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On the number of Abelian subvarieties of reduced degree on an Abelian variety

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Contents

Abstract

Due to Poincaré's Reducibility Theorem, we know that for an Abelian variety A there exists an isogeny decomposition of A into simple factors. However, this decomposition hides many interesting properties of the set of Abelian subvarieties of A. For example, it is not clear from the decomposition "how many" Abelian subvarieties A actually has. In this work, for a principally polarized Abelian variety $(A, \mathcal{L})$ of dimension g in the moduli space $\mathcal{A}_g$ and some positive integers t and n, we define

$$N_A(t,n) := \#\{S \leq A : S \text{ is an abelian subvariety with } \dim S = n, \chi(\mathcal{L}|_S) \leq t\},$$

which counts the number of Abelian subvarieties of dimension n and reduced degree bounded by t. After some reductions of the problem of obtaining a bound for this number, we characterize the set of Abelian subvarieties of dimension n on E^g, for E an elliptic curve, in terms of the Grassmannian variety via the Stiefel variety. Finally, we use machinery from Diophantine Geometry to obtain an asymptotic estimation for $N_A(t,n)$ for any $(A, \mathcal{L}) \in \mathcal{A}_g$.

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Introduction

To know or characterize the Abelian subvarieties of an Abelian variety can give us information without knowing the Abelian variety itself. For example, given that the $\mathbb{Q}-$endomorphisms algebra of an Abelian variety is a finite dimensional semisimple $\mathbb{Q}-$algebra, we have an explicit description of the possibilities through Albert's classification of rational division algebras with a positive involution. In another way, let C and C' be smooth projective curves of dimensions g, g', with Jacobians J and J', respectively. If $f : C \to C'$ is a finite morphism, since the pullback induces a homomorphism of the Jacobians with finite kernel, then J contains an Abelian subvariety of dimension g'.

For an Abelian variety A we study the number $N_A(n,t)$ of Abelian subvarieties of a given dimension n and degree bounded by t. Given a principally polarized Abelian variety $(A, \mathcal{L})$, let us denote by

$$N_A(t,n) := \#\{S \leq A : S \text{ is an abelian subvariety with } \dim S = n, \chi(\mathcal{L}|_S) \leq t\},$$

that is the number of Abelian subvarieties of A of dimension $n \geq 1$ and *reduced degree* at most t. We know that this number is finite for a polarized Abelian variety.

The modern point of view brought by Kani in [17] and [18] has inspired Guerra [11] to study the number of elliptic curves in a polarized Abelian variety A in terms of their degree, using an explicit description of the Néron-Severi group.

Our first attempt is to take a principally polarized Abelian variety, given that the asymptotic estimation of $N_A(n,t)$ can be controlled under isogenies (see Section 4.2.1). For the self product of a general elliptic curve, we find an asymptotic estimation of $N_A(n,t)$ by using Diophantine geometry.

In a succession of papers, Guerra has investigated the number of elliptic curves in a polarized Abelian variety A by their degrees, for example in [10], [11] and [12]. He denoted by $N_A(t)$ the number of elliptic curves in A with degree bounded by t. It is a classical result that for $(A, \mathcal{L})$ a polarized Abelian variety and a positive integer e, there are only finitely many Abelian subvarieties of A of exponent less or equal to e. See [4, 4.1]. This result implies that $N_A(t)$ is finite.

This strategy can not be emulated for higher dimensional Abelian subvarieties, given that the Néron-Severi group has a more complicated description, because although Abelian subvarieties of higher codimension can be described in terms of numerical divisor classes, the equations can be difficult to calculate (see [2, Theorem 1.3]).

Recently, Guerra in [12] generalized his result considering the collection of all Abelian subvarieties S of $(A, \mathcal{L})$ such that the Euler characteristic $\chi(L|_S)$ is bounded above by t.

Our approach is quite different from Guerra's in terms of the characterization of the (reduced) degree. Moreover, we will study Abelian subvarieties of a fixed dimension.

In this work, for a principally polarized Abelian variety of dimension g in the moduli space $\mathcal{A}_g$ and some integers t and n, we associate the counting function of Abelian subvarieties of dimension n and reduced degree bounded by t by the following function

$$\begin{aligned} N : \mathcal{A}_g \times \mathbb{N}^2 &\to \mathbb{N} \\ (A, t, n) &\mapsto N_A(t, n). \end{aligned}$$

We would like to know where the function N achieves its maximum in the moduli space $\mathcal{A}_g$. We have good reasons to suspect that the maximum is achieved for a self product of elliptic curves E^g. Unfortunately, we have been unable to prove this in a general case. This could be developed in the future in order to get a complete answer to this problem.

In the first section, we will show some reductions to the problem, then the study of the asymptotic behavior of the number of Abelian subvarieties to a principally polarized Abelian varieties. Moreover, we will concentrate on the expected maximum case.

We will associate to every self product of an elliptic curve E^g a Grassmannian whose rational points correspond to Abelian subvarieties of E^g of a fixed dimension. Moreover, the degree of the Abelian subvariety will be associated with a certain height on the Grassmannian. A similar idea was developed by Auffarth in his doctoral thesis [1], where he studied some homogeneous forms on the Néron-Severi group. Instead of following his path, we will study the algebra of endomorphims of E^g.

In the second chapter, for K an algebraically closed field, we will study the Grassmannian of a $g-$dimensional K-vector space. In our context, we will show the relation between the Grassmannian variety $\mathrm{Gr}(n, K^g)$ and the Abelian subvarieties of E^g. Then, we will characterize the Abelian subvarieties of reduced degree in terms of the endomorphisms of the initial elliptic curve, using the fact that we are looking for the Abelian subvarieties of a given dimension. Then we use some classical results from Diophantine Theory in order to find an estimate for the asymptotic behavior of $N_A(t, n)$.

Chapter 1

Abelian Varieties

We will start the first section by fixing some concepts and notation. This area is vast, so we put the focus on the algebra of endomorphisms of an Abelian variety. This emphasis is due to the fact that we will translate the problem of counting Abelian subvarieties to the study of endomorphisms of the Abelian variety.

For Abelian varieties over an arbitrary algebraically closed field, we consult [25] and [24]. For complex Abelian varieties, we have [3] as the standard reference.

1.1 Preliminaries.

We define our principal object of study from an algebraic point of view.

Definition 1.1. An *Abelian variety* over a field k is an irreducible projective variety that is also an algebraic group over k.

In the case of a reducible variety with an algebraic group law, the irreducible components are disjoint and form a finite group. The connected component of G containing e, denoted by G^0, is called the identity component of G and is a subgroup of G.

For any $g \in G$, the right and left translation maps are biregular maps. Moreover, algebraic groups are smooth varieties and it can be shown that Abelian varieties are commutative and divisible groups.

Given an Abelian variety A, we will denote by $t_x : A \to A$ translation by $x \in A$. A homomorphism between two Abelian varieties will be a morphism that sends 0 to 0 (it can be shown that such a morphism is in particular a group homomorphism). For two Abelian varieties A and B, we denote by $\mathrm{Hom}(A, B)$ the group of homomorphisms from A to B.

Let A and B be Abelian varieties, and $f : A \to B$ a surjective morphism. Then $k(A)$ is an extension of $k(B)$ via f^*, where $k(A)$ is the rational function field of A. With this notation, we define:

Definition 1.2. An *isogeny* between A and B is a homomorphism $f : A \to B$ that is surjective and has finite kernel. If such a map exists between A and B, we will write $A \sim B$ and say that they are *isogenous*. We will call $d = [k(A) : f^*k(B)]$ the *degree* of f. Define $e(f)$ to be the *exponent* of f, that is, the least common multiple of the orders of all the elements of $\ker f$.

If f is a separable map, i.e. $k(A)$ is separable over $k(B)$, then d is the cardinality of $f^{-1}(y)$ for all $y \in B$. For f inseparable, the separable degree $[k(A) : f^*k(B)]_s$ is the cardinality of $f^{-1}(y)$ for all $y \in B$. Since the cardinality of the kernel of f is the cardinality of $f^{-1}(y)$ for all $y \in B$, we have that $\# \ker f = [k(A) : f^*k(B)]_s$. In particular, in characteristic 0 we have that $\deg f = \# \ker f$.

For any isogeny $f : A \to B$ of exponent $e(f)$ there exists an isogeny $g : B \to A$, unique up to isomorphisms, such that $g \circ f = e_f$ on A and $f \circ g = e_f$ on B, where e_f is the multiplication map by the integer $e(f)$.

Let $\mathrm{End}(A)$ be the set of endomorphisms of an Abelian variety A, i.e. homomorphisms of A to itself. It has a natural ring structure, with composition of endomorphisms as the ring multiplication. For $f \in \mathrm{End}(A)$ we define $\deg(f)$ as before if f is an isogeny and we set $\deg(f) = 0$ otherwise.

We consider the category whose objects are Abelian varieties and where morphisms between A and B correspond to the elements of the $\mathbb{Q}-$vector space $\mathrm{Hom}_{\mathbb{Q}}(A, B) := \mathrm{Hom}(A, B) \otimes \mathbb{Q}$. In this category, we can identify isogenies with isomorphisms, via the natural inclusion from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}_{\mathbb{Q}}(A, B)$.

Definition 1.3. A *polarization* on an Abelian variety A is an ample line bundle $\mathcal{L}$ on A defined up to translation. The pair $(A, \mathcal{L})$ is called a *polarized Abelian variety*. We say that $\mathcal{L}$ is a *principal polarization* if $h^0(\mathcal{L}) := \dim H^0(A, \mathcal{L}) = 1$, in this case, the pair $(A, \mathcal{L})$ will be called a *principally polarized Abelian variety* (ppav).

We define $A^\vee = \mathrm{Pic}^0(A)$ as the dual Abelian variety of A. For any ample line bundle $\mathcal{L}$, we can define an isogeny between an Abelian variety and its dual

$$\begin{aligned} \phi_{\mathcal{L}} : A &\to A^\vee \\ x &\mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned}$$

which only depends on the class of $\mathcal{L}$ in the Nerón-Severi group $\mathrm{NS}(A)$ of A. We denote by $\psi_{\mathcal{L}}$ the unique isogeny from $A^\vee$ to A such that $\psi_{\mathcal{L}} = e_{\mathcal{L}}\phi_{\mathcal{L}}^{-1}$ in $\mathrm{Hom}_{\mathbb{Q}}(A^\vee, A)$, where $e_{\mathcal{L}}$ is the exponent of the finite group $\ker \phi_{\mathcal{L}}$.

Let us denote by $K(\mathcal{L}) := \ker \phi_{\mathcal{L}}$.

Proposition 1.4. *Let k be an algebraically closed field. Let A be an Abelian variety over k and let $\mathcal{L}$ be an ample line bundle on A such that $K(\mathcal{L})$ is finite and prime to* $\mathrm{char}(k)$*. Then there is a sequence of integers $d_1|d_2|\dots|d_g$, called* *the type of $\mathcal{L}$, such that $K(\mathcal{L}) = \mathscr{G}(\mathcal{L})/\mathbb{G}_{m,k}$ where $\mathscr{G}(\mathcal{L})$ is isomorphic to the Heisenberg group associated to the group $H = (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_g\mathbb{Z})$ and $\mathbb{G}_{m,k}$ is the multiplicative group over the field k.*

Proof. [7, 8.25] □

For $k = \mathbb{C}$ the previous theorem can be quite specific. If $\mathcal{L}$ is an ample line bundle on A, $K(\mathcal{L})$ is finite and thus $\phi_{\mathcal{L}}$ is separable and

$$K(\mathcal{L}) \simeq \bigoplus_{i=1}^{g} \left(\mathbb{Z}/d_i\mathbb{Z}\right)^2$$

for $d_i \in \mathbb{Z}_{>0}$ with $d_i | d_{i+1}$. See [3, 3.1.4].

Theorem 1.5 (Riemann-Roch)**.** *Let $(A, \mathcal{L})$ be a complex polarized Abelian variety of type $(d_1, \ldots, d_g)$. Then,*

$$h^0(\mathcal{L}) = \prod_{i=1}^{g} d_i.$$

In particular, $\mathcal{L}$ is a principal polarization if and only if $d_i = 1$ for all i.

Proof. See [3, 3.6.1]. □

Definition 1.6. Let k be an algebraically closed field. Let A be an Abelian variety over k and let $\mathcal{L}$ be an ample line bundle on A We define the *degree of the polarization* $\mathcal{L}$ to be the product $d_1 \cdots d_g$.

In the following, we will see that in this work the degree of polarization is chosen so that it is a square root of the degree of the polarizing isogeny.

1.2 Divisors and Intersection Numbers.

Let X be a smooth projective algebraic variety of dimension n. The *group of Weil divisors on X* is the free group generated by the closed subvarieties of codimension one on X. It is denoted by $\mathrm{Div}(X)$.

Let $D_1, \ldots, D_n \in \mathrm{Div}(X)$ be effective irreducible divisors with the property that $\dim(\cap_i D_i) = 0$. Choose local equations $f_1, \ldots, f_n$ for $D_1, \ldots, D_n$ in a neighborhood of a point $x \in X$. The (local) intersection index of $D_1, \ldots, D_n$ at x is

$$(D_1, \ldots, D_n)_x = \dim_k(\mathcal{O}_{X,x}/(f_1, \ldots, f_n))$$

This dimension is always finite and does not depend on the selected local equations. We define the *intersection index (or number)* of $D_1, \ldots, D_n$ to be

$$(D_1, \ldots, D_n) = \sum_{x \in X} (D_1, \ldots, D_n)_x.$$

If the D_i intersect transversally, that is, if each $(D_1, \dots, D_n)_x$ is either 0 or 1, then $(D_1, \dots, D_n)$ actually counts the number of points in $(D_1 \cap \dots \cap D_n)$.

We extend this definition linearly to any n divisors. This number depends only on the linear equivalence classes of the divisors, it is symmetric and multilinear. In addition, if the D_i are effective and they meet transversely, then $(D_1, \dots, D_n) = \# D_1 \cap \dots \cap D_n$.

Theorem 1.7. *Let X and Y be smooth projective varieties of dimension n, and let $f : X \to Y$ be a finite morphism. Let $D_1 \dots D_n \in \mathrm{Div}(Y)$. Then*

$$(f^* D_1, \dots, f^* D_n) = \deg(f) \cdot (D_1, \dots, D_n).$$

Proof. See [25, 2,II.6] □

If $Z \subset X$ is a subvariety of dimension s and $D_1, \dots, D_s$ are divisors on X, then we can define the intersection number $(D_1, \dots, D_s \cdot Z) := (D_1|_Z, \dots, D_s|_Z)$. Moreover, we denote by (D^n) the self intersection of the divisor D, n times. The most important case is when Z is a curve. Moreover, we define the degree of a subvariety with respect to a divisor:

Definition 1.8. Let X be a projective variety, let $i : Z \to X$ be a subvariety of dimension s, and let $D \in \mathrm{Div}(X)$. The degree of Z with respect to D is defined to be

$$\deg_D(Z) = (\underbrace{i^*(D), \dots, i^*(D)}_{s \text{ times}}) = (i^*(D)^s).$$

In terms of intersection numbers, we have the following geometric Riemann-Roch Theorem for Abelian varieties:

Theorem 1.9 (Riemann-Roch for Abelian varieties)**.** *Let A be an Abelian variety of dimension n, let D be a divisor on A and let $\chi(D) := \chi(\mathcal{O}_A(D))$ be its Euler characteristic. Then*

$$\chi(D) = \frac{(D^n)}{n!}$$
$$\chi(D)^2 = \deg \phi_{\mathcal{O}_A(D)}$$

Proof. See [25, p.140] □

1.3 Abelian subvarieties and the endomorphism algebra.

Let $f : A \to B$ be a homomorphism. By pulling back line bundles, we get a homomorphism $f^\vee : B^\vee \to A^\vee$. In particular, if $A = B$, for every endomorphism of A we obtain an endomorphism of $A^\vee$. Assume that A has a polarization $\mathcal{L}$ and let $f \in \mathrm{End}(A)$. We define an (anti)involution of $\mathrm{End}_{\mathbb{Q}}(A)$, called the *Rosati involution*:

$$\begin{aligned}\dagger : \mathrm{End}_{\mathbb{Q}}(A) &\to \mathrm{End}_{\mathbb{Q}}(A)\\ f &\mapsto f^\dagger := \phi_{\mathcal{L}}^{-1} f^\vee \phi_{\mathcal{L}}.\end{aligned}$$

In fact, $R = \mathrm{End}_{\mathbb{Q}}(A)$ is a semisimple finite-dimensional algebra over $\mathbb{Q}$. Moreover, a finite-dimensional $\mathbb{Q}$-algebra R is simple if and only if $R \simeq \mathcal{M}_n(\Delta)$ where $n \geq 1$ and Δ is a finite-dimensional division $\mathbb{Q}$-algebra.

There exists a result of Albert classifying the possibilities for Δ. See [35, 8.5.4].

In the next section, we will study the degree of an endomorphism. First, we need the following:

Definition 1.10. An *Abelian subvariety* X of an Abelian variety A is a closed irreducible subvariety of A that is also an algebraic group by inheriting the group structure of A. Over $\mathbb{C}$, if $A = \mathbb{C}^n/\Lambda$, an Abelian subvariety is given by a vector subspace $W \leq \mathbb{C}^n$ such that $W \cap \Lambda$ is a lattice in W; the subgroup $X = W/W \cap \Lambda$ is thus an Abelian subvariety of $\mathbb{C}^n/\Lambda$. If $\mathcal{L}$ is an ample line bundle on A, then the restriction $\mathcal{L}|_X$ is ample on X. Thus, every Abelian subvariety of an Abelian variety with a polarization inherits the structure of a polarized Abelian variety.

Definition 1.11. An Abelian variety is *simple* if it does not contain an Abelian subvariety distinct from itself and zero.

To every Abelian subvariety X of $(A, \mathcal{L})$ we can canonically associate an Abelian subvariety called the *Abelian complement* of X in A with respect to the polarization $\mathcal{L}$. This is an Abelian subvariety of A such that its intersection with X is finite.

Theorem 1.12 (Poincaré's Complete Reducibility Theorem)**.** *Let X be an Abelian subvariety of A, and let Y denote the Abelian complement of X. Then the addition map $X \times Y \to A$ is an isogeny whose kernel is isomorphic to $X \cap Y$. Moreover, this shows that A is isogenous to a product $X_1^{m_1} \times \cdots \times X_s^{m_s}$, where the X_i are simple Abelian varieties not isogenous to each other.*

Proof. See [3, 5.3.7]. □

As a consequence, $\dim X + \dim Y = \dim A$. Moreover, we have a description of endomorphisms of Abelian varieties, in terms of simple algebras.

Corollary 1.13. *For A simple, the ring $\mathrm{End}_{\mathbb{Q}}(A)$ is a division ring. For any Abelian variety A, if $A = X_1^{n_1} \times \cdots \times X_r^{n_r}$ with X_i simple and pairwise* *non-isogenous, then*

$$\mathrm{End}_{\mathbb{Q}}(A) = \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{Q}}(X_i^{n_i}, X_i^{n_i}) = \bigoplus_{i=1}^{r} \mathcal{M}_{n_i}(\mathrm{End}_{\mathbb{Q}}(X_i))$$

where $\mathrm{End}_{\mathbb{Q}}(X_i)$ *is a division ring.*

Remark 1.14. Let X be an Abelian subvariety of the principally polarized Abelian variety $(A, \mathcal{L})$, and let Y the Abelian complement of X with respect to $\mathcal{L}$. If $n = \dim X \leq \dim Y = m$ and the type of X is $(d_1, \dots, d_n)$ then for Y the type is $(1, \dots, 1, d_1, \dots, d_m)$ where the number of 1's is $m - n$. It follows that $\chi(\mathcal{L}|_X) = \chi(\mathcal{L}|_Y)$, are both equal to the product $d_1 \cdots d_m$.

Moreover, let X_1, X_2 be Abelian varieties with polarizations $\mathcal{L}_1, \mathcal{L}_2$ of degrees d_1, d_2, respectively. Then $p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$ is a polarization on $X_1 \times X_2$ of degree $d_1 d_2$. In the next section, we will generalize this fact.

To simplify our study, we define the following:

Definition 1.15. Let $(A, \mathcal{L})$ be a polarized Abelian variety and let S be an Abelian subvariety of A of dimension d. We define the *reduced degree* of S to be

$$\deg_r(S) := \chi(\mathcal{L}|_S) = \frac{(\mathcal{L}|_S{}^d)}{d!},$$

where $(\mathcal{L}|_S{}^d)$ is the intersection number $S \cdot \mathcal{L}^d$.

For simplicity of the calculations, we consider the Euler characteristic of the line bundle associated to the restricted line bundle. We recall that from Riemann-Roch Theorem, for a divisor D on A, its Euler $\chi(D)$ is the square root of $\deg \phi_{\mathcal{O}_A(D)}$.

1.4 Degree of an endomorphism and complex multiplication.

In this section, we briefly review some useful results on the degrees of endomorphisms of Abelian varieties. We are especially interested in the endomorphisms of the self-product of an Abelian variety, which are related to semisimple algebras and norms. For this theory, we consult [35].

The following results are the simplest cases. However, they are illustrative enough to be considered.

Proposition 1.16. *Let A be an Abelian variety of dimension g. For every positive integer n, the degree of multiplication $[n]_X : X \to X$ by an integer is n^{2g}.*

Proof. [20, Proposition 1.5.17] □

Proposition 1.17. *Let X be an Abelian variety of dimension g. If $\alpha \in M_n(\mathbb{Z})$ then the induced endomorphism $[\alpha]_X : X^n \to X^n$ has degree* $\det(\alpha)^{2g}$.

Proof. If $\det(\alpha) = 0$ then it is readily seen that $[\alpha]_X$ has an infinite kernel, so by convention we have that its degree is zero. Now assume $\det(\alpha) \neq 0$. Let $\{v_1, \ldots, v_n\}$ be the ordered basis of $\mathbb{Z}^n$ in which α is written. By the theory of elementary divisors, there exist a symplectic basis of X which induces an ordered basis $\{f_1, \ldots, f_n\}$ for $\mathbb{Z}^n$ and a sequence of nonzero integers $(m_1, \ldots, m_n)$ such that $\alpha(v_i) = m_i f_i$ and $S, T \in \mathrm{GL}_n(\mathbb{Z})$ be the matrices with $\alpha = SBT$, where $B = \mathrm{diag}(m_1, \ldots, m_n)$ be the diagonal matrix with coefficients m_i is the rational representation of $[\alpha]_X$. Then $[S]_X, [T]_X$ are automorphism of X^n and $[B]_X : X^n \to X^n$ is given by $(x_1, \ldots, x_n) \mapsto (m_1 x_1, \ldots, m_n x_n)$, has degree $(m_1 \cdots m_n)^{2g} = \det(\alpha)^{2g}$. □

This result is particularly useful for the endomorphisms of a general elliptic curve E. In fact, Kani in [18, Cor. 65] proved that $\deg([\alpha]_E) = \det(\alpha)^2$ when $g = 1$ and $n = 2$, using an explicit description of the degrees. This description can be obtained by using an explicit isomorphism between the Néron-Severi group of E^2 and the group $\mathbb{Z}^2 \oplus \mathrm{End}(E)$. Guerra in [11] used this idea to find the characterization of elliptic curves of bounded degree in an Abelian variety.

We will generalize Proposition 1.17 for an elliptic curve with complex multiplication and $n = 1$. Let $\mathrm{Nm}_{\mathbb{C}/\mathbb{Q}}$ be the norm defined as the determinant of a matrix, as follows

$$\mathrm{Nm}_{\mathbb{C}/\mathbb{Q}}(a + bi) = \begin{vmatrix} a & -b \\ b & a \end{vmatrix}.$$

Proposition 1.18. *Let E be an elliptic curve over $\mathbb{C}$ with complex multiplication by the ring of integers $R_K \cong \mathrm{End}(E)$ of a quadratic imaginary field K. For all $\alpha \in R_K$ the endomorphism $[\alpha] : E \to E$ has degree* $|\,\mathrm{Nm}_{K/\mathbb{Q}}(\alpha)|$

Proof. [32, Cor. 1.5] □

Let k be a field and X an Abelian variety defined over k. For m an integer, let X_m denote the group of m-torsion points of X. Let p be a prime number different from the characteristic of k. Let $T_p(X)$ denote the projective limit of the groups X_{p^l} with respect to the map

$$X_{p^{l+1}} \to X_{p^l},$$

induced by multiplication by p. That is,

$$T_p(X) = \varprojlim X_{p^l},$$

called the p-adic Tate module of X over the p-adic integers $\mathbb{Z}_p$.

Theorem 1.19. *Let f be an endomorphism of an Abelian variety X of dimension g, and $T_p(f)$ the induced endomorphism of $T_p(X)$ ($p \neq \operatorname{char} k$). Then*

$$\deg f = \det T_p(f),$$

hence

$$\deg(n \cdot 1_X - f) = P(n),$$

where $P(t)$ is the characteristic polynomial, $\det(t - T_p(f))$ of $T_p(f)$. The polynomial P is monic of degree $2g$, has integral coefficients, and $P(f) = 0$.

Proof. [25, Theorem IV.4] □

The function $f \to \det(T_p(f))$ extends uniquely to a norm Nm of degree $2g$ on the semisimple $\mathbb{Q}_p$-algebra $\mathbb{Q}_p \otimes_{\mathbb{Z}} \operatorname{End}(X)$, where $\mathbb{Q}_p$ is the field of fraction of $\mathbb{Z}_p$.

In general, we can relate the degree of a homomorphism with some induced determinant using that the algebra of endomorphisms of an Abelian variety is a semisimple algebra. The construction is a little bit trickier for non-commutative division algebras than for the commutative case, because it is not possible to define the determinant a priori. For a detailed construction, see [35, 8.5] or [25, IV].

Let R be a simple finite dimensional $\mathbb{Q}$-algebra with center Z, then $R \simeq \mathcal{M}_r(\Delta)$ for Δ a finite dimensional division ring over $\mathbb{Q}$ and we let m denote the Schur index of Δ.

First, we define the reduced norm for a finite dimensional simple $\mathbb{Q}-$algebra $R \simeq \mathcal{M}_r(\Delta)$ to $\mathbb{Q}$. Let Z be the center of Δ and choose an extension field F of Z, such that there is an isomorphism $\iota : F \otimes_Z R \to \mathcal{M}_{rm}(F)$ of $F-$algebras. Then for a $\alpha \in R$ let $\operatorname{Nrd}_{R/Z}(\alpha)$ the determinant of the image of $1 \otimes \alpha$ under the isomorphism ι.

We define the norm from Z to $\mathbb{Q}$ as follows: let $\sigma_1, \ldots, \sigma_t$ be the distinct embeddings of Z into $\mathbb{C}$. For $\alpha \in Z$ then $\operatorname{Nr}_{Z/R}(\alpha) := \prod_{i=1}^{t} \sigma_i(\alpha)$.

Finally, we define the *reduced norm* from R to $\mathbb{Q}$ by the composition $\operatorname{Nrd}_{R/\mathbb{Q}}(\cdot) := \operatorname{Nr}_{Z/\mathbb{Q}}(\circ) \operatorname{Nr}_{R/Z}(\cdot)$.

The decomposition of the abelian variety $A = X_1^{n_1} \times \cdots \times X_r^{n_r}$ with X_i simple and pairwise non-isogenous, allows us to decompose the division ring

$$\operatorname{End}_{\mathbb{Q}}(A) = R_1 \times \cdots \times R_r,$$

where $R_i = \mathcal{M}_{n_i}(\operatorname{End}_{\mathbb{Q}}(X_i))$ and $\operatorname{Nrd}_{R_i/\mathbb{Q}}(\cdot)$ is the reduced norm from R_i to $\mathbb{Q}$ as above, for every $1 \leq i \leq r$.

Corollary 1.20. *Let f be an endomorphism of A as above, then*

$$\deg(f) = \prod_{j=1}^{r} \operatorname{Nrd}_{R_i/\mathbb{Q}}(f_j)^{s_j} = \prod_{j=1}^{r} \det(f_j)^{s_j},$$

where $f = (f_1, \ldots, f_r)$ *and some* $s_j \geq 1$ *with* $1 \leq j \leq r$.

Corollary 1.21. *Let* X *be a simple Abelian variety of dimension* g, Z *the center of the algebra* $R = \mathrm{End}_{\mathbb{Q}}(X)$, $[Z : \mathbb{Q}] = e$, $[\mathrm{End}_{\mathbb{Q}}(X) : Z] = d^2$, *then* ed *divides* $2g$. *Moreover, for* $A = X^n$, *let* f *be an endomorphism of* A. *Then*

$$\deg(f) = \mathrm{Nrd}_{R/\mathbb{Q}}(f)^{2g/ed}.$$

Note that for an Abelian variety of dimension g such that $\mathrm{End}_{\mathbb{Q}}(X) = \mathbb{Q}$ we recover Proposition 1.17.

Proposition 1.22. *Let* X *be a simple principally polarized Abelian variety with endomorphism algebra* $\mathrm{End}_{\mathbb{Q}}(X)$ *and let* n *be a positive integer. If* $\alpha \in M_n(K)$ *then* $[\alpha]_X : X^n \to X^n$ *has degree* $|\,\mathrm{Nm}_{K/\mathbb{Q}}(\det(\alpha))|$.

Proof. See [9, 7] □

Chapter 2

The Grassmannian and the Stiefel manifold

In this section, we will introduce the Grassmannian. We use [31] and [13] as references. We will describe some properties and results that will be useful in what follows. While a projective space gives a geometric structure to the set of one-dimensional subspaces of a vector space, a Grassmann variety does this for the set of $n-$dimensional subspaces, for n fixed.

The Grassmannian may be viewed from differential and algebraic perspectives and appears in multiples contexts. Additionally, there are several ways to develop this object. In particular, we are interested in its construction from the Stiefel variety, which will appear naturally in our study of Abelian subvarieties.

Let K be an algebraically closed field, V a $g-$dimensional vector space over K.

Definition 2.1. Let n be a positive integer such that $n < g$. *The Grassmannian* $\mathrm{Gr}(n, V)$ is defined to be the set of $n-$dimensional linear subspaces of V. That is,

$$\mathrm{Gr}(n, V) := \{W \subset V : W \text{ is a linear subspace, } \dim W = n\}.$$

We will sometimes write $\mathrm{Gr}(n, K^g)$ instead.

Let us assume that some basis has been chosen in the vector space V. Let W be an $n-$dimensional linear subspace of V and let us choose an arbitrary basis $v_1, \ldots, v_n$ of the subspace. The vector v_i has coordinates $v_{i1}, \ldots, v_{ig}$ $(i = 1, \ldots, n)$ in the chosen basis of the space V, which can be arranged in the form of a matrix M of order $g \times n$. Then, given $W \in \mathrm{Gr}(n, V)$ we may represent it by a set of n column vectors in K^g spanning W, that is by a $g \times n$

matrix

$$[W] := \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{g1} & \cdots & v_{gn} \end{pmatrix} \in \mathcal{M}_{g\times n}(K).$$

Definition 2.2. We define the (non-compact) *Stiefel variety* as

$$\mathrm{St}(n, K^g) := \{M \in \mathcal{M}_{g\times n}(K) : \ \mathrm{rank}(M) = n\}.$$

Note that $\mathrm{St}(n, K^g)$ is an open subset of $\mathcal{M}_{g\times n}(K)$ as the preimage of the open set $K \setminus \{0\}$ under the continuous map $M \mapsto \det(M^t M)$. Clearly any such matrix represents an element of $\mathrm{Gr}(n, K^g)$ and any two such a matrices A, A' represent the same element of $\mathrm{Gr}(n, K^g)$ if and only if $A\sigma = A'$ for some $\sigma \in \mathrm{GL}(n, K)$. Then, there is a right action of the general linear group $\mathrm{GL}(n, K)$ on $\mathrm{St}(n, K^g)$.

Moreover, there is a natural projection

$$p : \mathrm{St}(n, K^g) \to \mathrm{Gr}(n, K^g)$$

from the Stiefel variety $\mathrm{St}(n, K^g)$ to the Grassmannian of $n-$dimensional subspaces in K^g which sends a matrix with maximal rank to the subspace spanned by the columns of the matrix and $\mathrm{Gr}(n, K^g)$ can be identified with the set of orbits $\mathrm{St}(n, K^g)/\mathrm{GL}(n, K)$. For more details on this construction, we recommend [31, 10].

Since the number of columns of a matrix M is equal to n, a minor of order n is uniquely defined by the indices of its rows. Let $I = \{i_1, \dots, i_n\} \subset \{1, \dots, g\}$ with $i_1 < \cdots < i_n$. We denote by W_I the $n \times n$ submatrix of $[W]$ formed with the ith rows, for $i \in I$. The determinant of these matrices are called the *Plücker coordinates* of the n-dimensional subspace $W \subset V$. Let us denote by $N = \binom{g}{n}$ the number of all the combinations of such indices.

We assign to a subspace W the vector $(\det(W_I))_I$. The condition of linear independence of the vectors v_i implies that the rank of the matrix M is equal to n, so at least one of its minors of order n is nonzero. These coordinates characterize the subspace:

Theorem 2.3. *The Plücker coordinates of a subspace $W \subset V$ uniquely determine the subspace.*

Proof. [31, Theorem 10.2] □

These minors cannot take arbitrary values. Therefore, it is possible to describe the relations satisfied by the Plücker coordinates for an arbitrary subspace of dimension n on K^g, although it may not be straightforward to calculate them.

Theorem 2.4. *For every n-dimensional subspace W of a g-dimensional space V and for any two sets $(j_1, \ldots, j_{n-1})$ and $(k_1, \ldots, k_{n+1})$ of indices taking the values $1, \ldots, g$, such that $j_1 < \cdots < j_n$ and $k_1 < \cdots < k_n$, the following relationships hold:*

$$\sum_{\ell=1}^{n+1} (-1)^{\ell} \det(W_{j_1,\ldots,j_{n-1},k_\ell}) \det(W_{k_1,\ldots,\widehat{k_\ell},\ldots k_{n+1}}) = 0,$$

where $j_1, \ldots, \widehat{j_\ell}, \ldots j_{n+1}$ denotes the sequence $j_1, \ldots, j_{n+1}$ with the term j_ℓ omitted. These are called the Plücker relations*.*

Proof. [31, 10.4]. □

The relations are homogeneous of degree 2 with respect to the numbers W_I. Moreover, they define a projective variety in $\mathbb{P}^N_K$ called the *Grassmannian variety*, where $N = \binom{g}{n} - 1$. In fact, we can define the embedding known as the *Plücker map or Plücker embedding*, given by

$$\begin{aligned} \mathcal{P} : \mathrm{Gr}(n, K^g) &\to \mathbb{P}^N_K, \\ \langle v_1, \ldots, v_n \rangle &\mapsto [(\det(W_I))_I], \end{aligned}$$

where $I = \{i_1, \ldots, i_n\}$ with $1 \le i_1 < \cdots < i_n \le g$.

When $\dim(V) = 4$, and $n = 2$, for example, the above reduces to a single equation. Denoting the coordinates of $\mathbb{P}^5_K$ by a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}, the image of $\mathrm{Gr}(2, V)$ under the Plücker map is defined by

$$a_{12}a_{34} - a_{13}a_{24} + a_{23}a_{14} = 0.$$

In general, as we saw in the theorem above, many more equations are needed to define the Plücker embedding of a Grassmannian in projective space. Despite that we can say more about the Grassmannian:

Theorem 2.5. $\mathrm{Gr}(n, V)$ *is a non-singular variety of dimension $n(g - n)$.*

Proof. [13, Part I.6] □

Proposition 2.6. *Let $G(n, V)$ be the Grassmann variety parametrizing subspaces of dimension n of a vector space V of dimension g. Then its degree $\delta(n, g)$ with respect to the Plücker embedding is given by*

$$\delta(n, g) = \frac{(n(g-n))! \prod_{1 \le k \le n-1} k!}{\prod_{1 \le k \le n} (g-k)!}.$$

Proof. [15, Corollary 0.3] □

Chapter 3

Topics in Diophantine Geometry

In this section, we briefly introduce some general results from Diophantine geometry. The relevance of this approach lies in the fact that our objective can be translated to the study of a norm in a division algebra, as we saw in Section 1.4. In the particular case of a number field, it reduces to the study of heights in projective space.

For the general theory of heights in Diophantine geometry, we consult [26], [19] and [22].

3.1 Height in projective space.

First, we need to introduce and fix some notation and definitions due to the existence of several definitions of height functions, without mentioning the exponential and logarithmic version of each one of them.

There is a natural way to measure the size of a rational point a/b written in lowest terms, as the maximum between $|a|$ and $|b|$. To move to projective space, we define the classical or naive height, which is defined in terms of ordinary absolute value on homogeneous coordinates. Namely, the height of a point $P \in \mathbb{P}^m_{\mathbb{Q}}$ is defined by writing $P = [x_0 : \cdots : x_m]$ with $x_0, \ldots, x_m \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_m) = 1$ and setting

$$H(P) = \max\{|x_0|, \ldots, |x_m|\}.$$

Let t a positive real number. It is easy to see that there are only finitely many points $P \in \mathbb{P}^m_{\mathbb{Q}}$ with height $H(P) \leq t$ by a simple calculation. Although this fact is simple, it is nevertheless important.

This notion of height can be generalized to number fields. Let K be a number field of degree d over $\mathbb{Q}$. If two absolute values define the same topology, they are called equivalent. A place v is an equivalence class of non-trivial absolute values.

Let L be an extension of K and v a place of K. For a place v of L, we write $v|u$ if the restriction to K of any representative of v is a representative of u. We say that v extends u.

We write K_v for the extension field of K which is the completion of K with respect to v. We recall that the places of $\mathbb{Q}$ are indexed by the prime number and ∞ and their associated completions are the p-adic numbers and the real numbers, respectively.

Then K_v is a local field and a finite extension of $\mathbb{Q}_u$. For the local degree at the place v we write $d_v = [K_v : \mathbb{Q}_u]$. We also write $\mathbb{Q}_v$ for the closure of $\mathbb{Q}$ in K_v. Moreover, we have

$$\sum_{v|u} d_v = d = [K : \mathbb{Q}]$$

(see [5, 1.3.2]).

Proposition 3.1. *Let L/K be a finite-dimensional Galois extension with Galois group $G = Gal(L/K)$ and let $|\cdot|_{w_0}$, $|\cdot|_w$ be absolute values on L extending $|\cdot|_v$. Then there is an element $\sigma \in G$ with $|x|_w = |\sigma(x)|_{w_0}$ for $x \in L$. The completions L_w and L_{w_0} are isomorphic over K_v. However, they need not be isomorphic over L.*

Proof. [5, 1.3.5]. □

For $\alpha \in K$ we define the norm as

$$N_{K/\mathbb{Q}}(\alpha) = \prod_{v|u} N_{K_v/\mathbb{Q}_u}(\alpha),$$

where the sum and product are over the set of all places v of K such that $v|u$. See [5, 1.3.7].

If u is a place of $\mathbb{Q}$ then at each place v of K such that $v|u$ we define

$$\|x\|_v := |N_{K_v/\mathbb{Q}_u}(x)|_u^{d_v} \text{ for } x \in K.$$

Let M_K be the set of all non-trivial inequivalent absolute values on K such that the set

$$\{|\cdot|_v \in M_K : |x|_v \neq 1\}$$

is finite for any $x \in K \setminus \{0\}$. Thus M_K contains an archimedean absolute value for each embedding of K into $\mathbb{R}$ or $\mathbb{C}$ and a $p-$adic absolute value $|\cdot|_p$ for each prime ideal in the ring of integers of K. We identify the elements of M_K with the corresponding places. For each non-zero rational number x, we have

$$\prod_{v \in M_K} |x|_v = 1,$$

This is known as the product formula.

Definition 3.2. In this context, the height of a point $P = [x_0 : \cdots : x_m] \in \mathbb{P}^m_K$ is defined by

$$H_K(P) = \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_m\|_v\}.$$

We call H_K the multiplicative height of P.

Lemma 3.3. *Let K be a number field and let $P \in \mathbb{P}^m_K$ be a point.*

1. *The height $H_K(P)$ is independent of the choice of homogeneous coordinates for P.*
2. *$H_K(P) \geq 1$ for all $P \in \mathbb{P}^m_K$.*
3. *Let K' be a finite extension of K. Then*
$$H_{K'}(P) = H_K(P)^{[K':K]}.$$
4. *When $K = \mathbb{Q}$, the multiplicative height $H_{\mathbb{Q}}(P) = H(P)$ is the naive height.*

Proof. See [5, Chap. 1]. □

Theorem 3.4 (Northcott)**.** *Given $t \in \mathbb{R}_{\geq 0}$, there are only finitely many $P \in \mathbb{P}^m_K$ with $H_K(P) \leq t$.*

Proof. See [26]. □

The set of $K-$rational points on $\mathbb{P}^m_K$ of height bounded by t, and asymptotics for its cardinality were found by Schanuel [28] in 1979. He obtained the estimate $C_{K,m}t^{m+1}$ for some constant depending only on m and on classical invariants of K: the discriminant, class number, regulator and roots of unity contained in K.

Theorem 3.5 (Schanuel)**.** *Let K be a number field of degree d over $\mathbb{Q}$, then*

$$\#\{P \in \mathbb{P}^m_{\mathbb{Q}} \colon H_K(P) \leq t\} = C_{K,m}t^{m+1} + \mathcal{O}(t^{m+1-1/d}).$$

The constant $C_{K,m}$ is given explicitly by

$$C_{K,m} = \frac{h_K R_K / w_K}{\zeta_K(m+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\Delta_K}} \right)^{m+1} (m+1)^{r_1+r_2-1}$$

where

- *Δ_K is the absolute value of the discriminant Disc_K of K, h_K is the number of ideal classes of K.*

- *R_K is the regulator of K.*
- *w_K is the order of the group of unity in K.*
- *r_1, r_2 are the number of real and complex embbedings, respectively, of $\mathbb{Q}$ to K, that is $[K : \mathbb{Q}] = r_1 + 2r_2$.*
- *ζ_K is the Dedekind zeta function.*

Proof. See [28, Theorem 1]. □

We recall that the $\mathcal{O}$-notation means that we can approximate the number of points by calculating $C_{K,m}t^{m+1}$ and the error will be bounded from above by a positive constant muliple of $t^{m+1-1/d}$ as t goes to infinity.

In the special case $K = \mathbb{Q}$ we have that $\Delta_{\mathbb{Q}} = r_1 = R_{\mathbb{Q}} = h_K = 1$, $w_K = 2$, $r_2 = 0$ and ζ_K is the Riemann zeta function. By the theorem,

$$\#\{P \in \mathbb{P}^m_{\mathbb{Q}} \colon H(P) \le t\} = \frac{2^m}{\zeta(m+1)} t^{m+1} + \mathcal{O}(t^m).$$

Since

$$\zeta(m+1) = 1 + 2^{-(m+1)} + 3^{-(m+1)} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^{m+1}},$$

then

$$\#\{P \in \mathbb{P}^m_{\mathbb{Q}} \colon H(P) \le t\} \le 2^m t^{m+1} + \mathcal{O}(t^m).$$

Also, we will be interested in the quadratic case, that is $d = 2$. Despite the fact that we have an estimate, the constant $C_{K,m}$ can be difficult to calculate. Fortunately, in some cases we can give a uniform constant, which avoids the terminology of algebraic number theory.

For instance, the constant can be rewritten as

$$C_{K,m} = \mathrm{Res}_{s=1}(\zeta_K) \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\Delta_K}} \right)^m \frac{(m+1)^{r_1+r_2-1}}{\zeta_K(m+1)},$$

where $\mathrm{Res}_{s=1}(\zeta_K)$ is the residue of ζ_K at $s = 1$. Then we can apply some inequalities from number theory, for example [21, Theorem 1]

$$\mathrm{Res}_{s=1}(\zeta_K) \le \frac{e \log(\Delta_K)}{2}.$$

However, Schmidt in [30] gives some explicit bounds, which may be more interesting in our context. Let K be a number field of degree d over $\mathbb{Q}$, we denote by $d(P)$ the degree of the extension of $\mathbb{Q}$ by adjoining the quotients x_i/x_j, for $0 \le i, j \le n$, $x_j \neq 0$, where $P = [x_0 : \cdots : x_m]$. Schmidt proved:

$$Z(K, d, m, t) := \#\{P \in \mathbb{P}^m(K) \colon d(P) = d, \quad H(P) \le t\} \le c_1 t^{d(d+m)}.$$

Theorem 3.6. *We have*

$$Z(K,d,m,t) \le c_2(K,d,m)t^{d+m}$$

$$c_3(K,m)t^{m+1} \le Z(K,d,m,t) \text{ when } t_1(K,d,m) < t,$$
$$c_4(K,d)t^{d+1} \le Z(K,d,m,t) \text{ when } t_2(K,d) < t.$$

with $c_2.c_3, c_4$ *some positive constants and for some numbers* $t_1(K,d,m), t_2(K,d,m)$. *For* $K = \mathbb{Q}$ *we have the explicit lower bounds*

$$\frac{1}{4}t^{m+1} \le Z(\mathbb{Q},1,m,t) \text{ when } 1 \le t,$$
$$6^{-d(d+1)}t^{d+1} \le Z(\mathbb{Q},d,m,t) \text{ when } 2 \le t.$$

Proof. [30, Theorem, p.170] □

Theorem 3.7. *Let* K *be a number field of degree* d *over* $\mathbb{Q}$*, then*

$$\#\{P \in \mathbb{P}^m_K : d(P) \le d, \quad H_K(P) \le t\} \le c_1 t^{d(m+d)},$$

with $c_1 = c_1(m,d) = 2^{(2d+m)(d+m+10)}$.

Proof. [30, Theorem, p.170] □

Now, we will show some technical results that we will need in the next chapter.

Proposition 3.8. *Let* $\phi : \mathbb{P}^n_K \to \mathbb{P}^m_K$ *be a rational map of degree* η *defined over* $\mathbb{Q}$*. Let* $Z \subset \mathbb{P}^n_K$ *be the closed subvariety of points where* ϕ *is not defined. Then*

1. $H_K(\phi(P)) \le CH_K(P)^\eta$*, for all* $P \in \mathbb{P}^n_{\bar{\mathbb{Q}}}$*, for some constant* C*.*
2. *Let* X *be a subvariety contained in* $\mathbb{P}^n_K \backslash Z$*, then* $H_K(\phi(P)) = C'H_K(P)^\eta$*, for all* $P \in X(\bar{\mathbb{Q}})$ *for some* C'*.*

The second statement does not hold in general on the whole of $\mathbb{P}^n_K \setminus Z$*.*

Proof. See [14, Theorem B.2.5] which is written in logarithmic notation. □

Corollary 3.9. *Let* $\tau : \mathbb{P}^n_K \to \mathbb{P}^m_K$ *be a linear map defined over* $\mathbb{Q}$*, given by* $(L_0, \dots, L_m)$*. Let* $Z \subset \mathbb{P}^n_K$ *be the linear subspace where the* $L_0, \dots, L_m$ *simultaneously vanish, and let* $X \subset \mathbb{P}^n_K$ *be a closed subvariety with* $X \cap Z = \emptyset$*. Then*

$$H_K(\tau(P)) = C_{\tau,X} H_K(P), \text{ for all } P \in X(\bar{\mathbb{Q}}).$$

With these arithmetic relations between heights and rational maps, we can state relations from a Diophantine point of view.

Proposition 3.10. *Let $X \hookrightarrow \mathbb{P}^n_K$ be a projective variety of degree δ (with respect to the universal bundle) and dimension ρ. Then*

$$N(X,t) := \#\{P \in X(\mathbb{Q})\colon H(P) \le t\} \le C_X \delta N(\mathbb{P}^\rho_{\mathbb{Q}}, t).$$

Proof. Let Z a linear subspace over $\mathbb{Q}$ of dimension $n-\rho-1$ and disjoint from X, and let $\pi : \mathbb{P}^n_{\mathbb{Q}} \setminus Z \to \mathbb{P}^\rho_{\mathbb{Q}}$ be the corresponding linear projection. Consider a linear automorphism $\sigma \in PGL(\rho, \mathbb{Q})$ such that $\sigma(Z)$ is the standard linear subspace Z_0 defined by the vanishing of the last $n-\rho-1$ coordinates.

Now, we can consider the projection $\pi' := \pi \circ \sigma^{-1} : \mathbb{P}^n_{\mathbb{Q}} \setminus Z_0 \to \mathbb{P}^\rho_{\mathbb{Q}}$. It is independent of X, hence all $P \in \mathbb{P}^n_{\mathbb{Q}}$ satisfy $H(\pi'(P)) \le C_X H(P)$, with a constant C_X from the previous corollary which is independent of P. The original projection, $\pi = \pi' \circ \sigma$, thus sends

$$\{P \in X(\mathbb{Q})\colon H(P) \le t\}$$

at most $\delta : 1$ to the set

$$\{P \in \mathbb{P}^\rho_{\mathbb{Q}}\colon H(\sigma' \cdot P) \le C_X \cdot t\}$$

where $\sigma' \in PGL(\rho, \mathbb{Q})$ is a certain element (induced by σ^{-1}). But the last set is in bijection $Q \longleftrightarrow \sigma' Q$ with

$$\{P \in \mathbb{P}^\rho_{\mathbb{Q}}\colon H(P) \le C_X \cdot t\} = C_X \#\{P \in \mathbb{P}^\rho_{\mathbb{Q}}\colon H(P) \le t\}.$$

□

3.2 Heights on Grassmannians.

Although, it is possible to count the number of rational points on the Grassmannian with bounded height using the proposition above, there are other choices for the height which can give us a different estimation using the fact that the Grassmannian is a Fano variety. See [29].

Let M denote the set of absolute values on $\overline{\mathbb{Q}}$ extending $M_{\mathbb{Q}}$.

Definition 3.11. For $x \in \overline{\mathbb{Q}}^{g+1}$ and $u \in M$, we set

$$H_u(x) := \begin{cases} \max_j |x_j|_u & \text{if } u \text{ is non-archimedean,} \\ \left(\sum_{j=0}^n |x_j|_u^2\right)^{1/2} & \text{if } u \text{ is archimedean} \end{cases}$$

Let $F \subset \overline{\mathbb{Q}}$ be a number field. Then, for $x \in F^{g+1}$ and $w \in M_F$, we define

$$H_w(x) := H_u(x)^{[F_w:\mathbb{Q}_p]/[F:\mathbb{Q}]},$$

where $p \in M_{\mathbb{Q}}$ and $u \in M$ are such that $w|p$ and $u|w$.

Now we define the *Arakelov height* for $P \in \mathbb{P}^g_{\overline{\mathbb{Q}}}(F)$ with representative $x \in F^{g+1}$ by

$$h_{Ar}(P) := \sum_{w \in M_F} \log H_w(x)$$

and the multiplicative Arakelov height by $H_{Ar}(P) := \exp(h_{Ar}(P))$.

Let W be an $n-$dimensional subspace of $\overline{\mathbb{Q}}^g$. The nth exterior power $\wedge^n W$ is a one-dimensional subspace of $\wedge^n \overline{\mathbb{Q}}^g$. Therefore, we may view W as a point P_W of the projective space $\mathbb{P}(\wedge^n \overline{\mathbb{Q}}^g)$. The latter may be identified with projective space of dimension $\binom{g}{n} - 1$ by using standard coordinates.

Definition 3.12. $h_{Ar}(W) := h_{Ar}(P_W)$ is called the *Arakelov height of* W.

Let A be a matrix in $\mathrm{St}(n, \overline{\mathbb{Q}}^g)$. Then the *Arakelov height* $h_{Ar}(A)$ of A is defined as the Arakelov height of the subspace of $\overline{\mathbb{Q}}^n$ spanned by the columns of A.

We can give a more explicit definition for $h_{Ar}(A)$. We already know that the point in $\mathbb{P}^N_{\overline{\mathbb{Q}}}$, where $N = \binom{g}{n} - 1$, corresponding to A is given by the coordinates $\det(A_I)$, where I ranges over all subsets of $\{1, \dots, g\}$ of cardinality n. Let $F \subset \overline{\mathbb{Q}}$ be a number field containing all entries of A and set

$$H_u(A) := \begin{cases} \max_I |\det(A_I)|_u & \text{if } u \text{ is non-archimedean,} \\ \left(\sum_I |\det(A_I)|^2\right)^{1/2} & \text{if } u \text{ is archimedean} \end{cases}$$

Now, for $w \in M_F$, we set for an $g \times n$ matrix of rank n

$$H_w(A) = H_u(A)^{[F_w:\mathbb{Q}_p]/[F:\mathbb{Q}]},$$

where $p \in M_{\mathbb{Q}}$ and $u \in M$ are such that $w|p$ and $u|w$. Then

$$h_{Ar}(A) = \sum_{w \in M_F} \log H_w(A).$$

It is clear that $h_{Ar}(AG) = h_{Ar}(A)$ for any matrix $G \in \mathrm{GL}(n, \overline{\mathbb{Q}})$.

Proposition 3.13. *Let $u \in M$ be an archimedean place. Then*

$$H_u(A) := |\det(A^*A)|_u^{1/2}$$

where $A^* = \overline{A}^t$ is the conjugate transpose.

Proof. This follows from the Binet formula

$$\det(A^*A) = \sum_I |\det(A_I)|^2.$$

□

The Picard group $\mathrm{Pic}(\mathrm{Gr}(n, K^g))$ is isomorphic to $\mathbb{Z}$ and it is generated by the line bundle $L = \mathcal{P}^*\mathcal{O}(1)$, where $\mathcal{P}$ is the Plücker map. The anticanonical bundle is $\omega^{-1} = L^{\otimes g}$. Then $\mathrm{Gr}(n, K^g)$ is Fano and allows one to define an anticanonical height function $H = H_{\omega^{-1}}$ on $\mathrm{Gr}(n, K^g)(\mathbb{Q})$ the set of $\mathbb{Q}$–rational points.

Let $B \in \mathrm{Gr}(n, K^g)$. The covolume of the lattice $\Lambda_B =< \{b_1, \dots, b_n\} >$ is given by

$$\mathrm{covol}(\Lambda_B) = \sqrt{\det(\langle b_i, b_j\rangle)_{i,j}} = \sqrt{\det(B^t B)},$$

where $B = (b_1 \dots b_n)$ is the matrix whose columns are the vectors b_i. Then we define $H(B) = \mathrm{covol}(\Lambda_B)^g$. Schmidt [29] has shown that:

Proposition 3.14. *For any* $1 \leq n \leq g-1$

$$\#\{B \in \mathrm{Gr}(n, g)(\mathbb{Q})\colon H(B) \leq t\} = c_{n,g} t^g + \mathcal{O}(t^{g - \max\{\frac{1}{n}, \frac{1}{(g-n)}\}}),$$

where

$$c_{n,g} = \frac{1}{g}\binom{g}{n}\frac{V(g)V(g-1)\cdots V(g-n+1)}{V(1)V(2)\dots V(n)}\frac{\zeta(2)\cdots\zeta(n)}{\zeta(g)\cdots\zeta(g-n+1)}$$

and $V(m)$ *denotes the volume of the unit ball in the Euclidean space* $\mathbb{R}^m$*, for any* $m \in \mathbb{N}$.

Thunder in [33] has shown a generalization of the Schmidt's asymptotic formula, where the subspaces of $\mathbb{Q}^m$ are replaced by d-dimensional subspaces of K^n, where K is an algebraic number field of degree d over $\mathbb{Q}$.

Proposition 3.15. *For any* $1 \leq n \leq g-1$

$$\#\{B \in \mathrm{Gr}(n, g)(K)\colon H(B) \leq |\Delta|^{n/2} t\} = a(n, g) t^g + \mathcal{O}(t^{g - \max\{\frac{1}{dn}, \frac{1}{d(g-n)}\}}),$$

where $a(n, g)$ *is a positive constant.*

Proof. [33, Theorem 1] □

Chapter 4

Counting Abelian Subvarieties

As we said in the previous chapter, the aim of this work is to study the behavior of the number of Abelian subvarieties for a polarized Abelian variety of a given dimension and bounded (reduced) degree.

In a succession of papers, Guerra has investigated the number of elliptic curves on a polarized Abelian variety A by their degrees, for example in [10], [11] and [12]. He denotes by $N_A(t)$ the number of elliptic curves in A with degree bounded by t. This problem was studied by Bolza and Poincaré [27] in dimension two, and by Tamme [34] and Kani for Jacobian varieties [16].

The reason to define this number for arbitrary Abelian varieties is due to the following classical result proved by Birkenhake and Lange:

Theorem 4.1. *For any positive integer e there are only finitely many Abelian subvarieties of A of exponent less than or equal to e.*

Proof. [4, 4.1] □

The theorem above implies that $N_A(t)$ is finite, since the exponent divides the degree of an Abelian subvariety.

Definition 4.2. Given a polarized Abelian variety $(A, \mathcal{L})$, let us denote by

$$N_A(t, n) = \#\{S \text{ is an abelian subvariety of } A\colon\ \dim S = n,\ \chi(\mathcal{L}|_S) \leq t\},$$

the number of Abelian subvarieties of A with dimension $n > 1$ and reduced degree at most t.

This function has some pleasant properties. It is clear that $N_A(t, n)$ is an increasing function with respect to t. Moreover, we have the following:

Proposition 4.3. *For a principally polarized Abelian variety,*

$$N_A(t, n) = N_A(t, g - n),$$

since an Abelian subvariety and its Abelian complement have the same reduced degree.

Proof. This is a consequence of Remark 1.14. □

For a polarized Abelian variety of dimension g in the moduli space $\mathcal{A}_g$ and integers t and n, we associate the counting function of Abelian subvarieties of dimension n and reduced degree bounded by t by the following function

$$\begin{aligned} N : \mathcal{A}_g \times \mathbb{N}^2 &\to \mathbb{N} \\ (A, t, n) &\mapsto N_A(t, n). \end{aligned}$$

We would like to know where the function N achieves its maximum in the moduli space $\mathcal{A}_g$. We have good reason to suspect that the maximum is achieved for a self product of elliptic curves E^g. Unfortunately, we have been unable to prove this in a general case. We hope to develop this in the future in order to get a complete answer to this problem.

In section 4.1 we briefly show how Guerra attacked the problem.

In the second section, we will show some reductions to the problem of determining bounds for $N_A(t, n)$, then we will study the asymptotic behavior of the number of Abelian subvarieties of a principally polarized Abelian variety. Moreover, we can give an upper bound for $N_A(t, n)$.

In the following, we will concentrate on the expected maximum case. From the self product of elliptic curves, we will develop a construction of some associated variety whose rational points correspond to its Abelian subvarieties with bounded degree. Auffarth developed a similar idea in his doctoral thesis, where he studied some homogeneous forms on the Néron-Severi group. Instead of following that path, we will study the algebra of endomorphisms of E^g.

As we saw in the previous chapter, in the study of $k-$linear subspaces of a $g-$dimensional linear space, we defined the Grassmannian. In our context, we show the relation between the Grassmannian variety $\mathrm{Gr}(n, K^g)$ and the Abelian subvarieties of E^g to characterize the Abelian subvarieties of reduced degree in terms of endomorphisms of the initial elliptic curve. Then we use some classical results from Diophantine theory in order to find an estimate for the asymptotic behavior of $N_A(t, n)$.

4.1 Guerra's Results.

In this section, we will see the principal results obtained by Guerra, counting elliptic curves and Abelian subvarieties.

For a polarized complex Abelian variety A Guerra studied the function $N_A(t)$ counting the number of elliptic curves in A with degree bounded by t. Then for $n = 1$, $N_A(t) := N_A(\sqrt{t}, 1)$.

4.1.1 Elliptic curves of bounded degree in a polarized Abelian variety.

The next result refers to the self product $A = E^k$ of an elliptic curve, where k is greater than 2, endowed with a split polarization

$$L = m_1\Theta_1 + \cdots + m_k\Theta_k,$$

where Θ_i denotes the pullback to A of the principal polarization on the ith factor and the coefficients m_i are positive integers. First, we need to define the following:

- let $\mathfrak{m}$ be the minimum of the coefficients m_i.
- let $\mathfrak{d}$ be the minimum degree of a non-trivial isogeny $E \to E$.
- let δ be the (negative) discriminant of the degree form on the endomorphism group $\mathrm{End}(E)$, when the elliptic curve has complex multiplication.

Theorem 4.4. *There is an asymptotic estimate*

$$N_{E^k}(t) = Ct^r + \mathcal{O}(t^i),$$

where $r = k$ if the curve admits no complex multiplication, and $r = 2k-1$ if the curve has complex multiplication, the constant C being given by

$$\frac{2^k/(k+1)}{(\sqrt{\mathfrak{d}})^{k-1}\mathfrak{m}^k} \text{ for } r = k, \qquad \frac{(2\pi)^{k-1}/k}{(\sqrt{-\delta})^{k-1}\mathfrak{m}^{2k-1}} \text{ for } r = 2k-1,$$

and the exponent i being $k-1$ for $r = k$, $2k-3+2e$ for $r = 2k-1$, where $e = 33/104 = 0.317\ldots$

Proof. See [11, 1.1] □

To get this result, Guerra studied the collection of elliptic curves in an Abelian variety A, denote by $\mathrm{EC}(A)$. He considers the product $S \times F$, where S is an Abelian variety of any dimension and F is an elliptic curve. He described the subcollection of elliptic curves which are not entirely contained in $S \times \{0\}$ or $\{0\} \times F$. This set is bijectively parametrized by a certain set of parameter data, which is related to the Néron-Severi group. In terms of this data, he characterized the degree of an elliptic curve in $S \times F$ with respect to a split polarization.

Then he applied this parametrization to the self product E^g of an elliptic curve. In this particular case, an estimate of the counting function $N_A(t)$ can be obtained from an asymptotic study of the degree form $f \mapsto \deg(f)$ on the group of endomorphisms of the elliptic curve. This counting problem is related to the number of lattice points in a bounded region in the real plane, which has been extensively studied in Number Theory.

More precisely, let S be an Abelian variety, let F be an elliptic curve, and consider the product Abelian variety $A = S \times F$. We denote, for an arbitrary Abelian variety, by $\mathrm{EC}(A)$ the collection of homology classes $\gamma = [C]$ in $H_2(A, \mathbb{Z})$ corresponding to elliptic curves C in A.

Denote by $S_h := S \times \{0\}$ and by $F_v := \{0\} \times F$ the horizontal and the vertical factors in A.

If C is an elliptic curve in A, different from F_v, then $D = pr_1(C)$ is an elliptic curve in S, corresponding to an element $\gamma = [C]$ in the Néron-Severi group $\mathrm{NS}(D \times F)$. This group is described by an isomorphism

$$\mathbb{Z}^2 \oplus \mathrm{Hom}(D, F) \xrightarrow{\sim} \mathrm{NS}(D \times F).$$

Then, he defined the set which parametrizes the set of elliptic curves in $S \times F$ by

$$\mathrm{D}(S \times F) = \Big\{(u, v; g) \in \mathbb{Z}^2 \times \mathrm{Hom}'(F, S) \colon (u, v; g) \text{ is primitive,}$$

$$uv = \deg(g), \text{ and } u + v > 0\Big\},$$

where $(u, v; g)$ is primitive in the module $\mathbb{Z}^2 \times \mathrm{Hom}(F, S)$. He showed [11, 5.1] that there is a bijective correspondence between $\mathrm{D}(S \times F)$ and $\mathrm{EC}(S \times F) \setminus \big(\mathrm{EC}(Sh) \cup \{[F_v]\}\big)$ induced by the correspondence C defined above.

Now, consider $S = E_1 \times \cdots \times E_k$ a product of elliptic curves, endowed with a split polarization

$$\Theta_S = m_1\Theta_1 + \cdots + m_k\Theta_k,$$

where Θ_i denotes the pullback to A of the principal polarization on the ith factor and the coefficients m_i are positive integers. Then we have the following description of the degree function on $D(S \times F)$:

Theorem 4.5. *The degree function* $\mathrm{D}(S \times F) \to \mathbb{Z}$ *is given by*

$$\deg C(u, v; g) = uD(g) \cdot \Theta + nv.$$

In the particular case $S = E_1 \times \cdots \times E_k$*, one has the expression*

$$\deg C(u, v; g) = \frac{m_1 \deg(h_1) + \cdots + m_k \deg(h_k)}{v} + nv$$

where the homomorphism $g : F \to S$ *is given by a sequence* $h_1, \ldots, h_k$ *of homomorphisms* $h_i : F \to E_i$.

Proof. [11, Theorem 5.2] □

Let $A = E^k$, with $k \geq 2$, be the kth self product of an elliptic curve E, endowed with a split polarization $L = m_1\Theta_1 + \cdots + m_k\Theta_k$, where Θ_i denotes the pullback to A of the principal polarization on the ith factor and the coefficients m_i are positive integers. Now, let $(E^{k-1})_h := E^{k-1} \times \{0\}$ and $E_v := \{(0, \ldots, 0)\} \times E$. Let $\mathfrak{m}$ be the minimum among the coefficients $m_1, \ldots, m_k$. Using the bijection given above, Guerra studied the number $N_{E^k}(t)$. He obtained

$$\#\big\{(u,v;g) \in D(E^k) \colon \deg C(u,v;g) \le t\big\} \le \#\big\{(v;h_1,\ldots,h_k) \colon 1 \le v \le \frac{t}{\mathfrak{m}},\ \deg(h_i) \le v\frac{t}{\mathfrak{m}}\big\}$$
$$\le \sum_{1 \le v \le \frac{t}{\mathfrak{m}}} \Phi\left(\frac{vt}{\mathfrak{m}}\right)^{k-1},$$

where $\Phi(t)$ is the function which counts endomorphisms of E having degree bounded by t, where

$$\Phi_{E^k}(t) = \sum_{1 \le v \le \frac{t}{\mathfrak{m}}} \Phi\left(\frac{vt}{\mathfrak{m}}\right)^{k-1}.$$

Then Guerra reduced the problem to the study of Φ.

Lemma 4.6. *There is an asymptotic estimate*

$$\Phi_{E^k}(t) = Ct^r + \mathcal{O}(t^i)$$

with the same constant C and the same exponents r and i as in Theorem 4.4.

Finally, he used that

$$N_{E^k}(t) \le N_{E^{k-1}}(t) + 1 + \Phi_{E^k}(t)$$

to get the asymptotic bound.

4.1.2 Abelian subvarieties of bounded degree.

Recently, Guerra generalized his result in the following sense: Denote by the symbol AS_A the collection of all Abelian subvarieties of A. For every positive integer or real number t, he defined

$$\mathcal{N}_A(t) := \#\{S \in AS_A :\ \chi(\mathcal{L}|S) \le t\},$$

which counts the number of Abelian subvarieties S in A such that the Euler characteristic $\chi(\mathcal{L}|_S)$ is bounded above by t, for the induced polarization $\mathcal{L}|_S$. He makes explicit the relation between the rate of growth of the counting function $\mathcal{N}_A(t)$ and the complete decomposition of A up to isogeny into a product of simple Abelian varieties.

Remark 4.7. Note that $\mathcal{N}_A(t)$ considers the number of Abelian subvarieties with *reduced degree* $\chi(\mathcal{L}|_S)$ bounded by t. Guerra considers the degree for the elliptic curves and the reduced degree to count the number of Abelian subvarieties.

In order to get the asymptotic estimation of those Abelian subvarieties with degree bounded by t, it is sufficient to consider $\mathcal{N}_A(\sqrt{t})$, given that

$$\deg(\phi_{\mathcal{L}|_S}) = \chi(\mathcal{L}|S)^2 \leq t \quad \text{if and only if} \quad \deg_r(S) = \chi(\mathcal{L}|S) \leq \sqrt{t}.$$

Assume that the complete decomposition of A is given by an isogeny

$$A \sim B_1^{k_1} \times \cdots \times B_q^{k_q},$$

where $B_1, \ldots, B_q$ are simple Abelian varieties, mutually non-isogenous, and where $k_1, \ldots, k_q$ are positive integers. Define the integers:

- k, the maximum among the multiplicities k_j , and
- h the maximum rank of an endomorphism group $\mathrm{End}(B_j)$.

In terms of this data, in [12] Guerra proved the following:

Theorem 4.8. *There is an asymptotic estimate* $\mathcal{N}_{B^k}(t) = \mathcal{O}(t^{q(kh+2)(k-1)})$.

Corollary 4.9. *Let E be an elliptic curve and g a positive integer. There is an asymptotic estimate* $\mathcal{N}_{E^g}(t) = \mathcal{O}(t^{(g+2)(g-1)})$.

The strategy used to obtain these results is adjusting the method of his previous papers to the general situation as follows: Let (A, Θ) and (A', Θ') be polarized Abelian varieties, with A simple. Let n be the dimension of A and consider the product $A \times A'$ with the natural projections p, p' and endowed the split polarization $L = p^*\Theta + p'^*\Theta'$.

In order to reduce the problem, he found the following inequality

$$\mathcal{N}_{A\times A'}(t) \leq \mathcal{N}_A(t)\mathcal{N}_{A\times A'/p}(t^2),$$

where

$$\mathcal{N}_{A\times A'/p}(t) := \#\{S \in AS_{A\times A'} \, : \, p|_S : S \to A \text{ is an isogeny and } \chi(\mathcal{L}|S) \leq t\}.$$

Then Guerra found an asymptotic estimation to the cardinality of this set in terms of the rank h of homomorphism from A to A'. Finally, given an asymptotic estimation $\mathcal{N}_{A'}(t) = \mathcal{O}(t^r)$ Guerra proved that $\mathcal{N}_{A\times A'}(t) = \mathcal{O}(t^{r+2(h+1)})$.

4.2 Reductions of the problem.

In this section, we will exhibit a few reductions of the problem in order to be able to estimate the number of Abelian subvarieties of a given decomposable Abelian variety.

4.2.1 Behavior under isogenies.

Let A and B be Abelian varieties, and let $\phi : B \to A$ be an isogeny of degree d. Let L be a polarization on A and consider on B the pullback polarization $\phi^*(\mathcal{L})$. There is a one to one correspondence between the Abelian subvarieties of A and Abelian subvarieties of B.

Proposition 4.10. *Let A and B be Abelian varieties as above, then*

$$N_A(t,n) \leq N_B(dt,n) \text{ and } N_B(t,n) \leq N_A(t,n).$$

Proof. Given an Abelian subvariety $S \subset A$ the corresponding S^* in B is the connected component of 0 in the pre-image $\phi^{-1}(S)$. The restricted isogeny $S^* \to S$ has degree d_S a divisor of d. With respect to the polarization, one has $\chi((\phi^*\mathcal{L})|_{S^*}) = d_S\chi(\mathcal{L}|_S)$. So there is the chain of inequalities,

$$\chi(\mathcal{L}|_S) \leq \chi((\phi^*\mathcal{L})|_{S^*}) \leq d\chi(\mathcal{L}|_S).$$

It follows that the functions counting Abelian subvarieties in A and in B are related by the inequalities $N_A(t,n) \leq N_B(dt,n)$ and $N_B(t,n) \leq N_A(t,n)$. □

In addition to the Poincaré Complete Reducibility Theorem 1.12, the following classical result concerning isogenies will give us necessary information to simplify our problem.

Proposition 4.11. *Every polarized Abelian variety is isogenous to a principally polarized Abelian variety.*

Proof. See [6, 6.17] □

Remark 4.12. As a consequence of this and Poincaré's Complete Reducibility Theorem, we can reduce the study of the asymptotic behavior of the number of Abelian subvarieties to a product of simple principally polarized Abelian varieties.

4.2.2 Simple Abelian varieties.

Let A be an Abelian variety. By the Poincaré Complete Reducibility Theorem and Proposition 4.11, there exist simple principally polarized Abelian varieties $(X_i, \mathcal{L}_i)$ and integers n_i such that A is isogenous to the product $X_1^{n_1} \times \cdots \times X_r^{n_r}$ which is endowed with a polarization $\Theta = \boxtimes\ \theta_i$, where $\theta_i = \pi_i^*(\mathcal{L}_i)$ and π_i is the projection to the i-th factor.

From now on, we reduce the problem to $A = X_1^{n_1} \times \cdots \times X_r^{n_r}$ where each X_i is simple and principally polarized. Additionally, we can reduce the study of Abelian subvarieties of dimension n to the particular case when $\dim X_i \leq n$ for $1 \leq i \leq r$, given that we are assuming X_i simple.

Therefore, we have the following equality:

Proposition 4.13. *Let* $(A,\mathcal{L}) = (X_1^{n_1}\times\cdots\times X_r^{n_r}, \boxtimes\, \theta_i)$ *be a principally polarized Abelian variety as above and* t, n *positive integers. Then*

$$N_A(t,n) \leq \sum_{\substack{(m_1,\dots,m_r)\\ \sum_{i=1}^r m_i \dim X_i = n\\ m_i\leq n_i}} \prod_{i=1}^r N_{X_i^{n_i}}(t, m_i \dim X_i)).$$

Proof. Let Y be an Abelian subvariety of A of dimension n and degree $d\leq t$. Then we can find some integers m_i such that

$$Y \sim X_1^{m_1}\times\cdots\times X_r^{m_r},$$

with $0\leq m_i\leq n_i$ for every i and $\sum_{i=1}^r m_i \dim X_i = n$. Then, the factor $X_i^{m_i}$ is an Abelian subvariety of $X_i^{n_i}$ whose reduced degree respect to $\mathcal{L}_i$ is denoted by t_i for some integers t_i for every $1\leq i\leq r$. Thus, Y will have reduced degree $t_1\dots t_r = d$, in particular $t_i\leq t$.

□

Remark 4.14. Let $(A,\mathcal{L})$ be a polarized Abelian variety. From the Poincaré Reducibility Theorem, there exist X_i simple and pairwise non-isogenous such that

$$f : (X_1^{n_1}\times\cdots\times X_r^{n_r}, f^*\mathcal{L}) \to (A,\mathcal{L})$$

is an isogeny of degree d and where $f^*\mathcal{L} = \mathcal{L}_1^{\boxtimes n_1}\times\cdots\times\mathcal{L}_r^{\boxtimes n_r}$ and $\mathcal{L}_i = \mathcal{L}|_{X_i}$. Now, from Proposition 4.11, there exists an isogeny of degree d' of polarized Abelian varieties

$$h : \tilde{X}_1^{n_1}\times\cdots\times\tilde{X}_r^{n_r} \to X_1^{n_1}\times\cdots\times X_r^{n_r},$$

such that $h^*(f^*\mathcal{L}) \equiv (\mathcal{L}')^p$ with $\mathcal{L}'$ a principal polarization and some integer p. Then, there exists an isogeny

$$\gamma : \tilde{X}_1^{n_1}\times\cdots\times\tilde{X}_r^{n_r} \to A,$$

of degree d' such that $\gamma^*\mathcal{L}\equiv(\mathcal{L}')^q$ for some integer q.

If $X\subset A$ is an Abelian subvariety of reduced degree t, that is $\chi(\mathcal{L}|_X) = t$, then

$$\chi(\gamma^*\mathcal{L}|_{\gamma^{-1}(X)_0}) = \deg\gamma|_{\gamma^{-1}(X)_0}\chi(\mathcal{L}|_X)\leq d'\chi(\mathcal{L}|_X).$$

Finally,

$$\begin{aligned}\frac{q^{\dim X}(\mathcal{L}'|_{\gamma^{-1}(X)_0})^{\dim X}}{(\dim X)!} &= \frac{(\gamma^*\mathcal{L}|_{\gamma^{-1}(X)_0})^{\dim X}}{(\dim X)!}\\ &= \chi(\gamma^*\mathcal{L}|_{\gamma^{-1}(X)_0})\\ &\leq d'\chi(\mathcal{L}|_X).\end{aligned}$$

In consequence,

$$q^{\dim X}\chi(\mathcal{L}'|_{\gamma^{-1}(X)_0}) \leq d'\chi(\mathcal{L}|_X),$$

the reduced degree of $\gamma^{-1}(X)_0$ with respect to $\mathcal{L}'$ is less or equal to

$$\frac{d'}{q^{\dim X}}t \leq d't.$$

In terms of the counting function,

$$N_A(t,n) \leq N_{\tilde{X_1}^{n_1}\times\cdots\times\tilde{X_r}^{n_r}}(d't,n),$$

where $N_{\tilde{X_1}^{n_1}\times\cdots\times\tilde{X_r}^{n_r}}$ is counting with respect to the principal polarization $\mathcal{L}'$.

Now, we can reduce the problem to the self-product of a simple principally polarized Abelian variety.

In particular, we will develop the study for $A = E^g$ an elliptic curve, which is the expected maximum case for the function $N_A(t,n)$.

4.3 Abelian subvarieties on E^g.

Let E be an elliptic curve and g a positive integer. We denote by R the ring of endomorphisms $\mathrm{End}(E)$ and by $K = \mathrm{End}(E)\otimes_{\mathbb{Z}}\mathbb{Q}$ the algebra of $\mathbb{Q}-$endomorphisms.We consider the self product E^g with projections and the inclusions $\pi_i : E^g \to E$ and $\iota_i : E \to E^g$, respectively. Let us denote by Θ the divisor $\sum\theta_i$, where $\theta_i = \pi_i^*(0)$.

Let S be an Abelian subvariety of E^g of dimension n. By the Poincaré Reducibility Theorem, this subvariety is isogenous to E^n and can be obtained as the image of some homomorphism in $\mathrm{Hom}(E^n, E^g)$ defined by

$$\begin{aligned} f_S : E^n &\to E^g \\ (x_1,\ldots,x_n) &\mapsto \big(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n,\ldots,\alpha_{g1}x_1+\cdots+\alpha_{gn}x_n\big). \end{aligned}$$

In consequence, f_S can be identified with a matrix $M_S := (\alpha_{ij})$ in $\mathcal{M}_{g\times n}(R)$. Moreover, the rank of M_S must be n, in order to obtain an isogeny between E^n and the image of f_S. In fact, we have the following result.

Lemma 4.15. *Every matrix in* $\mathrm{St}(n,K^g)$ *determines an Abelian subvariety of* E^g *of dimension* n. *Moreover,* $M, M' \in \mathrm{St}(n,K^g)$ *determine the same Abelian subvariety if and only if* $MT = M'$ *for some* $T \in \mathrm{GL}(n,K)$.

Proof. Let M be a matrix in $\mathrm{St}(n,K^g)$, then we can multiply by an element h of R in order to get a matrix in $\mathrm{St}(n,R^g)$.

Moreover, the matrix hM determines an Abelian subvariety of E^g as the image of the homomorphism from E^n to E^g given by

$$(x_1\ldots x_n) \mapsto hM(x_1\ldots x_n)^t.$$

Since elliptic curves are divisible groups, this image does not depend on h and so each element of $\mathrm{St}(n, K^g)$ determines an Abelian subvariety of E^g of dimension n.

Additionally, let $M = \big(\alpha_{ij}\big), M' = \big(\beta_{ij}\big)$ be two different matrices in $\mathrm{St}(n, K^g)$ such that they determine the same Abelian subvariety S. Therefore, by the Poincaré Reducibility Theorem, there exist an isogeny σ from E^n to itself such that $\big(\alpha_{ij}\big)\sigma(x_1 \dots x_n)^t = \big(\beta_{ij}\big)(x_1 \dots x_n)^t$, then σ can be thought as the non-singular $n \times n-$matrix T with entries in K, such that $\big(\alpha_{ij}\big)T = \big(\beta_{ij}\big)$. The converse is obvious, given that M and M' have the same image.

□

Now we will study $\mathrm{St}(n, K^g)$ in order to obtain information about the Abelian subvarieties of E^g using this result.

We recall that the number of columns of a matrix $M \in \mathrm{St}(n, K^g)$ is equal to n, a minor of order n is uniquely determined by the indices of the rows of M. Let $I = \{i_1, \dots, i_n\} \subset \{1, \dots, g\}$ with $i_1 < \cdots < i_n$. We denote by M_I the $n \times n$ submatrix of M formed with the ith rows, for $i \in I$. Let us denote by $J = \{I = \{i_1, \dots, i_n\} \subset \{1, \dots, g\} \colon i_1 < \cdots < i_n\}$ and by $\#J = N = \binom{g}{n}$, as in the second chapter.

We define the function

$$\begin{aligned} \psi : \mathrm{St}(n, K^g) &\to \mathbb{P}_K^{N-1}, \\ M &\mapsto \left[\Big(\det(M_I)\Big)_{I \in J}\right]. \end{aligned}$$

This function characterizes the matrices by their minors. Moreover, from the general fact that the Plücker coordinates of a subspace $L \subset V$ uniquely determine the subspace, we have:

Lemma 4.16. *Let $M, M' \in \mathrm{St}(n, K^g)$, then $\psi(M) = \psi(M')$ if and only if $MT = M'$ for some $T \in \mathrm{GL}(n, K)$.*

Proof. As we saw in the previous chapter, ψ is the Plücker embedding, given by the Plücker coordinates.

On the other hand, there exists a right action β of $\mathrm{GL}(n, K)$ on $\mathrm{St}(n, K^g)$ given by $\beta(M, T) = MT$ and since the determinant of the product is the product of determinants, we get that $\psi(MT) = \det(T)\psi(M)$. As a consequence, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{St}(n, K^g) & \xrightarrow{\quad\psi\quad} & \mathbb{P}_K^{N-1} \\ \big\downarrow{\scriptstyle pr} & & {\scriptstyle p}\big\uparrow \\ \mathrm{St}(n, K^g)/\,\mathrm{GL}(n, K^g) & \xrightarrow{\ \cong\ } & \mathrm{Gr}(n, K^g) \end{array}$$

Indeed, the image of ψ corresponds to the Grassmannian variety, given that $\psi(M)$ can be seen as $\mathrm{span}(M)$, where $\mathrm{span}(M)$ is the subspace generated by the columns of M, or alternatively, as the wedge product of the columns of M. □

We note that this result implies that we can relate the Abelian subvarieties with the image of ψ of the associated matrix.

Summarizing, we have the following result.

Proposition 4.17. *There exists a correspondence between Abelian subvarieties of E^g of dimension n and the K-rational points of $\mathrm{Gr}(n, K^g)$, given by*

$$S \to \psi(M_S),$$

where M_S is a matrix in $\mathrm{St}(n, K^g)$ as above and $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. We already showed that for an Abelian subvariety S we can find some matrix M_S, such that the image by ψ is a rational point in the Grassmannian and we proved that this correspondence is well-defined.

For the converse, we take a point in the Grassmannian $\mathrm{Gr}(m, K^g)$ and a representative M in $\mathrm{St}(m, R^g)$, then the homomorphism from E^n to E^g induced by M defines an Abelian subvariety of E^g of dimension n since the rank of M is maximal. □

Now, from this perspective the problem of estimating the number of Abelian subvarieties can be reduced to finding the rational points of bounded height in the Grassmannian variety $\mathrm{Gr}(n, K^g)$.

Proposition 4.18. *Let E be an elliptic curve with $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ and g a positive integer. For S an Abelian subvariety of (E^g, Θ) of dimension n, we have that*

$$\deg_r(S) = \frac{1}{\deg(f_S)} \sum_{i_1 < \cdots < i_n} |\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{i_1,\ldots,i_n}})|,$$

where $M_S \in \mathrm{St}(n, K^g)$ is the matrix of a homomorphism f_S associated to S.

Proof. From the correspondence above, for S an Abelian subvariety, there is a homomorphism $f_S : E^n \to E^g$ whose image is the Abelian subvariety S. For every ordered sequence $1 \le i_1 < \cdots < i_n \le g$ of n positive integers, let's consider the homomorphism from E^n to itself induced by f_S, given by

$$(x_1, \ldots, x_n) \mapsto (\pi_{i_1}(f(x_1, \ldots, x_n)), \ldots, \pi_{i_n}(f(x_1, \ldots, x_n)))$$

and let

$$M_{S_{i_1,\ldots,i_n}} := \begin{pmatrix} \alpha_{i_1 1} & \ldots & \alpha_{i_1 n} \\ \vdots & \ddots & \vdots \\ \alpha_{i_n 1} & \ldots & \alpha_{i_n n} \end{pmatrix} \in \mathcal{M}_{n\times n}(R),$$

be the induced matrix of this homomorphism. Using Proposition 1.17, we compute the self-intersection

$$
\begin{aligned}
(f_S^*\Theta)^n &= \left(f_S^* \sum_{i=1}^{g} \theta_i \right)^n \\
&= \left(f_S^* \sum_{i=1}^{g} \pi_i^*(0) \right)^n \\
&= \left(\sum_{i=1}^{g} (\pi_i \circ f_S)^*(0) \right)^n \\
&= \left(\sum_{i=1}^{g} f_S^*\pi_i^*(0) \right)^n \\
&= \sum_{i_1+\cdots+i_g=n} \binom{n}{i_1,\ldots,i_g} f_S^*\pi_1^*(0)^{i_1} \cdot \cdots \cdot f_S^*\pi_g^*(0)^{i_g},
\end{aligned}
$$

where the last equality is obtained by applying the multinomial theorem. The sum is taken over all combinations of nonnegative integer indices $i_1,\ldots,i_g$, such that the sum of all of them is n and

$$\binom{n}{i_1,\ldots,i_g} = \frac{n!}{i_1!\ldots i_g!},$$

denotes the multinomial coefficient.

Since the codimension of $f_S^*\pi_i^*(0)$ is equal to 1 and $f_S^*\pi_i^*(0)$ is algebraically equivalent to a multiple of an Abelian subvariety, for every $1 \leq i \leq g$, we have that $f_S^*\pi_i^*(0)^2 = 0$. Then every i_k is at most 1. Thus, for every ordered sequence $1 \leq j_1 < \cdots < j_n \leq g$ of n positive integers and the last line is rewritten as

$$(f_S^*\Theta)^n = \sum_{1\leq j_1<\cdots<j_n\leq g} n! f_S^*\pi_{j_1}^*(0)\cdots f_S^*\pi_{j_n}^*(0).$$

Since $\pi_j f_S = (\alpha_{j1},\ldots,\alpha_{jn})$, then

$$(\pi_j f_S)^*(0) = \{(x_1,\ldots,x_n) \in E^n \colon \alpha_{j1}x_1 + \cdots + \alpha_{jn}x_n = 0\}.$$

After that, we have the intersection number

$$f_S^*\pi_{j_1}^*(0)\cdots f_S^*\pi_{j_n}^*(0) = \#\ker(M_{S_{j_1,\ldots,j_n}}) = \deg(M_{S_{j_1,\ldots,j_n}}) = |\operatorname{Nm}_{K/\mathbb{Q}}(\det(M_{S_{j_1,\ldots,j_n}}))|,$$

where the last equality comes from Proposition 1.22. Therefore,

$$
\begin{aligned}
(f_S^*\Theta)^n &= \sum_{1\leq j_1<\cdots<j_n\leq g} n! f_S^*\pi_{j_1}^*(0)\cdots f_S^*\pi_{j_n}^*(0) \\
&= n! \sum_{1\leq j_1<\cdots<j_n\leq g} \deg\left(M_{S_{j_1,\ldots,j_n}}\right),
\end{aligned}
$$

and so

$$(f_S^*\Theta)^n = n! \sum_{1\le j_1<\cdots<j_n\le g} |\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{j_1,\ldots,j_n}})|.$$

On the other hand,

$$(f_S^*\Theta)^n = \deg(f_S)(\Theta|_S)^n = n! \deg(f_S)\deg_r(S).$$

Finally,

$$\deg_r(S) = \frac{1}{\deg(f_S)} \sum_{1\le j_1<\cdots<j_n\le g} |\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{j_1,\ldots,j_n}})|.$$

□

Corollary 4.19. *Let E be an elliptic curve with $K = \operatorname{End}(E)\otimes_{\mathbb{Z}}\mathbb{Q}$ and g a positive integer. Let S be an Abelian subvariety of E^g of dimension n that is identified with the matrix $M_S := (\alpha_{ij})$ in $\mathcal{M}_{g\times n}(K)$. Then $\deg(f_S)$ divides $H_{Ar}(\psi(M_S))^2 = \sum_{i_1<\cdots<i_n} |\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{i_1,\ldots,i_n}})|$.*

As we saw above, every Abelian subvariety of dimension n will be isogenous to E^n. Now, for a fixed Abelian subvariety, we want to know what is the minimum degree that a representative can have.

Definition 4.20. For S be an Abelian subvariety of E^g of dimension n isogenous to E^n, we define

$$d_E(S) = \min\{\deg f : E^n \mapsto S \colon f \in \operatorname{End}(E)\}.$$

We need to study the morphism in terms of the matrices associated.

Proposition 4.21. *With the assumptions as above, if $R = \operatorname{End}(E)$ is a principal ideal domain (PID), then the minimum possible degree $d_E(S)$ is equal to 1.*

Proof. Let M_S be the matrix associated to f_S, then $rank(M_S) = n$. By the Smith Normal Form [23, Theorem 6.6], we can find matrices $T \in \mathcal{M}_{g\times g}(R)$

and $P \in \mathcal{M}_{n\times n}(R)$ invertible over R such that

$$B = T^{-1}M_SP^{-1} = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \beta_r & \cdots & 0 \\ 0 & 0 & & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in \mathcal{M}_{g\times n}(R).$$

where the diagonal elements β_i can be computed (up to multiplication by a unit) as

$$\beta_i = \frac{d_i(M_S)}{d_{i-1}(M_S)}$$

where $d_i(A)$ are equal to the greatest common divisor of the determinants of all $i \times i$ minors of the matrix M_S and $d_0(A) := 1$ and the matrix TM_SP is zero elsewhere.

Given that T and P are invertible,

$$\operatorname{rank}(B) = \operatorname{rank}(M_S) = n.$$

These matrices T and P can be considered as bijections that mix coordinates, from $E^g \to E^g$ and $E^n \to E^n$, respectively.

Moreover, the variety S is the image of the morphism associated to B,

$$\begin{aligned} S = \operatorname{im}(B) &= \{(\beta_1 x_1, \cdots, \beta_n x_n, 0, \cdots, 0) \colon x_i \in E\} \\ &= \{(y_1, \cdots, y_n, 0, \cdots, 0) \colon y_i \in E\}. \end{aligned}$$

The last equality holds since elliptic curves are divisible groups.

□

Remark 4.22. Recall that for $K = \mathbb{Q}$, from Proposition 1.17, we have that

$$|\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{j_1,\dots,j_n}})| = |\det M_{S_{j_1,\dots,j_n}}|^2.$$

and for K a quadratic number field,

$$|\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{j_1,\dots,j_n}})| = |\det M_{S_{j_1,\dots,j_n}}\overline{\det M_{S_{j_1,\dots,j_n}}}|.$$

Corollary 4.23. *If E is an elliptic curve such that $\operatorname{End}(E)$ is a principal ideal domain, $K = \operatorname{End}_{\mathbb{Q}}(E)$ and $[K : \mathbb{Q}] = d$, then the following inequalities holds*

$$H_K(\psi(M_S)) \leq H_{Ar}(\psi(M_S)) \leq \deg_r(S) = H_{Ar}^2(\psi(M_S)) \leq NH_K^2(\psi(M_S)).$$

Proof. Let $f_S : E^g \to S$ be an isomorphism (which exists by Proposition 4.21). From Proposition 4.18

$$\deg_r(S) = \frac{1}{\deg(f_S)} \sum_{i_1<\cdots<i_n} |\operatorname{Nm}_{K/\mathbb{Q}}(\det M_{S_{i_1,\dots,i_n}})| = \frac{1}{\deg(f_S)} H^2_{Ar}(\psi(M_S)).$$

Then

$$H_{Ar}(\psi(M_S)) \le \deg_r(S) = H^2_{Ar}(\psi(M_S)).$$

Moreover, for the first inequality

$$H_K(\psi(M_S)) = \max_I\{|\operatorname{Nm}_{K/\mathbb{Q}}(\det(M_{S_I}))|\} \le \sqrt{\sum_I |\operatorname{Nm}_{K/\mathbb{Q}}(\det(B_I))|^2} = H_{Ar}(\psi(M_S))$$

and for the last one

$$H^2_{Ar}(\psi(M_S)) = \sum_{I\in J} |\operatorname{Nm}_{K/\mathbb{Q}}(\det(M_{S_I}))|^2 \le N \max_I\{|\operatorname{Nm}_{K/\mathbb{Q}}(\det(M_{S_I}))|^2\} = N H^2_K(\psi(M_S)).$$

□

Identify $\operatorname{Gr}(n, K^g)$ as a subvariety of $\mathbb{P}^{N-1}$ for $N = \binom{g}{n}$. We define the following set

$$\operatorname{Gr}(n, K^g, t) = \{P = [x_0 : \cdots : x_N] \in \operatorname{Gr}(n, K^g) \colon H_K(P) \le t\}.$$

Proposition 4.24. *Let E be an elliptic curve such that $[\operatorname{End}_{\mathbb{Q}}(E) : \mathbb{Q}] = d$ and $\operatorname{End}(E)$ is a principal ideal domain. The number of Abelian subvarieties of dimension n of (E^g, Θ) with reduced degree less than or equal to t is bounded by the number of rational points of $\operatorname{Gr}(n, K^g, t)$.*

Proof. By the proposition above, if $NH_K^{2/d}(\psi(M_S)) \le t$, then $\deg_r(S) \le t$. Therefore $H_K(\psi(M_S)) \le \left(\frac{t}{N}\right)^{d/2} \le t$. On the other hand, if $\deg_r(S)$ is bounded by t from the Corollary 4.23

$$H_K(\psi(M_S)) \le H_{Ar}(\psi(M_S)) \le \deg_r(S) \le t$$

Since there exists a correspondence between Abelian subvarieties of reduced degree bounded by t and rational points of $\operatorname{Gr}(n, K^g, t)$ given by

$$S \to \psi(M_S),$$

and the number of subvarieties is bounded for the number of rational points in the Grassmannian variety such that $H_K(x) \le t$. □

As a consequence of this result, we obtain an alternative proof to the classical finiteness result 4.1 by using Northcott's Theorem 3.4 for the maximum expected case (E^g, Θ):

Corollary 4.25. *For any positive integer t there are only finitely many Abelian subvarieties of (E^g, Θ) of degree less or equal to t.*

Moreover, we can exhibit some explicit expressions for the asymptotic estimate in both cases using the tools from Diophantine geometry that we showed in Chapter 3.

Theorem 4.26. *Let g be a positive integer and E elliptic curve such that $\mathrm{End}(E)$ is a principal ideal domain and $[\mathrm{End}_{\mathbb{Q}}(E) : \mathbb{Q}] = d$. There is an asymptotic estimate for the number of Abelian subvarieties of (E^g, Θ) of dimension n and reduced degree bounded by t:*

$$N_{E^g}(t,n) = \delta(n,g) C_{K,n(g-n)} t^{(n(g-n)+1)} + \mathcal{O}\left(t^{n(g-n)-1/d}\right),$$

The constant $C_{K,N}$ is given explicitly by

$$C_{K,N} = \frac{h_K R_K / w_K}{\zeta_K(n(g-n)+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\Delta_K}}\right)^{n(g-n)+1} (n(g-n)+1)^{r_1+r_2-1}$$

where Δ_K is the absolute value of the discriminant of K, h_K is the number of ideal classes of K, R_K is the regulator of K, w_K is the order of the group of unity in K, r_i is the number of v in the set of archimedean absolute values of K, the normalized to extend the ordinary absolute value on $\mathbb{Q}$ where $i = N_v$, the degree of the completion K_v over $\mathbb{R}$, ζ_K is the Dedekind zeta function.

Proof. From Proposition 4.24

$$N_{E^g}(t,n) \leq \#\left\{P \in \mathrm{Gr}(n, \mathbb{Q}^g) \colon H_K(P) \leq t\right\}.$$

The Grassmannian $\mathrm{Gr}(n, \mathbb{Q}^g)$ is a projective variety of dimension $n(g-n)$ and degree $\delta(n,g) = \frac{(n(g-n))! \prod_{1 \leq k \leq n-1} k!}{\prod_{1 \leq k \leq n} (g-k)!}$ from Proposition 2.6. In consequence, by Proposition 3.10

$$N_{E^g}(t,n) \leq \delta(n,g) \# \left\{P \in \mathbb{P}^{n(g-n)}(\mathbb{Q}) \colon H_K(P) \leq t\right\}.$$

By Theorem 3.5 we conclude that

$$N_{E^g}(t,n) \leq \delta(n,g) C t^{n(g-n)+1} + \mathcal{O}\left(t^{n(g-n)-1/d}\right),$$

where the constant $C := C_{K,n(g-n)}$ is given explicitly in Theorem 3.5, for $K = \mathbb{Q}$ the constant is $C = \frac{2^{n(g-n)}}{\zeta(n(g-n)+1)}$. □

Note that $N_{E^g}(t,n)$ and $N_{E^g}(t, g-n)$ have the same asymptotic estimate. This reflects the fact that complementary Abelian subvarieties of a principally Abelian variety have the same degree.

With this perspective in mind, we can give explicit upper and lower bounds, as a consequence of Theorem 3.7 and 3.6.

Theorem 4.27. *Let E be an elliptic curve and g a positive integer, such that $\mathrm{End}(E)$ is a principal ideal domain and $[\mathrm{End}_{\mathbb{Q}}(E):\mathbb{Q}] = d$. There is an upper bound for the number of Abelian subvarieties of E^g of dimension n and reduced degree bounded by t:*

$$N_{E^g}(t,n) \leq c_1 t^{d(n(g-n)+d)},$$

with $c_1 = c_1(n(g-n),d) = 2^{(2d+n(g-n))(d+n(g-n)+10)}$.

Theorem 4.28. *We have explicit lower bounds:*
For general elliptic curves

$$\frac{1}{4}t^2 \leq N_{E^g}(t,n),$$

when $1 \leq t$ and for elliptic curves with complex multiplication

$$\frac{1}{6^2}t^2 + \frac{1}{6^6}t^3 \leq N_{E^g}(t,n)$$

when $2 \leq t$.

Proof. From Corollary 4.23 we obtain

$$H_K(\Psi(M_S)) \leq \deg_r(S).$$

As a consequence, we have the following inequalities

$$\#\{P \in \mathbb{P}^1(K) \colon d(P) \leq d,\ H_K(P) \leq t\} \leq N_{E^g}(t,1) \leq N_{E^g}(t,n),$$

for every n.

For the first inequality, using Theorem 3.6, with $m = 1$

$$\frac{1}{4}t^2 \leq Z(K,d,1,t) = \#\{P \in \mathbb{P}^1(K) \colon d(P) = d,\ H_K(P) \leq t\}.$$

The second one is obtained by taking the sum

$$\frac{1}{6^2}t^2 + \frac{1}{6^6}t^3 \leq Z(\mathbb{Q},1,1,t) + Z(\mathbb{Q},2,1,t) = \#\{P \in \mathbb{P}^1(K) \colon d(P) \leq 2, H_K(P) \leq t\}.$$

□

As mentioned above, we can apply different results of Diophantine theory depending on the height considered. Due to Corollary 4.23, we could relate the reduced degree with the multiplicative height and the Arakelov height. Now we obtain an alternative estimation using the second one:

Theorem 4.29. *Let g be a positive integer and E elliptic curve such that $[\mathrm{End}_{\mathbb{Q}}(E):\mathbb{Q}] = d$ and $\mathrm{End}(E)$ is a principal ideal domain. The number of Abelian subvarieties of dimension n of (E^g, Θ) with reduced degree less than or equal to t is bounded by the number of rational points of $\mathrm{Gr}\,(n, K^g)$ with Arakelov height bounded by t.*

Moreover, there is an asymptotic estimate for the number of Abelian subvarieties of (E^g, Θ) of dimension n and reduced degree bounded by t:

$$N_{E^g}(t,n) = c_{n,g} t^g + \mathcal{O}(t^{g-\max\{\frac{1}{dn},\frac{1}{d(g-n)}\}}),$$

where $c_{n,g}$ is a positive constant.

Proof. We saw that

$$H_{Ar}(M_S) \leq \deg_r(S) \leq \sum_{I\in J} |\,\mathrm{Nm}_{K/\mathbb{Q}}(\det M_{S_I})| = H_{Ar}(M_S)^2.$$

If $\deg_r(S) \leq t$, then $H_{Ar}(M_S) \leq t$. On the other hand, if $H_{Ar}(M_S) \leq H_{Ar}(M_S)^2 \leq t$, then $\deg_r(S) \leq t$.

The rest of the result is direct from Proposition 4.17 and Proposition 3.15. The constant $c_{n,g} = a(n,g)|\Delta|^{gn/2}$ with $a(n,g)$ the constant from 3.15. □

Now, it is interesting to compare the asymptotic estimates obtained from the different heights. A simple calculation shows that the quadratic function $f(n) = n(g-n)+1$ achieves its maximum in $g/2$, getting that the maximum on

$$N_{E^g}(t, \lfloor g/2\rfloor) = \delta(\lfloor g/2\rfloor), g) C_{K,(\lfloor g/2\rfloor))^2} t^{(\lfloor \frac{g}{2}\rfloor)^2+1} + \mathcal{O}\left(t^{(\lfloor \frac{g}{2}\rfloor)^2-1/d}\right),$$

while it achieves its minimum at the extreme values $n = 1$ and $n = g-1$, that is $f(1) = g$. Therefore, the exponent in Theorem 4.26 in the best case

$$N_{E^g}(t,1) = \delta(1,g) C_{K,g-1} t^g + \mathcal{O}\left(t^{(g-1)-1/d}\right),$$

which is worse than the one obtained in Theorem 4.29

$$N_{E^g}(t,n) = c_{n,g} t^g + \mathcal{O}(t^{g-\max\{\frac{1}{dn},\frac{1}{d(g-n)}\}}).$$

Although, it does not depend on the dimension n of the Abelian subvariety, $N_{E^g}(t,n)$ and $N_{E^g}(t,g-n)$ have the same asymptotic estimate, trivially.

4.4 Examples and applications.

4.4.1 Number of elliptic curves of bounded degree.

Let us compare our results with the previous one. We will focus when the dimension of the Abelian subvariety is one, i.e., we are counting the number of elliptic curves with reduced degrees bounded by a number t. We resume the estimation in the following tables.

Table 4.1: General elliptic curves

Using Result	Theorem	Asymptotic estimation of $N_{E^g}(t,1)$
Guerra's	4.4	$Ct^{2g}+\mathcal{O}(t^{2g-2})$
Multiplicative height	4.26	$\frac{\delta(1,g)2^{g-1}}{\zeta(g)}t^g+\mathcal{O}(t^{g-1})$
Arakelov height	4.29	$c_{1,g}t^g+\mathcal{O}(t^{g-1})$

We specialize in the complex multiplication case:

Table 4.2: Elliptic curves with complex multiplication

Using Result	Theorem	Asymptotic estimation of $N_{E^g}(t,1)$
Guerra's	4.4	$Ct^{2(2g-1)}+\mathcal{O}(t^{2(2g-3+33/52)})$
Multiplicative height	4.26	$\frac{\delta(1,g)2^{g-1}}{\zeta(g)}t^g+\mathcal{O}(t^{g-2})$
Arakelov height	4.29	$c_{1,g}t^g+\mathcal{O}(t^{g-1/2})$

In the case $n=1$, we obtain the same order on t for both estimations. In general, for any dimension n, since the function is $h(n)=n(g-n)+1-g$ is positive for $1\leq n\leq g-1$, our estimation using Arakelov height is sharper than the result for the multiplicative height,

Even though, in either comparison, the estimation using the multiplicative height is worse than the Theorem 4.29, using it we obtain that for general elliptic curves

$$\frac{1}{4}t^2\leq N_{E^g}(t,1)\leq c_1t^g,$$

and for elliptic curves with complex multiplication such that $\mathrm{End}(E)$ is a principal ideal domain

$$\frac{1}{6^2}t^2 + \frac{1}{6^6}t^3 \leq N_{E^g}(t,1) \leq c_1 t^{2g+2}$$

when $2 \leq t$. So far, we do not have an explicit bound. This shows that both heights have interesting consequences.

4.4.2 Abelian Surfaces on E^4.

Let E be an elliptic curve, such that $[\mathrm{End}_{\mathbb{Q}}(E) : \mathbb{Q}] = d$ with $\mathrm{End}(E)$ and is a principal ideal domain. We consider the self product E^4.

From the complementary property of Abelian subvarieties we have

$$N_{E^4}(t,1) = N_{E^4}(t,3),$$

in other words, the estimation of elliptic curves with reduced degree bounded by t is equal to the estimation of Abelian subvarieties of dimension 3 with bounded reduced degree. We are left to estimate the number of Abelian subvarieties of dimension 2 with a bounded reduced degree. For that, we are going to use Proposition 4.17.

Note that, in this case, the Grassmannian reduces to a single equation

$$\mathrm{Gr}(2, K^4) = \{[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^5 : x_0x_5 - x_1x_4 + x_2x_3 = 0\}.$$

Proposition 4.30. *The number of Abelian subvarieties of dimension* 2 *of* (E^4, Θ) *with reduced degree less or equal to* t *is bounded by the number of* $\mathbb{Q}$*-rational points of* $\mathrm{Gr}\,(2, \mathbb{Q}^4, t)$.

Moreover, there is an asymptotic estimate given by

$$N_{E^4}(t,2) = c_{2,4}t^4 + \mathcal{O}(t^{7/2})$$

if E *is a general elliptic curve, and by*

$$N_{E^4}(t,2) = c_{2,4}t^4 + \mathcal{O}(t^{15/4})$$

if E *is an elliptic curve with complex multiplication, where* $c_{2,4}$ *is the constant in Theorem 4.29.*

Now, we can compare our result with Corollary 4.9 explicitly using the asymptotic estimate in Fano varieties, from Theorem 4.29

$$\mathcal{N}_{E^4}(t) = \sum_{n=1}^{3} N_{E^4}(t,n) = \mathcal{O}(t^4).$$

Clearly this estimation is lower than $\mathcal{N}_{E^4}(t) = \mathcal{O}(t^{18})$ in Corollary 4.9.

Additionally, we can give the explicit bound.

Proposition 4.31. *Let E be an elliptic curve such that* $\mathrm{End}(E)$ *is a principal ideal domain. The number of Abelian surfaces of E^4 and reduced degree bounded by t is bounded by*

$$\frac{1}{4}t^2 \leq N_{E^4}(t,2) \leq 2^{90}t^5,$$

for a general elliptic curve, and

$$6^{-2}t^2 + 6^{-6}t^3 \leq N_{E^4}(t,2) \leq 2^{128}t^{12}$$

for an elliptic curve with complex multiplication and $t \geq 2$.

Then, for E^4 we have a complete description of the number of Abelian subvarieties.

4.4.3 Comparison on the number of Abelian subvarieties.

We have already compared our results on dimension one, and now we will study for larger dimensions of Abelian subvarieties.

Note that the total number of Abelian subvarieties on E^g with reduced degree bounded by t is given by

$$\mathcal{N}_{E^g}(t) = \sum_{n=1}^{g-1} N_{E^g}(t,n).$$

Since

$$N_{E^g}(t,n) = \delta(n,g)C_{K,n(g-n)}t^{(n(g-n)+1)} + \mathcal{O}\left(t^{n(g-n)-1/d}\right),$$

has a maximum value at $n = \left\lfloor \frac{g}{2} \right\rfloor$, then

$$\mathcal{N}_{E^g}(t) = \mathcal{O}(t^{\lfloor \frac{g}{2} \rfloor^2+1}).$$

Clearly, the estimation is lower than the estimation given by Guerra in Corollary 4.9 whose order is $\mathcal{O}(t^{(g+2)(g-1)})$. Furthermore, if we proceed with the estimation with the Arakelov height for a general elliptic curve, we improve both of them

$$\mathcal{N}_{E^g}(t) = \sum_{n=1}^{g-1} N_{E^g}(t,n) = \mathcal{O}(t^g).$$

4.5 Final comments.

The questions one can immediately ask are if the bound is sharp and in which cases, or for what kind of elliptic curves achieve that bound. Although the

Arakelov height gives us a better asymptotic estimation, Schmidt's approach provides explicit bounds [30]. Moreover, in the literature, many more asymptotic estimations improve the constant and the order of the error.

On the other hand, the results in Chapter 4 are strongly related to the fact that the dimension of E is 1 and the codimension of $f_S^*\pi_{i_j}^*(0)$ is equal to 1. Then the self-intersection can be calculated by

$$\begin{aligned}(f_S^*\Theta)^n &= \left(f_S^*\sum_{i=1}^{g}\theta_i\right)^n \\ &= \sum_{\substack{0\le i_1,\dots,i_n \\ i_1+\cdots+i_n=n}} \binom{n}{i_1,\dots,i_n} f_S^*\pi_{i_1}^*(0)^{i_1}\cdot\cdots\cdot f_S^*\pi_{i_n}^*(0)^{i_n},\end{aligned}$$

in consequence

$$(f_S^*\Theta)^n = \sum_{i_1,\dots,i_n} n!\, f_S^*\pi_{i_1}^*(0)\cdots f_S^*\pi_{i_n}^*(0).$$

The last equality can not be emulated for A^g with A of dimension greater than 1. In general, we will have more terms and

$$\sum_{i_1,\dots,i_n} n!\, f_S^*\pi_{i_1}^*(0)\cdots f_S^*\pi_{i_n}^*(0) \le (f_S^*\Theta)^n.$$

From a moduli point of view, for a polarized Abelian variety of dimension g in the moduli space $\mathcal{A}_g$ and integers t and n, we have been unable to prove that

$$\begin{aligned} N : \mathcal{A}_g \times \mathbb{N}^2 &\to \mathbb{N} \\ (A,t,n) &\mapsto N_A(t,n)\end{aligned}$$

achieves its maximum in the moduli space $\mathcal{A}_g$ for a self product of elliptic curves E^g. We hope to develop this in the future to get a complete answer to this problem. One way could be to bound the terms different from $f_S^*\pi_{i_1}^*(0)\cdots f_S^*\pi_{i_n}^*(0)$, with some techniques used by Guerra.

Additionally, from the multiplicative property of the degree, that is, for $A = A_1^{r_1}\times\cdots\times A_k^{r_k}$ with the Abelian varieties A_i simple and pairwise non-isogenous. If $\alpha\in \mathrm{End}_{\mathbb{Q}}(A)$ and $(\alpha_1,\dots,\alpha_k)$ with $\alpha_i\in\mathrm{End}_{\mathbb{Q}}(A_i^{r_i})$, then

$$\deg(\alpha) = \prod_{i=1}^{k}\deg(\alpha_i),$$

we can reduce the study to $B = A^r$. In this case, a principally polarized Abelian variety with $\mathrm{End}(A)$ an order $\mathfrak{o}_K$ in K, a totally real number field of degree g over $\mathbb{Q}$. Then

$$\deg(\alpha) = \mathrm{Nr}_{K/\mathbb{Q}}(\det(\alpha)),$$

and we can use the same strategy for an elliptic curve, to bound $f_S^*\pi_{i_1}^*(0)\cdots f_S^*\pi_{i_n}^*(0)$.

Another interesting point of view comes from the relation with Fano varieties. We want to study Abelian varieties and moduli spaces using the machinery of this theory.

Bibliography

[1] Auffarth, R., Non-simple principally polarized abelian varieties. https://www.mat.uc.cl/archivos/dip/tesis-postgrado/doctorado-mat/tesis-robert-auffarth.pdf

[2] Auffarth, R. On a numerical characterization of non-simple principally polarized abelian varieties. *Math. Z.* 282 (2016), no. 3-4, 731–746.

[3] Lange, H; Birkenhake, Ch. Complex abelian varieties. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 302. *Springer-Verlag, Berlin*, 1992.

[4] Birkenhake, Ch; Lange, H. The exponent of an abelian subvariety. *Math. Ann.* 290 (1991), no. 4, 801–814.

[5] Bombieri, E; Gubler, W. Heights in Diophantine geometry. New Mathematical Monographs, 4. *Cambridge University Press, Cambridge*, 2006.

[6] Debarre, O. Tores et variétés abéliennes complexes. (French) [Complex tori and abelian varieties] Cours Spécialisés [Specialized Courses], 6. *Société Mathématique de France, Paris; EDP Sciences, Les Ulis*, 1999.

[7] Edixhoven, B; Van der Geer, G; Moonen, B. Abelian Varieties. http://van-der-geer.nl/~gerard/AV.pdf

[8] Fulton, W. Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. *Springer-Verlag, Berlin*, 1998.

[9] Grieve, N. Reduced norms and the Riemann-Roch theorem for Abelian varieties. *New York J. Math.* 23 (2017), 1087–1110.

[10] Guerra, L. On elliptic curves of bounded degree in a polarized Abelian surface. *Rend. Istit. Mat. Univ. Trieste* 48 (2016), 495–508.

[11] Guerra, L. On elliptic curves of bounded degree in a polarized Abelian variety. *Rend. Istit. Mat. Univ. Trieste* 51 (2019), 105–123.

[12] Guerra, L. On Abelian subvarieties of bounded degree in a polarized Abelian variety. *Preprint* https://arxiv.org/pdf/2104.11899.pdf(2021).

[13] Harris, J. Algebraic geometry. A first course. Graduate Texts in Mathematics, 133. *Springer-Verlag, New York*, 1992.

[14] Hindry, M; Silverman, J. H. Diophantine geometry. An introduction. Graduate Texts in Mathematics, 201. *Springer-Verlag, New York*, 2000.

[15] Kaji, H; Terasoma, T. Degree Formula for Grassmann Bundles. *Journal of pure and applied algebra*, ISSN 0022-4049, Vol. 219, Nº 12 (2015) 5426-5428.

[16] Kani, E. Bounds on the number of nonrational subfields of a function field. *Invent. Math.* 85 (1986), no. 1, 185–198.

[17] Kani, E. Elliptic curves on abelian surfaces. *Manuscripta Math.* 84 (1994), no. 2, 199–223.

[18] Kani, E. The moduli spaces of Jacobians isomorphic to a product of two elliptic curves. *Collect. Math.* 67 (2016), no. 1, 21–54.

[19] Lang, S. Fundamentals of Diophantine geometry. *Springer-Verlag, New York*, 1983.

[20] Lombardo, D. Abelian varieties, 2018. https://people.dm.unipi.it/lombardo/Teaching/VarietaAbeliane1718/Notes.pdf.

[21] Louboutin, S. Explicit bounds for residues of Dedekind zeta functions, values of L-functions at s=1, and relative class numbers. *J. Number Theory* 85 (2000), no. 2, 263–282.

[22] Masser, D; Vaaler, J. D. Counting algebraic numbers with large height. II. *Trans. Amer. Math. Soc.* 359 (2007), no. 1, 427–445.

[23] Matthews, K. R. Linear Algebra, *Lecture Notes, University of Queensland*, 1991. http://www.numbertheory.org/courses/MP274/smith.pdf.

[24] Milne. J. S. Abelian varieties, 2008. http://www.jmilne.org/math/CourseNotes/AV.pdf.

[25] Mumford, D. Abelian varieties. With appendices by C. P. Ramanujam and Yuri Manin. Corrected reprint of the second (1974) edition. Tata Institute of Fundamental Research Studies in Mathematics, 5. *Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi*, 2008.

[26] Northcott, D. G. An inequality in the theory of arithmetic on algebraic varieties. *Proc. Cambridge Philos. Soc.* 45 (1949), 502–509.

[27] Poincaré, H. Sur les Fonctions Abeliennes. (French) *Amer. J. Math.* 8 (1886), no. 4, 289–342.

[28] Schanuel, S. Heights in number fields. *Bull. Soc. Math. France* 107 (1979), no. 4, 433–449.

[29] Schmidt, W. M. Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height. *Duke Math. J.* 35 (1968), 327–339.

[30] Schmidt, W. M. Northcott's theorem on heights. I. A general estimate. *Monatsh. Math.* 115 (1993), no. 1-2, 169–181.

[31] Shafarevich, I. R; Remizov, A. O. Linear algebra and geometry. Translated from the 2009 Russian original by David Kramer and Lena Nekludova. *Springer, Heidelberg*, 2013.

[32] Silverman, J. Advanced topics in the arithmetic of elliptic curves. Graduate Texts in Mathematics, 151. *Springer-Verlag, New York*, 1994.

[33] Thunder, J. An Asymptotic estimate for heights of algebraic subspaces. *Trans. Amer. Math. Soc.* Vol. 331, No. 1 (May, 1992), pp. 395-424.

[34] Tamme, G. Teilkörper höheren Geschlechts eines algebraischen Funktionenkörpers. (German) *Arch. Math.* (Basel) 23 (1972), 257–259.

[35] Voight, J. Quaternion algebras. Graduate Texts in Mathematics, 288. *Springer, Cham*, (2021).