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CLASSIFICATION OF RIGID INTERVALS IN $\tilde{A}_2$

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Abstract

Let W be an affine Weyl group of type $\widetilde{A}_2$ with corresponding finite Weyl group W_f. In this thesis, we study certain family of Bruhat intervals by exploring the relationship between the geometry of alcoves associated with the Coxeter complex of W and Euclidean geometry.

First, we provide a characterization of lower intervals for each element of W in terms of convex sets, completing the characterization given in [LP23]. Next, we give a geometric description of a family intervals $[x, y]$ which allow us to associate a polygon to its intervals. On the other hand, using the alcove geometry, we introduce a new poset isomorphism between these intervals. Finally, by utilizing the combinatorial information encoded in the polygon associated with each interval, we classify the intervals satisfying mild restrictions up to poset isomorphism.

Dedicada a mis padres
Adriana y José quienes me enseñaron
la importancia del esfuerzo.

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Contents

Introduction

Motivation

The Bruhat orderings of Coxeter groups are mathematical objects that appear in a wide variety of mathematical contexts, including algebraic combinatorics and geometric representation theory. In the geometric context (perhaps the most prominent), if G is a simply connected semisimple algebraic group over an algebraically closed field, with a Borel subgroup B and W the associated Weyl group, one has the Bruhat decomposition of the generalized flag variety:

$$G/B = \bigsqcup_{w \in W} BwB/B,$$

where the Schubert cells BwB/B in this decomposition are governed by the Bruhat order on W.

It is well known that every Coxeter group ordered by the Bruhat order is a graded poset. Furthermore, there are infinitely many ranked posets of length 3 that appear in Bruhat orders on infinite Coxeter groups (see e.g. [BB05]).

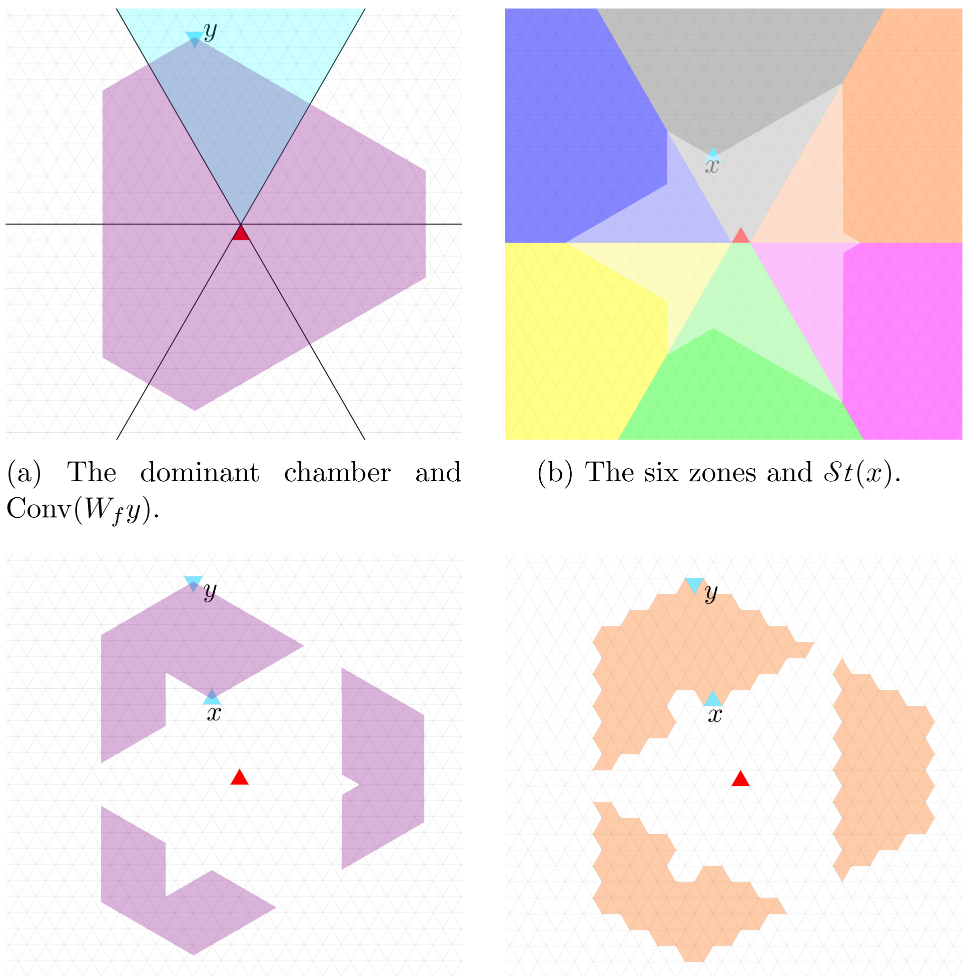
On the other hand, Björner [Bjö84a], asked whether only finitely many posets of each fixed length are Bruhat intervals. Dyer [Dye91] answered Björner's question affirmatively, which naturally leads to the following problem: "*The classification of Bruhat intervals of each fixed length, modulo isomorphism*".

While solutions are known for certain intervals — such as those of length 2 or length 3 — Hultman [Hul03] solved the classification problem for finite Weyl groups. The author found 24 distinct classes of intervals of length ≤ 4 and 36 classes of intervals of length ≤ 5 in the symmetric group S_6.

An alternative viewpoint on the classification problem involves studying the automorphisms of Bruhat ordering. First explored by van den Hombergh [vdH74] for any irreducible Coxeter system (W, S) and later independently rediscovered by Waterhouse [Wat89], this approach naturally leads to questions about the isomorphism of Bruhat intervals. In [LP23], an explicit result was presented for the first time that uses "*alcove geometry*" to explore the structure of intervals $[\mathrm{id}, y]$ in affine Weyl groups of type $\widetilde{A}_2$. Although this was not the central focus of the paper, it provided a strong starting point for research on how alcove geometry might be used to study intervals.

The idée-force of this thesis is that, at least for affine Weyl groups and with some modest restrictions on the intervals, this approach could provide a simple and elegant answer. We prove this in the $\widetilde{A}_2$ case and conjecture that something similar happens to all affine Weyl groups.

Visualizing Bruhat intervals



(a) The dominant chamber and $\mathrm{Conv}(W_f y)$.

(b) The six zones and $\mathcal{S}t(x)$.

(c) $\mathrm{Conv}(W_f y) \setminus \mathcal{S}t(x)$.

(d) Alcoves in $[x, y]$.

Figure 1: Let s_0, s_1 and s_2 be the simple reflections of W. The elements are $x = s_1s_2s_0s_1s_2s_0s_1s_2s_1s_0$ and $y = s_1s_2s_0s_1s_2s_0s_1s_2s_0s_1s_2s_0s_1s_2s_0s_1s_2s_0s_2s_1s_0s_2s_1s_0s_2$

A basic observation of this thesis is how to construct, using Euclidean geometry, all Bruhat intervals in the affine Weyl group W of type $\widetilde{A}_2$. Given that our main theorem is for dominant elements, i.e., elements in the dominant chamber C_+, the light-blue region in Figure 1a (note that the red alcove is the identity), we will restrict our attention to those elements.

The algorithm is as follows. Choose the center of an alcove y as in Figure 1a. Consider the hexagon constructed as the convex hull of this point's orbit under the Weyl group $W_f \cong S_3$ (this group is generated by the reflections through the black lines).

Choose the center of another alcove x as in Figure 1b. There are two equilateral triangles, denoted by $\triangleleft$ and $\triangleright$, with the following properties:

- $\triangleleft$ is a left-facing triangle centered at the bottom-right vertex of the red triangle, and its boundary contains the center of x.
- $\triangleright$ is a right-facing triangle centered at the bottom-left vertex of the red triangle, and its boundary also contains the center of x.

Consider the star defined by the union of these two triangles.

Figure 1c shows the intersection of the hexagon in Figure 1a and the complement (of the interior) of the star in Figure 1b. Finally, in Figure 1d, we see the interval $[x, y]$: it is the set of alcoves whose center belongs to Figure 1c.

Piecewise translations

Consider an interval $[x, y]$. Let gray$[x, y]$ be the set of alcoves in $[x, y]$ that are also in the gray zone.

Consider a vector v that satisfies the following three conditions:

- The sets $x + v$ and $y + v$ are alcoves.
- The set gray$[x, y] + v$ is completely contained in the gray zone.
- The sets gray$[x + v, y + v]$ and gray$[x, y]$ have the same cardinality.

If these conditions are satisfied, we define the *translation of the interval* $[x, y]$ *by* v. It is defined as a piecewise translation zone by zone. If you are, say, in the yellow zone, as this zone is inside the gray zone rotated counter-clockwise by $2\pi/3$, then in the yellow zone one has to translate every alcove by the vector v', which is the vector v rotated counter-clockwise by $2\pi/3$, see Figure 2. We can obtain the translation vector in each zone by applying reflections to the vector associated with an adjacent zone. For example, the vector used for translation in the blue zone is obtained by reflecting the vector v' (the translation vector in the yellow zone) across the line that contains the bottom edge of the red triangle. Whenever a vector v satisfies the above conditions, the resulting translation defines a poset isomorphism from $[x, y]$ to $[x + v, y + v]$.

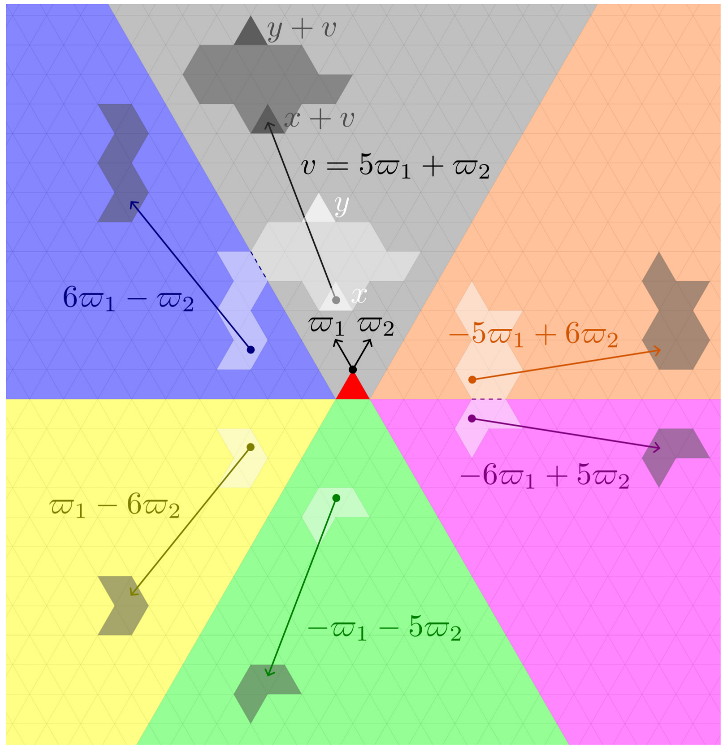


Figure 2: Visualization of the map $[x, y] \to [x + v, y + v] = [x', y']$. Where $x = s_1 s_2 s_0 s_1 s_0 s_2$, $y = s_1 s_2 s_0 s_1 s_2 s_0 s_1 s_2 s_1 s_0 s_2 s_1$, and $v = 5\varpi_1 + \varpi_2$.

Main result

We recall that every Coxeter system (W, S) has a well-understood group G of automorphisms that preserve the Bruhat order [BB05, Cor. 2.3.6], which we refer to as the group of *generic automorphisms* of (W, S). This group is generated by the map $x \mapsto x^{-1}$ and the automorphisms induced by the symmetries of the Dynkin diagram.

When the interval $[x, y]$ is geometrically very "thin", we say that the interval $[x, y]$ is *non-rigid* (for the precise meaning, see Definition 3.1.17). In that case, there might be a lot of additional isomorphisms. Being non-rigid is quite rare[1]. The following is the main result of the thesis.

Theorem A (Main result)**.** *If x, x', y, y' are dominant alcoves in $\widetilde{A}_2$, all the isomorphisms of posets between rigid intervals $[x, y]$ and $[x', y']$ are given by translations by a vector composed with a generic automorphism.*

We remark that an analogous but simpler version of this theorem is that of lower

[1] Indeed, let $D_{\leq n}$ be the set of dominant alcoves having length lesser or equal than $n \in \mathbb{N}$. Let $A_{\leq n}$ be the subset of $D_{\leq n} \times D_{\leq n}$ consisting of the pair of elements (x, y) such that $[x, y]$ is non empty. One can prove that as $n \to \infty$, the subset of $A_{\leq n}$ of pair of elements (x, y) such that $[x, y]$ is non-rigid, divided by the cardinality of $A_{\leq n}$ goes to zero.

intervals, i.e., when $x = x' = \mathrm{id}$. In this case, there is no possible translation by a vector because no vector satisfies the third bullet. In that case we prove the same result but for all elements in W. In other words, if u, v are arbitrary elements of W, then $[\mathrm{id}, u] \simeq [\mathrm{id}, v]$ if and only if v belongs to the G-orbit of u. The proof of this result, given in Proposition 3.3.1, is notably simpler than that of the main theorem.

Outline of Chapters

Chapter 1 is divided into two sections. In Section 1.1, we present the necessary background, notation, and conventions that will be used throughout this thesis. Section 1.2 introduces some definitions, notation and basic properties of the affine Weyl group of type $\widetilde{A}_2$. Additionally, we describe the construction of the Coxeter complex and define the $X\Theta$-partition.

Chapter 2 focuses on understanding intervals in geometric terms. In Section 2.1, we provide a geometric description of lower intervals $[\mathrm{id}, y]$, characterizing them as collections of elements whose alcoves lie within the convex hull C_y of a finite set. Section 2.2 is dedicated to establishing a geometric characterization of the intervals $[x, y]$ for $x, y \in D$, leading to the definition of the polygon $\mathrm{Pgn}_{x,y}$ associated with $[x, y]$. Finally, in Section 2.3, we introduce the order-preserving map τ_λ between intervals and determine conditions on $\mathrm{Pgn}_{x,y}$ that ensure the maps τ_λ define isomorphisms.

Chapter 3 consists of three sections. In Section 3.1, we examine the normalized lengths of the sides of $\mathrm{Pgn}_{x,y}$ and demonstrate that these lengths impose constraints on poset isomorphisms between *rigid intervals*. Section 3.2 establishes the invariance of $\mathrm{Pgn}_{x,y}$ and presents our main result (Theorem A). Finally, in Section 3.3, we show that all isomorphisms between lower intervals are induced by G (Theorem 3.3.1).

Chapter 1

Preliminaries

1.1 Basic notions in Coxeter systems

In this section, we recall some basic notions about Coxeter systems. We mainly follow [BB05, Chapter 2] for notation and terminology with some minor differences.

Let (W, S) be a Coxeter system with length ℓ and Bruhat order $\leq$. We denote $\ell(u, w) \coloneqq \ell(w) - \ell(u)$. A *Bruhat interval* is a set $[x, y] = \{z \in W \mid x \leq z \leq y\}$, with $x, y \in W$. Given an element $x \in W$, the *upper interval* of x is the set

$$\geq x \coloneqq \{z \in W \mid x \leq z\},$$

and the *lower interval* of x is the set $[\mathrm{id}, x]$.

If $u, z \in W$, we say that z *covers* u, and we denote it by $u \lhd z$, if $u < z$ and there is no element z' such that $u < z' < z$. If $x, y \in W$, we say that $z \in [x, y]$ is an *atom* (resp. *coatom*) of the interval $[x, y]$ if $x \lhd z$ (resp. $z \lhd y$).

We recall the fundamental property of the Bruhat ordering

Proposition 1.1.1. *[Hum90, Proposition 5.9] Let $u \leq v$ in W and $s \in S$ be such that $su > u$ and $sv < v$. Then the interval is stable under left multiplication.*

Let $x, y \in W$, with $x \leq y$. The *length counting sequence* $\mathrm{LC}(x, y)$ of the interval $[x, y]$ is the finite sequence

$$\mathrm{LC}(x, y) = (f_0^{x,y}, f_1^{x,y}, \ldots, f_{\ell(x,y)}^{x,y}) \in \mathbb{N}^{\ell(x,y)+1}$$

given by $f_i^{x,y} \coloneqq |\{z \in [x, y] \mid \ell(x, z) = i\}|$.

Proposition 1.1.2 ([BB05] pp. 35)**.** *The interval $[x, y]$ is a graded poset, with the rank function given by the length function $z \mapsto \ell(x, z)$.*

Proposition 1.1.3. *Let $\varphi\colon [x,y] \to [x',y']$ be an isomorphism of posets. Then, φ is an isomorphism of graded posets. In other words, it preserves the rank function or, in formulas, $\ell(x,z) = \ell(\varphi(x),\varphi(z))$. In particular,* $\mathrm{LC}(x,y) = \mathrm{LC}(x',y')$.

Proof. It is clear that if z covers u then $\varphi(z)$ covers $\varphi(u)$. If $z \in [x,y]$ is such that $\ell(x,z) = k$, by [BB05, Lemma 2.2.6] there is a chain $x = z_0 \lhd z_1 \lhd \cdots \lhd z_k = z$, so applying φ we obtain a chain $\varphi(x) = \varphi(z_0) \lhd \varphi(z_1) \lhd \cdots \lhd \varphi(z_k) = \varphi(z)$. By Proposition 1.1.2 this implies that $\ell(\varphi(x),\varphi(z)) = k$. □

An interval $[x,y]$ in W is *dihedral* if it is isomorphic to an interval of a dihedral Coxeter group. For each $y \in W$, we define $\mathcal{D}(y) := \{z \in [\mathrm{id}, y] \mid [z,y] \text{ is dihedral}\}$.

Proposition 1.1.4 ([Dye87, Theorem 7.25])**.** *Let $x, y \in W$ with $\ell(x,y) > 1$. The following statements are equivalent:*

1. *$[x,y]$ has two atoms;*
2. *$[x,y]$ has two coatoms;*
3. *$[x,y]$ is dihedral.*

The following lemma is immediate from Propositions 1.1.4 and 1.1.3.

Lemma 1.1.5. *Let $\phi\colon [x,y] \to [x',y']$ be an isomorphism of posets. If $[u,v] \subset [x,y]$ is dihedral, then $\phi([u,v])$ is dihedral.*

Definition 1.1.6. Given $y \in W$, we define the sets

$$\mathrm{L}(y) := \{z \in W \mid z \lhd y\}$$
$$\mathrm{U}(y) := \{z \in W \mid y \lhd z\}$$

of *lower covers* of y and *upper covers* of y respectively.

Definition 1.1.7. An interval $[x,y]$ is *non-degenerate* if $\mathrm{U}(x), \mathrm{L}(y) \subset [x,y]$. Otherwise, it is called *degenerate*.

Lemma 1.1.8. *Let $y \in W$ and $s \in S$ be such that $y < sy$. Then.*

$$[\mathrm{id}, sy] = [\mathrm{id}, y] \cup s[\mathrm{id}, y],$$

where $s[\mathrm{id}, y] \coloneqq \{sx \mid x \leq y\}$.

Proof. First, we prove the inclusion $\supseteq$. By hypothesis, it is clear that $[\mathrm{id}, y] \subset [\mathrm{id}, sy]$, so it remains to prove that $s[\mathrm{id}, y] \subset [\mathrm{id}, sy]$. Let $z \in s[\mathrm{id}, y]$, so $z = sw$ for some $w \in [\mathrm{id}, y]$.

- If $sw < w$. Since $w < y$ and $y \leq sy$, we have that $z = sw < w < y < sy$.
- If $w < sw$. Since $y < sy$, it follows that $z = sw < sy$

In both cases, we obtain that $z \leq sy$, so we conclude that $s[\mathrm{id}, y] \subset [\mathrm{id}, sy]$. Now we prove the inclusion $\subseteq$. Let $z \in [\mathrm{id}, sy]$, we will to prove that either $z \in [\mathrm{id}, y]$ or $z \in s[\mathrm{id}, y]$. We analyze two cases:

- If $sz < z$. Applying Lemma 1.1.1 to $u = sz$ and $v = sy$, we obtain that $z < y$.
- If $z < sz$. Applying Lemma 1.1.1 to $u = z$ and $v = sy$, we obtain that $sz < y$. This implies that $z \in s[\mathrm{id}, y]$

Thus, we conclude that either $z \in [\mathrm{id}, y] \cup s[\mathrm{id}, y]$, as desired. □

1.2 The affine Weyl group of type $\widetilde{A}_2$ and alcoves

From this point forward, W denotes the affine Weyl group of type $\widetilde{A}_2$. As a Coxeter system it has generators $S = \{s_0, s_1, s_2\}$. For simplicity, we will often denote s_0, s_1, s_2 by $0, 1, 2$ respectively. We will use "label mod 3" notation. For example, 1456 stands for $s_1 s_1 s_2 s_0$.

A *reflection* of W is an element belonging to the set

$$T = \bigcup_{z \in W} zSz^{-1}.$$

The finite subgroup $W_f := \langle s_1, s_2 \rangle \leq W$ is the *Weyl group* of type A_2. It is isomorphic to the dihedral group of order 6. The longest element of W_f is denoted by $w_0 = s_1 s_2 s_1$.

The Dynkin diagram of W has six symmetries (it can be seen as an equilateral triangle). Each one of them induces an automorphism of W. Let $\delta \colon S \to S$ be the map $\delta(i) = i + 1$, i.e. the map given by $\delta(s_0) = s_1, \delta(s_1) = s_2$, and $\delta(s_2) = s_0$. Similarly, we consider $\sigma \colon S \to S$ the map that fixes s_0 and permutes s_1 and s_2. The automorphisms of W induced by δ and σ will be denoted by the same symbols. The subgroup $\langle \delta, \sigma \rangle$ of $\mathrm{Aut}(W)$ generated by δ and σ is isomorphic to the dihedral group of order 6. Let $\iota \colon W \to W$ be the inversion group anti-automorphism that sends $x \mapsto x^{-1}$. We define G as the subgroup of the symmetric group of W generated by δ, σ, and ι. The order of G is 12.

Lemma 1.2.1 ([Hum90, Proposition 8.8])**.** *Suppose $g \in G$. The map $z \mapsto g(z)$ induces an isomorphism of posets $[x, y] \cong [g(x), g(y)]$.*

Let $(E, (-,-))$ be the real Euclidean plane and let $\Phi \subset E$ be the root system of type A_2, with simple roots $\{\alpha_1, \alpha_2\}$ spanning the root lattice $\mathbb{Z}\Phi$, and fundamental weights $\{\varpi_1, \varpi_2\}$ spanning the weight lattice Λ. The elements of Λ are called *weights*. A *dominant weight* is a weight that has non-negative coefficients when written as a linear combination of fundamental weights. Let $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the set of positive roots. For each $\alpha \in \Phi$ and $k \in \mathbb{Z}$ consider the affine line

$$H_{\alpha,k} := \{v \in E \mid (\alpha, v) = k\}.$$

Let $s_{\alpha,k}$ be the orthogonal reflection through $H_{\alpha,k}$. We denote by $\mathscr{H}$ the collection of all affine lines $H_{\alpha,k}$, with $\alpha \in \Phi, k \in \mathbb{Z}$.

Definition 1.2.2. The set $\mathscr{A}$ of *alcoves* of $\widetilde{A}_2$ is the set of connected components of

$$E - \bigcup_{H \in \mathscr{H}} H$$

The *fundamental alcove* is the following open subset of E

$$\mathscr{A}_{\mathrm{id}} := \{v \in E \mid -1 < (\alpha, v) < 0, \text{ for all } \alpha \in \Phi^+\} \in \mathscr{A}.$$

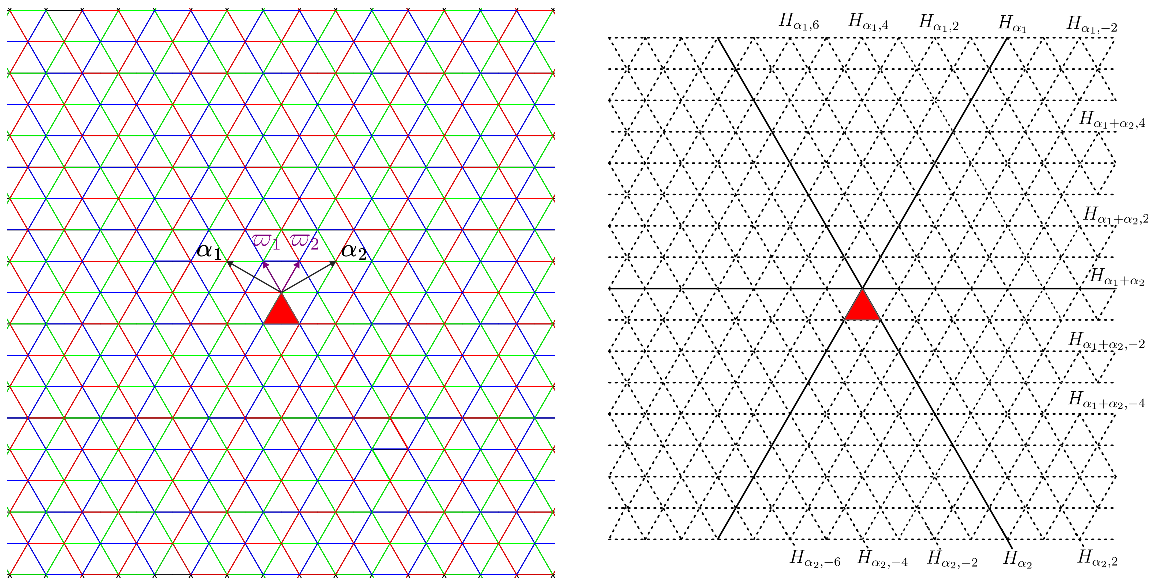
Let us recall a classical result.

Proposition 1.2.3 ([Bou68, Chapitre VI, §2, n^o 3.]). *Let $\mathscr{A}$ be the set of alcoves. The map $w \mapsto \mathscr{A}_w = w \cdot \mathscr{A}_{\mathrm{id}}$ gives a one-to-one correspondence between W and $\mathscr{A}$.*

The red triangle in Figure 1.1a is the fundamental alcove $\mathscr{A}_{\mathrm{id}}$ of $\widetilde{A}_2$. Each little triangle in the figure is an alcove. Via Proposition 1.2.3, we identify $\mathscr{A}_{\mathrm{id}}$ with the identity $\mathrm{id} \in W$. The colors on the edges of the triangles represent the following reflections: blue is s_0, green is s_1, and red is s_2.

We define the function $\mathrm{cen}\colon W \to E$, which maps an element $z \in W$ to the *center* of its alcove $\mathscr{A}_z$. When discussing the vertices, sides, faces, or area of an element $z \in W$, we refer to the corresponding property on the closure of its alcove, $\overline{\mathscr{A}_z}$. To avoid cumbersome notation, for $z \in W$ and a point $v \in E$ in the plane, we write $v \in z$ as a shortcut for $v \in \overline{\mathscr{A}_z}$. Since the left action of W on $\mathscr{A}$ is simply transitive, the equality $\mathscr{A}_x = \mathscr{A}_y$ implies that $x = y$. In particular, $\mathrm{cen}(x) = \mathrm{cen}(y)$ implies that $x = y$. An element $z \in W$ can be seen as a bijection $z\colon E \to E$, and its restriction $z|_{\mathrm{cen}(W)}\colon \mathrm{cen}(W) \to z(\mathrm{cen}(W)) = \mathrm{cen}(W)$ is a bijection. For any weight $\lambda \in \Lambda$, it is not hard to see that $\mathrm{cen}(W) + \lambda = \mathrm{cen}(W)$. Indeed, it is enough to check this for $\lambda = \varpi_1$ (by symmetry, this implies the case $\lambda = \varpi_2$), and one can check this directly in Figure 1.1a.

We say that two alcoves A, B have the same *orientation* if A can be obtained from B by a translation. If A has the same orientation as $\mathscr{A}_{\mathrm{id}}$, we say it is *up-oriented*; otherwise, we say *down-oriented*. The *orientation* of an element $z \in W$ is the orientation of its alcove $\mathscr{A}_z$.



(a) Coxeter complex of W.

(b) Set of (affine) lines $\mathscr{H}$.

Figure 1.1

Lemma 1.2.4. *Let $z \in W$.*

- *If $\ell(z)$ is odd, then z is down-oriented.*
- *If $\ell(z)$ is even, then z is up-oriented.*

Proof. It is standard that for $z \in W$, the length $\ell(z)$ measures the minimal number of hyperplanes of the form $H_{\alpha,k}$ that one must cross to go from the identity alcove to the alcove $\mathscr{A}_z$. Each crossing changes the orientation, allowing one to conclude by induction on $\ell(z)$. □

Notation 1.2.5. For $z \in W$ and a vector $v \in \Lambda$, we denote by $z + v$ the unique element in W with center $\text{cen}(z) + v$. In other words, $\text{cen}(z + v) = \text{cen}(z) + v$.

1.2.1 The $X\Theta$-partition of $\widetilde{A}_2$

In this section, we follow [BLP23] and [LP23] with slight modifications to their original notation.

Definition 1.2.6. Let $\underline{y} = (r_1, r_2, \cdots, r_k)$ be any expression of y (not necessarily reduced). If $1 < i < k$, we say that there is a *braid triplet in position* i if $r_{i-1} = r_{i+1} \neq r_i$. We define the *distance* between a braid triplet in position i and a braid triplet in position $j > i$ to be the number $j - i - 1$.

Lemma 1.2.7 ([LP23, Lemma 1.1]). *An expression of an element in W without adjacent simple reflections is reduced if and only if the distance between any two braid triplets is odd.*

For $n \in \mathbb{Z}_{\geq 1}$, consider the element $\mathbf{x}_n := 123\cdots n$. By Lemma 1.2.7, the expression $(1, 2, \ldots, n)$ is reduced, as there are no braid triplets, so $\ell(\mathbf{x}_n) = n$. Recall the group G from Section 1.2. Let us define

$$X^{\text{odd}} := \{g(\mathbf{x}_n) \mid n \text{ is odd}, g \in G\} \quad \text{and} \quad X^{\text{even}} := \{g(\mathbf{x}_n) \mid n \text{ is even}, g \in G\},$$

and we denote $X := X^{\text{odd}} \uplus X^{\text{even}}$, where $\uplus$ is the disjoint union.

For $m, n \in \mathbb{Z}_{\geq 0}$, we define

$$\theta(m,n) := 1234\cdots(2m+1)(2m+2)(2m+1)\cdots(2m-2n+1) \tag{1.1}$$

Since $(2m+1)(2m+2)(2m+1)$ is the unique braid triplet, Lemma 1.2.7 implies that the expression in (1.1) is reduced. In particular, we have that $\ell(\theta(m,n)) = 2m+2n+3$. We define

$$\Theta := \{g(\theta(m,n)) \mid m, n \geq 0, g \in G\}.$$

The elements in this set are precisely those elements in W having two elements in both their left and right descent sets. In particular, for any $\theta(m,n)$ there exists a unique $s_{m,n} \in S$ such that

$$\theta(m,n) < \theta(m,n)s_{m,n}.$$

To simplify formulas, we will use the notation $\theta^s(m,n) := \theta(m,n)s_{m,n}$. On the other hand, s_0 is the only simple reflection not included in the left descent set of $\theta(m,n)$. We define,

$$\Theta^s := \{g(\theta^s(m,n)) \mid m, n \geq 0, g \in G\},$$
$${}^s\Theta^s := \{g(s_0\theta^s(m,n)) \mid m, n \geq 0, g \in G\}.$$

Note that $\ell(\theta^s(m,n)) = 2m+2n+4$ and $\ell(s_0\theta^s(m,n)) = 2m+2n+5$.

Example 1.2.8. Consider $(m,n) = (2,4)$. In this case, we have $2m+2 = 6$ and $2m-2n-1 = -5$. The corresponding element $\theta(2,4)$ is then given by

$$\begin{aligned}\theta(2,4) &= 123456543210(-1)(-2)(-3)(-4)(-5)\\ &= \underbrace{1201}_{4}\underbrace{202}_{3}\underbrace{1021021021}_{8}.\end{aligned}$$

It follows that $\ell(\theta(2,4)) = 4+8+3 = 15$.

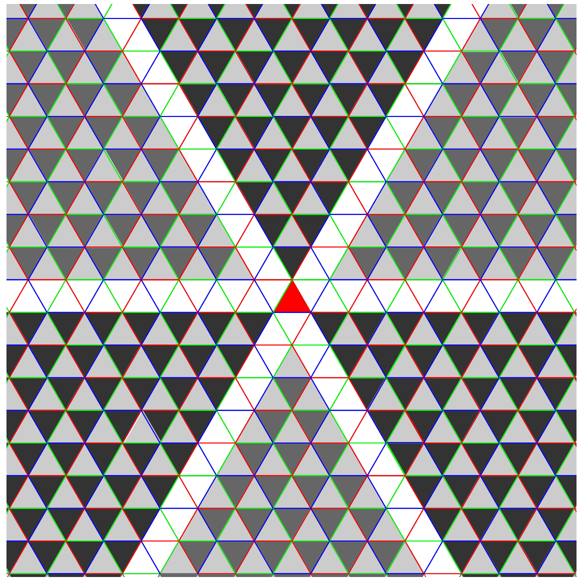


Figure 1.2: From lightest (white) to darkest gray: $X, \Theta^s, {}^s\Theta^s$, and Θ. The red triangle is $\mathcal{A}_{\mathrm{id}}$.

Remark 1.2.9. The element $\theta(m,n)$ can also be defined as the unique element in the affine Weyl group, such that $\mathcal{A}_{\theta(m,n)} = \mathcal{A}_{w_0} + m\varpi_1 + n\varpi_2$. Thus, for $m, m', n, n' \geq 0$ and $v = (m'-m)\varpi_1 + (n'-n)\varpi_2 \in \Lambda$, we have $\theta(m,n)+v = \theta(m',n')$. This last equation implies that $\theta^s(m,n) + v = \theta^s(m',n')$, because $\mathrm{cen}(\theta^s(m,n)) = \mathrm{cen}(\theta(m,n)) + \rho/3$.

Remark 1.2.10. For $k, k' \geq 0$ and $v = (k'-k)\varpi_1 \in \Lambda$, we have $\mathbf{x}_{2k} + v = \mathbf{x}_{2k'}$ and $\mathbf{x}_{2k+1} + v = \mathbf{x}_{2k'+1}$.

Remark 1.2.11. The elements in $X^{\mathrm{even}} \uplus \Theta^s$ have even length, while the elements in $X^{\mathrm{odd}} \uplus \Theta \uplus {}^s\Theta^s$ have odd length.

The $X\Theta$-*partition* of W is the decomposition $W \setminus \{\mathrm{id}\} = X \uplus \Theta \uplus \Theta^s \uplus {}^s\Theta^s$. This decomposition is illustrated in Figure 1.2.

Definition 1.2.12. We say that an element $x \in W$ is *dominant* if its alcove has a non-empty intersection with the dominant chamber (the light blue region in Figure 1a). Equivalently, x is dominant if it belongs to the set $D \coloneqq \{\theta(m,n), \theta^s(m,n) \mid m, n \geq 0\}$.

Remark 1.2.13. For any $y \in W \setminus \{\mathrm{id}\}$, there is an element $g \in \langle \delta, \sigma \rangle$ such that

$$g(y) \in \{\theta(m,n), s_0\theta(m,n), \theta^s(m,n), s_0\theta^s(m,n), \mathbf{x}_k \mid m, n \geq 0, k \geq 1\}.$$

Such a g is not necessarily unique for a fixed y. For example, $\sigma(\theta(3,4)) = \theta(4,3)$.

Using the equation $\ell(\mathbf{x}_n) = n$ and Remark 1.2.10, we obtain the following lemma.

Lemma 1.2.14. *If* $x, y \in \{\mathbf{x}_n \mid n \text{ is odd}\}$, *then* $\mathrm{cen}(y) - \mathrm{cen}(x) = (\ell(x,y)/2)\varpi_1$.

1.2.2 Upper and lower covers

In this section, we provide a complete list for the sets of lower covers of every element of W (the cases not considered in the list follow from Remark 1.2.13 and the fact that $L(gy) = gL(y)$ for $g \in G$) and a complete list for the sets of upper covers of every dominant element. In [BLP23, Lemma 2.2], there is a proof of the fifth case of the following lemma. One can prove the other cases similarly, so we omit this long and tedious proof.

Lemma 1.2.15. *Let m, n and k be positive integers with $m \geq n$, and $k \geq 2$. We have:*

$$\mathrm{L}(y) = \begin{cases} \{\theta^s(m-1,n), \theta^s(m,n-1), \delta(s_0\theta(m,n-1)), \\ \delta^2(s_0\theta(m-1,n))\} & \text{if } y = \theta(m,n), \\ \{\mathbf{x}_{2m+2}, \theta^s(m-1,0), \delta^2(s_0\theta(m-1,0))\} & \text{if } y = \theta(m,0), \\ \{\theta(m,n), \theta(m-1,n+1), \theta(m+1,n-1), \\ \delta(s_0\theta^s(m,n-1)), \delta^2(s_0\theta^s(m-1,n))\} & \text{if } y = \theta^s(m,n), \\ \{\mathbf{x}_{2m+3}, \theta(m,0), \theta(m-1,1), \delta^2(s_0\theta^s(m-1,0))\} & \text{if } y = \theta^s(m,0), \\ \{\theta(m,n), s_0\theta^s(m-1,n), s_0\theta^s(m,n-1) \\ \delta(\theta(m-1,n+1)), \delta^2(\theta(m+1,n-1))\} & \text{if } y = s_0\theta(m,n), \\ \{\theta(m,0), s_0\theta^s(m-1,0), \delta(\theta(m-1,1)), \\ \delta^2(\mathbf{x}_{2m+3})\} & \text{if } y = s_0\theta(m,0), \\ \{\theta^s(m,n), s_0\theta(m-1,n+1), s_0\theta(m+1,n-1), \\ s_0\theta(m,n), \delta(\theta^s(m-1,n+1)), \delta^2(\theta^s(m+1,n-1))\} & \text{if } y = s_0\theta^s(m,n), \\ \{\theta^s(m,0), s_0\theta(m,0), s_0\theta(m-1,1), \\ \delta^2(\mathbf{x}_{2m+4}), \delta^2(\theta^s(m-1,1))\} & \text{if } y = s_0\theta^s(m,0). \\ \{\mathbf{x}_{2k-1}, \theta(\frac{2k-4}{2},0), \delta(\mathbf{x}_{2k-1}), \delta^2(\theta(\frac{2k-4}{2},0))\} & \text{if } y = \mathbf{x}_{2k}, \\ \{\mathbf{x}_{2k}, \delta(\mathbf{x}_{2k}), \delta(s_0\theta(\frac{2k-4}{2},0)), \delta^2(\theta^s(\frac{2k-4}{2},0))\} & \text{if } y = \mathbf{x}_{2k+1}. \end{cases}$$

Although Lemma 1.2.15 assumes $m \geq n$, the remaining cases follow from the facts that $L(\sigma y) = \sigma L(y)$ and that σ swaps the values of m and n.

An immediate consequence of Lemma 1.2.15 is the following.

Lemma 1.2.16. *Let $y \in W$.*

1. *If $y \in \Theta \setminus \{\theta(0,0)\}$, then $|\,\mathrm{L}(y)| \in \{3,4\}$.*
2. *If $y \in \Theta^s \setminus \{\theta^s(0,0), s_0\theta(0,0)\}$, then $|\,\mathrm{L}(y)| \in \{4,5\}$.*
3. *If $y \in {}^s\Theta^s \setminus \{s_0\theta^s(0,0)\}$, then $|\,\mathrm{L}(y)| \in \{5,6\}$.*

The following lemma is a less obvious consequence of Lemma 1.2.15.

Lemma 1.2.17. *Let $x \in W$, we have:*

1. *If $x \in \Theta$, then $|\,\mathrm{U}(x)| = 6$.*
2. *If $x \in \Theta^s$, then $|\,\mathrm{U}(x)| = 5$.*
3. *If $x \in {}^s\Theta^s$, then $|\,\mathrm{U}(x)| = 4$.*

Proof. Let m, n be non-negative integers. Using Lemma 1.2.15 and the fact that $x \lhd y$ if and only if $gx \lhd gy$ for each $g \in G$, we obtain

$$\mathrm{U}(x) = \begin{cases} \{\theta^s(m,n), \theta^s(m+1,n-1), \theta^s(m-1,n+1), \\ s_0\theta(m,n), \delta(s_0\theta(m-1,n+1)), \delta^2(s_0\theta(m+1,n-1))\} & \text{if } x = \theta(m,n), \\ \{\theta(m+1,n), \theta(m,n+1), s_0\theta^s(m,n), \\ \delta(s_0\theta^s(m-1,n+1)), \delta^2(s_0\theta^s(m+1,n-1))\} & \text{if } x = \theta^s(m,n), \\ \{s_0\theta(m,n+1), s_0\theta(m+1,n), \\ \delta(\theta^s(m,n+1)), \delta^2(\theta^s(m+1,n))\} & \text{if } x = s_0\theta^s(m,n). \end{cases}$$

Thus, the lemma holds for elements in $\{\theta(m,n), \theta^s(m,n), s_0\theta^s(m,n) \mid m,n > 0\}$. For the remaining cases ($m = 0, n \geq 0$ or $m \geq 0, n = 0$), similar formulas for the set $U(x)$ can also be derived. Finally, by applying the fact $\mathrm{U}(gx) = g\,\mathrm{U}(x)$ for each $g \in G$, the result follows. □

Proposition 1.2.18. *Let $x, y \in W \setminus (X \cup \{\mathrm{id}\})$. If $[x,y]$ is non-degenerate, then the poset invariants $\mathrm{LC}(x,y)$ and $\ell(x,y)$ determine the parts of the $X\Theta$-partition containing x and y.*

Proof. We may assume that $y \notin \{\theta(0,0), \theta^s(0,0), s_0\theta(0,0), s_0\theta^s(0,0)\}$, since in each of these cases the possible interval $[x,y]$ is degenerate.

Since the interval $[x,y]$ is non-degenerate, the set of the atoms of $[x,y]$ is $U(x)$, and the set of coatoms of $[x,y]$ is $L(y)$. This implies that $f_1^{x,y} = |U(x)|$ and $f_{\ell(x,y)}^{x,y} = |L(y)|$. By Lemma 1.2.17 and the fact that $f_1^{x,y} = |U(x)|$, this the quantity determines the part of the $X\Theta$-partition containing x. By Remark 1.2.11, this determines whether $\ell(x)$ is odd or even. Now we can determine the $X\Theta$-partition containing y. There are two cases: If $\ell(x)$ and $\ell(x,y)$ have the same parity, then $\ell(y)$ is even, so by Remark 1.2.11 we have $y \in \Theta^s$ and we are done. If $\ell(x)$ and $\ell(x,y)$ have different parity, then $\ell(y)$ is odd so by Remark 1.2.11 we have $y \in \Theta \cup {}^s\Theta^s$. By Lemma 1.2.16 and the fact that $f_{\ell(x,y)-1}^{x,y} = |\mathrm{L}(y)|$, we can distinguish if $y \in \Theta$ or $y \in {}^s\Theta^s$. □

Corollary 1.2.19. *Let $[x,y], [x',y']$ be non-degenerate intervals and let x, x', y, y' be dominant elements. If $[x,y]$ and $[x',y']$ are isomorphic as posets, then x and x' belong to the same part of the $X\Theta$-partition, and similarly, y and y' belong to the same part of the $X\Theta$-partition. In particular, there are $\lambda, \mu \in \Lambda$ such that $x + \lambda = x'$ and $y + \mu = y'$.*

Proof. The first part follows from Proposition 1.2.18. The second part follows from Remark 1.2.9. □

Chapter 2

The alcove geometry of Bruhat intervals

2.1 The geometry of lower intervals

In this section we generalize the geometric descriptions of lower intervals given in [LP23, BLP23]. Applying these results, we give formulas for the cardinalities of all lower intervals.

2.1.1 Basic notions in planar convex geometry and a convention

A convex bounded polygon P is the convex hull of a finite set of points in the plane. If such a set is minimal, we call it the set of *vertices* of P, and we denote it by $\mathrm{V}(P)$. An *edge* of P is a closed segment obtained from the intersection of a closed half-plane and P. Given a polygon P, we say that two vertices a, b are *adjacent* if ab is an edge of P. For any subset Y of the plane, we denote its convex hull by $\mathrm{Conv}(Y)$.

To avoid cumbersome notation and when the context is clear, we will usually not distinguish between elements of W and the center of their respective alcoves. We will give a series of examples of this. For $A \subset W$, we define $\mathrm{Conv}(A) := \mathrm{Conv}(\mathrm{cen}(A))$. Similarly, if $\mathrm{V}(P) = \mathrm{cen}(A)$, we write $\mathrm{V}(P) = A$. If $x, y \in W$, we denote by $\mathrm{Sgm}(x, y)$ the line segment in the plane with end points $\mathrm{cen}(x)$ and $\mathrm{cen}(y)$, or in other words, $\mathrm{Sgm}(x, y) = \mathrm{Conv}(\{\mathrm{cen}(x), \mathrm{cen}(y)\})$. If $x \in W$ and $v \in E$, we denote by $\mathrm{Sgm}(x, v)$ or $\mathrm{Sgm}(v, x)$ the line segment in the plane with endpoints $\mathrm{cen}(x)$ and v. Similar conventions are used for the inner product $(-, -)$, so for instance, (v, x) means $(v, \mathrm{cen}(x))$ where $v \in E$ and $x \in W$. If $x, y, z \in W$, we write $z \in \mathrm{Sgm}(x, y)$ instead of $\mathrm{cen}(z) \in \mathrm{Sgm}(x, y)$. More generally, if $Y \subseteq E$ and $x \in W$, sometimes we write $x \in Y$ instead of $\mathrm{cen}(x) \in Y$. Finally, we denote $d(a, b) := |\,\mathrm{Sgm}(a, b)|$, where

a, b can be points of E or elements of W, or combinations of these.

2.1.2 Geometric description of lower intervals

Proposition 2.1.1. *For any $y \in W$ there is a bounded convex polygon $\mathcal{C}_y$ such that*

$$[\mathrm{id}, y] = \{x \in W \mid \mathrm{cen}(x) \in \mathcal{C}_y\}.$$

Proof. The proof is divided into cases. The case $y = \mathrm{id}$, follows immediately by considering the polygon $\mathcal{C}_{\mathrm{id}} := \mathrm{Conv}(\{\mathrm{id}\})$. If $s \in S$, then the polygon $\mathcal{C}_s := \mathrm{Conv}(\{\mathrm{id}, s\})$ ($s \in S$) satisfies the proposition. We now assume that $y \notin \{\mathrm{id}\} \cup S$. Consider the following two statements.

- If y is dominant, the hexagon $\mathcal{C}_y$ with vertex set $W_f \cdot y$ satisfies the proposition.
- Let $k \geq 2$ and $y = \mathbf{x}_k$. The quadrilateral $\mathcal{C}_{\mathbf{x}_k}$ with vertex set

  $$\{\mathbf{x}_k, s_1\mathbf{x}_k, s_2s_1\mathbf{x}_k, s_1s_2s_1\mathbf{x}_k\}$$

  satisfies the proposition.

The proof of the first bullet for $y = \theta(m, n)$ is done in [LP23, Lemma 1.4]. The remaining case of the first bullet, i.e., $y = \theta^s(m, n)$ and the second bullet, i.e., $\mathbf{x}_k$, have proofs similar to that of [LP23, Lemma 1.4], so we omit them. We now focus on the case $y \in {}^s\Theta^s$.

For m and n non-negative integers, let $y = s_0\theta^s(m, n)$. Define $z := \theta^s(m, n)$. By Lemma 1.1.8, we have $[\mathrm{id}, y] = [\mathrm{id}, z] \cup s_0[\mathrm{id}, z]$. As z belongs to the cases treated in the first bullet above, this is equivalent to

$$[\mathrm{id}, y] = \{x \in W \mid \mathrm{cen}(x) \in \mathcal{C}_z \cup s_0\mathcal{C}_z\}.$$

Claim 2.1.2. $s_0\mathcal{C}_z = \mathcal{C}_z - \rho$, *where* $\rho = \omega_1 + \omega_2$.

Proof. We have

$$\mathrm{V}(\mathcal{C}_z) = \{z, s_1z, s_2z, s_1s_2z, s_2s_1z, s_1s_2s_1z\}, \text{ and} \tag{2.1}$$
$$\mathrm{V}(s_0\mathcal{C}_z) = \{s_0z, s_0s_1z, s_0s_2z, s_0s_1s_2z, s_0s_2s_1z, s_0s_1s_2s_1z\}. \tag{2.2}$$

By definition $s_0 = s_{\rho,-1}$ and one can easily check that $s_{\rho,0} = s_1s_2s_1$ (for example, check what both elements do to the alcove $\mathcal{A}_{\mathrm{id}}$). This implies that for any $x \in W$, we have $s_0x = s_1s_2s_1x - \rho$. This last equation implies the claim as $s_0z = s_1s_2s_1z - \rho$, $s_0s_1z = s_1s_2z - \rho$, $s_0s_2z = s_2s_1z - \rho$, etc. □

Claim 2.1.3. $\mathrm{cen}^{-1}(\mathcal{C}_z \cup s_0\mathcal{C}_z) = \mathrm{cen}^{-1}(\mathrm{Conv}(\{z, s_1z, s_2z, s_0z, s_0s_1z, s_0s_2z\}))$

Proof. By Equation (2.1) and as $s_{\rho,0} = s_1s_2s_1$, we have that two edges of the hexagon $\mathcal{C}_z$ are parallel to the vector ρ and the same applies to the hexagon $s_0\mathcal{C}_z$.

Let $a = s_1z, b = s_1s_2z, c = s_2z, d = s_2s_1z$, so that a, b, c, d are four of the six vertices of $\mathcal{C}_z$ and the segments $\mathrm{Sgm}(a, b)$ and $\mathrm{Sgm}(c, d)$ are the edges of $\mathcal{C}_z$ parallel to ρ. By Claim 2.1.2, we have that $a-\rho, b-\rho, c-\rho, d-\rho \in \mathrm{V}(s_0\mathcal{C}_z)$. Note that it is possible for $a - \rho$ (resp. $c - \rho$) to either lie on $\mathrm{Sgm}(a, b)$ (resp. $\mathrm{Sgm}(c, d)$) or not. To clarify in which case we are, we will now provide a necessary and sufficient condition to determine if $a - \rho$ (resp. $c - \rho$) lies on $\mathrm{Sgm}(a, b)$ (resp. $\mathrm{Sgm}(c, d)$).

As $s_{\rho,0} = s_1s_2s_1$, we have that $b = s_{\rho,0}a$ and $d = s_{\rho,0}c$. This implies that $|\mathrm{Sgm}(a, b)| = 2d(a, H_{\rho,0})$ (resp. $|\mathrm{Sgm}(c, d)| = 2d(c, H_{\rho,0})$). Then $a - \rho \in \mathrm{Sgm}(a, b)$ if and only if $2d(a, H_{\rho,0}) \geq \|\rho\|$ i.e. $d(a, H_{\rho,0}) \geq \frac{1}{2}\|\rho\|$.

On the other hand, the distance between a and $H_{\rho,0}$ is given by $\mathrm{d}(a, H_{\rho,0}) = \frac{|(a,\rho)|}{\|\rho\|}$. Thus, $a - \rho \in \mathrm{Sgm}(a, b)$ if and only if $|(a, \rho)| \geq \frac{\|\rho\|^2}{2}$. Since $\rho = \varpi_1 + \varpi_2$ and, as we have that $\|\varpi_i\|^2 = \frac{2}{3}$ and the angle between ϖ_1 and ϖ_2 is $\frac{\pi}{3}$, we conclude $\|\rho\| = \sqrt{2}$. Hence, $a - \rho \in \mathrm{Sgm}(a, b)$ if and only if $|(a, \rho)| \geq 1$.

One can check that $y \in D$ if and only if $(\mathrm{cen}(y), \alpha) > 0$ for all $\alpha \in \{\alpha_1, \alpha_2\}$ Additionally, since $a = s_1z$ and $\rho = s_1\alpha_2$, we have that $|(a, \rho)| \geq 1$ if and only if $(\mathrm{cen}(z), \alpha_2) \geq 1$. We can rewrite the above condition as follows:

$$a - \rho \in \mathrm{Sgm}(a, b) \quad \text{if and only if} \quad (\mathrm{cen}(z), \alpha_2) \geq 1 \tag{2.3}$$

Similarly we have that

$$c - \rho \in \mathrm{Sgm}(c, d) \quad \text{if and only if} \quad (\mathrm{cen}(z), \alpha_1) \geq 1. \tag{2.4}$$

We split the rest of the proof into three cases. Recall that by definition $z = \theta^s(m, n)$.

1. Let us assume that $m, n \geq 1$ (see Figure 2.1). In this case, one can check that $(\mathrm{cen}(z), \alpha_1) \geq 1$ and $(\mathrm{cen}(z), \alpha_2) \geq 1$. To see this, it is enough to calculate these inequalities with $m = n = 1$, which is straightforward. Indeed by Remark 1.2.9, we have $\mathrm{cen}(\theta^s(1, 1)) = \mathrm{cen}(w_0) + \frac{4}{3}\rho$ (recall that $w_0 = s_1s_2s_1$ denotes the longest element in W_f). By Equation (2.3) and Equation (2.4), we have $a-\rho \in \mathrm{Sgm}(a, b)$ and $c-\rho \in \mathrm{Sgm}(c, d)$. This implies that $b \in \mathrm{Sgm}(a-\rho, b-\rho)$ and $d \in \mathrm{Sgm}(c - \rho, d - \rho)$. In other words, we have

$$\mathrm{Sgm}(a, b) \cup \mathrm{Sgm}(a - \rho, b - \rho) = \mathrm{Sgm}(a, b - \rho)$$
$$\mathrm{Sgm}(c, d) \cup \mathrm{Sgm}(c - \rho, d - \rho) = \mathrm{Sgm}(c, d - \rho).$$

   Furthermore, as $z = \theta^s(m, n)$ with $m, n \geq 1$, one can see that $z - \rho \in D$, so $z - \rho \in \mathcal{C}_z$. We also have that $s_1s_2s_1z \in s_0\mathcal{C}_z$, because $s_1s_2s_1z - \rho \in V(s_0\mathcal{C}_z)$. So we have $\{z - \rho, a - \rho, c - \rho\} \subset \mathcal{C}_z$ and $\{b, d, s_1s_2s_1z\} \subset s_0\mathcal{C}_z$. From this two inclusions, together with Equation (2.1), Equation (2.2), Claim (2.1.2) and the fact that $\mathrm{Sgm}(a,b), \mathrm{Sgm}(c,d)$ are parallel to ρ, we conclude that $\mathcal{C}_z \cup s_0\mathcal{C}_z$ is a hexagon with vertices $\{z, s_0z, a, c, b-\rho, d-\rho\}$. Therefore, we have

$$\mathcal{C}_z \cup s_0\mathcal{C}_z = \mathrm{Conv}(\{z, s_1z, s_2z, s_0z, s_0s_1z, s_0s_2z\}), \tag{2.5}$$

so the claim follows.

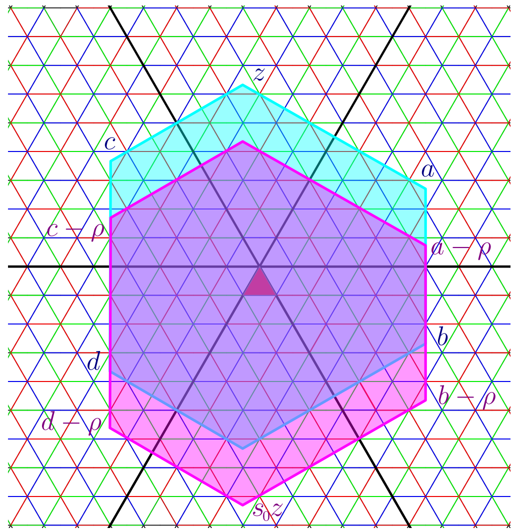
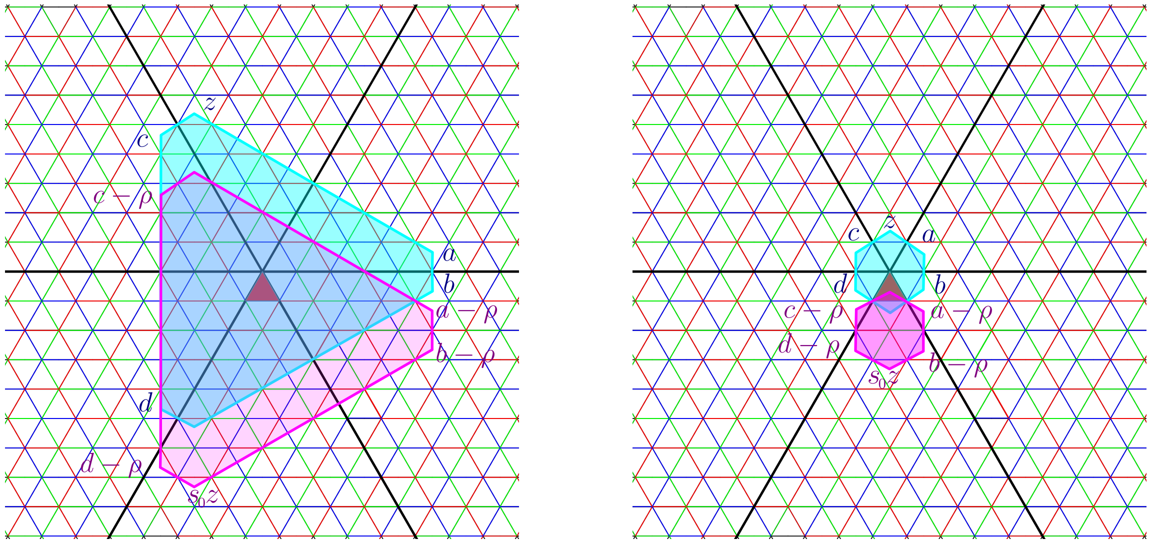


Figure 2.1: $\mathcal{C}_z$ and $s_0\mathcal{C}_z$, for $z = \theta^s(m,n)$.

2. In this case we assume $m = 0$ and $n \geq 1$ or the symmetric case $n = 0$ and $m \geq 1$, see Figure 2.2a for an illustration. Let us prove the case $z = \theta^s(m,0)$ with $m \geq 1$, the other one being symmetric. We have $(\mathrm{cen}(z), \alpha_1) \geq 1$ and $0 < (\mathrm{cen}(z), \alpha_2) < 1$. By Equation (2.4), we have that $c - \rho \in \mathrm{Sgm}(c,d)$, so $d \in \mathrm{Sgm}(c-\rho, d-\rho)$. However, by Equation (2.3), we have $a - \rho \notin \mathrm{Sgm}(a,b)$. Thus, Equation (2.5) does not hold, although the strict inclusion $\subsetneq$ still holds in this case. The difference between the set in the right and the set in the left of Equation (2.5) is a "small" equilateral triangle T, with two vertices of T being $a - \rho$ and b. It is easy to see that T contains no center of alcoves other than $a - \rho$ and b, as $a - \rho$ and b are the centers of two alcoves (call them A and B), the closures $\overline{A}$ and $\overline{B}$ share an edge, and $T \subset \overline{A} \cup \overline{B}$. This proves the claim for $n = 0$ and $m \geq 1$.

3. Let us assume $m = n = 0$, see Figure 2.2b. We have that $0 < (\mathrm{cen}(z), \alpha_1) < 1$ and $0 < (\mathrm{cen}(z), \alpha_2) < 1$. This case is similar to the previous one. However, instead of obtaining just one equilateral triangle, we now obtain two: one with $a - \rho$ and b as two of its vertices and the other with $c - \rho$ and d as two of its vertices. $\square$



(a) $\mathcal{C}_z$ and $s_0\mathcal{C}_z$, for $z = \theta^s(m, 0)$. (b) $\mathcal{C}_z$ and $s_0\mathcal{C}_z$, for $z = \theta^s(0, 0)$.

Figure 2.2

Back to the proof of the proposition, we conclude that the hexagon $\mathcal{C}_y$, for $y = s_0\theta^s(m, n)$ has vertex set $\{z, s_1z, s_2z, s_0z, s_0s_1z, s_0s_2z\}$, where $z = \theta^s(m, n)$.

The case $y = s_0\theta(m, n)$ can be proved in the same way as the case $y = s_0\theta^s(m, n)$, simply replacing $z = \theta^s(m, n)$ with $z = \theta(m, n)$ in the proof, so we omit the details. Finally, consider y and g as in Remark 1.2.13, then $x = gy$ belongs to one of the cases above. Since g acts as an isometry, for any set Y, we have that $g\,\mathrm{Conv}(Y) = \mathrm{Conv}(gY)$. So $\mathcal{C}_y := g^{-1}\mathcal{C}_x$ satisfies the proposition. □

Corollary 2.1.4. *Let $y \in W$, $g \in \langle \delta, \sigma \rangle$, and $k \geq 1$ then we have*

$$\mathrm{V}(\mathcal{C}_y) = \begin{cases} \delta^i W_f\, \delta^{-i} y & \text{if } y \in \delta^i D, \\ \{y, s_iy, s_{1+i}s_iy, s_{2+i}s_iy, s_is_{1+i}s_iy, s_is_{2+i}s_iy\} & \text{if } y \in \delta^i s_0 D, \\ g\{\mathbf{x}_k, s_1\mathbf{x}_k, s_2s_1\mathbf{x}_k, s_1s_2s_1\mathbf{x}_k\} & \text{if } y = g\mathbf{x}_k. \end{cases}$$

Where $D = \{\theta(m, n), \theta^s(m, n) \mid m, n \geq 0\}$ and $\mathcal{C}_y$ is the bounded convex polytope of Proposition 2.1.1 .

Corollary 2.1.4 gives a geometric characterization of the Bruhat order in W. The Bruhat order relation $x \leq y$ is equivalent to the geometric statement $\mathrm{cen}(x) \in \mathcal{C}_y$.

Corollary 2.1.5. *Let $y \in W$. Then $x \leq y$ if and only if $x \in \mathcal{C}_y$, where*

$$\mathcal{C}_y = \begin{cases} \mathrm{Conv}(\delta^i \langle s_1, s_2 \rangle \delta^{-i} y) & \text{if } y \in \delta^i D, \\ \mathrm{Conv}(\{y, s_iy, s_{1+i}s_iy, s_{2+i}s_iy, s_is_{1+i}s_iy, s_is_{2+i}s_iy\}) & \text{if } y \in \delta^i s_0 D, \\ \mathrm{Conv}(g\{\mathbf{x}_k, s_1\mathbf{x}_k, s_2s_1\mathbf{x}_k, s_1s_2s_1\mathbf{x}_k\}) & \text{if } y = g\mathbf{x}_k. \end{cases}$$

We give formulas of $|[\mathrm{id}, y]|$, for every $y \in W$. They are not hard to obtain from the geometric description above. For $m, n \geq 0$ and $i, j \in \{0, 1\}$, we denote $\theta(m, n, i, j) := (s_0)^i \theta(m, n)(s_{m,n})^j$ and $L(m, n, i, j) := |[\mathrm{id}, \theta(m, n, i, j)]|$.

Proposition 2.1.6. *For all $m, n \in \mathbb{Z}_{\geq 0}$ and $i, j \in \{0, 1\}$.*

$$L(m, n, i, j) = 3m^2 + 3n^2 + 12mn + 9m + 9n + 6 + (i + j)(6m + 6n + 6) + 4ij.$$

For $n \geq 1$, we have $|[\mathrm{id}, \mathbf{x}_{2n}]| = 3n^2 + n$ and $|[\mathrm{id}, \mathbf{x}_{2n+1}]| = 3n^2 + 5n$.

Most of these formulas already appear in [LP23, Lemma 1.5] and [BLP23, Equations 2.10, 2.11, and 2.12]. As far as we know, the formula for $L(m, n, 1, 0)$ is new.

2.1.3 Lower dihedral subintervals

Let y be dominant. In this section, we give a geometric description of the set $\mathcal{D}(y) = \{z \in [\mathrm{id}, y] \mid [z, y] \text{ is dihedral}\}$. The discussion of "upper" dihedral subintervals is more complex, so it has been postponed to Section 2.2.4.

First, we define $\mathcal{D}_{\mathrm{sgm}}(y) := \{z \in W \mid z \in \mathrm{Sgm}(y, s_1 y) \cup \mathrm{Sgm}(y, s_2 y)\}$.

Notation 2.1.7. We call an element $y \in W$ *left-dominant* if $y \in D$ and $y \in \mathbb{R}\rho$ or y lives in the left half-space determined by the line $\mathbb{R}\rho$. We define similarly *right-dominant.*

For the next definition, we suppose that y is left-dominant. In other words, $y = \theta(m, n)$ or $y = \theta^s(m, n)$ with $m \geq n \geq 0$. Let $W_{\geq 2, f, y} := \{w \in W_f \mid \ell(w) \geq 2\} y$.

$$\mathcal{D}_{\mathrm{rest}}(y) := \begin{cases} W_{\geq 2,f,y} \uplus \{\mathbf{x}_{2m}, \mathbf{x}_{2m+1}, \theta(m-1, 0)\} & \text{if } y = \theta(m, 0), \\ W_{\geq 2,f,y} \uplus \{\theta^s(m-1, n-1), \theta(m, n-1), \\ \theta(m-1, n), s_1\theta(m, n-1), s_2\theta(m-1, n)\} & \text{if } y = \theta(m, n), \\ W_{\geq 2,f,y} \uplus \{\mathbf{x}_{2m+2}, s_1\mathbf{x}_{2m+3}, \theta(m, 0), \\ \theta^s(m-1, n), s_2\theta(m-1, n)\} & \text{if } y = \theta^s(m, 0), \\ W_{\geq 2,f,y} \uplus \{\theta(m, n), \theta^s(m, n-1), \theta^s(m-1, n), s_1\theta(m, n), \\ s_2\theta(m, n), s_2\theta(m-1, n+1), s_1\theta(m+1, n-1)\} & \text{if } y = \theta^s(m, n). \end{cases}$$

The proof of the following lemma is a case-by-case verification.

Lemma 2.1.8. *We have $\ell(z, y) \leq 3$, for all $z \in \mathcal{D}_{\mathrm{rest}}(y)$.*

Proposition 2.1.9. *If y is left-dominant then $\mathcal{D}(y) = \mathcal{D}_{\mathrm{sgm}}(y) \uplus \mathcal{D}_{\mathrm{rest}}(y)$. If y is right-dominant then $\mathcal{D}(y) = \mathcal{D}_{\mathrm{sgm}}(y) \uplus \sigma(\mathcal{D}_{\mathrm{rest}}(y))$.*

Proof. We start with a technical claim.

Claim 2.1.10. *If $y \in D$, we have an inclusion of sets $L(y) \subset D \cup \{s_1 y, s_2 y\}$.*

Proof. Without loss of generality, we assume that y is left-dominant. If $y = \theta(0,0)$, the claim is obvious. Otherwise, Lemma 1.2.15 gives the list of elements of $L(y)$.

- If $y \in \{\theta(m,n) \mid m \geq n > 0\}$ the claim follows from the equalities

$$\begin{aligned} s_1\theta(m,n) &= \delta^2(s_0\theta(m-1,n)) \\ s_2\theta(m,n) &= \delta(s_0\theta(m,n-1)). \end{aligned}$$

- If $y \in \{\theta^s(m,n) \mid m \geq n > 0\}$ the claim follows from the equalities

$$\begin{aligned} s_1\theta^s(m,n) &= \delta^2(s_0\theta^s(m-1,n)) \\ s_2\theta^s(m,n) &= \delta(s_0\theta^s(m,n-1)). \end{aligned}$$

- If $y \in \{\theta(m,0) \mid m > 0\}$ the claim follows from the equalities

$$\begin{aligned} s_1\theta(m,0) &= \delta^2(s_0\theta(m-1,0)) \\ s_2\theta(m,0) &= \mathbf{x}_{2m+2}. \end{aligned}$$

- If $y \in \{\theta^s(m,0) \mid m > 0\}$ the claim follows from the equalities

$$\begin{aligned} s_1\theta^s(m,0) &= \delta^2(s_0\theta^s(m-1,0)) \\ s_2\theta^s(m,0) &= \mathbf{x}_{2m+3}. \end{aligned}$$

□

We will now prove the proposition for a left-dominant element, the argument for a right-dominant element being symmetric. Let $z \in \mathcal{D}(y)$ be such that $\ell(z,y) > 1$. By Proposition 1.1.4, the interval $[z,y]$ contains exactly two coatoms. If c is a coatom of $[z,y]$, then $c \in \mathrm{L}(y) \subset D \cup \{s_1 y, s_2 y\}$.

Let us label the elements of $\mathrm{L}(y)$ by $c_1, c_2, \ldots, c_k$. By Proposition 1.1.4, and Corollary 2.1.5 there are two different indexes i,j in $\{1,\ldots,k\}$, such that $\mathrm{cen}(z) \in \mathcal{C}_{c_i} \cap \mathcal{C}_{c_j}$ and $\mathrm{cen}(z) \notin \mathcal{C}_{c_t}$, for any $t \notin \{i,j\}$. So, if we define

$$A_{i,j} := (\mathcal{C}_{c_i} \cap \mathcal{C}_{c_j}) \setminus \left(\bigcup_{y \notin \{i,j\}} \mathcal{C}_{c_t} \right),$$

we have that

$$\mathcal{D}(y) = L(y) \cup \left(\bigcup_{i,j} A_{i,j} \right). \tag{2.6}$$

Claim 2.1.11. *Let y left-dominant. If $c \in L(y)$, then we have*

$$\mathcal{C}_c = \begin{cases} \mathrm{Conv}(W_f \cdot c) & \text{if } c \in D, \\ \mathrm{Conv}(\{c, y-\alpha_1, s_2(y-\alpha_1), s_1s_2y, s_2s_1y, s_1s_2s_1y\}) & \text{if } c = s_1y, \\ \mathrm{Conv}(\{c, y-\alpha_2, s_1(y-\alpha_2), s_1s_2y, s_2s_1y, s_1s_2s_1y\}) & \text{if } c = s_2y \text{ and } c \notin X, \\ \mathrm{Conv}\{c, s_1s_2y, s_1s_2s_1y, s_2s_1y\} & \text{if } c = s_2y \text{ and } c \in X. \end{cases}$$

Proof. By Claim 2.1.10, we have that $c \in D \cup \{s_1y, s_2y\}$. If $c \in D$, the result follows directly from Corollary 2.1.5. It remains the case when $c \in \{s_1y, s_2y\}$.

First, assume that $c = s_1y$. By Corollary 2.1.5, we have that

$$\mathcal{C}_c = \mathrm{Conv}\{c, s_2c, s_0s_2c, s_1s_2c, s_2s_0s_2c, s_2s_1s_2c\}.$$

The result follows from the identities $s_0 = s_{\rho,-1} = s_1s_2s_1 - \rho$ and $s_2\alpha_1 = \rho$ (for example, we observe that $s_0s_2c = s_1s_2s_1s_2c - \rho = s_2s_1c - \rho = s_2y - \rho = s_3(y - \alpha_1)$).

Now, assume that $c = s_2y$. Since y is left dominant, we have either $c \in \delta s_0 D$ or $c \in \{\mathbf{x}_{2m+2}, \mathbf{x}_{2m+3}\}$. The first case is analogous to the case $c = s_1y$. Therefore, it remains to consider that case $c \in \{\mathbf{x}_{2m+2}, \mathbf{x}_{2m+3}\}$, which follows directly from Corollary 2.1.5. □

Returning to the proof of the proposition, let us first prove the case $y = \theta(5, 2)$ (we will later prove that this case is generic for $\theta(m, n)$). In this case $|L(y)| = k = 4$. In Figures 2.3a, 2.3b and 2.3c, we have drawn the hexagons $\mathcal{C}_c$ for all $c \in L(y) = \{c_1, c_2, c_3, c_4\}$. Also, in Figure 2.3c, the purple hexagon corresponds to the intersection $\mathcal{C}_{c_2} \cap \mathcal{C}_{c_3}$. In Figure 2.3d, the orange alcoves represent the elements of $\mathcal{D}_{\mathrm{rest}}(y)$. A quick inspection shows that:

i $\mathrm{cen}(z) \in A_{1,2}$ if and only if $z \in \mathrm{Sgm}(y, s_2y) \setminus \{c_1, c_2\}$.

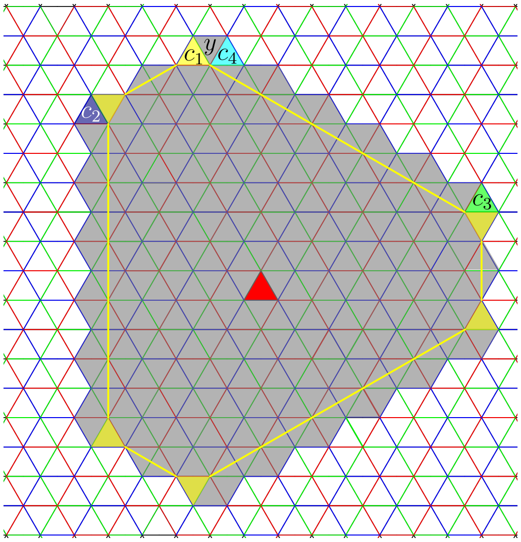
ii $\mathrm{cen}(z) \in A_{3,4}$ if and only if $z \in \mathrm{Sgm}(y, s_1y) \setminus \{c_3, c_4\}$.

iii In the remaining cases, we have:

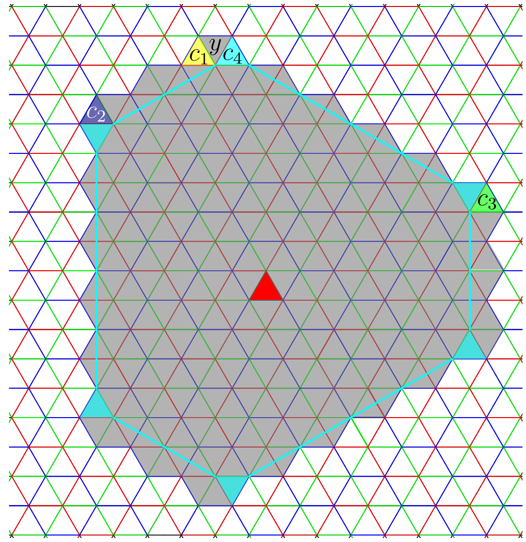
$$A_{1,4} = \{z_1, z_2, z_8\}, \quad A_{1,3} = \{z_7\}, \quad A_{2,4} = \{z_3\}, \quad A_{2,3} = \{z_4, z_5, z_6\}.$$

Therefore, using Equation (2.6) we have proved in this case that $z \in \mathcal{D}(y)$ if and only if $z \in \mathcal{D}_{\mathrm{sgm}}(y) \uplus \mathcal{D}_{\mathrm{rest}}(y)$.

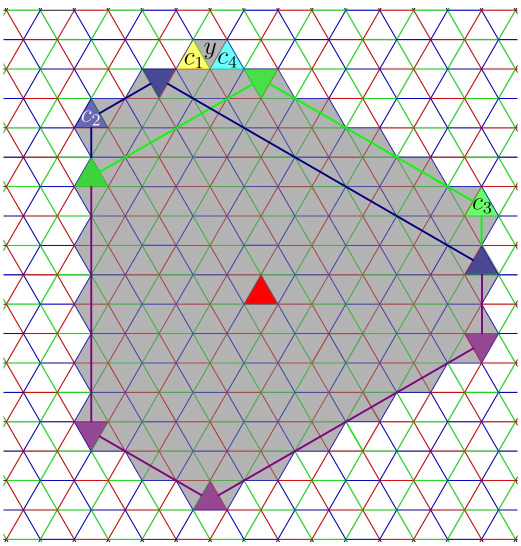
We claim that a general $\theta(m, n)$ is essentially the same argument. Claim 2.1.10 tell us that $\mathrm{L}(\theta(m, n)) = \{s_1y, s_2y, \theta^s(m-1, n), \theta^s(m, n-1)\}$. This means that $c_1 := \theta^s(m-1, n)$ and $c_4 := \theta^s(m, n-1)$ are the two alcoves sharing an edge with y different from $\theta^s(m, n)$, $c_2 := s_2y$ and $c_3 := s_1y$. We will still have that z_2 (resp. z_8) is the alcove below c_1 (resp. c_4). We have that z_1 is the only alcove sharing an edge



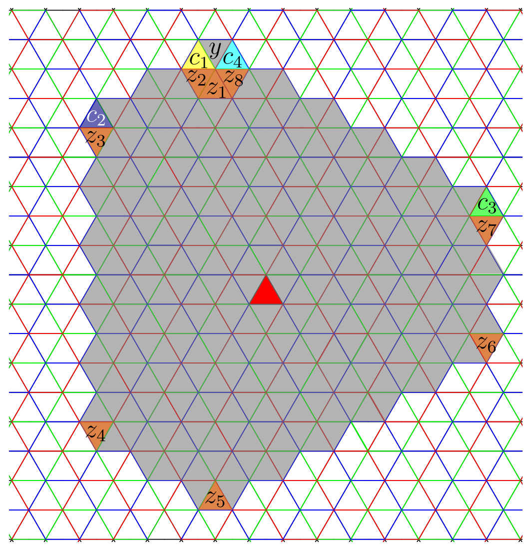
(a) The hexagon $\mathcal{C}_{c_1}$.



(b) The hexagon $\mathcal{C}_{c_4}$.



(c) The hexagons $\mathcal{C}_{c_2}$, $\mathcal{C}_{c_3}$, and their intersection.



(d) In orange, the eight elements of $\mathcal{D}_{\text{rest}}(y)$.

Figure 2.3: Polygons $\mathcal{C}_z$ for $z \in \mathrm{L}(\theta(5,2))$.

with both z_2 and z_8, the alcove z_7 is the one below c_3, etc. All of this implies that the calculations of $A_{i,j}$ give the exact same results as in i, ii, iii above.

The proof of the case $\theta^s(m,n)$ follows similar lines. We do not write it in detail as it is not very different from the case just treated. $\square$

2.2 The geometry of Bruhat intervals in $\widetilde{A_2}$

The running assumption in this section is that x, y are dominant elements satisfying $x < y$. We will describe geometrically the intervals $[x, y]$ for these elements.

2.2.1 Description of $\geq x$

For $w \in W_f$, we define F_w as the convex closed unbounded subset of E depicted in Figure 2.4. Each zone will be either called by its label or by its color.

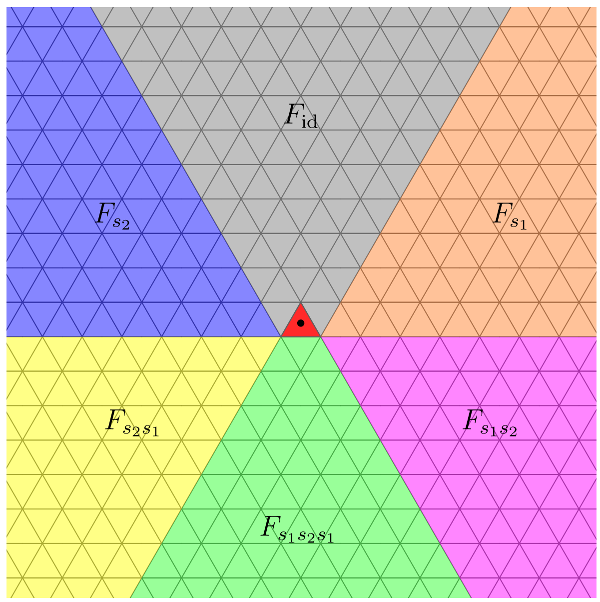


Figure 2.4: The six zones F_w. The red alcove, $\mathscr{A}_{\mathrm{id}}$ is part of F_{id}.

Some zones are contiguous to others and delimited by unbounded rays. There are 6 of these rays. Each of them is contained in a line of $\mathscr{H}$. The corresponding affine reflections of these lines are $s_{\alpha_1,-1}, s_0, s_2, s_1, s_0$, and $s_{\alpha_2,-1}$. For convenience, we denote them by r_1, r_2, r_3, r_4, r_5, and r_6 respectively. We use "label mod 6" notation, so for example $r_{11} = r_5$. Note that $r_2 = r_5 = s_0$. We also relabel the zones by $F_1, \ldots, F_6$ (again "label mod 6" notation) in a "clockwise" way starting from F_{id}, in other words, F_1 is gray, F_2 is orange, F_3 is pink, F_4 is green, F_5 is yellow, and F_6 is blue. From Figure 2.4, it is immediate that $r_i \cdots r_1(D) \subset F_{i+1}$ for $i \geq 1$.

Definition 2.2.1. [Construction of $\mathcal{S}t(x)$] See Figure 2.5a for an example of the construction that follows. Let $x \in D \subset W$. Define $x_1 := \mathrm{cen}(x) \in E$ and $x_{i+1} := r_i x_i$ for all $1 \leq i \leq 6$. As noted above, $x_i \in F_i$. The segment $\overline{x_i x_{i+1}}$ divides the ray corresponding to r_i into two distinct parts: bounded and unbounded. A unique point u_{i+1} satisfying that $x_i u_{i+1} x_{i+1}$ is an equilateral triangle, lies in the ray's unbounded part.

Note that $x_7 = x_1$ because $r_6 \cdots r_1 = \mathrm{id}$ (this is clear from Figure 2.4 since $r_6 \cdots r_1$ fixes $\mathcal{A}_{\mathrm{id}}$). For convenience, we use label mod 6 also for x_i and u_i. The union of the segments $x_i u_{i+1}$ and $u_{i+1} x_{i+1}$, for $1 \leq i \leq 6$, forms a closed loop which is the boundary of a (non-convex) bounded 12-sided polygon which we denote by $\mathcal{S}t(x)$. It is easy to see that $\mathcal{S}t(x)$ is a 6-pointed star with acute inner angles $\pi/3$ and obtuse outer angles $2\pi/3$.

We relabel the x_i's in the following way: For $w \in W_f$, there is a unique $i \mod 6$ such that $F_w = F_i$, so we write $x_w := x_i$. Let us denote by $\mathrm{IVS}_x := \{x_w \mid w \in W_f\}$ and $\mathrm{EVS}_x := \{u_i \mid 1 \leq i \leq 6\}$ the sets of inner and exterior vertices of $\mathcal{S}t(x)$ respectively.

Proposition 2.2.2. *If x is dominant*

$$\geq x = \{z \in W \mid \mathrm{cen}(z) \notin \mathcal{S}t^{\circ}(x)\},$$

where $\mathcal{S}t^{\circ}(x)$ denotes the interior of the star-shaped polygon $\mathcal{S}t(x)$.

Before we prove this proposition, we need some preliminary lemmas. For the rest of this section, we fix an element $x \in D$.

Consider the following regions (see Figure 2.5b)

$$H_1 := F_1 \setminus \{v \in E \mid -1 \leq (v, \alpha_2) < 0\},$$
$$H_2 := F_2, H_3 := F_3,$$
$$H_4 := F_4 \setminus \{v \in E \mid -1 \leq (v, \alpha_1) < 0\}.$$

For $1 \leq j \leq 4$, the set $H_j \setminus \partial(\mathcal{S}t(x))$ has two connected components, one in the interior of the star and one in the exterior of the star. Let us call H_j^{in} and H_j^{ex} the connected components in the interior and in the exterior, and $H_j^{\mathrm{bor}} := H_j \cap \partial(\mathcal{S}t(x))$. We emphasize that these sets depend on x, but we omit x from the notation. From the description of $\mathcal{S}t(x)$, one can deduce that for $1 \leq j \leq 3$ the reflection $r_j \in W$ induces a bijection $r_j \colon H_j \to H_{j+1}$ such that $r_j(H_j^*) = H_{j+1}^*$, for $* \in \{\mathrm{in}, \mathrm{ex}, \mathrm{bor}\}$.

Let us denote the union of these sets $H := \biguplus_{j=1}^{4} H_j$. Let $v \in H_i$. If $i = 1$, we define $v(1) = v \in H_1$. If not, we define $v(1) = r_1 r_2 \cdots r_{i-1} v \in H_1$. For $2 \leq k \leq 4$ we define $v(k) := r_{i-1} \cdots r_2 r_1(v(1)) \in H_k$.

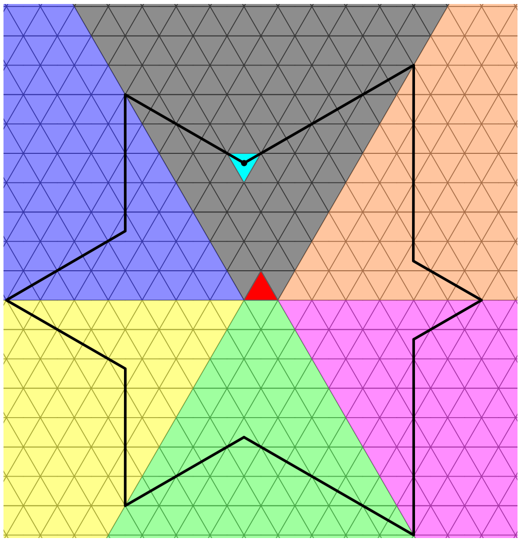
The following lemma is immediate.

Lemma 2.2.3. *We have $H_j^*(1) = H_1^*$ for every $1 \leq j \leq 4$ and $* \in \{\mathrm{in}, \mathrm{ex}, \mathrm{bor}\}$.*

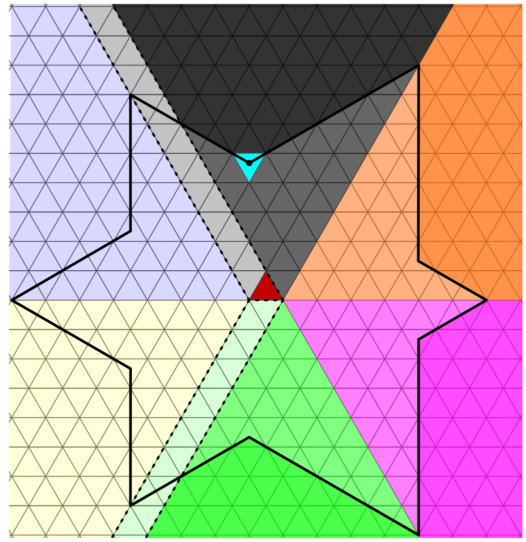
Lemma 2.2.4. *Let $z \in W$. If $z \in H$, then $z(1) \in \mathrm{V}(\mathcal{C}_z)$.*

Proof. We split the proof into four cases.

- Case $z \in H_1$. By Corollary 2.1.4, we have $z(1) = z \in \mathrm{V}(\mathcal{C}_z)$.



(a) The star $\mathcal{S}t(\theta(2,1))$.



(b) Regions $H_i^{in}, H_i^{ex}, 1 \leq i \leq 4$.

Figure 2.5: Illustration of $\mathcal{S}t(x)$ and the regions H_i^{in}, H_i^{ex} for $x = \theta(2,1)$, $1 \leq i \leq 4$. The teal-colored alcove corresponds to x. In the figure on the right, the complement of H is colored lighter than in the figure on the left, and the four exterior regions H_i^{ex} are depicted with darker colors than their corresponding interior regions H_i^{in}.

- Case $z \in H_2$. There are two possibilities:
  - $\delta z \in s_0 D$. Then $z \in \delta^2 s_0 D$. By Corollary 2.1.4, we have $z(1) = s_2 s_0 s_2 z \in \mathrm{V}(C_z)$.
  - $z \in X$. Then $z = \delta \mathbf{x}_k$ for some $k \geq 2$. By Corollary 2.1.4, we have $z(1) = s_2 s_0 s_2 z = \delta(s_1 s_2 s_1 \mathbf{x}_k) \in \mathrm{V}(C_z)$.
- Case $z \in H_3$. By Corollary 2.1.4, we have $z(1) = s_2 s_0 z \in \mathrm{V}(C_z)$.
- Case $z \in H_4$. The proof is similar to the second case. □

Definition 2.2.5. Let $w \in W_f$. Let $\mathrm{Cone}^w(\alpha_1, \alpha_2)$ be the positive cone generated by $w(\alpha_1)$ and $w(\alpha_2)$. In formulas, $\mathrm{Cone}^w(\alpha_1, \alpha_2) := \{aw(\alpha_1) + bw(\alpha_2) \mid a, b \in \mathbb{R}_{\geq 0}\}$. For simplicity, we write $\mathrm{Cone}(\alpha_1, \alpha_2) := \mathrm{Cone}^{\mathrm{id}}(\alpha_1, \alpha_2)$.

Lemma 2.2.6. *Let $z \in W$. If $z \in H$, then*

$$\mathcal{C}_z \cap H_1 \cap \mathrm{cen}(W) = \Big(z(1) - \mathrm{Cone}(\alpha_1, \alpha_2)\Big) \cap H_1 \cap \mathrm{cen}(W).$$

Proof. Let $X^{\mathrm{ne}} \subset E$ (resp. $X^{\mathrm{e}}, X^{\mathrm{se}} \subset E$) be the strip of the plane starting at the identity and going northeast (resp. east, southeast) intersected with H_1 (resp.

H_2, H_4). In formulas,

$$X^{\mathrm{ne}} := \{v \in H_1 \mid -1 \le (v, \alpha_1) < 0)\},$$
$$X^{\mathrm{e}} := \{v \in H_2 \mid -1 \le (v, \rho) < 0\},$$
$$X^{\mathrm{se}} := \{v \in H_4 \mid -1 \le (v, \alpha_2) < 0\}.$$

Let $Y^{\mathrm{ne}} \subset H_1$ and $Y^{\mathrm{se}} \subset H_4$ be the strips directly to the left of X^{ne} and X^{se} respectively. In formulas,

$$Y^{\mathrm{ne}} := \{v \in H_1 \mid 0 \le (v, \alpha_1) < 1\},$$
$$Y^{\mathrm{se}} := \{v \in H_4 \mid -2 \le (v, \alpha_2) < -1\}.$$

By Corollary 2.1.4 and case-by-case inspection, if $z \in H \setminus (X^{\mathrm{ne}} \cup X^{\mathrm{e}} \cup X^{\mathrm{se}} \cup Y^{\mathrm{ne}} \cup Y^{\mathrm{se}})$, then $\mathcal{C}_z$ is a hexagon with only one vertex in H_1 (for instance, the blue hexagon in Figure 2.6). If $z \in X^{\mathrm{e}}$, then $\mathcal{C}_z$ is a quadrilateral with only one vertex in H_1 (for instance, the yellow quadrilateral in Figure 2.6). In both cases, we have

$$\mathcal{C}_z \cap H_1 = \Big(z(1) - \mathrm{Cone}(\alpha_1, \alpha_2)\Big) \cap H_1, \tag{2.7}$$

so the lemma follows.

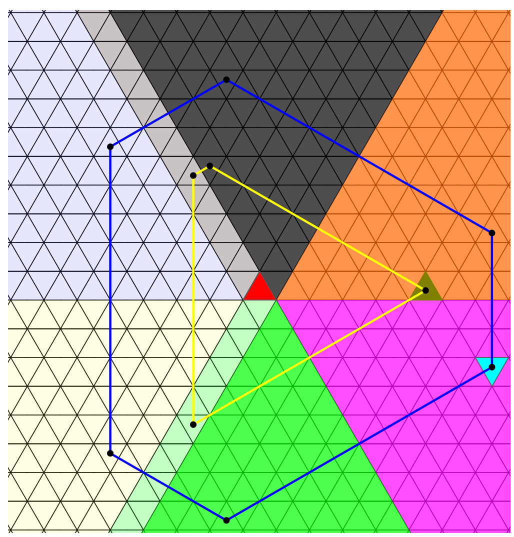
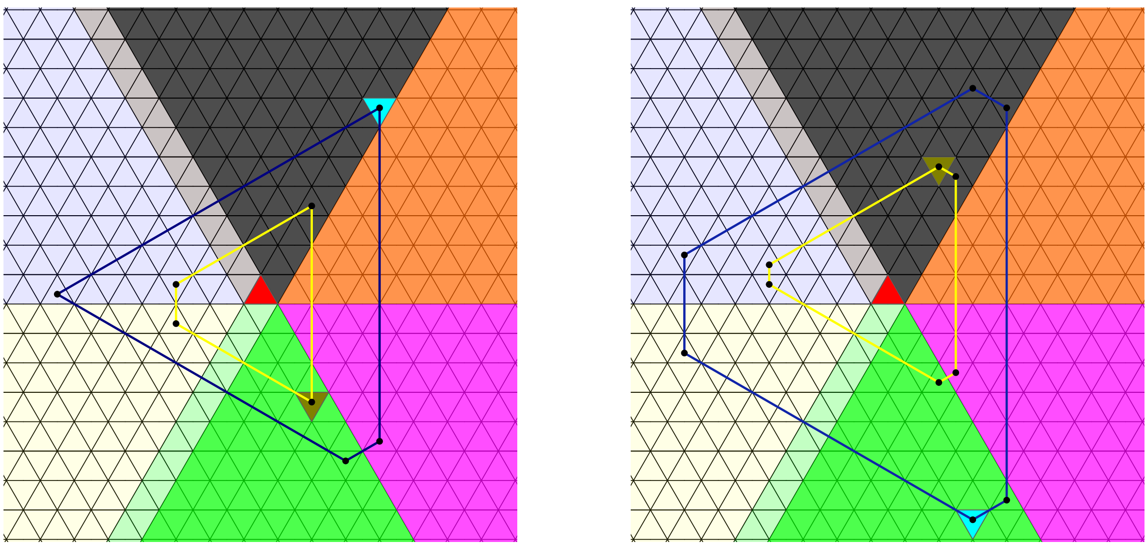


Figure 2.6: ∂C_z for $z \notin X^{\mathrm{ne}} \cup X^{\mathrm{se}} \cup Y^{\mathrm{ne}} \cup Y^{\mathrm{se}}$.

It remains to prove the case $z \in X^{\mathrm{ne}} \cup X^{\mathrm{se}} \cup Y^{\mathrm{ne}} \cup Y^{\mathrm{se}}$. Suppose that z is as in Figures 2.7a or 2.7b. By Corollary 2.1.4 and case-by-case inspection, Equation (2.7) does not hold, although the strict inclusion $\subsetneq$ always holds in this case. The difference between the two sets in this strict inclusion (denoted by Dif_z) is a right triangle minus its hypotenuse. The set Dif_z contains no center of alcoves (although along its hypotenuse, there are either one or two elements of $\mathrm{cen}(W)$.) This proves the lemma for the elements z appearing in Figures 2.7a and 2.7b. Let us argue why this is enough to prove the lemma.

In Figure 2.7a, we see a yellow quadrilateral corresponding to a specific element $z \in X^{\mathrm{se}}$. If one chooses a different element z in X^{se}, then Corollary 2.1.4 gives a quadrilateral $\mathcal{C}_z$ with the same inner angles as the yellow quadrilateral. In this case, $z(1)$ also belongs to X^{ne}. Modulo a translation of the plane, the right triangle Dif_z depends only on the parity of $\ell(z)$. If the parity of $\ell(z)$ is different from the one considered in Figure 2.7a, the area of Dif_z is much smaller (it is one-sixth of the area of an alcove) than in the case depicted there. In both cases, $\mathrm{Dif}_z \cap \mathrm{cen}(W) = \emptyset$. The remaining three cases, $z \in X^{\mathrm{ne}} \cup Y^{\mathrm{ne}} \cup Y^{\mathrm{se}}$ are similar. This concludes the proof of the Lemma. □



(a) $z \in X^{\mathrm{ne}} \cup X^{\mathrm{se}}$. (b) $z \in Y^{\mathrm{ne}} \cup Y^{\mathrm{se}}$.

Figure 2.7: Other examples of ∂C_z.

A similar result holds if we replace H_1 with $\sigma(H_1)$.

Lemma 2.2.7. *Let $z \in F_{\mathrm{id}}$. Then $\mathcal{C}_z \cap F_{\mathrm{id}} \cap \mathrm{cen}(W) = \big(z - \mathrm{Cone}(\alpha_1, \alpha_2)\big) \cap F_{\mathrm{id}} \cap \mathrm{cen}(W)$.*

As $F_{\mathrm{id}} = H_1 \cup \sigma(H_1)$, the case $z \in H_1 \cap \sigma(H_1)$ is proved by using Lemma 2.2.6 with $z(1) = z$ and the version of Lemma 2.2.6 where H_1 is replaced by $\sigma(H_1)$.

The remaining cases to consider are $z = \mathbf{x}_n$ or $z = \sigma(\mathbf{x}_n)$.

These cases are proved in a similar way as in the part of the proof of Lemma 2.2.6 following Figure 2.7. The reason for this is that, although a different z is considered, the set $\mathcal{C}_z$ remains essentially the same as depicted in Figure 2.7a and the arguments apply in the same way. In other words (see Figure 2.7a) the difference between $\big(z - \mathrm{Cone}(\alpha_1, \alpha_2)\big) \cap F_{\mathrm{id}}$ and $\mathcal{C}_z \cap F_{\mathrm{id}}$ is a right triangle minus its hypotenuse included in $\overline{\mathcal{A}_z}$ that contains no element of $\mathrm{cen}(W)$.

Corollary 2.2.8. *Let $z, z' \in F_{\mathrm{id}}$. Then $z \in \mathcal{C}_{z'}$ if and only if $(\varpi_1, \mathrm{cen}(z') - \mathrm{cen}(z)) \geq 0$ and $(\varpi_2, \mathrm{cen}(z') - \mathrm{cen}(z)) \geq 0$.*

Proof. By Lemma 2.2.7, $z \in \mathcal{C}_{z'}$ if and only if $\mathrm{cen}(z) = \mathrm{cen}(z') - t_1\alpha_1 - t_2\alpha_2$, for some $t_1, t_2 \in \mathbb{R}_{\geq 0}$ if and only if for $i \in \{1, 2\}$

$$(\varpi_i, \mathrm{cen}(z') - \mathrm{cen}(z)) = (\varpi_i, t_1\alpha_1 + t_2\alpha_2) = t_i \geq 0.$$

□

Lemma 2.2.9. *Let z be dominant and $w \in W_f$. We have,*

$$\mathcal{C}_z \cap F_w \cap \mathrm{cen}(W) = \big(wz - \mathrm{Cone}^w(\alpha_1, \alpha_2)\big) \cap F_w \cap \mathrm{cen}(W).$$

Proof. If each element of $\mathrm{V}(\mathcal{C}_z)$ belongs to a different zone, then it is clear that $\mathcal{C}_z \cap F_w = \big(wz - \mathrm{Cone}^w(\alpha_1, \alpha_2)\big) \cap F_w$, and we are done. This happens if $z \in D + \rho$. The complement by this set in D is composed of two strips, and by symmetry we can restrict to the case $z \in \{\theta(0, n), \theta^s(0, n) \mid n \in \mathbb{N}\}$. For the case $z = \theta(0, n)$ see Figure 2.7b, and use a similar argument as in the part of the proof of Lemma 2.2.6 following Figure 2.6. The case $z = \theta^s(0, n)$ is similar. □

The next lemma is an analogue of Lemma 2.2.9 for $z \in \{\mathbf{x}_k \mid k > 3\}$.

Lemma 2.2.10. *For $k \geq 4$, we have that*

$$\mathcal{C}_{\mathbf{x}_k} \cap F_w \cap \mathrm{cen}(W) = \begin{cases} \big(w\mathbf{x}_k - \mathrm{Cone}^w(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) & \text{if } w \notin \{s_2, s_1 s_2\}, \\ \big(ws_2\mathbf{x}_k - \mathrm{Cone}^{ws_2}(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) & \text{if } w \in \{s_2, s_1 s_2\}. \end{cases}$$

Proof. By Corollary 2.1.4, we have $V(\mathcal{C}_{\mathbf{x}_k}) = \{\mathbf{x}_k, s_1\mathbf{x}_k, s_2 s_1 \mathbf{x}_k, s_1 s_2 s_1 \mathbf{x}_k\}$. Recall that $s_1 = s_{\alpha_1, 0}, s_2 = s_{\alpha_2, 0}$ and $s_1 s_2 s_1 = s_{\rho, 0}$. Note that

$$s_1\mathbf{x}_k \in F_{s_1}, \quad s_2 s_1 \mathbf{x}_k \in F_{s_1 s_2 s_1}, \quad s_1 s_2 s_1 \mathbf{x}_k \in F_{s_2 s_1}. \tag{2.8}$$

Moreover, we have the following

$$\begin{aligned} s_1(\mathbf{x}_k) &= \mathbf{x}_k - 2t_1\alpha_1, \\ s_2 s_1(\mathbf{x}_k) &= s_1(\mathbf{x}_k) - 2t_2\alpha_2, \\ s_1 s_2 s_1(\mathbf{x}_k) &= s_2 s_1(\mathbf{x}_k) + 2t_3\alpha_1, \\ s_1 s_2 s_1(\mathbf{x}_k) &= s_{\rho,0}(\mathbf{x}_k) = \mathbf{x}_k - 2t\rho. \end{aligned} \tag{2.9}$$

for some $t_1, t_2, t_3, t \in \mathbb{R}_{\geq 0}$.

We will first prove the equation in the second case, i.e. $w \in \{s_2, s_1s_2\}$. From Equations (2.8), it follows that there is no vertex on F_w. Also, we have that the zones adjacent to F_{s_2} are F_{id} and $F_{s_2s_1}$. By Equation (2.8), the vertices of $\mathcal{C}_{\mathbf{x}_k}$ in F_{id} and $F_{s_2s_1}$ are $\mathbf{x}_k$ and $s_1s_2s_1\mathbf{x}_k$, respectively. By Equation (2.9), we have that $s_1s_2s_1\mathbf{x}_k = \mathbf{x}_k - 2t\rho$, so

$$\mathcal{C}_{\mathbf{x}_k} \cap F_{s_2} = (\mathbf{x}_k - \mathrm{Cone}(\alpha_1, \rho)) \cap F_{s_2},$$

and after intersecting both sides with $\mathrm{cen}(W)$ this concludes the proof of the lemma in the case $w = s_2$.

The adjacent zones to $F_{s_1s_2}$ are F_{s_1} and $F_{s_1s_2s_1}$. By Equation (2.8), the vertices of $\mathcal{C}_{\mathbf{x}_k}$ in F_{s_1} and $F_{s_2s_1s_2}$ are $s_1\mathbf{x}_k$ and $s_2s_1\mathbf{x}_k$, respectively. By Equation (2.9), we have that $s_2s_1\mathbf{x}_k = s_1\mathbf{x}_k - 2t_2\alpha_2$ and $s_1(\mathbf{x}_k) = \mathbf{x}_k - 2\tau_1\alpha_1$, so

$$\begin{aligned} \mathcal{C}_{\mathbf{x}_k} \cap F_{s_1s_2} &= (s_1\mathbf{x}_k - \mathrm{Cone}(-\alpha_1, \alpha_2)) \cap F_{s_1s_2} \\ &= (s_1\mathbf{x}_k - \mathrm{Cone}^{s_1}(\alpha_1, \rho)) \cap F_{s_1s_2}, \end{aligned}$$

and after intersecting both sides with $\mathrm{cen}(W)$ this concludes the proof of the lemma in the case $w = s_1s_2$.

The result for the remaining cases follows similarly, with the caveat that for $F_{s_2s_1}$ and $F_{s_1s_2s_1}$ the equation is not true before intersecting with $\mathrm{cen}(W)$. As in several previous arguments, a small triangle (without intersection with $\mathrm{cen}(W)$) is the difference between the left-hand side and the right-hand side that after intersecting with $\mathrm{cen}(W)$ disappears. For brevity, we omit further details. □

The following lemma is the analog of Equation (2.7) for the set $E \setminus \mathcal{S}t(x)$ in place of the set $\mathcal{C}_z$. It follows directly from the definition of $\mathcal{S}t(x)$, and we will omit its proof. Unlike Equation (2.7), an equality holds in each possible case.

Lemma 2.2.11. *We have $H_1 \setminus \mathcal{S}t(x)^\circ = H_1 \cap (x + \mathrm{Cone}(\alpha_1, \alpha_2))$. In particular, if $z \in W$ and $z \in H_1$, then $z \notin \mathcal{S}t(x)^\circ$ if and only if $x \in z - \mathrm{Cone}(\alpha_1, \alpha_2)$.*

Proof of Proposition 2.2.2. Let us first prove the proposition for $z \in W$ with $\mathrm{cen}(z) \in H_i$ for some $1 \leq i \leq 4$. By Corollary 2.1.5, $x \leq z$ if and only if $x \in \mathcal{C}_z$. Note that $x \in H_1$. By Lemma 2.2.6, $x \in \mathcal{C}_z$ if and only if $x \in z(1) - \mathrm{Cone}(\alpha_1, \alpha_2)$. By Lemma 2.2.11, $x \in z(1) - \mathrm{Cone}(\alpha_1, \alpha_2)$ if and only if $z(1) \notin \mathcal{S}t(x)^\circ$ if and only if $z(1) \in H_1^{\mathrm{ex}} \cup H_1^{\mathrm{bor}}$. By Lemma 2.2.3, $z(1) \in H_1^{\mathrm{ex}} \cup H_1^{\mathrm{bor}}$ if and only if $z \in H_i^{\mathrm{ex}} \cup H_i^{\mathrm{bor}}$ if and only if $z \notin \mathcal{S}t(x)$. This proves the proposition for $z \in H$.

If one replaces everywhere in the argument H by $\sigma(G)$ (swap s_1 and s_2, v and σv, etc.) we obtain a proof for the proposition for $z \in W$ such that $z \in \sigma H$. The only elements $z \in W$ such that $z \notin H \cup \sigma H$ are id and s_0. However $\mathrm{id}, s_0 \notin (\geq x)$. The proposition is proved. □

2.2.2 Elements that differ by a multiple of simple root in F_{id}

Lemma 2.2.12. *Let $z, z' \in F_{\mathrm{id}}$ such that $\mathrm{cen}(z') - \mathrm{cen}(z) \in \mathbb{R}_{>0}\alpha_i$ for some $i \in \{1,2\}$ and $\overline{\mathcal{A}_z} \cap \overline{\mathcal{A}_{z'}} \neq \emptyset$. Then $z < z'$ and $\ell(z, z') = 1$.*

Proof. By hypothesis we have that $z \in z' - \mathrm{Cone}(\alpha_1, \alpha_2)$. By Lemma 2.2.7, $z \in \mathcal{C}_{z'}$. By Corollary 2.1.5, this implies that $z \leq z'$. It remains to prove that $\ell(z, z') = 1$.

Let us suppose that z' is down-oriented. In this case we have that $\mathrm{cen}(z') - \mathrm{cen}(z) = \frac{1}{3}\alpha_i$ and z, z' share an edge. It is a known fact from Coxeter complexes that if two elements z, z' share a face, then there is $s \in S$ such that $zs = z'$. Since $z \leq z'$, we have that $\ell(z, z') = 1$.

Let us suppose that z' is up-oriented. In this case, $\overline{\mathcal{A}_z}$ and $\overline{\mathcal{A}_{z'}}$ intersect in a vertex and

$$\mathrm{cen}(z) = \mathrm{cen}(z') - \frac{2}{3}\alpha_i. \tag{2.10}$$

Now we study all the possibilities for z and z'. In each case, we verify that $\ell(z, z') = 1$ by using the formulas for the length of the elements of W provided in Section 1.2.1.

- $z' = \theta^s(m, n)$, for integers $m, n > 0$. In this case we have that $\ell(z') = 2m + 2n + 4$. Then Equation (2.10) implies that either $z = \theta(m-1, n+1)$ or $z = \theta(m+1, n-1)$. In either case we have that $\ell(z) = 2m + 2n + 3$, so $\ell(z, z') = 1$.

- $z' = \mathbf{x}_{2k}$ or $\sigma(\mathbf{x}_{2k})$ for $k \geq 2$. By symmetry, we can assume $z' = \mathbf{x}_{2k}$. In this case we have $\ell(z') = 2k$. Since $\mathrm{cen}(z') - \frac{2}{3}\alpha_2 \notin F_{\mathrm{id}}$, we have that $i = 1$, so $z = \theta(k-2, 0)$. Thus $\ell(z) = 2(k-2) + 3 = 2k - 1$ and $\ell(z, z') = 1$.

- $z' = \theta^s(m, 0)$ or $z' = \theta^s(0, n)$. By symmetry we consider only the first case. In this case we have $\ell(z') = 2m + 4$. Equation (2.10) implies that either $z = \theta(m-1, 1)$ or $z = \mathbf{x}_{2m+3}$. In both cases we have $\ell(z) = 2m + 3$, so $\ell(z, z') = 1$. ∎

Lemma 2.2.13. *Let $x, y \in F_{\mathrm{id}}$ and $t \in \mathbb{R}_{\geq}0$. If $\ell(z, y) \geq 4$ and $z = y - t\alpha_i \in F_{\mathrm{id}}$, then $t \geq 2$. Similarly, $\ell(x, z) \geq 4$ and $z = x + t\alpha_i$, then $t \geq 2$.*

Proof. We will prove only the first statement, the second is analog. Let us define $C := \{u \in W \mid u \in \mathrm{Sgm}(y, z)\}$. Lemma 2.2.7 implies that $\mathrm{Sgm}(y, z) \subset [z, y]$, and also each pair of elements of W in $\mathrm{Sgm}(y, z)$ are comparable. This implies that $C \subset [z, y]$, and each pair of elements in C are comparable.

We will enumerate the elements of C as follows: we define $u_0 := z$. Since z is the end-point of $\mathrm{Sgm}(y, z)$, we define u_1 as the unique element in C such that $\overline{\mathcal{A}_{u_1}} \cap \overline{\mathcal{A}_{u_0}} \neq \emptyset$. For $k \geq 1$, if u_k is defined and $u_k \neq y$, then we define u_{k+1} as the unique element in $C \setminus \{u_{k-1}\}$ such that $\overline{\mathcal{A}_{u_{k+1}}} \cap \overline{\mathcal{A}_{u_k}} \neq \emptyset$. We define $n \in \mathbb{N}$ such that $u_n = y$.

By Lemma 2.2.12 we have that C is a maximal chain in $[z, y]$. In other words, we have $n = \ell(z, y)$ and

$$z = u_0 \lhd u_1 \lhd \ldots \lhd u_n = y.$$

For $1 \leq k \leq n$, if u_{k-1} is up-oriented then u_k is down-oriented and $u_{k-1} = u_k - \frac{1}{3}\alpha_i$. If u_{k-1} is down-oriented then u_k is up-oriented and $u_{k-1} = u_k - \frac{2}{3}\alpha_i$. This implies the following facts.

- If $u_{k-1} = u_k - \frac{1}{3}\alpha_i$ then $u_k = u_{k+1} - \frac{2}{3}\alpha_i$.
- If $u_{k-1} = u_k - \frac{2}{3}\alpha_i$ then $u_k = u_{k+1} - \frac{1}{3}\alpha_i$.

By hypothesis $n \geq 4$. From the facts above, we have $u_4 = z + 2\alpha_i$, so the lemma follows. $\square$

Remark 2.2.14. We note the following simple generalization, which we do not prove here as it is not required for this thesis. If $z, z' \in F_{\mathrm{id}}$ are such that $\mathrm{cen}(z') - \mathrm{cen}(z) = r\alpha_i$ for some $r > 0$ and $i \in \{1, 2\}$, then $\ell(z, z') = \lfloor r \rfloor + \lceil r \rceil$.

2.2.3 Types of intervals and the polygon $\mathrm{Pgn}_{x,y}$

Recall that, at the beginning of Section 2.2, we are assuming that x and y are dominant elements. As $[x, y] = [\mathrm{id}, y] \cap (\geq x)$, Corollary 2.1.5 and Proposition 2.2.2 give the following geometric description of the interval

$$\begin{aligned} [x, y] &= \{z \in W \mid \mathrm{cen}(z) \in \mathcal{C}_y \setminus \mathcal{S}t^\circ(x)\} \\ &= \mathrm{cen}^{-1}(\mathcal{C}_y \setminus \mathcal{S}t^\circ(x)) \end{aligned}$$

This description was illustrated in Figure 1 of the introduction. Figure 2.8 illustrates the next two definitions.

Definition 2.2.15. We define the polygon $\mathrm{Par}^w_{x,y}$ as the parallelogram with opposite vertices $\{\mathrm{cen}(x_w), w\,\mathrm{cen}(y)\}$ and edges parallel to $w\alpha_1$ and $w\alpha_2$. In the degenerate case, i.e. when $\mathrm{cen}(x_w)$ and $w\,\mathrm{cen}(y)$ differ by a multiple of $w\alpha_1$ or $w\alpha_2$, we denote $\mathrm{Par}^w_{x,y} := \mathrm{Sgm}(\mathrm{cen}(x_w), w\,\mathrm{cen}(y))$ and still call it a parallelogram. We denote $\mathrm{Par}_{x,y} := \mathrm{Par}^{\mathrm{id}}_{x,y}$.

Definition 2.2.16. For each $w \in W_f$, we define the following convex bounded polygon

$$\mathrm{Pgn}^w_{x,y} = \mathrm{Par}^w_{x,y} \cap F_w.$$

As before, we denote $\mathrm{Pgn}_{x,y} := \mathrm{Pgn}^{\mathrm{id}}_{x,y}$.

Remark 2.2.17. 1. It is not hard to see that each element of $\mathrm{V}(\mathrm{Par}^w_{x,y})$ is either the center or a vertex of an alcove.

2. Vertices of $\mathrm{Pgn}_{x,y}$ adjacent to $\mathrm{cen}(x)$ can be either a vertex or the center of an alcove.

3. Vertices of $\mathrm{Pgn}_{x,y}$ adjacent to $\mathrm{cen}(y)$ can be the midpoint of an edge, the center, or a vertex of an alcove.

Remark 2.2.18. In some cases, $w\,\mathrm{cen}(y)$ is not contained in F_w, so it is not a vertex of $\mathrm{Pgn}^w_{x,y}$.

Lemma 2.2.19. $\mathrm{cen}^{-1}(\mathrm{Pgn}^w_{x,y}) = \mathrm{cen}^{-1}(F_w \cap \mathcal{C}_y \setminus \mathcal{S}t^\circ(x))$.

Proof. The lemma follows by proving $F_w \cap \big(\mathcal{C}_y \setminus \mathcal{S}t^\circ(x)\big) \cap \mathrm{cen}(W) = \mathrm{Pgn}^w_{x,y} \cap \mathrm{cen}(W)$. Note that $F_w \setminus \mathcal{S}t^\circ(x) = F_w \cap \big(x_w + \mathrm{Cone}^w(\alpha_1,\alpha_2)\big)$. Thus we obtain

$$\begin{aligned}
F_w \cap \big(\mathcal{C}_y \setminus \mathcal{S}t^\circ(x)\big) \cap \mathrm{cen}(W) &= (F_w \setminus \mathcal{S}t^\circ(x)) \cap \mathcal{C}_y \cap \mathrm{cen}(W)\\
&= F_w \cap \big(x_w + \mathrm{Cone}^w(\alpha_1,\alpha_2)\big) \cap \mathcal{C}_y \cap \mathrm{cen}(W)\\
&= \big(F_w \cap \mathcal{C}_y \cap \mathrm{cen}(W)\big) \cap \big(x_w + \mathrm{Cone}^w(\alpha_1,\alpha_2)\big)\\
&= \big(F_w \cap \big(wy - \mathrm{Cone}^w(\alpha_1,\alpha_2)\big) \cap \mathrm{cen}(W)\big) \cap\\
&\quad \big(x_w + \mathrm{Cone}^w(\alpha_1,\alpha_2)\big)\\
&= F_w \cap \mathrm{Par}^w_{x,y} \cap \mathrm{cen}(W)\\
&= \mathrm{Pgn}^w_{x,y} \cap \mathrm{cen}(W).
\end{aligned}$$

The fourth equality is given in Lemma 2.2.9. The fifth and sixth equalities follow by definition of $\mathrm{Par}^w_{x,y}$ and $\mathrm{Pgn}^w_{x,y}$, respectively. ☐

Therefore, we have

$$[x,y] = \biguplus_{w \in W_f} \mathrm{cen}^{-1}(\mathrm{Pgn}^w_{x,y}). \tag{2.11}$$

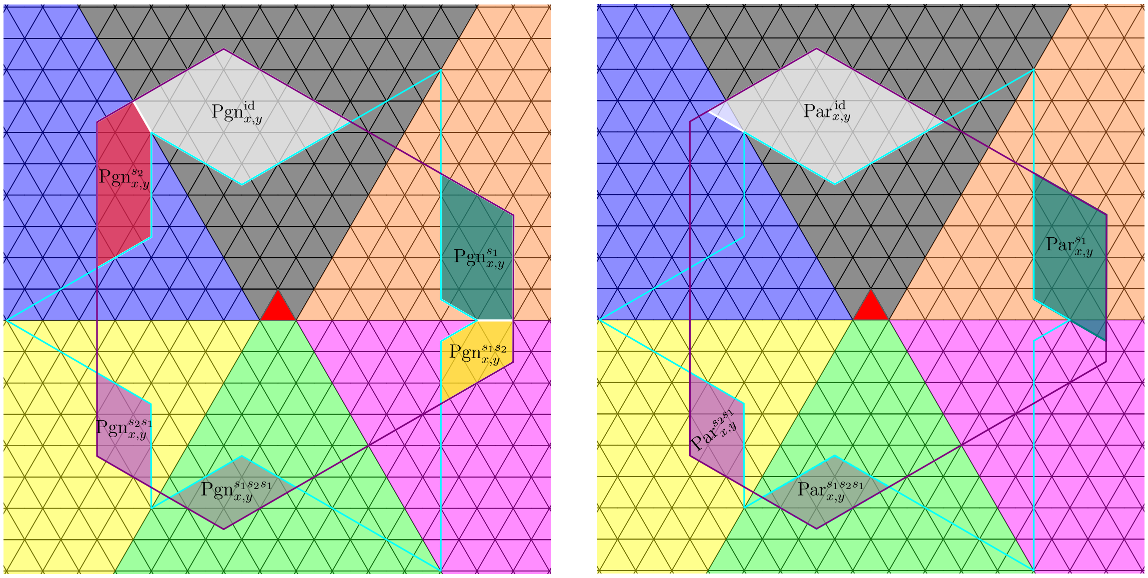
In other words, $[x,y]$ is completely determined by the six polygons $\mathrm{Pgn}^w_{x,y}$. We can now state an important definition, although we will not use it until Section 2.3.

Definition 2.2.20. We call $[x,y]$ a *parallelogram interval* (resp. *pentagon interval*, *hexagon interval*) if $\mathrm{Pgn}_{x,y}$ is a parallelogram (resp. pentagon, hexagon). If the area of $\mathrm{Pgn}_{x,y}$ is zero, i.e. $\mathrm{Pgn}_{x,y}$ is a line segment, we also say that $[x,y]$ is a *parallelogram interval*.

We will now describe $\mathrm{Pgn}^w_{x,y}$ in terms of $\mathrm{Pgn}_{x,y}$. The polygons $\mathrm{Pgn}^w_{x,y}$ have equal or smaller area than $\mathrm{Pgn}_{x,y}$. We will shortly see that one can obtain them by subtracting a particular set $S^{x,y}_w$ from $\mathrm{Pgn}_{x,y}$ before applying w. Before proving this, we need some definitions.

Consider the strips in the plane:

$$\begin{aligned}
S^{x,y}_{\alpha_1} &:= \{\mathrm{cen}(x) + t\alpha_1 + t'\alpha_2 \in \mathrm{Par}_{x,y} \mid t \in [0,1), t' \in \mathbb{R}\},\\
S^{x,y}_{\alpha_2} &:= \{\mathrm{cen}(x) + t\alpha_2 + t'\alpha_1 \in \mathrm{Par}_{x,y} \mid t \in [0,1), t' \in \mathbb{R}\}.
\end{aligned}$$



(a) The polygons $\mathrm{Pgn}^w_{x,y}$.

(b) The parallelograms $\mathrm{Par}^{\mathrm{id}}_{x,y}$, $\mathrm{Par}^{s_1}_{x,y}$ $\mathrm{Par}^{s_2 s_1}_{x,y}$ and $\mathrm{Par}^{s_1 s_2 s_1}_{x,y}$.

Figure 2.8: Polygons and parallelograms associated with the interval $[\theta(2,0)s, \theta(5,2)]$.

Furthermore, for $w \in W_f$, we define

$$S^{x,y}_w := \begin{cases} \emptyset & \text{if } w = \mathrm{id}, \\ S^{x,y}_{\alpha_1} & \text{if } w = s_1, \\ S^{x,y}_{\alpha_2} & \text{if } w = s_2, \\ S^{x,y}_{\alpha_1} \cup S^{x,y}_{\alpha_2} & \text{otherwise,} \end{cases} \quad \text{and} \quad v_w := \begin{cases} 0 & \text{if } w = \mathrm{id}, \\ \alpha_1 & \text{if } w = s_1, \\ \alpha_2 & \text{if } w = s_2, \\ \rho & \text{otherwise.} \end{cases}$$

We have the following equalities.

$$\mathrm{cen}(x_w) = w\,\mathrm{cen}(x) + wv_w = w\,\mathrm{cen}(x + v_w) \in F_w. \tag{2.12}$$

The first equality follows from the definitions of x_w and v_w (it is indeed the reason to define v_w as so). The second equality is a tautology, using Notation 1.2.5.

Proposition 2.2.21. *For $w \in W_f$, we have*

$$\mathrm{Pgn}^w_{x,y} = w(\mathrm{Par}_{x,y} \setminus S^{x,y}_w) \cap F_w = w(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w) \cap F_w.$$

Proof. Since $\mathrm{Pgn}^w_{x,y} = \mathrm{Par}^w_{x,y} \cap F_w$, to prove the first equality, it is sufficient to prove that

$$\mathrm{Par}^w_{x,y} = w(\mathrm{Par}_{x,y} \setminus S^{x,y}_w). \tag{2.13}$$

By definition, $\mathrm{Par}^w_{x,y}$ is completely determined by its opposite vertices $\mathrm{cen}(x_w)$ and $w\,\mathrm{cen}(y)$. It is easy to check that the set $\mathrm{Par}_{x,y} \setminus S^{x,y}_w$ is a parallelogram with opposite vertices $\mathrm{cen}(x)+v_w$ and $\mathrm{cen}(y)$, and edges parallel to α_1 and α_2. Then $w(\mathrm{Par}_{x,y} \setminus S_w^{x,y})$ is a parallelogram with opposite vertices $w(\mathrm{cen}(x)+v_w)$ and $w(\mathrm{cen}(y))$. By Equation (2.12), we have $w(\mathrm{cen}(x) + v_w) = \mathrm{cen}(x_w)$, so $w(\mathrm{Par}_{x,y} \setminus S_w^{x,y})$ and $\mathrm{Par}_{x,y}^w$ have the same opposite vertices and edges parallel to $w\alpha_1$ and $w\alpha_2$ and thus we have proved Equation (2.13).

For the second equality, we have

$$\begin{aligned} w(\mathrm{Pgn}_{x,y} \setminus S_w^{x,y}) \cap F_w &= w((\mathrm{Par}_{x,y} \cap F_{\mathrm{id}}) \setminus S_w^{x,y}) \cap F_w \\ &= w((\mathrm{Par}_{x,y} \setminus S_w^{x,y}) \cap F_{\mathrm{id}}) \cap F_w \\ &= w(\mathrm{Par}_{x,y} \setminus S_w^{x,y}) \cap w(F_{\mathrm{id}}) \cap F_w \\ &= \mathrm{Par}_{x,y}^w \cap F_w \end{aligned}$$

The last equality follows from the fact that $F_w \subset w(F_{\mathrm{id}})$. □

2.2.4 The geometry of maximal dihedral subintervals

This section is a continuation of Section 2.1.3 where we described geometrically the sets $\mathcal{D}^u(y) := \mathcal{D}(y) = \{w \in W \mid [w, y] \text{ is dihedral}\}$ for y dominant. Here u stands for "upper".

Definition 2.2.22. Let $x \in D$ and $\{u_i\}_{1 \leq i \leq 6} = \mathrm{EVS}_x$ be the exterior vertices of $\mathcal{S}t(x)$. We define

$$\begin{aligned} \mathcal{D}^l(x) &:= \{w \in W \mid [x, w] \text{ is dihedral}\} \\ \mathcal{D}^l_{\mathrm{sgm}}(x) &:= \{z \geq x \mid z \in \mathrm{Sgm}(u_1, u_3) \cup \mathrm{Sgm}(u_2, u_6)\}. \end{aligned}$$

Let $\mathcal{D}^l_{\mathrm{rest},\leq 2}(x) := \{z \in W \setminus \mathcal{D}^l_{\mathrm{sgm}}(x) \mid x \leq z \text{ and } \ell(x, z) \leq 2\}$, we define

$$\mathcal{D}^l_{\mathrm{rest}}(x) := \begin{cases} \mathcal{D}^l_{\mathrm{rest},\leq 2}(x) & \text{if } x = \theta(m, n), \\ \mathcal{D}^l_{\mathrm{rest},\leq 2}(x) \uplus \{\theta(m+1, n+1)\} & \text{if } x = \theta(m, n)s. \end{cases}$$

In these definitions, the symbol l stands for "lower".

The following is the analog of Proposition 2.1.9 for $\mathcal{D}^l_{\mathrm{sgm}}(x)$.

Proposition 2.2.23. *Let $x \in D$. Then $\mathcal{D}^l(x) = \mathcal{D}^l_{\mathrm{sgm}}(x) \uplus \mathcal{D}^l_{\mathrm{rest}}(x)$.*

Sketch of proof. We will assume that $x \in \{\theta(m, n), \theta^s(m, n) \mid m, n > 0\}$. The remaining cases corresponding to $x \in \{\theta(m, 0), \theta^s(m, 0), \theta(0, m), \theta^s(0, m) \mid m > 0\}$ are similar so we will not explain them.

Let $z \in \mathcal{D}^l(x)$ be such that $\ell(x,z) > 1$. By Proposition 1.1.4, the interval $[x,z]$ contains exactly two atoms. By Lemma 1.2.17, we have that

$$\mathrm{U}(x) = \begin{cases} \{\theta^s(m,n), \theta^s(m+1,n-1), \theta^s(m-1,n+1), \\ s_0\theta(m,n), \delta(s_0\theta(m-1,n+1)), \delta^2(s_0\theta(m+1,n-1))\} & \text{if } x = \theta(m,n), \\ \{\theta(m+1,n), \theta(m,n+1), s_0\theta^s(m,n), \\ \delta(s_0\theta^s(m-1,n+1)), \delta^2(s_0\theta^s(m+1,n-1))\} & \text{if } x = \theta^s(m,n). \end{cases}$$

It is evident that if a is an atom of $[x,z]$, then $a \in U(x)$. We will now proceed by outlining the main ideas of the proof:

The inclusion $\subset$.

1. Show that $U(x) \subset \mathcal{C}_{x+\rho}$. This observation implies that $[x, x+\rho]$ is not dihedral.
2. Define the sets
$$A = \mathcal{D}^l_{\mathrm{sgm}}(x) \uplus \mathcal{D}^l_{\mathrm{rest}}(x) \quad \text{and}$$
$$D^w(x) = \big(\mathcal{S}t^\circ(x+\rho) \setminus (\mathcal{S}t^\circ(x) \cup A)\big) \cap F_w.$$
3. For $b \in D^{\mathrm{id}}(x)$, show that $|U(x) \cap \mathcal{C}_b| \geq 3$.
4. Let $w \in W_f \setminus \{\mathrm{id}\}$ and let $b \in D^w(x)$. By Lemma 2.2.4, we know that $b(1) \in V(\mathcal{C}_b)$. Since $b(1) \in D^{\mathrm{id}}(x)$, it follows that $[x, b(1)]$ is not dihedral. Consequently, $[x,b]$ is also not dihedral. This proves that if $b \in \cup_{w \in W_f} D^w(x)$, then $b \notin \mathcal{D}^l(x)$. Equivalently, as $\cup_{w\in W_f} D^w(x)$ is the complement of A in $\mathcal{S}t^\circ(x+\rho)$ we have that $\mathcal{D}^l(x) \subset \mathcal{D}^l_{\mathrm{sgm}}(x) \uplus \mathcal{D}^l_{\mathrm{rest}}(x)$.

The inclusion $\supset$. We consider two cases

1. Suppose that $z \in \mathcal{D}^l_{\mathrm{rest}}(x)$.
   - If $\ell(x,z) = 1$, then $[x,z]$ is dihedral.
   - If $\ell(x,z) = 2$ then $[x,z]$ is also dihedral because any interval of length 2 contains exactly two atoms.
   - In the case where $z = \theta(m+1, n+1)$. The atoms of $[x,z]$ are $\theta(m+1,n)$ and $\theta(m,n+1)$.
2. Suppose that $z \in \mathcal{D}^l_{\mathrm{sgm}}(x)$.
   - If $z \in \mathrm{Sgm}(u_1,u_3)$. The of atoms of $[x,z]$ are

$$\begin{cases} \{\theta^s(m+1,n-1), \delta^2(s_0\theta(m+1,n-1))\} & \text{if } x = \theta(m,n), \\ \{\theta(m+1,n), \delta^2(s_0\theta^s(m+1,n-1))\} & \text{if } x = \theta^s(m,n). \end{cases}$$

- If $z \in \mathrm{Sgm}(u_2, u_6)$. The atoms of $[x, z]$ are

$$\begin{cases} \{\theta^s(m-1, n+1), \delta(s_0\theta(m-1, n+1))\} & \text{if } x = \theta(m, n), \\ \{\theta(m, n+1), \delta(s_0\theta^s(m-1, n+1))\} & \text{if } x = \theta^s(m, n). \end{cases}$$

In both cases $[x, z]$ has two atoms, which implies that $z \in \mathcal{D}^l(x)$.

□

2.3 Translations

In this section, $x, y \in W$ and $\lambda \in \Lambda$ are assumed to be dominant unless otherwise stated. We will also assume that $x < y$. We study an important class of morphisms of the partially ordered sets $[x, y] \to [x + \lambda, y + \lambda]$ given by "translations" in the Euclidean plane by the weight λ. These play a major role in the main theorem of this thesis. The type of the interval (Definition 2.2.20) tells us exactly which of these translations are isomorphisms of partially ordered sets (Proposition 2.3.32).

2.3.1 Basic facts about translations

Let $i \in \{1, 2\}$, we define the *translation by the i-th fundamental weight* $\tau_i\colon W \to W$ as follows: if $z \in W$ there is a unique $w \in W_f$ such that $z \in F_w$, so we set $\tau_i(z) := z + w\varpi_i$ (see Notation 1.2.5) where ϖ_i is the i-th fundamental weight. For $\lambda = a\varpi_1 + b\varpi_2$ dominant weight, i.e., $a, b \geq 0$, we define $\tau_\lambda := \tau_1^a \circ \tau_2^b$.

Lemma 2.3.1. *For $w \in W_f$ and $i \in \{1, 2\}$, we have $\tau_i(F_w) \subset F_w$. Furthermore, $\tau_\lambda(F_w) \subset F_w$.*

Proof. For any $w \in W_f$, the set F_w is a convex and unbounded polygon with two or three edges. The two unbounded edges of F_w are rays in the direction of $w\varpi_1$ and $w\varpi_2$. The first part follows. The second part is a direct consequence of the first. □

Lemma 2.3.2. *For $i \in \{1, 2\}$, the map τ_i is injective.*

Proof. Suppose $\tau_i(u) = \tau_i(v) \in F_w$ for some $w \in W_f$ and $u, v \in W$. By Lemma 2.3.1, we have $u, v \in F_w$ so the previous equality becomes $u + w\varpi_i = v + w\varpi_i$. This implies $\mathrm{cen}(u) = \mathrm{cen}(v)$, so $u = v$. □

Lemma 2.3.3. *We have $(x + \lambda)_w = x_w + w\lambda$ for all $w \in W_f$.*

Proof. By Equation (2.12), we have $\mathrm{cen}((x + \lambda)_w) = \mathrm{cen}(x_w) + w\lambda = \mathrm{cen}(x_w + w\lambda)$ so the result follows. □

Lemma 2.3.4. *For all $w \in W_f$, we have:*

(i) $\mathrm{Par}^w_{x,y} + w\lambda = \mathrm{Par}^w_{x+\lambda,y+\lambda}$.

(ii) $\mathrm{Pgn}^w_{x,y} + w\lambda \subset \mathrm{Pgn}^w_{x+\lambda,y+\lambda}$.

Proof. We prove both claims:

(i) By definition of $\mathrm{Par}^w_{x,y}$ and Lemma 2.3.3, $\mathrm{Par}^w_{x,y} + w\lambda$ is the parallelogram with opposite vertices $w\,\mathrm{cen}(y) + w\lambda = w\,\mathrm{cen}(y+\lambda)$ and $\mathrm{cen}(x_w) + w\lambda = (x+\lambda)_w$ and sides parallel to $w\alpha_1$ and $w\alpha_2$. By definition of $\mathrm{Par}^w_{x+\lambda,+\lambda y}$, we conclude $\mathrm{Par}^w_{x,y} + w\lambda = \mathrm{Par}^w_{x+\lambda,y+\lambda}$.

(ii) $\mathrm{Pgn}^w_{x,y} + w\lambda = (\mathrm{Par}^w_{x,y} \cap F_w) + w\lambda \subset (\mathrm{Par}^w_{x,y} + w\lambda) \cap F_w = \mathrm{Pgn}^w_{x+\lambda,y+\lambda}$. This is because $F_w + w\lambda \subset F_w$ (Lemma 2.3.1.)

□

Lemma 2.3.5. *For $i \in \{1,2\}$, we have $\tau_i([x,y]) \subset [\tau_i(x), \tau_i(y)]$.*

Proof. Let $x \leq z \leq y$ and $w \in W_f$ be such that $z \in F_w$. We need to prove that $x + \varpi_i \leq z + w\varpi_i \leq y + \varpi_i$. By Equation (2.11), it is enough to show that $\mathrm{cen}(z + w\varpi_i) \in \mathrm{Pgn}^w_{x+\varpi_i,y+\varpi_i}$. Note that $\mathrm{cen}(z) \in \mathrm{Pgn}^w_{x,y}$. By Lemma 2.3.4(ii),

$$\mathrm{cen}(z + w\varpi_i) = \mathrm{cen}(z) + w\varpi_i \in \mathrm{Pgn}^w_{x,y} + w\varpi_i \subset \mathrm{Pgn}^w_{x+\varpi_i,y+\varpi_i}.$$ □

Proposition 2.3.6. *Let $\mathrm{Pgn}_{x,y}$ be a hexagon and $\mu \in \Lambda$ be any weight. The equality of sets $\mathrm{Pgn}_{x,y} + \mu = \mathrm{Pgn}_{x+\mu,y+\mu}$ implies that $\mu = 0$.*

Proof. Recall that every element of $\mathrm{V}(\mathrm{Pgn}_{x,y})$ different from $\mathrm{cen}(x)$ and $\mathrm{cen}(y)$ belongs to $\mathbb{R}\varpi_1 + \alpha_1$ or $\mathbb{R}\varpi_2 + \alpha_2$. The analogous statement holds for $\mathrm{V}(\mathrm{Pgn}_{x+\mu,y+\mu})$. Let a, b be the two vertices of $\mathrm{Pgn}_{x,y}$ adjacent to $\mathrm{cen}(x)$ such that $a \in \mathbb{R}\varpi_1 + \alpha_1$ and $b \in \mathbb{R}\varpi_2 + \alpha_2$. Since $a+\mu, b+\mu \in \mathrm{V}(\mathrm{Pgn}_{x+\mu,y+\mu})$ and both points are different from $\mathrm{cen}(x+\mu)$ and $\mathrm{cen}(y+\mu)$, we have:

- $a + \mu \in \mathbb{R}\varpi_1 + \alpha_1$ and $b + \mu \in \mathbb{R}\varpi_2 + \alpha_2$, or
- $a + \mu \in \mathbb{R}\varpi_2 + \alpha_2$ and $b + \mu \in \mathbb{R}\varpi_1 + \alpha_1$.

Let us prove that the second case is impossible. Let $c \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ be the other vertex living in the affine line $\mathbb{R}\varpi_1 + \alpha_1$, so $c = a + t\varpi_1$ for some $t > 0$. Since $a + \mu \in \mathbb{R}\varpi_2 + \alpha_2$, we have that $\mu \notin \mathbb{R}\varpi_1$. Since $c + \mu$ is a vertex of $\mathrm{Pgn}_{x+\mu,y+\mu}$ different from $x+\mu$ and $y+\mu$, it belongs to $\mathbb{R}\varpi_2+\alpha_2$ (it can not belong to $\mathbb{R}\varpi_1+\alpha_1$ because $\mu \notin \mathbb{R}\varpi_1$). Then $c+\mu-(a+\mu) \in \mathbb{R}\varpi_2$. This implies that $c-a \in \mathbb{R}\varpi_1 \cap \mathbb{R}\varpi_2$, so $a = c$, which is a contradiction. So, we can suppose the first case. The condition $a + \mu \in \mathbb{R}\varpi_1 + \alpha_1$ holds only if $\mu \in \mathbb{R}\varpi_1$ and the condition $b + \mu \in \mathbb{R}\varpi_2 + \alpha_2$ holds only if $\mu \in \mathbb{R}\varpi_2$ so $\mu = 0$. □

Proposition 2.3.7. *Let $\mu \in \Lambda$ be any weight. If $\mathrm{Pgn}_{x,y}$ is a pentagon with two adjacent vertices on $\mathbb{R}_{\geq 0}\varpi_i + \alpha_i$ for some $i \in \{1,2\}$, then $\mathrm{Pgn}_{x,y} + \mu = \mathrm{Pgn}_{x+\mu,y+\mu}$ only if $\mu \in \mathbb{R}\varpi_i$.*

Proof. Let $x', y' \in D$ be dominant elements, such that $\mathrm{Pgn}_{x',y'}$ is a hexagon with vertices lying in $\mathbb{R}\varpi_2 + \alpha_2$. It is easy to prove, by comparing the interior angles of both polygons, that if $\mathrm{Pgn}_{x,y}$ has two vertices lying on $\mathbb{R}\varpi_1 + \alpha_1$, then $\mathrm{Pgn}_{x',y'}$ cannot be a translation of $\mathrm{Pgn}_{x,y}$. Since this remains true for any choice of dominant elements x', y', the result follows. □

Proposition 2.3.8. *The following statements hold:*

1. *If $\mathrm{Pgn}_{x,y}$ is a parallelogram, then $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$.*
2. *If $\mathrm{Pgn}_{x,y}$ is a pentagon with two adjacent vertices on $\mathbb{R}_{\geq 0}\varpi_i + \alpha_i$ for some $i \in \{1,2\}$, then $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$ if and only if $\lambda \in \mathbb{R}_{\geq 0}\varpi_i$.*
3. *If $[x,y]$ is a hexagon interval, then $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$ only if $\lambda = 0$.*

Proof.

1. It is not hard to prove (using that $\mathrm{cen}(x), \mathrm{cen}(y) \in \mathrm{V}(\mathrm{Pgn}_{x,y})$) that $\mathrm{Pgn}_{x,y}$ is a parallelogram if and only if $\mathrm{Par}_{x,y} = \mathrm{Pgn}_{x,y}$. By Lemma 2.3.1, we have $\mathrm{Pgn}_{x,y} + \lambda \subset F_{\mathrm{id}}$. By Lemma 2.3.4(i), we have $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Par}_{x+\lambda,y+\lambda}$, so
$$\mathrm{Pgn}_{x+\lambda,y+\lambda} = \mathrm{Par}_{x+\lambda,y+\lambda} \cap F_{\mathrm{id}} = (\mathrm{Pgn}_{x,y} + \lambda) \cap F_{\mathrm{id}} = \mathrm{Pgn}_{x,y} + \lambda.$$
2. The implication $\Rightarrow$ is given by Proposition 2.3.7.

   Let us prove the implication $\Leftarrow$. Without loss of generality we can assume that $i = 1$. Consider the following general result of planar geometry:

   Let H be a closed half-plane of the plane E. For any subset $X \subset E$ we use the notation $X_H := X \cap H$. If v is a vector such that $\partial H + v = \partial H$, then for any subset $S \subset E$ we have $S_H + v = (S + v)_H$.

   We apply this result to the special case where $H = \{u \in E \mid (\alpha_2, u) \geq -1\}$, $v = \lambda \in \mathbb{R}_{\geq 0}\varpi_1$, and $S = \mathrm{Par}_{x,y}$ (note that $\partial H = \mathbb{R}\varpi_1 + \alpha_1$ and $\lambda + \partial H = \partial H$.) Thus, we have $(\mathrm{Par}_{x,y})_H + \lambda = (\mathrm{Par}_{x,y} + \lambda)_H$. We have $(\mathrm{Par}_{x,y})_H \subset F_{\mathrm{id}}$, so $(\mathrm{Par}_{x,y})_H = \mathrm{Pgn}_{x,y}$. By combining these identities, we get $\mathrm{Pgn}_{x,y} + \lambda = (\mathrm{Par}_{x,y} + \lambda)_H$. By Lemma 2.3.4(i) and the fact that $F_{\mathrm{id}} \subset H$, we have
$$\begin{aligned}\mathrm{Pgn}_{x+\lambda,y+\lambda} &= (\mathrm{Par}_{x+\lambda,y+\lambda})_H \cap F_{\mathrm{id}} \\ &= (\mathrm{Par}_{x,y} + \lambda)_H \cap F_{\mathrm{id}} \\ &= (\mathrm{Pgn}_{x,y} + \lambda) \cap F_{\mathrm{id}} \\ &= \mathrm{Pgn}_{x,y} + \lambda.\end{aligned}$$
   The last equality follows from $\mathrm{Pgn}_{x,y} \subset F_{\mathrm{id}}$ and Lemma 2.3.1.
3. This is a particular case of Proposition 2.3.6. □

2.3.2 Translations preserving the cardinality of the Bruhat interval

The goal of this section is to prove Propositions 2.3.25.

Notation 2.3.9. If $A \subseteq E$, we denote $cA := \text{cen}(W) \cap A$.

Let $A \subset E$. For any weight $\mu \in \Lambda$, recall from Section 1.2 that $\text{cen}(W) = \text{cen}(W) + \mu$. This implies that $\text{cen}(W) \cap (A + \mu) = (\text{cen}(W) \cap A) + \mu$, or in compressed notation, $c(A + \mu) = cA + \mu$. Similarly, for $z \in W$, we have $z(\text{cen}(W)) = \text{cen}(W)$. Thus, for any subset $A \subset E$, we have $\text{cen}(W) \cap zA = z(\text{cen}(W) \cap A)$, or in compressed notation, $czA = zcA$. We usually prefer the expression czA over zcA to emphasize that we deal with a subset of $\text{cen}(W)$.

Let $z \in W$. As discussed in Section 1.2, $z \colon E \to E$ is a bijection. In particular, for any subset $A \subset E$, we have $z|_{cA} \colon cA \to czA$ is a bijection, so $|cA| = |czA|$. Similarly, for $\mu \in \Lambda$ and any subset $A \subset E$, we have $|cA| = |cA + \mu|$. Summarizing, for any $z \in W$, $\mu \in \Lambda$ and $A \subset E$ we have the following identities:

$$c(A + \mu) = cA + \mu \tag{2.14}$$

$$|cA| = |cA + \mu|. \tag{2.15}$$

By Equation (2.11), we have

$$|[x, y]| = \sum_{w \in W_f} |\,\text{cPgn}^w_{x,y}\,|. \tag{2.16}$$

Lemma 2.3.10. *Let $w \in W_f$, then $|\,\text{cPgn}^w_{x,y}\,| \leq |\,\text{cPgn}^w_{x+\lambda,y+\lambda}\,|$ for all $w \in W_f$. So $|[x, y]| \leq |[x + \lambda, y + \lambda]|$.*

Proof. By Equation (2.14) and Lemma 2.3.4(ii) we have

$$c\,\text{Pgn}^w_{x,y} + w\lambda \subset c\,\text{Pgn}^w_{x+\lambda,y+\lambda}.$$

By Equation (2.15) we have $|c\,\text{Pgn}^w_{x,y} + w\lambda| = |c\,\text{Pgn}^w_{x,y}\,|$, which allows us to conclude $|\,\text{cPgn}^w_{x,y}\,| \leq |\,\text{cPgn}^w_{x+\lambda,y+\lambda}\,|$. By Equation (2.16), this implies $|[x, y]| \leq |[x + \lambda, y + \lambda]|$. □

Proposition 2.3.11. *If $\text{Pgn}_{x,y} + \lambda = \text{Pgn}_{x+\lambda,y+\lambda}$, then $\text{Pgn}^w_{x,y} + w\lambda = \text{Pgn}^w_{x+\lambda,y+\lambda}$ for all $w \in W_f$.*

Proof. **Case A.** If $\text{Pgn}_{x,y}$ is a hexagon, by Proposition 2.3.8 we have $\lambda = w\lambda = 0$ and the proposition follows.

To prove the cases where $\mathrm{Pgn}_{x,y}$ is a pentagon or a parallelogram, we need much more work. Let $w \in W_f$. By Proposition 2.2.21, we have

$$\begin{aligned}\mathrm{Pgn}^w_{x,y} + w\lambda &= \big(w(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w) \cap F_w\big) + w\lambda \\ &= \big[w\big(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w\big) + w\lambda\big] \cap \big[F_w + w\lambda\big] \\ &\subset w\big((\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w) + \lambda\big) \cap F_w \\ &= w\big(\mathrm{Pgn}_{x+\lambda,y+\lambda} \setminus S^{x+\lambda,y+\lambda}_w\big) \cap F_w \\ &= \mathrm{Pgn}^w_{x+\lambda,y+\lambda}.\end{aligned} \tag{2.17}$$

The inclusion in the third line of (2.17) follows from Lemma 2.3.1. The fourth equality is a consequence of the hypothesis and $S^{x+\lambda,y+\lambda}_w = S^{x,y}_w + \lambda$ (this last equality easily follows from the definition of $S^{x,y}_w$ and Lemma 2.3.4(i)). To finish the proof of the proposition we need to prove the inverse inclusion $\mathrm{Pgn}^w_{x+\lambda,y+\lambda} \subset \mathrm{Pgn}^w_{x,y} + w\lambda$, in other words, we need to prove that

$$A := \big(w(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w) + w\lambda\big) \cap F_w \subset \big(w(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_w) \cap F_w\big) + w\lambda =: B.$$

Let $p \in A$, then $p = wp_1 + w\lambda \in F_w$ for some $p_1 \in \mathrm{Pgn}_{x,y} \setminus S^{x,y}_w$. Then $p \in B$ if and only if $wp_1 \in F_w$. So we need to prove that $wp_1 \in F_w$.

Claim 2.3.12. *Let $\mathrm{Pgn}_{x,y}$ be a pentagon with two adjacent vertices on $\mathbb{R}\varpi_1 + \alpha_1$.*

(i) $s_1(\mathrm{Pgn}_{x,y} \setminus S^{x,y}_{s_1}) \subset F_{s_1}$.

(ii) Let $w \in W_f \setminus \{\mathrm{id}, s_1\}$ and $a \in \mathbb{Z}_{\geq 0}$. If $v \in \mathrm{Pgn}_{x,y} \setminus S^{x,y}_w$ and $w(v + a\varpi_1) \in F_w$, then $wv \in F_w$.

Proof. (i) From Figure 2.4 we get

$$F_{s_1} = s_1\{v \in F_{\mathrm{id}} \mid (\alpha_1, v) \geq 1\}. \tag{2.18}$$

Let b be the only vertex of $\mathrm{Pgn}_{x,y}$ adjacent to both $\mathrm{cen}(x)$ and $\mathrm{cen}(y)$. In particular, $(\alpha_1, b + \alpha_1) \geq 1$. From the definitions of $S^{x,y}_{s_1}$ and $\mathrm{Pgn}_{x,y}$, every element of $\mathrm{Pgn}_{x,y} \setminus S^{x,y}_{s_1}$ can be written as $b + \alpha_1 + c\alpha_1 - d\alpha_2$, for some $c, d \geq 0$. Hence $\mathrm{Pgn}_{x,y} \setminus S^{x,y}_{s_1} \subset \{v \in F_{\mathrm{id}} \mid (\alpha_1, v) \geq (\alpha_1, b + \alpha_1)\}$. The claim follows from Equation (2.18).

(ii) We only prove the case $w = s_2$; the other cases are similar, so we omit them. By hypothesis $v \in \mathrm{Pgn}_{x,y} \setminus S^{x,y}_{s_2}$. Let us prove the claim by contrapositive, so let us suppose that $s_2 v \notin F_{s_2}$. Thus $s_2 v \in s_2(F_{\mathrm{id}}) \setminus F_{s_2}$. From Figure 2.4, we note that $s_2(F_{\mathrm{id}}) \setminus F_{s_2} \subset \{u \in E \mid -1 < (\alpha_2, u) \leq 1\}$ and $F_{s_2} \subset \{u \in E \mid (\alpha_2, u) \leq -1\}$. In particular, $-1 < (\alpha_2, s_2 v + a\varpi_1)$, so $s_2(v + a\varpi_1) \notin F_{s_2}$ (recall that $s_2\varpi_1 = \varpi_1$). This proves the claim. ∎

Case B. If $\text{Pgn}_{x,y}$ is a pentagon, without loss of generality, assume that $\text{Pgn}_{x,y}$ has two adjacent vertices in $\mathbb{R}\varpi_1 + \alpha_1$. By Proposition 2.3.8, we have $\lambda = a\varpi_1$ for some $a \in \mathbb{Z}_{\geq 0}$. We have two cases:

- $w = s_1$. By Claim 2.3.12(i) we have $s_1 p_1 \in F_{s_1}$ and the proposition follows.
- $w \in W_f \setminus \{\text{id}, s_1\}$. Note that $w(p_1 + a\varpi_1) = p \in F_w$, so by Claim 2.3.12(ii) we have $wp_1 \in F_w$ and the proposition follows.

Claim 2.3.13. *Let* $\text{Pgn}_{x,y}$ *be a parallelogram. We have* $w(\text{Pgn}_{x,y} \setminus S_w^{x,y}) \subset F_w$.

Proof. The proof is similar to the first part of Claim 2.3.12, so we will omit it. □

Case C. If $\text{Pgn}_{x,y}$ is a parallelogram, Claim 2.3.13 gives $wp_1 \in F_w$ and the proposition follows. This finishes the proof of the proposition. □

Lemma 2.3.14. *If* $\text{Pgn}_{x,y} + \lambda = \text{Pgn}_{x+\lambda,y+\lambda}$, *then* $|[x,y]| = |[x+\lambda, y+\lambda]|$.

Proof. By Equation (2.16), Proposition 2.3.11, Equation (2.14), and again Equation (2.16) we obtain the following sequence of equalities.

$$\begin{aligned}
|[x+\lambda, y+\lambda]| &= \sum_{w \in W_f} |c\,\text{Pgn}^w_{x+\lambda,y+\lambda}| \\
&= \sum_{w \in W_f} |c(\text{Pgn}^w_{x,y} + w\lambda)| \\
&= \sum_{w \in W_f} |c\,\text{Pgn}^w_{x,y}| \\
&= |[x,y]|
\end{aligned}$$ □

Definition 2.3.15. Let $x < y$ and $1 \leq i \leq 2$ and let $v_i^{x,y}(y) \in \text{V}(\text{Pgn}_{x,y})$ be the vertex adjacent to $\text{cen}(y)$ such that $\text{cen}(y) - v_i^{x,y}(y) \in \mathbb{R}_{>0}\alpha_i$. Similarly, let $v_i^{x,y}(x) \in \text{V}(\text{Pgn}_{x,y})$ be the vertex adjacent to $\text{cen}(x)$ such that $v_i^{x,y}(x) - \text{cen}(x) \in \mathbb{R}_{>0}\alpha_i$. If the interval $[x,y]$ is clear from the context, the superscript $(-)^{x,y}$ will be omitted.

Lemma 2.3.16. *Let* $\{i,j\} = \{1,2\}$. *We have that* $v_i^{x,y}(x) = v_j^{x,y}(y)$ *implies the equality*

$$v_i^{x,y}(x) = \text{cen}(x) + (\varpi_i, \text{cen}(y) - \text{cen}(x))\alpha_i.$$

Proof. Note that $v = v_i^{x,y}(x) = v_j^{x,y}(y)$ is in the intersection between the affine lines $\text{cen}(x) + \mathbb{R}\alpha_i$ and $\text{cen}(y) + \mathbb{R}\alpha_j$. Thus, there is $a_i \in \mathbb{R}$ such that $v = \text{cen}(x) + a_i\alpha_i \in \text{cen}(y) + \mathbb{R}\alpha_j$. This implies that $(\varpi_i, \text{cen}(x) + a_i\alpha_i) = (\varpi_i, \text{cen}(y))$ so we obtain $a_i = (\varpi_i, \text{cen}(y) - \text{cen}(x))$. □

Lemma 2.3.17. *Let* $\{i,j\} = \{1,2\}$. *If* $v = v_i^{x,y}(x) = v_j^{x,y}(y)$, *then*

$$v + \lambda = v_i^{x+\lambda,y+\lambda}(x+\lambda) = v_j^{x+\lambda,y+\lambda}(y+\lambda).$$

Proof. The equality $v_i^{x,y}(x) = v_j^{x,y}(y)$ is equivalent to the inclusion

$$(x + \mathbb{R}\alpha_i) \cap (y + \mathbb{R}\alpha_j) \in F_{\mathrm{id}}.$$

This inclusion and Lemma 2.3.1 imply that

$$\lambda + \big((x + \mathbb{R}\alpha_i) \cap (y + \mathbb{R}\alpha_j)\big) = (x + \lambda + \mathbb{R}\alpha_i) \cap (y + \lambda + \mathbb{R}\alpha_j) \in F_{\mathrm{id}}.$$

This in turn implies that $v_i^{x+\lambda,y+\lambda}(x+\lambda) = v_j^{x+\lambda,y+\lambda}(y+\lambda) = v + \lambda$. □

Lemma 2.3.18. *Let* $\{i,j\} = \{1,2\}$.

(i) If $v_i^{x,y}(x) \notin \partial F_{\mathrm{id}}$, *then*

$$v_i^{x,y}(x) = v_j^{x,y}(y) = \mathrm{cen}(x) + (\varpi_i, \mathrm{cen}(y) - \mathrm{cen}(x))\alpha_i.$$

(ii) If $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$, *then*

$$v_i^{x,y}(x) = \mathrm{cen}(x) + (1 + (\alpha_j, \mathrm{cen}(x)))\alpha_i.$$

Proof. (i) If $v_i^{x,y}(x) \notin \partial F_{\mathrm{id}}$ then $v_i^{x,y}(x) = v_j^{x,y}(y)$. The second equality follows from Lemma 2.3.16.

(ii) By definition of $v_i^{x,y}(x)$, there is $a_i > 0$ such that $\mathrm{cen}(x) + a_i\alpha_i = v_i^{x,y}(x)$. Since $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$, we have either $v_i^{x,y}(x)$ is in $H_{\alpha_j,-1}$, $H_{\alpha_i,-1}$, or $H_{\rho,-1}$. The second case is impossible since a_i, $(\alpha_i, \mathrm{cen}(x))$, and (α_i, α_i) are positive. Similarly, the third case is impossible since a_i, $(\rho, \mathrm{cen}(x))$, and (ρ, α_i) are positive. Therefore $(\alpha_j, \mathrm{cen}(x) + a_i\alpha_i) = -1$, so we obtain $a_i = 1 + (\alpha_j, \mathrm{cen}(x))$. □

Lemma 2.3.19. *Suppose* $\mathrm{Pgn}_{x,y} + \lambda \neq \mathrm{Pgn}_{x+\lambda,y+\lambda}$. *Then* $|\,\mathrm{cPgn}_{x,y}\,| < |\,\mathrm{cPgn}_{x+\lambda,y+\lambda}\,|$ *and* $|[x,y]| < |[x+\lambda, y+\lambda]|$.

Proof. We start the proof by replacing the condition $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$ with a simpler statement. Let $n_i \in \mathbb{Z}_{\geq 0}$ be maximal such that $\mathrm{cen}(x) + n_i\alpha_i \in \mathrm{Pgn}_{x,y}$ and $m_i \in \mathbb{N}$ maximal such that $\mathrm{cen}(x) + \lambda + m_i\alpha_i \in \mathrm{Pgn}_{x+\lambda,y+\lambda}$ for $i \in \{1,2\}$. Similarly, let $a_i \in \mathbb{R}_{\geq 0}$ be maximal such that $\mathrm{cen}(x) + a_i\alpha_i \in \mathrm{Pgn}_{x,y}$ and $b_i \in \mathbb{R}_{\geq 0}$ be maximal such that $\mathrm{cen}(x) + \lambda + b_i\alpha_i \in \mathrm{Pgn}_{x+\lambda,y+\lambda}$ for $i \in \{1,2\}$. Clearly, $n_i = \lfloor a_i \rfloor$ and $m_i = \lfloor b_i \rfloor$. By Lemma 2.3.4(ii), we have $\mathrm{Pgn}_{x,y} + \lambda \subset \mathrm{Pgn}_{x+\lambda,y+\lambda}$ which implies that

$$m_i \geq n_i \text{ and } b_i \geq a_i. \tag{2.19}$$

In particular, $v_i^{x,y}(x) = \mathrm{cen}(x) + a_i\alpha_i$ and $v_i^{x+\lambda,y+\lambda}(x+\lambda) = \mathrm{cen}(x+\lambda) + b_i\alpha_i$.

Claim 2.3.20. *The condition* $a_i = b_i$ *for* $i \in \{1, 2\}$ *implies* $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$.

Proof. Let us assume that $a_i = b_i$ for $i \in \{1, 2\}$. There exist $c_i \in \mathbb{R}_{\geq 0}$ such that

$$\{x, y, x + a_1\alpha_1, x + a_2\alpha_2, x + a_1\alpha_1 + c_1\varpi_1, x + a_2\alpha_2 + c_2\varpi_2\} = \mathrm{V}(\mathrm{Pgn}_{x,y}),$$

where if one or two of the c_i's are zero, we are in the pentagon or parallelogram case. As $x + a_i\alpha_i + c_i\varpi_i \in y + \mathbb{R}\alpha_j$, by pairing this inclusion with ϖ_i we obtain

$$c_i = \frac{3}{2}((y - x, \varpi_i) - a_i). \tag{2.20}$$

The set $\mathrm{V}(\mathrm{Pgn}_{x+\lambda,y+\lambda})$ is given by

$$\{x+\lambda, y+\lambda, x+\lambda+a_1\alpha_1, x+\lambda+a_2\alpha_2, x+\lambda+a_1\alpha_1+c_1'\varpi_1, x+\lambda+a_2\alpha_2+c_2'\varpi_2\}.$$

As the right-hand side of Equation (2.20) only depends on $y - x = (y + \lambda) - (x + \lambda)$ and $a_i = b_i$, we obtain that $c_i = c_i'$ and we conclude. □

Claim 2.3.21. *Let* $1 \leq i \leq 2$ *and suppose that* $n_i = m_i$. *If* $a_i \neq b_i$, *then* $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$ *and* $v_i^{x+\lambda,y+\lambda}(x + \lambda) \notin \partial F_{\mathrm{id}}$.

Proof. Recall that $v_i^{x,y}(x) \in \mathrm{V}(\mathrm{Pgn}_{x,y}) \subset F_{\mathrm{id}}$.

- Suppose that $v_i^{x,y}(x) \notin \partial F_{\mathrm{id}}$. By Lemma 2.3.18(i) we have that $v_i^{x,y}(x) = v_j^{x,y}(y)$. By Lemma 2.3.17 we have that $v_i^{x,y}(x) + \lambda = v_i^{x+\lambda,y+\lambda}(x + \lambda)$, so $a_i = b_i$, a contradiction.
- Suppose that $v_i^{x+\lambda,y+\lambda}(x + \lambda) \in \partial F_{\mathrm{id}}$. As $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$, by Lemma 2.3.18(ii), we have that $b_i - a_i = (\alpha_j, \lambda) \in \mathbb{Z}$ is the difference between two numbers having the same floor, so $a_i = b_i$, a contradiction. □

Claim 2.3.22. *Let* $1 \leq i \leq 2$ *and suppose that* $n_i = m_i$. *If* $a_i \neq b_i$ *then* $v_i^{x+\lambda,y+\lambda}(x + \lambda) = \mathrm{cen}(z)$ *for some* $z \in W$. *Furthermore* $\mathrm{cen}(z) - \lambda \in \mathrm{V}(\mathrm{Par}_{x,y})$ *and* $\mathrm{cen}(z) - \lambda \notin \mathrm{Pgn}_{x,y}$

Proof. By Claim 2.3.21, we have that $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$ and $v_i^{x+\lambda,y+\lambda}(x + \lambda) \notin \partial F_{\mathrm{id}}$. From Lemma 2.3.18(i), we obtain that

$$v \coloneqq v_i^{x+\lambda,y+\lambda}(x + \lambda) = v_j^{x+\lambda,y+\lambda}(y + \lambda).$$

This implies that v is a vertex of both $\mathrm{Par}_{x+\lambda,y+\lambda}$ and $\mathrm{Pgn}_{x+\lambda,y+\lambda}$, because it is the intersection of the lines $x + \lambda + \mathbb{R}\alpha_i$ and $y + \lambda + \mathbb{R}\alpha_j$. Hence $v - \lambda \in V(\mathrm{Par}_{x,y})$ because it is the intersection of the lines $x + \mathbb{R}\alpha_i$ and $y + \mathbb{R}\alpha_j$.

Now, recall that the floor of a_i is n_i, and $v_i^{x,y}(x) = \mathrm{cen}(x) + a_i\alpha_i$. By Remark 2.2.17, we have that a_i is either n_i, $n_i + \frac{1}{3}$ or $n_i + \frac{2}{3}$. Similarly b_i is either m_i, $m_i + \frac{1}{3}$ or $m_i + \frac{2}{3}$.

If $a_i = n_i$, since $\mathrm{cen}(W) = \mathrm{cen}(W) + n_i\alpha_i$ we have that $\mathrm{cen}(x) + a_i\alpha_i \in \mathrm{cen}(W) \cap \partial F_{\mathrm{id}} = \emptyset$, a contradiction. So we conclude that $a_i \neq n_i$. From hypothesis and Equation (2.19), it follows that $a_i < b_i$.

By hypothesis $n_i = m_i$, thus, $a_i < b_i$ implies that $a_i = n_i + \frac{1}{3}$ and $b_i = n_i + \frac{2}{3}$. This implies that clearly,

$$v - \lambda = \mathrm{cen}(x) + \left(n_i + \frac{2}{3}\right)\alpha_i \in \mathrm{V}(\mathrm{Par}_{x,y}). \tag{2.21}$$

The point $v_i^{x,y}(x) = \mathrm{cen}(x) + a_i\alpha_i \in \partial F_{\mathrm{id}}$ is a vertex of an alcove because $v_i^{x,y}(x) \in \partial F_{\mathrm{id}}$. Since $a_i \in \mathbb{N} + \frac{1}{3}$, we have that x is down-oriented. By Equation (2.21) we have that $v - \lambda = \mathrm{cen}(z_v)$ for some up-oriented element $z_v \in W$. The claim follows by taking $z = z_v + \lambda$, since

$$\mathrm{cen}(z) = \mathrm{cen}(z_v) + \lambda = v \in \mathrm{Pgn}_{x+\lambda, y+\lambda}.$$

Moreover, by Equation (2.21)

$$\mathrm{cen}(z) - \lambda = v_i^{x,y}(x) + \frac{1}{3}\alpha_i \notin F_{\mathrm{id}},$$

which implies that $\mathrm{cen}(z) - \lambda \notin \mathrm{Pgn}_{x,y}$. □

Claim 2.3.23. *If $n_i = m_i$ for all $i \in \{1, 2\}$, then either $a_i = b_i$ for all $i \in \{1, 2\}$ or $|\,\mathrm{cPgn}_{x+\lambda,y+\lambda}\,| - |\,\mathrm{cPgn}_{x,y}\,| \geq 1$*

Proof. Without loss of generality, we can suppose $a_1 \neq b_1$. By Claim 2.3.22, we have $v_1^{x+\lambda,y+\lambda}(x + \lambda) = \mathrm{cen}(z) \in \mathrm{Pgn}_{x+\lambda,y+\lambda}$ for some $z \in W$. Furthermore we have that $\mathrm{cen}(z) - \lambda \notin \mathrm{Pgn}_{x,y}$. Lemma 2.3.4(ii) states that $\mathrm{Pgn}_{x,y} + \lambda \subset \mathrm{Pgn}_{x+\lambda,y+\lambda}$, and this inclusion is strict because $\mathrm{cen}(z) \in \mathrm{Pgn}_{x+\lambda,y+\lambda} \setminus (\mathrm{Pgn}_{x,y} + \lambda)$. □

Now we can finish the proof of Lemma 2.3.19. By the previous claim, if $n_i = m_i$ for all $i \in \{1, 2\}$, we have two possibilities. The first one is that $a_i = b_i$ for all $i \in \{1, 2\}$. In this case by Claim 2.3.20 we have that $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda,y+\lambda}$ and this contradicts the hypothesis. The second possibility, by Claim 2.3.23, we have $|\,\mathrm{cPgn}_{x,y}\,| < |\,\mathrm{cPgn}_{x+\lambda,y+\lambda}\,|$. By Lemma 2.3.10 we know that $|\,\mathrm{cPgn}^w_{x,y}\,| \leq |\,\mathrm{cPgn}^w_{x+\lambda,y+\lambda}\,|$ for all $w \in W_f$. Equation (2.16) implies that $|[x, y]| < |[x + \lambda, y + \lambda]|$, which proves the lemma in this case.

It remains the case when there is $1 \leq i_0 \leq 2$ such that $n_{i_0} < m_{i_0}$. Then $\mathrm{cen}(x) + m_{i_0}\alpha_{i_0} \notin \mathrm{Pgn}_{x,y}$ and the injective map $\mathrm{cPgn}_{x,y} \to \mathrm{cPgn}_{x+\lambda,y+\lambda}$ given by $z \mapsto z + \lambda$ is not surjective because $\mathrm{cen}(x) + \lambda + m_{i_0}\alpha_{i_0}$ has no preimage. This proves that $|\,\mathrm{cPgn}_{x,y}\,| < |\,\mathrm{cPgn}_{x+\lambda,y+\lambda}\,|$. By the same reasoning as before, we have $|[x, y]| < |[x + \lambda, y + \lambda]|$. □

Lemmas 2.3.14 and 2.3.19 are summarized as follows

Lemma 2.3.24. *$|[x,y]| = |[x+\lambda, y+\lambda]|$ if and only if $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x+\lambda, y+\lambda}$.*

The following Proposition is a direct consequence of Proposition 2.3.8, Lemma 2.3.10, and Lemma 2.3.24.

Proposition 2.3.25. *The following statements hold:*

1. *If $[x,y]$ is a parallelogram interval, then $|[x,y]| = |[x+\lambda, y+\lambda]|$, for $i \in \{1,2\}$.*
2. *If $[x,y]$ is a pentagon interval, then there is $\{i_0, j_0\} \in \{1,2\}$ such that for every $a \in \mathbb{Z}_{>0}$, the following holds*

$$|[x,y]| = |[x + a\varpi_{i_0}, y + a\varpi_{i_0}]| \text{ and } |[x,y]| < |[x + a\varpi_{j_0}, y + a\varpi_{j_0}]|.$$

   Moreover, $\mathrm{Pgn}_{x,y}$ is a pentagon with two adjacent vertices lying on $\mathbb{R}_{\geq 0}\varpi_{i_0} + \alpha_{i_0}$.

3. *If $[x,y]$ is a hexagon interval, then $|[x,y]| < |[\tau_i(x), \tau_i(y)]|$, for $i \in \{1,2\}$.*

2.3.3 Translations preserving the Bruhat order

The goal of this section is to prove Propositions 2.3.30 and 2.3.32.

Let $z, z' \in W$. For $i \in \{1,2\}$, consider the following property

$$P_i(z', z) : z' \leq z \iff \tau_i(z') \leq \tau_i(z).$$

We say that the property $P(z', z)$ holds if $P_i(z', z)$ for all $1 \leq i \leq 2$.

Lemma 2.3.26. *The property $P(z', z)$ holds for $z' \in W$ and $z \in D \cup s_0 D$.*

Proof. Let $z \in D$. Corollary 2.1.4, says that $\mathrm{V}(\mathcal{C}_{\tau_i(z)}) = \{w\tau_i(z)\}_{w \in W_f}$. We have:

$$\begin{aligned}
z' \leq z &\iff z' \in \mathcal{C}_z \cap F_w \cap \mathrm{cen}(W) && \text{(Corollary 2.1.5)}\\
&\iff z' \in \big(wz - \mathrm{Cone}^w(\alpha_1, \alpha_2)\big) \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.9)}\\
&\iff \tau_i(z') \in \big(w\tau_i(z) - \mathrm{Cone}^w(\alpha_1, \alpha_2)\big) \cap F_w \cap \mathrm{cen}(W) &&\\
&\iff \tau_i(z') \in \mathcal{C}_{\tau_i(z)} \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.9)}\\
&\iff \tau_i(z') \leq \tau_i(z) && \text{(Corollary 2.1.5)}
\end{aligned}$$

The case $z \in s_0 D$ is proved in the same way, with the only difference being that it requires an analog of Lemma 2.2.9 for these elements, where the same equation holds but with wz replaced by $ww_0 z$. Since the proof of this analog follows the same reasoning, we omit it. □

Lemma 2.3.27. *Suppose that $\tau_i \colon [x,y] \to [\tau_i(x), \tau_i(y)]$ is a bijection. The property $P_i(z', z)$ holds for $z' \in [x,y]$ and $z \in [x,y] \cap \{\mathbf{x}_n, \sigma(\mathbf{x}_n)\}_{n \in \mathbb{N}}$.*

Proof. We prove the result only for $z \in [x,y] \cap \{\mathbf{x}_n\}_{n\in\mathbb{N}}$, as the case $z \in [x,y] \cap \{\sigma(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ follows analogously. To continue, we examine each translation individually.

Case $i = 1$. In this case, we observe that $\tau_1(z) \in \{\mathbf{x}_n \mid n \geq 3\}$.

Suppose that $z' \in F_w$, for $w \notin \{s_2, s_1s_2\}$. We have:

$$\begin{aligned} z' \leq z &\iff z' \in \mathcal{C}_z \cap F_w \cap \mathrm{cen}(W) && \text{(Corollary 2.1.5)} \\ &\iff z' \in \big(wz - \mathrm{Cone}^w(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.10)} \\ &\iff \tau_1(z') \in \big(w\tau_1 z - \mathrm{Cone}^w(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) && \\ &\iff \tau_1(z') \in \mathcal{C}_{\tau_1 z} \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.10)} \\ &\iff \tau_1(z') \leq \tau_1(z) && \text{(Corollary 2.1.5)} \end{aligned}$$

Suppose that $z' \in F_w$, for $w \in \{s_2, s_1s_2\}$. In the third equivalence below, we use that $s_2\varpi_1 = \varpi_1$:

$$\begin{aligned} z' \leq z &\iff z' \in \mathcal{C}_z \cap F_w \cap \mathrm{cen}(W) && \text{(Corollary 2.1.5)} \\ &\iff z' \in \big(ws_2 z - \mathrm{Cone}^{ws_2}(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.10)} \\ &\iff \tau_1(z') \in \big(ws_2\tau_1 z - \mathrm{Cone}^{ws_2}(\alpha_1, \rho)\big) \cap F_w \cap \mathrm{cen}(W) && \\ &\iff \tau_1(z') \in \mathcal{C}_{\tau_1 z} \cap F_w \cap \mathrm{cen}(W) && \text{(Lemma 2.2.10)} \\ &\iff \tau_1(z') \leq \tau_1(z) && \text{(Corollary 2.1.5)} \end{aligned}$$

Case $i = 2$. In this case, we note that $\tau_2(z) \in D$. By Lemma 2.3.5, and since τ_2 is a bijection, we have that $|[x,y]| = |[\tau_2(x), \tau_2(y)]|$. By Proposition 2.3.25, the polygon $\mathrm{Pgn}_{x,y}$ has no edge supported on $\mathbb{R}_{\geq 0}\varpi_1 + \alpha_1$. It is not hard to see that if $v_1^{x,y}(x) \notin \partial F_{\mathrm{id}}$, then $v_1^{x,y}(x) = \mathbf{x}_m$ for some $m \in \mathbb{N}$. Similarly, it is not hard to see that if $v_1^{x,y}(x) \in \partial F_{\mathrm{id}}$, then $v_1^{x,y}(x) = \mathbf{x}_m - \frac{1}{3}\alpha_2$ for some $m \in \mathbb{N}$.

Claim 2.3.28. $\mathcal{S}t^\circ(x) \cap \mathrm{V}(\mathcal{C}_z) = \{s_2s_1z, s_1s_2s_1z\}$.

Proof. The proof of this claim is straightforward and will become clear to the reader once we illustrate a single case (see Figure 2.9). Consider the case $v_1^{x,y} \in \partial F_{\mathrm{id}}$. The green, yellow, and purple alcoves are the only possible choices for z that live in $\mathrm{Pgn}_{x,y} \cap \{\mathbf{x}_n\}_{n\in\mathbb{N}}$. In this example, we chose z to be the green alcove. The aqua quadrilateral is $\mathcal{C}_z$. In this case, s_2s_1z is the blue dot, and $s_1s_2s_1z$ is the red dot. □

In a similar manner, it can be shown that

$$\mathcal{S}t(\tau_2(x)) \cap \mathrm{V}(\mathcal{C}_{\tau_2(z)}) = \mathrm{V}(\mathcal{C}_{\tau_2(z)}) \setminus \{\tau_2(z), \tau_2(s_1z)\}.$$

Since the proof is longer and tedious, it will be omitted.

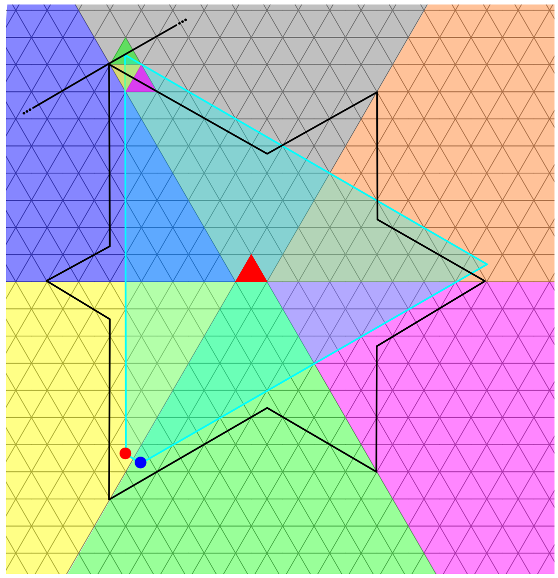


Figure 2.9: $\mathcal{C}_z$ and $\mathcal{S}t(x)$

Returning to the proof of the lemma, by Claim 2.3.28, we know that $\{z, s_1z\} \notin \mathcal{S}t^\circ(x)$. Let P_z denote the parallelogram with opposite vertices $\{x, z\}$ and edges parallel to α_1 and α_2. Similarly, let P_{s_1z} denote the parallelogram with opposite vertices $\{x_{s_1}, s_1z\}$ and edges parallel to α_1 and ρ. We have two cases:

- $z \in \mathcal{S}t(x)$ (for example, the purple or the yellow alcove in Figure 2.9). The two parallelograms are degenerate and can be expressed by the formulas $P_z = \text{Sgm}(z, x)$ and $P_{s_1z} = \text{Sgm}(x_{s_1}, s_1z)$. By Corollary 2.1.5 we have that $[x, z] = \{u \in [x, z] \mid u \in P_z \uplus P_{s_1z}\}$. Furthermore, since

$$\begin{aligned}\tau_2(\text{Sgm}(z,x) \cup (\text{Sgm}(x_{s_1}, s_1z)) &= \text{Sgm}(z,x) \uplus (\text{Sgm}(x_{s_1}, s_1z) + \varpi_2. \\ &= \text{Sgm}(\tau_2(z), \tau_2(x)) \uplus \text{Sgm}(\tau_2(x_{s_1}), \tau_2(s_1(z))).\end{aligned}$$

  From this, it follows that $[\tau_2x, \tau_2z] = \{\tau_2u \mid u \in P_z \uplus P_{s_1z}\}$ so the claim is proved.

- If $z \in \partial\mathcal{C}_z \setminus \mathcal{S}t(x)$ (for example, see the green alcove in Figure 2.9.) Since $z = v_1^{x,y}(x) + \frac{1}{3}\alpha_2$ and $s_1z = u_3 + \frac{1}{3}\rho$, it follows that

$$\text{V}(P_z) = \{x, z, v_1^{x,y}(x), x + \frac{1}{3}\alpha_2\} \text{ and}$$

$$\text{V}(P_{s_1z}) = \{x_{s_1}, s_1z, s_1v_1^{x,y}(x), x_{s_1} + \frac{1}{3}\rho\}.$$

  Additionally, we have

$$\text{c}P_z = \{\text{cen}(u) \mid u \in \text{Sgm}(v_1^{x,y}(x), x) \uplus \text{Sgm}(y, x + \frac{1}{3}\alpha_2)\}$$

$$\text{c}P_{s_1z} = \{\text{cen}(u) \mid u \in \text{Sgm}(x_{s_1}, s_1v_1^{x,y}(x)) \uplus \text{Sgm}(x_{s_1} + \frac{1}{3}\rho, s_1y)\}.$$

By Corollary 2.1.5, we note that $[x, z] = \{u \mid u \in P_z \uplus P_{s_1 z}\}$. It is an easy computation to check that in fact $[\tau_2 x, \tau_2 z] = \{\tau_2 u \mid u \in P_z \uplus P_{s_1 z}\}$ so the claim is proved. □

The following lemma generalize Lemmas 2.3.26 and 2.3.27. Its proof relies on ideas analogous to those in the given on these lemmas, but in the interest of brevity, the proof will be omitted.

Lemma 2.3.29. *Let $j \in \{0, 1, 2\}$. The property $P(z', z)$ holds for $z' \in W$ and $z \in \delta^j D \cup \delta^j s_0 D$. Furthermore, if $z', z \in [x, y]$ and $\tau_i : [x, y] \to [\tau_i(x), \tau_i(y)]$ is a bijection, then the property $P_i(z', z)$ holds for $z' \in W$ and $z \in X$*

Proposition 2.3.30. *Suppose that $\tau_i \colon [x, y] \to [\tau_i(x), \tau_i(y)]$ is a bijection. Then τ_i defines an isomorphism of posets between $[x, y]$ and $[\tau_i(x), \tau_i(y)]$.*

Proof. Recall that for $z, z' \in W$, we have property:

$$P(z', z) : z' \leq z \iff \tau_i(z') \leq \tau_i(z).$$

The Proposition is equivalent to stating that $P(z', z)$ holds for all $z, z' \in [x, y]$. By Lemma 2.3.26, $P(z', z)$ holds when $z \in D \cup s_0 D$ and $z' \in W$. By Lemma 2.3.27, $P_i(z', z)$ holds for $z \in \{\mathbf{x}_n, \sigma \mathbf{x}_n\}_{n \in \mathbb{N}}$ and $z' \in W$. The remaining cases follow by Lemma 2.3.29. □

The following result is of fundamental importance for this paper.

Lemma 2.3.31. *Let $[x, y]$ be an interval such that x, y are dominant.*

1. *If $[x, y]$ is a parallelogram interval, then $\tau_i \colon [x, y] \to [\tau_i(x), \tau_i(y)]$ is an isomorphism of posets, for $i \in \{1, 2\}$.*

2. *If $[x, y]$ is a pentagon interval, then for some $i \neq j \in \{1, 2\}$, we have $\tau_i \colon [x, y] \to [\tau_i(x), \tau_i(y)]$ is an isomorphism of posets and $[x, y] \not\simeq [\tau_j(x), \tau_j(y)]$.*

3. *If $[x, y]$ is a hexagon interval, then $[x, y] \not\simeq [\tau_i(x), \tau_i(y)]$, for $i \in \{1, 2\}$.*

Proof. The result follows directly from Lemmas 2.3.2 and 2.3.5, and Propositions 2.3.25 and 2.3.30. □

Proposition 2.3.32. *Let $[x, y]$ be an interval such that x, y are dominant. We have:*

1. *If $[x, y]$ is a parallelogram interval, then $\tau_\lambda \colon [x, y] \to [\tau_\lambda(x), \tau_\lambda(y)]$ is an isomorphism of posets for any dominant weight λ.*

2. *If $[x, y]$ is a pentagon interval (so $\mathrm{Pgn}_{x,y}$ has two adjacent vertices on $\mathbb{R}\varpi_{i_0} + \alpha_{i_0}$ for some $i_0 \in \{1, 2\}$), then $\tau_\lambda \colon [x, y] \to [\tau_\lambda(x), \tau_\lambda(y)]$ is an isomorphism of posets for $\lambda = a\varpi_{i_0}$ and $a \geq 0$. For any other λ, we have $[x, y] \not\simeq [x + \lambda, y + \lambda]$.*

3. *If $[x, y]$ is a hexagon interval, then $[x, y] \not\simeq [x + \lambda, y + \lambda]$ for all $\lambda \neq 0$.*

Chapter 3

A Geometric approach to the intervals classification

3.1 The sides of $\mathrm{Pgn}_{x,y}$ and rigid intervals

In this section, x, y are dominant elements such that $[x, y]$ is a non-degenerate interval.

3.1.1 A normalization of the distance

To avoid heavy use of square roots, sometimes we will use a normalized version of the standard distance $d(-,-)$. Given $a, b \in E$, we define

$$\mathrm{dist}(a,b) := \frac{\sqrt{2}}{3} d(a,b).$$

In this normalization, the height of an alcove is $3/2$, its sides have length $\sqrt{3}$, and the distance from any vertex of an alcove to its center is 1.

3.1.2 The edges of $\mathrm{Pgn}_{x,y}$

Let $v \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ be adjacent to $\mathrm{cen}(y)$. There is $z \in W$ such that $z \in \mathrm{Sgm}(y, v)$ and

$$d(y,z) = \max\{d(y,w) \mid w \in W, w \in \mathrm{Sgm}(y,v)\}. \tag{3.1}$$

Such a z is unique. By Equation (2.11), we have that $z \in [x, y]$. Equivalently, z can be characterized as the unique element in W such that $z \in \mathrm{Sgm}(y, v)$ and $v \in z$ (we recall that by $v \in z$ we mean $v \in \overline{\mathcal{A}_z}$.) Similarly, if $v \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ is adjacent to cen(x) there is a unique $z \in W$ such that $z \in \mathrm{Sgm}(v, x)$ and

$$d(z, x) = \max\{d(w, x) \mid w \in W, w \in \mathrm{Sgm}(v, x)\}, \tag{3.2}$$

or equivalently, such that $z \in \mathrm{Sgm}(v, x)$ and $v \in z$.

Lemma 3.1.1. *Let $v \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ be adjacent to* cen(y)*. Let $z \in [x, y]$ be the unique element such that $z \in \mathrm{Sgm}(y, v)$ and $v \in z$ as above. Suppose that $z \neq y$, then we have*

$$\mathrm{dist}(y, v) = \begin{cases} \frac{3}{2}(\ell(z, y) - 1) + \epsilon(y) + 1 & \text{if } v = \mathrm{cen}(z) \text{ and } \ell(z) \text{ is odd,} \\ \frac{3}{2}(\ell(z, y) - 1) + \epsilon(y) + \frac{1}{2} & \text{if } v = \mathrm{cen}(z) \text{ and } \ell(z) \text{ is even,} \\ \frac{3}{2}(\ell(z, y) - 1) + \epsilon(y) + \frac{3}{2} & \text{otherwise.} \end{cases}$$

Where $\epsilon(y) = 1/2$ if $y \in \Theta$ and $\epsilon(y) = 1$ if $y \in \Theta^s$.

Proof. Let $p_y, p_z \in \mathrm{Sgm}(y, v)$ be such that $p_y \in \partial(\mathcal{A}_y)$, $p_z \in \partial(\mathcal{A}_z)$ and

$$d(y, p_y) = \max\{d(y, p) \mid p \in \mathrm{Sgm}(y, v) \cap \overline{\mathcal{A}_y}\}$$
$$d(y, p_z) = \min\{d(y, p) \mid p \in \mathrm{Sgm}(y, v) \cap \overline{\mathcal{A}_z}\}.$$

We will break the problem into three computations using the following formula.

$$\mathrm{dist}(y, v) = \mathrm{dist}(y, p_y) + \mathrm{dist}(p_y, p_z) + \mathrm{dist}(p_z, v).$$

Since $\mathrm{Sgm}(y, v)$ is parallel to a simple root, we have that either p_y (resp. p_z) is a vertex or the midpoint of an edge of y (resp. z). If $y \in \Theta$, y is down-oriented, so p_y is the mid-point of an edge of y. If $y \in \Theta^s$, y is up-oriented, p_y is one of the bottom vertices of y. In both cases, $\mathrm{dist}(y, p_y) = \epsilon(y)$.

Similarly, when z is down-oriented, p_z is a vertex of z, and when z is up-oriented we have that p_z is a mid-point of one of the edges of z. By Lemma 1.2.4, we have

$$\mathrm{dist}(p_z, z) = \begin{cases} \frac{1}{2} & \text{if } \ell(z) \text{ is even,} \\ 1 & \text{if } \ell(z) \text{ is odd.} \end{cases}$$

If $v = \mathrm{cen}(z)$, then obviously $\mathrm{dist}(p_z, z) = \mathrm{dist}(p_z, v)$. If $v \neq \mathrm{cen}(z)$, then $\mathrm{Sgm}(p_z, v)$ is a height of $\mathcal{A}_z$, so $\mathrm{dist}(p_z, v) = \frac{3}{2}$.

By Corollary 2.1.5 and Lemma 2.2.7, an element w whose center is in $\mathrm{Sgm}(y, v)$ is also in $[z, y]$. As in the proof of Lemma 2.2.13, we have that $\{u \in W \mid u \in \mathrm{Sgm}(y, v)\}$ is a maximal chain in $[z, y]$. As

$$|\{u \in W \mid u \in \mathrm{Sgm}(y, v)\}| = \ell(z, y) + 1,$$

one has

$$|\{u \in W \mid u \in \mathrm{Sgm}(p_y, p_z)\}| = \ell(z, y) - 1.$$

Since for any $u \in W$ with $\mathrm{cen}(u) \in \mathrm{Sgm}(p_y, p_z)$, we have that $\mathcal{A}_u$ intersects $\mathrm{Sgm}(p_y, p_z)$ along one of its heights, the equality $\mathrm{dist}(p_y, p_z) = \frac{3}{2}(\ell(z, y) - 1)$ follows. □

The next lemma is proved in the same way as the previous one.

Lemma 3.1.2. *Let $v \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ be adjacent to $\mathrm{cen}(x)$. Let $z \in [x, y]$ be the unique element such that $z \in \mathrm{Sgm}(v, x)$ and $v \in z$. If $z \neq x$, we have*

$$\mathrm{dist}(v, x) = \begin{cases} \frac{3}{2}(\ell(x, z) - 1) + \eta(x) + \frac{1}{2} & \text{if } v = \mathrm{cen}(z) \text{ and } \ell(z) \text{ is odd,} \\ \frac{3}{2}(\ell(x, z) - 1) + \eta(x) + 1 & \text{if } v = \mathrm{cen}(z) \text{ and } \ell(z) \text{ is even,} \\ \frac{3}{2}(\ell(x, z) - 1) + \eta(x) + \frac{3}{2} & \text{otherwise.} \end{cases}$$

Where $\eta(x) = 1$ if $x \in \Theta$ and $\eta(x) = 1/2$ if $x \in \Theta^s$.

Lemma 3.1.3. *Let $v_0, v_1 \in \mathrm{V}(\mathrm{Pgn}_{x,y})$ be vertices that are either identical or adjacent, such that v_0 is adjacent to $\mathrm{cen}(x)$ and v_1 is adjacent to $\mathrm{cen}(y)$. Let $z_0, z_1 \in [x, y]$ be the elements such that $z_0 \in \mathrm{Sgm}(x, v_0)$, $v_0 \in z_0$, $z_1 \in \mathrm{Sgm}(v_1, y)$, and $v_1 \in z_1$. We have*

$$\mathrm{dist}(v_0, v_1) = \begin{cases} \frac{\sqrt{3}}{2}(\ell(z_0, z_1) - 1) & \text{if } v_0 \neq \mathrm{cen}(z_0), \\ 0 & \text{if } v_0 = \mathrm{cen}(z_0). \end{cases}$$

Proof. If $v_0 = \mathrm{cen}(z_0)$, then $z_0 = z_1$ so $v_0 = v_1$ and $d(v_0, v_1) = 0$.

Let us suppose that $v_0 \neq \mathrm{cen}(z_0)$. By Remark 2.2.17 and since $z_0 \in \mathrm{Sgm}(x, v_0)$, it follows that v_0 is one of the top vertices of the down-oriented z_0. By Lemma 1.2.4, this can only happen if $\ell(z_0)$ is odd, that is, $z_0 = \theta(m, n)$ for some $m, n \geq 0$ or $z_0 \in X^{\mathrm{odd}}$. In the first case, we have $z_1 = \theta^s(m, n)$ which leads to $v_0 = v_1$ and $d(v_0, v_1) = 0 = \ell(z_0, z_1) - 1$.

For the remainder of the proof, we address the remaining case by assuming that $z_0 \in X^{\mathrm{odd}}$. Note that $v_0 \in \partial F_{\mathrm{id}}$. Since v_0 and v_1 are adjacent, we have $\mathrm{Sgm}(v_0, v_1) \subset \partial F_{\mathrm{id}}$. This implies that $\mathscr{A}_{z_0} \cup \mathscr{A}_{z_1}$ is contained in a strip $\{v \in E : -1 < (x, \alpha_i) < 0\}$ for some $i \in \{1, 2\}$. Without loss of generality, we can assume $i = 2$. So $z_0 = \mathbf{x}_{2k-1}$ for some $k \geq 3$. There is $j \geq 3$ such that either $z_1 = \mathbf{x}_{2j}$ or $z_1 = \mathbf{x}_{2j-1}$.

- $z_1 = \mathbf{x}_{2j-1}$. From Lemma 1.2.14, we have $z_1 - z_0 = (j - k)\varpi_1$. Let c_1 be the top vertex of z_1 that belongs to ∂F_{id}, then $c_1 - v_0 = (j - k)\varpi_1$. Since z_0 is down-oriented, $v_1 \in \partial F_{\mathrm{id}}$ is the mid-point of the left edge of z_1. More precisely, we have that $v_1 = c_1 - (1/2)\varpi_1$, hence

$$v_1 - v_0 = (j - k - 1/2)\varpi_1 = \frac{\ell(z_0, z_1) - 1}{2}\varpi_1.$$

  Since $\mathrm{dist}(0, \varpi_1) = \sqrt{3}$, the lemma follows.

- $z_1 = \mathbf{x}_{2j}$. In this case, z_1 is up-oriented, so $v_1 \in \partial F_{\mathrm{id}}$ is the bottom-left vertex of z_1. Note that v_1 is also the top-left vertex of $\mathbf{x}_{2j-1}$. From the same computation as in the previous case, we have that

$$v_1 - v_0 = (j - k)\varpi_1 = \frac{\ell(z_0, z_1) - 1}{2}\varpi_1,$$

and the lemma follows. ∎

The following lemma is a straightforward consequence of Proposition 2.2.7.

Lemma 3.1.4. *Let $z_0, z_1 \in [x, y]$ and $v_0, v_1 \in \mathrm{V}(\mathrm{Pgn}_{x,y}) \setminus \{\mathrm{cen}(x), \mathrm{cen}(y)\}$ be vertices that are either the same or adjacent, such that v_0 is adjacent to x and v_1 is adjacent to y. If $v_i \in z_i$ for $i \in \{0, 1\}$, $z_0 \in \mathrm{Sgm}(v_0, x)$, and $z_1 \in \mathrm{Sgm}(y, v_1)$. Then $z_0 \leq z_1$.*

Definition 3.1.5. Let $z \in W$. We define $C(z)$ as the singleton containing the element $\mathrm{cen}(\mathcal{A}_z), \mathrm{V}(z)$ as the three-element set of vertices of $\overline{\mathcal{A}_z}$, and $\mathrm{ME}(z)$ as the three-element set of midpoints of the edges of $\overline{\mathcal{A}_z}$.

Proposition 3.1.6. *We will proceed using the assumptions of Lemma 3.1.4.*

- *If $\ell(z_0, z_1) = 0$, then $v_0 \in C(z_0)$ and $v_1 \in C(z_1)$.*
- *If $\ell(z_0, z_1) = 1$, then $v_0 \in \mathrm{V}(z_0)$, $v_1 \in \mathrm{V}(z_1)$ and $v_0 = v_1$ is a vertex of the segment $\overline{\mathcal{A}_{z_0}} \cap \overline{\mathcal{A}_{z_1}}$.*
- *If $\ell(z_0, z_1) > 1$, then v_0 and v_1 are different points of ∂F_{id}. Furthermore:*
  - *If $\ell(z_0, z_1)$ is even, $v_0 \in \mathrm{V}(z_0)$ and $v_1 \in \mathrm{V}(z_1)$.*
  - *If $\ell(z_0, z_1)$ is odd, $v_0 \in \mathrm{V}(z_0)$ and $v_1 \in \mathrm{ME}(z_1)$.*

Proof. Recall that z_0 (resp. z_1) is the unique element in $[x, y]$, such that $z_0 \in \mathrm{Sgm}(v_0, x)$ and $v_0 \in z_0$ (resp. $z_1 \in \mathrm{Sgm}(y, v_1)$ and $v_1 \in z_1$).

- $\ell(z_0, z_1) = 0$. This is equivalent to the equality $z_0 = z_1$. By remark 2.2.17, $v_0 \in \{C(z_0), \mathrm{V}(z_0)\}$, but if $v_0 \in \mathrm{V}(z_0)$ then $z_1 \neq z_0$, which is a contradiction, so $v_0 \in C(z_0)$. This implies that $v_0 = \mathrm{cen}(z_0) = v_1 = \mathrm{cen}(z_1)$, thus proving this bullet.

- $\ell(z_0, z_1) = 1$. Once again, by remark 2.2.17, we have $v_0 \in \{C(z_0), \mathrm{V}(z_0)\}$. In either case, applying the formulas from Lemma 3.1.3, we find $d(v_0, v_1) = 0$, which implies $v_0 = v_1$. If $v_0 = \mathrm{cen}(z_0)$, then $z_0 = z_1$, which is a contradiction, so $v_0 \in \mathrm{V}(z_0)$. A vertex of an alcove cannot be the center or the midpoint of an edge of another alcove, which implies that $v_1 \in \mathrm{V}(z_1)$.

- $\ell(z_0, z_1) > 1$. As in the last bullet, if $v_0 \in C(z_0)$ then $z_0 = z_1$ which is a contradiction. Then, by Lemma 3.1.3, we have $d(v_0, v_1) > 0$. By definition of $\mathrm{Pgn}_{x,y}$, this can only happen if both v_0, v_1 lie on ∂F_{id}. Once again, by Remark 2.2.17, we have that $v_0 \in \mathrm{V}(z_0)$, with z_0 down-oriented.

  Similarly, as $v_1 \in \overline{\mathcal{A}_{z_1}} \cap \partial F_{\mathrm{id}}$, we observe that

$$\overline{\mathcal{A}_{z_1}} \cap \partial F_{\mathrm{id}} = \begin{cases} \text{a vertex of } z_1 & \text{if } z_1 \text{ is up-oriented;} \\ \text{an edge of } z_1 & \text{if } z_1 \text{ is down-oriented.} \end{cases} \tag{3.3}$$

As z_0 is down-oriented, by Lemma 1.2.4, we have that $\ell(z_0)$ is odd. Additionally, by applying Lemma 1.2.4 again, we have that:

- If $\ell(z_0, z_1)$ is odd, then z_1 is up-oriented. By Equation (3.3), $v_1 \in \mathrm{V}(z_1)$.
- If $\ell(z_0, z_1)$ is even, then z_1 is down-oriented. Here v_1 is the mid-point of the edge $\overline{\mathcal{A}_{z_1}} \cap \partial F_{\mathrm{id}}$, or rephrasing, $v_1 \in \mathrm{ME}(z_1)$. □

We will rewrite the results of this section using a more convenient notation, which is compatible with Definition 2.3.15.

Notation 3.1.7. Let $z_i^{x,y}(y) \in W$ be the element as in Equation (3.1) such that $z_i^{x,y}(y) \in \mathrm{Sgm}(y, v_i^{x,y}(y))$ and

$$d(y, z_i^{x,y}(y)) = \max\{d(y, w) \mid w \in W, w \in \mathrm{Sgm}(y, v_i^{x,y}(y))\}.$$

Similarly, let $z_i^{x,y}(x) \in W$ be the element as in Equation (3.2) such that $z_i^{x,y}(x) \in \mathrm{Sgm}(v_i^{x,y}(x), x)$ and

$$d(z_i^{x,y}(x), x) = \max\{d(w, x) \mid w \in W, w \in \mathrm{Sgm}(v_i^{x,y}(x), x)\}.$$

Furthermore, if the interval $[x, y]$ is clear from the context, we remove the superscript $(-)^{x,y}$ from the notation, so for $1 \leq i \leq 2$, we write $z_i(x)$, $z_i(y)$, $v_i(x)$, and $v_i(y)$ instead.

If $\{i, j\} = \{1, 2\}$, note that it is possible that $z_j(y) = z_i(x)$ and/or $v_j(y) = v_j(x)$. In the latter case, the edge $\{v_j(y), v_i(x)\}$ is degenerate. We have the set

$$\mathrm{V}(\mathrm{Pgn}_{x,y}) \setminus \{\mathrm{cen}(x), \mathrm{cen}(y)\} = \{v_1(x), v_1(y), v_2(x), v_2(y)\},$$

consists of $2, 3$, or 4 elements; these are precisely the cases where $\mathrm{Pgn}_{x,y}$ is a parallelogram, a pentagon, or a hexagon.

Remark 3.1.8. Let $\{i, j\} = \{1, 2\}$. We have:

$$\begin{aligned} \mathrm{cen}(y) &= v_j(y) + d(y, v_j(y))\alpha_j/||\alpha_j|| \\ v_j(y) &= v_i(x) + d(v_j(y), v_i(x))\varpi_i/||\varpi_i|| \\ v_i(x) &= \mathrm{cen}(x) + d(v_i(x), x)\alpha_i/||\alpha_i|| \end{aligned}$$

The next two results are a summary of the Lemmas 3.1.1, 3.1.2, 3.1.3, and 3.1.4, in terms of Notation 3.1.7.

Corollary 3.1.9. *Let $\{i, j\} = \{1, 2\}$, then $z_i(x) \leq z_j(y)$.*

Proposition 3.1.10. *Let $\{i,j\}=\{1,2\}$, then we have:*

$$\operatorname{dist}(y,v_i(y))=\begin{cases}\frac{3}{2}(\ell(z_i(y),y)-1)+\epsilon(y)+\gamma(z_i(y)) & \text{if } v_i(y)=\operatorname{cen}(z_i(y)),\\ \frac{3}{2}(\ell(z_i(y),y)-1)+\epsilon(y)+\frac{3}{2} & \text{otherwise.}\end{cases}$$

$$\operatorname{dist}(v_j(y),v_i(x))=\begin{cases}\frac{\sqrt{3}}{2}(\ell(z_i(x),z_j(x))-1) & \text{if } v_i(x)\neq\operatorname{cen}(z_i(x)),\\ 0 & \text{otherwise.}\end{cases}$$

$$\operatorname{dist}(v_i(x),x)=\begin{cases}\frac{3}{2}(\ell(x,z_i(x))-1)+\eta(x)+\frac{3}{2}-\gamma(z_i(x)) & \text{if } v_i(x)=\operatorname{cen}(z_i(x)),\\ \frac{3}{2}(\ell(x,z_i(x))-1)+\eta(x)+\frac{3}{2} & \text{otherwise.}\end{cases}$$

Where

$$\epsilon(y)=\begin{cases}\frac{1}{2} & \text{if } y\in\Theta\\ 1 & \text{if } y\in\Theta^s\end{cases};\quad \eta(x)=\begin{cases}\frac{1}{2} & \text{if } x\in\Theta^s\\ 1 & \text{if } x\in\Theta\end{cases};\quad \gamma(z)=\begin{cases}\frac{1}{2} & \text{if } \ell(z) \text{ is even}\\ 1 & \text{if } \ell(z) \text{ is odd.}\end{cases}$$

Proposition 3.1.11. $\{i,j\}=\{1,2\}$.

- *If $\ell(z_i(x),z_j(y))=0$, then $v_i(x)\in C(z_i(x))$ and $v_j(y)\in C(z_j(y))$.*
- *If $\ell(z_i(x),z_j(y))=1$, then $v_i(x)\in \mathrm{V}(z_i(x))$, $v_j(y)\in\mathrm{V}(z_j(y))$, and $v_i(x)=v_j(y)$ is a vertex of the segment $\overline{\mathcal{A}_{z_i(x)}}\cap\overline{\mathcal{A}_{z_j(x)}}$.*
- *If $\ell(z_i(x),z_j(y))>1$, then $v_i(x)$ and $v_j(y)$ are different points of ∂F_{id}. Furthermore:*
  - *If $\ell(z_i(x),z_j(y))$ is even, $v_i(x)\in\mathrm{V}(z_i(x))$ and $v_j(y)\in\mathrm{V}(z_j(y))$.*
  - *If $\ell(z_i(x),z_j(y))$ is odd, $v_i(x)\in\mathrm{V}(z_i(x))$ and $v_j(y)\in\mathrm{ME}(z_j(y))$.*

3.1.3 Rigid intervals

The main goal of this section is to prove Proposition 3.1.19, which imposes constraints on the possible poset isomorphisms between Bruhat intervals under certain mild assumptions. Recall that $x,y\in D$ and $[x,y]$ is a non-degenerate interval.

Definition 3.1.12. The set of maximal length elements in $\mathcal{D}^l(x)\cap[x,y]$ is denoted by $\mathcal{D}^{l,\max}_{[x,y]}$. The set of minimal length elements in $\mathcal{D}^u(y)\cap[x,y]$ is denoted by $\mathcal{D}^{u,\min}_{[x,y]}$.

Definition 3.1.13. For $z\in\mathcal{D}^{l,\max}_{[x,y]}$, the subset of maximal length elements of $(\mathcal{D}^l(x)\cap[x,y])\setminus[x,z]$ is denoted $A_{x,z}$. For $z\in\mathcal{D}^{u,\min}_{[x,y]}$, the subset of minimal length elements of $(\mathcal{D}^u(y)\cap[x,y])\setminus[z,y]$ is denoted $B_{z,y}$.

Remark 3.1.14. Since $[x,y]$ is a non-degenerate interval, the sets $A_{x,z}$ and $B_{z,y}$ are nonempty. Indeed, since $[x,z]$ has at most two upper covers of x, it follows from Lemma 1.2.17 that $\mathcal{D}^l(x)\setminus[x,z]$ is nonempty. An analogous argument applies to $B_{z,y}$, using Lemma 1.2.16.

Lemma 3.1.15. *Let* $z_1 \in \mathcal{D}^{l,\max}_{[x,y]}$ *and* $z_2 \in A_{x,z_1}$*. If* $\ell(x, z_i) \geq 4$ *for* $i \in \{1, 2\}$*, then* $\{z_1(x), z_2(x)\} = \{z_1, z_2\}$*. In particular,*

$$\mathcal{D}^{l,\max}_{[x,y]} \subset \{z_1(x), z_2(x)\}.$$

Proof. Let u_1, u_2, u_3, u_6, and x_2 be the vertices of $\mathcal{S}t(x)$ as in Definition 2.2.1. By Proposition 2.2.23, $z_1, z_2 \in \mathcal{D}^l_{\text{sgm}}(x)$, because the elements $z \in \mathcal{D}^l_{\text{rest}}(x)$ satisfy $\ell(x, z) \leq 3$ (even $\ell(x, z) \leq 2$ in the case $x \in \Theta$). Thus,

$$z_1, z_2 \in \text{Sgm}(u_1, u_3) \cup \text{Sgm}(u_2, u_6). \tag{3.4}$$

Without loss of generality, assume that $z_1 \in \text{Sgm}(u_1, u_3)$. In this case, we have that $z_1 \in F_{\text{id}} \cup F_{s_1}$. Note that

$$\text{Sgm}(u_1, u_3) \setminus \mathcal{S}t^\circ(x) = \text{Sgm}(u_1, x) \uplus \text{Sgm}(x_2, u_3).$$

Since $z_1 \in [x, y]$, by Equation (2.11), we have that $z_1 \in \text{Pgn}_{x,y} \cup \text{Pgn}^{s_1}_{x,y}$. From Proposition 2.2.21, it is not hard to see that

$$\begin{aligned}\text{cPgn}_{x,y} \cap \text{Sgm}(u_1, u_3) &= \text{cSgm}(z_1(x), x)\\ \text{cPgn}^{s_1}_{x,y} \cap \text{Sgm}(u_1, u_3) &= \text{cSgm}(x_2, s_1 z_1(x)).\end{aligned}$$

This implies that $z_1 \in \text{Sgm}(z_1(x), x) \uplus \text{Sgm}(x_2, s_1 z_1(x))$.

Case A. $z_1 \in \text{Sgm}(x_2, s_1 z_1(x))$. Note that

$$s_1 z_1(x) \in F_{s_1} = \delta^2 s_0 D \cup \delta\{\mathbf{x}_n \mid n \geq 2\}.$$

By Corollary 2.1.4, we have that $s_2 s_0 s_2 \cdot s_1 z_1(x)$ is a vertex of $\mathcal{C}_{s_1 z_1(x)}$. Since $s_2 H_{\rho,-1} = H_{\alpha_1,-1}$, it follows that $s_2 s_0 s_2 = s_{\alpha_1,-1} = r_1$. By definition of $\mathcal{S}t(x)$ we have that $r_1(\text{Sgm}(x_2, u_3)) \subset \text{Sgm}(x, u_1)$, so $r_1 s_1 z_1(x) \in \text{Sgm}(x, u_1)$. As

$$r_1 s_1 z_1(x), s_1 z_1(x), \in \text{V}(\mathcal{C}_{s_1 z_1(x)}),$$

by convexity of $\mathcal{C}_{s_1 z_1(x)}$ we have that $\text{Sgm}(x_2, s_1 z_1(x)) \subset \mathcal{C}_{s_1 z_1(x)}$, so $z_1 \leq s_1 z_1(x) < z_1(x)$, the strict inequality holds because $z_1(x) \in F_{\text{id}}$.

Since $z_1(x) \in \mathcal{D}^l_{\text{sgm}}(x)$, Proposition 2.2.23 implies that $z_1(x) \in \mathcal{D}^l(x)$. This contradicts the maximality of $\ell(z_1)$ in $\mathcal{D}^l(x)$.

Case B. $z_1 \in \text{Sgm}(z_1(x), x)$. Suppose by contradiction that $z_1 \neq z_1(x)$. Clearly z_1 and $z_1(x)$ differ by a multiple of α_1. By Lemma 2.2.12, there exists a chain $z_1 = t_0 \lhd t_1 \lhd \ldots \lhd t_n = z_1(x)$ of elements in $\text{Sgm}(z_1(x), x) \subset F_{\text{id}}$, similar to the one described in the proof of Lemma 2.2.13. In particular $z_1 < z_1(x)$. Since $z_1(x) \in$

$\mathcal{D}^l_{\text{sgm}}(x)$, Proposition 2.2.23 implies $z_1(x) \in \mathcal{D}^l(x)$. This contradicts the maximality of z_1 in $\mathcal{D}^l(x)$, so $z_1 = z_1(x)$.

It remains to prove that $z_2 = z_2(x)$. Recall that by Equation (3.4) we have that $z_2 \in \text{Sgm}(u_1, u_3) \cup \text{Sgm}(u_2, u_6)$. Let us show that $z_2 \in \text{Sgm}(u_2, u_6)$. Suppose, by contradiction, that $z_2 \in \text{Sgm}(u_1, u_3)$. As in Case A, if $z_2 \in \text{Sgm}(x_2, s_1 z_1(x))$ we have that $z_2 \leq s_1 z_1(x) < z_1(x)$ which yields a contradiction. Otherwise, $z_2 \in \text{Sgm}(z_1(x), x)$ and by the same reasoning as before, there exists a chain $z_2 = t_0 \lhd t_1 \lhd \ldots \lhd t_n = z_1(x)$ of elements in $\text{Sgm}(z_1(x), x) \subset F_{\text{id}}$, so $z_2 \leq z_1(x)$, and since $z_1 = z_1(x)$, this contradicts the assumption that $z_2 \in A_{x,z_1}$. Therefore, we conclude that $z_2 \in \text{Sgm}(u_2, u_6)$. The proof of the fact that $z_2 = z_2(x)$ is analogous to that of $z_1 = z_1(x)$, relying on the maximality of $\ell(z_2)$, so we will omit the details. □

The next lemma is the analog of Lemma 3.1.15 for the elements $z_1(y)$ and $z_2(y)$. Since the proof follows similarly to that of Lemma 3.1.15, we will omit certain details that are not essential.

Lemma 3.1.16. *Let $z_1 \in \mathcal{D}^{u,\min}_{[x,y]}$ and $z_2 \in B_{x,z_2}$. If $\ell(x, z_i) \geq 4$ for $i \in \{1, 2\}$, then $\{z_1(y), z_2(y)\} = \{z_1, z_2\}$. In particular,*

$$\mathcal{D}^{u,\min}_{[x,y]} \subset \{z_1(y), z_2(y)\}.$$

Proof. By Proposition 2.1.9, $z_1, z_2 \in \mathcal{D}^u_{\text{sgm}}(y)$ because by Lemma 2.1.8 the elements $z \in \mathcal{D}^u_{\text{rest}}(y)$ satisfy $\ell(z, y) \leq 3$. Thus, $z_1, z_2 \in \text{Sgm}(y, s_1 y) \cup \text{Sgm}(y, s_2 y)$. This implies that $z_1, z_2 \in F_{\text{id}} \cup F_{s_1} \cup F_{s_2}$. Without loss of generality, assume that $z_1 \in \text{Sgm}(y, s_1 y)$. Recall from Section 2.2.1 that $r_1 = s_{\alpha_1,-1}$ and $r_2 = s_{\alpha_2,-1}$. Since $z_1 \in [x, y]$, by Equation (2.11) and as $z_1 \in F_{\text{id}} \cup F_{s_1}$, we have that $z_1 \in \text{Pgn}_{x,y} \cup \text{Pgn}^{s_1}_{x,y}$. From Proposition 2.2.21, it is not hard to see that

$$\begin{aligned} &\text{cPgn}_{x,y} \cap \text{Sgm}(y, s_i y) \cap F_{\text{id}} = \text{cSgm}(y, z_i(y)), \\ &\text{cPgn}^{s_i}_{x,y} \cap \text{Sgm}(y, s_i y) \cap F_{s_i} = \text{cSgm}(r_i z_i(y), s_i y), \text{ for } i \in \{1, 2\}. \end{aligned} \tag{3.5}$$

As in Case A of Lemma 3.1.15, we conclude that $z_1 \in F_{\text{id}}$. Furthermore, since z_1 and $z_1(y)$ differ by a multiple of α_1, and both are in F_{id}, we apply the reasoning from Case B of Lemma 3.1.15, to conclude that $z_1 = z_1(y)$.

It remains to prove that $z_2 = z_2(x)$. Before proceeding, we will show that $z_2 \in \text{Sgm}(y, s_2 y) \cap F_{\text{id}}$. First, suppose that $z_2 \notin F_{\text{id}}$. Then

- $z_2 \in \delta s_0 D \cup (\sigma\delta)\{\mathbf{x}_n \mid n \geq 2\}$, or
- $z_2 \in \delta^2 s_0 D \cup \delta\{\mathbf{x}_n \mid n \geq 2\}$.

In the first case, by Corollary 2.1.4, we have that $r_2 z_2 \in \text{V}(\mathcal{C}_{z_2})$. In the second case, by Corollary 2.1.4, $r_1 z_2 \in \text{V}(\mathcal{C}_{z_2})$. In both cases, since $\text{Sgm}(r_i z_i(y), y) \subset$

$r_i \operatorname{Sgm}(r_i z_i(y), s_i y)$, we obtain that $z_i(y) < z_2$. Since $z_i(y) \in \mathcal{D}^u_{\operatorname{Sgm}}(y)$, Proposition 2.2.23 implies that $z_i(y) \in \mathcal{D}^u_{[x,y]}$. This contradicts the minimality of z_2. Hence $z_2 \in F_{\mathrm{id}}$.

Now, suppose by contradiction that $z_2 \in \operatorname{Sgm}(y, s_1 y)$. Clearly, $\ell(z_2) \leq \ell(z_1)$. By the same reasoning as Case B of Lemma 3.1.15, we can prove that $z_2 \leq z_1$. This contradicts the assumption that $z_2 \in B_{x,z_1}$. Therefore, we conclude that $z_2 \in \operatorname{Sgm}(y, s_2 y) \cap F_{\mathrm{id}}$. By Equation (3.5) we have that $z_2 \in \operatorname{Sgm}(y, z_2(y))$. Since z_2 and $z_2(y)$ differ by α_2 and both are in F_{id}, we can apply the reasoning from Case B of Lemma 3.1.15 to conclude that $z_2 = z_2(y)$. □

Definition 3.1.17. We say that $[x,y]$ (x, y dominant, $[x,y]$ non-degenerate and nonempty) is *rigid* if $\ell(x, z_i(x)) \geq 4$ and $\ell(z_i(y), y) \geq 4$ for $i \in \{1,2\}$.

The next lemma is a direct consequence of Lemma 2.2.13.

Lemma 3.1.18. *If $\ell(z_i(y), y) \geq 4$ then $z_i(y) = y - t\alpha_i$ for $t \geq 2$. Similarly, $\ell(x, z_i(y)) \geq 4$ implies $z_i(x) = x + t\alpha_i$ for $t \geq 2$.*

Proposition 3.1.19. *Let $\phi\colon [x,y] \to [x',y']$ be an isomorphism of posets.*

- *If $\ell(x, z_i(x)) \geq 4$ for $i \in \{1,2\}$, then $\phi(\{z_1(x), z_2(x)\}) = \{z_1(x'), z_2(x')\}$.*
- *If $\ell(z_i(y), y) \geq 4$ for $i \in \{1,2\}$, then $\phi(\{z_1(y), z_2(y)\}) = \{z_1(y'), z_2(y')\}$.*

Proof. By Lemma 1.1.5, ϕ induces bijections

$$\mathcal{D}^l(x) \cap [x,y] \longleftrightarrow \mathcal{D}^l(x') \cap [x',y']$$
$$\mathcal{D}^u(y) \cap [x,y] \longleftrightarrow \mathcal{D}^u(y') \cap [x',y'].$$

The first bijection, plus Proposition 1.1.3, implies that

$$\phi(\mathcal{D}^{l,\max}_{[x,y]}) = \mathcal{D}^{l,\max}_{[x',y']}. \tag{3.6}$$

By Lemma 3.1.15, there are distinct indexes $i, j \in \{1,2\}$ such that $z_i(x) \in \mathcal{D}^{l,\max}_{[x,y]}$ and $z_j(x) \in A_{x,z_i(x)}$. By Equation (3.6), it follows that

$$\phi(z_i(x)) \in \mathcal{D}^{l,\max}_{[x',y']} \text{ and } \phi(z_j(x)) \in A_{x',\phi(z_i(x))}.$$

By Lemma 3.1.15, we conclude that $\phi(\{z_1(x), z_2(x)\}) = \{z_1(x'), z_2(x')\}$.

By a similar reasoning, but using Lemma 3.1.16 instead of Lemma 3.1.15, we conclude that $\phi(\{z_1(y), z_2(y)\}) = \{z_1(y'), z_2(y')\}$. □

Remark 3.1.20. If the hypothesis of Proposition 3.1.19 does not hold, neither does its conclusion. This is the primary reason for including the rigidity hypothesis in the main theorem of this thesis.

3.2 Congruence of polygons and proof of the main theorem

In this section, we consider $x, x', y, y' \in D$ such that the intervals $[x, y]$ and $[x', y']$ are rigid.

Proposition 3.2.1. *If $[x, y]$ and $[x', y']$ are isomorphic as posets, then there is a weight μ, such that $\mathrm{Pgn}_{x',y'} + \mu$ is equal to $\mathrm{Pgn}_{x,y}$ or to $\sigma \mathrm{Pgn}_{x,y}$. In particular, $\mathrm{Pgn}_{x,y}$ and $\mathrm{Pgn}_{x',y'}$ are congruent.*

Before embarking on the proof, we need some technical lemmas.

Lemma 3.2.2. *If $\phi\colon [x, y] \to [x', y']$ is a poset isomorphism, then $\phi(x + \rho) = x' + \rho$.*

Proof. Let $Z(x)$ be the set of elements $z \in [x, y]$ such that

1. $\ell(x, z) = 4$.
2. $U(x) \subset [x, z]$.
3. $B_2(x, z) := |\{[a, b] \subset [x, z] \mid [a, b] \text{ is a 2-crown}\}| = 2$.

We define $Z(x') \subseteq [x', y']$ in a similar manner. Let $Z_{12}(x)$ be the set of elements that satisfy conditions (1) and (2). In the proof of Lemma 1.2.17, the elements of $U(x)$ are given explicitly, depending on whether x is an element of the form $\theta(m, n)$ or $\theta(m, n)^s$, where m, n are positive integers. The other cases ($m = 0, n > 0$ and $m = 0, n = 0$) are not written there but are easy to calculate. From this small set of elements, one can check which ones have the correct length. Thus we have:

- If $x = \theta(m, n)$,
$$Z_{12}(x) = \{\theta(m + 1, n + 1), \delta(s_0\theta^s(m, n + 1)), \delta^2((s_0\theta^s(m + 1, n))\}.$$
- If $x = \theta^s(m, n)$,
$$Z_{12}(x) = \{\theta^s(m + 1, n + 1), \delta(s_0\theta(m + 1, n + 1)), \delta^2(s_0\theta(m + 1, n + 1))\}.$$

In [LP23], explicit formulas are given for the Kazhdan–Lusztig polynomials $P_{w,w'}(q)$, for any pair of elements $w, w' \in W$. We can apply this to obtain the following formulas:

$$\begin{aligned} P_{\theta^s(m,n),\theta^s(m+1,n+1)} &= 1 + 2q, \\ P_{\theta^s(m,n),\delta(s_0\theta(m+1,n+1))} &= 1 + q, \\ P_{\theta^s(m,n),\delta^2(s_0\theta(m+1,n+1))} &= 1 + q, \end{aligned}$$

and for the specific cases

$$\begin{aligned} &\text{If } m>0, n>0: && \text{If } m>0, n=0: \\ P_{\theta(m,n),\theta(m+1,n+1)} &= 1+q, & P_{\theta(m,0),\theta(m+1,1)} &= 1+q; \\ P_{\theta(m,n),\delta(s_0\theta^s(m,n+1))} &= 1+2q, & P_{\theta(m,n),\delta(s_0\theta^s(m,1))} &= 1+2q; \\ P_{\theta(m,n),\delta^2(s_0\theta^s(m+1,n))} &= 1+2q, & P_{\theta(m,0),\delta^2(s_0\theta^s(m+1,0))} &= 1+q. \end{aligned}$$

Similarly, the remaining cases ($m = 0, n = 0$ or $m = 0, n > 0$) can also be addressed, but we will omit the formulas. By [BB05, Exercise 5.8] we have

$$P_{w,w'}(q) = 1 + \left(|L(w)| + \frac{B_2(w,w')}{2} - 4\right) q \tag{3.7}$$

From Equation (3.7), Lemma 1.2.16, and the explicit Kazhdan–Lusztig polynomials calculated above, we obtain

$$\begin{aligned} Z(x) &= \begin{cases} \{\theta(m+1,n+1)\} & \text{if } x = \theta(m,n), \\ \{\theta^s(m+1,n+1)\} & \text{if } x = \theta^s(m,n). \end{cases} \\ &= \{x+\rho\}. \end{aligned}$$

By the same argument, $Z(x') = \{x' + \rho\}$. Since the defining properties of $Z(x)$ are preserved by any poset isomorphism, we have $\phi(Z(x)) \subset Z(x')$, so $\phi(x + \rho) = x' + \rho$. □

For the rest of this section, we use the following shortcuts for $1 \leq i \leq 2$:

$$\begin{aligned} z_i(x) &:= z_i^{x,y}(x) & v_i(x) &:= v_i^{x,y}(x) \\ z_i(y) &:= z_i^{x,y}(y) & v_i(y) &:= v_i^{x,y}(y) \end{aligned}$$

The same notations apply if we replace x, y with x', y'.

Lemma 3.2.3. *Suppose that $z_{i_0}(x) \leq z_{i_0}(y)$, for some $i_0 \in \{1, 2\}$. Then $\mathrm{Pgn}_{x,y}$ is a hexagon.*

Proof. Without loss of generality, we assume $z_1(x) \leq z_1(y)$. We need to prove that $v_2(y) \neq v_1(x)$ and $v_1(y) \neq v_2(x)$.

Suppose that $v_2(y) = v_1(x)$. According to Remark 2.2.17, $v_1(x)$ is either the center of an alcove or a vertex of an alcove. In the first case, we have $v_1(x) = \mathrm{cen}(z_1(x))$, while in the second case, it is straightforward to see that $v_1(x) = \frac{1}{3}\alpha_1 + \mathrm{cen}(z_1(x))$. In any case, $(\varpi_1, v_2(y) - z_1(x)) \leq 1/3$. As $v_2(y) \in y + \mathbb{R}\alpha_2$, we have $(\varpi_1, y) = (\varpi_1, v_2(y))$. By Lemma 3.1.18, there is $t \geq 2$ such that $z_1(y) = y - t\alpha_1$. We have,

$$(\varpi_1, z_1(y) - z_1(x)) = (\varpi_1, v_2(y) - t\alpha_1 - z_1(x)) \leq \frac{1}{3} - 2 < 0.$$

By Corollary 2.2.8 we get $z_1(x) \not\leq z_1(y)$, a contradiction.

Suppose $v_1(y) = v_2(x)$. The reasoning here is similar, so we omit the details. We have either $v_2(x) = \mathrm{cen}(z_1(y)) - \frac{1}{3}\alpha_1$ or $v_2(x) = \mathrm{cen}(z_1(y))$. In either case, $(\varpi_1, z_1(y) - v_2(x)) \leq \frac{1}{3}$ and $(\varpi_1, v_2(x)) = (\varpi_1, x)$. By Lemma 3.1.18, as before, we deduce $(\varpi_1, x - z_1(x)) \leq -2$. Combining these inequalities yields $(\varpi_1, z_1(y) - z_1(x)) \leq -\frac{5}{3} < 0$, which, once again, leads to a contradiction. □

In the following three lemmas, we explore the subintervals $[x+\rho, y] \subset [x, y]$ and $[x'+\rho, y'] \subset [x', y']$. For convenience, we replace the superscripts $(-)^{x+\rho,y}$ and $(-)^{x'+\rho,y'}$ by $(-)^{\rho}$. That is, for $1 \leq i \leq 2$:

$$z_i^{\rho}(x+\rho) := z_i^{x+\rho,y}(x+\rho) \qquad v_i^{\rho}(x+\rho) := v_i^{x+\rho,y}(x+\rho)$$
$$z_i^{\rho}(y) := z_i^{x+\rho,y}(y) \qquad v_i^{\rho}(y) := v_i^{x+\rho,y}(y).$$

The same notations apply if we replace x, y with x', y'.

Lemma 3.2.4. *Let $\{i, j\} = \{1, 2\}$, then $z_i(x) < z_i^{\rho}(x+\rho)$ and $z_i(x) \not\leq z_j^{\rho}(x+\rho)$.*

Proof. It is not hard to deduce the first part using Lemma 2.2.7.

We now prove the second part. By rigidity we have that $z_i(x) = x + t\alpha_i$ for $t \geq 2$ leads to the inequality $(\varpi_i, z_i(x) - x) \geq 2$. Note that $(\varpi_i, x+\rho) = (\varpi_i, z_j^{\rho}(x+\rho))$ and $(\varpi_i, x+\rho) = 1 + (\varpi_i, x)$. So

$$(\varpi_i, z_i(x)) \geq (\varpi_i, x) + 2 = (\varpi_i, x+\rho) + 1 > (\varpi_i, z_j^{\rho}(x+\rho)).$$

By Corollary 2.2.8, we have $z_i(x) \not\leq z_j^{\rho}(x+\rho)$. □

Lemma 3.2.5. *Suppose that $z_{i_0}(x) \leq z_{i_0}(y)$, for some $i_0 \in \{1, 2\}$. Then $z_i^{\rho}(y) = z_i(y)$ for $1 \leq i \leq 2$.*

Proof. Without loss of generality, we assume $z_1(x) \leq z_1(y)$. By Lemma 3.2.3, $\mathrm{Pgn}_{x,y}$ is a hexagon. Since $\mathrm{Pgn}_{x+\rho,y} \subset \mathrm{Pgn}_{x,y}$, we have $z_i^{\rho}(y) \in \mathrm{Sgm}(y, z_i(y))$. By Lemma 2.2.7, we have $z_i^{\rho}(y) \in [z_i(y), y]$ for $1 \leq i \leq 2$.

Part A. We prove that $z_1^{\rho}(y) = z_1(y)$. Suppose by contradiction that $z_1^{\rho}(y) \neq z_1(y)$. Then $z_1(y) < z_1^{\rho}(y)$, so $v_1^{\rho}(y) = v_2^{\rho}(x+\rho) \notin \partial F_{\mathrm{id}}$. By Remark 2.2.18, we have that $v_2^{\rho}(x+\rho)$ is either the center of $z_2^{\rho}(x+\rho)$ or a vertex of $z_2^{\rho}(x+\rho)$.

1. In the first case, we have that $z_1^{\rho}(y) = z_2^{\rho}(x+\rho)$. This implies $z_1(x) \leq z_1(y) < z_2^{\rho}(x+\rho)$, which is not possible by Lemma 3.2.4.

2. The remaining case corresponds to the second case of Proposition 3.1.6, so we have:

- $\ell(z_2^\rho(x+\rho), z_1^\rho(y)) = 1$ and $z_2^\rho(x+\rho)$ and $z_1^\rho(y)$ share an horizontal edge.
- $z_2^\rho(x+\rho) + \frac{1}{3}\alpha_2 = v_1^\rho(y)$, so $(\varpi_1, z_2^\rho(x+\rho) - v_1^\rho(y)) = 0$.

We also have $v_1^\rho(y) - t\alpha_1 = \mathrm{cen}(z_1(y))$, where $t > 0$. Hence $(\varpi_1, z_1(y)) \leq (\varpi_1, v_1^\rho(y)) = (\varpi_1, z_2^\rho(x+\rho))$ by the second bullet above. Since $\mathrm{Pgn}_{x,y}$ is a hexagon, there is $k \geq 5$ such that $z_1(x) = \mathbf{x}_k$. By Corollary 2.2.8, $0 \leq (\varpi_1, z_1(y) - \mathbf{x}_k) \leq (\varpi_1, z_2^\rho(x+\rho) - \mathbf{x}_k)$.

It is not hard to check that if $z \in F_{\mathrm{id}}$ and $0 \leq (\varpi_1, z - \mathbf{x}_k)$ then $0 \leq (\varpi_2, z - \mathbf{x}_k)$. Thus we have that $0 \leq (\varpi_2, z_2^\rho(x+\rho) - \mathbf{x}_k)$ and by Corollary 2.2.8 we have that $z_1(x) \in \mathcal{C}_{z_2^\rho(x+\rho)}$. By Corollary 2.1.5, $z_1(x) \leq z_2^\rho(x+\rho)$ contradicting again Lemma 3.2.4.

Part B. We prove that $z_2^\rho(y) = z_2(y)$. We base the proof on the following geometric observation: Let $u_1(x)$ and $u_1(x+\rho)$ be the exterior vertices of $\mathcal{S}t(x)$ and $\mathcal{S}t(x+\rho)$, which are on the left boundary of F_{id}. It is easy to see that $u_1(x) + 3\varpi_1 = u_1(x+\rho)$. Since $[x, y]$ is a hexagon interval, $u_1(x) = v_1(x)$, so

$$v_1(x) + 3\varpi_1 = u_1(x+\rho)$$

Claim 3.2.6. $\mathrm{dist}(v_2(y), v_1(x)) \geq \frac{5}{2}\sqrt{3}$.

Proof. Since $\mathrm{Pgn}_{x,y}$ is a hexagon, there is $t > 0$ such that $v_2(y) - v_1(x) = t\varpi_1$. From $v_2(y) \in y + \mathbb{R}\alpha_2$, we get $(\varpi_1, y) = (\varpi_1, v_2(y))$. By Lemma 3.1.18, we have $(\varpi_1, z_1(y)) \leq (\varpi_1, y) - 2$. At the beginning of the proof, we assumed $z_1(x) \leq z_1(y)$, so by Corollary 2.2.8, $(\varpi_1, z_1(y) - z_1(x)) \geq 0$. Then

$$(\varpi_1, z_1(x)) \leq (\varpi_1, v_2(y)) - 2 = (\varpi_1, v_1(x)) + t(\varpi_1, \varpi_1) - 2.$$

Since $v_1(x) = z_1(x) + \frac{1}{3}\alpha_1$, we have $(\varpi_1, z_1(x) - v_1(x)) = -1/3$. Then

$$\frac{5}{3} = 2 - \frac{1}{3} \leq (\varpi_1, \varpi_1)t = \frac{2}{3}t,$$

or equivalently, $t \geq 5/2$. □

We have two scenarios:

- $\mathrm{dist}(v_2(y), v_1(x)) = \frac{5}{2}\sqrt{3}$. This implies that $v_1(x) + \frac{5}{2}\varpi_1 = v_2(y)$. In this case $v_2(y)$ must be the midpoint of the left edge of $z_2(y)$ which must be down-oriented, and since $v_1(x) + 3\varpi_1 = u_1(x+\rho)$, we have $u_1(x+\rho) = v_2(y) + \frac{1}{2}\varpi_1$, so $u_1(x+\rho)$ is the top-left vertex of $z_2(y)$. This implies that

$$\mathrm{Sgm}(u_1(x+\rho), x+\rho) \cap \mathrm{Sgm}(y, v_2(y)) = \{\mathrm{cen}(z_2(y))\}.$$

In particular, $\mathrm{cen}(z_2(y)) \in \mathrm{V}(\mathrm{Pgn}_{x+\rho,y})$, so $v_1^\rho(x+\rho) = v_2^\rho(y) = \mathrm{cen}(z_2(y))$. The second equality implies that $z_2^\rho(y) = z_2(y)$.

- $\text{dist}(v_2(y), v_1(x)) > \frac{5}{2}\sqrt{3}$. As $v_2(y) - v_1(x) \in \frac{1}{2}\mathbb{Z}\varpi_1$, we have that $v_1(x) + a\varpi_1 = v_2(y)$, where $a \geq 3$. Furthermore, since $v_1(x) + 3\varpi_1 = u_1(x+\rho)$, it follows that $u_1(x+\rho) = v_1^\rho(x+\rho) \in \partial F_{\text{id}}$. Therefore $v_2^\rho(y) \in \partial F_{\text{id}}$, so $v_2(y) = v_2^\rho(y)$, and we conclude that $z_2(y) = z_2^\rho(y)$. □

Lemma 3.2.7. *Assume $z_{i_0}(x) \leq z_{i_0}(y)$ for some $i_0 \in \{1,2\}$. If $1 \leq i \leq 2$, then we have that $v_i(x) + 3\varpi_i = u_i(x+\rho)$ and $u_i(x+\rho)$ is a top vertex of $z_i(x+\rho)$. In particular we have that $z_i(x) + 3\varpi_i = z_i^\rho(x+\rho)$.*

Proof. By Lemma 3.2.3, we know that $\text{Pgn}_{x,y}$ is an hexagon. Consequently, we have that $v_i(x) = u_i(x)$. The first equation follows from the fact that $u_i(x) + 3\varpi_i = u_i(x) + \rho + \alpha_i = u_i(x+\rho)$. We now proceed to prove that $u_i(x+\rho)$ is a top vertex of $z_i^\rho(x+\rho)$. Without loss of generality, suppose that $i_0 = 1$. By Claim 3.2.6, we have that $\text{dist}(v_2(y), v_1(x)) \geq \frac{5}{2}\sqrt{3}$. By Lemma 3.1.18, we have that

$$z_i(x) = x + t\alpha_i \tag{3.8}$$

for some $t \geq 2$. By Equation (3.8) and since $z_1(x) \leq z_1(y)$ and $v_1(y) \in v_2(x) + \mathbb{R}_{\geq 0}\varpi_2$, it is not hard to check that $\text{dist}(v_1(y), v_2(x)) \geq \frac{5}{2}\sqrt{3}$. By following the argument in the bullets of Part B in the proof of Lemma 3.2.5, we have that $u_i(x+\rho)$ is a top vertex of $z_i^\rho(x+\rho)$. The equation $z_i(x) + 3\varpi_i = z_i^\rho(x+\rho)$, follows from the fact that $u_i(x+\rho)$ is a top vertex of $z_i(x+\rho)$. The reason for this is that since $u_i(x)$ is a top vertex of $z_i(x)$, and $u_i(x) + 3\varpi_i = u_i(x+\rho)$, we deduce that $z_i(x) + 3\varpi_i = z_i^\rho(x+\rho)$. □

Lemma 3.2.8. *Suppose that $z_{i_0}(x) \leq z_{i_0}(y)$, for some $i_0 \in \{1,2\}$. Then $[x+\rho, y]$ is rigid.*

Proof. Let $1 \leq i \neq j \leq 2$. By Lemma 3.2.5, we have that $z_i^\rho(y) = z_i(y)$. As $[x,y]$ is rigid, it suffices to show that $\ell(x+\rho, z_i^\rho(x+\rho)) \geq 4$. By Proposition 3.1.10, as $\gamma(z_i(x+\rho)) > 0$, this inequality is implied by the following inequality.

$$\text{dist}(v_i^\rho(x+\rho), x+\rho) \geq 6 + \eta(x+\rho)$$

Since $[x,y]$ is rigid, it follows that $\ell(x, z_i(x)) \geq 4$. By Lemma 3.2.3, we have that $\text{Pgn}_{x,y}$ is a hexagon. Therefore, $v_i(x) \in H_{\alpha_j,-1}$, which means that $v_i(x)$ is a top vertex of $z_i(x)$, in particular $v_i(x) \neq \text{cen}(z_i(x))$. We obtain the following inequality by applying Proposition 3.1.10.

$$\text{dist}(v_i(x), x) \geq 6 + \eta(x). \tag{3.9}$$

Recall that $u_i(*)$ be the exterior vertex of $\mathcal{S}t(*)$ in $H_{\alpha_j,-1}$, where $* \in \{x, x+\rho\}$. Since $v_i(x) \in H_{\alpha_j,-1}$, it follows that $v_i(x) = u_i(x)$. Consequently, $u_i(x) + 3\varpi_i$ is a top vertex of $z_i(x) + 3\varpi_i$. By Lemma 3.2.7, we have that $u_i(x) + 3\varpi_i = u_i(x+\rho)$ and $z_i(x) + 3\varpi_i = z_i^\rho(x+\rho)$. Thus, $u_i(x+\rho)$ is a top vertex of $z_i^\rho(x+\rho)$, which implies that

$u_i(x+\rho) = z_i^\rho(x+\rho)+\frac{1}{3}\alpha_i$. Since $\rho+\alpha_i = 3\varpi_i$, it follows that $v_i(x)+\rho+\alpha_i = u_i(x+\rho)$ which implies that $v_i(x) + \rho + \frac{2}{3}\alpha_i = z_i^\rho(x + \rho)$. Therefore, we conclude that

$$\mathrm{Sgm}(v_i(x) + \rho, x + \rho) \subset \mathrm{Sgm}(z_i^\rho(x + \rho), x + \rho) \subset \mathrm{Sgm}(u_i(x + \rho), x + \rho).$$

One has that $\mathrm{dist}(z_i^\rho(x + \rho), v_i(x) + \rho) = \mathrm{dist}(0, \frac{2}{3}\alpha_i) = 2$. By Lemma 3.2.7, we can deduce that $v_i^\rho(x + \rho)$ is either equal to $z_i^\rho(x + \rho)$ or to $u_i(x + \rho)$. Therefore, the result follows from the following inequalities:

$$\begin{aligned}\mathrm{dist}(v_i^\rho(x + \rho), x + \rho) &= \mathrm{dist}(v_i^\rho(x + \rho), v_i(x) + \rho) + \mathrm{dist}(v_i(x) + \rho, x + \rho)\\ &\geq 2 + \mathrm{dist}(v_i(x), x)\\ &\geq 2 + (6 + \eta(x))\\ &= 8 + \eta(x + \rho),\end{aligned}$$

Where the inequality in the third line follows from Equation (3.9) and the final equality is a consequence of the fact that both x and $(x + \rho)$ belongs to the same $X\Theta$-partition. □

Lemma 3.2.9. *Let $\{i, j\} = \{1, 2\}$. Let $\phi\colon [x, y] \to [x', y']$ be a poset isomorphism. We have:*

• $[x, y]$ *is rigid* • $\phi(z_i(x)) = z_i(x')$ • $\phi(z_i(y)) = z_j(y')$ • $z_i(x) \leq z_i(y)$	$\Longrightarrow$	• $[x + \rho, y]$ *is rigid* • $\phi(z_i^\rho(x + \rho)) = z_i^\rho(x' + \rho)$ • $\phi(z_i^\rho(y)) = z_j^\rho(y')$ • $z_i^\rho(x + \rho) \leq z_i^\rho(y)$.

Proof. Without loss of generality, let us suppose that $i = 1$ and $j = 2$. From Lemma 3.2.2, we know that $\phi(x + \rho) = x' + \rho$. In particular, we have that $\phi([x + \rho, y]) = [x' + \rho, y']$.

- By Lemma 3.2.8 and the fourth bullet, the interval $[x + \rho, y]$ is rigid.
- By Lemma 3.2.4, we know that

  $$z_1(x) \leq z_1^\rho(x + \rho) \quad \text{and} \quad z_1(x) \not\leq z_2^\rho(x + \rho) \tag{3.10}$$

  Applying ϕ in the inequalities (3.10), we obtain

  $$z_1(x') \leq \phi(z_1^\rho(x + \rho)) \quad \text{and} \quad z_1(x') \not\leq \phi(z_2^\rho(x + \rho)) \tag{3.11}$$

  Since $[x + \rho, y]$ is rigid, Proposition 3.1.19 implies that

  $$\{\phi(z_1^\rho(x + \rho)), \phi(z_2^\rho(x + \rho))\} = \{z_1^\rho(x' + \rho), z_2^\rho(x' + \rho)\}.$$

  Thus, applying Lemma 3.2.4 to $[x', y']$, we obtain that $z_1(x') \not\leq z_2^\rho(x' + \rho)$, so one can not have $\phi(z_1^\rho(x+\rho)) = z_2^\rho(x'+\rho)$, because the first inequality of (3.11) would yield a contradiction. So we conclude that $\phi(z_1^\rho(x + \rho)) = z_1^\rho(x' + \rho)$.

- By Corollary 3.1.9 we have that $z_1(x) \leq z_2(y)$, which implies $z_1(x') \leq z_1(y')$. By the fourth bullet and Lemma 3.2.5, we have that $z_1^\rho(y') = z_1(y')$. By Lemma 3.2.5, we know that $z_1^\rho(y) = z_1(y)$ and $z_2^\rho(y) = z_2(y)$, so $\phi(z_2^\rho(y)) = \phi(z_2(y)) = z_1(y') = z_1^\rho(y')$.

- Suppose now, that $z_1^\rho(x'+\rho) \not\leq z_1(y')$. Applying ϕ^{-1}, we obtain that $z_1^\rho(x+\rho) \not\leq z_2(y)$, which contradicts Corollary 3.1.9. □

Corollary 3.2.10. *Let $\{i, j\} = \{1, 2\}$, and let $\phi\colon [x, y] \to [x', y']$ be a poset isomorphism. The four conditions listed on the left side of Lemma 3.2.9 cannot all hold simultaneously.*

Proof. If these conditions were satisfied simultaneously, then by Lemma 3.2.9 and induction on n, the four conditions would also hold when x is replaced by $x + n\rho$, x' by $x' + n\rho$, and ϕ by its restriction

$$\phi|_{[x+n\rho,y]}\colon [x + n\rho, y] \to [x' + n\rho, y'].$$

However, this leads to a contradiction because for sufficiently large n, $x + n\rho$ will no longer satisfy $x + n\rho \leq y$, so $[x + n\rho, y]$ will not be rigid. □

Lemma 3.2.11. *Let $1 \leq i \leq 2$. Let $\phi\colon [x, y] \to [x', y']$ be a poset isomorphism. If $\phi(z_i(x)) = z_i(x')$ then $\phi(z_i(y)) = z_i(y')$.*

Proof. Without loss of generality, let us suppose that $i = 1$ and $j = 2$. Let us suppose that $\phi(z_1(y)) = z_2(y')$. This implies that $z_1(x) \leq z_1(y)$, because if $z_1(x) \not\leq z_1(y)$, we have that $z_1(x') \not\leq z_2(y')$ which is a contradiction by Corollary 3.1.9. By Corollary 3.2.10, this is not possible, so by Proposition 3.1.19, we conclude that $\phi(z_1(y)) = z_1(y')$. □

Lemma 3.2.12. *Let $\phi\colon [x, y] \to [x', y']$ be a poset isomorphism. There exists a bijection $f\colon \mathrm{V}(\mathrm{Pgn}_{x,y}) \to \mathrm{V}(\mathrm{Pgn}_{x',y'})$ such that $f(x) = x'$, $f(y) = y'$, and f preserves adjacency relations. Moreover, for $i \in \{1, 2\}$, we have $f(v_i(x)) \in \phi(z_i(x))$ and $f(v_i(y)) \in \phi(z_i(y))$.*

Proof. By Proposition 3.1.19, either $\phi(z_1(x))$ is equal to $z_1(x')$ or to $z_2(x')$. Without loss of generality, we consider the first case. Then $\phi(z_2(x)) = z_2(x')$. We define $f\colon \mathrm{V}(\mathrm{Pgn}_{x,y}) \to \mathrm{V}(\mathrm{Pgn}_{x',y'})$ by the formulae[1] $f(x) = x', f(y) = y'$, and $f(v_i(*)) = v_i(*')$. Where $* \in \{x, y\}$ and $i \in \{1, 2\}$. The fact that f preserves the adjacency relations is obvious from the definition of f. That $f(v_i(x)) \in \phi(z_i(x))$ is trivial. Lemma 3.2.11 implies that $\phi(z_1(y)) = z_1(y')$ and this implies by Proposition 3.1.19 that $\phi(z_2(y)) = z_2(y')$. □

[1] If $\phi(z_1(x)) = z_2(x')$ this formula reads $f(v_i(*)) = v_j(*')$ with $\{i, j\} = \{1, 2\}$.

Definition 3.2.13. Let $\phi\colon [x,y] \to [x',y']$ be a poset isomorphism. We denote by $f_\phi\colon \mathrm{V}(\mathrm{Pgn}_{x,y}) \to \mathrm{V}(\mathrm{Pgn}_{x',y'})$ the bijection from Lemma 3.2.12.

Lemma 3.2.14. *Let $\phi\colon [x,y] \to [x',y']$ be a poset isomorphism. If $v_i(*) \in A(z_i(*))$, then $f_\phi(v_i(*)) \in A(\phi(z_i(*)))$ where $A \in \{C, \mathrm{V}, \mathrm{ME}\}$, $* \in \{x,y\}$, and $i \in \{1,2\}$.*

Proof. Let $\{i,j\} = \{1,2\}$. By Lemma 3.2.12, we know that:

(a) $f_\phi(v_i(x)) \in \phi(z_i(x))$ and $f_\phi(v_i(y)) \in \phi(z_i(y))$,

(b) $f_\phi(v_i(x))$ and $f_\phi(v_j(y))$ are adjacent or equal.

By (a), there are $A, B, A', B' \in \{C, \mathrm{V}, \mathrm{ME}\}$ such that $v_i(x) \in A(z_i(x))$, $v_j(y) \in B(z_j(y))$, $f_\phi(v_i(x)) \in A'(\phi(z_i(x)))$ and $f_\phi(v_j(y)) \in B'(\phi(z_j(y)))$. By Proposition 3.1.11, A and B are determined by $\ell(z_i(x), z_j(y))$. By (b), A' and B' are determined by $\ell(\phi(z_i(x)), \phi(z_j(y)))$. By Proposition 1.1.3, it follows that

$$\ell(z_i(x), z_j(y)) = \ell(\phi(z_i(x)), \phi(z_j(y)))$$

so $A = A'$ and $B = B'$. □

Lemma 3.2.15. *Let $\phi\colon [x,y] \to [x',y']$ be a poset isomorphism., we have for $\{i,j\} = \{1,2\}$:*

$$\begin{aligned} d(y, v_j(y)) &= d(y', f_\phi(v_j(y))), \\ d(v_j(y), v_i(x)) &= d(f_\phi(v_j(y)), f_\phi(v_i(x))), \\ d(v_i(x), x) &= d(f_\phi(v_i(x)), x'). \end{aligned}$$

Proof. This is immediate from Corollary 1.2.19, Proposition 3.1.10 and Lemma 3.2.14. □

We are ready to prove the combinatorial nature of $\mathrm{Pgn}_{x,y}$.

Proof of Proposition 3.2.1. From Proposition 3.1.19, we have two cases:

- $\phi(z_1(x)) = z_1(x')$ and $\phi(z_2(x)) = z_2(x')$. By rigidity of $[x',y']$ one has that $z_1(x') \cap z_2(x') = \emptyset$, so $f_\phi(v_1(x)) = v_1(x')$ and $f_\phi(v_2(x)) = v_2(x')$. Since f_ϕ is a bijection which preserves the adjacency relations, and $v_i(x)$ is either adjacent or equal to $v_j(y)$ for $i \neq j$, it follows that $f_\phi(v_1(y)) = v_1(y')$ and $f_\phi(v_2(y)) = v_2(y')$. By Corollary 1.2.19, x and x' both belong either to Θ or to Θ^s. This implies that there exists a weight μ such that $x' = x - \mu$. Note that $\mathrm{V}(\mathrm{Pgn}_{x,y} - \mu) = \mathrm{V}(\mathrm{Pgn}_{x,y}) - \mu$. By Remark 3.1.8 and Lemma 3.2.15, for distinct $i, j \in \{1,2\}$, the relation
$$\mathrm{cen}(x') = \mathrm{cen}(x) - \mu,$$

implies the following:

$$\begin{aligned}
v_i(x') &= \mathrm{cen}(x') + d(v_i(x'), x')\alpha_i/||\alpha_i|| \\
&= \mathrm{cen}(x') + d(f_\phi(v_i(x)), x')\alpha_i/||\alpha_i|| \\
&= \mathrm{cen}(x) - \mu + d(v_i(x), x)\alpha_i/||\alpha_i|| \\
&= v_i(x) - \mu.
\end{aligned}$$

Similarly for $v_j(y')$, we have that

$$\begin{aligned}
v_j(y') &= v_i(x') + d(v_j(y'), v_i(x'))\varpi_i/||\varpi_i|| \\
&= v_i(x') + d(f_\phi(v_j(y)), f_\phi(v_i(x)))\varpi_i/||\varpi_i|| \\
&= v_i(x) - \mu + d(v_j(y), v_i(x))\varpi_i/||\varpi_i|| \\
&= v_j(y) - \mu,
\end{aligned}$$

Finally, for $\mathrm{cen}(y')$, we have that

$$\begin{aligned}
\mathrm{cen}(y') &= v_j(y') + d(y', v_j(y'))\alpha_j/||\alpha_j|| \\
&= v_j(y') + d(y', f_\phi(v_j(y)))\alpha_j/||\alpha_j|| \\
&= v_j(y) - \mu + d(y, v_j(y))\alpha_j/||\alpha_j|| \\
&= \mathrm{cen}(y) - \mu.
\end{aligned}$$

This proves the equality of polygons $\mathrm{Pgn}_{x,y} - \mu = \mathrm{Pgn}_{x',y'}$.

- $\phi(z_1(x)) = z_2(x')$ and $\phi(z_2(x)) = z_1(x')$. Recall that σ is an automorphism of the Bruhat order. In this case, we compose ϕ with the map $z \mapsto \sigma(z)$ to reduce to the previous case. We now explain this in more detail. Obviously, $f_\phi(v_1(x)) = v_2(x')$ and $f_\phi(v_2(x)) = v_1(x')$. By the adjacency assertion in Lemma 3.2.12, we have $f_\phi(v_1(y)) = v_2(y')$ and $f_\phi(v_2(y)) = v_1(y')$.

  The equality of polygons $\sigma\,\mathrm{Pgn}_{x,y} - \mu = \mathrm{Pgn}_{x',y'}$ requires similar calculations (omitted here) to those in the previous case, using $\mu \coloneqq \mathrm{cen}(\sigma x) - \mathrm{cen}(x')$. □

3.2.1 Main result

Let λ be a dominant weight. Let τ_λ be as in Section 2.3.1.

Definition 3.2.16. We say that $[x, y]$ and $[x', y']$ are *comparable* if there are u, v dominant elements and λ, λ' dominant weights such that

- τ_λ induces an isomorphism of posets between $[x, y]$ and $[u, v]$.
- $\tau_{\lambda'}$ induces an isomorphism of posets between $[x', y']$ and $[u, v]$.

The following result is a restatement of Theorem A.

Theorem 3.2.17. *Let $[x, y]$ and $[x', y']$ be rigid intervals. Then, $[x, y]$ and $[x', y']$ are isomorphic as partially ordered sets if and only if there is $g \in \{\mathrm{id}, \sigma\}$ such that $[x, y]$ and $g[x', y']$ are comparable.*

Proof. The sufficient condition follows directly from Lemma 1.2.1, since g is an automorphism of Bruhat order. Thus, it only remains to prove the necessary condition. By Proposition 3.2.1, we have $\mathrm{Pgn}_{x,y} + \mu = \mathrm{Pgn}_{x',y'}$ or $\mathrm{Pgn}_{x,y} + \mu = \sigma(\mathrm{Pgn}_{x',y'})$ for some (not necessarily dominant) weight μ. Suppose that the first equality is satisfied. In this case, we will prove that $[x, y]$ and $[x', y']$ are comparable, i.e., we will take $g = \mathrm{id}$ in the statement of the theorem. By Lemma 3.2.12, we have that $x + \mu = x'$ and $y + \mu = y'$. There are three possibilities:

- $\mathrm{Pgn}_{x,y}$ is a hexagon. By Proposition 2.3.6, $\mu = 0$ which proves this case.
- $\mathrm{Pgn}_{x,y}$ is a pentagon. Let $i \in \{1, 2\}$ be such that $\mathrm{Pgn}_{x,y}$ contains two adjacent vertices lying on $\mathbb{R}_{\geq 0}\varpi_i + \alpha_i$. By Proposition 2.3.7, $\mu \in \mathbb{R}\varpi_i$. We may assume that μ is dominant. If it is not, then $-\mu$ is dominant, in which case we can exchange the roles of $[x, y]$ and $[x', y']$. By Proposition 2.3.32, τ_μ is an isomorphism of the posets $[x, y]$ and $[x + \mu, y + \mu]$.
- $\mathrm{Pgn}_{x,y}$ are parallelograms. As $[x, y]$ and $[x', y']$ are isomorphic, x and x' both belong either to Θ or to Θ^s. This implies that there exist λ, λ' dominant weights, such that $x + \lambda = x' + \lambda'$. This implies that $\mathrm{Pgn}_{x,y} + \lambda = \mathrm{Pgn}_{x',y'} + \lambda'$. Consider $u \coloneqq x + \lambda = x' + \lambda'$ and $v \coloneqq y + \lambda = y' + \lambda'$. By Proposition 2.3.32, we have $[u, v]$ is isomorphic to $[x, y]$ (resp. $[x', y']$) as posets under the map τ_λ (resp. $\tau_{\lambda'}$). So $[x, y]$ and $[x', y']$ are comparable.

Finally, suppose that $\mathrm{Pgn}_{x,y} + \mu = \sigma(\mathrm{Pgn}_{x',y'})$. In this case, we consider $[\sigma(x'), \sigma(y')]$ which is isomorphic to $[x', y']$ as posets. By the way in which the polygons Pgn are constructed, it is not hard to see that $\mathrm{Pgn}_{\sigma(x'),\sigma(y')} = \sigma(\mathrm{Pgn}_{x',y'})$. Now we can use the previous case to get that $[x, y]$ and $[\sigma(x'), \sigma(y')] = \sigma([x', y'])$ are comparable. □

3.3 Further results and conjectures

3.3.1 Classification of lower intervals in $\widetilde{A}_2$

Let W be the Weyl group of affine type A_2. In this section, we classify the lower Bruhat intervals $[\mathrm{id}, y]$ for $y \in W$ up to poset isomorphism.

Theorem 3.3.1. *Let u, v be elements in W. Then $[\mathrm{id}, u] \simeq [\mathrm{id}, v]$ if and only if v belongs to the G-orbit of u.*

We need to deal first with "exceptional" intervals $[\mathrm{id}, y]$ for small values of $\ell(y)$ which cannot be told apart from their cardinalities or $\mathrm{LC}(\mathrm{id}, y)$.

Lemma 3.3.2. *We have* $[\mathrm{id}, u] \not\simeq [\mathrm{id}, v]$ *in the following cases:*

1. $\{u, v\} = \{\theta(0,0,1,1), \mathbf{x}_5\}$
2. $\{u, v\} = \{\theta(1,0,0,1), \mathbf{x}_6\}$
3. $\{u, v\} = \{\theta(0,1,0,1), \mathbf{x}_6\}$
4. $\{u, v\} = \{\theta(1,1,0,0), \mathbf{x}_7\}$

Proof. Let $\phi\colon [\mathrm{id}, u] \to [\mathrm{id}, v]$ be an isomorphism. By Lemma 1.1.5, we have $[a, b] \subset [\mathrm{id}, u]$ is a dihedral subinterval if and only if $\phi([a, b]) \subset [\mathrm{id}, v]$ is a dihedral subinterval.

From Proposition 1.1.4 and Corollary 2.1.5, it is an easy (but tedious task) to see that:

$$\mathscr{D}(u) = \begin{cases} \{312, 321, 323, 313, 121, 123, 213\} & \text{if } u = \theta(0,0,1,1), \\ \{213, 1321, 1213, 2313, 12312, 31323, 31321, \\ 13231, 23123, 23121, 12313, 31231\} & \text{if } u = \mathbf{x}_5. \\ \{312, 1231, 1213, 1323, 2313, 2312, 2132, 3132\} & \text{if } u = \theta(1,0,0,1), \\ \{132, 213, 1231, 3132, 1323, 2312, 2313, 1213, 3123\} & \text{if } u = \mathbf{x}_6, \\ \{321, 2132, 1213, 2313, 1323, 1321, 1231, 3231\} & \text{if } u = \theta(0,1,0,1), \\ \{1213, 3121, 12312, 21323, 12313, 13231, 23132, \\ 23121, 21321, 31321\} & \text{if } u = \theta(1,1,0,0), \\ \{213, 1321, 1213, 2313, 12312, 31323, 31321, \\ 13231, 23123, 23121, 12313, 31231\} & \text{if } u = \mathbf{x}_7. \end{cases}$$

In each case, we observe that $|\mathscr{D}(u)| \neq |\mathscr{D}(v)|$ so $[\mathrm{id}, u] \not\simeq [\mathrm{id}, v]$. □

We recall from Section 2.1.2 that for $m, n \geq 0$ and $i, j \in \{0, 1\}$, we have the notation $L(m, n, i, j) = |[\mathrm{id}, \theta(m, n, i, j)]|$ where $\theta(m, n, i, j) := (s_0)^i \theta(m, n)(s_{m,n})^j$.

Lemma 3.3.3. *If* $[\mathrm{id}, u] \simeq [\mathrm{id}, v]$ *and* $\ell(u) = \ell(v)$ *is odd, then* $u, v \in \Theta$ *or* $u, v \in {}^s\Theta^s$ *or* $u, v \in X^{odd}$.

Proof. By Remark 1.2.11, we have $u, v \in \Theta \uplus {}^s\Theta^s \uplus X^{\mathrm{odd}}$. We will suppose by contrapositive that u, v belong to different regions, then we will prove that $[\mathrm{id}, u] \not\simeq [\mathrm{id}, v]$. Modulo swapping u and v, there are only three possibilities.

- $u \in \Theta$ and $v \in X^{\mathrm{odd}}$. By Remark 1.2.13, we can assume that $u = \theta(m, n, 0, 0)$ and $v = \mathbf{x}_{2(m+n+2)+1}$. By Proposition 2.1.6, we have

  $$L(m, n, 0, 0) = |[\mathrm{id}, \mathbf{x}_{2(m+n+1)+1}]| \implies 6mn - 2m - 2n - 2 = 0.$$

  The unique solution over the non-negative integers of that equation is $(m, n) = (1, 1)$. By Lemma 3.3.2(4), we have $[\mathrm{id}, u] \not\simeq [\mathrm{id}, v]$.

- $u \in {}^s\Theta^s$ and $v \in X^{\text{odd}}$. The proof is similar to the previous case.
- $u \in \Theta$ and $v \in {}^s\Theta^s$. By Remark 1.2.13, we can assume that $u = \theta(m, n, 0, 0)$ and $v = \theta(h, k, 1, 1)$. If $\ell(u) = \ell(v)$ then $m + n = h + k + 1$. If in addition $L(m, n, 0, 0) = L(h, k, 1, 1)$ we have
$$3(mn - hk - h - k) = 2,$$
which does not have solutions in $\mathbb{Z}$, so $[\text{id}, u] \not\simeq [\text{id}, v]$. □

The proof of the following lemma is similar to the previous one, so we will skip it.

Lemma 3.3.4. *If* $[\text{id}, u] \simeq [\text{id}, v]$ *and* $\ell(u)$ *is even, then* $u, v \in \Theta^s$ *or* $u, v \in X^{even}$.

Proof of Theorem 3.3.1. The only if part follows from the definition of the Bruhat order and the group G. Conversely, if $[\text{id}, u] \simeq [\text{id}, v]$, we have $\ell(u) = \ell(v)$. We can suppose that $\ell(u)$ is even, as the proof of the other case is similar. By Remark 1.2.11 and Lemma 3.3.4 either $u, v \in X^{\text{even}}$ or $u, v \in \Theta^s$.

If $u, v \in X^{\text{even}}$, the equality $\ell(u) = \ell(v)$ immediately implies $v \in G \cdot u$.

If $u, v \in \Theta^s$, by Remark 1.2.13, we can assume $u = \theta(m, n, 0, 1)$, $v = \theta(h, k, 0, 1)$. The equality $\ell(u) = \ell(v)$ implies $m + n = h + k$. From $L(m, n, 0, 1) = L(h, k, 0, 1)$ and Proposition 2.1.6, we get $\{m, n\} = \{h, k\}$. Then either $u = \sigma v$ or $u = v$. □

3.3.2 A naive question

It is natural to ask whether the analog of Theorem 3.3.1 holds in general Coxeter groups. With the aid of a computer, we establish the following conjecture for type A affine Weyl groups.

Conjecture 3.3.5. *Let* W *be an affine Weyl group of type* $\widetilde{A}_n$, *and let* G *be the group of automorphisms of the Bruhat ordering of* W. *Consider an element* $w \in W$, *and denote by* $\mathcal{C}\ell_{[\text{id},w]}$ *the class of intervals isomorphic (as partially ordered sets) to* $[\text{id}, w]$. *If* $\ell(w) \geq \frac{n^2+3n-2}{2}$, *then* $|\mathcal{C}\ell_{[\text{id},w]}| = |G \cdot w|$.

The conjecture has been tested in the following cases

- $n = 2$. Theorem 3.3.1.
- $n = 3$. Tested for every w with $\ell(w) \leq 31$.
- $n = 4$. Tested for every w with $\ell(w) \leq 24$.
- $n = 5$. Tested for every w with $\ell(w) \leq 20$.

Due to computational constraints, the conjecture remains unconfirmed for $n \geq 6$.

3.3.3 A conjecture for arbitrary affine Weyl groups

Let $W = \mathbb{Z}\Phi^\vee \rtimes W_f$ be an arbitrary affine Weyl group with root system Φ and finite Weyl group W_f. Let $\Lambda^\vee$ be the coweight lattice, so $W_{\text{ext}} = \Lambda^\vee \rtimes W_f$ is the corresponding extended Weyl group.

Definition 3.3.6. Let $x, x' \in W$. We says that x and x' has the same orientation if there is a coweight $\mu \in \Lambda^\vee$, such that $\mathscr{A}_x + \mu = \mathscr{A}_{x'}$.

Definition 3.3.7. We define

$$G_x^{ori}(x') = \{g \in G \mid gx' \text{ and } x \text{ has the same orientation}\}.$$

Conjecture 3.3.8. *Let x, x', y, y' be elements dominant elements in W satisfying:*

- $|G_x^{ori}(x')| \neq 0$
- $[x, y]$ *and* $[x', y']$ *are non -degenerate and*
- $[x, y] \simeq [x', y']$

Then exist $g \in G_x^{ori}(x, x')$ such that $y + \lambda_{x,gx'} = gy'$, where $\lambda_{x,gx'} = gx' - x$

This conjecture has been tested in the following cases:

- rank$(\Phi) \leq 2$.
  - **Type** A**.** Proved in Theorem 3.2.17.
  - **Type** B**.** Tested for all pairs of isomorphic intervals $[x, y]$ such that $\ell(x, y) \leq 15$ and $0 \leq \ell(x) \leq 14$
- rank$(\Phi) = 3$.
  - **Type** A**.** Tested for all pairs of isomorphic intervals $[x, y]$ such that $\ell(x, y) \leq 10$ and $\ell(y) \leq 20$.

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