, s_k)$ an **expression** for w, of **length** k. We write $\underline{w}$ for the sequence $(s_1, \ldots, s_k)$, when the product $s_1 \cdots s_k$ is equal to w. That is, the notation $\underline{w}$ indicates both an element $w \in W$ and a particular choice of expression for w.

The **Coxeter graph** of a Coxeter system (W, S) is the following labeled graph which efficiently encodes the data of the Coxeter system. Its vertex set is S, and vertices s and t are joined by an edge if $m_{s,t} > 2$. The edge is labeled $m_{s,t}$ if $m_{s,t} > 3$. Thus, $m_{s,t} = 2$ if s and t are not joined, and $m_{s,t} = 3$ if they are joined by an unlabeled edge.

Example 1.2 The **Coxeter system of type** A_{n-1}, $n \geq 2$, is given by the following Coxeter graph.

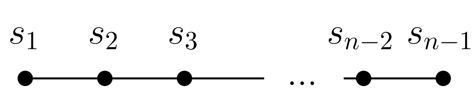


Figure 1.1

This Coxeter group is isomorphic to the symmetric group S_n, the group of permutations of $\{1, 2, ..., n\}$. It has as generator set $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where the generators s_i correspond to the adjacent transposition $(i, i+1)$. They satisfy the relations

$$s_i^2 = \mathrm{Id}, \; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; s_i s_j = s_j s_i \text{ if } |i-j| \geq 2. \tag{1.4}$$

Definition 1.2 By saying "type A" we refer to the family of symmetric groups.

Now, we present two common ways to describe a permutation $w \in S_n$.

- The *one-line notation* for w is the sequence $w(1)w(2)\ldots w(n)$. For example, $w = 3\ 2\ 4\ 1$ means that $w(1) = 3, w(2) = 2, w(3) = 4, w(4) = 1$.

- The *strand diagram notation*, which will be useful later. A permutation $w \in S_n$ is depicted as a diagram with n strands, in which each strand connects i on the bottom horizontal line to $w(i)$ on the top horizontal line. For example, the following diagram describes $w = 3\ 2\ 4\ 1$.

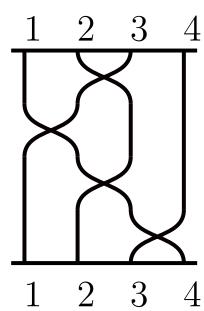
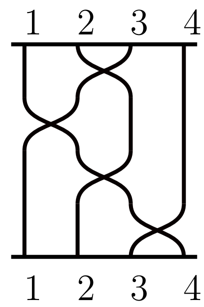


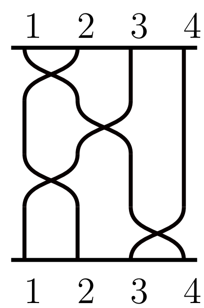
Figure 1.2: Strand diagram

Multiplication is given by stacking diagrams vertically. Note that xy is x on top of y.

The strand diagram notation is the one which relates the most to the Coxeter presentation of S_n (note that a strand diagram represents not only a permutation $w \in S_n$, but also an expression $\underline{w}$ for that permutation). Any strand diagram which is suitably generic (see Observation 1.2 below) can be built by vertically stacking crossings, which are diagrams with a single crossing that correspond to the simple reflections $(i, i+1)$. Thus the length of the expression is the number of crossings. For example, the strand diagram



corresponds to the expression $\underline{w} = (s_2, s_1, s_2, s_3)$. Here we have a different expression for the same permutation



which corresponds to $\underline{w} = (s_1, s_2, s_1, s_3)$.

Observation 1.2 A strand diagram is **generic** if it has only transverse intersections with no triple intersections, and if two crossings never occur at the same height. We obtain an expression from a generic strand diagram by reading off the crossings from bottom to top. Thus an isotopy of diagrams which does not change the order of the crossings will not change the resulting expression, and we typically think of two diagrams which are isotopic in this way as being the same. An isotopy which does not create triple intersections but does change the order of the crossings will produce a different expression related to the original by applications of braid relations as in Figure 1.4 and Figure 1.5 below. If the precise expression is not relevant then a non-generic strand diagram can often be much more compact to draw. In this case, one can perturb the diagram slightly to obtain a generic strand diagram if desired.

The Coxeter relations in Example 1.2 become the following equivalences of strand diagrams, representing equalities in the symmetric group.

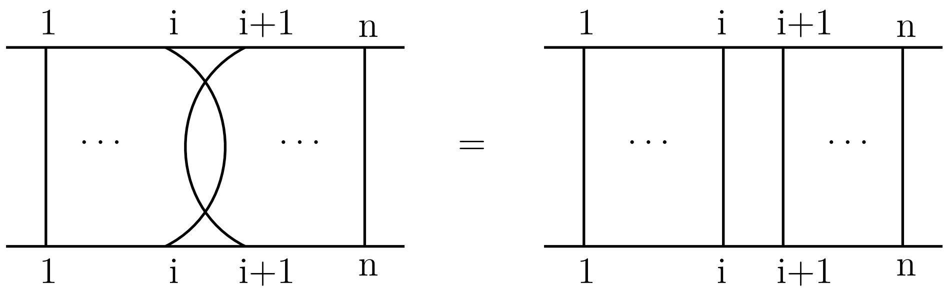


Figure 1.3

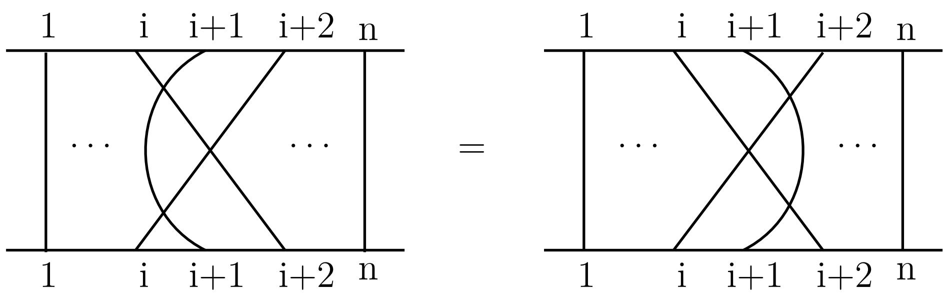


Figure 1.4

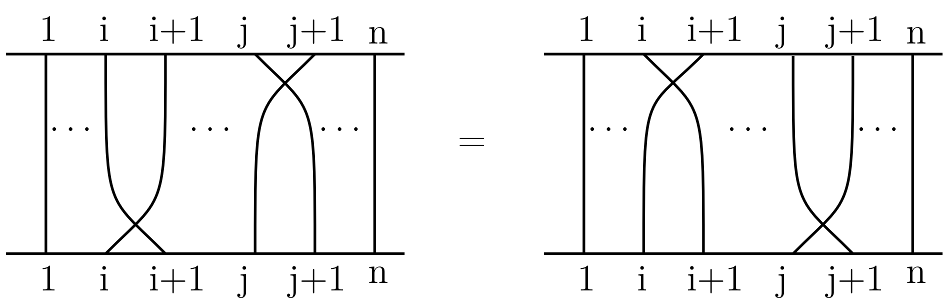


Figure 1.5

It is a theorem by Matsumoto (see Theorem 1.2 below) that, given a permutation, with a sequence of the diagrammatic equalities above (Figures 1.3, 1.4, and 1.5) we can go from any strand diagram of that permutation, to any other (see Section 1.2).

1.1.2. The geometric representation

Any finite Coxeter group can be embedded in some orthogonal group, by means of its geometric representation.

Definition 1.3 The **geometric representation** of a Coxeter system (W, S) is the representation V of W defined as follows. Let V be the real vector space with basis $\{\alpha_s \mid s \in S\}$ indexed by S. Equip V with the symmetric bilinear form $(-, -)$ determined by

$$(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{s,t}}\right). \tag{1.5}$$

When $m_{s,t} = \infty$ we use the convention that $\frac{\pi}{m_{s,t}} = 0$. Note that $(\alpha_s, \alpha_s) = 1$. We may define an action of W on V, where each $s \in S$ acts by reflecting along the hyperplane orthogonal to α_s. That is,

$$s(\lambda) = \lambda - 2(\lambda, \alpha_s)\alpha_s. \tag{1.6}$$

Recall that a **faithful representation** ρ of a group W on a vector space V is a linear representation in which different elements $w \in W$ are represented by distinct linear mappings $\rho(w)$. In other words, this means that the group homomorphism $\rho \colon W \to GL(V)$ is injective. We have the following important fact about the geometric representation.

Proposition 1.1 For any Coxeter system the geometric representation is faithful.

Proof. We refer the interested reader to [7, Chapter 5]. $\square$

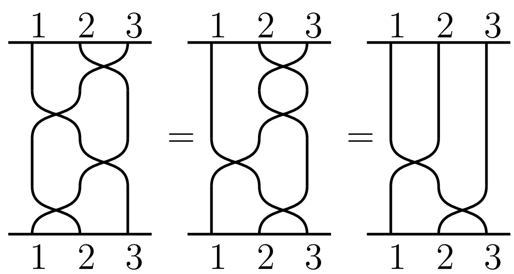
1.1.3. The length function

As we mentioned, each element $w \in W$ admits an expression $\underline{w} = (s_1, \ldots, s_k)$ for some $s_1, \ldots, s_k \in S$. The length $l(\underline{w})$ of this expression is k.

Definition 1.4 The **length** of w, denoted by $l(w)$, is the minimal k for which w admits an expression of length k. Any expression for w with this minimal length $l(w)$ is called a **reduced expression**. In particular, $l(w) = 0$ if and only if $w = \mathrm{Id}$.

Fact 1.1 ([5], Example 1.39.) Let $\underline{w}$ be an expression for $w \in W = S_n$, and consider its corresponding strand diagram. If two strands cross twice, then we can remove these two crossings and obtain a shorter expression for the same permutation.

Example 1.3 Here we present an example:



Consider $w \in S_n$. An **inversion** of w is a pair $\{i, j\}$ with $1 \leq i < j \leq n$, such that $w(i) > w(j)$. If $\{i, j\}$ is an inversion of w, we have that the strand with i on the bottom and the strand with j on the bottom must cross at least once in any expression for w. Thus $l(w)$ is at least the number of inversions of w. This way, we see that if no two strands cross twice, then this expression is reduced, and therefore the length of this expression is precisely the number of inversions. Using Fact 1.1, we thus conclude that $\underline{w}$ is reduced if and only if any two strands of its strand diagram cross each other at most once. In this case, $l(w)$ is the number of crossings in the aforementioned diagram.

Example 1.4 Let $W = S_3$, and $\underline{w} = (s_1, s_2, s_1)$.

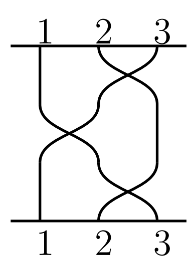


Figure 1.6

Then $\underline{w}$ is a reduced expression with $l(w) = 3$. This can be seen by checking that no strands cross each other twice in the above diagram. In fact, the element $w = s_1 s_2 s_1$ is the longest element of S_3, whose set of inversions consists of all pairs $\{i, j\}$ with $1 \leq i < j \leq n$.

Proposition 1.2 ([5], Proposition 1.41.) The length function satisfies the following properties for $s \in S$ and $v, w \in W$:

1. $l(w) = 1$ if and only if $w \in S$,
2. $l(w) = l(w^{-1})$,
3. $l(wv) \leq l(w) + l(v)$,
4. $l(wv) \geq l(w) - l(v)$,
5. $l(ws) = l(w) \pm 1$.

PROOF. Properties 1,2, and 3 are straightforward. Applying property 3 to wv and v^{-1} yields $l(w) \leq l(wv) + l(v^{-1})$. Using property 2 and rewriting the inequality gives property 4. For $v = s$, properties 3, 4, and parity imply property 5. □

1.1.4. The descent set

In the last proposition we saw that either $l(ws) = l(w)+1$ or $l(ws) = l(w)-1$ for $w \in W$ and $s \in S$.

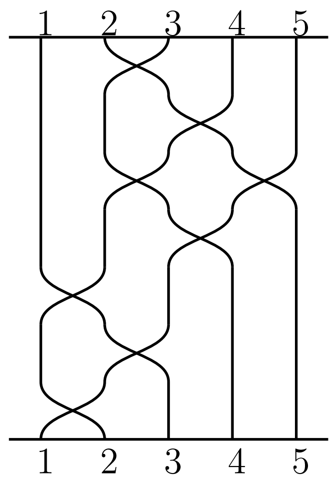
Definition 1.5 Given $w \in W$, its **right descent set** $\mathcal{R}(w)$ is the set defined as $\{s \in S \mid l(ws) < l(w)\}$. Analogously, its **left descent set** $\mathcal{L}(w)$ is the set defined as $\{s \in S \mid l(sw) < l(w)\}$.

In type A (see Definition 1.2), the right and left descent sets of $w \in S_n$ are easy to determine from the strand diagram of a reduced expression $\underline{w}$. The simple reflection s_i is in $\mathcal{R}(w)$ if the strands with bottom label i and $i+1$ eventually cross in $\underline{w}$. This is because if they do not cross, then adding this crossing s_i will still produce a reduced expression, so $l(ws_i) = l(w) + 1$. On the other hand, if they do cross, then adding s_i will produce a double crossing, hence a non-reduced expression. Then $l(ws_i) < l(w)$, meaning that $l(ws_i) = l(w) - 1$. Similarly, $s_i \in \mathcal{L}(w)$ if the strands with top label i and $i+1$ eventually cross in $\underline{w}$.

Example 1.5 As an example, let $W = S_5$, and consider the reduced expression

$$\underline{w} = (s_2, s_3, s_4, s_2, s_3, s_1, s_2, s_1),$$

corresponding to the following (non-generic) strand diagram.



Then $\mathcal{L}(w) = \{s_2, s_3, s_4\}$ and $\mathcal{R}(w) = \{s_1, s_2, s_4\}$.

1.1.5. The exchange condition

Theorem 1.1 (Exchange condition)[[6], Theorem 1.4.3.] Consider the Coxeter system (W, S) and $w \in W$. Let $\underline{w} = (s_1, s_2, \ldots, s_k)$ be a reduced expression for w, and $s \in S$. If $l(ws) < l(w)$, then there exists i such that $1 \leq i \leq k$ and $ws = s_1 s_2 \cdots \widehat{s_i} \cdots s_k$.

This is a general theorem, but we are only interested in the idea behind the exchange condition in type A.

Proof. When $l(ws) < l(w)$, it is because adding the crossing s will re cross two strands which have already crossed in $\underline{w}$. That previous crossing is s_i. By removing the new crossing s and the previous crossing s_i, we obtain an expression for ws. $\square$

We state two very important corollaries of the exchange condition.

Corollary 1.1 For $w \in W$, the descent set $\mathcal{R}(w)$ is equal to the set

$$\{s \in S \mid w \text{ admits a reduced expression ending in } s\}$$

Proof. Let $l(w) = k$. If w admits a reduced expression ending in s, so that $w = s_1 s_2 \cdots s_{k-1} s$, then $ws = s_1 s_2 \cdots s_{k-1}$ by the quadratic relation. Thus $l(ws) = l(w) - 1$ and $s \in \mathcal{R}(w)$. Conversely, if $s \in \mathcal{R}(w)$, then ws is the reduced expression $s_1 s_2 \cdots \widehat{s_i} \cdots s_k$ for some i, by the exchange condition. Then $w = wss = s_1 s_2 \cdots \widehat{s_i} \cdots s_k s$ is an expression of lenght k and is thus a reduced expression ending in s. $\square$

In type A, the corollary states that if adjacent strands (i.e. with labels i and $i+1$ on bottom) are going to cross at some point in a permutation, it is possible to choose a reduced expression such that they are the first strands to cross.

Proposition 1.3 (Deletion condition) Let $\underline{w} = (s_1, s_2, \ldots, s_k)$ be an expression for $w \in W$ with $l(w) < k$. Then there exist a pair of indexes $i < j$ such that $w = s_1 s_2 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k$.

1.1.6. The longest element

Proposition 1.4 ([6], Prop. 2.3.1) Let (W, S) be a Coxeter system. Then there exists an element $w_0 \in W$ with $\mathcal{R}(w_0) = S$ if and only if W is finite. Such an element w_0 is unique, satisfies $l(w) < l(w_0)$ for all $w \neq w_0 \in W$, and is called the **longest element** of W. It also satisfies, and is determined by the condition $\mathcal{L}(w_0) = S$. When $W = S_n$, we denote the longest element by $w_{0,n}$.

Example 1.6 In general, the longest element of S_n is the permutation sending $i \to n + 1 - i$, and its inversion set is all pairs $\{i, j\}$ with $1 \leq i < j \leq n$. The longest elements in S_n for $n = 2, 3, 4$ are respectively $s_1, s_1 s_2 s_1, s_1 s_2 s_1 s_3 s_2 s_1$. We can obtain $w_{0,n+1}$ inductively, by joining the sequence $s_n s_{n-1} \ldots s_2 s_1$ on the right of $w_{0,n}$.

1.1.7. Matsumoto's theorem

Let $(s_{i_1}, \ldots, s_{i_k})$ and $(s_{j_1}, \ldots, s_{j_k})$ be two arbitrary expressions of the same length. If we can apply a sequence of braid relations (see Eq.1.3) to obtain $(s_{j_1}, \ldots, s_{j_k})$ from $(s_{i_1}, \ldots, s_{i_k})$, we say that they are **related by braid relations**. The following theorem will be used later.

Theorem 1.2 (Matsumoto's theorem [8]) Any two reduced expressions for $w \in W$ are related by braid relations.

1.1.8. Bruhat order

The exchange condition says that if $w = s_1 s_2 \cdots s_k$, $s \in S$, and $l(ws) < l(w)$, then $ws = s_1 s_2 \cdots \widehat{s_i} \cdots s_k$ for some i. However, not every i in the set $1 \leq i \leq k$ can appear this way. Let T denote the set of all reflections, i.e., the set of all elements of W conjugate to an element of S. Note that any element of the form $s_1 s_2 \cdots \widehat{s_i} \cdots s_k$, for any $1 \leq i \leq k$, can be expressed as wt where $t \in T$. Indeed, the element $t = s_k s_{k-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{k-1} s_k$ is a conjugate of s_i, and will suffice.

Definition 1.6 For $x, y \in W$, we write $x \to y$ if $l(x) < l(y)$ and $xt = y$ for some $t \in T$. The **Bruhat graph** is the directed graph with vertices the elements of W, and arrows given by the relation $\to$. The **Bruhat order** on W is the partial order $\leq$ obtained as the transitive closure of the relation $\to$. Thus, $x \leq y$ if and only if there is a directed path in the Bruhat graph from x to y. Equivalently, we have that $x \leq y$ if there exists a chain $x = x_0, x_1, \ldots, x_m = y$ such that $l(x_i) < l(x_{i+1})$ and $x_i^{-1} x_{i+1} \in T$ for all $0 \leq i < m$.

Observation 1.3 In the definition of the relation $\to$, an thus of the Bruhat order, we use right multiplication by reflections. However, since $xt = (xtx^{-1})x$ and xtx^{-1} is a reflection if t is a reflection, left multiplication defines the same relation.

We now state the theorem used in most application of the Bruhat order.

Theorem 1.3 (Subword property) Let $w \in W$ and $\underline{w} = s_1 s_2 \ldots s_q$ be a fixed reduced expression for w. Then,

$$\begin{aligned} u \leq w \Leftrightarrow & \text{ there exists a reduced expression} \\ & \qquad u = s_{i_1} s_{i_2} \cdots s_{i_k},\ 1 \leq i_1 < i_2 < \cdots < i_k \leq q. \end{aligned}$$

Observation 1.4 In this case, we say that u may be obtained as a **subex-**

pression of the expression $\underline{w}$.

1.2. Hecke algebra

Definition 1.7 The **Hecke algebra** $\mathcal{H}$ of a Coxeter system (W, S) is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators h_s for $s \in S$, and the following relations:

- $h_s^2 = (v^{-1} - v)h_s + 1$ (quadratic relation). (1.7)
- $\underbrace{h_s h_t h_s \ldots}_{m_{s,t}} = \underbrace{h_t h_s h_t \ldots}_{m_{s,t}}$ for all $s, t \in S$, with $m_{s,t} < \infty$ (braid relation) (1.8)

Observation 1.5 The quadratic relation is equivalent to

$$(h_s - v^{-1})(h_s + v) = 0. \tag{1.9}$$

Thus the eigenvalues of the action of h_s on $\mathcal{H}$ are v^{-1} and $-v$. (This lead us to a useful mnemotecnic for the quadratic relation. Both eigenvalues have a minus sign somewhere: one in the exponent, and one in the coefficient). When v is specialized to 1, we recover the usual statement that an involution s has eigenvalues +1 and -1, and we obtain the algebra $\mathbb{Z}W$. This way, it is possible to see the Hecke algebra as a deformation of the group algebra.

1.2.1. The standard basis

For any reduced expression $\underline{w} = sr \ldots t$ of an element $w \in W$ define the element $h_{\underline{w}} = h_s h_r \ldots h_t$. Thanks to Matsumoto's theorem we know that $h_{\underline{w}}$ does not depend on the reduced expression $\underline{w}$, it just depends on w. We call this element h_w. We define $h_e = 1$. We have the following result (for a proof, we refer the reader to [7], Ch 7).

Lemma 1.1 (Nagayoshi Iwahori) The set $\{h_w\}_{w \in W}$ is a basis of $\mathcal{H}$ when viewed as a $\mathbb{Z}[v, v^{-1}]$-algebra. It is called the **standard basis**.

Now we give some basic formulae for multiplying an arbitrary standard basis element h_x and h_s for some simple reflection s.

Let $w \in W$, with $\underline{w} = (s_1, \dots, s_n)$ being a reduced expression for w, and $s \in S$. Recall the Bruhat order on W defined in Section 1.1.8. If $ws > w$, then $(s_1, \dots, s_n, s)$ is reduced, and we have $h_w h_s = h_{ws}$. On the other hand, if $ws < w$, then by Corollary 1.1 we may replace $\underline{w}$ with a reduced expression ending in $s_n = s$. Applying the quadratic relation (1.7), we find that

$$h_w h_s = h_{s_1} \cdots h_{s_{n-1}} h_s^2 = (v^{-1} - v) h_w + h_{ws}. \tag{1.10}$$

A similar analysis holds for left multiplication by h_s, leading to the following two formulae:

$$h_w h_s = \begin{cases} h_{ws} & \text{if } w < ws, \\ (v^{-1} - v) h_w + h_{ws} & \text{if } w > ws, \end{cases} \tag{1.11}$$

and

$$h_s h_w = \begin{cases} h_{sw} & \text{if } w < sw, \\ (v^{-1} - v) h_w + h_{sw} & \text{if } w > ws. \end{cases} \tag{1.12}$$

1.2.2. Inversion

The element h_s has an inverse, namely $(h_s + v - v^{-1})$ as we show in the next equation:

$$h_s(h_s + v - v^{-1}) = [(v^{-1} - v) h_s + 1] + h_s(v - v^{-1}) = 1 \tag{1.13}$$

This implies that h_w has an inverse for every $w \in W$. It will be useful to have an idea of how h_w^{-1} looks like when expanded in the standard basis.

Lemma 1.2 For all $w \in W$, the standard basis element h_w is invertible in H. Moreover, we have

$$h_{w^{-1}}^{-1} = h_w + \sum_{x < w} a_x h_x \tag{1.14}$$

for some coefficients $a_x \in \mathbb{Z}[v, v^{-1}]$.

In order to prove this we will need the following important concept:

Definition 1.8 Let $\underline{x} = (s_1, \dots, s_m)$ be an arbitrary expression. A **subexpression** of $\underline{x}$ is a string $\underline{e} = e_1 \dots e_m$ of length m, where each $e_i \in \{0, 1\}$. We write $\underline{e} \subset \underline{x}$ to mean that $\underline{e}$ is a subexpression of $\underline{x}$.

We think of a subexpression as an expression obtained by "crossing out" (resp. keeping) some of the terms in x, where $e_i = 0$ (resp $e_i = 1$) means that s_i is "crossed out" (resp. kept) in our subexpression.

Proof of Lemma 1.2. Suppose that $\underline{w} = (s_1, \dots, s_m)$ is a reduced expression for w, and $\underline{e} \subset \underline{w}$ is a subexpression. We first show that

$$h_{s_1}^{e_1} h_{s_2}^{e_2} \cdots h_{s_m}^{e_m} \in \sum_{x \leq w} \mathbb{Z}[v, v^{-1}] h_x. \tag{1.15}$$

Consider the expression $(r_1, \dots, r_k)$ obtained by omitting the terms in $\underline{w}$ where $e_i = 0$. If this expression is already reduced, then $h_{r_1} h_{r_2} \cdots h_{r_k}$ is equal to $h_{r_1 r_2 \cdots r_k}$ and we are done. Otherwise, let $1 \leq i < k$ be maximal such that $(r_1, r_2, \dots, r_i)$ is reduced. Then $(r_1, r_2, \dots, r_i, r_{i+1})$ is not reduced and so by the deletion condition (Prop. 1.3) there exist $1 \leq a < b \leq i + 1$ such that $(r_1, \dots, \widehat{r}_a, \dots, \widehat{r}_b, \dots, r_{i+1})$ is reduced. Using Eq.(1.10) we obtain

$$\begin{aligned} h_{r_1} h_{r_2} \cdots h_{r_k} &= (h_{r_1 r_2 \cdots r_i} h_{r_{i+1}}) h_{r_{i+2}} \cdots h_{r_k} \\ &= ((v^{-1} - v) h_{r_1 r_2 \cdots r_i} + h_{r_1 r_2 \cdots r_i r_{i+1}}) h_{r_{i+2}} \cdots h_{r_k} \\ &= (v^{-1} - v) h_{r_1} h_{r_2} \cdots \widehat{h_{r_{i+1}}} \cdots h_{r_k} + h_{r_1} h_{r_2} \cdots \widehat{h_{r_a}} \cdots \widehat{h_{r_b}} \cdots h_{r_k}. \end{aligned}$$

From here, Eq.(1.15) follows by induction on k, using Theorem 1.3.
Now, let $(s_1, \dots, s_m)$ be a reduced expression for w. Then

$$(h_{w^{-1}})^{-1} = h_{s_1}^{-1} \cdots h_{s_m}^{-1} = (h_{s_1} + (v - v^{-1})) \cdots (h_{s_m} + (v - v^{-1})). \tag{1.16}$$

This product expands to h_w plus a sum of products of h_{s_i}'s with coefficients which all give proper subexpressions of w. The lemma follows from Eq.(1.15).
□

1.2.3. The Kazhdan-Lusztig basis

The Hecke algebra admits a basis known as the **Kazhdan-Lusztig basis**. We introduce an involution on $\mathcal{H}$, which will be used to characterize this basis.

Definition 1.9 We define a $\mathbb{Z}$-module morphism $\varphi : \mathcal{H} \to \mathcal{H}$ which transforms the generators v and h_x by the rules $\varphi(v) = v^{-1}$ and $\varphi(h_x) = (h_{x^{-1}})^{-1}$. This function is called the **duality function** in the Hecke algebra.

The duality function is also known as the **Kazhdan-Lusztig involution** or **bar involution**, since in the literature is frequently written as

$$\begin{aligned} \mathcal{H} &\to \mathcal{H} \\ h &\mapsto \overline{h}. \end{aligned}$$

The following is the main theorem of the Kazhdan-Lusztig theory.

Theorem 1.4 (D. Kazhdan and G. Lusztig) For every element $x \in W$ there is a unique element $b_x \in \mathcal{H}$, satisfying $\varphi(b_x) = b_x$, such that

$$b_x \in h_x + \sum_{y<x} v\mathbb{Z}[v]h_y. \tag{1.17}$$

Definition 1.10 The **Kazhdan-Lusztig basis** is the set $\{b_x \mid x \in W\} \subseteq \mathcal{H}$. It is a basis of $\mathcal{H}$ as a $\mathbb{Z}[v, v^{-1}]$-module.

1.2.4. Kazhdan-Lusztig polynomials

Any set $\{b_x \mid x \in W\}$ satisfying the Eq.(1.17) is automatically a $\mathbb{Z}[v, v^{-1}]$ basis of $\mathcal{H}$. This is readily seen by considering the $\mathbb{Z}[v, v^{-1}]$-linear map $\phi : \mathcal{H} \to \mathcal{H}$, $\phi(h_x) = b_x$. In the standard basis (with any total order on W refining the Bruhat order), the matrix of ϕ will be triangular with 1's along the diagonal (and hence ϕ is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules), and Kazhdan-Lusztig polynomials are essentially the entries of the change of basis matrix from the standard basis to a Kazhdan-Lusztig basis. It is also worth noting for this argument that only finitely many elements of W are less than a given x in the Bruhat order.

Definition 1.11 If we write $b_x = h_x + \sum_{y \in W} a_{y,x} h_y$ then the **Kazhdan-Lusztig polynomials** $p_{y,x}$ are defined by the formula $p_{y,x} = v^{l(x)-l(y)} a_{y,x}$.

1.3. Soergel bimodules

The Hecke algebra can be categorified by means of the Soergel bimodules. In this section we provide an introduction to these bimodules. First, we consider the symmetric algebra on the geometric representation of the Coxeter group and state the Chevalley theorem concerning its invariant subrings. Then, we introduce the Demazure operators. Finally, we discuss Bott–Samelson bimodules and Soergel bimodules, and state Soergel's categorification theorem.

1.3.1. Gradings

From now on, graded objects play a fundamental role, so here we fix our conventions. Unless otherwise specified, "graded" will mean $\mathbb{Z}$-graded. A **graded vector space** is a vector space M equipped with a decomposition

$$M \coloneqq \bigoplus_{i \in \mathbb{Z}} M^i \tag{1.18}$$

into subspaces M^i. The M^i are the **graded pieces** of M. It can be useful to imagine the graded pieces of M as arranged by height, with M^i occupying height i. An element m of M contained in some M^i will be called **homogeneous**, in which case its **degree** is i. The grading is **bounded above** (resp. **bounded below**) if $M^i = 0$ for $i \gg 0$ (resp. $i \ll 0$).
More generally, it is possible to consider gradings for all sorts of structures like rings, modules, algebras, bimodules, etc., usually with some additional compatibility conditions. For example, a **graded ring** is a ring R with a decomposition $R = \oplus_{i \in \mathbb{Z}} R^i$ into subgroups R^i of the additive group of R, such that $R^i R^j \subseteq R^{i+j}$. A typical example is the polynomial ring $K[x_1, \dots, x_n]$ with the usual grading.

A **graded module** over a graded ring R is an R-module M with a decomposition $M = \oplus_{i \in \mathbb{Z}} M^i$ into subgroups M^i of M such that $R^i M^j \subseteq M^{i+j}$. Notice

that a graded vector space over a field K is the same as a graded K-module when considering K with the trivial grading, i.e. $K^0 = K$.
Given a graded object M (vector space, module, ring, etc.) and $i \in \mathbb{Z}$ we can define a new object $M(i)$ with graded pieces

$$M(i)^j := M^{i+j}. \tag{1.19}$$

If we imagine the graded pieces of M arranged vertically as above, then $M(1)$ is obtained by shifting M down by one. We say that $M(i)$ is obtained from M via a shift in the grading.

A morphism $M \to N$ between graded objects is typically assumed to be **homogeneous of degree 0**, meaning that it sends M^i to N^i for each $i \in \mathbb{Z}$. We might also consider maps which are homogeneous of degree k for some $k \in \mathbb{Z}$. They send M^i to N^{i+k}. Note that a degree k map $M \to N$ is the same data as a degree 0 map $M \to N(k)$, or a degree 0 map $M(-k) \to N$. Two graded objects M and N are **isomorphic up to shift** if $M \simeq N(i)$ for some i. The **graded Hom** space between graded objects M and N is defined to be the direct sum of morphism spaces of all degrees:

$$\mathrm{Hom}^\bullet(M, N) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(M, N(k)). \tag{1.20}$$

Given an object M in an additive category, and $m \in \mathbb{Z}_{\geq 0}$, we can consider the iterated direct sum $m \cdot M := M^{\oplus m}$. If M is graded, then one can take a number of copies of M, shift them, and add them together. Given a Laurent polynomial $p = \sum n_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ with positive integer coefficients we set

$$p \cdot M := \bigoplus_{i \in \mathbb{Z}} n_i \cdot M(i). \tag{1.21}$$

Thus "multiplication by v" corresponds to a shift by 1.

1.3.2. Polynomials

The data needed to define a Soergel bimodule will be a Coxeter system (W, S) together with its geometric representation V over $\mathbb{R}$ (see Definition 1.3). There is a version of Soergel bimodules for more general representations of W (see [9]). Recall that V is a real vector space of dimension $|S|$ with basis $\{\alpha_s \mid s \in S\}$ indexed by simple reflections. These basis elements α_s are known as **simple roots**. We defined an action of W on V where the simple reflections act by

$$s(\alpha_t) = \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s. \tag{1.22}$$

Here $(-,-)\colon V \times V \to \mathbb{R}$ is the symmetric bilinear form in Eq.(1.5). i.e.,

$$(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{s,t}}\right).$$

When $m_{s,t} = \infty$, this is interpreted as $(\alpha_s, \alpha_t) = -1$.
Let $Sym(V)^i$ be the symmetrized tensorial product of order i defined on V. Let

$$R = Sym(V) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} Sym^i(V) \tag{1.23}$$

be the **symmetric algebra** of V. We view R as a graded algebra in which $\deg(V) = 2$. That is, R is the polynomial ring $R = \mathbb{R}[\alpha_s \mid s \in S]$ with grading $\deg \alpha_s = 2$ for all $s \in S$. Note that $Sym^i(V)$ lives in degree $2i$. Soergel bimodules, to be defined below, will be certain graded bimodules over this graded algebra R. Note that the W-action on V induces a W-action on R via

$$w \cdot \prod_{s \in S} \alpha_s^{k_s} = \prod_{s \in S} (w(\alpha_s))^{k_s} \tag{1.24}$$

on monomials, and then extended $\mathbb{R}$-linearly to polynomials.

1.3.2.1. Invariant polynomials

For a subset $I \subset S$, define $W_I \coloneqq \langle I \rangle \subset W$, the **(standard) parabolic subgroup** generated by I. We say that I is **finitary** if W_I is a finite group. Let R^I be the ring of W_I-invariants of R, i.e.

$$R^I = \{f \in R \mid w \cdot f = f \text{ for all } w \in W_I\}. \tag{1.25}$$

We sometimes write R^W instead of R^S for the invariants under the entire

Coxeter group. We also write R^s instead of $R^{\{s\}}$, $R^{s,t}$ instead of $R^{\{s,t\}}$, etc. The following theorem is one of the algebraic foundations upon which the theory of Soergel bimodules is built.

Theorem 1.5 (Chevalley-Shephard-Todd)[[7], Ch.3.] Let $I \subset S$ be finitary. Then R^I is a polynomial ring. Moreover, R is a graded free module of finite rank over R^I.

A more refined version of the Chevalley-Shephard-Todd theorem also specifies the degrees of the generators of R^I in terms of the Poincaré polynomial of W_I.

Example 1.7 Let $W = S_5$ act on $\mathbb{R}[x_1, \ldots, x_5]$. Let I be the set $\{s_1, s_3, s_4\}$, so $W_I \simeq S_2 \times S_3$. Then we have the following equality

$$R^I = \mathbb{R}[x_1 + x_2, x_1x_2, x_3 + x_4 + x_5, x_3x_4 + x_3x_5 + x_4x_5, x_3x_4x_5].$$

Here, the Chevalley-Shephard-Todd theorem implies that R^I has 5 algebraically independent generators (just like R), though in different degrees.

Observation 1.6 Note that $\mathbb{R}[x_1, \ldots, x_5]$ is not the ring R we associated to the Coxeter group S_5, since it comes from the permutation representation, not the geometric representation considered above. However, we can safely use this action and the theory of Soergel bimodules will work in the same way.

Example 1.8 Let $W = S_2, S = \{s\}$. Let $R = \mathbb{R}[\alpha]$, with the action of W given by $s(\alpha) = -\alpha$. We have that $R^s = \mathbb{R}[\alpha^2]$. This way, R has a direct sum decomposition as $R = R^s \oplus (R^s \cdot \alpha)$ into s-invariants and s-antiinvariants (it is said that an element is **s-antiinvariant** if $sf = -f$).

Example 1.9 In general, we can consider an arbitrary Coxeter system (W, S). Then for every $s \in S$, R^s is generated by α_s^2 and the elements

$$\alpha_t + \cos\left(\frac{\pi}{m_{s,t}}\right)\alpha_s \text{ for all } t \in S \setminus \{s\}.$$

Moreover, $R = R^s \oplus R^s\alpha_s$ is a splitting of R into s-invariants and s-antiinvariants.

In other words, any polynomial $f \in R$ can be written uniquely in the form $f = g + h\alpha_s$, where $g, h \in R^s$ (it is possible to describe g and h explicitly and it will be done in the next section). Since α_s has degree 2, we have an isomorphism

$$R \cong R^s \oplus R^s(-2) \tag{1.26}$$

as graded R^s-bimodules.

1.3.2.2. Demazure operators

Definition 1.12 For $s \in S$, the **Demazure operator** ∂_s (known as divided difference operator as well) is the graded map

$$\partial_s : R \to R^s(-2),$$

$$f \to \frac{f - s(f)}{\alpha_s}. \tag{1.27}$$

The operator ∂_s is well defined because $f - s(f)$ is zero in R^s, the hyperplane defined by α_s, so $f - s(f)$ is divisible by α_s. Note that de numerator and the denominator are both s-antiinvariant, so this resulting fraction is s-invariant.

The Demazure operator can be used to make explicit the R^s-module splitting $R \cong R^s \oplus R^s\alpha_s$ according to what we saw in Eq.(1.26). In fact, we have that $f + s(f)$ is s-invariant for any $f \in R$. Meanwhile $\alpha_s\partial_s(f) = f - s(f)$ is s-antiinvariant for any $f \in R$. This way,

$$f = \frac{f + s(f)}{2} + \frac{\alpha_s}{2}\partial_s(f) \tag{1.28}$$

and we have the decomposition of f as a sum of its invariant and antiinvariant parts. The isomorphism $R \to R^s \oplus R^s(-2)$ can be given explicitly by

$$f \to \left(\frac{f + s(f)}{2}, \frac{\partial_s(f)}{2}\right) \tag{1.29}$$

with inverse

$$(g, h) \to g + h\alpha_s \tag{1.30}$$

1.3.3. Bimodules and tensor products

Recall our conventions on gradings in Section 1.3.1. Let R-$gbim_{qc}$ denote the category of graded R-bimodules . The "qc" stands for quasi-coherent, and is intended to remind the reader that we make no assumptions on finite generation for bimodules in this category. It has a shift functor (n) for each integer n which sends M to $M(n)$. It also has a tensor product $- \otimes_R -$ which makes R-$gbim_{qc}$ into a monoidal category. The tensor product is graded. The graded part $(M \otimes_R N)^k$ is the image of $\bigoplus_{i+j=k} M^i \otimes_{\mathbb{Z}} N^j$ in $M \otimes_R N$. The monoidal identity is the bimodule R. Tensor product and grading shift commute, i.e., given graded R-bimodules M, N and $n \in \mathbb{Z}$ we have the following canonical identifications

$$(M(n)) \otimes_R N = M \otimes_R (N(n)) = (M \otimes_R N)(n). \tag{1.31}$$

For two modules M and N in R-$gbim_{qc}$, we usually abbreviate

$$MN := M \otimes_R N. \tag{1.32}$$

We view R-$gbim_{qc}$ as a graded category, where the morphisms are those graded R-bimodule maps $M \to N$ which are homogeneous of degree 0. A map of degree k sending M to N represents the same data as a degree 0 morphism from M to $N(k)$. Let R-$gbim$ denote the category of graded R-bimodules that are finitely generated both as a left and as a right R-module. It is a full monoidal subcategory of R-$gbim_{qc}$.

1.3.4. Bott-Samelson bimodules

Definition 1.13 For $s \in S$, we let B_s denote the graded R-bimodule

$$B_s := R \otimes_{R^s} R(1). \tag{1.33}$$

We will prove that B_s belongs to R-$gbim$. An element in B_s can be represented as $\sum_i f_i \otimes g_i$ for some f_i and g_i in R. Ben Elias recommends replacing the tensor product symbol with a wall, as in

$$\sum_i f_i \underset{s}{|} g_i. \tag{1.34}$$

The symbol $\underset{s}{|}$ is meant to represent a porous wall that can only be trespassed by appropriate polynomials. In this case, $f\underset{s}{|}1 = 1\underset{s}{|}f$ if and only if f is s-invariant. Recall that, from our grading convention, the element $1\underset{s}{|}1$ has degree -1.

Definition 1.14 The **Bott-Samelson bimodule** corresponding to an expression $\underline{w} = (s, r, \ldots, t)$, and denoted by $BS(\underline{w})$, is the graded R-bimodule given by

$$BS(\underline{w}) \coloneqq B_s B_r \cdots B_t. \tag{1.35}$$

That is, it is an iterated tensor product of the bimodules B_{s_i}, that we saw in Definition 1.13. By convention, if $\underline{w}$ is the empty expression, then $BS(\underline{w}) = R$. It is easy to see that we have a canonical isomorphism

$$BS(\underline{w}) = R \otimes_{R^s} R \otimes_{R^r} \cdots \otimes_{R^t} R(l(\underline{w})). \tag{1.36}$$

Thus, an element in $BS(\underline{w})$ has the form $\sum_i f_i \otimes g_i \otimes \cdots \otimes h_i$ for $f_i, g_i, \ldots, h_i \in R$ which we may similarly denote by $\sum_i f_i \underset{s}{|} g_i \underset{r}{|} \cdots \underset{t}{|} h_i$. The element $1\underset{s}{|}1\underset{r}{|}\cdots\underset{t}{|}1$ has degree $-l(\underline{w})$. We will refer to it as the 1-tensor and simbolize it by $1^{\otimes}$.

Given two expressions $\underline{u}$ and $\underline{v}$ we have

$$BS(\underline{u})BS(\underline{v}) = BS(\underline{uv}) \tag{1.37}$$

where $\underline{uv}$ denotes the concatenation of $\underline{u}$ and $\underline{v}$. Thus Bott-Samelson bimodules are closed under tensor product. Note that Bott-Samelson bimodules are not closed under taking grading shifts or direct sums.

We have seen in Eq.(1.26) above that R is graded free as an R^s-module. By Eq.(1.26) we deduce that, as graded left R-modules,

$$B_s \simeq R \otimes_{R^s} (R^s \oplus R^s(-2))(1) \simeq R(1) \oplus R(-1). \tag{1.38}$$

In particular, B_s is graded free as a left R-module. A similar argument shows that the same is true when B_s is regarded as a right R-module. Because tensor products of bimodules which are free of finite rank as left (resp. right) modules are free of finite rank as left (resp. right) modules, we conclude the following.

Lemma 1.3 Any Bott-Samelson bimodule is graded free of finite rank as a left (resp. right) R-module.

1.3.5. Soergel bimodules

Recall our conventions about direct summands of graded modules in Section 1.3.1. We can now define Soergel bimodules.

Definition 1.15 A **Soergel bimodule** is a direct summand of a finite direct sum of grading shifts of Bott-Samelson bimodules. The **category of Soergel bimodules**, denoted by $\mathbb{S}\mathrm{Bim}$, is the strictly full subcategory of *R-gbim* consisting of Soergel bimodules.

Observation 1.7 The words **strictly full** mean that $\mathbb{S}\mathrm{Bim}$ is closed under isomorphism. Being closed under isomorphism is not of essential importance, and is a matter of categorical taste.

It is not difficult to see that $\mathbb{S}\mathrm{Bim}$ is closed under tensor products. Equivalently, the category of Soergel bimodules is the smallest strictly full subcategory of *R-gbim* containing R and B_s for all $s \in S$ that is closed under tensor products, directs sums, direct summands and shifts.

Lemma 1.4 Any Soergel bimodule is graded free as a left or right R-module.

This is a consequence of Lemma 1.3 and the following result.

Proposition 1.5 Let M denote a graded left R-module which is free of finite rank. Any graded summand N of M is also graded free.

Proof. We refer the reader to [10, Theorem 3.7, Section 3, Chapter XXI]. □

Note that morphisms between Soergel bimodules are assumed to be homogeneous of degree 0, that is

$$\mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(B, B') = \mathrm{Hom}_{R\text{-}gbim}(B, B').$$

The space of homogeneous morphisms of degree k from B to B' can still be studied within $\mathbb{S}$Bim, being isomorphic to the space $\mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(B, B'(k))$. However, there is a notational difference between Soergel bimodules and Bott-Samelson bimodules, since the latter are not closed under grading shifts. We still want to remember the space of homogeneous morphisms of all degrees between Bott-Samelson bimodules. This is why we make the following definition.

Definition 1.16 The **category of Bott-Samelson bimodules**, denoted $\mathbb{BS}$Bim, is the monoidal category defined as follows. Its objects are Bott-Samelson bimodules. The morphism space between two objects B and B' is the graded vector space

$$\mathrm{Hom}_{\mathbb{BS}\mathrm{Bim}}(B, B') = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{R\text{-}gbim}(B, B'(k)), \tag{1.39}$$

whose degree k piece is the space of homogeneous bimodule maps of degree k. The monoidal structure is the tensor product of R-bimodules.

1.3.6. Examples of Soergel bimodules

Recall that an object M of an additive category is called **indecomposable** if it cannot be expressed as a direct sum $M' \oplus M'' \subseteq M$ for nonzero subobjects M', M''. We will be busy finding indecomposable Soergel bimodules. For this task, the following lemma is useful.

Lemma 1.5 Suposse that M is a graded R-bimodule which is generated as an R-bimodule by a homogeneous element $m \in M$. Then M is indecomposable.

Proof. Let d denote the degree of m. Given that M is generated by m and $R^0 = \mathbb{R}$, we have $M^d = \mathbb{R}m$. Suposse that $M = L \oplus N$. Then $M^d = L^d \oplus N^d$, and we may assume that $m \in L$. Then $M = R\cdot m\cdot R \subset L$ and hence $N = 0$. $\square$

It follows from the lemma that the bimodules R and B_s are indecomposable, because they are generated by $1^{\otimes}$.

Example 1.10 (Soergel bimodules of type A_1) . Let $W = S_2$, generated by the simple reflection s. The bimodules R and B_s are indecomposable Soergel bimodules. The B_s squared is

$$\begin{aligned} B_s B_s &\simeq R \otimes_{R^s} R \otimes_{R^s} R(2) && \text{by Eq.(1.36)} \\ &\simeq R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) && \text{(see Example 1.9)} \\ &\simeq R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R \\ &\simeq B_s(1) \oplus B_s(-1) \end{aligned}$$

Thus, it follows that R and B_s give representatives for all indecomposable Soergel bimodules in type A_1, up to shift and isomorphism.

Observation 1.8 The isomorphism in the above example is very similar to the equation

$$b_s b_s = (v + v^{-1}) b_s = b_s v + b_s v^{-1} \tag{1.40}$$

in the Hecke algebra $\mathcal{H}$ of type A_1, where b_s is the Kazhdan-Lusztig basis element in $\mathcal{H}$.

Example 1.11 We may use the calculations in Example 1.9 to conclude that R is generated by the subrings R^s and R^t, whenever $s \neq t$ and $m_{s,t} \neq \infty$. In particular, the bimodules

$$B_s B_t = R \otimes_{R^s} R \otimes_{R^t} R(2) \text{ and } B_t B_s = R \otimes_{R^t} R \otimes_{R^s} R(2)$$

are indecomposable, because they are generated by the 1-tensor $1^{\otimes}$. This implies that B_s and B_t are not isomorphic as we have seen that $B_s B_t$ is indecomposable, whereas $B_s B_s$ is not. Henceforth we write

$$B_{st} := B_s B_t \text{ and } B_{ts} := B_t B_s. \tag{1.41}$$

1.3.7. A little brushstroke on categorification

In the previous section we saw parallels between the behavior of indecomposable Soergel bimodules and that of the Kazhdan-Lusztig basis of the Hecke algebra. We are now in a position to see the first concrete connection between Soergel bimodules and the Hecke algebra.

Let us consider the **split Grothendieck group** $[\mathbb{S}\mathrm{Bim}]_\oplus$ of the category of Soergel bimodules. By definition, this is an abelian group generated by symbols $[B]$ for each object B in $\mathbb{S}\mathrm{Bim}$, subject to the relations

$$[B] = [B'] + [B''] \text{ whenever } B \simeq B' \oplus B''. \tag{1.42}$$

Because $\mathbb{S}\mathrm{Bim}$ is a monoidal category, $[\mathbb{S}\mathrm{Bim}]_\oplus$ is a ring, via

$$[B][B'] := [BB']. \tag{1.43}$$

We may also consider $[\mathbb{S}\mathrm{Bim}]_\oplus$ as a $\mathbb{Z}[v, v^{-1}]$-algebra via

$$v[B] := [B(1)]. \tag{1.44}$$

The following is known as the **Soergel categorification theorem**.

Theorem 1.6 The assignments $v \mapsto [R(1)]$, and $b_s \mapsto [B_s]$ for $s \in S$ induce a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism

$$\mathcal{H} \to [\mathbb{S}\mathrm{Bim}]_\oplus.$$

Chapter 2

Diagrammatic Calculus

For the symmetric group, Elias and Khovanov introduced a diagrammatic presentation of the category of Bott-Samelson bimodules in [3], which is extended by Elias in [2] to a presentation of the category of generalized Bott-Samelson bimodules. In that work, Elias obtains an explicit description of the idempotent which picks out a generalized Bott-Samelson bimodule as a summand inside a Bott-Samelson bimodule. This description uses a deep analysis of the reduced expression graph of the longest element of S_n, and the semi-orientation on this graph given by the higher Bruhat order of Manin and Schechtman. We will take advantage of that analysis in the next chapter, that is why we present here a brief summary of the tools developed and the results obtained in that paper. This chapter relies strongly on color figures. Some references to color may not be meaningful in a printed version, and we refer the reader to the digital version which includes the colored figures.

2.1. Context

The regular representation of the Hecke algebra $\mathcal{H}$ associated to a Dynking diagram can be interpreted as a decategorification of the category of "semisimple" B-equivariant perverse sheaves on the flag variety, or as the decategorification of the associated category $\mathcal{O}$. In other words, the Hecke algebra encodes numerics associated to those categories. At the beginning of the 1990s, Soergel presented an additional categorification of the Hecke algebra that we already know, the **category of Soergel bimodules** $\mathbb{S}\mathrm{Bim}$ (see Definition 1.15), which are bimodules over a polynomial ring.

As we have been doing, we will restrict ourselves to work with finite type A Dynkin diagrams. In [3], Ben Elias and Mikhail Khovanov gave a diagrammatic presentation of $\mathbb{S}$Bim. Being more precise, they gave a diagrammatic presentation of the subcategory $\mathbb{BS}$Bim $\subset$ $\mathbb{S}$Bim of our familiar **Bott-Samelson bimodules**. In that description, every morphism can be represented by a linear combination of planar graphs with boundary (modulo graphical relations), and composing morphisms is done by stacking planar graphs on top of each other. We now present the relevant.

For $n \in \mathbb{N}$, we will consider (W, S) the Coxeter system with $W = S_n$ the symmetric group on $\{1, \dots, n\}$, the set of generators $S = \{s_i \mid i = 1, 2, \dots, n-1\}$ where each s_i is the transposition $(i \; i+1)$.

Let R be the polynomial ring over $\mathbb{R}$ in variables $x_1, \dots, x_n$, together with an action of W where s_i permutes the variables x_i and x_{i+1}. As we did in Section 1.3.2.1, from a subset J of the simple reflections S, which we call a **parabolic subset**, we define R^J as the subring of R consisting of polynomials invariants under the simple reflections in J.

Example 2.1 Consider the polynomial ring $R = \mathbb{R}[x_1, x_2, x_3, x_4]$. We have a natural action of S^4 on R. The simple reflection s_1 interchanges x_1 and x_2, s_2 interchanges x_2 and x_3 and, s_3 interchanges x_3 and x_4. For a polynomial f, we have in formulas that

$$\begin{aligned} s_1 \cdot f(x_1, x_2, x_3, x_4) &= f(x_2, x_1, x_3, x_4) \\ s_2 \cdot f(x_1, x_2, x_3, x_4) &= f(x_1, x_3, x_2, x_4) \\ s_3 \cdot f(x_1, x_2, x_3, x_4) &= f(x_1, x_2, x_4, x_3). \end{aligned}$$

So R^{s_1} (the subset of R fixed by the action of s_1) is the polynomial ring $\mathbb{R}[x_1 + x_2, x_1x_2, x_3, x_4]$ and $\mathbb{R}^{s_2}$ is $\mathbb{R}[x_1, x_2 + x_3, x_2x_3, x_4]$.
The subring fixed by both simple reflections s_1 and s_2 is

$$R^{s_1,s_2} = \mathbb{R}[x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_1x_3, x_1x_2x_3, x_4]$$

Let B_J be the R-bimodule $B_J := R \otimes_{R^J} R$, and let w_J be the longest element (see Section 1.1.6) of the parabolic subgroup generated by J. We call

$B_i := R \otimes_{R^{s_i}} R$. Tensor products of various B_i are the Bott-Samelson bimodules, which form a full monoidal subcategory $\mathbb{BS}\mathrm{Bim}$ of R-bimodules. Similarly, tensor products of B_J are generalized Bott-Samelson bimodules, and form a category $g\mathbb{BS}\mathrm{Bim}$.

Observation 2.1 Recall that the R is graded (Section 1.3.1) with $deg(x_i) = 2$ (if $M = \bigoplus M^i$ is a graded R-module, then the grading shift convention will be $M(i)^j = M^{i+j}$), and (generalized) Bott-Samelson bimodules are graded R-bimodules which differ from the above by certain grading shifts.

The category $\mathbb{S}\mathrm{Bim}$ of Soergel bimodules is the full (additive monoidal graded) subcategory of R-bimodules generated by all direct summands of Bott–Samelson bimodules. Soergel proved in [9] that the isomorphism classes of indecomposable Soergel bimodules (up to grading shift) are parameterized by W; we denote them by $\{B_w\}$. The bimodule B_w appears as a summand inside $B_{i_1} \otimes \ldots \otimes B_{i_{l(w)}}$ for any reduced expression $s_{i_1} \cdots s_{i_{l(w)}}$ for w, and does not appear in any "shorter" Bott-Samelson bimodule.
It is possible to show that $B_{w_J} \cong B_J$ (see [11, Theorem 1.4]). Thus B_J will appear as a summand of $B_s B_r \ldots B_t$ whenever $sr \cdots t$ is a reduced expression for w_J, the longest element of J.

Given a reduced expression $w = s_{i_1} \cdots s_{i_d}$, we know that B_w is a summand inside $B_{i_1} \otimes \cdots \otimes B_{i_d}$, but finding a formula for this idempotent is an interesting and extremely difficult problem, for which a complete solution is out of reach. We should expect that the idempotents may become arbitrarily complex for arbitrary $w \in W$, but that they might be computable for certain classes of $w \in W$.

Observation 2.2 To understand why this could be important and useful, it is worth to mention that Elias and Williamson [12] have proved the Soergel conjecture, which states that indecomposable Soergel bimodules descend to the Kazhdan-Lustig basis, when the category is defined over $\mathbb{R}$. However, in finite characteristic the sizes of the indecomposable bimodules will change, and so will their images in the Grothendieck group. Finding the idempotents explicitly will tell one which primes need to be inverted for the indecomposable bimodule to have its "generic" size, and can help answer several questions in modular representation theory. For example, in a recent work [13] Williamson constructs idempotents requiring certain Fibonacci numbers to be invertible, and uses this to disprove the Lusztig's conjecture on the characters of simple rational modules for SL_n over a field of positive characteristic.

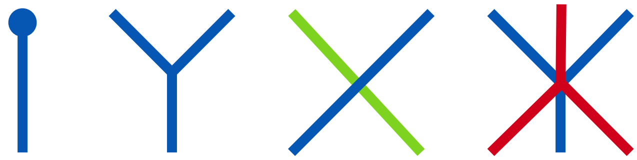
Elias presents a diagrammatic construction of this projection (to find the aforementioned summand) for the longest element, and we use that construction to achieve our goal.
An application of the Forking Path Conjecture (see Conjecture 1) is that it can be seen as a tool to help in this task.

2.2. Soergel diagrammatics

In [3], Elias and Khovanov give a diagrammatic presentation of a category $\mathcal{D}$ by generators and relations. A functor $\mathcal{F} : \mathcal{D} \to \mathbb{BS}\mathrm{Bim}$ was constructed, and it was shown that $\mathcal{F}$ is an equivalence when Soergel bimodules "behave well", i.e., categorify the Hecke algebra.

What follows is a brief summary of [14] (subsections 2.3 and 2.4). If this is the reader's first encounter with Soergel diagrammatics, then we recommend reading those sections instead as a better introduction. One can also see [3] for a version where the equivalence $\mathcal{F}$ between $\mathcal{D}$ and $\mathbb{BS}\mathrm{Bim}$ is explicitly defined.

An object in $\mathcal{D}$ is given by a sequence $\underline{\mathbf{i}} \coloneqq (i_1, i_2, \ldots, i_d) \in S^d$, which is visualized as d points on the real line $\mathbb{R}$, labeled or "colored" by elements of S. Morphisms from $\underline{\mathbf{i}}$ to $\underline{\mathbf{j}}$ are planar graphs in $\mathbb{R} \times [0,1]$, modulo some local relations (Figures 2.3 to 2.17) with each edge colored by an element of S, with bottom boundary $\underline{\mathbf{i}}$ and top boundary $\underline{\mathbf{j}}$. The allowed vertices (see Figure 2.1) are univalent vertices (degree +1), trivalent vertices joining three edges of the same color (degree -1), 4-valent vertices joining edges of alternating colors (degree 0), and 6-valent vertices joining edges of alternating adjacent colors (degree 0).
From now on, the colors blue, red and green have the same adjacency as $\{i, i+1, i+2\}$, respectively.



a) *b*) *c*) *d*)

Figure 2.1: *a*) univalent, *b*) trivalent, *c*) 4-valent, *d*) 6-valent.

We will occasionally use a shorthand to represent double dots (two univalent vertices connected by an edge). For example, see Figure 2.2, where there are three double dots on the left side of the equality. We identify a double dot colored i with the polynomial $\alpha_i \in R$, and to a linear combination of disjoints unions of double dots in the same region of a graph, we associate the appropiate linear combination of products of α_i. For any polynomial $f \in R$, a square box with a polynomial f in a region will represent the corresponding linear combination of graphs with double dots. We have a bimodule action of R on morphisms by placing boxes (that is, double dots) in the leftmost or rightmost regions of a graph. The functor $\mathcal{F}$ respects this R-bimodule action. For instance,

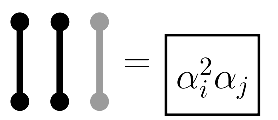


Figure 2.2

In the following relations, blue represents a generic index.

Observation 2.3 You will notice that some liberties are taken in the drawings. We can draw a horizontal line colored i, and even though this can not be constructed using our generators, it is isotopically equivalent to other figures that can be constructed. For more on this, see ([3], Section 3.)

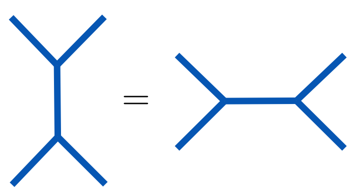


Figure 2.3

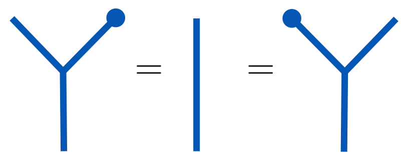


Figure 2.4

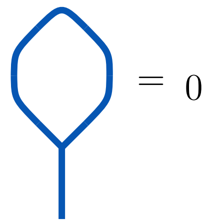


Figure 2.5

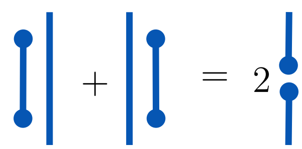


Figure 2.6

In the following relations, the two colors are distant, i.e., the respective indices are not adjacent ($i \neq j \pm 1$), where i is blue and j is green.

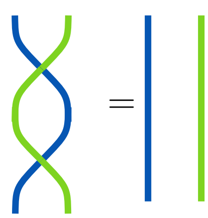


Figure 2.7

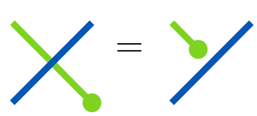


Figure 2.8

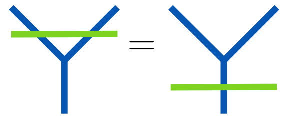


Figure 2.9

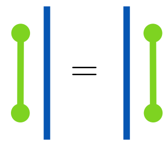


Figure 2.10

In the following relation, red and green are adjacent, and both are distant to purple.

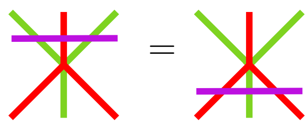


Figure 2.11

In the following relation, all three color are mutually distant.

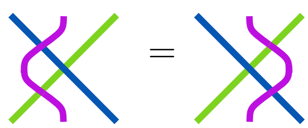


Figure 2.12

Observation 2.4 Relations 2.7 through 2.12 indicate that any part of the graph colored i and any part of the graph colored j "do not intereact" for i and j distant. We may visualize sliding the j-colored part past the i-colored part, and it will not change the morphism. This is the **distant sliding property**.

In the following relations, blue and red are adjacent.

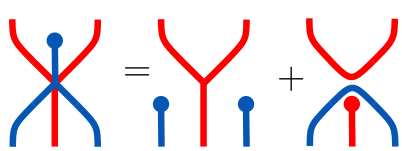


Figure 2.13

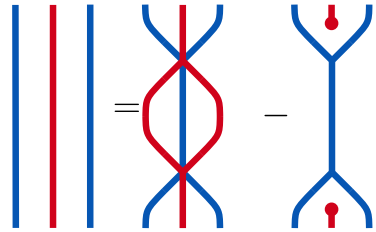


Figure 2.14

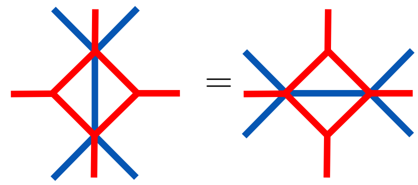


Figure 2.15

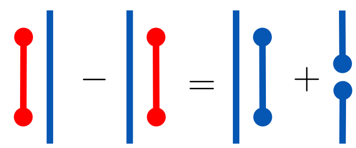


Figure 2.16

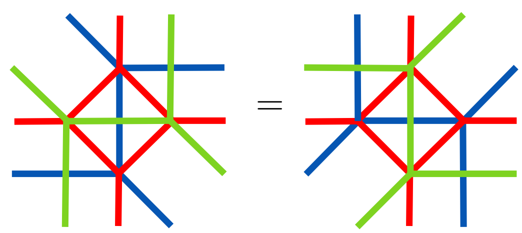


Figure 2.17

We have the following implication from Figure 2.6.

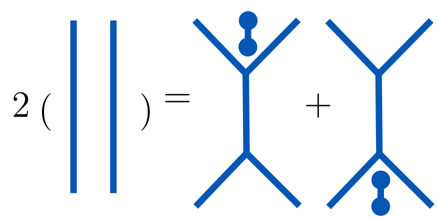


Figure 2.18

We also have the following implication from Figures 2.13 and 2.15.

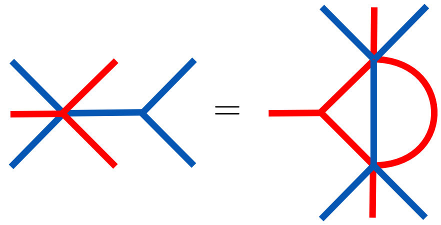


Figure 2.19

Theorem 2.1 (Main Theorem of [3]) There is a functor $\mathcal{F}$ from $\mathcal{D}$ to $\mathbb{BS}Bim$, which is an equivalence of categories. Thus, the indecomposable objects in the idempotent completion of $\mathcal{D}$ are parametrized by $w \in W$.

By abuse of notation, we will also denote such indecomposable objects by B_w.

We conclude this review with several remarks and bits of notation that will be important in the future. It will be significant that the 6-valent vertex and the 4-valent vertex both send 1-tensors ($1^{\otimes}$) to 1-tensors. So does the dot, positioned so that it represents a map $B_i \to R$, and the trivalent vertex, positioned so that it represents a map from $B_i \to B_i \otimes B_i$.
When the trivalent vertex is positioned in such a way that represents a map from $B_i \otimes B_i \to B_i$, the corresponding map of bimodules will simply apply the Demazure operator ∂_i to the middle term in $R \otimes_{R^i} R \otimes_{R^i} R$. That is, $a \otimes b \otimes c \to a\partial_i(b) \otimes c$.

The relation in Figure 2.14 is sent under $\mathcal{F}$ to the first direct sum decomposition below. Flipping the colors yields the second. Here, blue is i, red is $i+1$, and $J = \{i, i+1\}$.

$$B_i B_{i+1} B_i = B_J \oplus B_i, \tag{2.1}$$

$$B_{i+1} B_i B_{i+1} = B_J \oplus B_{i+1} \tag{2.2}$$

That is, the identity $1_{i(i+1)i}$ is decomposed into orthogonal idempotents. The first idempotent, which we call a **doubled 6-valent vertex**, is the projection from $B_i B_{i+1} B_i$ to its summand B_J. The 6-valent vertex itself is the projection from $B_i B_{i+1} B_i$ to B_J and then the inclusion into $B_{i+1} B_i B_{i+1}$.

We call the following map, which is the projection from $i(i+1)i$ to the "wrong" summand, by the name **failed 6-valent vertex**.

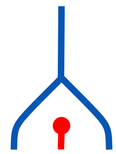


Figure 2.20

Because projections to different summands are orthogonal, we have the following key equation, a simple consequence of Figures 2.3, 2.5, and 2.13.

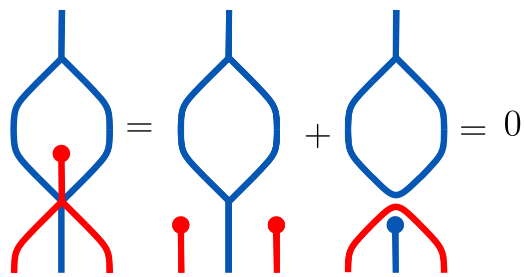


Figure 2.21

2.3. Expression graphs and path morphisms

Definition 2.1 The **reduced expression graph** of an element $w \in W$, usually abreviated **rex graph** and denoted by $Rex(w)$, is the graph defined as follows. Its vertices are the reduced expressions of w, with an edge between two reduced expressions if they differ by a single braid relation. These relations are $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i \in \{1, 2 \ldots, n-2\}$ and $s_i s_j = s_j s_i$ when $|i - j| \geq 2$. We call the edges determined by the former identity **adjacent edges**, and the ones determined by the latter, **distant edges**. We simplify notation writing ijk in place of $s_i s_j s_k$

Example 2.2 The reduced expression graph of 21321.

21321 — 23121 — 23212 — 32312 — 32132

Figure 2.22

Definition 2.2 Given a rex graph of $w \in W$, we can draw the distant edges with dashed lines. With this convention, we name this colored graph the **expanded expression graph of** w. We symbolize it by $\widetilde{\Gamma}_w$.

Example 2.3 The expanded expression graph of 12321.

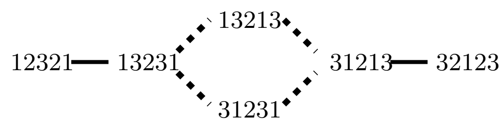


Figure 2.23

Example 2.4 The expanded expression graph of $w_{0,4}$ (see Example 1.6).

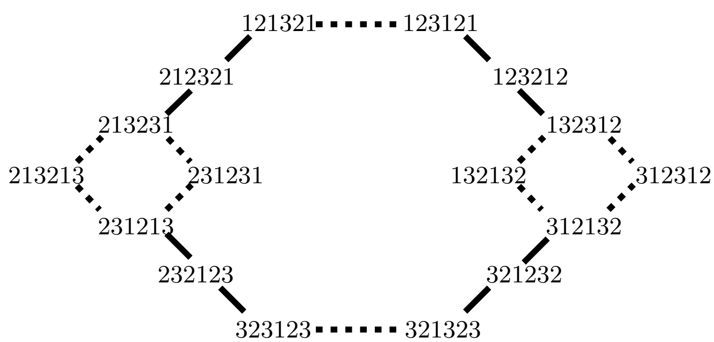


Figure 2.24

There are different kinds of cycles appearing in Figure 2.24. For instance, a square is formed between 213231 and 231213, because there are two **disjoint** distant moves connecting them. In other words, these movements can be applied in either order. Any square of this kind in any graph is called a **disjoint square**. A disjoint square can involve distant or adjacent edges. For example, there is a disjoint square of adjacent edges from 121343 to 212434. See Example 2.8.

Example 2.5 The expanded expression graph of 1214, a **distant Octagon**.

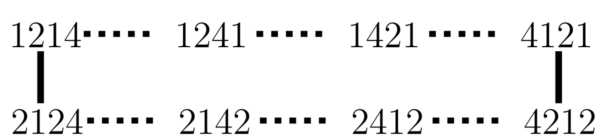


Figure 2.25: $\widetilde{\Gamma}_{1214}$

2.3.1. Braid morphisms f_{sr}

Recall that we restrict ourselves to work with type A groups, so let (W, S) be a Coxeter group, with $W = S_n$ and transpositions $(i\ i{+}1)$ as generators in S.

Consider the bimodules $X_{sr} = B_s B_r B_s \ldots$ and $X_{rs} = B_r B_s B_r \ldots$, each product having $m_{s,r}$ terms. The morphism f_{sr} is defined as the only degree 0 morphism from X_{sr} to X_{rs} sending $1^{\otimes}$ to $1^{\otimes}$ (see [4], Section 6.2.1). We write $f_{s_i s_j}$ as f_{ij}.
We describe these maps in terms of certain generators (as an (R, R)-bimodule) of the corresponding Bott-Samelson bimodules. We have to consider three cases:

First case: If $|i - j| \geq 2$. The morphism $f_{ij} : B_i B_j \to B_j B_i$ is determined by the formula $f_{ij}(1^{\otimes}) = 1^{\otimes}$, because $1^{\otimes}$ generates $B_i B_j$ as a bimodule.

Second case: The morphism $f_{i(i+1)} : B_i B_{i+1} B_i \to B_{i+1} B_i B_{i+1}$ is determined by the formulae $f_{i(i+1)}(1^{\otimes}) = 1^{\otimes}$ and

$$f_{i(i+1)}(1 \otimes x_i \otimes 1 \otimes 1) = (x_i + x_{i+1}) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2}. \tag{2.3}$$

Third case: The morphism $f_{i(i-1)} : B_i B_{i-1} B_i \to B_{i-1} B_i B_{i-1}$ is determined by the formulae $f_{i(i-1)}(1^{\otimes}) = 1^{\otimes}$ and

$$f_{i(i-1)}(1 \otimes x_{i+1} \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes (x_i + x_{i+1}) - x_{i-1} \otimes 1 \otimes 1 \otimes 1. \tag{2.4}$$

The diagrams for the second and the third case, with the usual coloring convention, are the following.

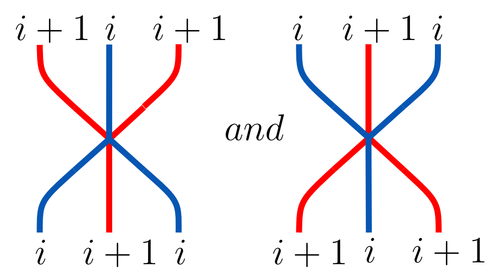


Figure 2.26: Diagrams for the second and the third case, respectively.

2.3.2. Path morphisms

Let $G = (V, E, \varphi)$ be a graph. Here, V denotes the set of vertices, E denotes the set of edges and φ is the incidence function

$$\varphi : E \to \{\{x, y\} \mid x, y \in V \text{ and } x \neq y\}.$$

Definition 2.3 Let $G = (V, E, \varphi)$ be a graph. A *path* p is a sequence of edges $(e_1, e_2, \ldots, e_{n-1})$ for which there is a sequence of vertices $[v_1, v_2, \ldots, v_n]$ such that $\varphi(e_i) = \{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n-1$. The sequence $[v_1, v_2, \ldots, v_n]$ is the **vertex sequence** of the path. Note that it is possible to recover the edges of a path from its vertex sequence, so we will work with vertex sequences and paths indistinctly. For any path p we denote by $[p]$ the associated sequence of vertices. We say that the **length** of p is n.

Definition 2.4 We give a semi-orientation to the rex graph. We orient adjacent edges with the lexicographic order. This way, these edges go from $i(i+1)i$ to $(i+1)i(i+1)$. The distant edges remain unoriented. When we speak of an **oriented path** in a semi-oriented graph, we refer to a path which may follow unoriented edges freely, but can only follow oriented edges along the orientation. A **reverse-oriented path** is a path oriented backwards. When we say **path** with no specification, we refer to any path. The starting point (vertex) and the ending point of a path p will be referred as p_a and p_z respectively. A **subpath** is a path that makes up part of a larger path.

For a pair of Bott-Samelson bimodules B, B' whose expressions differ by a single braid relation, we have a morphism of the type

$$\mathrm{Id} \otimes \ldots \otimes f_{sr} \otimes \ldots \otimes \mathrm{Id} \in \mathrm{Hom}(B, B')$$

where s and r depend on the aforementioned braid relation.

Example 2.6 In S_4, the expressions 212321 and 213231 are reduced expressions of the same element, and they differ by the braid relation $232 = 323$. The aforementioned morphism from 212321 to 213231 has the following form.

$$\mathrm{Id}^2 \otimes f_{23} \otimes \mathrm{Id} \colon B_2B_1(B_2B_3B_2)B_1 \to B_2B_1(B_3B_2B_3)B_1.$$

Definition 2.5 For each path p in the rex graph $Rex(w)$ we call $f(p)$ the associated morphism between the Bott-Samelson bimodules B_{p_a} and B_{p_z}. We call $f(p)$ a **path morphism**.

Note that for expressions related by distant edges (first case), the morphisms f_{sr} are isomorphisms. We will see in Section 3.1 that the path morphism associated to a composition of distant edges only depends on the starting point and the ending point. In this way, we can collapse the dashed lines obtaining a new graph that we now define.

Definition 2.6 The **conflated expression graph**, denoted by Γ_w, is the quotient of $\widetilde{\Gamma}_w$ (or $Rex(w)$) by all its distant edges. In other words, if p is a path such that all its edges are distant, then we identify p_a and p_z. We remark that there are no possible adjacent edges between p_a and p_z because the sum of the indices of a reduced expression remains unchanged when applied to a distant edge and varies when applied to an adjacent edge. When identifying the vertices we must choose a representative, which usually will be a specific one depending on the path morphisms we are working with. When the representative is not explicit, by convention, we will consider that it is the lower in the lexicographical order among the identified elements. We remark that there might be multiple edges between two vertices in this graph (see Example 2.5), as opposed to the expanded expression graph. Here we choose a representative following the same criteria, avoiding multigraphs.

Observation 2.5 If e is an edge (resp. v is a vertex) of the expanded expression graph we call $\pi(e)$ (resp. $\pi(v)$) its image in the conflated expression graph. In particular, if e is a distant edge, $\pi(e) = \emptyset$. For a sequence of edges $p = (e_1, \dots, e_n)$ we denote by $\pi(p)$ the sequence $(\pi(e_1), \dots, \pi(e_n))$ omitting $\pi(e_j)$ when it is empty.

Example 2.7 The conflated expression graph for 12321 in S_4 has three vertices.

Figure 2.27: Γ_{12321}

Example 2.8 The following figure is the conflated expression graph of 121343. This configuration is also known as **disjoint square**.

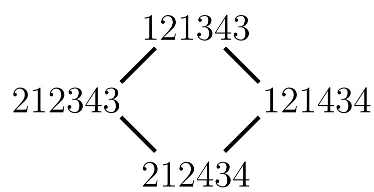


Figure 2.28: Γ_{121343}

Example 2.9 The expanded expression graph for 246 in S_7 and its conflated expression graph. A configuration like the first one is known as **distant hexagon**.

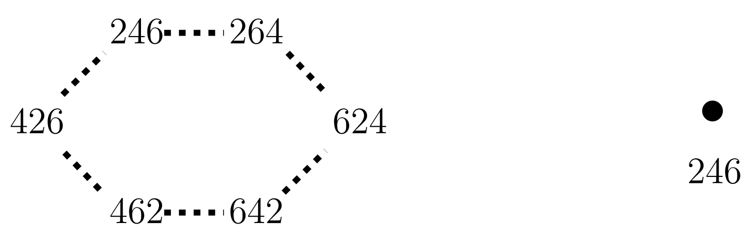


Figure 2.29: $\widetilde{\Gamma}_{246}$ and Γ_{246}

Definition 2.7 Considering the semiorientation in Definition 2.4, and the quotient in Definition 2.6, we obtain a proper orientation in Γ_w. This orientation is known as the **Manin-Schechtman orientation** [15].

Example 2.10 The conflated expression graph of $w_{0,4}$ with the Manin-Schechtman orientation. We refer to this cycle in any of its forms (i.e., in its reduced, expanded, or conflated expression graph) as a **Zamolodchikov cycle**.

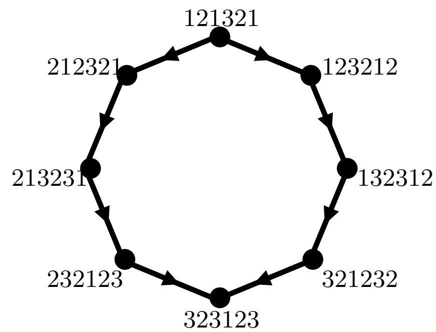


Figure 2.30: Manin-Schechtman oriented Zamolodchikov cycle.

For any $w \in S_n$, the Manin-Schechtman orientation determines a unique source and a unique sink in Γ_w. We refer to them as $\mathbf{s}$ and $\mathbf{t}$ respectively. In [2, Section 3], it is proven that the Manin-Schechtman orientation satisfies the following properties.

1. It is **BS-consistent** or **consistent with Bott-Samelson bimodules** ([2, Prop. 3.10]). This means that for any pair of oriented (or reverse-oriented) paths p and q, with $p_a = q_a$ and $p_z = q_z$, we have $f(p) = f(q)$.

2. For $w_{0,n}$, the orientation is said to be **idempotent-magical**. This means that the morphism associated to an oriented path from $\mathbf{s}$ to $\mathbf{t}$ composed with the morphism associated to a reverse-oriented path from $\mathbf{t}$ to $\mathbf{s}$ is an idempotent.

A **complete path** in a graph is a path passing through every vertex of the graph at least once. Recall from the introduction the Forking Path Conjecture.

Conjecture 2 Let $w \in S_n$, and let p, q be two complete paths in $Rex(w)$, with $p_a = q_a$ and $p_z = q_z$. Then $f(p) = f(q)$.

2.3.3. Conflated expression graph of the longest element

We now restrict our attention to the graph $\Gamma = \Gamma_{w_{0,n}}$ with the Manin-Schechtman orientation.

Definition 2.8 For $\mathbf{x}, \mathbf{y} \in \Gamma$, we denote $\mathbf{x} \searrow \mathbf{y}$ (resp. $\mathbf{y} \nearrow \mathbf{x}$) for some oriented (resp. reverse-oriented) path from $\mathbf{x}$ to $\mathbf{y}$ (resp. $\mathbf{y}$ to $\mathbf{x}$), presuming that one exists. We use $f_{\mathbf{x}\searrow\mathbf{y}}$ and $f_{\mathbf{y}\nearrow\mathbf{x}}$ for the induced path morphisms, which do not depend on the choice of oriented path by the BS-consistency.

The following is Proposition 3.16 in [2].

Proposition 2.1 There is a unique source $\mathbf{s}$, and a unique sink $\mathbf{t}$ in Γ. Let m be the length of the shortest (not necessarily oriented) path from $\mathbf{s}$ to $\mathbf{t}$. Then every vertex lies on some oriented path $\mathbf{s} \searrow \mathbf{t}$ of length m, and every oriented path $\mathbf{x} \searrow \mathbf{y}$ can be extended to a length m path $\mathbf{s} \searrow \mathbf{x} \searrow \mathbf{y} \searrow \mathbf{t}$.

Definition 2.9 For $\mathbf{x}, \mathbf{y} \in \Gamma$, let $DUD_{\mathbf{x},\mathbf{y}} = f_{\mathbf{s}\searrow\mathbf{y}} \circ f_{\mathbf{t}\nearrow\mathbf{s}} \circ f_{\mathbf{x}\searrow\mathbf{t}}$. That is, $DUD_{\mathbf{x},\mathbf{y}}$ corresponds to any oriented path which goes from $\mathbf{x}$ down to the sink, up to the source, and down to $\mathbf{y}$. Let $UDU_{\mathbf{x},\mathbf{y}} = f_{\mathbf{t}\nearrow\mathbf{y}} \circ f_{\mathbf{s}\searrow\mathbf{t}} \circ f_{\mathbf{x}\nearrow\mathbf{s}}$ corresponds to any path which goes from $\mathbf{x}$ up to the source, down to the sink, and up to $\mathbf{y}$.

Theorem 2.2 [2, Theorem 3.18] For all $\mathbf{x}, \mathbf{y} \in \Gamma$, we have $DUD_{\mathbf{x},\mathbf{y}} = UDU_{\mathbf{x},\mathbf{y}}$. Its image is the indecomposable object B_{w_0} corresponding to the longest element of S_n.

Definition 2.10 Let $Z = f_{\mathbf{s}\searrow\mathbf{t}}$ denote the unique oriented path morphism from source to sink. Let $\overline{Z} = f_{\mathbf{t}\nearrow\mathbf{s}}$ denote the unique reverse-oriented path morphism from sink to source.

Note that $DUD_{\mathbf{t},\mathbf{s}} = \overline{Z}$, $UDU_{\mathbf{s},\mathbf{t}} = Z$, and $DUD_{\mathbf{s},\mathbf{s}} = UDU_{\mathbf{s},\mathbf{s}} = \overline{Z} \circ Z$. Also, note that considering $\mathbf{x} = \mathbf{s}$ and $\mathbf{y} = \mathbf{t}$, Theorem 2.2 says that

$$Z \circ \overline{Z} \circ Z = Z. \tag{2.5}$$

Analogously, we have

$$\overline{Z} \circ Z \circ \overline{Z} = \overline{Z}. \tag{2.6}$$

Chapter 3

The Forking Path Conjecture.

In this chapter we show the Forking Path Conjecture (FPC) veracity for all except one element in S_4, where we provide explicitly a path which works as a counterexample.

3.1. Distant edges identification

If $\underline{w} \in \Gamma_w$ is a vertex in the conflated expression graph, the set $\pi^{-1}(\underline{w}) \in \widetilde{\Gamma}_w$ is called a **cloud.** If C is a cloud, then by definition, every two vertices in C are connected by a sequence of distant edges. If we consider the statistic $N(\underline{w})$ given by adding all the indexes of the reduced expression (for example $N(s_1 s_3 s_2) = 1 + 3 + 2 = 6$) we can see that the function N is constant in the vertices of a cloud.

Definition 3.1 Consider any $w \in S_n$. Let p be a path in the conflated expression graph Γ_w. The path morphism $f(p)$ defined by p is $f(\tilde{p})$, where $\tilde{p} = (e_1, e_2, \ldots, e_{n-1})$ is any path in the expanded expression graph $\widetilde{\Gamma}_w$ with $\tilde{p}_a = p_a$, $\tilde{p}_z = p_z$, and such that one obtains p from $\tilde{p}$ by applying π (see Observation 2.5) to this sequence.

Proposition 3.1 Path morphisms in conflated expression graphs are well-defined. In other words, given two paths $\tilde{p}$, $\tilde{p}'$ in $\widetilde{\Gamma}_w$ satisfying the conditions in Definition 3.1, we have $f(\tilde{p}) = f(\tilde{p}')$.

Proof. Any two paths in $\widetilde{\Gamma}_w$ defining $f(p)$ will only differ on their distant edges connecting two successive adjacent edges. So, for a fixed pair of successive adjacent edges, each sequence of distant edges will have the same starting vertex and the same ending vertex. These sequences represent oriented paths, and therefore, their induced path morphisms are the same (see [2, Prop. 3.10]). Applying this argument to each sequence of distant edges, we have the proof. □

Observation 3.1 Definition 3.1 does not depend on π. If we change the choices of distant edges e such that $\pi(e) = \phi$, by [2, Prop 3.10] we obtain the same path morphism.

The next two propositions show the equivalence between working with paths in rex graphs and working with paths in conflated expression graphs. We will therefore deduce that there is an equivalence between the Forking Path Conjecture (that we call FPC in the rest of this document) for Γ_w and for $\widetilde{\Gamma}_w$.

Proposition 3.2 For any $w \in S_n$, finding paths in its conflated expression graph giving a counterexample for the FPC gives a counterexample for the FPC in its rex graph.

Proof. For any $w \in S_n$, let p, q be complete paths in Γ_w, with $p_a = q_a$ and $p_z = q_z$, such that $f(p) \neq f(q)$. By definition, $f(p)$ is equal to $f(\tilde{p})$, where $\tilde{p}$ is a path in $\widetilde{\Gamma}_w$ satisfying the requirements in Definition 3.1. Similarly for $f(q)$ and $f(\tilde{q})$. As p and q are complete, $\tilde{p}$ and $\tilde{q}$ pass through every cloud in $\widetilde{\Gamma}_w$. We can modify $\tilde{p}$ and $\tilde{q}$ to pass through every vertex in $\widetilde{\Gamma}_w$ as follows. Each time $\tilde{p}$ or $\tilde{q}$ passes through a cloud, add a complete closed path in that cloud and then continue as before (this does not alter the path morphism). Let us call $\tilde{p}_0$ and $\tilde{q}_0$ the new paths in $Rex(w)$, then they are complete paths such that $f(\tilde{p}_0) \neq f(\tilde{q}_0)$. □

Proposition 3.3 For any $w \in S_n$, the FPC in Γ_w implies the FPC in $Rex(w)$.

Proof. Suppose that the FPC is true in Γ_w for some element $w \in S_n$. Let $\tilde{p}, \tilde{q}$ be complete paths in $Rex(w)$, with $\tilde{p}_a = \tilde{q}_a$ and $\tilde{p}_z = \tilde{q}_z$. When applying the projection π to these paths, it is possible to obtain $\pi(e) = \phi$ for some adjacent edges of these paths. This means that these edges vanish while doing the identifications of vertices and choices of edges in the construction of Γ_w. If this is the case, it is possible to replace each of these edges e at a time. We can do that replacement with a path that goes through the corresponding cloud from the same starting vertex of e to the starting vertex of the edge that does not vanish, then follows that edge, and then goes back through the corresponding cloud again to the same ending vertex of e, returning to the original path. These local replacements (one for each vanishing edge e) do not alter the resulting path morphism, because the involved subpaths are oriented paths (see [2, Prop. 3.10]). So, by modifying the paths as so, the resulting projections will satisfy the hypothesis of the FPC in Γ_w. Therefore $f(\tilde{p}) = f(\pi(\tilde{p})) = f(\pi(\tilde{q})) = f(\tilde{q})$ and we have the FPC for $Rex(w)$. $\square$

So we obtain an equivalent conjecture.

Conjecture 3 (FPC for conflated expression graphs) Let $w \in S_n$. Let p, q be two complete paths in Γ_w, with $p_a = q_a$ and $p_z = q_z$. Then $f(p) = f(q)$.

3.1.1. Calculating path morphisms

Definition 3.2 Consider $w \in S_n$ and Γ_w with the Manin-Schechtman orientation. We say that a path is **straight** if it goes in an oriented or reverse-oriented fashion from one vertex $\mathbf{x}$ to another vertex $\mathbf{y}$. We denote that by $\mathbf{x} \to \mathbf{y}$. We use this notation when we do not want to specify if $x \nearrow y$ or $x \searrow y$. In particular, straight paths $\mathbf{s} \searrow \mathbf{t}$ or $\mathbf{t} \nearrow \mathbf{s}$ will be called **direct** paths, and we denote them by the letter d. We say that a pair of paths p, q are **equivalent** and we write $p \simeq q$ if $f(p) = f(q)$, i.e., if they define the same path morphism. If a direct path (resp. straight path) is a subpath of a larger path, we call it a **direct (resp. straight) subpath of the larger path**.

Now we restrict our attention to $\Gamma_{w_{0,n}}$. Let p, q be two complete paths with $p_a = q_a$ and $p_z = q_z$, both containing a direct subpath d. We will show that $p \simeq q$. The main idea is to construct equivalent paths that lead us to a reduced problem, i.e., studying equivalences between a small set of paths. We divide p into three parts: the path before d, from p_a to d_a, which we call the p^α **subpath**, the direct subpath d, and the path after d, from d_z to p_z, which we call the p^β **subpath**. If there exist more than one direct subpath, it does not matter which one we choose to work with. We now focus on the p^α subpath.

Proposition 3.4 Let p be a complete path in $\Gamma_{w_{0,n}}$ with a direct subpath d. Then p^α is equivalent to a path p' of the form $p_a \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \ldots \rightarrow d_a$, or $p_a \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow \ldots \rightarrow d_a$.

Proof. We assume without loss of generality that $d_a = \mathbf{s}$. Then, p^α ends in the vertex $\mathbf{s}$, that is, $p^\alpha_z = \mathbf{s}$. If $p^\alpha = \mathrm{id}$ (i.e. the empty sequence), we are done. If not, the path p^α has a straight subpath from a vertex x_1 to $\mathbf{s}$ (with $x_1 \neq \mathbf{s}$) which we take maximal in this sense, i.e. the straight subpath $x_1 \nearrow \mathbf{s}$ is not contained in any larger straight subpath.

If $x_1 \nearrow \mathbf{s} = p^\alpha$, we have the desired path p'; if this is not the case, there exists a vertex x_2 (maximal in the same sense) such that $x_2 \searrow x_1$ is a subpath of p. We have that

$$x_2 \searrow x_1 \nearrow \mathbf{s} \searrow \mathbf{t} \tag{3.1}$$

is a subpath of the path p corresponding to $p^\alpha = x_2 \searrow x_1 \nearrow \mathbf{s}$, followed by $d = \mathbf{s} \searrow \mathbf{t}$. Using equation (2.5), we rewrite (3.1) as $x_2 \searrow x_1 \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t}$. Now, by Theorem 2.2, we apply $UDU_{x_1,\mathbf{t}} = DUD_{x_1,\mathbf{t}}$, to obtain

$$x_2 \searrow x_1 \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t}.$$

Using again (2.5) to simplify, we see that (3.1) has the same path morphism as $x_2 \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t}$. We now consider the subpath $\mathbf{t} \nearrow \mathbf{s}$ as our new direct subpath d. Using repeatedly this process we obtain the equivalent path p' of the prescribed form. □

The same arguments work for the β subpath, *mutatis mutandis*. This way, after simplifications using the identities (2.5) and (2.6) if needed, from p it is possible to obtain a new path $\hat{p}$ consisting of the following: a straight path from p_a to $\mathbf{s}$ or $\mathbf{t}$, followed by one or two direct paths, and then a straight path from $\mathbf{s}$ or $\mathbf{t}$ to p_z, satisfying $f(\hat{p}) = f(p)$. If $\hat{p}_a$ or $\hat{p}_z$ are $\mathbf{s}$ or $\mathbf{t}$, then $\hat{p}$ does not have the α or the β subpaths (both cases may occur at the same time).

Definition 3.3 The path $\hat{p}$ obtained from the application of Proposition 3.4 to p will be called a **simplified path**.

Proposition 3.5 In Γ_{w_0}, consider any pair of simplified complete paths, p and q, both containing a direct path d. Suppose that $p_a = q_a$ and $p_z = q_z$. Then $f(p)$=$f(q)$.

PROOF. • First case: $p_a = q_a = \mathbf{s}$. If $p_z = q_z = \mathbf{t}$, then they are necessarily equivalent to $\mathbf{s} \searrow \mathbf{t}$. If $p_z = q_z = \mathbf{s}$, they will be equivalent to $\mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s}$. If $p_z = q_z = u$, with u being a vertex that is not $\mathbf{s}$ or $\mathbf{t}$, we have two possibilities: $\mathbf{s} \searrow \mathbf{t} \nearrow u$ and $\mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s} \searrow u$. By Theorem 2.2, $UDU_{\mathbf{s},u} = DUD_{\mathbf{s},u}$, so they are equivalent.
• Second case: If $p_a = q_a = \mathbf{t}$, or, $p_z = q_z = \mathbf{s}$ or $p_z = q_z = \mathbf{t}$, we repeat a similar analysis as in the first case, and conclude that $f(p) = f(q)$.
• Third case: Now we study the case $p_a = q_a = u$ and $p_z = q_z = v$, with u and v being vertices that are neither $\mathbf{s}$ nor $\mathbf{t}$. There are four possible cases for the paths p, q.

1. $u \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow \mathbf{s} \searrow v$
2. $u \searrow \mathbf{t} \nearrow \mathbf{s} \searrow v$
3. $u \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow v$
4. $u \searrow \mathbf{t} \nearrow \mathbf{s} \searrow \mathbf{t} \nearrow v$

The equation $UDU_{u,\mathbf{s}} = DUD_{u,\mathbf{s}}$ implies that the first is equivalent to the second, $UDU_{u,v} = DUD_{u,v}$ implies that the second is equivalent to the third, and $UDU_{u,\mathbf{t}} = DUD_{u,\mathbf{t}}$ implies that the third is equivalent to the fourth. □

The FPC would follow if we could guarantee the existence of a direct subpath in any path, but this is not the case. Despite this, we will show that in $\Gamma_{w_{0,4}}$, any complete path will always contain a subpath that is equivalent to a direct path, proving the conjecture for this element.

3.1.2. The Forking Path Conjecture in S_4

We begin by considering a complete path p in $\Gamma_{w_{0,4}}$ and its path morphism $f(p)$. Since the path is complete, the vertices $\mathbf{s}$ and $\mathbf{t}$ are part of the path. Note that it could be possible to visit these points multiple times. So there is at least one subpath, that we will call **candidate path**, starting in $\mathbf{s}$ and ending in $\mathbf{t}$, or starting in $\mathbf{t}$ and ending in $\mathbf{s}$, minimal with this property. This means that there are no proper subpaths of the candidate path starting in $\mathbf{s}$ and ending in $\mathbf{t}$, or starting in $\mathbf{t}$ and ending in $\mathbf{s}$.

Without loss of generality we will assume that the candidate path starts in $\mathbf{s}$ and ends in $\mathbf{t}$. Since the Zamolodchikov cycle has a "ring shape" (see Figure 2.30), our conditions imply that the candidate path will be hosted either in the left or in the right half of this cycle. Let us suppose without loss of generality that the candidate path is in the left half. Let us consider the following path which represents the desired direct subpath.

$$\mathbf{s} \searrow A \searrow B \searrow C \searrow \mathbf{t} \tag{3.2}$$

This is a path from $\mathbf{s}$ to $\mathbf{t}$ where $\mathbf{s} = 121321$, $A = 212321$, $B = 213231 = 231231 = 213213 = 231213$, $C = 232123$, $\mathbf{t} = 323123$.

Recall from Definition 2.3 that for any path p we denote by $[p]$ the associated sequence of vertices. The beginning of our candidate path k is from $\mathbf{s}$ to A. We cannot return to $\mathbf{s}$ by the minimality of the candidate path. So we have $[k] = [\mathbf{s}, A, B, \ldots, \mathbf{t}]$. Once we are in B we can go back to A or go forward to C. If we go back to A, since we cannot return to $\mathbf{s}$, we have to return to B.

Lemma 3.1 For $\mathbf{s}, A, B$ as in path (3.2), we have $[A, B, A, B] \simeq [A, B]$ (so $[\mathbf{s}, A, B] \simeq [\mathbf{s}, A, B, A, B] \simeq [\mathbf{s}, A, B, A, B, A, B]$, and so on). Also, $[B, A, B, A] \simeq [B, A]$.

Proof. This is a well-known identity. Such composition of morphisms can be represented and decomposed as illustrated in Figure 3.1 below. We will use black rectangular frames to highlight spots where we use local relations. Blue, red, and green correspond to indexes 1, 2, and 3 respectively. The following is a consequence of Fig.2.14.

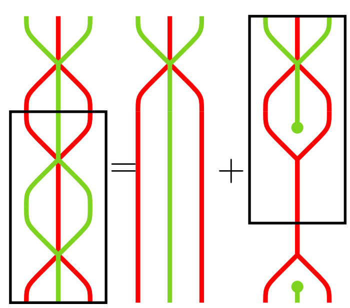


Figure 3.1

The part inside the rectangle in the second summand in Figure 3.1 is decomposed as in Figure 3.2, by Figure 2.13.

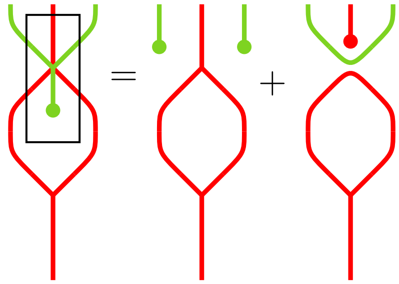


Figure 3.2

Each summand in the right-hand side is zero by Figure 2.21.
Reading the diagrams upside down we conclude that $[B, A, B, A] \simeq [B, A]$. $\square$

Note that, using the same diagrams (but with different colors), we can also find the equivalence

$$[B, C] \simeq [B, C, B, C].$$

Without loss of generality we will assume that our candidate path has minimal length when compared to all its equivalent paths. Because of this, the candidate path $[\mathbf{s}, A, B, \ldots, \mathbf{t}]$ has no subsequences of the forms $[A, B, A, B]$ and $[B, C, B, C]$. So we can assume that our candidate path starts with $[\mathbf{s}, A, B, C]$. Being at C, if we go to $\mathbf{t}$ we are done. We will find a contradiction if this is not the case.

If we don't go to $\mathbf{t}$, the path returns to B. From B we can not return to C, because we would have $[B, C, B, C]$ as a subpath. Thus, from B we go to A. Since we cannot return to $\mathbf{s}$ we have to go to B. So our path starts as follows $[\mathbf{s}, A, B, C, B, A, B]$. Again, by minimality of the length of the candidate path, the next vertex has to be C. The following proposition proves the contradiction.

Proposition 3.6 For A, B, C as in path 3.2, we have

$$[A, B, C, B, A, B, C] \simeq [A, B, C]$$

Proof. The equivalence is proved diagrammatically in Figure 3.3.

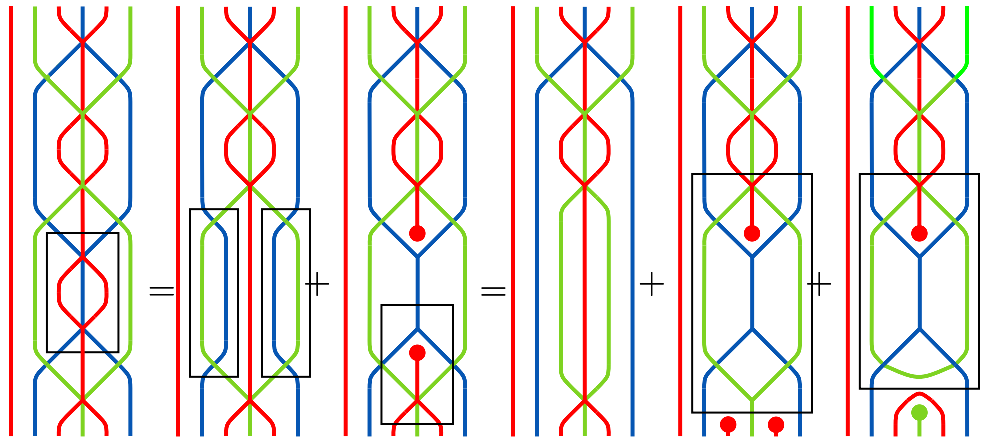


Figure 3.3

The local relations we use are all listed in Section 2. In particular, from 1) to 2) we apply Fig. 2.13. From 2) to 4) we apply Fig. 2.7 twice. We obtain 5) and 6) from 3) by means of Fig. 2.13. Applying Lemma 3.1 to the term 4) we obtain Figure 3.4. Summands 5) and 6) are zero, as we see in Figure 3.5.

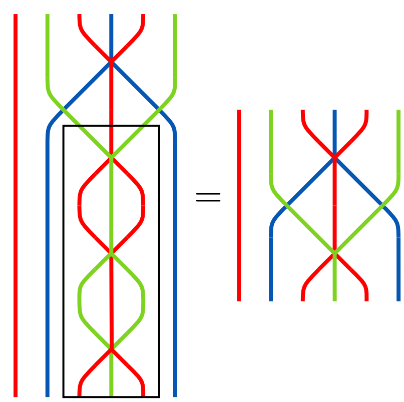


Figure 3.4: Application of Lemma 3.1.

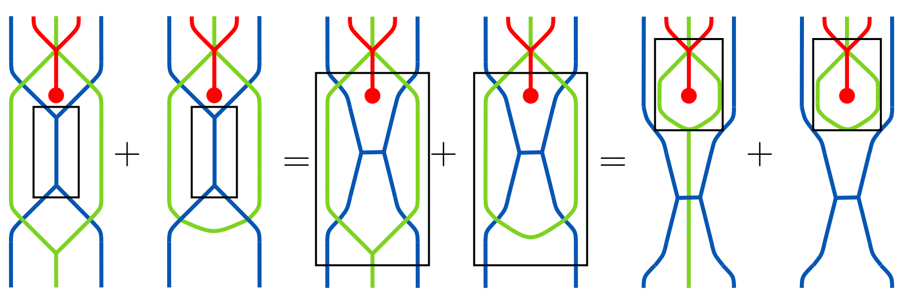


Figure 3.5

From 5) and 6), to 7) and 8) we apply Fig. 2.3. From 7) and 8) to 9) and 10), we repeatedly apply Fig. 2.7 and Fig. 2.10. As in Figure 3.2, we recognize in 9) and 10) compositions equivalent to the zero morphism. □

Proposition 3.7 The Forking Path Conjecture is true for $w_{0,4}$.

Proof. By Lemma 3.1 and Proposition 3.6, we have that any path starting at $\mathbf{s}$ and ending in $\mathbf{t}$, that does not visit $\mathbf{s}$ or $\mathbf{t}$ in the rest of the path, and that is located on the left half of the Zamolodchikov cycle (See Figure 2.30) is equivalent to a direct path. The same result is true if one considers paths from $\mathbf{s}$ to $\mathbf{t}$ or from $\mathbf{t}$ to $\mathbf{s}$, located on the left or on the right half of the cycle.

The reason for this is that the proofs for the four cases will be the same as before, but turning the diagrams upside down for the case $\mathbf{t}$ to $\mathbf{s}$ in the left half, applying a vertical axial symmetry to the diagrams for the right half for the case $\mathbf{s}$ to $\mathbf{t}$ and turning the diagrams upside down and applying a vertical axial symmetry to the diagrams for the case $\mathbf{t}$ to $\mathbf{s}$ on the right[1].

So we conclude that any complete path in $\Gamma_{w_{0,4}}$ has a subpath that is equivalent to a direct path. Therefore, by Proposition 3.5 the proof of the FPC for w_0 in S_4 is complete. □

Now we verify the conjecture for the remaining elements in S_4 different from 12321. The elements w and their Γ_w graphs oriented according to Manin-Schechtman are given in the following table.

e $\bullet$	1 $\bullet$	2 $\bullet$	21 $\bullet$	12 $\bullet$	121 $\bullet \to \bullet$
3 $\bullet$	31 $\bullet$	32 $\bullet$	321 $\bullet$	312 $\bullet$	3121 $\bullet \to \bullet$
23 $\bullet$	231 $\bullet$	232 $\bullet \to \bullet$	2321 $\bullet \to \bullet$	2312 $\bullet$	23121 $\bullet \to \bullet \to \bullet$
123 $\bullet$	1231 $\bullet \to \bullet$	1232 $\bullet \to \bullet$	12321 $\bullet \to \bullet \to \bullet$	12312 $\bullet \to \bullet \to \bullet$	123121 Zam

Figure 3.6: Elements in S_4 and their conflated expression graphs.

[1] A deeper reason for this is that we are implicitly applying some equivalences of categories. There is a contravariant equivalence of monoidal categories $\mathbb{S}$Bim$\to$ $\mathbb{S}$Bim, given by the flip (that sends a diagram to its horizontal flip) and also an auto-equivalence of $\mathbb{S}$Bim associated to the only non-trivial automorphism of the Dynkin diagram of type A_n, sending $s_i \mapsto s_{n-i+1}$.

The last entry Zam is the Zamolodchikov cycle, as introduced in Example 2.10. There is no need to verify the trivial graphs Γ_w ($\bullet$), since the only possible morphism is the id. For any $\bullet \to \bullet$ case, the proof of the Forking Path Conjecture follows easily from Lemma 3.1. It remains to check the $\bullet \to \bullet \to \bullet$ cases. We will concentrate in the cases 23121 and 12312 because the other case is the one giving the counterexample to the Forking Path Conjecture. We now study the element 23121.

Proposition 3.8 The Forking path conjecture is true for the element 23121.

Proof. In this case we can speak of a **simplified path** p similar to that of Definition 3.3. Consider $\{x, y\} = \{\mathbf{s}, \mathbf{t}\}$. These paths will be of the form $p_a \to x \to y \to p_z$, where p_a could be equal to x, and p_z could be equal to y, or alternatively, of the form $p_a \to x \to y \to x \to p_z$, where p_a and p_z could be equal to x. The path $p_a \to x$ (resp. $y \to p_z$ in the first case, and $x \to p_z$ in the second) will have length one only when $p_a = c$ (resp. $p_z = c$), where c is the only vertex different to $\mathbf{s}$ and $\mathbf{t}$. We consider simplified paths p and q.

• We first study the case $p_a, p_z \in \{\mathbf{s}, \mathbf{t}\}$. It is immediate that $p \simeq q$, since there is only one possible path, i.e., $p = q$.

• Consider the case $p_a = \mathbf{s}$ and $p_z = c$. There are two possible simplified paths, $P_1 := [\mathbf{s}, c, \mathbf{t}, c]$ and $P_2 := [\mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c]$. We will prove that P_1 is equivalent to P_2. In the following figure, part 1) represents the path morphism of P_2.

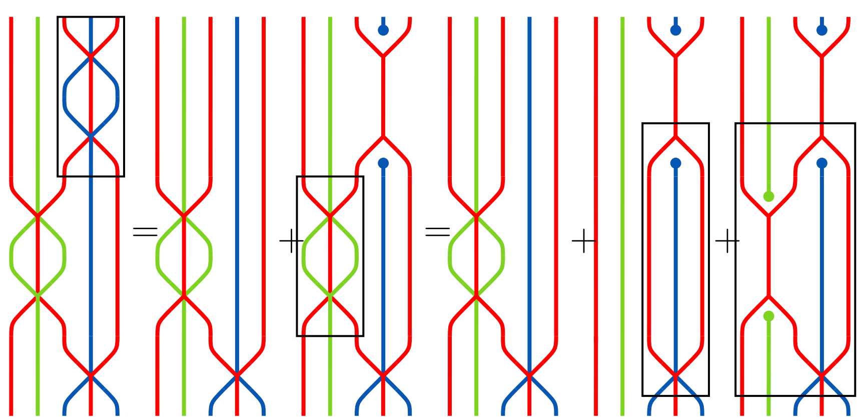


Figure 3.7

We begin in 1) applying the relation seen in Figure 3.1. Then, we apply the same relation in 3). Diagram 5) is zero by the same reason as diagram 9) in Figure 3.5. In 6) we focus on the red and blue strands because we can retract the green dots. We have the following diagrams.

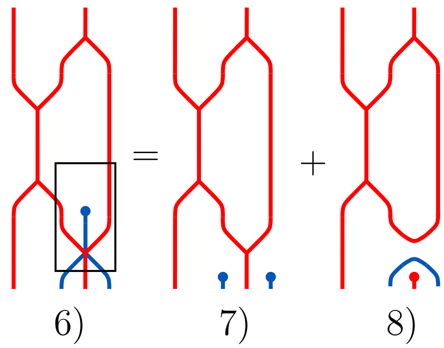


Figure 3.8

So diagram 6) is zero. Thus, diagram 1) is equal to diagram 4), or in other words, the path morphism of P_2 is equal to the path morphism of P_1.

• We have proved the proposition when $p_a = \mathbf{s}$ and p_z is any vertex. One can prove similarly the proposition for $p_a = \mathbf{t}$ and p_z any vertex, by symmetry. By flipping the diagrams we can prove the proposition for any $p_z \in \{\mathbf{s}, \mathbf{t}\}$.

• The only case that remains to show is the equivalence for p and q such that $p_a = q_a = p_z = q_z = c$. There are four possible simplified paths

$$Q_1 := [c, \mathbf{s}, c, \mathbf{t}, c], Q_2 := [c, \mathbf{t}, c, \mathbf{s}, c], Q_3 := [c, \mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c], Q_4 := [c, \mathbf{t}, c, \mathbf{s}, c, \mathbf{t}, c].$$

In the left hand-side of the following figure, we draw the path morphism corresponding to Q_1, which can be seen to be equal to the path morphism corresponding to Q_2 after an application of the local relation in Figure 2.3.

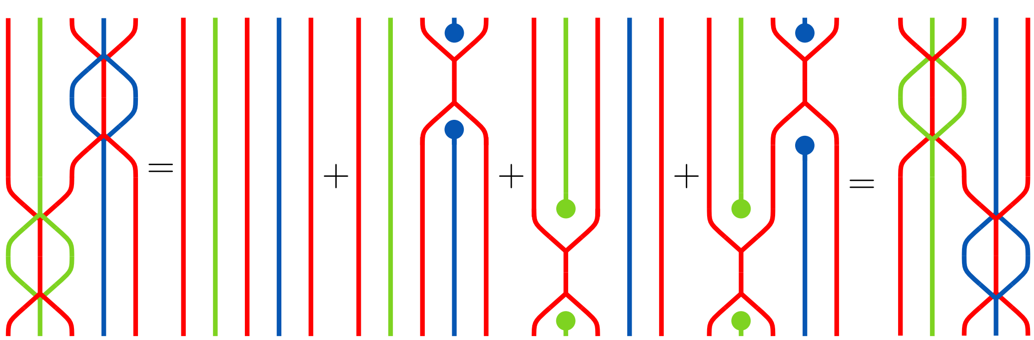


Figure 3.9: Proof that $f(Q_1) = f(Q_2)$.

By reading Figure 3.7 upside down, we have that $[\mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c] \simeq [\mathbf{s}, c, \mathbf{t}, c]$. This way, using that $Q_1 \simeq Q_2$ and simplifying, we deduce that

$$Q_4 \simeq [c, \mathbf{t}, c, \mathbf{s}, c, \mathbf{t}, c] \simeq [c, \mathbf{t}, c, \mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c] \simeq [c, \mathbf{t}, c, \mathbf{s}, c] \simeq Q_2.$$

Analogously, $[c, \mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c] \simeq [c, \mathbf{s}, c, \mathbf{t}, c]$, so $Q_3 \simeq Q_1$. $\square$

The proof of the Forking Path Conjecture for the element $12132 = 12312$ is essentially the same as the proof given in Proposition 3.8, after applying the auto-equivalence of $\mathbb{S}$Bim given by the unique non-trivial automorphism of the Dynkin diagram, i.e., applying a vertical symmetry to all diagrams.

3.2. The counterexample

Let us consider the element $\sigma = 12321 \in S_4$. According to this notation, it has the following strand diagram:

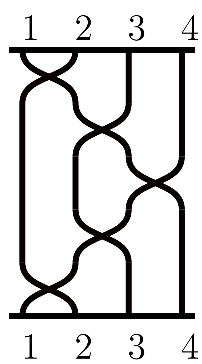


Figure 3.10: Strand diagram of σ.

Equivalently, in the one-line notation, this element would be 4231. Its rex graph $Rex(\sigma)$ corresponds to the following figure:

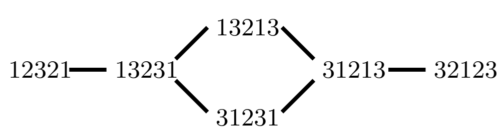


Figure 3.11: Reduced expression graph of σ.

Note that the four vertices in the middle are the same vertex in the conflated expression graph. Let us consider the element

$$x := 1 \otimes_{s_1} 1 \otimes_{s_3} 1 \otimes_{s_2} x_3 \otimes_{s_3} 1 \otimes_{s_1} 1$$

in the Bott-Samelson bimodule $B_1B_3B_2B_3B_1$. Let v_1 and v_2 be the following paths respectively:

$$v_1 := 13231 \to 31231 \to 31213 \to 32123 \to 31213 \to 13213 \to 13231 \to 12321 \to 13231.$$

$$v_2 := 13231 \to 12321 \to 13231 \to 31231 \to 31213 \to 32123 \to 31213 \to 13213 \to 13231.$$

To simplify calculations we need the following. From Equation (2.3) we obtain

$$f_{i(i+1)}(1 \otimes_{s_i} x_{i+1} \otimes_{s_{i+1}} 1 \otimes_{s_i} 1) = 1 \otimes_{s_{i+1}} 1 \otimes_{s_i} 1 \otimes_{s_{i+1}} x_{i+2}. \tag{3.3}$$

Similarly, from Equation (2.4) we obtain

$$f_{i(i-1)}(1 \otimes_{s_i} 1 \otimes_{s_{i-1}} x_i \otimes_{s_i} 1) = x_{i-1} \otimes_{s_{i-1}} 1 \otimes_{s_i} 1 \otimes_{s_{i-1}} 1. \tag{3.4}$$

We will use a diagrammatic method to evaluate homomorphisms in $\mathbb{S}\mathrm{Bim}$. The next figure shows the evaluation of $f(v_1)$ (left) and $f(v_2)$ (right) in the element x defined above.

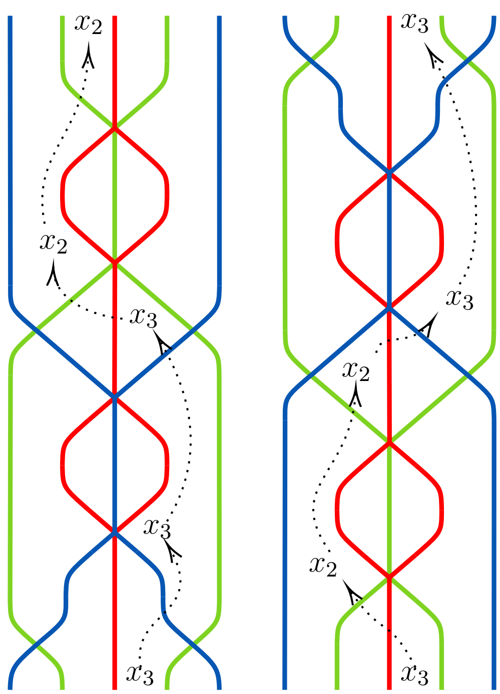


Figure 3.12: Diagrams for v_1 and v_2, evaluated in x.

In $B_1B_3B_2B_3B_1$, we have

$$f(v_1)(x) = 1 \otimes_{s_1} 1 \otimes_{s_3} 1 \otimes_{s_2} x_3 \otimes_{s_3} 1 \otimes_{s_1} 1$$

and

$$f(v_2)(x) = 1 \otimes_{s_1} x_2 \otimes_{s_3} 1 \otimes_{s_2} 1 \otimes_{s_3} 1 \otimes_{s_1} 1.$$

To check that the elements obtained are different, it is enough to apply dot morphisms over B_1, B_3, B_2, B_3, and B_1, obtaining x_2 on the left side and x_3 on the right side. Since applying the same morphism over $f(v_1)(x)$ and $f(v_2)(x)$ gives us different results, then $f(v_1)(x)$ and $f(v_2)(x)$ must be different.

3.3. A family of counterexamples

The element $\sigma = 12321$ is the only one where the FPC fails for the group S_4. We proved this by showing that the diagrams in Figure 3.13 (the same diagrams as in Figure 3.12) are not equal.

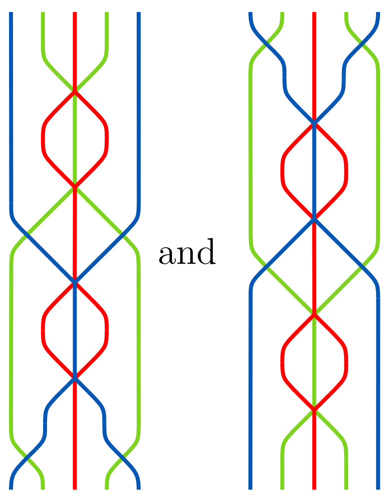


Figure 3.13: Diagrams for $f(v_1)$ and $f(v_2)$.

The first diagram decomposes as in Figure 3.14, while the second decomposes as in Figure 3.15. The only summand that is different is the last one.

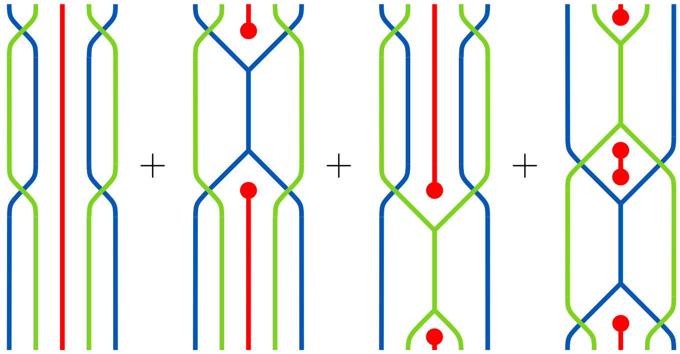


Figure 3.14: Decomposition of $f(v_1)$.

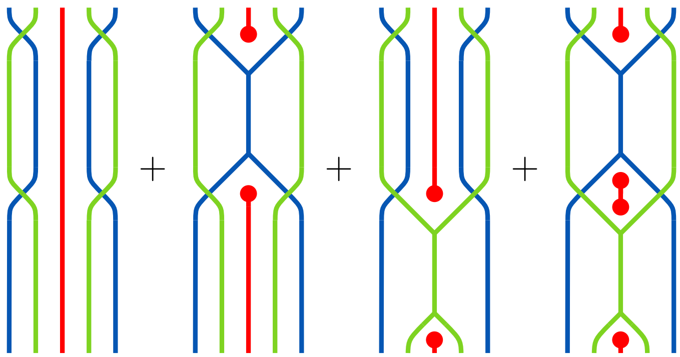


Figure 3.15: Decomposition of $f(v_2)$.

This allowed us to find the correct element to find the counterexample. Repeating the same idea, we see that the elements of the form

$$\tau = 12\ldots(n-1)n(n-1)\ldots 21$$

have a line as conflated expression graph. Let's say this line is the following.

$$E_1 \text{——} \; E_2 \text{——} \; E_3 \text{——} \cdots \text{——} \; E_{n-1} \text{——} \; E_n$$

Figure 3.16: Conflated expression graph of τ.

Considering p the path $[E_2, E_1, E_2, E_3, \ldots, E_n, E_{n-1}, \ldots, E_2]$ and q the path $[E_2, E_3, \ldots, E_n, E_{n-1}, \ldots, E_1, E_2]$. We can check in general that $f(p) \neq f(q)$ by evaluating these path morphisms in particular elements. We will not give a rigorous proof of this fact, but the general strategy can be inferred from Figure 3.17. The purple strand is related to the index 4. The black, to the index 5.

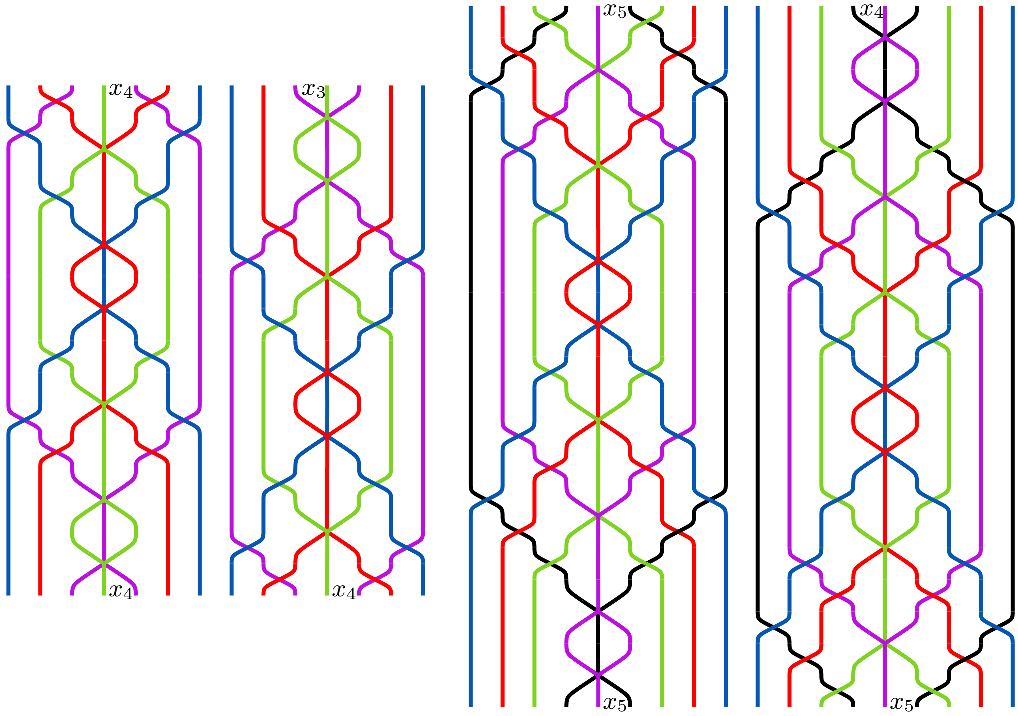


Figure 3.17: Induced morphisms in $\Gamma_{1234321}$ and $\Gamma_{123454321}$.

Observation 3.2 There are also elements of symmetric groups which can be used to produce counterexamples, and whose conflated expression graphs are not linear, i.e., elements whose conflated expression graph is different from Figure 3.16. The reader can verify that one such element is 12134325 in S_6.

Note that all elements in our family of counterexamples are paths with $p_a \notin \{\mathbf{s}, \mathbf{t}\}$ and $p_z \notin \{\mathbf{s}, \mathbf{t}\}$. This could make us think that the behavior for complete paths with $p_a \in \{\mathbf{s}, \mathbf{t}\}$ or $p_z \in \{\mathbf{s}, \mathbf{t}\}$ is different. That is not the case! Consider the same element $\sigma = 12321$, and Γ_σ, the paths $[\mathbf{s}, c, \mathbf{t}, c, \mathbf{s}, c]$ and $[\mathbf{s}, c, \mathbf{t}, c]$ give a counterexample. It is enough to evaluate both path morphisms in the element

$$1 \otimes_{s_1} x_2 \otimes_{s_2} 1 \otimes_{s_3} 1 \otimes_{s_2} 1 \otimes_{s_1} 1.$$

These counterexamples (and some others that we do not show here) have in common particular choices of elements and paths, however, verification for other families of elements show that there is a phenomenon hidden underneath. To be precise, we have observed that for the longest element in S_n, any path p with $p_a = \mathbf{s}$ and $p_z = \mathbf{t}$ will be equivalent to an oriented path from $\mathbf{s}$ to $\mathbf{t}$ (we proved this for S_4 in Section 3.1.2). In other words, we do not need to follow the Manin-Schechtman orientation as long as we start and end in the right vertices. The same when we start in $\mathbf{t}$ and end in $\mathbf{s}$. We propose the following strengthening of the FPC for w_0.

Conjecture 4 Let $w_{0,n} \in S_n$ and $\mathbf{s}$, $\mathbf{t}$ be the source and sink of the Manin-Schechtman orientation. Let p, q be two paths in $\Gamma_{w_{0,n}}$, which pass through $\mathbf{s}$ and $\mathbf{t}$, satisfying $p_a = q_a$, and $p_z = q_z$. Then $f(p) = f(q)$.

We also conjecture the same for other choices of $\mathbf{s}$ and $\mathbf{t}$ obtained from other orientations different from the one given by Manin and Schechtman. Of course, this conjecture implies the FPC for $w_{0,n}$.

Chapter 4

Conclusions

4.1. Summary

We applied the Diagrammatic Calculus to compute induced morphisms corresponding to a specific family of paths to establish the veracity of the Forking Path Conjecture. In that attempt, we succeeded when working with all but one element in S_4.

The element $\sigma = 12321 \in S_4$ has as Rex graph a linear graph, where it is possible to find two specific complete paths (as we described in Section 3.2) inducing different path morphisms.

Regardless, we have seen intriguing behaviors when analyzing paths in Rex-graphs of a particular family of elements, namely the longest elements in each S_n. In particular, we have seen that some outstanding couples of vertices work similarly to the source and the sink determined by the Manin-Schechtman orientation. In other words, complete paths are no longer needed as long as we visit those couples of vertices at some point in the involved path.

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